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Instantons, special cycles and knot concordance

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We introduce a framework for defining concordance invariants of knots using equivariant singular instanton Floer theory with Chern–Simons filtration. It is demonstrated that many of the concordance invariants defined using instantons in recent years can be recovered from our framework. This relationship allows us to compute Kronheimer and Mrowka's $s^\sharp$-invariant and fractional ideal invariants for two-bridge knots, and more. In particular, we prove a quasiadditivity property of $s^\sharp$, answering a question of Gong. We also introduce invariants that are formally similar to the Heegaard Floer τ-invariant of Ozsváth and Szabó and the ε-invariant of Hom. We provide evidence for a precise relationship between these latter two invariants and the $s^\sharp$-invariant.

Some new topological applications that follow from our techniques are as follows. First, we produce a wide class of patterns whose induced satellite maps on the concordance group have the property that their images have infinite rank, giving a partial answer to a conjecture of Hedden and Pinzón-Caicedo. Second, we produce infinitely many two-bridge knots K which are torsion in the algebraic concordance group and yet have the property that the set of positive $1/n$-surgeries on K is a linearly independent set in the homology cobordism group. Finally, for a knot which is quasipositive and not slice, we prove that any concordance from the knot admits an irreducible SU(2)-representation on the fundamental group of the concordance complement.

While much of the paper focuses on constructions using singular instanton theory with the traceless meridional holonomy condition, we also develop an analogous framework for concordance invariants in the case of arbitrary holonomy parameters, and some applications are given in this setting.

57K18, 57R58

1 Introduction

Homological knot invariants provide many useful tools to study 4-dimensional aspects of knots. For instance, there are several knot homology theories that can be used to prove a conjecture due to Milnor [1968], a variation of which says that the slice genus of any torus knot $T_{p,q}$ is given by

$$g_4(T_{p,q}) = \tfrac{1}{2}(p-1)(q-1).$$

The original proof, due to Kronheimer and Mrowka [1993], used instanton gauge theory with respect to connections on 4-manifolds that are singular along embedded surfaces. Later, alternative proofs were given by Ozsváth and Szabó [2003b] using Heegaard knot Floer homology, and Rasmussen [2010] using Khovanov homology. Motivated by Rasmussen's work, Kronheimer and Mrowka [2013] introduced a concordance invariant $s^\sharp$, further studied by Gong [2021]. Kronheimer and Mrowka's proof of the Milnor conjecture can be recast in terms of $s^\sharp$.

Equivariant singular instanton Floer theory is a package of invariants, studied in [Daemi and Scaduto 2024b; 2024a], which is roughly the S^1-equivariant Morse–Floer theory of a Chern–Simons functional defined on a space of connections which are singular along a knot. As is explained in more detail below, the $s^\sharp$-invariant, and in fact all of the more recent concordance invariants constructed in [Kronheimer and Mrowka 2021b], can be recovered from equivariant singular instanton theory in a systematic way. This new perspective allows us to compute all of these concordance invariants for two-bridge knots, prove a quasiadditivity property of $s^\sharp$ answering a question of Gong [2021], and more.

In this setting, we also introduce concordance invariants $\tilde{s}$ and $\tilde{\varepsilon}$, which can be recovered from $s^\sharp$. We provide evidence that $\tilde{s}$ and $\tilde{\varepsilon}$ are equal to the Heegaard Floer τ-invariant [Ozsváth and Szabó 2003b] and ε-invariant [Hom 2014], respectively. Further incorporating the Chern–Simons filtration structure into the equivariant singular instanton package, we construct a suite of numerical concordance invariants, generalizing the Γ-invariants studied in [Daemi 2020; 2024b] and the r_s-invariants of [Nozaki et al. 2024]. The combination of these techniques leads to new applications involving satellite operations in concordance, the homology cobordism group, and the existence of nonabelian SU(2)-representations for concordance complements.

Concordance invariants from equivariant singular instanton theory

Daemi and Scaduto [2024b] associate, to a knot in the 3-sphere, a certain algebraic object, called an $\mathcal{S}$-*complex*, up to chain homotopy. Let $\mathcal{C}$ be the smooth knot concordance group. Following a general strategy, such as in [Stoffregen 2020], equivariant singular instanton Floer theory gives rise to a homomorphism

$$\mathcal{C} \to \Theta^{\mathcal{S}}_R, \tag{1}$$

where $\Theta^{\mathcal{S}}_R$, an algebraic object, is the *local equivalence group* of $\mathcal{S}$-complexes over the coefficient ring R. To define homology concordance invariants, one may use (1) and then work to algebraically extract information from $\Theta^{\mathcal{S}}_R$. Some progress in this direction was given in [Daemi and Scaduto 2024b; 2024a].

In search of a relationship between these types of invariants and Kronheimer and Mrowka's $s^\sharp$-invariant, we consider (1) in the case that R is the power series ring $\mathbb{Q}[\![\Lambda]\!]$ which appears in the construction of $s^\sharp$ [Kronheimer and Mrowka 2013]. We define several local equivalence invariants in this setting, two of which, when composed with (1), give rise to concordance maps, denoted by

$$\tilde{s}\colon \mathcal{C} \to \mathbb{Z}, \quad \tilde{\varepsilon}\colon \mathcal{C} \to \{-1, 0, 1\}.$$

Theorem 1.1 *The invariant $\tilde{s}$ defines a homomorphism from the smooth knot concordance group $\mathcal{C}$ to $\mathbb{Z}$. For any knot K in the 3-sphere, we have an inequality*

$$\tilde{s}(K) \le g_4(K). \tag{2}$$

This inequality is sharp for any given positive torus knot $T_{p,q}$, in that we have

$$\tilde{s}(T_{p,q}) = \tfrac{1}{2}(p-1)(q-1). \tag{3}$$

Moreover, Kronheimer and Mrowka's $s^\sharp$-invariant is determined by $\tilde{s}$ and $\tilde{\varepsilon}$ as follows:

$$s^\sharp(K) = 2\tilde{s}(K) - \tilde{\varepsilon}(K). \tag{4}$$

As an immediate consequence of Theorem 1.1, we see that $s^\sharp(K)$ factors through the local equivalence construction of (1). This gives an affirmative answer to [Daemi and Scaduto 2024b, Question 8.44]. Theorem 1.1 also implies that $2\tilde{s}(K)$ is a slice-torus invariant in the sense of [Livingston 2004; Lewark 2014]. The results of [Livingston 2004, Proposition 3.3; Lewark 2014, Corollary 5.9.] allow us to compute $\tilde{s}$ for several families of knots.

Corollary 1.2 *If K is a quasipositive knot, then* (2) *is an equality*:

$$\tilde{s}(K) = g_4(K).$$

If K is an alternating knot, then

$$\tilde{s}(K) = -\tfrac{1}{2}\sigma(K),$$

where $\sigma(K)$ denotes the knot signature.

Gong [2021] posed the question of whether there exists a constant C such that

$$|s^\sharp(K \# K') - s^\sharp(K) - s^\sharp(K')| \le C \tag{5}$$

for any knots K and K' in the 3-sphere. Relation (4) and additivity of $\tilde{s}$ can be used to show that (5) holds with $C = 3$. Using our techniques, we obtain the optimal version of quasiadditivity for $s^\sharp$:

Theorem 1.3 *For any pair of knots K and K' in the 3-sphere, we have*

$$|s^\sharp(K \# K') - s^\sharp(K) - s^\sharp(K')| \le 1.$$

Gong [2021] studies a pair of concordance invariants $s^\sharp_+(K)$ and $s^\sharp_-(K)$ which satisfy

$$s^\sharp(K) = s^\sharp_+(K) + s^\sharp_-(K).$$

We also exhibit these as local equivalence invariants, prove connected sum inequalities, and derive slice genus bounds which improve on the ones in [Gong 2021]. See Section 4.2 and Theorem 6.1 for details.

The above properties of the instanton concordance invariants $s^\sharp$, $\tilde{s}$ and $\tilde{\varepsilon}$ are reminiscent of similar properties for certain invariants defined in Heegaard Floer theory. Recall the concordance invariants τ, ν defined by Ozsváth and Szabó [2003b; 2011], and the ε-invariant of Hom [2014].

Conjecture 1.4 *For any knot $K \subset S^3$, we have $s^\sharp_+(K) = \nu(K)$. In particular,*

$$\tilde{s}(K) = \tau(K), \quad s^\sharp(K) = \nu(K) - \nu(K^*), \quad \tilde{\varepsilon}(K) = \varepsilon(K).$$

Here we write K^* for the mirror of a knot K with the reverse orientation. As further evidence towards this conjecture, we have the following result. For an integer homology 3-sphere Y, we write $h(Y) \in \mathbb{Z}$ for the instanton Frøyshov invariant of Y [2002].

Theorem 1.5 *For any knot $K \subset S^3$ with $\tilde{s}(K) > 0$, we have $h(S^3_1(K)) < 0$.*

A folklore conjecture asserts that the Frøyshov instanton invariant h is equal to half the d-invariant of Ozsváth and Szabó [2003a] for integer homology 3-spheres. Given this and Conjecture 1.4, the relation between $\tilde{s}$ and h in Theorem 1.5 corresponds to an analogous relation between τ and d. Indeed, if $\tau(K) > 0$ then $d(S^3_1(K)) < 0$. This latter relation can be understood via the ν^+-invariant [Hom and Wu 2016]; for details, see for example [Sato 2018, Section 2.2].

Recently, Baldwin and Sivek [2021] introduced concordance invariants $\tau^\sharp$, $\nu^\sharp$, $\varepsilon^\sharp$ derived from the behavior of the framed instanton homology groups $I^\sharp(S^3_r(K))$ with respect to the surgery coefficient $r \in \mathbb{Q}$. Perhaps a more accessible variation of Conjecture 1.4 is the following:

Conjecture 1.6 *For any knot $K \subset S^3$, we have*

$$\tilde{s}(K) = \tau^\sharp(K), \quad s^\sharp(K) = \nu^\sharp(K), \quad \tilde{\varepsilon}(K) = \varepsilon^\sharp(K).$$

Further evidence is an analogue of Theorem 1.5 for $\tau^\sharp$ and h which is given in [Baldwin and Sivek 2022, Section 9]. Note that the Baldwin–Sivek invariants are defined in the context of nonsingular, nonequivariant instanton homology for 3-manifolds, and, as such, the manner in which they are constructed is entirely different from the methods of the current paper. We remark that an analogue of τ was also defined using sutured instanton theory by Li [2021], and was shown to agree with $\tau^\sharp$ in [Ghosh et al. 2024].

Kronheimer and Mrowka [2021b] recently introduced new concordance invariants that are constructed using certain versions of singular instanton homology for knots with local coefficients. An example is the fractional ideal invariant $z^\natural(K)$, which is an $\mathscr{S}$-submodule

$$z^\natural(K) \subset \mathrm{Frac}(\mathscr{S}), \quad \mathscr{S} := \boldsymbol{F}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}].$$

They also define numerical invariants $f_\sigma(K)$, derived from such ideal invariants, for a choice of homomorphism σ from $\mathscr{S}$ to a valuation ring. These invariants fit into our framework as well:

Theorem 1.7 *All of the concordance invariants of* [Kronheimer and Mrowka 2021b], *such as* $z^{\natural}(K)$ *and* $f_\sigma(K)$, *factor through the map* (1) *for an appropriate choice of coefficient ring* R. *In other words, they are determined by the local equivalence class of the equivariant singular instanton* $\mathcal{S}$*-complex.*

For more details, see Section 5. It should be noted that, while Kronheimer and Mrowka's invariants are defined using cobordism maps, our characterization of the invariants is entirely in terms of the equivariant singular instanton $\mathcal{S}$-complex of the knot. The same remark holds for $s^\sharp$.

As the equivariant singular instanton theory of two-bridge knots is partially understood (see [Daemi and Scaduto 2024a]), we may use our new perspective on these invariants to carry out computations for this class of knots.

Theorem 1.8 *For a two-bridge knot* K, *the invariant* $s^\sharp(K)$, *as well as the concordance invariants of* [Kronheimer and Mrowka 2021b], *are determined by the knot signature* $\sigma(K)$.

See Section 4.5 for more detailed statements. Note that Kronheimer and Mrowka previously computed the concordance invariants of [loc. cit.] for special families of two-bridge knots. Theorem 1.8 implies:

Corollary 1.9 *Conjecture 1.4 is true for two-bridge knots.*

We also verify Conjecture 1.4 for positive knots; see Corollary 6.2.

All of the invariants discussed thus far, and in fact all constructions in this paper, work more generally for knots in integer homology 3-spheres. In particular, we extend the definitions of $s^\sharp(K)$ and the concordance invariants of [loc. cit.] to knots in integer homology 3-spheres. However, not all of the properties of these invariants given above are established in this more general setting.

One of the main technical (algebraic) tools used in our constructions is that of *special cycles*. The notion of a special cycle can be traced back to Frøyshov's definition [2002] of the h-invariant in the setting of integer homology 3-spheres; see in particular (21)–(22). In this paper we systematically develop special cycles. All of our concordance invariants are obtained by utilizing special cycles in various incarnations of the equivariant singular instanton Floer complex of a knot.

Topological applications

There is an additional layer of structure on the equivariant singular instanton Floer $\mathcal{S}$-complex of a knot which comes in the form of a real-valued filtration, which is roughly obtained by keeping track of the Chern–Simons values of flat singular connections. Incorporating this into the above framework leads to a plethora of numerical concordance invariants, which we describe below. However, for the benefit of the reader, we first describe some of the topological applications that come out of this story.

Satellites and concordance Consider a knot $P \subset S^1 \times D^2$, called a *pattern*. Given a knot K in the 3-sphere, cutting out a solid torus neighborhood of K and gluing back in $(S^1 \times D^2, P)$ gives the *satellite* knot $P(K)$. The assignment $K \mapsto P(K)$ descends to define a map on the smooth concordance group,

$$P : \mathcal{C} \to \mathcal{C}.$$

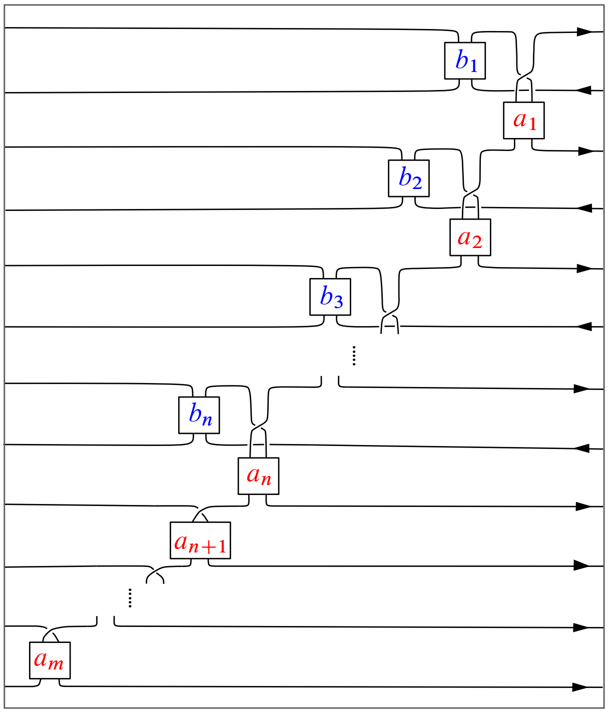

Figure 1: Tangle corresponding to a pattern P determined by $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^n$.

In some cases this is a constant map. For example, if the pattern P is contained in a 3-ball embedded inside $S^1 \times D^2$, then, for all knots K, we have $P(K) = P(U_1)$, where U_1 is the unknot. At the other extreme, Hedden and Pinzón-Caicedo make the following conjecture:

Conjecture 1.10 [Hedden and Pinzón-Caicedo 2021, Conjecture 2] *If for a given pattern P the satellite operator $P : \mathcal{C} \to \mathcal{C}$ is nonconstant, then its image generates an infinite rank subgroup of $\mathcal{C}$.*

The following result, proved by our methods, verifies Conjecture 1.10 for a large class of patterns:

Theorem 1.11 *Suppose a pattern P satisfies the following*:

(i) *There exists a knot K that can be unknotted by a sequence of positive-to-negative crossing changes, and is such that $P(K)$ is concordant to a nonslice quasipositive knot.*

(ii) *$P(U_1)$ is the unknot.*

Then the image of the induced map $P : \mathcal{C} \to \mathcal{C}$ generates an infinite rank subgroup of $\mathcal{C}$.

We prove a more general result; see Theorem 6.7.

As concrete examples, we may consider a pattern P associated to two sequences of negative integers $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^n$, where $m \geq n-1$ and $\max\{m, n\} > 0$, obtained from the tangle diagram in Figure 1 by identifying the vertical edges of the rectangle. Included in this family of patterns are Whitehead doubles, $(m+1, 1)$-cables, and Yasui's pattern $P_{0,k}$ [2015, Figure 10]. We show in Section 6.2 that this general class of patterns satisfies the assumptions of Theorem 1.11.

In fact, any pattern P as in Figure 1 satisfies a stronger version of (i) in Theorem 1.11: P preserves strong quasipositivity, in the sense that, if K is strongly quasipositive, then so too is $P(K)$. Thus, if $P_1, \dots, P_l$ are constructed as in Figure 1, then the composition $P_l \circ \dots \circ P_1$ also satisfies the assumptions of Theorem 1.11. Note that the patterns of Figure 1 have winding number zero when $m = n - 1$.

The above remarks imply that our result is independent of the ones in [Hedden and Pinzón-Caicedo 2021, Theorem 3]. There, Conjecture 1.10 is proved for nonzero winding number patterns, and also for certain patterns satisfying an assumption involving nontriviality of rational linking numbers. By an argument similar to one in [ibid., Section 5.3], if at least one (resp. two) of $P_1, \dots, P_l$ from Figure 1 has winding number zero, then the composition has zero winding number (resp. vanishing rational linking numbers).

For a pattern P as described in Figure 1 (or a composition thereof), we specifically show that

$$\{P(T_{p,q+np})\}_{n=0}^{\infty}$$

is a linearly independent set in $\mathcal{C}$. Here p, q are coprime positive integers. More generally, this is true for any pattern P satisfying Theorem 1.11 where in (i) the knot K can be taken as $T_{p,q}$. In particular, taking P to be an iterated Whitehead double, we obtain the following:

Corollary 1.12 *Let p and q be coprime integers with $p, q > 1$. Then, for any integer $r > 0$, the r^{th} iterated Whitehead doubles $\{\mathrm{Wh}^r(T_{p,q+np})\}_{n=0}^{\infty}$ are linearly independent in $\mathcal{C}$.*

This result generalizes [Nozaki et al. 2024, Theorem 1.12] (the case $r = 1$), which in turn is a generalization of a result in [Hedden and Kirk 2012]. See also [Pinzón-Caicedo 2019; Park 2018] for related results.

Two-bridge knots and homology cobordism The study of the homology cobordism group $\Theta^3_{\mathbb{Z}}$ of integer homology 3-spheres is a central topic in 4-dimensional topology. See [Manolescu 2018] for a survey. The techniques of this paper lead to the following:

Theorem 1.13 *There exist infinitely many two-bridge knots K which are torsion in the algebraic concordance group and for which the set $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ is linearly independent in $\Theta^3_{\mathbb{Z}}$.*

Recall that K is torsion in the algebraic concordance group if and only if the Tristram–Levine signature function of K is identically zero. Write $K(p, q)$ for the two-bridge knot of type (p, q). Our convention is such that $K(3, 1)$ is the right-handed trefoil. Define a family of a two-bridge knots by

$$K_{m,n} := K(212mn - 68n + 53, 106m - 34). \tag{6}$$

We show, more specifically, for integers $m \geq 7$ and $n \geq 0$, that $K_{m,n}$ has vanishing Tristram–Levine signature function, and the set of homology spheres $\{S^3_{1/k}(K^*_{m,n})\}_{k=1}^{\infty}$ is linearly independent in $\Theta^3_{\mathbb{Z}}$.

The simplest example is $K^*_{m,0}$ (independent of m), which is the knot 10^*_{28}. Theorem 1.13 also applies to two-bridge knots outside of the family $K^*_{m,n}$. For example, we show that it is true for

$$K(65,51) = 11a^*_{333}, \quad K(81,52) = 12a^*_{596}.$$

These two examples have the additional feature that the Rokhlin invariants of their $1/n$-surgeries all vanish.

While the methods of this paper, together with previous work, can produce many variations of Theorem 1.13, this particular result is of interest because it seems difficult to prove using other flavors of Floer theory. Moreover, in the context of instanton theory, a result proved in [Nozaki et al. 2024] says that

$$h(S^3_1(K)) < 0 \implies \{S^3_{1/n}(K)\}_{n=1}^{\infty} \text{ is linearly independent in } \Theta^3_{\mathbb{Z}}, \tag{7}$$

where h is the instanton Frøyshov invariant [2002]. However, we expect that all of our examples satisfying Theorem 1.13 have $h(S^3_1(K)) = 0$. We remark that h is conjecturally equal to half the d-invariant of [Ozsváth and Szabó 2003a]. Our results, to the best of our knowledge, give the first examples of knots for which the $1/n$-surgeries for all positive integers n are linearly independent and their d-invariants vanish.

In the other direction, our methods give new information about the instanton Frøyshov invariant h, and, combined with the result (7) from [Nozaki et al. 2024], we can prove the following:

Theorem 1.14 *Let K belong to one of the following two classes of knots*:

(i) *alternating knots with negative signature*;

(ii) *quasipositive knots which are not slice.*

Then $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ is a linearly independent set in the homology cobordism group.

Recently, Baldwin and Sivek [2022] independently gave an alternative proof of Theorem 1.14 using a relationship between $\tau^\sharp$ and the Frøyshov invariant h, and also passing through (7) from [Nozaki et al. 2024]. Our method uses $\tilde{s}$ instead of $\tau^\sharp$. Also note that, when K is a positive torus knot, Theorem 1.14 recovers linear independence results proven in [Furuta 1990; Fintushel and Stern 1992].

In the course of proving the above results, various concordance properties of knots are established. Here is one such result. A knot $K \subset S^3$ is called *H-slice* in a given closed 4-manifold X if there is a properly and smoothly embedded null-homologous disk in $X \setminus \operatorname{int} D^4$ bounded by K.

Theorem 1.15 *Consider the two-bridge knots $K_{m,n}$ defined in* (6). *Let m be a positive integer, and let K be a knot whose concordance class is represented by*

$$[K] = a_1[K_{m,0}] + a_2[K_{m,1}] + \cdots + a_N[K_{m,N}],$$

where a_i are integers and $a_N > 0$. Then K is not smoothly H-slice in any positive definite smooth closed 4-manifold with $b_1 = 0$. For $m \geq 7$, the knot K has vanishing Tristram–Levine signature function.

We remark that Heegaard Floer theory [Ozsváth and Szabó 2003b] and Seiberg–Witten theory [Baraglia 2024] provide several obstructions to H-sliceness in definite 4-manifolds. However, it is known that these obstructions reduce to the classical knot signature for two-bridge knots.

Existence of nonabelian SU(2)-representations There is a strong relationship between signatures of knots and SU(2)-representations of knot complements. For example, suppose the Tristram–Levine signature $\sigma_\omega(K)$ of a knot K is nonzero for some $\omega \in \left(0, \frac{1}{2}\right)$, where $e^{4\pi i \omega}$ is not a root of the Alexander polynomial. It then follows from [Herald 1997b] that there exists a nonabelian (or equivalently an irreducible) SU(2)-representation of $\pi_1(S^3 \setminus K)$ which sends a meridian to the conjugacy class of

$$\begin{bmatrix} e^{2\pi i \omega} & 0 \\ 0 & e^{-2\pi i \omega} \end{bmatrix} \in \mathrm{SU}(2). \tag{8}$$

See also [Herald 1997a; Heusener and Kroll 1998]. The following is a 4-dimensional extension of this result, applying to complements of concordances:

Theorem 1.16 *Let K be a knot with nonvanishing Tristram–Levine signature $\sigma_\omega(K) \neq 0$ for some $\omega \in \left(0, \frac{1}{2}\right)$, where $e^{4\pi i \omega}$ is not a root of the Alexander polynomial. Then, for any smooth knot concordance $S \subset [0,1] \times S^3$ from K to a knot K', there exists a nonabelian* SU(2)*-representation on the fundamental group of the concordance complement which sends meridians to the conjugacy class of* (8).

This generalizes an analogous result involving the ordinary knot signature $\sigma(K) = \sigma_{1/4}(K)$, proved in [Daemi and Scaduto 2024a, Theorem 5]. Versions of Theorem 1.16 for the $\omega = \frac{1}{4}$ case hold with the Tristram–Levine signature replaced by many of the concordance invariants considered in this paper. For example:

Theorem 1.17 *If any of $\tilde{s}(K)$, $s^\sharp(K)$ or $\tilde{\varepsilon}(K)$ is nonzero, then, for any knot concordance $S \subset [0,1] \times S^3$ from K to K', there exists a nonabelian traceless* SU(2)*-representation on the concordance complement.*

In particular, Theorem 1.17 and Corollary 1.2 imply the following:

Corollary 1.18 *Let K be a nonslice quasipositive knot. Then, for any knot concordance $S \subset [0,1] \times S^3$ from K to another knot K', there exists a nonabelian traceless* SU(2)*-representation on the concordance complement. In particular, for any smooth slice disk D of $K \# K^*$ in D^4, the fundamental group of $D^4 \setminus D$ is not isomorphic to $\mathbb{Z}$.*

To see that the second statement follows from the first, combine (D^4, D) with the standard cobordism of pairs (X, F) from $(S^3, K \# K^*)$ to $(S^3, K) \sqcup (S^3, K^*)$ to obtain a knot concordance $S \subset [0,1] \times S^3$ from K to K. Using Seifert–Van Kampen, we obtain an isomorphism $\pi_1(S^3 \setminus K \# K^*) \cong \pi_1(X \setminus F)$ and a surjection from $\pi_1(D^4 \setminus D)$ to $\pi_1([0,1] \times S^3 \setminus S)$ (in fact, this is also an isomorphism). In this way, if $\pi_1([0,1] \times S^3 \setminus S)$ is nonabelian, then $\pi_1(D^4 \setminus D)$ is not isomorphic to $\mathbb{Z}$.

For any topologically slice knot K with trivial Alexander polynomial, Freedman [1982] constructed a locally flat slice disk D bounded by K whose fundamental group of the complement is isomorphic to $\mathbb{Z}$. Corollary 1.18 implies that such a phenomenon fails in the smooth category, ie there is a topologically slice knot K_0 with trivial Alexander polynomial such that, for any smooth slice disk D of K_0, the fundamental group of $D^4 \setminus D$ is not isomorphic to $\mathbb{Z}$. For example, one can take such a K_0 as $\mathrm{Wh}^r(T_{p,q})\#\mathrm{Wh}^r(T_{p,q})^*$ for positive p, q and r. Ruberman [2016], using Taubes's diagonalization theorem [1987], provided examples of doubly slice knots with trivial Alexander polynomial which are not superslice. Our technique provides examples of such knots as well.

Concordance invariants and the Chern–Simons filtration

We now give a brief overview of the invariants that are used to prove the above applications. As already alluded to above, these invariants are obtained by incorporating the Chern–Simons filtration into the strategy, following (1), of extracting concordance invariants by probing the local equivalence group of $\mathcal{S}$-complexes. First we broaden our topological viewpoint.

As already mentioned, all of the constructions in this paper are carried out for knots in integer homology 3-spheres. A (smooth) cobordism of pairs $(W,S)\colon (Y,K)\to(Y',K')$ between knots in integer homology 3-spheres is a *homology concordance* if W is an integer homology cobordism and S is a properly embedded annulus. The collection of knots in integer homology 3-spheres modulo homology concordance gives rise to an abelian group $\Theta^{3,1}_{\mathbb{Z}}$, called the *homology concordance group*. There is a natural homomorphism from the smooth concordance group $\mathcal{C}$ to the group $\Theta^{3,1}_{\mathbb{Z}}$.

Daemi and Scaduto [2024b; 2024a] define and study a homology concordance invariant $\Gamma_{(Y,K)}\colon\mathbb{Z}\to[0,\infty]$, modeled after the homology cobordism invariant Γ_Y from [Daemi 2020]. Homology cobordism invariants $r_s(Y)\in[0,\infty]$, where $s\in[-\infty,0]$, were defined in [Nozaki et al. 2024]. All of these invariants utilize the Chern–Simons filtration on instanton homology. In this paper, we adapt the construction of $r_s(Y)$ to knots and define homology concordance invariants $r_s(Y,K)$. In fact, we define a homology concordance invariant

$$\mathcal{N}_{(Y,K)}\colon \mathbb{Z}\times[-\infty,0)\to[0,\infty] \tag{9}$$

which simultaneously generalizes $\Gamma_{(Y,K)}$ and $r_s(Y,K)$. That is, for certain choices of coefficient rings,

$$\Gamma_{(Y,K)}(k)=\mathcal{N}_{(Y,K)}(k,-\infty),\quad r_s(Y,K)=-\min\bigl\{\inf\{r\in[-\infty,0)\mid \mathcal{N}_{(Y,K)}(0,r)\le -s\},0\bigr\}.$$

The construction of $\mathcal{N}_{(Y,K)}$ roughly uses the equivariant singular instanton $\mathcal{S}$-complex of (Y,K), together with its Chern–Simons filtration structure, and the algebraic machinery of *filtered special cycles*.

All of the properties proved for $r_s(Y)$ in [Nozaki et al. 2024] have analogues for $r_s(Y,K)$. More generally, we show that the invariant $\mathcal{N}_{(Y,K)}$ satisfies certain inequalities with respect to cobordisms. See Theorems 5.47 and 5.50. These inequalities generalize the inequalities of $\Gamma_{(Y,K)}$ given in [Daemi and Scaduto 2024a], and the inequalities for $r_s(Y,K)$, which are analogous to the ones for $r_s(Y)$ in [Nozaki et al. 2024].

The behavior of these invariants with respect to connected sums is also studied:

Theorem 1.19 *Given knots in integer homology* 3*-spheres* (Y,K) *and* (Y',K'), *suppose* $s^{\otimes}<0$, *where* $s^{\otimes}:=\max\{\mathcal{N}_{(Y,K)}(k,s)+s',\mathcal{N}_{(Y',K')}(k',s')+s\}$. *Then*

$$\mathcal{N}_{(Y\#Y',K\#K')}(k+k',s^{\otimes})\le\mathcal{N}_{(Y,K)}(k,s)+\mathcal{N}_{(Y',K')}(k',s').$$

Consequently, *the invariants* Γ *and* r_s *satisfy the inequalities*

$$\Gamma_{(Y\#Y',K\#K')}(k+k')\le\Gamma_{(Y,K)}(k)+\Gamma_{(Y',K')}(k'),$$
$$r_{s+s'}(Y\#Y',K\#K')-s-s'\ge\min\{r_s(Y,K)-s,r_{s'}(Y',K')-s'\}.$$

To make contact with 3-manifold invariants, we have the following result. It is proved using a natural cobordism from $(Y_1(K),U_1)$, which is 1-surgery with an unknot, to the pair (Y,K).

Theorem 1.20 *Let* K *be a knot in an integer homology* 3*-sphere* Y *satisfying* $\sigma(Y,K)\le 0$. *Suppose* $\frac{1}{8}<\Gamma_{(Y,K)}\bigl(-\frac{1}{2}\sigma(Y,K)\bigr)$. *Then* $\Gamma_{Y_1(K)}(0)>0$ *and* $r_0(Y_1(K))<\infty$.

Theorem 1.13 is an application of this result.

The topological applications described earlier in this introduction all make use of $\mathcal{N}_{(Y,K)}$, to varying degrees. We note that $\mathcal{N}_{(Y,K)}$, and thus $\Gamma_{(Y,K)}$ and $r_s(Y,K)$, are defined in the setting of singular instanton theory with holonomy around shrinking meridians of K having trace limiting to zero. The same is true for the invariants $\tilde{s}$, $s^{\sharp}$, $\tilde{\varepsilon}$ and those of [Kronheimer and Mrowka 2021b], discussed above.

More generally, we consider the case in which the meridional holonomies have limiting trace $2\cos(2\pi\omega)$, where $\omega\in\bigl(0,\frac{1}{2}\bigr)$, the traceless case corresponding to $\omega=\frac{1}{4}$. The analytical framework here was established by Kronheimer and Mrowka [1993; 2011b]. Knot invariants defined using singular instanton theory with general holonomy parameter ω were studied in [Echeverria 2019; Imori 2024].

The cases in which $\omega\neq\frac{1}{4}$ are distinguished from the traceless case in that the coefficient ring used must take into account the possibility of infinitely many isolated instantons interpolating between two singular flat connections. In Section 7, we develop the local equivalence story, with Chern–Simons filtration, for arbitrary holonomy parameter $\omega\in\bigl(0,\frac{1}{2}\bigr)$. For example, generalizing (9), we define

$$\mathcal{N}^{\omega}_{(Y,K)}\colon\mathbb{Z}\times[-\infty,0)\to[0,\infty]$$

for each $\omega\in\bigl(0,\frac{1}{2}\bigr)$ such that $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of the knot. This invariant is constant with respect to homology concordances in its domain of definition. Theorem 1.16 is an application of this more general framework.

Organization

In Section 2, we review aspects of equivariant singular instanton theory, following [Daemi and Scaduto 2024b; 2024a]. Furthermore, certain deformed versions of the instanton $\mathcal{S}$-complex, called (*un*)*reduced framed* homology theories, are studied. In Section 3, *special cycles* are introduced and studied. In Section 4, we use the machinery of special cycles to define concordance invariants. In particular, $\tilde{s}$ and $\tilde{\varepsilon}$ are defined, and we prove Theorem 1.1 (recovering $s^{\sharp}$), Corollary 1.2 (computations of $\tilde{s}$) and Theorem 1.3

(quasiadditivity of $s^\sharp$). Here we also prove Theorems 1.7 and 1.8 (two-bridge knot computations). In Section 5, we incorporate the Chern–Simons filtration into our constructions, and study *filtered special cycles*. We introduce $\mathcal{N}_{(Y,K)}$, and prove Theorems 1.5 and 1.19. In Section 6, all of the topological applications are proved, except for Theorem 1.16. In this section, we also prove Theorem 1.20. Finally, in Section 7, the theory for general holonomy parameters is developed, and Theorem 1.16 is proved.

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2 Singular instanton theory

This section provides background for the rest of the paper. We first review relevant constructions and results from [Daemi and Scaduto 2024b; 2024a] on $\mathcal{S}$-complexes for knots derived from equivariant singular instanton Floer theory, with meridional holonomy parameter $\omega = \frac{1}{4}$. (More general holonomy parameters are considered in Section 7.) Next, equivariant complexes from [Daemi and Scaduto 2024b, Section 4] are reviewed. Finally, the *framed* complexes that are central to the concordance invariants defined in Section 4 are introduced and studied.

2.1 $\mathcal{S}$-complexes from singular instantons

We begin by reviewing material from [Daemi and Scaduto 2024b; 2024a].

Definition 2.1 An $\mathcal{S}$*-complex* over a ring R consists of a finitely generated free graded R-module $\widetilde{C}_*$ and endomorphisms $\tilde{d}$ and χ of $\widetilde{C}_*$ such that the following hold:

- (i) $\tilde{d}$ decreases the grading by 1 and χ increases the grading by 1.
- (ii) $\tilde{d}^2 = 0$, $\chi^2 = 0$ and $\chi \circ \tilde{d} + \tilde{d} \circ \chi = 0$.
- (iii) $H(\widetilde{C}_*, \chi) = \text{Ker}\,(\chi)/\text{Im}\,(\chi)$ is isomorphic to $R_{(0)}$, a copy of R in grading 0.

The *differential* of $\widetilde{C}$ is the endomorphism $\tilde{d}$. Our $\mathcal{S}$-complexes will typically be graded by $\mathbb{Z}/4$, at least up until Section 5, where certain types of filtered $\mathcal{S}$-complexes are studied. For a chain complex C_* we typically omit the grading from the subscript and simply write C.

The *trivial $\mathcal{S}$-complex* is given by $\widetilde{C} = R_{(0)}$ with $\tilde{d} = \chi = 0$. Given two $\mathcal{S}$-complexes $(\widetilde{C}, \tilde{d}, \chi)$ and $(\widetilde{C}', \tilde{d}', \chi')$, the tensor product complex is naturally an $\mathcal{S}$-complex,

$$(\widetilde{C}^{\otimes}, \tilde{d}^{\otimes}, \chi^{\otimes}) = (\widetilde{C} \otimes \widetilde{C}', d \otimes 1 + \epsilon \otimes d', \chi \otimes 1 + \epsilon \otimes \chi').$$

Here ϵ is the sign map that multiplies an element in grading i by $(-1)^i$. The dual $\mathcal{S}$-complex $(\widetilde{C}^{\dagger}, \tilde{d}^{\dagger}, \chi^{\dagger})$ is defined by $\widetilde{C}_i^{\dagger} = \mathrm{Hom}(\widetilde{C}_{-i}, R)$, with $\tilde{d}^{\dagger}(f) = -\epsilon(f) \circ \tilde{d}$ and $\chi^{\dagger}(f) = -\epsilon(f) \circ \chi$ for $f \in C^{\dagger}$.

A typical $\mathcal{S}$-complex has the form $\widetilde{C}_* = C_* \oplus C_{*-1} \oplus R_{(0)}$ with differential

$$\tilde{d} = \begin{bmatrix} d & 0 & 0 \\ v & -d & \delta_2 \\ \delta_1 & 0 & 0 \end{bmatrix} \tag{10}$$

and where χ maps C_* isomorphically to the summand C_{*-1}, and is otherwise zero. Every $\mathcal{S}$-complex is isomorphic to one of this form, and most $\mathcal{S}$-complexes that we encounter come with such a decomposition. We refer to a choice of such a decomposition as a *splitting* of the $\mathcal{S}$-complex. The summand $R_{(0)} \subset \widetilde{C}$ will be referred to as the *reducible summand* of the $\mathcal{S}$-complex.

Definition 2.2 Given $\mathcal{S}$-complexes $(\widetilde{C}, \tilde{d}, \chi)$ and $(\widetilde{C}', \tilde{d}', \chi')$, a graded R-module map $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ is an *$\mathcal{S}$-morphism* (or simply *morphism*) if $\widetilde{\lambda}\tilde{d} = \tilde{d}'\widetilde{\lambda}$ and $\widetilde{\lambda}\chi = \chi'\widetilde{\lambda}$. An *$\mathcal{S}$-chain homotopy* $\widetilde{K}$ between morphisms $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ is an R-module map $\widetilde{K}\colon \widetilde{C} \to \widetilde{C}'$ satisfying $\widetilde{\lambda} - \widetilde{\lambda}' = \widetilde{K}\tilde{d} + \tilde{d}'\widetilde{K}$ and $\chi'\widetilde{K} + \widetilde{K}\chi = 0$.

With respect to decompositions $\widetilde{C} = C_* \oplus C_{*-1} \oplus R_{(0)}$ and $\widetilde{C}' = C'_* \oplus C'_{*-1} \oplus R_{(0)}$ of our $\mathcal{S}$-complexes as described above, an $\mathcal{S}$-morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ has the form

$$\widetilde{\lambda} = \begin{bmatrix} \lambda & 0 & 0 \\ \mu & \lambda & \Delta_2 \\ \Delta_1 & 0 & \eta \end{bmatrix}. \tag{11}$$

An $\mathcal{S}$-chain homotopy has a similar shape, but with a sign appearing in the middle entry.

Remark 2.3 The morphisms defined here are more general than those of [Daemi and Scaduto 2024b; 2024a], which require either $\eta = 1$ or that η is invertible. Morphisms of the latter type are *strong height* 0 morphisms, in terminology introduced below.

Let Y be an integer homology 3-sphere, and $K \subset Y$ a knot. In [Daemi and Scaduto 2024b], singular instanton gauge theory is used to construct a $\mathbb{Z}/4$-graded $\mathcal{S}$-complex associated to (Y, K), denoted by

$$\widetilde{C}(Y, K) = C_*(Y, K) \oplus C_{*-1}(Y, K) \oplus R_{(0)}. \tag{12}$$

We proceed to summarize the construction in what follows, ignoring various technicalities.

Choose an orbifold metric on Y which is singular along K with cone angle π. Associated to (Y, K), we have the space $\mathcal{A}(Y, K)$ of singular SU(2) connections on Y that are singular along K. Roughly speaking, any such singular connection is a connection on $Y \setminus K$ which is compatible with a preferred reduction near K and, around shrinking meridians of K, has asymptotic traceless holonomy. There is a Chern–Simons functional

$$\mathrm{cs}\colon \mathcal{A}(Y,K) \to \mathbb{R} \tag{13}$$

which descends to $\mathrm{cs}\colon \mathcal{B}(Y,K) \to \mathbb{R}/\mathbb{Z}$, where $\mathcal{B}(Y,K)$ is the quotient of $\mathcal{A}(Y,K)$ by a group of gauge transformations. The critical set of cs in $\mathcal{B}(Y,K)$ consists of flat singular connections with traceless holonomy around meridians μ of K. As such, $\mathrm{Crit}(\mathrm{cs}) \subset \mathcal{B}(Y,K)$ can be identified with

$$\{\rho \in \mathrm{Hom}(\pi_1(Y \setminus K), \mathrm{SU}(2)) \mid \mathrm{tr}\rho(\mu) = 0\}/\mathrm{SU}(2),$$

where the SU(2)-action is by conjugation. There is a unique reducible representation class in this set which factors through $\pi_1(Y \setminus K) \to H_1(Y \setminus K;\mathbb{Z})$ and sends the generator of $H_1(Y \setminus K;\mathbb{Z})$ to a traceless element. The corresponding reducible connection class in $\mathcal{B}(Y,K)$ is typically denoted by θ.

The $\mathbb{Z}/4$-graded chain complex $(C(Y,K),d)$ is roughly the Morse–Floer complex with respect to the functional cs, having discarded θ. In general, perturbation data π is chosen to ensure that transversality holds. The underlying R-module of $C(Y,K)$ is defined by

$$C(Y,K) = \bigoplus_{\alpha \in \mathfrak{C}^*_\pi(Y,K)} R \cdot \alpha,$$

where $\mathfrak{C}_\pi(Y,K) = \mathfrak{C}^*_\pi(Y,K) \sqcup \{\theta\}$ is the critical set of the cs perturbed by π. The differential d counts (perturbed) instantons on $\mathbb{R} \times Y$ that have the prescribed singularity type along $\mathbb{R} \times K$. The underlying technical work here relies heavily on [Kronheimer and Mrowka 2011b].

Now fix a basepoint $p \in K$ and an orientation of K. Consider the framed configuration space $\widetilde{\mathcal{B}}(Y,K)$ consisting of singular connections that, around a family of meridians of K shrinking around p, oriented compatibly with K, have limiting holonomy exactly equal (and not just conjugate) to the element

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathrm{SU}(2). \tag{14}$$

Then $\widetilde{\mathcal{B}}(Y,K)$ is a principal S^1-bundle over $\mathcal{B}(Y,K)$. The (perturbed) Chern–Simons functional gives an S^1-invariant function on $\widetilde{\mathcal{B}}(Y,K)$, and the $\mathcal{S}$-complex (12) is a model for its Morse–Floer complex with some additional structure induced by the S^1-action. The summand $R_{(0)}$ is generated by the reducible θ. The maps δ_1, δ_2 in (10) involve counting singular instantons on $\mathbb{R} \times Y$ with one reducible limit, and the map v involves cutting down moduli spaces by holonomy along the path $\mathbb{R} \times \{p\}$. For more details, see [Daemi and Scaduto 2024b], where the following is proved:

Theorem 2.4 [Daemi and Scaduto 2024b, Theorem 3.34] *The $\mathcal{S}$-chain homotopy type of the $\mathbb{Z}/4$-graded $\mathcal{S}$-complex $\widetilde{C}(Y,K)$ is an invariant of (Y,K). In particular, it is independent of metric, perturbation and basepoint.*

This invariant is an enhancement of Kronheimer and Mrowka's $I^\natural(Y,K)$ [2011a]:

Theorem 2.5 [Daemi and Scaduto 2024b, Theorem 8.9] *The total homology of $\widetilde{C}(Y, K)$, over $\mathbb{Z}$-coefficients, is naturally isomorphic to the $\mathbb{Z}/4$-graded abelian group $I^\natural(Y, K)$, by an isomorphism of degree $\sigma(Y, K)$* (mod 4).

The $\mathcal{S}$-complex (12) can also be defined using a local coefficient system which is modeled on a construction from [Kronheimer and Mrowka 2011b]. In its simplest form, this is a $\mathbb{Z}/4$-graded $\mathcal{S}$-complex over the ring $\mathbb{Z}[T^{\pm 1}]$,

$$\widetilde{C}(Y, K; \Delta) = C_*(Y, K; \Delta) \oplus C_{*-1}(Y, K; \Delta) \oplus \mathbb{Z}[T^{\pm 1}]_{(0)}.$$

If R is an algebra over $\mathbb{Z}[T^{\pm 1}]$, we write $\widetilde{C}(Y, K; \Delta_R) = \widetilde{C}(Y, K; \Delta) \otimes R$.

We next turn to morphisms induced by cobordisms. It will be important in the sequel to classify the types of morphisms we obtain by how they behave with respect to the reducible summands of the $\mathcal{S}$-complexes. To this end, we introduce the following algebraic definition, which is a minor variation of a definition that appears in [Daemi and Scaduto 2024a]:

Definition 2.6 Let $i \in \mathbb{Z}_{\geq 0}$. Let $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ be a morphism as in (11) and set $c_0 = \eta$ and

$$c_j := \delta_1'(v')^{j-1}\Delta_2(1) + \Delta_1 v^{j-1}\delta_2(1) + \sum_{l=0}^{j-2} \delta_1'(v')^l \mu v^{j-2-l}\delta_2(1) \tag{15}$$

for $j \in \mathbb{Z}_{>0}$. Then $\widetilde{\lambda}$ is a *height i morphism* if it has homological degree $2i$ and satisfies $c_j = 0$ for $j < i$. It is a *strong height i morphism* if, in addition, the element c_i is invertible. A strong height 0 morphism is also called a *local morphism*.

Now consider a cobordism of pairs $(W, S)\colon (Y, K) \to (Y', K')$, where $K \subset Y$ and $K' \subset Y'$ are knots in integer homology 3-spheres. Here W is a cobordism from Y to Y' and S is an orientable surface cobordism from K to K', embedded in W. Let E be a $U(2)$-bundle over W. Then we consider connections on W which are singular along S, with traceless holonomy around shrinking meridians of S. The *topological energy* of such a connection A is given by

$$\kappa(A) = \frac{1}{8\pi^2}\int_{W \setminus S} \operatorname{tr}(F_{\mathrm{ad}(A)} \wedge F_{\mathrm{ad}(A)}),$$

where $F_{\mathrm{ad}(A)}$ is the traceless part of the curvature. We note that the Chern–Simons functional in (13), for $A_0 \in \mathcal{A}(Y, K)$, can be defined by choosing a connection on $[0,1] \times Y$ restricting to A_0 at 0 and $A_1 = \theta$ at 1 and then setting $\mathrm{cs}(A_0) = 2\kappa(A)$.

A cobordism map can be constructed from (W, S, E), under some topological assumptions. One first fixes an orbifold metric on W with cone angle π along the surface S. Roughly, the map counts singular instantons on E, or, more precisely, E with cylindrical ends attached. If $b_1(W) = b^+(W) = 0$, reducible singular instantons on E (for any metric) are in bijection with elements of $H^2(W; \mathbb{Z})$; an instanton A_L compatible with a splitting $E = L \oplus L^* \otimes \det(E)$ is sent to $c_1(L) \in H^2(W; \mathbb{Z})$. Its topological energy is

$$\kappa(A_L) = -\big(c_1(L) + \tfrac{1}{4}S - \tfrac{1}{2}c_1(E)\big)^2. \tag{16}$$

The *index*, $\mathrm{ind}(A)$, of a singular connection A on a cobordism with cylindrical ends refers to the index of the linearized anti-self-dual operator defined with weighted Sobolev spaces that give exponential decay at the ends. For A_L, this formula for the index is

$$\mathrm{ind}(A_L) = 8\kappa(A_L) + \tfrac{3}{2}(\sigma(W) + \chi(W)) + \tfrac{1}{2} S \cdot S + \chi(S) + \sigma(Y, K) - \sigma(Y', K') - 1.$$

Any reducible which has minimal index among all reducibles is called a *minimal reducible*. The minimal topological energy over all reducibles is defined as follows, where $c := c_1(E)$:

$$\kappa_{\min}(W, S, c) := \min\{\kappa(A_L) \mid c_1(L) \in H^2(W; \mathbb{Z})\}.$$

For the following definition, let R be any algebra over the ring $\mathbb{Z}[T^{\pm 1}]$.

Definition 2.7 For a nonnegative integer i, the data (W, S, c), where $(W, S)\colon (Y, K) \to (Y', K')$, is a cobordism of pairs and $c \in H^2(W; \mathbb{Z})$ is *negative definite of height* i if the following hold:

(i) $b_1(W) = b^+(W) = 0$.

(ii) The index of one (and hence all) minimal reducibles is $2i - 1$.

Furthermore, define the following element of R:

$$\eta(W, S, c) := \sum_{A_L \text{ minimal}} (-1)^{c_1(L)^2} T^{(2c_1(L) - c)\cdot S}. \tag{17}$$

If in addition to (i) and (ii), $\eta(W, S, c)$ is invertible in R, then (W, S, c) is of *strong* height i over R.

Proposition 2.8 [Daemi and Scaduto 2024a, Propositions 2.24 and 4.17] *Suppose (W, S, c) as above is negative definite of (strong) height $i \geq 0$. Then there is an associated (strong) height i morphism $\widetilde{C}(Y, K; \Delta_R) \to \widetilde{C}(Y', K'; \Delta_R)$. The term c_i as defined in* (15) *is equal to $\eta(W, S, c)$.*

The construction of the above morphism depends on some auxiliary choices, including a metric and perturbation, but different choices lead to homotopy equivalent morphisms. The construction also depends, a priori, on a choice of path between the basepoints of K and K' used to define the $\mathcal{S}$-complexes, but this is not important in the sequel.

There are also morphisms induced by cobordisms where the surface is immersed, with transverse double points. For simplicity, suppose in what follows that $W : Y \to Y'$ is a homology cobordism, and $S : K \to K'$ is a connected orientable surface with $s_\pm$ many $\pm$-double points, and genus $g(S)$. Then there is an induced morphism of $\mathcal{S}$-complexes $\widetilde{C}(Y, K; \Delta_R) \to \widetilde{C}(Y', K'; \Delta_R)$ with height

$$i := -g(S) + \tfrac{1}{2}\sigma(Y, K) - \tfrac{1}{2}\sigma(Y', K'), \tag{18}$$

assuming $i \geq 0$. Furthermore, for this morphism, the term c_i as given in (15) is, up to a unit, equal to

$$c_i = (T^2 - T^{-2})^{s_+}.$$

See [ibid., Section 4.3] for the construction. This type of morphism is used at several points in the sequel.

If $(W,S)\colon (Y,K)\to(Y',K')$ is a homology concordance, there is a unique minimal reducible of index -1, and we obtain a local morphism $\widetilde{C}(Y,K;\Delta_R)\to\widetilde{C}(Y',K';\Delta_R)$ for any R. By reversing the direction and orientation of this cobordism, we obtain a local morphism $\widetilde{C}(Y',K';\Delta_R)\to\widetilde{C}(Y,K;\Delta_R)$. These observations motivate the following construction.

Let $\widetilde{C}$ and $\widetilde{C}'$ be two $\mathbb{Z}/4$-graded $\mathcal{S}$-complexes over some ring R. We say that $\widetilde{C}$ and $\widetilde{C}'$ are *locally equivalent*, and write $\widetilde{C}\sim\widetilde{C}'$, if there are local morphisms $\widetilde{C}\to\widetilde{C}'$ and $\widetilde{C}'\to\widetilde{C}$. Write

$$\Theta^{\mathcal{S}}_R=\{\mathbb{Z}/4\text{-graded }\mathcal{S}\text{-complexes}\}/{\sim}.$$

Then $\Theta^{\mathcal{S}}_R$ is an abelian group, where the identity element is the trivial $\mathcal{S}$-complex $R_{(0)}$, the group operation is tensor product of $\mathcal{S}$-complexes, and inverses are dual $\mathcal{S}$-complexes. For any algebra R over $\mathbb{Z}[T^{\pm1}]$, the assignment described above, $(Y,K)\mapsto\widetilde{C}(Y,K;\Delta_R)$, induces a group homomorphism

$$\Theta^{3,1}_{\mathbb{Z}}\to\Theta^{\mathcal{S}}_R, \tag{19}$$

where $\Theta^{3,1}_{\mathbb{Z}}$ is the homology concordance group of knots in integer homology 3-spheres.

In light of (19), homology concordance invariants with values in a set R' can be defined, in a purely algebraic manner, by constructing some map $\Theta^{\mathcal{S}}_R\to R'$ (not necessarily a homomorphism), from the local equivalence group to R'. The simplest example, for R an integral domain, is a homomorphism

$$h\colon\Theta^{\mathcal{S}}_R\to\mathbb{Z}, \tag{20}$$

called the *Frøyshov invariant*, essentially introduced in [Frøyshov 2002]. The Frøyshov invariant of an $\mathcal{S}$-complex $\widetilde{C}$ is uniquely determined by the following conditions, where k is a nonnegative integer:

$$h(\widetilde{C})>k \iff \text{there exists } \alpha\in C_* \text{ satisfying } d\alpha=0,\ \delta_1 v^i(\alpha)=0 \text{ for } 0\le i<k, \text{ and } \delta_1 v^k(\alpha)\neq0, \tag{21}$$

$$h(\widetilde{C})\ge -k \iff \text{there exist } a_0,\dots,a_{-k}\in R \text{ satisfying } d\alpha=\sum_{i=0}^{-k} v^i\delta_2(a_i) \text{ and } a_{-k}\neq0. \tag{22}$$

Moreover, if there is a local map $\widetilde{C}\to\widetilde{C}'$, then $h(\widetilde{C})\le h(\widetilde{C}')$. Finally, (20) is an isomorphism if R is a field. Thus the Frøyshov invariant for a general integral domain R factors as $\Theta^{\mathcal{S}}_R\to\Theta^{\mathcal{S}}_{\mathrm{Frac}(R)}\cong\mathbb{Z}$.

The following result computes the Frøyshov invariants for most of the $\mathcal{S}$-complexes used in this paper:

Theorem 2.9 [Daemi and Scaduto 2024a, Theorem 7] *Let K be a null-homotopic knot in an integer homology 3-sphere Y. If $T^4\neq1$ in the $\mathbb{Z}[T^{\pm1}]$-algebra R, then the Frøyshov invariant of $\widetilde{C}(Y,K;\Delta_R)$ is*

$$h(\widetilde{C}(Y,K;\Delta_R))=-\tfrac{1}{2}\sigma(Y,K)+4h(Y), \tag{23}$$

where $\sigma(Y,K)$ is the knot signature and $h(Y)$ is Frøyshov's instanton invariant [2002].

In the sequel, we often write $h(Y,K)$ for the quantity appearing in (23).

2.2 Equivariant complexes

We next review several algebraic constructions that can be applied to $\mathcal{S}$-complexes. We start with *equivariant theories* that one can associate to an $\mathcal{S}$-complex $(\widetilde{C}, \tilde{d}, \chi)$ over a ring R. There are two models for these equivariant theories: *large model* and *small model* [Daemi and Scaduto 2024b]. Each of these models has some advantages over the other one. In the following, $R[\![x^{-1}, x]$ denotes the ring of Laurent power series in the variable x^{-1} and coefficients in R. This ring is an algebra over the polynomial ring $R[x]$ in the obvious way, and the quotient algebra is denoted by $R[\![x^{-1}, x]/R[x]$.

The large equivariant complexes associated to $(\widetilde{C}, \tilde{d}, \chi)$ are given by $(\widehat{\boldsymbol{C}}, \hat{d})$, $(\check{\boldsymbol{C}}, \check{d})$ and $(\overline{\boldsymbol{C}}, \bar{d})$, where

$$\widehat{\boldsymbol{C}} = \widetilde{C} \otimes_R R[x], \quad \check{\boldsymbol{C}} = \widetilde{C} \otimes_R R[\![x^{-1}, x]/R[x], \quad \overline{\boldsymbol{C}} = \widetilde{C} \otimes_R R[\![x^{-1}, x]$$

are $R[x]$-modules and the corresponding differentials are given as

$$\hat{d} = -\tilde{d} \otimes 1 + \chi \otimes x, \quad \check{d} = \tilde{d} \otimes 1 - \chi \otimes x, \quad \bar{d} = -\tilde{d} \otimes 1 + \chi \otimes x.$$

If we equip $\widehat{\boldsymbol{C}}$, $\check{\boldsymbol{C}}$ and $\overline{\boldsymbol{C}}$ with the $\mathbb{Z}/4$-grading induced by that of $\widetilde{C}$ while requiring that x^{2j} has degree $-2j$, then the differentials of these complexes decrease the grading by 1. The subspace of $\widehat{\boldsymbol{C}}$ given by elements with $\mathbb{Z}/4$-grading is denoted by $\widehat{\boldsymbol{C}}_i$, and a similar convention is used for $\check{\boldsymbol{C}}$ and $\overline{\boldsymbol{C}}$. In particular, $\widehat{\boldsymbol{C}}_i$ is a module over $R[x^2]$ and the same comment applies to $\check{\boldsymbol{C}}$ and $\overline{\boldsymbol{C}}$.

We write $H(\widehat{\boldsymbol{C}})$, $H(\check{\boldsymbol{C}})$ and $H(\overline{\boldsymbol{C}})$ for the homology groups of the large equivariant complexes. These homology groups fit into an exact triangle of the form

$$\begin{array}{ccc} H(\check{\boldsymbol{C}}) & \xrightarrow{\quad \boldsymbol{j}_* \quad} & H(\widehat{\boldsymbol{C}}) \\ & {\scriptstyle \boldsymbol{p}_*}\nwarrow \qquad \swarrow{\scriptstyle \boldsymbol{i}_*} & \\ & H(\overline{\boldsymbol{C}}) & \end{array} \tag{24}$$

where the module homomorphisms are induced by the inclusion map $\boldsymbol{i} : \widehat{\boldsymbol{C}} \to \overline{\boldsymbol{C}}$, the map $\boldsymbol{j}$ given as

$$\boldsymbol{j}\left(\sum_{i=-\infty}^{-1} \zeta_i x^i\right) = -\chi(\zeta_{-1}),$$

and the map $\boldsymbol{p}\colon \overline{\boldsymbol{C}} \to \check{\boldsymbol{C}}$ given as the composition of the projection map and the sign map ϵ. Note that $\boldsymbol{i}$ and $\boldsymbol{p}$ are $R[x]$-module homomorphisms while $\boldsymbol{j}$ is only an R-module homomorphism. However, we have

$$x\boldsymbol{j} - \boldsymbol{j}x = \hat{d}K + K\check{d}$$

for the homomorphism $K\colon \check{\boldsymbol{C}} \to \widehat{\boldsymbol{C}}$ that sends an element $\sum_{i=-\infty}^{-1} \zeta_i x^i$ to ζ_{-1}. In particular, the induced map $\boldsymbol{j}_*$ is an $R[x]$-module homomorphism.

Proposition 2.10 *Let R be an integral domain and $n := \mathrm{rank}_R(C_{k-2})$ for $k \in \mathbb{Z}/4$. Then, for any element $\xi \in H_k(\check{\boldsymbol{C}})$, there exists a nonzero polynomial $f(x) \in R[x]$ with $\deg_x(f(x)) \le n$ such that*

$$f(x^2) \cdot \boldsymbol{j}_*(\xi) = 0.$$

In particular, Im $\boldsymbol{j}_*$ *is a torsion submodule of* $H(\widehat{\boldsymbol{C}})$.

Proof Let $\zeta = \sum_{i=-\infty}^{-1} \zeta_i x^i \in \check{\boldsymbol{C}}_k$ with $\zeta_i = (\alpha_i, \beta_i, a_i) \in \widetilde{C}_{k+2i}$ be a representative for ξ. Then

$$\boldsymbol{j}(R \cdot \langle \zeta, x^2\zeta, \dots, x^{2n}\zeta \rangle) = \chi(R \cdot \langle \zeta_{-1}, \zeta_{-3}, \dots, \zeta_{-2n-1} \rangle) = R \cdot \langle \alpha_{-1}, \alpha_{-3}, \dots, \alpha_{-2n-1} \rangle \subset C_{k-2}.$$

Since $n = \operatorname{rank}_R C_{k-2}$, we have a nontrivial linear relation $\sum_{i=0}^{n} b_i \alpha_{-2i-1} = 0$ with $b_i \in R$. Therefore, for the nonzero polynomial $f(x) := \sum_{i=0}^{n} b_i x^i$, we have

$$f(x^2) \cdot \boldsymbol{j}_*(\xi) = [\boldsymbol{j}(f(x^2) \cdot \zeta)] = \left[-\chi\left(\sum_{i=0}^{n} b_i \zeta_{-2i-1}\right)\right] = -\left[\sum_{i=0}^{n} b_i \alpha_{-2i-1}\right] = 0. \qquad \square$$

Small equivariant complexes provide smaller models for the equivariant homology groups of the $\mathcal{S}$-complex $\widetilde{C}$. Pick a splitting $\widetilde{C} = C_* \oplus C_{*-1} \oplus R_{(0)}$ and let d, v, δ_1 and δ_2 be the homomorphisms associated to $\tilde{d}$ with respect to this splitting. The three versions of small equivariant complexes are denoted by $(\widehat{\mathfrak{C}}, \hat{\mathfrak{d}})$, $(\check{\mathfrak{C}}, \check{\mathfrak{d}})$ and $(\bar{\mathfrak{C}}, \bar{\mathfrak{d}})$, where

$$\widehat{\mathfrak{C}} = C_{*-1} \oplus R[x], \quad \check{\mathfrak{C}} = C_* \oplus R[\![x^{-1}, x]/R[x], \quad \bar{\mathfrak{C}} = R[\![x^{-1}, x],$$

with differentials defined by

$$\hat{\mathfrak{d}}\left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \left(d\alpha - \sum_{i=0}^{N} v^i \delta_2(a_i), 0\right), \quad \check{\mathfrak{d}}\left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) = \left(d\alpha, \sum_{i=-\infty}^{-1} \delta_1 v^{-i-1}(\alpha) x^i\right), \quad \bar{\mathfrak{d}} = 0.$$

The $R[x]$-module structures on $\widehat{\mathfrak{C}}$ and $\check{\mathfrak{C}}$ are respectively given by the chain maps

$$x \cdot \left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \left(v\alpha, \delta_1(\alpha) + \sum_{i=0}^{N} a_i x^{i+1}\right), \quad x \cdot \left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) = \left(v\alpha + \delta_2(a_{-1}), \sum_{i=-\infty}^{-2} a_i x^{i+1}\right),$$

whereas the module structure on $\bar{\mathfrak{C}}$ is the obvious one. We equip each of these complexes with a $\mathbb{Z}/4$-grading using the gradings of the summands, where again the degree of x^{2j} is equal to $-2j$.

The analogue of the exact triangle in (24) for the homology groups $H(\widehat{\mathfrak{C}})$, $H(\check{\mathfrak{C}})$ and $H(\bar{\mathfrak{C}})$ is given by

(25)
$$\begin{array}{ccc} H(\check{\mathfrak{C}}) & \xrightarrow{\ \mathfrak{j}_*\ } & H(\widehat{\mathfrak{C}}) \\ & {\scriptstyle \mathfrak{p}_*} \nwarrow \quad \swarrow {\scriptstyle \mathfrak{i}_*} & \\ & H(\bar{\mathfrak{C}}) & \end{array}$$

where the maps are defined at the chain level by

$$\mathfrak{i}\left(\alpha, \sum_{i=0}^{N} a_i x^i\right) := \sum_{i=-\infty}^{-1} \delta_1 v^{-i-1}(\alpha) x^i + \sum_{i=0}^{N} a_i x^i,$$

$$\mathfrak{j}\left(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\right) := (-\alpha, 0),$$

$$\mathfrak{p}\left(\sum_{i=-\infty}^{N} a_i x^i\right) := \left(\sum_{i=0}^{N} v^i \delta_2(a_i), \sum_{i=-\infty}^{-1} a_i x^i\right).$$

The following relation between large and small complexes is proved in [Daemi and Scaduto 2024b, Lemma 4.11]:

Proposition 2.11 *There are R-module homomorphisms*

$$\widehat{\Phi}\colon \widehat{\boldsymbol{C}} \to \widehat{\mathfrak{C}}, \quad \check{\Phi}\colon \check{\boldsymbol{C}} \to \check{\mathfrak{C}}, \quad \overline{\Phi}\colon \overline{\boldsymbol{C}} \to \overline{\mathfrak{C}}$$

that are chain maps and commute with the action of x up to chain homotopy. Moreover, these homomorphisms are chain homotopy equivalences with the chain homotopy inverses

$$\widehat{\Psi}\colon \widehat{\mathfrak{C}} \to \widehat{\boldsymbol{C}}, \quad \check{\Psi}\colon \check{\mathfrak{C}} \to \check{\boldsymbol{C}}, \quad \overline{\Psi}\colon \overline{\mathfrak{C}} \to \overline{\boldsymbol{C}}.$$

In particular, $\widehat{\Phi}_*\colon H(\widehat{\boldsymbol{C}}) \to H(\widehat{\mathfrak{C}})$ is an isomorphism of $R[x]$-modules with the inverse $\widehat{\Psi}_*$, and similar claims hold for $(\check{\Phi}_*, \check{\Psi}_*)$ and $(\overline{\Phi}_*, \overline{\Psi}_*)$. Moreover, these maps define a homomorphism between (24) *and* (25) *commuting with the homomorphisms in the exact triangle.*

Sketch of the proof The splitting $\widetilde{C} = C_* \oplus C_{*-1} \oplus R_{(0)}$ induces a splitting of large equivariant complexes. With respect to these splittings, the maps from the large complexes to the small complexes are

$$\widehat{\Phi}\Bigl(\sum_{i=0}^{N}\alpha_i x^i, \sum_{i=0}^{N}\beta_i x^i, \sum_{i=0}^{N} a_i x^i\Bigr) := \Bigl(\sum_{i=0}^{N} v^i(\beta_i), \sum_{i=0}^{N} a_i x^i + \sum_{i=1}^{N}\sum_{j=0}^{i-1}\delta_1 v^j(\beta_i) x^{i-j-1}\Bigr),$$

$$\check{\Phi}\Bigl(\sum_{i=-\infty}^{-1}\alpha_i x^i, \sum_{i=-\infty}^{-1}\beta_i x^i, \sum_{i=-\infty}^{-1} a_i x^i\Bigr) := \Bigl(\alpha_{-1}, \sum_{i=-\infty}^{-1} a_i x^i + \sum_{i=-\infty}^{-1}\sum_{j=0}^{\infty}\delta_1 v^j(\beta_i) x^{i-j-1}\Bigr),$$

$$\overline{\Phi}\Bigl(\sum_{i=-\infty}^{N}\alpha_i x^i, \sum_{i=-\infty}^{N}\beta_i x^i, \sum_{i=-\infty}^{N} a_i x^i\Bigr) := \sum_{i=-\infty}^{N} a_i x^i + \sum_{i=-\infty}^{N}\sum_{j=0}^{\infty}\delta_1 v^j(\beta_i) x^{i-j-1}.$$

The maps in the reverse direction are given by

$$\widehat{\Psi}\Bigl(\alpha, \sum_{i=0}^{N} a_i x^i\Bigr) := \Bigl(\sum_{i=1}^{N}\sum_{j=0}^{i-1} v^j\delta_2(a_i) x^{i-j-1}, \alpha, \sum_{i=0}^{N} a_i x^i\Bigr),$$

$$\check{\Psi}\Bigl(\alpha, \sum_{i=-\infty}^{-1} a_i x^i\Bigr) := \Bigl(\sum_{i=-\infty}^{-1} v^{-i-1}(\alpha) x^i + \sum_{i=-\infty}^{-1}\sum_{j=0}^{\infty} v^j\delta_2(a_i) x^{i-j-1}, 0, \sum_{i=-\infty}^{-1} a_i x^i\Bigr),$$

$$\overline{\Psi}\Bigl(\sum_{i=-\infty}^{N} a_i x^i\Bigr) := \Bigl(\sum_{i=-\infty}^{N}\sum_{j=0}^{\infty} v^j\delta_2(a_i) x^{i-j-1}, 0, \sum_{i=-\infty}^{N} a_i x^i\Bigr). \qquad \square$$

Remark 2.12 The maps in Proposition 2.11 satisfy a few additional useful properties. The equivariant complexes $\overline{\mathfrak{C}}$ and $\overline{\boldsymbol{C}}$ are $R[\![x^{-1}, x]$-modules in the obvious way, and the maps $\overline{\Phi}$ and $\overline{\Psi}$ are $R[\![x^{-1}, x]$-module homomorphisms. The maps $\widehat{\Phi}$, $\check{\Phi}$ and $\overline{\Phi}$ are in fact left inverses to $\widehat{\Psi}$, $\check{\Psi}$ and $\overline{\Psi}$:

$$\widehat{\Phi}\circ\widehat{\Psi} = 1_{\widehat{\mathfrak{C}}}, \quad \check{\Phi}\circ\check{\Psi} = 1_{\check{\mathfrak{C}}}, \quad \overline{\Phi}\circ\overline{\Psi} = 1_{\overline{\mathfrak{C}}}.$$

We also have the relations

$$\overline{\Phi}\circ \boldsymbol{i} = \mathfrak{i}\circ\widehat{\Phi},\quad \widehat{\Phi}\circ\boldsymbol{j} = \mathfrak{j}\circ\check{\Phi},\quad \widehat{\Psi}\circ\mathfrak{j} = \boldsymbol{j}\circ\check{\Psi},\quad \check{\Psi}\circ\mathfrak{p} = \boldsymbol{p}\circ\overline{\Psi}.$$

Corollary 2.13 *The equivariant homology groups $H(\widehat{\mathfrak{C}})$ and $H(\check{\mathfrak{C}})$ fit into the exact sequences*

$$0 \to \operatorname{Im} \mathfrak{j}_* \hookrightarrow H(\widehat{\mathfrak{C}}) \xrightarrow{\mathfrak{i}_*} \operatorname{Im} \mathfrak{i}_* \to 0, \tag{26}$$

$$0 \to R[\![x^{-1}, x]/\operatorname{Im} \mathfrak{i}_* \xrightarrow{\mathfrak{p}_*} H(\check{\mathfrak{C}}) \xrightarrow{\mathfrak{j}_*} \operatorname{Im} \mathfrak{j}_* \to 0, \tag{27}$$

where $\operatorname{Im}\mathfrak{j}_*$ *is a torsion $R[x]$-submodule of $H(\widehat{\mathfrak{C}})$. If R is an integral domain, then* $\operatorname{Im}\mathfrak{j}_*$ *agrees with the torsion submodule of $H(\widehat{\mathfrak{C}})$. For a field R, the $R[x]$-module* $\operatorname{Im}\mathfrak{i}_*$ *is isomorphic to $R[x]$.*

Proof The short exact sequences (26) and (27) immediately follow from (25). Proposition 2.10 implies that $\operatorname{Im} \mathfrak{j}_*$ is a torsion $R[x]$-submodule of $H(\widehat{\mathfrak{C}})$. In the other direction and if R is an integral domain, then a torsion element of $H(\widehat{\mathfrak{C}})$ is in the kernel of the map $\mathfrak{i}_*$ because $R[\![x^{-1}, x]$ has trivial torsion.

Now let R be a field. The submodule $\operatorname{Im} \mathfrak{i}_*$ contains elements of the form $\sum_{i=0}^{N} a_i x^i$ for which

$$a_0\delta_2(1) + a_1 v\delta_2(1) + \cdots + a_N v^i \delta_2(a_N)$$

is trivial. In particular, $\operatorname{Im} \mathfrak{i}_*$ is not trivial. Since the $\mathcal{S}$-complex $\widetilde{C}$ is finitely generated over R, the submodule $\operatorname{Im} \mathfrak{i}_*$ of $R[\![x^{-1}, x]$ has an element $Q(x)$ with minimal x-degree. It is straightforward to check that any element of $\operatorname{Im} \mathfrak{i}_*$ can be written uniquely as a multiple of $Q(x)$ by some element of $R[x]$. □

Any morphism of $\mathcal{S}$-complexes $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ induces a morphism of equivariant theories. In the case of large equivalent complexes, we have the $R[x]$-module homomorphisms

$$\widehat{\boldsymbol{\lambda}}\colon \widehat{\boldsymbol{C}} \to \widehat{\boldsymbol{C}}', \quad \check{\boldsymbol{\lambda}}\colon \check{\boldsymbol{C}} \to \check{\boldsymbol{C}}', \quad \overline{\boldsymbol{\lambda}}\colon \overline{\boldsymbol{C}} \to \overline{\boldsymbol{C}}',$$

each given by $\widetilde{\lambda}\otimes 1$. These are compatible with (24) in that we have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\boldsymbol{j}} & \widehat{\boldsymbol{C}} & \xrightarrow{\boldsymbol{i}} & \overline{\boldsymbol{C}} & \xrightarrow{\boldsymbol{p}} & \check{\boldsymbol{C}} & \xrightarrow{\boldsymbol{j}} & \cdots \\
 & & \downarrow{\scriptstyle\widehat{\boldsymbol{\lambda}}} & & \downarrow{\scriptstyle\overline{\boldsymbol{\lambda}}} & & \downarrow{\scriptstyle\check{\boldsymbol{\lambda}}} & & \\
\cdots & \xrightarrow{\boldsymbol{j}'} & \widehat{\boldsymbol{C}}' & \xrightarrow{\boldsymbol{i}'} & \overline{\boldsymbol{C}} & \xrightarrow{\boldsymbol{p}'} & \check{\boldsymbol{C}}' & \xrightarrow{\boldsymbol{j}'} & \cdots
\end{array} \tag{28}$$

We also remark that chain homotopic morphisms of $\mathcal{S}$-complexes induce chain homotopic homomorphisms of large equivariant complexes.

Proposition 2.11 can be used to obtain homomorphisms of small equivalent complexes

$$\widehat{\lambda} := \widehat{\Phi}'\circ\widehat{\boldsymbol{\lambda}}\circ\widehat{\Psi}\colon \widehat{\mathfrak{C}} \to \widehat{\mathfrak{C}}', \quad \check{\lambda} := \check{\Phi}'\circ\check{\boldsymbol{\lambda}}\circ\check{\Psi}\colon \check{\mathfrak{C}} \to \check{\mathfrak{C}}', \quad \overline{\lambda} := \overline{\Phi}'\circ\overline{\boldsymbol{\lambda}}\circ\overline{\Psi}\colon \overline{\mathfrak{C}} \to \overline{\mathfrak{C}}'.$$

We may describe the induced map $\overline{\lambda}\colon \overline{\mathfrak{C}} \to \overline{\mathfrak{C}}'$ more explicitly as an endomorphism of $R[\![x^{-1}, x]$ given as multiplication by a Laurent power series of the form

$$c_0 + c_1 x^{-1} + c_2 x^{-2} + c_3 x^{-3} + \cdots, \tag{29}$$

where c_j is as introduced in Definition 2.6. (In the case $c_0 = 1$, this computation is done in [Daemi and Scaduto 2024b, Section 4.2], and the verification in the more general case is similar.) If $\widetilde{\lambda}$ has height i, then (29) has degree at most $-i$, and the equality holds if $\widetilde{\lambda}$ is a strong height i morphism. The commutative diagram in (28) together with Proposition 2.11 gives rise to the following diagram for the small equivariant theories, which is commutative up to chain homotopy:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\mathfrak{j}} & \widehat{\mathfrak{C}}_* & \xrightarrow{\mathfrak{i}} & \overline{\mathfrak{C}}_* & \xrightarrow{\mathfrak{p}} & \check{\mathfrak{C}}_* & \xrightarrow{\mathfrak{j}} & \cdots \\ & & \downarrow{\scriptstyle\hat{\lambda}} & & \downarrow{\scriptstyle\bar{\lambda}} & & \downarrow{\scriptstyle\check{\lambda}} & & \\ \cdots & \xrightarrow{\mathfrak{j}} & \widehat{\mathfrak{C}}'_* & \xrightarrow{\mathfrak{i}} & \overline{\mathfrak{C}}_* & \xrightarrow{\mathfrak{p}} & \check{\mathfrak{C}}'_* & \xrightarrow{\mathfrak{j}} & \cdots \end{array} \tag{30}$$

Next, let $(\widehat{\boldsymbol{C}}^{\otimes}, \overline{\boldsymbol{C}}^{\otimes}, \check{\boldsymbol{C}}^{\otimes})$ be the large equivariant complexes of the $\mathcal{S}$-complex $(\widetilde{C}^{\otimes}, \tilde{d}^{\otimes}, \chi^{\otimes})$ obtained by taking the tensor product of $\mathcal{S}$-complexes $(\widetilde{C}, \tilde{d}, \chi)$ and $(\widetilde{C}', \tilde{d}', \chi')$. The isomorphisms

$$R[x] \otimes_{R[x]} R[x] \cong R[x], \quad R[\![x^{-1}, x] \otimes_{R[\![x^{-1},x]} R[\![x^{-1}, x] \cong R[\![x^{-1}, x]$$

induce the following chain maps, which are module isomorphisms:

$$\widehat{T} \colon \widehat{\boldsymbol{C}} \otimes_{R[x]} \widehat{\boldsymbol{C}}' \to \widehat{\boldsymbol{C}}^{\otimes}, \quad \overline{T} \colon \overline{\boldsymbol{C}} \otimes_{R[\![x^{-1},x]} \overline{\boldsymbol{C}}' \to \overline{\boldsymbol{C}}^{\otimes}.$$

The following claim is essentially proved in the proof of [Daemi and Scaduto 2024b, Lemma 4.27]:

Lemma 2.14 *The composition*

$$\overline{\Phi}^{\otimes} \circ \overline{T} \circ (\overline{\Psi} \otimes_{R[\![x^{-1},x]} \overline{\Psi}') \colon \overline{\mathfrak{C}} \otimes_{R[\![x^{-1},x]} \overline{\mathfrak{C}}' \to \overline{\mathfrak{C}}^{\otimes}$$

is equal to the multiplication map $R[\![x^{-1}, x] \otimes_{R[\![x^{-1},x]} R[\![x^{-1}, x] \to R[\![x^{-1}, x]$.

Proof Noting that the maps $\overline{\Phi}^{\otimes}$, $\overline{\Psi} \otimes_{R[\![x^{-1},x]} \overline{\Psi}'$ and $\overline{T}$ are all $R[\![x^{-1}, x]$-module homomorphisms, it suffices to verify the claim for $1 \otimes 1$:

$$\begin{aligned} \overline{\Phi}^{\otimes} \circ \overline{T} \circ [\overline{\Psi} \otimes_{R[\![x^{-1},x]} \overline{\Psi}'](1 \otimes 1) &= \overline{\Phi}^{\otimes} \circ \overline{T}\left(\left(\sum_{j=0}^{\infty} v^j \delta_2(1) x^{-j-1}, 0, 1\right) \otimes \left(\sum_{j=0}^{\infty} v^j \delta_2(1) x^{-j-1}, 0, 1\right)\right) \\ &= \overline{\Phi}^{\otimes}(A, 0, 1) = 1 \end{aligned}$$

for some chain $A \in C_*^{\otimes} \otimes R[\![x^{-1}, x]$. □

The following proposition provides a naturality result for the maps $\widehat{T}$ and $\overline{T}$:

Proposition 2.15 *The diagram of* $R[x]$*-modules*

$$\begin{array}{ccccc} H(\widehat{\boldsymbol{C}}) \otimes_{R[x]} H(\widehat{\boldsymbol{C}}') & \xrightarrow{i_* \otimes i'_*} & H(\overline{\boldsymbol{C}}) \otimes_{R[\![x^{-1},x]} H(\overline{\boldsymbol{C}}') & \xrightarrow{\overline{\Phi}_* \otimes_{R[\![x^{-1},x]} \overline{\Phi}'_*} & \overline{\mathfrak{C}} \otimes_{R[\![x^{-1},x]} \overline{\mathfrak{C}}' \\ \downarrow & & \downarrow & & \downarrow \\ H(\widehat{\boldsymbol{C}}^{\otimes}) & \xrightarrow{i_*^{\otimes}} & H(\overline{\boldsymbol{C}}^{\otimes}) & \xrightarrow{\overline{\Phi}_*^{\otimes}} & \overline{\mathfrak{C}}^{\otimes} \end{array}$$

is commutative, where the vertical maps are respectively induced by $\widehat{T}$, $\overline{T}$ *and multiplication.*

Proof The commutativity of the left square follows readily from the relation

$$\overline{T}\circ(\boldsymbol{i}\otimes\boldsymbol{i}')=\boldsymbol{i}^{\otimes}\circ\widehat{T}. \tag{31}$$

The commutativity of the right square follows from Lemma 2.14 and the fact that $\overline{\Psi}_*$ and $\overline{\Psi}'_*$ are the inverses of $\overline{\Phi}_*$ and $\overline{\Phi}'_*$, respectively. □

Let S be an algebra over R. Define the $\mathcal{S}$-complex $\widetilde{C}^S$ over S as the base change $\widetilde{C}\otimes_R S$. Associated to $\widetilde{C}^S$ are equivariant complexes $(\widehat{\boldsymbol{C}}^S,\overline{\boldsymbol{C}}^S,\widecheck{\boldsymbol{C}}^S)$ and $(\widehat{\mathfrak{C}}^S,\overline{\mathfrak{C}}^S,\widecheck{\mathfrak{C}}^S)$. We have natural isomorphisms

$$\widehat{\boldsymbol{C}}^S\cong\widehat{\boldsymbol{C}}\otimes_R S,$$

and similar isomorphisms hold for the other versions of equivariant theories. The chain maps in the exact triangles (24)–(25) and the homomorphisms in the proof of Proposition 2.11 commute with these isomorphisms. A morphism of $\mathcal{S}$-complexes $\widetilde{\lambda}\colon\widetilde{C}\to\widetilde{C}'$ induces a morphism $\widetilde{\lambda}^S\colon\widetilde{C}^S\to\widetilde{C}'^S$ of the base changes. This in turn induces morphisms of equivariant theories. For instance, we have $\widehat{\boldsymbol{\lambda}}^S\colon\widehat{\boldsymbol{C}}^S\to\widehat{\boldsymbol{C}}'^S$ that is equal to $\widetilde{\lambda}^S\otimes_S 1_{R[x]}=\widehat{\boldsymbol{\lambda}}\otimes_R 1_S$.

The constructions of equivariant theories from above can be applied to the $\mathbb{Z}/4$-graded $\mathcal{S}$-complex $\widetilde{C}(Y,K;\Delta_R)$ of a knot K in an integer homology sphere Y, where R is any algebra over $\mathbb{Z}[T^{\pm1}]$. In particular, after using the package of either large or small equivariant complexes and then passing to homology, we obtain equivariant instanton knot homology groups that fit into an exact triangle

$$\begin{array}{ccc} \widecheck{I}(Y,K;\Delta_R) & \xrightarrow{\;j_*\;} & \widehat{I}(Y,K;\Delta_R) \\ & {\scriptstyle p_*}\nwarrow \quad \swarrow{\scriptstyle i_*} & \\ & \overline{I}(Y,K;\Delta_R) & \end{array} \tag{32}$$

Moreover, $\overline{I}(Y,K)\cong R[\![x^{-1},x]\!]$. The exact triangle in (32) is functorial with respect to negative definite cobordisms of pairs. That is to say, if the cobordism of pairs $(W,S)\colon(Y,K)\to(Y',K')$ together with $c\in H^2(W;\mathbb{Z})$ is negative definite of height $i\geq 0$, then there is a cobordism map

$$\widehat{\lambda}_{(W,S,c)}\colon\widehat{I}(Y,K;\Delta_R)\to\widehat{I}(Y',K';\Delta_R),$$

and similarly $\widecheck{\lambda}_{(W,S,c)}$, $\overline{\lambda}_{(W,S,c)}$. These maps commute with the maps in (32). Furthermore, after identification of $\overline{I}(Y,K;\Delta_R)$ and $\overline{I}(Y',K';\Delta_R)$ with $R[\![x^{-1},x]\!]$ using the small model, the map $\overline{\lambda}_{(W,S,c)}$ is multiplication by an expression of the form (29), where $c_j=0$ for $j<i$ and c_i is given in (17).

2.3 Deformed complexes and framed singular instanton homology

The equivariant complex $\widehat{\boldsymbol{C}}$ of an $\mathcal{S}$-complex $\widetilde{C}$ can be regarded as a deformation of the chain complex $(\widetilde{C},-\tilde{d})$ after applying the base change of $R[x]$ to R, by evaluation of x at 1. This definition of equivariant complexes from this viewpoint can be generalized in the following way. Suppose $\varphi\colon R[x]\to\boldsymbol{S}$ is a ring homomorphism. In particular, $\boldsymbol{S}$ can be regarded as an algebra over R. We define the *φ-deformed complex* associated to $\widetilde{C}$ by

$$\widetilde{C}^{\varphi} := \widetilde{C} \otimes_R \boldsymbol{S}, \quad \tilde{d}^{\varphi} := \tilde{d} \otimes 1 + \chi \otimes \varphi(x).$$

In the case that $\widetilde{C}$ is the $\mathcal{S}$-complex of (Y, K), the homology of the φ-deformed complex $(\widetilde{C}^{\varphi}(Y, K), \tilde{d}^{\varphi})$ is denoted by $\widetilde{I}^{\varphi}(Y, K)$. Previously, several knot invariants under the general name of *singular instanton Floer homology* were constructed by Kronheimer and Mrowka [2011a; 2013; 2021a; 2021b]. It is shown in [Daemi and Scaduto 2024b, Section 8] that any of these knot invariants can be characterized as $\widetilde{I}^{\varphi}(Y, K)$ for an appropriate choice of φ.

Suppose R is an algebra over $\mathbb{Q}[T^{\pm 1}]$ and $\Lambda := T - T^{-1} \in R$. For an $\mathcal{S}$-complex $\widetilde{C}$ over R, we can associate the *unreduced framed complex* $C^{\#}$ defined by

$$C^{\#} := \widetilde{C}_{*} \oplus \widetilde{C}_{*+2}, \quad d^{\#} := \begin{bmatrix} \tilde{d} & 2\Lambda^2\chi \\ 2\chi & \tilde{d} \end{bmatrix}.$$

We write $(C^{\#}_{*}(Y, K; \Delta_R), d^{\#})$ for the unreduced framed complex of the $\mathcal{S}$-complex $\widetilde{C}(Y, K; \Delta_R)$. The motivation to consider unreduced framed complexes comes from the following result:

Theorem 2.16 [Daemi and Scaduto 2024b, Theorem 8.20] *The total homology of $(C^{\#}_{*}(Y, K; \Delta_R), d^{\#})$, defined with coefficients $R = \mathbb{Q}[T^{\pm 1}]$, is naturally isomorphic to Kronheimer and Mrowka's singular instanton homology $I^{\#}(Y, K)$* [2013], *by an isomorphism of degree $\sigma(Y, K) + 1 \pmod 4$.*

When comparing the notation in this paper to [Kronheimer and Mrowka 2013], the reader should note that u and λ in [loc. cit.] are equal to T^2 and Λ in our notation. The degree of the isomorphism in the theorem comes from comparing the grading conventions in [Daemi and Scaduto 2024b] and [Kronheimer and Mrowka 2013]. We may also let $R = \mathbb{Q}$ be the algebra over $\mathbb{Q}[T^{\pm 1}]$ where we set $T = 1$. Then it is shown in [Daemi and Scaduto 2024b, Theorem 8.13] that the homology of $(C^{\#}_{*}(Y, K; \Delta_R), d^{\#})$ agrees with the flavor of $I^{\#}(Y, K)$ defined in [Kronheimer and Mrowka 2011a].

The unreduced framed complex can be identified as a deformed complex. Suppose R is an algebra over $\mathbb{Q}[T^{\pm 1}]$, $\Lambda := T - T^{-1} \in R$ and $\boldsymbol{S}$ is the quotient of $R[x]$ by the ideal generated by $x^2 - 4\Lambda^2$. Let $\varphi\colon R[x] \to \boldsymbol{S}$ be the quotient map. Then $C^{\#}$ is the φ-deformed complex associated to $\widetilde{C}$. More concretely, we identify $(\zeta_1, \zeta_2) \in C^{\#}$ with $\zeta_1 + \frac{1}{2}x\zeta_2 \in \widetilde{C}^{\varphi}$. We may also identify $(C^{\#}, d^{\#})$ with $(\hat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}, -\hat{d} \otimes 1_{\boldsymbol{S}})$ using the isomorphism that sends $(\zeta_1, \zeta_2) \in C^{\#}$ to $\zeta_1 - \frac{1}{2}x\zeta_2$. Composing the inverse of this isomorphism and the sign map ϵ gives an isomorphism $\boldsymbol{f}^{\#}\colon (\hat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}, \hat{d} \otimes 1_{\boldsymbol{S}}) \to (C^{\#}, d^{\#})$ of chain complexes. If we define the $R[x]$-module structure on $(C^{\#}, d^{\#})$ by

$$x \cdot (\zeta_1, \zeta_2) = (-2\Lambda^2\zeta_2, -2\zeta_1), \tag{33}$$

then $\boldsymbol{f}^{\#}$ is an $R[x]$-module homomorphism. We have the exact sequence of $R[x]$-modules

$$\cdots \to H(\hat{\boldsymbol{C}}) \xrightarrow{x^2 - 4\Lambda^2} H(\hat{\boldsymbol{C}}) \xrightarrow{\boldsymbol{q}^{\#}_{*}} H(C^{\#}) \to H(\hat{\boldsymbol{C}}) \xrightarrow{x^2 - 4\Lambda^2} \cdots, \tag{34}$$

where $\boldsymbol{q}^{\#}_{*}$ is induced by composing the quotient map with the chain isomorphism $\boldsymbol{f}^{\#}$.

Remark 2.17 For any knot K in S^3, the singular instanton Floer homology $I^\sharp(S^3,K)$ has rank 1 over the ring $\mathbb{Q}[T^{\pm 1}]$ in degrees 0 and 2 and rank 0 in degrees 1 and 3 [Kronheimer and Mrowka 2013]. This property is special to the unreduced framed homology of $\mathcal{S}$-complexes that are given by classical knots, and it does not hold for an arbitrary $\mathcal{S}$-complex.

A morphism $\tilde{\lambda}\colon \tilde{C} \to \tilde{C}'$ with homological degree $2i$ induces a chain map $\lambda^\sharp\colon C^\sharp \to C'^\sharp$,

$$\lambda^\sharp := \begin{bmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\lambda} \end{bmatrix},$$

which has the same homological degree as $2i$.

Proposition 2.18 *For any morphism of $\mathcal{S}$-complexes $\tilde{\lambda}\colon \tilde{C} \to \tilde{C}'$ as above, we have the commutative diagram of $R[x]$-modules*

$$\begin{array}{ccc} H_*(\hat{\boldsymbol{C}}) & \xrightarrow{\ \boldsymbol{q}^\sharp\ } & H_*(C^\sharp) \\ \big\downarrow{\scriptstyle \hat{\boldsymbol{\lambda}}_*} & & \big\downarrow{\scriptstyle \lambda^\sharp_*} \\ H_{*+2i}(\hat{\boldsymbol{C}}') & \xrightarrow{\ \boldsymbol{q}'^\sharp\ } & H_{*+2i}(C'^\sharp) \end{array}$$

Next, we discuss unreduced framed complexes for tensor products. Let $(\tilde{C}^\otimes, \tilde{d}^\otimes, \chi^\otimes)$ be the tensor product of $\mathcal{S}$-complexes $(\tilde{C}, \tilde{d}, \chi)$ and $(\tilde{C}', \tilde{d}', \chi')$. Let $(\hat{\boldsymbol{C}}^\otimes, \overline{\boldsymbol{C}}^\otimes, \check{\boldsymbol{C}}^\otimes)$ and $C^{\otimes\sharp}$ be the large equivariant complexes and the unreduced framed complex of $\tilde{C}^\otimes$, respectively. Then we have canonical chain isomorphisms

$$C^\sharp \otimes_{R[x]} C'^\sharp \cong (\hat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}) \otimes_{R[x]} (\hat{\boldsymbol{C}}' \otimes_{R[x]} \boldsymbol{S}) \cong \hat{\boldsymbol{C}}^\otimes \otimes_{R[x]} \boldsymbol{S} \cong C^{\otimes\sharp}.$$

We denote by $T^\sharp\colon C^\sharp \otimes_{R[x]} C'^\sharp \to C^{\otimes\sharp}$ the resulting isomorphism. By the definition of $T^\sharp$ and arguments in Section 2.2, the following proposition holds:

Proposition 2.19 *We have the commutative diagram of $R[x]$-modules*

$$\begin{array}{ccccc} H(\hat{\boldsymbol{C}}) \otimes_{R[x]} H(\hat{\boldsymbol{C}}') & \longrightarrow & H(\hat{\boldsymbol{C}} \otimes_{R[x]} \hat{\boldsymbol{C}}') & \xrightarrow{\ \hat{T}_*\ } & H(\hat{\boldsymbol{C}}^\otimes) \\ \big\downarrow{\scriptstyle \boldsymbol{q}^\sharp_* \otimes_{R[x]} \boldsymbol{q}'^\sharp_*} & & \big\downarrow{\scriptstyle (\boldsymbol{q}^\sharp \otimes_{R[x]} \boldsymbol{q}'^\sharp)_*} & & \big\downarrow{\scriptstyle \boldsymbol{q}^{\otimes\sharp}_*} \\ H(C^\sharp) \otimes_{R[x]} H(C'^\sharp) & \longrightarrow & H(C^\sharp \otimes_{R[x]} C'^\sharp) & \xrightarrow{\ T^\sharp_*\ } & H(C^{\otimes\sharp}) \end{array}$$

Finally, we discuss the duality of $C^\sharp$. We begin with a few remarks about dual complexes. Let (V_*, d) be a $\mathbb{Z}/4$-graded chain complex freely generated over R, and $(V_*^\dagger, d^\dagger)$ be its dual complex. This is defined as follows: for any i, we have $V_i^\dagger = \mathrm{Hom}(V_{-i}, R)$, and for any $f \in V^\dagger$, we have $d^\dagger(f) := -\epsilon(f) \circ d$, where ϵ is the sign map. (Note that the dual of an $\mathcal{S}$-complex from Section 2.1 is defined using a similar convention.) We have a natural pairing $\langle \cdot , \cdot \rangle\colon V_{-i}^\dagger \otimes V_i \to R$ given as

$$\langle f, v \rangle \mapsto f(v). \tag{35}$$

The sign convention above is chosen so that if we equip $V_*^\dagger \otimes V_*$ with the differential $d^\dagger \otimes 1 + \epsilon \otimes d$, then (35) determines a chain map. In particular, if R is an integral domain, we obtain the nondegenerate *evaluation* bilinear form $\langle \cdot, \cdot \rangle \colon (H(V_*^\dagger)/\mathrm{Tor}_R) \times (H(V_*)/\mathrm{Tor}_R) \to R$.

Let $(C^{\dagger\sharp}, d^{\dagger\sharp})$ denote the unreduced framed complex of $\widetilde{C}^\dagger$. Note that

$$C_i^{\dagger\sharp} = \widetilde{C}_i^\dagger \oplus \widetilde{C}_{i+2}^\dagger = \mathrm{Hom}_R(\widetilde{C}_{-i}, R) \oplus \mathrm{Hom}_R(\widetilde{C}_{-i-2}, R). \tag{36}$$

Similarly, denote by $(C^{\sharp\dagger}, d^{\sharp\dagger})$ the dual complex of $C^\sharp$. Then

$$C_i^{\sharp\dagger} = \mathrm{Hom}_R(C_{-i}^\sharp, R) = \mathrm{Hom}_R(\widetilde{C}_{-i}, R) \oplus \mathrm{Hom}_R(\widetilde{C}_{-i+2}, R). \tag{37}$$

Proposition 2.20 *The map $C_*^{\dagger\sharp} \to C_{*+2}^{\sharp\dagger}$, defined with respect to the splittings in* (36) *and* (37) *by*

$$(\varphi, \psi) \mapsto (\psi, \varphi),$$

is a chain isomorphism. In particular, for any integral domain R, we have the isomorphism

$$H_*(C^{\dagger\sharp})/\mathrm{Tor}_R \cong \mathrm{Hom}_R(H_{-*-2}(C^\sharp), R).$$

The *reduced framed complexes* of the $\mathcal{S}$-complex $\widetilde{C}$ are given by the $\mathbb{Z}/2$-graded deformed complexes

$$(\widetilde{C}^\pm, \tilde{d}^\pm) := (\widetilde{C}, \tilde{d} \pm 2\Lambda\chi).$$

Since $\widetilde{C}$ is $\mathbb{Z}/4$-graded, and $\tilde{d}$ and χ have, respectively, degrees -1 and 1, the homology groups $H(\widetilde{C}^+, \tilde{d}^+)$ and $H(\widetilde{C}^-, \tilde{d}^-)$ are isomorphic. The reduced theories are related to the unreduced framed complex $C^\sharp$ in the following way. First define R-module homomorphisms

$$\iota_\pm \colon \widetilde{C}^\pm \to C_*^\sharp, \quad \iota_\pm := \begin{bmatrix} \Lambda \\ \pm 1 \end{bmatrix}, \tag{38}$$

$$\pi_\pm \colon C^\sharp \to \widetilde{C}^\pm, \quad \pi_\pm := \begin{bmatrix} 1 & \pm\Lambda \end{bmatrix}. \tag{39}$$

These are chain maps, and we have a short exact sequence

$$0 \to \widetilde{C}^- \xrightarrow{\iota_-} C^\sharp \xrightarrow{\pi_+} \widetilde{C}^+ \to 0 \tag{40}$$

of $\mathbb{Z}/2$-graded chain complexes over R. There is a similar exact sequence with $+$ and $-$ interchanged. Note that the maps $\iota_\pm$ and $\pi_\pm$ are $R[x]$-module homomorphisms, where the action of x on $\widetilde{C}^\pm$ is set to be multiplication by $\mp 2\Lambda$. Furthermore, $\pi_+ \circ \iota_+ = 2\Lambda$, and hence the exact sequence (40) splits if the element Λ is invertible in R.

Similar to $C^\sharp$, we may identify $\widetilde{C}^\pm$ with appropriate base changes of the equivariant complex $(\widehat{\boldsymbol{C}}, \hat{d})$. Suppose $\boldsymbol{S}^\pm$ is the ring R regarded as an algebra over $R[x]$, where the action of x is given by multiplication by $\mp 2\Lambda$. Then $(\widetilde{C}^\pm, \tilde{d}^\pm)$ is naturally isomorphic to $(\widehat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}^\pm, -\hat{d} \otimes 1_{\boldsymbol{S}^\pm})$. Composing the inverse of this isomorphism and the sign map determines a chain isomorphism $\boldsymbol{f}^\pm \colon (\widehat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}^\pm, \hat{d} \otimes 1_{\boldsymbol{S}^\pm}) \to (\widetilde{C}^\pm, \tilde{d}^\pm)$. From this description, we obtain the long exact sequence of $R[x]$-modules

$$\cdots \to H(\widehat{\boldsymbol{C}}) \xrightarrow{x \pm 2\Lambda} H(\widehat{\boldsymbol{C}}) \xrightarrow{\tilde{\boldsymbol{q}}_*^\pm} H(\widetilde{C}_*^\pm) \to H(\widehat{\boldsymbol{C}}) \xrightarrow{x \pm 2\Lambda} \cdots. \tag{41}$$

Here $\tilde{\boldsymbol{q}}_*^\pm$ is induced by composing the quotient map with the chain isomorphism $\boldsymbol{f}^\pm$.

Remark 2.21 For any knot K in S^3, the singular instanton Floer homology $\widetilde{I}^{\pm}(S^3,K) := H(\widetilde{C}^{\pm})$ associated to $\widetilde{C}(S^3,K;\Delta_{\mathbb{Q}[T^{\pm 1}]})$, defined over the ring $\mathbb{Q}[T^{\pm 1}]$, has rank 1 in degree 0 and rank 0 in degree 1. This follows from Remark 2.17 and the exact sequence induced by (40).

For the remainder of this section, we focus on basic properties of $\widetilde{C}^+$. Similar arguments hold for $\widetilde{C}^-$. We start with the counterpart of Proposition 2.18.

Proposition 2.22 *For any morphism of $\mathcal{S}$-complexes $\tilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ of even degree, we have the commutative diagram of $\mathbb{Z}/2$-graded $R[x]$-modules*

$$\begin{array}{ccc} H_*(\widehat{\boldsymbol{C}}) & \xrightarrow{\tilde{\boldsymbol{q}}_*^+} & H(\widetilde{C}^+) \\ \big\downarrow{\scriptstyle \hat{\boldsymbol{\lambda}}_*} & & \big\downarrow{\scriptstyle \tilde{\lambda}_*^+} \\ H(\widehat{\boldsymbol{C}}') & \xrightarrow{\tilde{\boldsymbol{q}}_*'^+} & H(\widetilde{C}'^+) \end{array}$$

where $\tilde{\lambda}^+\colon \widetilde{C}^+ \to \widetilde{C}'^+$ is equal to $\tilde{\lambda}$, and $\tilde{\lambda}_*^+$ is induced by $\tilde{\lambda}^+$.

Next, we discuss the tensor product of reduced framed complexes. Let $(\widetilde{C}^\otimes, \tilde{d}^\otimes, \chi^\otimes)$ be the tensor product of $\mathcal{S}$-complexes $(\widetilde{C}, \tilde{d}, \chi)$ and $(\widetilde{C}', \tilde{d}', \chi')$, and $(\widehat{\boldsymbol{C}}^\otimes, \overline{\boldsymbol{C}}^\otimes, \check{\boldsymbol{C}}^\otimes)$ be the large equivariant complexes of $\widetilde{C}^\otimes$. Analogous to the unreduced case, we have canonical chain isomorphisms

$$\widetilde{C}^+ \otimes_{R[x]} \widetilde{C}'^+ \cong (\widehat{\boldsymbol{C}} \otimes_{R[x]} \boldsymbol{S}^+) \otimes_{R[x]} (\widehat{\boldsymbol{C}}' \otimes_{R[x]} \boldsymbol{S}^+) \cong \widehat{\boldsymbol{C}}^\otimes \otimes_{R[x]} \boldsymbol{S}^+ \cong \widetilde{C}^{\otimes +},$$

where $\widetilde{C}^{\otimes+}$ is the reduced framed complex of $\widetilde{C}^\otimes$. Denote by $\widetilde{T}^+\colon \widetilde{C}^+ \otimes_{R[x]} \widetilde{C}'^+ \to \widetilde{C}^{\otimes+}$ the resulting isomorphism over $R[x]$. As the x-action on $\widetilde{C}$ is multiplication by -2Λ, we have a canonical isomorphism

$$\widetilde{C}^+ \otimes_{R[x]} \widetilde{C}'^+ \cong \widetilde{C}^+ \otimes_R \widetilde{C}'^+.$$

Under this identification, the map $\widetilde{T}^+$ coincides with the identity. The following result is the counterpart of Proposition 2.19:

Proposition 2.23 *We have the commutative diagram of $R[x]$-modules*

$$\begin{array}{ccccc} H(\widehat{\boldsymbol{C}}) \otimes_{R[x]} H(\widehat{\boldsymbol{C}}') & \longrightarrow & H(\widehat{\boldsymbol{C}} \otimes_{R[x]} \widehat{\boldsymbol{C}}') & \xrightarrow{\widehat{\boldsymbol{T}}_*} & H(\widehat{\boldsymbol{C}}^\otimes) \\ \big\downarrow{\scriptstyle \tilde{\boldsymbol{q}}_*^+ \otimes_{R[x]} \tilde{\boldsymbol{q}}_*'^+} & & \big\downarrow{\scriptstyle (\tilde{\boldsymbol{q}}^+ \otimes_{R[x]} \tilde{\boldsymbol{q}}'^+)_*} & \swarrow{\scriptstyle \tilde{\boldsymbol{q}}_*^{\otimes+}} & \\ H(\widetilde{C}^+) \otimes_R H(\widetilde{C}'^+) & \longrightarrow & H(\widetilde{C}^{\otimes+}) & & \end{array}$$

3 Special cycles

The characterization of the Frøyshov invariant of an $\mathcal{S}$-complex given in (21)–(22) highlights the importance of cycles satisfying certain conditions when constructing local equivalence invariants. In this section we systematically study such *special cycles* in the (large) equivariant complexes associated to the $\mathcal{S}$-complex. We describe the behavior of these cycles in the context of the associated unreduced and reduced framed complexes. These algebraic constructions will be applied in later sections to define homology concordance invariants of knots.

3.1 Special cycles of $\mathcal{S}$-complexes

Fix a $\mathbb{Z}/4$-graded $\mathcal{S}$-complex $\widetilde{C}$ over a ring R, and use the same notation for its associated equivariant complexes as was used in Section 2.2. Let $\mathfrak{z} = \left(\alpha, \sum_{i=0}^{N} a_i x^i\right)$ be an element of the small equivariant complex $\widehat{\mathfrak{C}}$. If $\mathfrak{z}$ is a cycle, ie $\hat{\mathfrak{d}}(\mathfrak{z}) = 0$, then $\mathfrak{i}(\mathfrak{z}) \in R[\![x^{-1}, x]$ defines an element in the image of the homology map $\mathfrak{i}_*$. Noting the expressions

$$\hat{\mathfrak{d}}(\mathfrak{z}) = \left(d\alpha - \sum_{i=0}^{N} v^i \delta_2(a_i), 0\right), \quad \mathfrak{i}(\mathfrak{z}) = \sum_{i=0}^{N} a_i x^i + \sum_{i=-\infty}^{-1} \delta_1 v^{-i-1}(\alpha) x^i, \tag{42}$$

we see that there is some $\mathfrak{z} \in \widehat{\mathfrak{C}}$ that produces an element in $\mathrm{Im}\ \mathfrak{i}_*$ of x-degree equal to $-k-1 < 0$ if and only if condition (21) holds; and an element in $\mathrm{Im}\ \mathfrak{i}_*$ of x-degree $-k \geq 0$ if and only if condition (22) holds. Thus we have the alternative characterization of the Frøyshov invariant

$$h(\widetilde{C}) = -\min\{\deg_x(Q(x)) \mid 0 \neq Q(x) \in \mathrm{Im}\ \mathfrak{i}_* \subset R[\![x^{-1}, x]\} \in \mathbb{Z}. \tag{43}$$

To explore this structure further, following [Daemi and Scaduto 2024b, Section 4.7], we associate to each $k \in \mathbb{Z}$ the ideal

$$J_k(\widetilde{C}) := \{c_{-k} \in R \mid c_{-k}x^{-k} + c_{-k-1}x^{-k-1} + \cdots \in \mathrm{Im}\ \mathfrak{i}_* \subset R[\![x^{-1}, x]\}.$$

The Frøyshov invariant is the support of these ideals in the sense that $J_k(\widetilde{C}) \neq 0$ if and only if $h(\widetilde{C}) \geq k$. Since $\mathrm{Im}\ \mathfrak{i}_*$ is an $R[x]$-module, the nontrivial instances of $J_k(\widetilde{C})$ form a nested sequence of ideals in R,

$$J_{h(\widetilde{C})}(\widetilde{C}) \subseteq J_{h(\widetilde{C})-1}(\widetilde{C}) \subseteq \cdots \subseteq R.$$

Moreover, the collection $\{J_k(\widetilde{C})\}_{k \in \mathbb{Z}}$ is a local equivalence invariant of the $\mathcal{S}$-complex $\widetilde{C}$.

The cycles of interest will be those that realize the elements of the ideals $J_k(\widetilde{C})$. We would like to view these as in $\widehat{\boldsymbol{C}}$, as many of the constructions in subsequent sections are tied more directly to the large equivariant complexes. To this end, we introduce the following terminology. Recall that $\widehat{\Psi}\colon \widehat{\mathfrak{C}} \to \widehat{\boldsymbol{C}}$ is a chain homotopy equivalence.

Definition 3.1 For $k \in \mathbb{Z}$ and $f \in R$, a chain $z \in \widehat{\boldsymbol{C}}$ is a *special* (k, f)*-cycle* if there exists a cycle $\mathfrak{z} \in \widehat{\mathfrak{C}}$ such that $\widehat{\Psi}(\mathfrak{z}) = z$ and $\mathfrak{i}(\mathfrak{z}) = f x^{-k} + \sum_{i=-\infty}^{-k-1} b_i x^i$.

We often call z simply a *special cycle*, even though the data (k, f) is essential. Note that every cycle in the image of $\widehat{\Psi}$ is a special cycle for some (k, f): take $f = 0$ and $k \ll 0$. Interesting special cycles have $f \neq 0$. In fact, for any $k \in \mathbb{Z}$ and $f \in R$, there exists a special (k, f)-cycle in $\widehat{\boldsymbol{C}}$ if and only if $f \in J_k(\widetilde{C})$.

We remark that the cycle $\mathfrak{z}$ in the above definition is uniquely determined by the special cycle $z \in \widehat{\boldsymbol{C}}$, since $\widehat{\Phi}\colon \widehat{\boldsymbol{C}} \to \widehat{\mathfrak{C}}$ satisfies $\mathfrak{z} = \widehat{\Phi} \circ \widehat{\Psi}(\mathfrak{z}) = \widehat{\Phi}(z)$.

The following construction plays a central role in subsequent sections:

Proposition 3.2 *Given a $\mathbb{Z}/4$-graded $\mathcal{S}$-complex $\widetilde{C}$ over R, let $h := h(\widetilde{C})$ and $f \in J_h(\widetilde{C})$. Then there is a special (h, f)-cycle $z \in \widehat{\boldsymbol{C}}_{2h}$. Moreover, the homology class $[z]$ is uniquely determined by f up to $R[x^2]$-torsion. Consequently, we have an injective R-homomorphism*

$$\widehat{\xi}\colon J_h(\widetilde{C}) \to H_{2h}(\widehat{\boldsymbol{C}})/\mathrm{Tor}_{R[x^2]}, \quad f \mapsto \widehat{\xi}(f) := [z]. \tag{44}$$

Proof That there exists a special (h, f)-cycle z follows from the remarks preceding the proposition. To prove the second statement of the proposition, let z and z' be two distinct special (h, f)-cycles, whose associated cycles are $\mathfrak{z}$ and $\mathfrak{z}'$, respectively. Then $\mathfrak{i}(\mathfrak{z}) = \mathfrak{i}(\mathfrak{z}')$. Indeed, if not, then $\mathfrak{i}(\mathfrak{z} - \mathfrak{z}')$ would have x-degree strictly smaller than $-h$, contradicting (43). Noting that

$$\boldsymbol{i}_*[z] = \boldsymbol{i}_*\widehat{\Psi}_*[\mathfrak{z}] = \overline{\Psi}_*\mathfrak{i}_*[\mathfrak{z}],$$

we have $\boldsymbol{i}_*[z] - \boldsymbol{i}_*[z'] = 0$, and hence $[z] - [z'] = 0 \in \mathrm{Im}\ \boldsymbol{j}_*$. As $\mathrm{Im}\ \boldsymbol{j}_*$ is exactly the $R[x^2]$-torsion in $H(\widehat{\boldsymbol{C}})$, we conclude that $[z]$ is uniquely determined up to $R[x^2]$-torsion.

To prove injectivity, note that, by construction, the composition of maps

$$J_h(\widetilde{C}) \xrightarrow{\widehat{\xi}} H_{2h}(\widehat{\boldsymbol{C}})/\mathrm{Tor}_{R[x^2]} \xrightarrow{\widehat{\Phi}_* \circ \mathfrak{i}_*} \mathrm{Im}\ \mathfrak{i}_* \xrightarrow{\mathfrak{c}_h} J_h(\widetilde{C})$$

is the identity, where $\mathfrak{c}_h$ is the map which sends $\sum_{-\infty}^{N} b_j x^j$ to b_{-h}. □

We now describe the behavior of special cycles under various operations involving $\mathcal{S}$-complexes.

Lemma 3.3 *Let $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ be a height i morphism. Then, for a special (k, f)-cycle $z \in \widehat{\boldsymbol{C}}$, the chain*

$$\widehat{\Psi}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z) \in \widehat{\boldsymbol{C}}'$$

is a special $(k+i, c_i f)$-cycle, where c_i is defined by the expression in (15).

Proof Let $z = \widehat{\Psi}(\mathfrak{z})$ and $\mathfrak{z}' = \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z) = \widehat{\lambda}(\mathfrak{z})$. Recall that $\overline{\lambda}$ is multiplication by $\sum_{j=-\infty}^{-i} c_j x^j$. Then

$$\mathfrak{i}'(\mathfrak{z}') = \mathfrak{i}' \circ \widehat{\lambda}(\mathfrak{z}) = \overline{\lambda} \circ \mathfrak{i}(\mathfrak{z}) = \Bigl(\sum_{j=-\infty}^{-i} c_{-j} x^j\Bigr) \cdot \Bigl(f x^{-k} + \sum_{j=-\infty}^{-k-1} b_j x^j\Bigr).$$

Thus $\mathfrak{z}'$ is a cycle such that $\mathfrak{i}'(\mathfrak{z}')$ has leading term $c_i f x^{-k-i}$. □

Let $(\widetilde{C}^{\otimes}, \tilde{d}^{\otimes}, \chi^{\otimes})$ be the tensor product of $(\widetilde{C}, \tilde{d}, \chi)$ and $(\widetilde{C}', \tilde{d}', \chi')$, and write $(\widehat{\boldsymbol{C}}^{\otimes}, \overline{\boldsymbol{C}}^{\otimes}, \check{\boldsymbol{C}}^{\otimes})$ for the large equivariant complexes associated to $\widetilde{C}^{\otimes}$.

Lemma 3.4 *Let $z \in \widehat{\boldsymbol{C}}$ (resp. $z' \in \widehat{\boldsymbol{C}}'$) be a special (k, f)-cycle (resp. (k', f')-cycle). Then the chain*

$$\widehat{\Psi}^{\otimes} \circ \widehat{\Phi}^{\otimes} \circ \widehat{T}(z \otimes_{R[x]} z') \in \widehat{\boldsymbol{C}}^{\otimes}$$

is a special $(k+k', ff')$-cycle.

Proof Recall the commutative diagram in Proposition 2.15. Since $\overline{\mathfrak{C}}^{\otimes}_*$ has trivial differential, we have

$$\mathfrak{i}^{\otimes} \circ \hat{\Phi}^{\otimes} \circ \hat{T}(z \otimes_{R[x]} z') = \overline{\Phi}^{\otimes} \circ \boldsymbol{i}^{\otimes} \circ \hat{T}(z \otimes_{R[x]} z') = \overline{\Phi}^{\otimes}_* \circ \boldsymbol{i}^{\otimes}_* \circ \hat{T}_*([z] \otimes_{R[x]} [z'])$$
$$= (\overline{\Phi}_* \circ \boldsymbol{i}_*[z]) \cdot (\overline{\Phi}'_* \circ \boldsymbol{i}'_*[z']) = (\overline{\Phi} \circ \boldsymbol{i}(z)) \cdot (\overline{\Phi}' \circ \boldsymbol{i}'(z'))$$
$$= \left(f x^{-k} + \sum_{i=-\infty}^{-k-1} b_i x^i \right) \cdot \left(f' x^{-k'} + \sum_{i=-\infty}^{-k'-1} b'_i x^i \right).$$

The leading term is $f f' x^{-k-k'}$, and this completes the proof. □

Let S be an algebra over R. For an $\mathcal{S}$-complex $\widetilde{C}$ over R, let $(\hat{\boldsymbol{C}}^S, \overline{\boldsymbol{C}}^S, \check{\boldsymbol{C}}^S)$ denote the large equivariant complexes of the $\mathcal{S}$-complex $\widetilde{C} \otimes_R S$ over the coefficient ring S. The following is immediate:

Lemma 3.5 *Let $z \in \hat{\boldsymbol{C}}$ be a special (k, f)-cycle. Then the chain*

$$z \otimes_R 1 \in \hat{\boldsymbol{C}}^S \cong \hat{\boldsymbol{C}} \otimes_R S$$

is a special (k, f)-cycle, where $f = f \cdot 1 \in S$.

3.2 Behavior in unreduced framed complexes

Throughout, we suppose that R is an integral domain algebra over $\mathbb{Q}[T^{\pm 1}]$. Let $\widetilde{C}$ be an $\mathcal{S}$-complex over R and $h := h(\widetilde{C})$. Recall from (34) that we have a map

$$H(\hat{\boldsymbol{C}}) \xrightarrow{\boldsymbol{q}^\sharp_*} H(C^\sharp).$$

In what follows, it will be convenient to quotient $H(C^\sharp)$ by the image of $R[x^2]$-torsion under $\boldsymbol{q}^\sharp_*$. For brevity, we use the following notation for this quotient:

$$H(C^\sharp)^{\boldsymbol{q}} := H(C^\sharp)/\boldsymbol{q}^\sharp_*(\mathrm{Tor}_{R[x^2]}).$$

In particular, we have a well-defined induced map

$$H_i(\hat{\boldsymbol{C}})/\mathrm{Tor}_{R[x^2]} \xrightarrow{\boldsymbol{q}^\sharp_*} H_i(C^\sharp)^{\boldsymbol{q}}.$$

We now apply this map to the special cycle construction of Proposition 3.2.

Proposition 3.6 *Let $h := h(\widetilde{C})$. Then the following maps are injective R-module homomorphisms:*

$$\xi^\sharp_+ : J_h(\widetilde{C}) \to H_{2h}(C^\sharp)^{\boldsymbol{q}}, \qquad f \mapsto \xi^\sharp_+(f) := \boldsymbol{q}^\sharp_* \hat{\xi}(f), \tag{45}$$

$$\xi^\sharp_- : J_h(\widetilde{C}) \to H_{2h-2}(C^\sharp)^{\boldsymbol{q}}, \quad f \mapsto \xi^\sharp_-(f) := x \cdot \boldsymbol{q}^\sharp_* \hat{\xi}(f). \tag{46}$$

In particular, $\xi^\sharp_+$ and $\xi^\sharp_-$ map any nontrivial element in $J_h(\widetilde{C})$ into a nontorsion element.

Proof Suppose $\xi^\sharp_+(f) \in H(C^\sharp)^{\boldsymbol{q}}$ is zero. Then there is a representative $[y] \in H(\hat{\boldsymbol{C}})$ of $\hat{\xi}(f) \in H(\hat{\boldsymbol{C}})/\mathrm{Tor}_{R[x]}$ such that $\boldsymbol{q}^\sharp_*[y] = 0$. By the exact sequence (34), we have $[y] = (x^2 - 4\Lambda^2)[z]$ for some $[z] \in H(\hat{\boldsymbol{C}})$. Now, if $f \neq 0$, by definition of $\hat{\xi}(f)$, we have

$$\overline{\Psi}_* \boldsymbol{i}_*((x^2 - 4\Lambda^2)[z]) = f x^{-h} + \sum_{j=-\infty}^{-h-1} b_j x^j.$$

On the other hand, $\overline{\Psi}_* \boldsymbol{i}_*[z]$ then has leading term of x-degree equal to $-2-h$, contradicting (43). Thus $f=0$ and $\xi_+^\sharp$ is injective. For $\xi_-^\sharp$ the argument is similar. The last claim follows from injectivity and the fact that $J_h(\widetilde{C})$ is an ideal in an integral domain R.

An alternative proof follows by applying the pairing result given below as Proposition 3.11. □

Remark 3.7 We can make the following observation about the module $\boldsymbol{q}_*^\sharp(\mathrm{Tor}_{R[x^2]})$ used in the definition of $H_i(C^\sharp)^{\boldsymbol{q}}$. Suppose $[z] \in H_i(\widehat{\boldsymbol{C}})$ satisfies $f(x^2)[z]=0$ for some $f(x) \in R[x]$. As $C^\sharp$ is isomorphic to the quotient of $\widehat{\boldsymbol{C}}$ by the relation $x^2 - 4\Lambda^2$, we have $0 = \boldsymbol{q}_*^\sharp(f(x^2)\cdot[z]) = f(4\Lambda^2)\cdot \boldsymbol{q}_*^\sharp[z]$. Thus, in the case that $f(4\Lambda^2) \neq 0$, the element $[z]$ is R-torsion in $H(C^\sharp)$.

To make the above construction more explicit, if $\xi_+^\sharp(f) = [(\zeta_1,\zeta_2)]$ for chains $\zeta_1 \in \widetilde{C}_{2h}$ and $\zeta_2 \in \widetilde{C}_{2h+2}$, then the description of the x-module structure on $C^\sharp$ in (33) gives the expression

$$\xi_-^\sharp(f) = x\cdot[(\zeta_1,\zeta_2)] = [(-2\Lambda^2\zeta_2, -2\zeta_1)]. \tag{47}$$

We now describe the behavior of $\xi_\pm^\sharp$ under some basic operations.

Lemma 3.8 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over R with $h := h(\widetilde{C})$ and $h' := h(\widetilde{C}')$. If $h' \geq h$, then, for any height $h'-h$ morphism $\widetilde{\lambda}\colon \widetilde{C}_* \to \widetilde{C}'$, the induced map $\lambda_*^\sharp\colon H(C^\sharp)^{\boldsymbol{q}} \to H(C'^\sharp)^{\boldsymbol{q}}$ satisfies*

$$\lambda^\sharp(\xi_\pm^\sharp(f)) = \xi_\pm^\sharp(c_{h'-h} f).$$

Here $c_{h'-h}$ is defined by the expression in (15).

Proof Let $z \in \widehat{\boldsymbol{C}}_{2h}$ be a special (h,f)-cycle. Then it follows from Lemma 3.3 that $\widehat{\Psi}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z) \in \widehat{\boldsymbol{C}}_{2h'}$ is a special $(h', c_{h'-h} f)$-cycle. Then we compute

$$\lambda_*^\sharp(\xi_+^\sharp(f)) = \lambda_*^\sharp \circ \boldsymbol{q}_*^\sharp([z]) = \boldsymbol{q}_*'^\sharp \circ \widehat{\boldsymbol{\lambda}}_*([z]) = \boldsymbol{q}_*'^\sharp\big([\widehat{\Psi}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z)]\big) = \xi_+^\sharp(c_{h'-h} f).$$

The second identity follows from the commutative diagram in Proposition 2.18. Since $\lambda^\sharp$ is an $R[x]$-module homomorphism, this completes the proof. □

Let $\widetilde{C}^\otimes$ be the tensor product of $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over R, and $(\widehat{\boldsymbol{C}}^\otimes, \overline{\boldsymbol{C}}^\otimes, \check{\boldsymbol{C}}^\otimes)$ (resp. $C^{\otimes\sharp}$) be the large equivariant complexes (resp. unreduced framed complex) of $\widetilde{C}^\otimes$. Let $h := h(\widetilde{C})$ and $h' := h(\widetilde{C}')$. Recall the chain isomorphism $T^\sharp\colon C^\sharp \otimes_{R[x]} C'^\sharp \to C^{\otimes\sharp}$ from Proposition 2.19. We have an induced map

$$t^\sharp\colon H(C^\sharp)^{\boldsymbol{q}} \otimes_{R[x]} H(C'^\sharp)^{\boldsymbol{q}} \to H(C^\sharp \otimes_{R[x]} C'^\sharp)^{\boldsymbol{q}}.$$

Lemma 3.9 *For any $f \in J_h(\widetilde{C})$ and $f' \in J_{h'}(\widetilde{C}')$, the composition of $t^\sharp$ with $T_*^\sharp$ maps the tensor products of $\xi_\pm^\sharp(f) \in H(C^\sharp)^{\boldsymbol{q}}$ and $\xi_\pm^\sharp(f') \in H(C'^\sharp)^{\boldsymbol{q}}$ as follows*:

$$\xi_+^\sharp(f)\otimes\xi_+^\sharp(f') \mapsto \xi_+^\sharp(ff'), \quad \xi_+^\sharp(f)\otimes\xi_-^\sharp(f') \mapsto \xi_-^\sharp(ff'),$$
$$\xi_-^\sharp(f)\otimes\xi_+^\sharp(f') \mapsto \xi_-^\sharp(ff'), \quad \xi_-^\sharp(f)\otimes\xi_-^\sharp(f') \mapsto \xi_+^\sharp(4\Lambda^2 ff').$$

Proof Since $x \cdot \xi_+^\sharp(ff') = \xi_-^\sharp(ff')$ and $x^2 \cdot \xi_+^\sharp(ff') = 4\Lambda^2 \cdot \xi_+^\sharp(ff') = \xi_+^\sharp(4\Lambda^2 ff')$, we only need to check $\xi_+^\sharp(f) \otimes \xi_+^\sharp(f') \mapsto \xi_+^\sharp(ff')$. Let $z \in \widehat{\boldsymbol{C}}_{2h}$ (resp. $z' \in \widehat{\boldsymbol{C}}'_{2h'}$) be a special (h, f)-cycle (resp. (h', f')-cycle). Then, from Lemma 3.4, $\widehat{\Psi}^\otimes \circ \widehat{\Phi}^\otimes \circ \widehat{T}(z \otimes z') \in \widehat{\boldsymbol{C}}^\otimes_{2h+2h'}$ is a special $(h+h', ff')$-cycle. The result now follows from

$$\begin{aligned} T_*^\sharp \circ t^\sharp(\xi_+^\sharp(f) \otimes \xi_+^\sharp(f')) &= T_*^\sharp \circ t^\sharp \circ (\boldsymbol{q}_*^\sharp \otimes_{R[x]} \boldsymbol{q}_*'^\sharp)([z] \otimes [z']) = \boldsymbol{q}_*^{\otimes\sharp} \circ \widehat{T}_*([z \otimes z']) \\ &= \boldsymbol{q}_*^{\otimes\sharp}([\widehat{\Psi}^\otimes \circ \widehat{\Phi}^\otimes \circ \widehat{T}(z \otimes z')]) = \xi_+^\sharp(ff'), \end{aligned}$$

where we have used the commutative diagram of Proposition 2.19. □

Finally, we discuss a duality result involving the two maps $\xi_+^\sharp$ and $\xi_-^\sharp$. We start with the following lemma. In the formulas below, we are using the splitting $\widetilde{C}_* = C_* \oplus C_{*-1} \oplus R_{(0)}$.

Lemma 3.10 *Let $\widetilde{C}$ be an $\mathcal{S}$-complex over R, and $h := h(\widetilde{C})$. The following assertions hold*:

(i) *If $h > 0$, then any representative of the homology class of $\xi_+^\sharp(f)$ is a cycle $((0, \alpha, 0), 0) \in \widetilde{C}_{2h} \oplus \widetilde{C}_{2h+2}$ for some $\alpha \in C_{2h-1}$ satisfying*

$$\delta_1 v^i(\alpha) = 0 \quad (0 \le i < h-1) \qquad \text{and} \qquad \delta_1 v^{h-1}(\alpha) = f.$$

(ii) *If $h = 0$, any representative of $\xi_+^\sharp(f)$ has the form $((0, \alpha, f), 0) \in \widetilde{C}_{2h} \oplus \widetilde{C}_{2h+2}$ for some $\alpha \in C_{-1}$.*

(iii) *If $h < 0$, then any representative of $\xi_+^\sharp(f)$ is given by a cycle in $\widetilde{C}_{2h} \oplus \widetilde{C}_{2h+2}$ in the form of*

$$\biggl((v^{-h-1}\delta_2(f) + \sum_{i=0}^{-h-2} v^i \delta_2(b_i), \alpha, b), \biggl(\sum_{i=0}^{-h-2} v^i \delta_2(b_i'), 0, b'\biggr)\biggr).$$

Proof Here we consider the case $h < 0$. Let $z = \widehat{\Psi}\bigl(\alpha, \sum_{i=0}^{-h} a_i x^i\bigr) \in \widehat{\boldsymbol{C}}_{2h}$ be a special (h, f)-cycle. In particular, $a_{-h} = f$. Then

$$\xi_+^\sharp(f) = \boldsymbol{q}_*^\sharp([z]) = \biggl[\boldsymbol{q}^\sharp \circ \widehat{\Psi}\biggl(\alpha, \sum_{i=0}^{-h} a_i x^i\biggr)\biggr].$$

Here, by the definitions of $\widehat{\Psi}$ and $\boldsymbol{q}^\sharp$, we have

$$\begin{aligned} &\boldsymbol{q}^\sharp \circ \widehat{\Psi}\biggl(\alpha, \sum_{i=0}^{-h} a_i x^i\biggr) \\ &\quad= \boldsymbol{q}^\sharp\biggl(\sum_{i=1}^{-h}\sum_{j=0}^{i-1} v^j \delta_2(a_i) x^{i-j-1}, \alpha, \sum_{i=0}^{-h} a_i x^i\biggr) \\ &\quad= \biggl(\sum_{i=1}^{-h}\sum_{j=0}^{i-1} v^j \delta_2(a_i)(4\Lambda^2)^{\lfloor (i-j-1)/2 \rfloor} x^{(i-j-1)-2\lfloor (i-j-1)/2 \rfloor}, \alpha, \sum_{i=0}^{-h} a_i (4\Lambda^2)^{\lfloor i/2 \rfloor} x^{i-2\lfloor i/2 \rfloor}\biggr) \\ &\quad= \biggl(v^{-h-1}(a_{-h}) + \sum_{i=0}^{-h-2} v^i \delta_2(b_i) + \sum_{i=0}^{-h-2} v^i \delta_2(b_i') x, \alpha, b + b'x\biggr) \end{aligned}$$

for some elements $b_i, b_i', b, b' \in R$ $(0 \le i \le -k-2)$. The remaining parts are proved similarly. □

Let $\widetilde{C}^\dagger$ denote the dual complex of an $\mathcal{S}$-complex $\widetilde{C}$ over R, and $C^{\sharp\dagger}$ denote the framed complex of $\widetilde{C}^\dagger$. Set $h := h(\widetilde{C}) = -h(\widetilde{C}^\dagger)$. Then the duality pairing for $\xi^\sharp_\pm$ is stated as follows:

Proposition 3.11 *For any nonzero elements $f \in J_h(\widetilde{C})$ and $f' \in J_{-h}(\widetilde{C}^\dagger)$, we have*

$$\langle \xi^\sharp_+(f'), \xi^\sharp_-(f)\rangle = \langle \xi^\sharp_-(f'), \xi^\sharp_+(f)\rangle = -2ff',$$

where $\xi^\sharp_\pm(f) \in H_*(C^\sharp)^{\boldsymbol{q}}$ and $\xi^\sharp_\pm(f') \in H_*(C^{\dagger\sharp})^{\boldsymbol{q}}$, and the above pairing is induced by the identification of Proposition 2.20. In particular, the pairing is independent of the choice of representatives for $\xi^\sharp_\pm(f)$.

The proof of this proposition uses Lemma 3.10. First we give some remarks on dual $\mathcal{S}$-complexes following [Daemi and Scaduto 2024b, Section 4.4]. For any $\mathcal{S}$-complex $\widetilde{C}_* = C_* \oplus C_{*-1} \oplus R_{(0)}$, the dual complex $\widetilde{C}^\dagger_*$ has a splitting of the form $C^\dagger_* \oplus C^\dagger_{*-1} \oplus R_{(0)}$, where $C^\dagger_* = \mathrm{Hom}_R(C_{-*-1}, R)$. With respect to this splitting, the components of the differential $\tilde{d}^\dagger$ are given by $d^\dagger$, $v^\dagger$, $\delta^\dagger_1$ and $\delta^\dagger_2$, where, for any $f \in C^\dagger_i$, we have

$$d^\dagger(f) = (-1)^i f \circ d, \quad v^\dagger(f) = f \circ v, \quad \delta^\dagger_1(f) = -f \circ \delta_2(1), \quad \delta^\dagger_2(1) = \delta_1.$$

Furthermore, the pairing in (35) for $\zeta = (\alpha, \beta, a) \in \widetilde{C}_i$ and $\zeta^\dagger = (f, g, b) \in \widetilde{C}^\dagger_{-i}$ is given by

$$\langle \zeta^\dagger, \zeta\rangle = (-1)^i g(\alpha) + f(\beta) + ab. \tag{48}$$

Proof of Proposition 3.11 Here we prove $\langle \xi^\sharp_+(f'), \xi^\sharp_-(f)\rangle = -2ff'$ for the case $h > 0$. By Lemma 3.10, the homology class $\xi^\sharp_+(f) \in H_{2h}(C^\sharp)^{\boldsymbol{q}}$ is represented by a cycle $((0,\alpha,0),0) \in \widetilde{C}_{2h} \oplus \widetilde{C}_{2h+2}$ such that

$$\delta_1 v^i(\alpha) = 0 \quad (0 \le i < h-1) \qquad \text{and} \qquad \delta_1 v^{h-1}(\alpha) = f,$$

while $\xi^\sharp_+(f') \in H_{-2h}(C^{\dagger\sharp})^{\boldsymbol{q}}$ is represented by a cycle in $\widetilde{C}^\dagger_{-2h} \oplus \widetilde{C}^\dagger_{-2h+2}$ that has the form

$$\left(\left((v^\dagger)^{h-1}\delta^\dagger_2(f') + \sum_{i=0}^{h-2}(v^\dagger)^i\delta^\dagger_2(b_i), \varphi, b\right), \left(\sum_{i=0}^{h-2}(v^\dagger)^i\delta^\dagger_2(b'_i), 0, b'\right)\right).$$

Moreover, the formula (47) shows that $\xi^\sharp_-(f) \in H_{2h}(C^\sharp)$ is represented by $(0,(0,-2\alpha,0)) \in \widetilde{C}_{2h-2} \oplus \widetilde{C}_{2h}$. Therefore, from Proposition 2.20 and (48), we have

$$\begin{aligned}\langle \xi^\sharp_+(f'), \xi^\sharp_-(f)\rangle &= \left\langle \left((v^\dagger)^{h-1}\delta^\dagger_2(f') + \sum_{i=0}^{h-2}(v^\dagger)^i\delta^\dagger_2(b_i), \varphi, b\right), (0,-2\alpha,0)\right\rangle \\ &= \left(f'\delta_1 v^{h-1} + \sum_{i=0}^{h-2} b_i\delta_1 v^i\right)(-2\alpha) = -2ff'.\end{aligned}$$

The remaining parts are proved similarly. □

3.3 Behavior in reduced framed complexes

Let R be an integral domain algebra over $\mathbb{Q}[T^{\pm 1}]$, $\widetilde{C}$ an $\mathcal{S}$-complex over R and $h := h(\widetilde{C}_*)$. Parallel to what was done for the unreduced complexes, we now consider special cycles in the reduced framed complexes $\widetilde{C}^{\pm}$ associated to $\widetilde{C}$. First, recall from (41) that we have maps

$$H(\widehat{\boldsymbol{C}}) \xrightarrow{\tilde{\boldsymbol{q}}^{\pm}_*} H(\widetilde{C}^{\pm}).$$

Similar to our convention in the unreduced case, we write $H(\widetilde{C}^{\pm})^{\boldsymbol{q}}$ for the quotient of $H(\widetilde{C}^{\pm})$ by the submodule generated by the image of $R[x^2]$-torsion under $\tilde{\boldsymbol{q}}^{\pm}_*$. Thus we have induced maps

$$H_i(\widehat{\boldsymbol{C}})/\mathrm{Tor}_{R[x^2]} \xrightarrow{\tilde{\boldsymbol{q}}^{\pm}_*} H_i(\widetilde{C}^{\pm})^{\boldsymbol{q}}.$$

As $\widetilde{C}^{+}$ and $\widetilde{C}^{-}$ are isomorphic complexes, for the remainder of this section we focus on the case of $\pm = +$, and drop the symbol $+$ from most of the notation. The proof of the following is analogous to that of Proposition 3.6:

Proposition 3.12 *Let $h := h(\widetilde{C})$. Then the following map is an injective R-module homomorphism*:

$$\tilde{\xi}\colon J_h(\widetilde{C}) \to H_{2h}(\widetilde{C}^{+})^{\boldsymbol{q}}, \quad f \mapsto \tilde{\xi}(f) := \tilde{\boldsymbol{q}}_*\hat{\xi}(f). \tag{49}$$

In particular, $\tilde{\xi}$ maps any nontrivial element in $J_h(\widetilde{C})$ into a nontorsion element.

In this context, the behavior of special cycles under nonnegative height morphisms is described as follows. The proof is entirely analogous to that of Lemma 3.8.

Lemma 3.13 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over R with $h := h(\widetilde{C})$ and $h' := h(\widetilde{C}')$. If $h' \geq h$, then, for any height $h'-h$ morphism $\tilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$, the induced map $\tilde{\lambda}^{+}_*\colon H(\widetilde{C}^{+})^{\boldsymbol{q}} \to H(\widetilde{C}'^{+})^{\boldsymbol{q}}$ satisfies*

$$\tilde{\lambda}^{+}_*(\tilde{\xi}(f)) = \tilde{\xi}(c_{h-h'} f).$$

Here $c_{h'-h}$ is defined by the expression in (15).

For tensor products, we have the following. Let $\widetilde{C}^{\otimes}$ be the tensor product of $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over R. Let $h := h(\widetilde{C})$ and $h' := h(\widetilde{C}')$. We have an induced map

$$\tilde{t}\colon H(\widetilde{C}^{+})^{\boldsymbol{q}} \otimes_R H(\widetilde{C}'^{+})^{\boldsymbol{q}} \to H(\widetilde{C}^{\otimes +})^{\boldsymbol{q}}.$$

Lemma 3.14 *For any $f \in J_h(\widetilde{C})$ and $f' \in J_{h'}(\widetilde{C}')$, we have*

$$\tilde{t}(\tilde{\xi}(f) \otimes \tilde{\xi}(f')) = \tilde{\xi}(ff').$$

Proof Let $z \in \widehat{\boldsymbol{C}}_{2h}$ (resp. $z' \in \widehat{\boldsymbol{C}}'_{2h'}$) be a special (h, f)-cycle (resp. (h', f')-cycle). Then it follows from Lemma 3.4 that $\widehat{\Psi}^{\otimes} \circ \widehat{\Phi}^{\otimes} \circ \widehat{T}(z \otimes z') \in \widehat{\boldsymbol{C}}^{\otimes}_{2h+2h'}$ is a special $(h+h', ff')$-cycle. Now we see, using the commutative diagram from Proposition 2.23, that

$$\begin{aligned}\tilde{t}(\tilde{\xi}(f) \otimes \tilde{\xi}(f')) &= \tilde{t} \circ (\tilde{\boldsymbol{q}}_* \otimes_{R[x]} \tilde{\boldsymbol{p}}'_*)([z] \otimes [z']) = \tilde{\boldsymbol{q}}^{\otimes}_* \circ \widehat{T}_*([z \otimes z']) \\ &= \tilde{\boldsymbol{q}}_*([\widehat{\Psi}^{\otimes} \circ \widehat{\Phi}^{\otimes} \circ \widehat{T}(z \otimes z')]) = \tilde{\xi}(ff').\end{aligned}$$

□

4 Concordance invariants from special cycles

In this section, we construct several local equivalence invariants of $\mathcal{S}$-complexes using the machinery of special cycles. As in the previous sections, the element $\Lambda = T - T^{-1}$ of $\mathbb{Q}[T^{\pm 1}]$ plays an important role for us. The formal power series ring of Λ may be identified with the completion of the localization of $\mathbb{Q}[T^{\pm 1}]$ at $T = 1$. An explicit identification comes from noting that $\Lambda = a(T-1)$, where $a = 1 + T^{-1}$ is a unit in $\mathbb{Q}[T^{\pm 1}]$. Thus, as a power series in $T - 1$, we can write

$$\Lambda = 2(T-1) - (T-1)^2 + (T-1)^3 - (T-1)^4 + \cdots .$$

With this identification, $\mathbb{Q}[\![\Lambda]\!]$ is a $\mathbb{Q}[T^{\pm 1}]$-algebra. Recall that $\Theta^{\mathcal{S}}_{\mathbb{Q}[\![\Lambda]\!]}$ denotes the local equivalence group of $\mathbb{Z}/4$-graded $\mathcal{S}$-complexes over $\mathbb{Q}[\![\Lambda]\!]$. The first set of invariants are of the form

$$\tilde{s}, s^{\sharp}_{\pm}, s^{\sharp}, \tilde{\varepsilon} \colon \Theta^{\mathcal{S}}_{\mathbb{Q}[\![\Lambda]\!]} \to \mathbb{Z}.$$

Applying these constructions to the $\mathcal{S}$-complex $\widetilde{C}(Y, K; \Delta_{\mathbb{Q}[\![\Lambda]\!]})$ of a knot in an integer homology 3-sphere gives corresponding homology concordance invariants. For knots in S^3 we show: $\tilde{s}$ is a homomorphism and half a slice-torus invariant in the sense of [Lewark 2014]; $s^{\sharp}$ recovers Kronheimer and Mrowka's invariant [2013]; and $s^{\sharp}_{\pm}$ recover Gong's refinements of $s^{\sharp}$ [2021]. In particular, we prove Theorems 1.1 and 1.3 and Corollary 1.2.

We go on to define, for any $\mathbb{Z}/4$-graded $\mathcal{S}$-complex over $\mathbb{Z}[T^{\pm 1}]$, a $\mathbb{Z}[T^{\pm 1}]$-submodule

$$\hat{z}(\widetilde{C}) \subset \operatorname{Frac}(\mathbb{Z}[T^{\pm 1}]),$$

which is a local equivalence invariant. Applying this to knots, we show that this invariant recovers all of the concordance invariants defined in [Kronheimer and Mrowka 2021b], proving Theorem 1.7. Finally, we compute all of the invariants introduced in this section for two-bridge knots, and in particular prove Theorem 1.8.

4.1 The invariant $\tilde{s}$

Let $\widetilde{C}$ be an $\mathcal{S}$-complex over $\mathbb{Q}[\![\Lambda]\!]$ and $h := h(\widetilde{C})$. Since any ideal of $\mathbb{Q}[\![\Lambda]\!]$ is the form of (Λ^n), we have $J_h(\widetilde{C}) = (\Lambda^{n_0})$ for some $n_0 \in \mathbb{Z}_{\geq 0}$. Now $\tilde{s}(\widetilde{C})$ is defined as follows:

Definition 4.1 We define

$$\tilde{s}(\widetilde{C}) := \min\{n - m \mid \Lambda^n \in J_h(\widetilde{C}), \tilde{\xi}(\Lambda^n) = \Lambda^m \cdot y \text{ for some } y \in H(\widetilde{C}^+)^{\boldsymbol{q}}\}, \tag{50}$$

where $\tilde{\xi} \colon J_h(\widetilde{C}) \to H(\widetilde{C}^+)^{\boldsymbol{q}}$ is the R-homomorphism in (49).

An equivalent definition of $\tilde{s}$ uses the induced map $\tilde{\xi} \colon J_h(\widetilde{C}) \to H(\widetilde{C}^+)^{\boldsymbol{q}}/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$. In this version, we can fix any nonnegative integer n such that $\Lambda^n \in J_h(\widetilde{C})$ and take the minimum over all m for which the condition in (50) holds.

The following proposition implies the invariance of $\tilde{s}$ under local equivalence:

Proposition 4.2 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over $\mathbb{Q}[\![\Lambda]\!]$ with $i := h(\widetilde{C}') - h(\widetilde{C}) \geq 0$. If there exists a height i morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ with $c_i = a\Lambda^k$ for some unit $a \in \mathbb{Q}[\![\Lambda]\!]$, we have*

$$\tilde{s}(\widetilde{C}') \leq \tilde{s}(\widetilde{C}) + k.$$

In particular, if $\widetilde{\lambda}$ is a local map (or, *equivalently*, $i = 0$ *and* $k = 0$), *then* $\tilde{s}(\widetilde{C}') \leq \tilde{s}(\widetilde{C})$.

Proof Pick $n \in \mathbb{Z}_{\geq 0}$ and $y \in H(\widetilde{C}^+)^{\boldsymbol{q}}$ such that $\widetilde{\xi}(\Lambda^n) = \Lambda^{n-\tilde{s}(\widetilde{C})} \cdot y$. By applying $\widetilde{\lambda}^+_*$ to this identity and then using Lemma 3.13, we have

$$\widetilde{\xi}(a\Lambda^{n+k}) = \Lambda^{n-\tilde{s}(\widetilde{C})} \cdot \widetilde{\lambda}^+_*(y).$$

Since a is a unit, this implies that $\tilde{s}(\widetilde{C}') \leq (n+k) - (n - \tilde{s}(\widetilde{C})) = \tilde{s}(\widetilde{C}) + k$. □

Next, we prove subadditivity of $\tilde{s}$.

Proposition 4.3 *For any two $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over $\mathbb{Q}[\![\Lambda]\!]$, we have*

(51) $$\tilde{s}(\widetilde{C} \otimes \widetilde{C}') \leq \tilde{s}(\widetilde{C}) + \tilde{s}(\widetilde{C}').$$

In particular, for any $\mathcal{S}$-complex $\widetilde{C}$ and its dual $\mathcal{S}$-complex $\widetilde{C}^\dagger$, we have

(52) $$\tilde{s}(\widetilde{C}^\dagger) \geq -\tilde{s}(\widetilde{C}).$$

Proof Let $\tilde{s} := \tilde{s}(\widetilde{C})$, $\tilde{s}' := \tilde{s}(\widetilde{C}')$. Pick $n, n' \in \mathbb{Z}_{\geq 0}$, $[y] \in H(\widetilde{C}^+)$, $[y'] \in H(\widetilde{C}^+)$ and representatives $[z] \in H(\widetilde{C}^+)$ and $[z'] \in H(\widetilde{C}'^+)$ for $\widetilde{\xi}(\Lambda^n) \in H(\widetilde{C}^+)^{\boldsymbol{q}}$ and $\widetilde{\xi}(\Lambda^{n'}) \in H(\widetilde{C}'^+)^{\boldsymbol{q}}$ such that

$$[z] = \Lambda^{n-\tilde{s}} \cdot [y], \quad [z'] = \Lambda^{n'-\tilde{s}'} \cdot [y'].$$

By Lemma 3.14, the image $[z^\otimes]$ of $[z] \otimes [z']$ with respect to the map

(53) $$H(\widetilde{C}^+) \otimes_R H(\widetilde{C}'^+) \to H(\widetilde{C}^{\otimes +})$$

is a representative for $\widetilde{\xi}(\Lambda^{n+n'}) \in H(\widetilde{C}^{\otimes +})^{\boldsymbol{q}}$, and we have

(54) $$[z^\otimes] = \Lambda^{n+n'-\tilde{s}-\tilde{s}'}[y^\otimes],$$

where $[y^\otimes]$ is the image of $[y] \otimes [y']$ with respect to (53). This gives (51). The inequality in (52) follows from (51) and the fact that the tensor $\mathcal{S}$-complex $\widetilde{C} \otimes \widetilde{C}^\dagger$ is locally equivalent to the trivial $\mathcal{S}$-complex. □

Additivity of $\tilde{s}$ will be established under the following condition, which is a condition which holds for $\mathcal{S}$-complexes of knots in the 3-sphere; see Remark 2.21.

Assumption 4.4 $H_k(\widetilde{C}^+)$ has rank 1 for $k \equiv 0 \pmod 2$, and rank 0 for $k \equiv 1 \pmod 2$.

Remark 4.5 It is straightforward to verify, using the Künneth formula, that if the $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ satisfy Assumption 4.4, then so too does the tensor product $\mathcal{S}$-complex $\widetilde{C} \otimes \widetilde{C}'$. Similarly, if $\widetilde{C}$ satisfies Assumption 4.4, then so does the dual $\widetilde{C}^\dagger$.

Note that under Assumption 4.4, the natural quotient map $H(\widetilde{C}^+) \to H(\widetilde{C}^+)^{\mathfrak{q}}$ is an isomorphism mod $\mathbb{Q}[\![\Lambda]\!]$-torsion. This follows from the injectivity of the map $\widetilde{\xi}$. Thus $H(\widetilde{C}^+)/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$ can be used as the homology group appearing in the definition of $\tilde{s}$.

Proposition 4.6 *If $\widetilde{C}$ and $\widetilde{C}'$ satisfy Assumption 4.4, then*

$$\tilde{s}(\widetilde{C} \otimes \widetilde{C}') = \tilde{s}(\widetilde{C}) + \tilde{s}(\widetilde{C}'). \tag{55}$$

In particular, if $\widetilde{C}$ satisfies Assumption 4.4, then $\tilde{s}(\widetilde{C}^\dagger) = -\tilde{s}(\widetilde{C})$.

Proof With the remark preceding the statement of the proposition in mind, in the proof of Proposition 4.3 we replace the instances of $H(\widetilde{C}^+)^{\mathfrak{q}}$, $H(\widetilde{C}'^+)^{\mathfrak{q}}$ and $H(\widetilde{C}^{\otimes +})^{\mathfrak{q}}$ with the respective $\mathbb{Q}[\![\Lambda]\!]$-modules $H(\widetilde{C}^+)$, $H(\widetilde{C}'^+)$ and $H(\widetilde{C}^{\otimes +})$ modulo $\mathbb{Q}[\![\Lambda]\!]$-torsion. Now $[y]$, $[y']$ are generators of the free modules $H(\widetilde{C}^+)/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$, $H(\widetilde{C}'^+)/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$, respectively, and $[y^{\otimes}]$ is a generator of $H(\widetilde{C}^{\otimes +})/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$. This last point, together with (54), computes (55). □

Now, if K is a knot in an integer homology 3-sphere Y, we define

$$\tilde{s}(Y, K) := \tilde{s}(\widetilde{C}(Y, K; \Delta_{\mathbb{Q}[\![\Lambda]\!]})), \tag{56}$$

which by construction induces a map from the homology concordance group to $\mathbb{Z}$. In particular, restricting to knots in the 3-sphere induces a concordance invariant

$$\tilde{s} : \mathcal{C} \to \mathbb{Z} \tag{57}$$

which is a homomorphism by Remark 2.21 and Proposition 4.6.

Remark 4.7 The construction of $\tilde{s}$ does not require the completion of a ring: we could have simply used $\mathbb{Q}[T^{\pm 1}]$ localized at $T-1$. We take the completion as a matter of convenience, to more easily align our constructions with those of [Kronheimer and Mrowka 2013], as is done in the proof of Theorem 4.17. Similar remarks hold for the invariants defined in the next section. See also the discussion of coefficients in Section 4.2.2.

Remark 4.8 The reader may find the choice of coefficient ring $\mathbb{Q}[\![\Lambda]\!]$ and the element Λ used in this section mysterious. The definition of $\tilde{s}$, together with Propositions 4.2 and 4.3, in fact work more generally for $\mathcal{S}$-complexes over an integral domain R localized at a prime principal ideal, such as (Λ) above. The particular choice of $R = \mathbb{Q}[\![\Lambda]\!]$ with its prime ideal (Λ) is motivated by Remark 2.21, which says that the corresponding instanton homology for knots in S^3 satisfies Assumption 4.4.

4.2 Kronheimer and Mrowka's $s^\sharp$-invariant

We now define local equivalence invariants $s^\sharp_+$ and $s^\sharp_-$ that induce maps

$$s^\sharp_\pm : \Theta^{\mathcal{S}}_{\mathbb{Q}[\![\Lambda]\!]} \to \mathbb{Z}.$$

Similar to the case of $\tilde{s}$, these are defined using special cycles. We write the sum as $s^\sharp := s_+^\sharp + s_-^\sharp$. Applying these constructions to the $\mathcal{S}$-complex of a knot defines $\mathbb{Z}$-valued homology concordance invariants. For knots K in the 3-sphere, we show that our $s^\sharp$ recovers Kronheimer and Mrowka's invariant [2013], and that $s_+^\sharp$ and $s_-^\sharp$ recover Gong's refinements of $s^\sharp(K)$ [2021].

4.2.1 $s^\sharp$-invariants of $\mathcal{S}$-complexes Let $\widetilde{C}$ be an $\mathcal{S}$-complex over $\mathbb{Q}[\![\Lambda]\!]$ and $h := h(\widetilde{C})$. Recall that $\Lambda^n \in J_h(\widetilde{C})$ for $n \gg 0$.

Definition 4.9 We define $s_\pm^\sharp(\widetilde{C})$ by

$$s_+^\sharp(\widetilde{C}) := \min\{n-m \mid \Lambda^n \in J_h(\widetilde{C}),\ \xi_+^\sharp(\Lambda^n) = \Lambda^m y_+ \text{ for some } y_+ \in H_{2h}(C^\sharp)^{\boldsymbol{q}}\}, \tag{58}$$

$$s_-^\sharp(\widetilde{C}) := \min\{n-m \mid \Lambda^n \in J_h(\widetilde{C}),\ \xi_-^\sharp(\Lambda^n) = \Lambda^m y_- \text{ for some } y_- \in H_{2h-2}(C^\sharp)^{\boldsymbol{q}}\}. \tag{59}$$

Here $\xi_\pm^\sharp\colon J_h(\widetilde{C}) \to H_*(C^\sharp)^{\boldsymbol{q}}$ are the homomorphisms in (45)–(46). The $s^\sharp$-*invariant* of $\widetilde{C}$ is defined by

$$s^\sharp(\widetilde{C}) := s_+^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}).$$

Remark 4.10 Our description in Definition 4.9 of the $s^\sharp$-invariant, using the divisibility of certain canonical classes, is reminiscent of an alternative description of the Rasmussen invariant given in [Sano 2020].

The equalities $x \cdot \xi_+^\sharp(\Lambda^n) = \xi_-^\sharp(\Lambda^n)$ and $\frac{1}{4}x \cdot \xi_-^\sharp(\Lambda^n) = \xi_+^\sharp(\Lambda^{n+2})$ imply the inequality

$$0 \le s_+^\sharp(\widetilde{C}) - s_-^\sharp(\widetilde{C}) \le 2. \tag{60}$$

Proposition 4.11 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over $\mathbb{Q}[\![\Lambda]\!]$ with $i := h(\widetilde{C}') - h(\widetilde{C}) \ge 0$. If there exists a height i morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ with $c_i = a\Lambda^k$ for some unit $a \in \mathbb{Q}[\![\Lambda]\!]$, we have*

$$s_+^\sharp(\widetilde{C}') \le s_+^\sharp(\widetilde{C}) + k \quad \text{and} \quad s_-^\sharp(\widetilde{C}') \le s_-^\sharp(\widetilde{C}) + k.$$

In particular, if $\widetilde{\lambda}$ is a local map (or, equivalently, $i = 0$ and $k = 0$), then $s_\pm^\sharp(\widetilde{C}') \le s_\pm^\sharp(\widetilde{C})$ for each sign.

Proof The proof is the same as Proposition 4.2, except we use Lemma 3.8 instead of Lemma 3.13. □

Proposition 4.12 *For any two $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over $\mathbb{Q}[\![\Lambda]\!]$, we have the inequalities*

$$s_+^\sharp(\widetilde{C} \otimes \widetilde{C}') \le \min\{s_+^\sharp(\widetilde{C}) + s_+^\sharp(\widetilde{C}'),\ s_-^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}') + 2\},$$
$$s_-^\sharp(\widetilde{C} \otimes \widetilde{C}') \le \min\{s_+^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}'),\ s_-^\sharp(\widetilde{C}) + s_+^\sharp(\widetilde{C}')\}.$$

Proof Let $s_\pm := s_\pm^\sharp(\widetilde{C})$, $s'_\pm := s_\pm^\sharp(\widetilde{C}')$, and $C_*^{\otimes\sharp}$ denote the framed complex of the tensor product of $\widetilde{C}$ and $\widetilde{C}'$. Then there exist elements

$$y_\pm \in H_{2h(\widetilde{C})-1\pm 1}(C^\sharp)^{\boldsymbol{q}} \quad \text{and} \quad y'_\pm \in H_{2h(\widetilde{C}')-1\pm 1}(C_*'^\sharp)^{\boldsymbol{q}}$$

such that $\Lambda^{n_\pm - s_\pm} y_\pm = \xi^\sharp_\pm(\Lambda^{n_\pm})$ and $\Lambda^{n'_\pm - s'_\pm} y'_\pm = \xi^\sharp_\pm(\Lambda^{n'_\pm})$. By Lemma 3.9, we have

$$\begin{aligned}
\Lambda^{n_+ + n'_+ - s_+ - s'_+} T^\sharp_* \circ t^\sharp(y_+ \otimes y'_+) &= \xi^\sharp_+(\Lambda^{n_+ + n'_+}),\\
\Lambda^{n_+ + n'_- - s_+ - s'_-} T^\sharp_* \circ t^\sharp(y_+ \otimes y'_-) &= \xi^\sharp_-(\Lambda^{n_+ + n'_-}),\\
\Lambda^{n_- + n'_+ - s_- - s'_+} T^\sharp_* \circ t^\sharp(y_- \otimes y'_+) &= \xi^\sharp_-(\Lambda^{n_- + n'_+}),\\
\tfrac{1}{4}\Lambda^{n_- + n'_- - s_- - s'_-} T^\sharp_* \circ t^\sharp(y_- \otimes y'_-) &= \xi^\sharp_+(\Lambda^{n_- + n'_- + 2}).
\end{aligned}$$

These complete the proof. □

To proceed, we impose Assumption 4.4, which in the current setting has the following characterization:

Lemma 4.13 *Assumption 4.4 for an $\mathcal{S}$-complex $\widetilde{C}$ is equivalent to the condition that $H_k(C^\sharp)$ has rank* 1 *if $k \pmod 4$ is even, and is zero if $k \pmod 4$ is odd.*

Proof The equivalence follows from the short exact sequence of R-chain complexes in (40), which splits over the field of fractions of $R = \mathbb{Q}[[\Lambda]]$. □

It follows from Lemma 4.13 and the injectivity of the maps $\xi^\sharp_\pm$ from Proposition 3.6 that, under Assumption 4.4, the natural quotient map $H(C^\sharp) \to H(C^\sharp)^{\boldsymbol{q}}$ is an isomorphism mod $\mathbb{Q}[[\Lambda]]$-torsion. In particular, $H(C^\sharp)/\mathrm{Tor}_{\mathbb{Q}[[\Lambda]]}$ can be used as the homology group appearing in the definitions of $s^\sharp_\pm$. This fact is used in the proof of the following:

Proposition 4.14 *Let $\widetilde{C}^\dagger$ be the dual of an $\mathcal{S}$-complex $\widetilde{C}$ over $\mathbb{Q}[[\Lambda]]$. If $\widetilde{C}$ satisfies Assumption 4.4, then*

$$s^\sharp_+(\widetilde{C}^\dagger) = -s^\sharp_-(\widetilde{C}) \quad \textit{and} \quad s^\sharp_-(\widetilde{C}^\dagger) = -s^\sharp_+(\widetilde{C}).$$

Proof Here we prove $s^\sharp_+(\widetilde{C}^\dagger) = -s^\sharp_-(\widetilde{C})$. Let $s := s^\sharp_-(\widetilde{C})$ and $s^\dagger := s^\sharp_+(\widetilde{C}^\dagger)$, and let $C^{\dagger\#}$ denote the unreduced framed complex of $\widetilde{C}^\dagger$. By the assumption, there exist generators

$$y_- \in H_{2h-2}(C^\sharp)/\mathrm{Tor}_{\mathbb{Q}[[\Lambda]]} \quad \text{and} \quad y^\dagger_+ \in H_{-2h}(C^{\dagger\#}_*)/\mathrm{Tor}_{\mathbb{Q}[[\Lambda]]}$$

such that $\Lambda^{n-s} y_- = \xi^\sharp_-(\Lambda^n)$ and $\Lambda^{n'-s^\dagger} y^\dagger_+ = \xi^\sharp_+(\Lambda^{n'})$. By Proposition 2.20, $H_{-2h}(C^{\dagger\sharp})/\mathrm{Tor}_{\mathbb{Q}[[\Lambda]]}$ is isomorphic to $\mathrm{Hom}(H_{2h-2}(C^\sharp), \mathbb{Q}[[\Lambda]])$. Under this identification, we see that $\langle y^\dagger_+, y_-\rangle = a$ for some unit $a \in \mathbb{Q}[[\Lambda]]$, and we have

$$\langle \xi^\sharp_+(\Lambda^{n'}), \xi^\sharp_-(\Lambda^n)\rangle = \langle \Lambda^{n'-s^\dagger} y^\dagger_+, \Lambda^{n-s} y_-\rangle = a\Lambda^{n+n'-s-s^\dagger}.$$

On the other hand, Proposition 3.11 gives

$$\langle \xi^\sharp_+(\Lambda^{n'}), \xi^\sharp_-(\Lambda^n)\rangle = -2\Lambda^{n+n'}.$$

These imply $s^\sharp_-(\widetilde{C}) + s^\sharp_+(\widetilde{C}^\dagger) = s + s^\dagger = 0$. Similarly, we can prove $s^\sharp_+(\widetilde{C}) + s^\sharp_-(\widetilde{C}^\dagger) = 0$. □

Proposition 4.15 *For $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4, we have*

$$\max\{s_+^\sharp(\widetilde{C})+s_-^\sharp(\widetilde{C}'), s_-^\sharp(\widetilde{C})+s_+^\sharp(\widetilde{C}')\} \le s_+^\sharp(\widetilde{C}\otimes\widetilde{C}') \le \min\{s_+^\sharp(\widetilde{C})+s_+^\sharp(\widetilde{C}'), s_-^\sharp(\widetilde{C})+s_-^\sharp(\widetilde{C}')+2\}, \tag{61}$$

$$\max\{s_-^\sharp(\widetilde{C})+s_-^\sharp(\widetilde{C}'), s_+^\sharp(\widetilde{C})+s_+^\sharp(\widetilde{C}')-2\} \le s_-^\sharp(\widetilde{C}\otimes\widetilde{C}') \le \min\{s_+^\sharp(\widetilde{C})+s_-^\sharp(\widetilde{C}'), s_-^\sharp(\widetilde{C})+s_+^\sharp(\widetilde{C}')\}. \tag{62}$$

Proof Combining Proposition 4.14 with Remark 4.5, we have

$$\begin{aligned} s_+^\sharp(\widetilde{C}\otimes\widetilde{C}') = -s_-^\sharp(\widetilde{C}^\dagger\otimes\widetilde{C}'^\dagger) &\ge -\min\{s_+^\sharp(\widetilde{C}^\dagger)+s_-^\sharp(\widetilde{C}'^\dagger), s_-^\sharp(\widetilde{C}^\dagger)+s_+^\sharp(\widetilde{C}'^\dagger)\} \\ &= \max\{s_+^\sharp(\widetilde{C})+s_-^\sharp(\widetilde{C}'), s_-^\sharp(\widetilde{C})+s_+^\sharp(\widetilde{C}')\}. \end{aligned}$$

The remaining part is proved similarly. □

Now we prove the $\mathcal{S}$-complex version of Theorem 1.3. (Note that for proving Theorem 1.3, we need the agreement of $s^\sharp(\widetilde{C}(S^3,K;\mathbb{Q}[\![\Lambda]\!]))$ with Kronheimer and Mrowka's $s^\sharp(K)$, discussed below.)

Theorem 4.16 *For two $\mathcal{S}$-complexes $\widetilde{C}$ and $\widetilde{C}'$ over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4, we have*

$$|s^\sharp(\widetilde{C}\otimes\widetilde{C}')-s^\sharp(\widetilde{C})-s^\sharp(\widetilde{C}')| \le 1.$$

Proof As a consequence of Proposition 4.15, we have the inequalities

$$\begin{aligned} s^\sharp(\widetilde{C})+s^\sharp(\widetilde{C}') \le 2s_+^\sharp(\widetilde{C}\otimes\widetilde{C}') &\le s^\sharp(\widetilde{C})+s^\sharp(\widetilde{C}')+2, \\ -2+s^\sharp(\widetilde{C})+s^\sharp(\widetilde{C}') \le 2s_-^\sharp(\widetilde{C}\otimes\widetilde{C}') &\le s^\sharp(\widetilde{C})+s^\sharp(\widetilde{C}'). \end{aligned}$$

These give the desired inequality. □

4.2.2 Recovering Kronheimer and Mrowka's $s^\sharp$

Write $s^\sharp(K)$ for Kronheimer and Mrowka's concordance invariant [2013], and $s_\pm^\sharp(K)$ for the refinements satisfying $s^\sharp(K)=s_+^\sharp(K)+s_-^\sharp(K)$, studied by Gong [2021]. Here we prove:

Theorem 4.17 *For any knot K in S^3, we have*

$$s_\pm^\sharp(K)=s_\pm^\sharp(\widetilde{C}(S^3,K;\Delta_{\mathbb{Q}[\![\Lambda]\!]})).$$

Consequently, $s^\sharp(K)$ agrees with $s^\sharp(\widetilde{C}(S^3,K;\Delta_{\mathbb{Q}[\![\Lambda]\!]}))$.

Before giving the proof, we review the definition of the concordance invariants $s_\pm^\sharp(K)$. The main ingredient in the definition is the framed version $I^\sharp(K)$ of singular instanton Floer homology defined in [Kronheimer and Mrowka 2013]. First we give some remarks on coefficient rings.

Kronheimer and Mrowka [2013] first define $I^\sharp(K)$ over the coefficient ring $\mathbb{Q}[u^{\pm 1}]$, and then over the ring $\mathbb{Q}[\![\lambda]\!]$. This latter ring is identified with the completion of the ring $\mathbb{Q}[u^{\pm 1}]$ localized at the prime ideal $(u-1)$. This completion is taken so that there is a square root for the expression

$$u+u^{-1}-2 \tag{63}$$

in the ring, one of which corresponds to λ. Our variable T is related to their u by $u = T^2$, and our $\Lambda = T - T^{-1}$, which is already a square root of (63) (so that a completion is in fact unnecessary), replaces the role of λ. More precisely, there is a natural map from $\mathbb{Q}[u^{\pm 1}]_{(u-1)} = \mathbb{Q}[T^{\pm 2}]_{(T^2-1)}$ to $\mathbb{Q}[T^{\pm 1}]_{(T-1)}$, which induces an isomorphism on completions, and which in turn is used to identify $\mathbb{Q}[\![\lambda]\!]$ with $\mathbb{Q}[\![\Lambda]\!]$. All homology groups $I^\sharp(K)$ below will be taken with the coefficient ring $\mathbb{Q}[\![\Lambda]\!]$. We also write $I^\sharp(K)' = I^\sharp(K)/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$.

Kronheimer and Mrowka [2013] prove that for any knot K in the 3-sphere, $I^\sharp(K)$ has rank 2 over $\mathbb{Q}[\![\Lambda]\!]$ in degrees 1 and $-1 \pmod 4$. In the case of the unknot U_1, these generators are respectively called u_+ and u_-. Next, take a normally immersed oriented surface cobordism S from U_1 to K in $I \times S^3$ with genus g and s_+ positive double points. Then S induces a map

$$m^\sharp(S)\colon \mathbb{Q}[\![\Lambda]\!]\langle u_+\rangle \oplus \mathbb{Q}[\![\Lambda]\!]\langle u_+\rangle = I^\sharp(U_1)' \to I^\sharp(K)'. \tag{64}$$

Then the invariants $s^\sharp_\pm(K)$ defined in [Gong 2021] are given by

$$s^\sharp_+(K) := g + s_+ - m_+(S), \quad s^\sharp_-(K) := g + s_+ - m_-(S),$$

where $m_\pm(S)$ are the maximal nonnegative integers satisfying, for some $y_\pm \in I^\sharp(K)'$ in degree ± 1,

$$m^\sharp(S)u_+ = \begin{cases} \Lambda^{m_+(S)} y_+ & \text{if } g \text{ is even,} \\ \Lambda^{m_+(S)} y_- & \text{if } g \text{ is odd,} \end{cases} \qquad m^\sharp(S)u_- = \begin{cases} \Lambda^{m_-(S)} y_- & \text{if } g \text{ is even,} \\ \Lambda^{m_-(S)} y_+ & \text{if } g \text{ is odd.} \end{cases}$$

The invariant $s^\sharp(K)$ from [Kronheimer and Mrowka 2013] is defined to be the sum $s^\sharp_+(K) + s^\sharp_-(K)$.

Proposition 4.18 [Kronheimer and Mrowka 2013] *For any knot K in S^3, we have*

$$s^\sharp_\pm(K^*) = -s^\sharp_\mp(K).$$

Proof We first take an embedded cobordism S from U to K appearing in the definition of $s^\sharp_\pm(K)$. We obtain a cobordism S^* from K^* to U which is the same as S, but the incoming and the outgoing ends are switched. This gives a cobordism map $m^\sharp(S^*)$ from $I^\sharp(K^*)'$ to $I^\sharp(U_1)'$. As is explained in [ibid., Lemma 3.2], this map is the dual of the map (64) where the generators $y^*_\pm$ of $I^\sharp(K^*)'$ in degrees ± 1 are identified with the dual basis using the relation $\langle y^*_\pm, y_\mp\rangle = 1$. Also, we take an embedded cobordism T from U to K^*, and, for simplicity, we assume that $g(S)$ and $g(T)$ are even. Then, since the composition $S^* \circ T$ is a cobordism from the unknot to itself,

$$0 = s^\sharp_\pm(U) = g(S^* \circ T) - m_\pm(S^* \circ T).$$

Now $m_\pm(S^* \circ T) = m_\pm(T) + m_\mp(S)$. We then compute

$$0 = s^\sharp_\pm(U) = g(T) + g(S) - m_\pm(T) - m_\mp(S) = s^\sharp_\mp(K) + s^\sharp_\pm(K^*). \qquad \square$$

For a knot $K \subset S^3$, write $C^\sharp(K)$ for the unreduced framed complex associated to the $\mathcal{S}$-complex $\widetilde{C}(S^3, K; \Delta_{\mathbb{Q}[\![\Lambda]\!]})$. Analogous to $I^\sharp(K)'$, the quotient of $H(C^\sharp(K))$ by $\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$ is denoted by $H(C^\sharp(K))'$.

Lemma 4.19 *Let $S\colon U \to K$ be an orientable surface cobordism in $I \times S^3$ with genus zero and s_+ positive double points. Suppose $\sigma(K) \le 0$. Then S induces a homomorphism*

$$\lambda_S^\sharp\colon H(C^\sharp(U))' \to H(C^\sharp(K))'. \tag{65}$$

This homomorphism sends the generators $u_\pm$ to $\xi_\pm^\sharp(\Lambda^{s^+})$.

Proof This follows from Lemma 3.8, and the fact that such a cobordism induces a morphism of height $h(S^3,K) = -\frac{1}{2}\sigma(K) \ge 0$, as explained in Section 2.1; see (18). Note that since Assumption 4.4 is satisfied for knots in the 3-sphere, in this setting $H(C^\sharp)^{\mathfrak{q}}/\mathrm{Tor}_{\mathbb{Q}[\![\Lambda]\!]}$ is isomorphic to $H(C^\sharp)'$. □

Now we give a proof of Theorem 4.17:

Proof of Theorem 4.17 From Proposition 4.18, one can assume $h(S^3,K) \ge 0$. Let $S\colon U_1 \to K$ be a negative definite cobordism with genus 0. Then S also induces the commutative diagram

$$\begin{array}{ccc} I^\sharp(U)' & \xrightarrow{m^\sharp(S)} & I^\sharp(K)' \\ \cong\downarrow & & \cong\downarrow \\ H(C^\sharp(U))' & \xrightarrow{\lambda_S^\sharp} & H(C^\sharp(K))' \end{array}$$

The vertical isomorphisms are from Theorem 2.16. By definition, $s_\pm^\sharp(K) = s_+ - m_\pm$, where $m_\pm$ are the maximal nonnegative integers such that $m^\sharp(S)u_\pm = \Lambda^m y_\pm$ for some elements $y_\pm$ in $I^\sharp(K)'$. Lemma 4.19 implies

$$\Lambda^{m_\pm} y_\pm = m^\sharp(S)u_\pm = \lambda_S^\sharp u_\pm = \xi_\pm^\sharp(\Lambda^{s^+}),$$

and hence we obtain

$$s_\pm^\sharp(\widetilde{C}(S^3,K;\Delta_{\mathbb{Q}[\![\Lambda]\!]})) = s^+ - m_\pm = s_\pm^\sharp(K). \qquad \square$$

The following is a restatement of Theorem 1.3:

Theorem 4.20 *For any pair of knots K and K' in the 3-sphere, we have*

$$|s^\sharp(K \# K') - s^\sharp(K) - s^\sharp(K')| \le 1.$$

Proof This follows from Theorems 4.17 and 4.16. □

Similarly, the inequalities of Section 4.2.1 give rise to connected sum inequalities for $s_\pm^\sharp$.

4.3 Relationship between $\tilde{s}$ and $s^\sharp$

We next study the relationship between the concordance homomorphism $\tilde{s}$ and the invariants $s_\pm^\sharp$. In the course of doing so, we establish that $2\tilde{s}$ is a slice-torus invariant, in the sense of [Lewark 2014]. We begin by studying the relationship between these invariants at the level of $\mathcal{S}$-complexes.

Proposition 4.21 *For any $\mathcal{S}$-complex $\widetilde{C}$ over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4, we have the inequalities*

$$\max\{s_+^\sharp(\widetilde{C})-1, s_-^\sharp(\widetilde{C})\} \le \tilde{s}(\widetilde{C}) \le \min\{s_+^\sharp(\widetilde{C}), s_-^\sharp(\widetilde{C})+1\}. \tag{66}$$

In particular,

$$|s^\sharp(\widetilde{C}) - 2\tilde{s}(\widetilde{C})| \le 1.$$

Proof By Propositions 4.3 and 4.14, we only need to prove the right-hand inequality in (66).

Recall the map $\pi_+ \colon C^\sharp \to \widetilde{C}^+$ in (39), defined over $\mathbb{Q}[\![\Lambda]\!]$ by the formula $[1\ \Lambda]$ with respect to the decomposition of $\widetilde{C}^+$. Regard a special $(h(\widetilde{C}), \Lambda^n)$-cycle in $\hat{\boldsymbol{C}}$ as a polynomial $Q(x)$ whose coefficients are chains of $\widetilde{C}$, and consider the decomposition $Q(x) = Q'(x^2) + xQ''(x^2)$. Using (33), we have

$$\begin{aligned}
(\pi_+)_*(\xi_+^\sharp(\Lambda^n)) &= (\pi_+)_* \circ \boldsymbol{p}_*^\sharp\big([Q'(x^2)+xQ''(x^2)]\big) = (\pi_+)_*\big([(Q'(4\Lambda^2), -2Q''(4\Lambda^2))]\big)\\
&= [Q'(4\Lambda^2) - 2\Lambda Q''(4\Lambda^2)] = \tilde{\boldsymbol{p}}_*\big([Q'(x^2)+xQ''(x^2)]\big) = \tilde{\xi}(\Lambda^n),\\
(\pi_+)_*(\xi_-^\sharp(\Lambda^n)) &= (\pi_+)_* \circ \boldsymbol{p}_*^\sharp\big([xQ'(x^2)+x^2Q''(x^2)]\big) = (\pi_+)_*\big([(4\Lambda^2 Q''(4\Lambda^2), -2Q'(4\Lambda^2))]\big)\\
&= [4\Lambda^2 Q''(4\Lambda^2) - 2\Lambda Q'(4\Lambda^2)] = -2\Lambda\, \tilde{\boldsymbol{p}}_*\big([Q'(x^2)+xQ''(x^2)]\big) = -2\tilde{\xi}(\Lambda^{n+1}).
\end{aligned}$$

These complete the proof. □

We give two applications of Proposition 4.21. The first is a description of $\tilde{s}$ as a limit.

Proposition 4.22 *For any $\mathcal{S}$-complex $\widetilde{C}$ over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4, we have*

$$\tilde{s}(\widetilde{C}) = \tfrac{1}{2} \lim_{n\to\infty} \frac{s^\sharp(\widetilde{C}^{\otimes n})}{n},$$

where $\widetilde{C}^{\otimes n}$ denotes the tensor product of n copies of $\widetilde{C}$.

Proof This immediately follows from the inequality

$$|s^\sharp(\widetilde{C}^{\otimes n}) - n \cdot 2\tilde{s}(\widetilde{C})| \le 1$$

given by Propositions 4.6 and 4.21. □

As the $\mathcal{S}$-complexes of knots in the 3-sphere satisfy Assumption 4.4, we obtain the following:

Corollary 4.23 *For a knot K in the 3-sphere, we have*

$$\tilde{s}(K) = \tfrac{1}{2} \lim_{n\to\infty} \frac{s^\sharp(\#_n K)}{n}.$$

This description allows us to prove the slice-torus property [Lewark 2014] of $2\tilde{s}$, where $\tilde{s} \colon \mathcal{C} \to \mathbb{Z}$ is the concordance homomorphism of (57). Write $g_4(K)$ for the smooth 4-ball genus of K.

Theorem 4.24 *The map $\tilde{s}: \mathcal{C} \to \mathbb{Z}$ is half a slice-torus invariant, ie $\tilde{s}$ satisfies the following properties*:

(i) $\tilde{s}(K)$ *is a homomorphism.*

(ii) $\tilde{s}(K) \le g_4(K)$.

(iii) $\tilde{s}(T_{p,q}) = g_4(T_{p,q})$, *where $T_{p,q}$ denotes the (p,q) torus knot.*

The equality $\tilde{s}(K) = g_4(K)$ holds more generally for any quasipositive knot.

Proof Property (i) immediately follows from Theorem 5.8 and Proposition 4.3. Property (ii) follows from Corollary 4.23 and the genus bound for $s^\sharp$ given by

$$s^\sharp(K) \le 2g_4(K),$$

which is a direct consequence of Kronheimer and Mrowka's construction of $s^\sharp(K)$ [2013]. Indeed,

$$g_4(K) \ge \frac{g_4(\#_n K)}{n} \ge \frac{1}{2}\left(\frac{s^\sharp(\#_n K)}{n}\right) \to \tilde{s}(K) \quad (n \to \infty).$$

Property (iii) and the last statement, regarding quasipositive knots, follow from Corollary 4.23 and [Gong 2021, Proposition 1.8]: for any quasipositive knot,

$$2g_4(K) - 1 \le s^\sharp(K) \le 2g_4(K). \tag{67}$$

Indeed, since the connected sum of two quasipositive knots is also quasipositive, and g_4 is additive among quasipositive knots [Rudolph 1993], the inequalities (67) imply

$$2g_4(K) - \frac{1}{n} \le \frac{s^\sharp(\#_n K)}{n} \le 2g_4(K)$$

for any $n \in \mathbb{Z}_{>0}$. Thus, $\tilde{s}(K) = \frac{1}{2}\lim_{n\to\infty} s^\sharp(\#_n K)/n = g_4(K)$. □

Using [Lewark 2014, Corollary 5.9], we also obtain the following. Note that this result together with Theorem 4.24 proves Corollary 1.2 from the introduction.

Corollary 4.25 *For an alternating knot K, we have*

$$\tilde{s}(K) = -\tfrac{1}{2}\sigma(K).$$

In particular,

$$|s^\sharp(K) + \sigma(K)| \le 1.$$

As the second application of Proposition 4.21, we give a new $\{-1, 0, 1\}$-valued concordance invariant $\tilde{\varepsilon}$, which behaves similarly to Hom's ε-invariant [2014, Definition 3.4] in Heegaard Floer theory.

Definition 4.26 For an $\mathcal{S}$-complex $\widetilde{C}$ over $\mathbb{Q}[\![\Lambda]\!]$, we define

$$\tilde{\varepsilon}(\widetilde{C}) := 2\tilde{s}(\widetilde{C}) - s^\sharp(\widetilde{C}).$$

By Proposition 4.21, if the $\mathcal{S}$-complex $\widetilde{C}$ satisfies Assumption 4.4, then $\tilde{\varepsilon}(\widetilde{C}) \in \{-1, 0, 1\}$. For any knot K in an integer homology 3-sphere Y, we define

$$\tilde{\varepsilon}(Y, K) := \tilde{\varepsilon}(\widetilde{C}(Y, K; \Delta_{\mathbb{Q}[\![\Lambda]\!]})).$$

In the case that $Y = S^3$, we abbreviate $\tilde{\varepsilon}(S^3, K)$ to $\tilde{\varepsilon}(K)$. In particular,

$$\tilde{\varepsilon}(K) \in \{-1, 0, 1\}.$$

Proof of Theorem 1.1 The result follows from Theorem 4.24 and the above properties of $\tilde{\varepsilon}$. □

We show the following properties of $\tilde{\varepsilon}$, analogous to [Hom 2014, Proposition 3.6]:

Proposition 4.27 *The invariant $\tilde{\varepsilon}$ satisfies the following properties for knots in S^3:*

(i) *If K is smoothly slice, then $\tilde{\varepsilon}(K) = 0$.*

(ii) $\tilde{\varepsilon}(-K) = -\tilde{\varepsilon}(K)$.

(iii) (a) *If $\tilde{\varepsilon}(K) = \tilde{\varepsilon}(K')$, then $\tilde{\varepsilon}(K \# K') = \tilde{\varepsilon}(K) = \tilde{\varepsilon}(K')$.*

(b) *If $\tilde{\varepsilon}(K) = 0$, then $\tilde{\varepsilon}(K \# K') = \tilde{\varepsilon}(K')$.*

To prove the proposition, we introduce three types of $\mathcal{S}$-complex over $\mathbb{Q}[\![\Lambda]\!]$ according to (60). See also the discussion in [Gong 2021, Proposition 1.7].

Definition 4.28 Let $\widetilde{C}$ be an $\mathcal{S}$-complex over $\mathbb{Q}[\![\Lambda]\!]$. We call $\widetilde{C}$ *Type O*, *Type I* and *Type II* if the value of $s^\sharp_+(\widetilde{C}) - s^\sharp_-(\widetilde{C})$ is equal to 0, 1 and 2, respectively.

Lemma 4.29 *For an $\mathcal{S}$-complex $\widetilde{C}$ over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4, we have the following:*

(i) *$\widetilde{C}$ is Type O if and only if $\tilde{s}(\widetilde{C}) = s^\sharp_+(\widetilde{C}) = s^\sharp_-(\widetilde{C})$.*

(ii) *$\widetilde{C}$ is Type I if and only if $\tilde{s}(\widetilde{C}) = s^\sharp_+(\widetilde{C}) = s^\sharp_-(\widetilde{C}) + 1$ or $\tilde{s}(\widetilde{C}) = s^\sharp_+(\widetilde{C}) - 1 = s^\sharp_-(\widetilde{C})$.*

(iii) *$\widetilde{C}$ is Type II if and only if $\tilde{s}(\widetilde{C}) = s^\sharp_+(\widetilde{C}) - 1 = s^\sharp_-(\widetilde{C}) + 1$.*

In particular, $\tilde{\varepsilon}(\widetilde{C}) = 0$ if and only if $\widetilde{C}$ is Type O or Type II.

Proof All assertions directly follow from the inequalities (66). □

Lemma 4.30 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over $\mathbb{Q}[\![\Lambda]\!]$ satisfying Assumption 4.4.*

(i) *If $\widetilde{C}$ is Type O, then $s^\sharp_+(\widetilde{C} \otimes \widetilde{C}') = s^\sharp_+(\widetilde{C}) + s^\sharp_+(\widetilde{C}')$ and $s^\sharp_-(\widetilde{C} \otimes \widetilde{C}') = s^\sharp_-(\widetilde{C}) + s^\sharp_-(\widetilde{C}')$.*

(ii) *If $\widetilde{C}$ is Type II, then $s^\sharp_+(\widetilde{C} \otimes \widetilde{C}') = s^\sharp_+(\widetilde{C}) + s^\sharp_-(\widetilde{C}')$ and $s^\sharp_-(\widetilde{C} \otimes \widetilde{C}') = s^\sharp_-(\widetilde{C}) + s^\sharp_+(\widetilde{C}')$.*

In particular, if $\tilde{\varepsilon}(\widetilde{C}) = 0$, then $s^\sharp(\widetilde{C} \otimes \widetilde{C}') = s^\sharp(\widetilde{C}) + s^\sharp(\widetilde{C}')$.

Proof Suppose that $\widetilde{C}$ is Type O. Then Proposition 4.15 gives

$$s^\sharp_+(\widetilde{C}) + s^\sharp_+(\widetilde{C}') = s^\sharp_-(\widetilde{C}) + s^\sharp_+(\widetilde{C}') \le s^\sharp_+(\widetilde{C} \otimes \widetilde{C}') \le s^\sharp_+(\widetilde{C}) + s^\sharp_+(\widetilde{C}'),$$
$$s^\sharp_-(\widetilde{C}) + s^\sharp_-(\widetilde{C}') \le s^\sharp_-(\widetilde{C} \otimes \widetilde{C}') \le s^\sharp_+(\widetilde{C}) + s^\sharp_-(\widetilde{C}') = s^\sharp_-(\widetilde{C}) + s^\sharp_-(\widetilde{C}').$$

Similarly, if $\widetilde{C}$ is Type II, then

$$s_+^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}') \le s_+^\sharp(\widetilde{C} \otimes \widetilde{C}') \le s_-^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}') + 2 = s_+^\sharp(\widetilde{C}) + s_-^\sharp(\widetilde{C}'),$$
$$s_-^\sharp(\widetilde{C}) + s_+^\sharp(\widetilde{C}') = s_+^\sharp(\widetilde{C}) + s_+^\sharp(\widetilde{C}') - 2 \le s_-^\sharp(\widetilde{C} \otimes \widetilde{C}') \le s_-^\sharp(\widetilde{C}) + s_+^\sharp(\widetilde{C}').$$

Finally, Lemma 4.29 completes the proof. □

Now we prove Proposition 4.27:

Proof of Proposition 4.27 If K is smoothly slice, then $\tilde{\varepsilon}(K) = 2\tilde{s}(K) - s^\sharp(K) = 0$ follows from the slice genus bounds for $\tilde{s}$ and $s^\sharp$ (see Theorem 1.1). By Propositions 4.6 and 4.14, we have property (ii):

$$\tilde{\varepsilon}(-K) = 2\tilde{s}(-K) - s^\sharp(-K) = -(2\tilde{s}(K) - s^\sharp(K)) = -\tilde{\varepsilon}(K).$$

Let us consider property (iii). By Lemma 4.30 and property (ii), it suffices to prove the assertion (a) for the case $\tilde{\varepsilon}(K) = \tilde{\varepsilon}(K') = 1$. Indeed, Proposition 4.6 and Theorem 4.16 imply

$$\tilde{\varepsilon}(K \# K') = 2\tilde{s}(K \# K') - s^\sharp(K \# K') \ge (2\tilde{s}(K) - s^\sharp(K)) + (2\tilde{s}(K') - s^\sharp(K')) - 1 = 1. \qquad \square$$

Motivated by [Hom 2014, Proposition 3.6], we expect a positive answer to the following question:

Question 4.31 *Does $\tilde{\varepsilon}$ satisfy the following properties?*

(i) *If $\tilde{\varepsilon}(K) = 0$, then $\tilde{s}(K) = 0$.*

(ii) *If $|\tilde{s}(K)| = g(K)$, where $g(K)$ is the Seifert genus of K, then $\tilde{\varepsilon}(K) = \operatorname{sgn} \tilde{s}(K)$.*

(iii) *If K is homologically thin in the sense of Heegaard Floer theory, then $\tilde{\varepsilon}(K) = \operatorname{sgn} \tilde{s}(K)$.*

We expect that one can define a corresponding homologically thinness for instanton theory in terms of $\widetilde{C}$. Item (iii) is proved below for the special case of two-bridge knots in Proposition 4.53, and the proof gives some indication of how one might define homological thinness in terms of $\mathcal{S}$-complexes.

4.4 Fractional ideal invariants from instantons

Several concordance invariants for knots $K \subset S^3$ were introduced recently by Kronheimer and Mrowka [2021b], using a version of singular instanton homology defined for webs. Building on [Daemi and Scaduto 2024b], we define an $R[x]$-submodule $\hat{z}(K)$ inside the field of fractions of $R[x]$ which is constructed from the $\mathcal{S}$-complex of the knot, using special cycles. We show that $\hat{z}(K)$ specializes, upon changing coefficients, to Kronheimer and Mrowka's fractional ideal $z_\sigma^\natural(K)$ [2021b]. As a consequence of this relationship, all of the concordance invariants defined in [loc. cit.] depend only on the local equivalence class of the $\mathcal{S}$-complex $\widetilde{C}(K; \Delta)$.

Let R be an integral domain and M be an arbitrary R-module. For two R-submodules N_1 and N_2 in M, recall that the ideal $[N_1 : N_2]$ is the R-submodule of $\mathrm{Frac}(R)$ given by

$$[N_1 : N_2] = \left\{ \frac{a}{b} \;\middle|\; aN_2 \subset bN_1 \right\}.$$

For $m \in M$, we write $[N_1 : m] := [N_1 : N_2]$, where N_2 is the submodule generated by m. Note that if m is altered by a torsion element, then $[N_1 : m]$ is unchanged. For the remainder of this section, R will denote the ring $\mathbb{Z}[T^{\pm 1}]$.

We now review the construction of the ideals $z_\sigma^\natural(K)$ for any knot K in the 3-sphere. Let $I^\natural(K;\mathscr{S})$ be the instanton homology as defined in [Kronheimer and Mrowka 2021b], using local coefficients over

$$\mathscr{S} = \boldsymbol{F}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}],$$

where $\boldsymbol{F}$ is the field with two elements. In what follows, we will often ignore torsion. Thus we write

$$I^\natural(K;\mathscr{S})' = I^\natural(K;\mathscr{S})/\mathrm{Tor}_{\mathscr{S}}.$$

Kronheimer and Mrowka show that $I^\natural(K;\mathscr{S})' \cong \mathscr{S}$ as an $\mathscr{S}$-module. Let $S : U_1 \to K$ be an orientable surface cobordism in $[0,1] \times S^3$, possibly immersed, with transverse double points. Let g be its genus, and s_+ be the number of positive double points. There is an induced $\mathscr{S}$-module homomorphism

$$I^\natural(S) : I^\natural(U_1;\mathscr{S})' \to I^\natural(K;\mathscr{S})'.$$

With these choices, define $z^\natural(K) \subset \mathrm{Frac}(\mathscr{S})$ by

$$z^\natural(K) := P^g (T_1^2 - T_1^{-2})^{s_+} [I^\natural(K;\mathscr{S})' : I^\natural(S)(1)]. \tag{68}$$

Here, the element P is the symmetric Laurent polynomial defined by

$$P = T_1T_2T_3 + T_1T_2^{-1}T_3^{-1} + T_1^{-1}T_2T_3^{-1} + T_1^{-1}T_2^{-1}T_3.$$

If $\sigma : \mathscr{S} \to \mathscr{S}'$ is a ring homomorphism, we obtain an ideal $z_\sigma^\natural(K) \subset \mathrm{Frac}(\mathscr{S}')$ by base change. Kronheimer and Mrowka prove that $z_\sigma^\natural(K)$ depends only on the knot K, and is a concordance invariant.

Remark 4.32 For defining cobordism maps of immersed surfaces, we are following the conventions of [Daemi and Scaduto 2024a, Section 2.5]. The definition of maps for immersed surfaces in [Kronheimer and Mrowka 2021b] is slightly different and is given as the sum of maps of the form $I^\natural(\overline{W}, \overline{S}, c)$, where W is the blow-up of $[0,1] \times S^3$ at the double points, $\overline{S}$ is the proper transform of S, and c runs over a set of degree two cohomology classes of exceptional spheres. The definition of $z^\natural(K)$ given above, at least when $s_+ \neq 0$, is slightly different from the one in [loc. cit.] because of this difference in convention. However, the above definition gives the same invariant $z^\natural(K)$ as in [loc. cit.] because we can use a surface cobordism S with $s_+ = 0$, and then use the invariance of both definitions from the choice of S.

The theory $I(K;\mathscr{S})'$ used in the above construction is related to the $\mathcal{S}$-complex of a knot as follows:

Theorem 4.33 [Daemi and Scaduto 2024b, Corollary 8.41] *There is a natural isomorphism of $\mathscr{S}$-modules*

$$I^\natural(K) \cong \widehat{I}(K;\Delta) \otimes_{\mathbb{Z}[T^{\pm 1}, x]} \mathscr{S},$$

where on the right the $\mathbb{Z}[T^{\pm 1}, x]$-algebra structure of $\mathscr{S}$ is given by reducing coefficients mod 2, and

$$T \mapsto T_1, \quad x \mapsto P. \tag{69}$$

We now introduce a local equivalence invariant associated to any $\mathcal{S}$-complex, which we will show below can be used to recover Kronheimer and Mrowka's invariants $z^{\natural}_{\sigma}(K)$.

Definition 4.34 Let R be an integral domain, $\widetilde{C}$ an $\mathcal{S}$-complex over R, $h := h(\widetilde{C})$, $f \in J_h(\widetilde{C}) \setminus \{0\}$, and $\hat{\xi}\colon J_h(\widetilde{C}) \to H_{2h}(\hat{C})/\mathrm{Tor}_{R[x^2]}$ the R-homomorphism in (44). Define

$$\hat{z}(\widetilde{C}) := f[H(\hat{C}) : \hat{\xi}(f)] \subset \mathrm{Frac}(R[x]).$$

Proposition 4.35 *$\hat{z}(\widetilde{C})$ is independent of the choice of $f \in J_h(\widetilde{C})$.*

Proof This follows from the general property $[N_1 : N_2] = r[N_1 : rN_2]$ for any $r \in R$. □

Proposition 4.36 *Let $\widetilde{C}$ and $\widetilde{C}'$ be $\mathcal{S}$-complexes over R with $i := h(\widetilde{C}') - h(\widetilde{C}) \geq 0$. If there exists a height i morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$, then, with c_i as defined in* (15), *we have*

$$c_i \cdot \hat{z}(\widetilde{C}) \subset \hat{z}(\widetilde{C}').$$

In particular, if $\widetilde{\lambda}$ is a local map (*or, equivalently, $i = 0$ and c_i is a unit of R*), *then $\hat{z}(\widetilde{C}) \subset \hat{z}(\widetilde{C}')$.*

Proof Lemma 3.3 implies $\hat{\boldsymbol{\lambda}}_*(\hat{\xi}(f)) = \hat{\xi}(c_i f) \in H_*(\hat{\boldsymbol{C}}'_*)$. Hence,

$$\begin{aligned} c_i \hat{z}(\widetilde{C}) = c_i f \left\{ \frac{a}{b} \;\middle|\; a\hat{\xi}(f) \in bH_*(\hat{\boldsymbol{C}}_*) \right\} &\subset c_i f \left\{ \frac{a}{b} \;\middle|\; a\hat{\boldsymbol{\lambda}}_*(\hat{\xi}(f)) \in b\hat{\boldsymbol{\lambda}}_*(H_*(\hat{\boldsymbol{C}}_*)) \right\} \\ &\subset c_i f \left\{ \frac{a}{b} \;\middle|\; a\hat{\xi}(c_i f) \in bH_*(\hat{\boldsymbol{C}}'_*) \right\} = \hat{z}(\widetilde{C}'). \end{aligned}$$ □

It follows that the $R[x]$-submodule $\hat{z}(\widetilde{C}) \subset \mathrm{Frac}(R[x])$ is a local equivalence invariant of $\widetilde{C}$.

Definition 4.37 Let K be a knot in an integer homology 3-sphere Y. With $R = \mathbb{Z}[T^{\pm 1}]$, define

$$\hat{z}(Y, K) := \hat{z}(\widetilde{C}(Y, K; \Delta)).$$

This is an $R[x]$-submodule $\hat{z}(Y, K) \subset \mathrm{Frac}(R[x])$. If $Y = S^3$, we simply write $\hat{z}(K)$.

From above, we see that $\hat{z}(Y, K)$ is a homology concordance invariant of (Y, K). For the remainder of this section we focus on the case of knots in the 3-sphere.

To relate $\hat{z}(K)$ to $z^{\natural}_{\sigma}(K)$, we give a cobordism interpretation of $\hat{z}(\widetilde{C})$, similar to (68). Take an immersed cobordism $S\colon U_1 \to K$ as above, with $g = 0$. If $h := -\frac{1}{2}\sigma(K) \geq 0$, we obtain the induced morphism $\widetilde{\lambda}_S\colon \widetilde{C}(U_1; \Delta) \to \widetilde{C}(K; \Delta)$ of height h, with c_h equal to $(T^2 - T^{-2})^{s_+}$, up to a unit. Thus,

$$\hat{I}(S)(1) = \hat{I}(S)(\hat{\xi}(1)) = \hat{\xi}((T^2 - T^{-2})^{s_+}), \tag{70}$$

where $\hat{I}(S) = (\hat{\lambda}_S)_*$. From this, we immediately obtain

$$\hat{z}(K) = (T^2 - T^{-2})^{s_+}[\hat{I}(K; \Delta) : \hat{I}(S)(1)]. \tag{71}$$

Using Theorem 4.33 and its naturality with respect to cobordism maps, we see that $z^{\natural}(K)$ is obtained from $\hat{z}(K)$ by tensoring with $\mathscr{S}$, using the base change described in (69).

The above argument leading to (71) only works for knots with nonpositive signature. To work around this, we extend our discussion to cobordisms $S\colon K' \to K$ inside $[0,1]\times S^3$ which, as above, have genus g and some double points, but for which K' is not necessarily the unknot. From (18), if

$$h := \tfrac{1}{2}\sigma(K') - \tfrac{1}{2}\sigma(K) - g \geq 0,$$

then such a cobordism induces a morphism of $\mathcal{S}$-complexes of height h, with c_h equal to $(T^2 - T^{-2})^{s_+}$ up to a unit. Now, if $-\frac{1}{2}\sigma(K) < 0$, S has genus zero and $K' = \#_l\, T^*_{2,3}$, where $l \geq \frac{1}{2}\sigma(K)$, then $h \geq 0$, and so S induces such a morphism. Having replaced U_1 by a connected sum of left-handed trefoils, we require the following:

Lemma 4.38 *Let $l \geq 0$. Then $J_{-l}(\#_l\, T^*_{2,3}) = R$, and $\hat{\xi}(1)$ is a generator of $\hat{I}(\#_l\, T^*_{2,3};\Delta)' \cong R[x]$.*

Proof The $\mathcal{S}$-complex for $T^*_{2,3}$ is $\widetilde{C} = C_* \oplus C_{*-1} \oplus R_{(0)}$, where C_* has one generator α in degree 2 (mod 4); further, $d = \delta_1 = v = 0$ and $\delta_2(1) = (T^2 - T^{-2})\alpha$. For details, see [Daemi and Scaduto 2024b, Section 9]. Thus, $\hat{\mathfrak{C}} = \langle\alpha\rangle \oplus R[x]$ with differential

$$\hat{\mathfrak{d}}\Bigl(a\alpha, \sum_{i=0}^{N} a_i x^i\Bigr) = -(a_0(T^2 - T^{-2})\alpha, 0).$$

The $R[x]$-module structure on $\hat{\mathfrak{C}}$ is the obvious one on $\langle\alpha\rangle \oplus R[x]$, where $x\cdot\alpha = 0$. Thus $\hat{I}(T^*_{2,3};\Delta)$ is isomorphic to $R[x] \oplus R/(T^2 - T^{-2})R$ via mapping the cycle $(0, x^i)$ for $i \geq 1$ to $x^{i-1} \in R[x]$, and mapping $(\alpha, 0)$ to $R/(T^2 - T^{-2})R$. In particular, $\hat{I}(T^*_{2,3};\Delta)' \cong R[x]$. The map $\mathfrak{i}$ sends $(0, x^i)$ to x^i. Thus $\operatorname{Im}\mathfrak{i}_* = xR[x] \subset R[\![x^{-1}, x]$. From this we obtain $h(T^*_{2,3}) = -1$ and $J_{-1}(\widetilde{C}) = R$. A special $(-1,1)$-cycle is given by $\hat{\Psi}(\mathfrak{z})$ where $\mathfrak{z} = (0, x)$, and so $\hat{\xi}(1)$ maps to the generator $x \in xR[x] \cong \hat{I}(T^*_{2,3};\Delta)'$.

The result for $l > 1$ follows from the $l = 1$ case using the connected sum theorem. □

We will also make use of the following:

Lemma 4.39 *Let $S\colon T^*_{2,3} \to U_1$ be the immersed cobordism induced by the movie of a crossing change, from negative to positive. Then there is an identification $\hat{I}(T^*_{2,3})' \cong R[x]$ such that the map $\hat{I}(S)\colon \hat{I}(T^*_{2,3})' \to \hat{I}(U_1)' = R[x]$ is multiplication by $T^2 - T^{-2}$.*

Proof The cobordism S induces a height 1 morphism $\widetilde{\lambda}_S\colon \widetilde{C}(T^*_{2,3};\Delta) \to \widetilde{C}(U_1;\Delta)$, which is determined by the component Δ_1. By the arguments in [Daemi and Scaduto 2024b], $\Delta_1\delta_2(1)$ is equal to a count of two reducibles, the latter of which is $(T^2 - T^{-2})^{s_+} = T^2 - T^{-2}$. Thus $\widetilde{\lambda}_S$ is a height 1 morphism with $c_1 = \Delta_1\delta_2(1)$ equal to $T^2 - T^{-2}$ up to a unit. The result follows. □

With these preliminaries we now prove the main result of this section.

Theorem 4.40 *Kronheimer and Mrowka's invariant $z^\natural_\sigma(K) \subset \operatorname{Frac}(\mathscr{S})$ can be recovered from $\hat{z}(K)$ as*

$$z^\natural_\sigma(K) = \hat{z}(K) \otimes_{\mathbb{Z}[T^{\pm 1}, x]} \mathscr{S}',$$

where the module structure defining the tensor product uses (69) *and the base change $\sigma\colon \mathscr{S} \to \mathscr{S}'$.*

Proof We prove the result for $\sigma = \mathrm{id}_{\mathscr{S}}$. The more general case follows easily.

As already indicated, if $-\frac{1}{2}\sigma(K) \geq 0$, the result follows from (71) and the isomorphisms in Theorem 4.33, which are natural with respect to cobordism maps (when they are defined).

Suppose $-\frac{1}{2}\sigma(K) < 0$. Choose a cobordism $S\colon \#_l\, T^*_{2,3} \to K$ in the cylinder with genus 0 and s_+ double points, and $l \geq \frac{1}{2}\sigma(K)$. Specifically, choose S so that it is a composition of $S'\colon \#_l\, T^*_{2,3} \to U_1$ and $S''\colon U_1 \to K$, where S' is a connected sum of cobordisms $T^*_{2,3} \to U_1$ each induced by a negative to positive crossing change.

By Lemma 4.38, we still have (70), and thus (71) for $\hat{z}(K)$ is true for the cobordism S. By Lemma 4.39 and Theorem 4.33, the map $I^\natural(S')\colon I^\natural(\#_l\, T^*_{2,3})' \to I^\natural(U_1)'$ is up to a unit equal to multiplication by $(T_1^2 - T_1^{-2})^l$. Using this input, we now compute

$$\begin{aligned}\hat{z}(K) \otimes_{\mathbb{Z}[T^{\pm 1},x]} \mathscr{S} &= (T_1^2 - T_1^{-2})^{s_+}[I^\natural(K;\mathscr{S})' : I^\natural(S)(1)]\\ &= (T_1^2 - T_1^{-2})^{s_+}[I^\natural(K;\mathscr{S})' : (T_1^2 - T_1^{-2})^l I^\natural(S'')(1)]\\ &= (T_1^2 - T_1^{-2})^{s_+ - l}[I^\natural(K;\mathscr{S})' : I^\natural(S'')(1)].\end{aligned}$$

As $s_+ - l$ is the number of positive double points of $S''\colon U_1 \to K$, this last expression is $z^\natural(K)$. □

The cobordism interpretation for $\hat{z}(K)$ as in (71) extends to the nonzero genus case. First we need to study further the invariants of the connected sums of $T_{2,3}$ and $T^*_{2,3}$.

Lemma 4.41 *Let $S\colon T^*_{2,3} \to U_1$ be a genus 1 embedded cobordism. Then $\widehat{I}(S)$ is multiplication by x.*

Proof Such a cobordism induces a morphism $\widetilde{C}(T^*_{2,3};\Delta) \to \widetilde{C}(U_1;\Delta)$ of height $\frac{1}{2}\sigma(T^*_{2,3}) - g = 0$, with $c_0 = 1$. This morphism is determined by mapping the reducible summand $R_{(0)} \subset \widetilde{C}(T_{2,3};\Delta)$ isomorphically to $R_{(0)} = \widetilde{C}(U_1;\Delta)$. From this and the isomorphism of $\widehat{I}(T^*_{2,3})' \cong R[x]$ given in the proof of Lemma 4.38, we see that the induced map on $\widehat{I}$ is multiplication by x. □

Lemma 4.42 *For $l > 0$, we have $\widehat{I}(\#_l\, T_{2,3};\Delta) \cong \mathcal{I}^l$, where*

$$\mathcal{I}^l = (x^l, x^{l-1}(T^2 - T^{-2}), x^{l-2}(T^2 - T^{-2})^2, \dots, (T^2 - T^{-2})^l). \tag{72}$$

Moreover, $J_l(\#_l\, T_{2,3}) = ((T^2 - T^{-2})^l)$ and $\hat{\xi}((T^2 - T^{-2})^l) = (T^2 - T^{-2})^l \in \mathcal{I}^l$.

Proof The $\mathcal{S}$-complex for $T_{2,3}$ is given by $\widetilde{C} = C_* \oplus C_{*-1} \oplus R_{(0)}$, where C_* has a single generator α in degree 1 (mod 4); further, $d = \delta_2 = v = 0$ and $\delta_1(\alpha) = T^2 - T^{-2}$. See [Daemi and Scaduto 2024b, Section 9]. Thus $\widehat{\mathfrak{C}}(T_{2,3}) = \langle\alpha\rangle \oplus R[x]$ with zero differential, and so $\widehat{I}(T_{2,3};\Delta) = \widehat{\mathfrak{C}}(T_{2,3})$. The $R[x]$-module structure is

$$x \cdot \left(a\alpha, \sum_{i=0}^N a_i x^i\right) = \left(0, (T^2 - T^{-2})a + \sum_{i=0}^N a_i x^i\right).$$

That means that $\widehat{I}(T_{2,3};\Delta)$ is isomorphic as an $R[x]$-module to $\mathcal{I}^1 = (x, T^2 - T^{-2})$ by sending $(\alpha, 0)$ to $T^2 - T^{-2}$ and $(0, x^i)$ to x^{i+1}. Furthermore, $\mathfrak{i}\big(a\alpha, \sum_{i=0}^N a_i x^i\big) = a(T^2 - T^{-2})x^{-1} + \sum_{i=0}^N a_i x^i$. Thus $h(T_{2,3}) = 1$ and $J_1(\widetilde{C}) = (T^2 - T^{-2}) \subset R$. A special $(1, T^2 - T^{-2})$-cycle is given by $\widehat{\Psi}(\mathfrak{z})$, where $\mathfrak{z} = (\alpha, 0)$, and so $\hat{\xi}(T^2 - T^{-2})$ corresponds under the isomorphism $\widehat{I}(T_{2,3};\Delta) \cong \mathcal{I}^1$ to the element $T^2 - T^{-2}$. The completes the proof in the case $l = 1$. The more general case follows in a similar fashion, making use of the connected sum theorem. □

The following lemma can be proved in the same way as Lemmas 4.39 and 4.41:

Lemma 4.43 *Let $S\colon U_1 \to T_{2,3}$ be the immersed cobordism induced by the movie of a crossing change, from negative to positive. With respect to the identification of $\widehat{I}(T_{2,3})$ with $\mathcal{I}$ in Lemma 4.42, the cobordism map $\widehat{I}(S)\colon \widehat{I}(U_1) = R[x] \to \widehat{I}(T_{2,3})$ is given by multiplication by $T^2 - T^{-2}$. If $S'\colon U_1 \to T_{2,3}$ is an embedded cobordism of genus 1, then the cobordism map $\widehat{I}(S)\colon \widehat{I}(U_1) = R[x] \to \widehat{I}(T_{2,3})$ is given by multiplication by x.*

Now suppose $S\colon \#_l\, T^*_{2,3} \to K$ is an immersed cobordism with genus g and s_+ positive double points, and for which $l \geq \frac{1}{2}\sigma(K) + g$. Then

$$\hat{z}(K) = x^g (T^2 - T^{-2})^{s_+} [\widehat{I}(K;\Delta)' : \widehat{I}(S)(1)]. \tag{73}$$

To see this, we first assume that $l \geq g$, and that S is the composition of $S'\colon \#_l\, T^*_{2,3} \to \#_{l-g}\, T^*_{2,3}$ and $S''\colon \#_{l-g}\, T^*_{2,3} \to K$ where S' is embedded and has genus g, and S'' has genus zero; further assume that S' is the connected sum of g genus 1 cobordisms $T^*_{2,3} \to U_1$ and the product cobordism $\#_{l-g}\, T^*_{2,3} \to \#_{l-g}\, T^*_{2,3}$. The cobordisms S' and S'' respectively induce morphisms of $\mathcal{S}$-complexes with levels 0 and $l - g - \frac{1}{2}\sigma(K)$. In particular, $\widehat{I}(S')$ and $\widehat{I}(S'')$ are both well defined and $\widehat{I}(S) = \widehat{I}(S'') \circ \widehat{I}(S')$. By Lemma 4.41 and naturality of the connected sum theorem, $\widehat{I}(S')$ is multiplication by x^g. Then (73) follows from $\widehat{I}(S) = \widehat{I}(S'') \circ \widehat{I}(S') = x^g\, \widehat{I}(S'')$ and the fact it already has been established for the cobordism S''. In the case $g > l$, we may still obtain cobordisms S' and S'' as above except that $\#_{l-g}\, T^*_{2,3}$ should be replaced with $\#_{g-l}\, T_{2,3}$. Lemmas 4.41 and 4.43 imply that $\widehat{I}(S')$ is still multiplication by x^g. Therefore,

$$\begin{aligned}
x^g (T^2 - T^{-2})^{s_+} [\widehat{I}(K;\Delta)' : \widehat{I}(S)(1)] &= x^g (T^2 - T^{-2})^{s_+} [\widehat{I}(K;\Delta)' : \widehat{I}(S'')(x^g)] \\
&= x^g (T^2 - T^{-2})^{s_+ + g - l} [\widehat{I}(K;\Delta)' : \widehat{I}(S'')((T^2 - T^{-2})^{g-l} x^g)] \\
&= (T^2 - T^{-2})^{s_+ + g - l} \big[\widehat{I}(K;\Delta)' : \widehat{I}(S'')\big(\hat{\xi}((T^2 - T^{-2})^{g-l})\big)\big] \\
&= (T^2 - T^{-2})^{s_+ + g - l} [\widehat{I}(K;\Delta)' : \hat{\xi}((T^2 - T^{-2})^{s_+ + g - l})] \\
&= \hat{z}(K).
\end{aligned}$$

Note that we used Lemma 4.42 for the third identity. More generally, for any $S\colon \#_l\, T^*_{2,3} \to K$ with $l \geq \frac{1}{2}\sigma(K) + g$, we can homotope S to a surface of the type just considered through a sequence of twists and finger moves, and appeal to [Daemi and Scaduto 2024b, Proposition 2.30], to see that (73) still holds.

Proposition 4.44 *Let $S\colon U_1 \to K$ be an orientable immersed cobordism with genus g and s_+ positive double points. Let $i = \max\{0, g + \frac{1}{2}\sigma(K)\}$. Then, for any $j, k \in \mathbb{Z}_{\geq 0}$ satisfying $j + k = i$, we have*

$$x^{g+j}(T^2 - T^{-2})^{s_+ + k} \in \hat{z}(K). \tag{74}$$

Proof If $g + \frac{1}{2}\sigma(K) \leq 0$, so that $i = j = k = 0$, this follows directly from (73). If $i = g + \frac{1}{2}\sigma(K) > 0$, form a cobordism $S_1\colon \#_l\, T^*_{2,3} \to K$ by precomposing S with a cobordism $S_2\colon \#_l\, T^*_{2,3} \to U_1$, where S_2 is formed by a connected sum of j genus 1 cobordisms and k standard crossing change cobordisms. Applying (73) to the cobordism S_1 yields the result. □

Remark 4.45 For an immersed orientable cobordism $S\colon U_1 \to K$ with genus g and s_+ positive double points, it follows from Kronheimer and Mrowka's definition of $z^\natural(K)$ that

$$P^g(T_1^2 - T_1^{-2})^{s_+} \in z^\natural(K).$$

This is recovered by (74) when $g + \frac{1}{2}\sigma(K) \leq 0$. One might expect that $x^g(T^2 - T^{-2})^{s_+} \in \hat{z}(K)$ holds with no condition on $g + \frac{1}{2}\sigma(K)$. This shortcoming is likely a deficiency of the above trick involving trefoils. We expect the stronger claim to follow once cobordism maps are defined in more generality.

We now turn to some computations.

Lemma 4.46 *For $l \geq 0$, we have $\hat{z}(\#_l\, T^*_{2,3}) = R[x] \subset \mathrm{Frac}(R[x])$.*

Proof As explained above, the equality (73) is valid for the identity cobordism $S\colon \#_l\, T^*_{2,3} \to \#_l\, T^*_{2,3}$. This has $g = s_+ = 0$, and of course $\hat{I}(S)(1)$ is a generator of $\hat{I}(\#_l\, T^*_{2,3})'$. The result follows. □

For connected sums of the right-handed trefoil, we have the following:

Lemma 4.47 *For $l > 0$, we have $\hat{z}(\#_l\, T_{2,3}) = \mathcal{I}^l \subset \mathrm{Frac}(R[x])$.*

Proof From Definition 4.34 and Lemma 4.42, we compute

$$\hat{z}(\#_l\, T_{2,3}) = (T^2 - T^{-2})^l [\hat{I}(\#_l\, T_{2,3}; \Delta) : \hat{\xi}((T^2 - T^{-2})^l)] = (T^2 - T^{-2})^l [\mathcal{I}^l : (T^2 - T^{-2})^l] = \mathcal{I}^l. \ \square$$

Remark 4.48 For simplicity, we have restricted our discussion above to the invariant $z^\natural_\sigma(K)$. Kronheimer and Mrowka [2021b] also define a concordance invariant $z^\sharp(K)$ which is a fractional ideal for the ring $\boldsymbol{F}[T_0^{\pm 1}, T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}]$. Following the discussion in [Daemi and Scaduto 2024b, Section 8.8], we have

$$z^\sharp(K) = \hat{z}(K) \otimes_{R[x]} \boldsymbol{F}[T_0^{\pm 1}, T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}],$$

where $R[x]$ acts on the right-hand ring by reducing modulo 2, $T \mapsto T_0$ and $x \mapsto P$.

Kronheimer and Mrowka [2021b] define several concordance invariants using constructions in singular instanton homology. For example, for any homomorphism $\sigma\colon \mathscr{S} \to \mathscr{S}'$ where $\mathscr{S}'$ is a valuation ring with valuation group G, they define a homomorphism from the knot concordance group to G,

$$f_\sigma\colon \mathcal{C} \to G.$$

Specific choices of homomorphisms σ are given in [ibid., Section 5]. Each such invariant f_σ is determined by $z^\natural_\sigma(K)$. Since the latter is determined by the local equivalence invariant $\hat{z}(K)$ by Theorem 4.40, we arrive at the following, which proves Theorem 1.7:

Theorem 4.49 *All of the concordance invariants defined in* [Kronheimer and Mrowka 2021b], *such as* $z^\#_\sigma(K)$, $z^\natural_\sigma(K)$, $f_\sigma(K), \dots,$ *depend only on the local equivalence class of the* $\mathcal{S}$*-complex* $\widetilde{C}(K;\Delta)$.

4.5 Two-bridge knots

In the case of two-bridge knots, many of the invariants considered thus far are determined by the signature. The results in this section imply Theorem 1.8.

Proposition 4.50 *Let K be a two-bridge knot and F any field. Then the local equivalence class of*

$$\widetilde{C}(K;\Delta\otimes_R F[T^{\pm 1}]) \tag{75}$$

is determined by $\sigma(K)$. In particular, if $h := -\frac{1}{2}\sigma(K) \geq 0$, it is locally equivalent to the $\mathcal{S}$-complex over $F[T^{\pm1}]$ of both the knots $T_{2,2h+1}$ and $\#_h T_{2,3}$.

Proof Assume $h = h(\widetilde{C}_0) \geq 0$; the result for $h < 0$ follows by duality. As explained in [Daemi and Scaduto 2024a, Section 3.1], the $\mathcal{S}$-complex $\widetilde{C}$ in (75) for a two-bridge knot K has a free basis over $F[T^{\pm1}]$ such that its differential $\tilde{d}$ can be written, with respect to this basis, as $T^2 - T^{-2}$ times a matrix with integer entries. As a consequence,

$$\widetilde{C} = \widetilde{C}_0 \otimes_F F[T^{\pm1}], \quad \tilde{d} = \tilde{d}_0 \otimes (T^2 - T^{-2}),$$

where $(\widetilde{C}_0, \tilde{d}_0)$ is an $\mathcal{S}$-complex over the field F. The local equivalence class of any $\mathcal{S}$-complex over a field is determined by its Frøyshov invariant; see [Daemi and Scaduto 2024b, Proposition 4.30]. Concretely, having assumed $h(\widetilde{C}) = h(\widetilde{C}_0) \geq 0$, the $\mathbb{Z}/4$-graded $\mathcal{S}$-complex $\widetilde{C}_0$ over F is locally equivalent to the $\mathcal{S}$-complex

$$\widetilde{C}' = C'_* \oplus C'_{*-1} \oplus F_{(0)}$$

with C' freely generated over F by elements $\alpha_1, \dots, \alpha_h$ with the $\mathbb{Z}/4$-grading of α_i given by $2i-1$ (mod 4); the only nontrivial differentials are $v(\alpha_i) = \alpha_{i-1}$ for $i \geq 2$ and $\delta_1(\alpha_1) = 1$.

Let $\widetilde{\lambda}\colon \widetilde{C}_0 \to \widetilde{C}'$ and $\widetilde{\lambda}'\colon \widetilde{C}' \to \widetilde{C}_0$ be local morphisms realizing the local equivalence between $\widetilde{C}_0$ and $\widetilde{C}'$. Define an $\mathcal{S}$-complex over $F[T^{\pm1}]$,

$$\widetilde{C}'' = \widetilde{C}' \otimes_F F[T^{\pm1}], \quad \tilde{d}'' = \tilde{d}' \otimes (T^2 - T^{-2}).$$

Then $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ extend in the obvious way to local morphisms $\widetilde{C} \to \widetilde{C}''$ and $\widetilde{C}'' \to \widetilde{C}$, respectively. Finally, $h(\widetilde{C}) = -\frac{1}{2}\sigma(K)$ by [Daemi and Scaduto 2024a, Theorem 7]. □

Remark 4.51 The above argument does not work with $F = \mathbb{Z}$, because $\mathcal{S}$-complexes over $\mathbb{Z}$ are not classified by Frøyshov invariants. On the other hand, the authors have no counterexample to the general statement of Proposition 4.50 with F replaced by $\mathbb{Z}$.

Remark 4.52 An additional layer of structure on $\mathcal{S}$-complexes for knots comes from the Chern–Simons filtration, which is studied in the next section. The analogue of Proposition 4.50 is not true in the filtered setting. This is evident from our applications of the theory to two-bridge knots; see also [Daemi and Scaduto 2024b; 2024a].

Using Proposition 4.50, we can compute the concordance invariants $s^\sharp_\pm$, $s^\sharp$, $\tilde{\varepsilon}$ for two-bridge knots.

Proposition 4.53 *The invariants $s^\sharp_\pm$ for a two-bridge knot K are*

$$s^\sharp_+(K) = \begin{cases} -\frac{1}{2}\sigma(K) & \text{if } \sigma(K) < 0, \\ 0 & \text{if } \sigma(K) = 0, \\ -\frac{1}{2}\sigma(K) + 1 & \text{if } \sigma(K) > 0, \end{cases} \qquad s^\sharp_-(K) = \begin{cases} -\frac{1}{2}\sigma(K) - 1 & \text{if } \sigma(K) < 0, \\ 0 & \text{if } \sigma(K) = 0, \\ -\frac{1}{2}\sigma(K) & \text{if } \sigma(K) > 0. \end{cases}$$

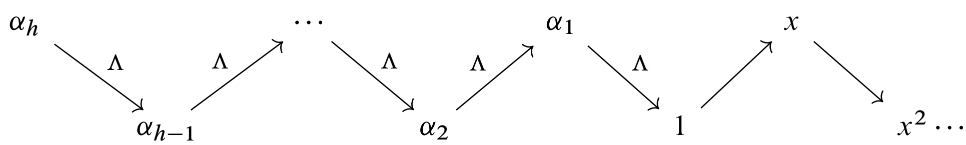
Thus, Kronheimer and Mrowka's invariant $s^\sharp$ and the invariant $\tilde{\varepsilon}$ are as follows (where $\operatorname{sign}(0) = 0$*):*

$$s^\sharp(K) = -\sigma(K) + \operatorname{sign}(\sigma(K)), \quad \tilde{\varepsilon}(K) = \operatorname{sign}(\sigma(K)).$$

Proof By Proposition 4.18, it suffices to treat the case $h := -\frac{1}{2}\sigma(K) \geq 0$. If $h = 0$, by Proposition 4.50, the $\mathcal{S}$-complex for K with coefficients in $\mathbb{Q}[\![\Lambda]\!]$ is locally trivial, and thus $s^\sharp_\pm(K) = \tilde{\varepsilon}(K) = 0$. So henceforth we assume $h > 0$.

By Proposition 4.50, and the fact that $s^\sharp_\pm$ are local equivalence invariants, we may assume that the $\mathcal{S}$-complex of K over $\mathbb{Q}[\![\Lambda]\!]$ is of the form $\widetilde{C}_* = C_* \oplus C_{*-1} \oplus \mathbb{Q}[\![\Lambda]\!]$ where C_* is freely generated by $\alpha_1, \dots, \alpha_h$ with nontrivial components of the differential given by $v(\alpha_i) = \Lambda\alpha_{i-1}$ for $i \geq 2$ and $\delta_1(\alpha_1) = \Lambda$. To choose such a basis we use that $T^2 - T^{-2}$ is equal to Λ times a unit in the ring $\mathbb{Q}[\![\Lambda]\!]$.

The complex $\widehat{\mathfrak{C}} = C_{*-1} \oplus \mathbb{Q}[\![\Lambda]\!][x]$ has zero differential and is therefore equal to $H(\widehat{\boldsymbol{C}})$. For simplicity, write α_i for the generator in $H(\widehat{\boldsymbol{C}})$ which is $(\alpha_i, 0)$, and x^i for $(0, x^i)$. The x-action on $H(\widehat{\boldsymbol{C}})$ is depicted as follows:



More precisely, $x \cdot \alpha_i = \Lambda\alpha_{i-1}$ for $i \geq 2$, $x \cdot \alpha_1 = \Lambda$ and $x \cdot x^i = x^{i+1}$. In the diagram, each vertex generates a copy of $\mathbb{Q}[\![\Lambda]\!]$. The elements in the top row are in grading 2 (mod 4), and, in the bottom row, 0 (mod 4). In the diagram we have assumed that h is odd, the even case being similar. We have an isomorphism of $\mathbb{Q}[\![\Lambda]\!][x]$-modules

$$H(\widehat{\boldsymbol{C}}) \cong (x^h, x^{h-1}\Lambda, \dots, x\Lambda^{h-1}, \Lambda^h) = \mathcal{I}^h \otimes_{R[x]} \mathbb{Q}[\![\Lambda]\!][x].$$

The isomorphism sends α_i to $\Lambda^i x^{h-i}$, and x^i to x^{i+h}. The map $\mathrm{i}\colon \widehat{\mathfrak{C}} = H(\widehat{\boldsymbol{C}}) \to \mathbb{Q}[\![\Lambda]\!][\![x^{-1}, x]$ sends $\alpha_i \in \widehat{\mathfrak{C}}$ to $\Lambda^i x^{-i}$ and sends x^i to x^i. From this we obtain $J_h(\widetilde{C}) = (\Lambda^h) \subset \mathbb{Q}[\![\Lambda]\!]$. We also have

$$\hat{\xi}\colon J_h(\widetilde{C}) = (\Lambda^h) \to H(\widehat{\boldsymbol{C}}), \quad \hat{\xi}(\Lambda^h) = \alpha_h.$$

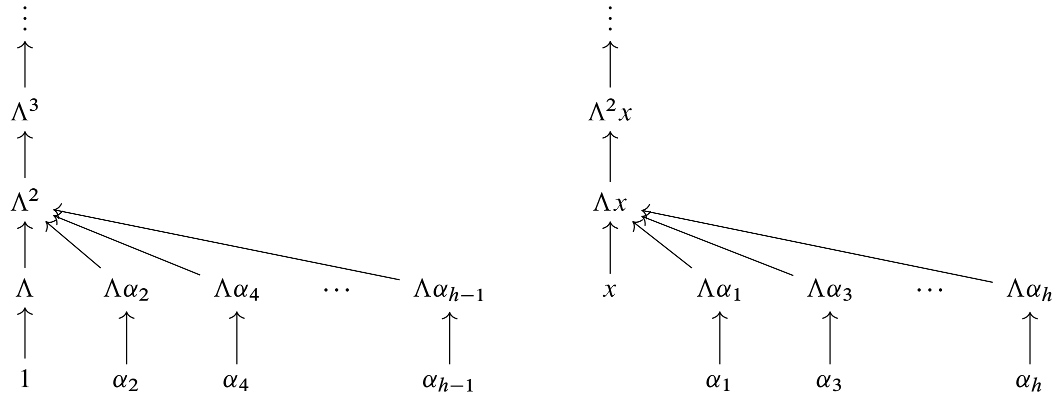


Figure 2: The $\mathbb{Q}[\![\Lambda]\!]$-module $H(C^\sharp)$ in the proof of Proposition 4.53.

Next we turn to $H(C^\sharp)$. The action of $x^2-4\Lambda^2$ on $H(\widehat{C})$ is injective, and so $H(C^\sharp)$ may be identified with the quotient of $H(\widehat{C})$ by $(x^2-4\Lambda^2)H(\widehat{C})$. To describe $H(C^\sharp)$ as this quotient, we have the diagram in Figure 2, assuming h is odd. As a $\mathbb{Q}$-vector space, $H(C^\sharp)$ is the direct sum of the $\mathbb{Q}$-spans of the vertices. (We abuse notation and write α_i also for the equivalence class of α_i in $H(C^\sharp)$, and so on.) The Λ-action is described by the arrows, each of which is multiplication by Λ up to a unit in $\mathbb{Q}$. More precisely,

$$\Lambda\cdot\alpha_i=\Lambda\alpha_i,\qquad \Lambda\cdot\Lambda\alpha_i=\begin{cases}\frac14\Lambda^2 & \text{if } i \text{ is even,}\\ \frac14\Lambda x & \text{if } i \text{ is odd,}\end{cases}$$

and the obvious Λ-action holds for the two $\mathbb{Q}[\![\Lambda]\!]$ towers generated by 1 and x. (The $\mathbb{Z}/4$-gradings are induced from $H(\widehat{C})$ and not represented in the diagram.) We obtain

$$\xi_+^\sharp(\Lambda^h)=\alpha_h,\quad \xi_-^\sharp(\Lambda^h)=\Lambda\alpha_{h-1}, \tag{76}$$

where, if $h=1$, we set the convention $\alpha_0:=1$. From the description of $H(C^\sharp)$ it is easy to see that the minima appearing in the definitions for $s_\pm^\sharp(\widetilde{C})$ are realized by the expressions (76). Thus $s_+^\sharp(\widetilde{C})=h=-\frac12\sigma(K)$ and $s_-^\sharp(\widetilde{C})=h-1=-\frac12\sigma(K)-1$. □

Corollary 4.54 *Let K be a two-bridge knot, and F a field. Then $\hat z(K)$ with coefficients $F[T^{\pm1}]$ is*

$$\hat z(K)\otimes_{R[x]}F[T^{\pm1},x]=\begin{cases}F[T^{\pm1},x] & \text{if } \sigma(K)\ge 0,\\ \mathcal{I}^{-\sigma(K)/2}\otimes_{R[x]}F[T^{\pm1},x] & \text{if } \sigma(K)<0,\end{cases}$$

where $\mathcal{I}^l$ is as defined in (72). *As a consequence, for a two-bridge knot, Kronheimer and Mrowka's $z_\sigma^\natural(K)$, and in fact all concordance invariants defined in* [Kronheimer and Mrowka 2021b], *are determined by $\sigma(K)$ via these formulas.*

Proof This follows from Proposition 4.50, Lemmas 4.46 and 4.47, and Theorem 4.49. □

Question 4.55 *Is Proposition 4.50 true for alternating knots?*

5 Concordance invariants from filtered special cycles

The $\mathcal{S}$-complexes that arise from singular instanton Floer theory possess additional structure which roughly comes in the form of an $\mathbb{R}$-valued filtration, defined using values of the Chern–Simons functional. In this section we use this additional structure to define more homology concordance invariants, generalizing both the invariant $\Gamma_{(Y,K)}$ in the singular setting from [Daemi and Scaduto 2024b; 2024a], as well as the invariants Γ_Y and $r_s(Y)$ in the setting of integer homology 3-spheres Y from [Daemi 2020] and [Nozaki et al. 2024], respectively.

We first introduce the notion of an enriched complex, following [Daemi and Scaduto 2024b] with minor variations. This is the algebraic enhancement of an $\mathcal{S}$-complex that singular instanton theory outputs when keeping track of the Chern–Simons filtration. We then adapt the material of Section 3 to this setting, and define *filtered* special cycles. These are then used to define various homology concordance invariants. In this section, Theorems 1.5 and 1.19 are proved.

5.1 Enriched $\mathcal{S}$-complexes

Let R be an integral domain over $\mathbb{Z}[T^{\pm 1}]$. Let δ be a nonnegative real number. An *I-graded $\mathcal{S}$-complex* over $R[U^{\pm 1}]$ is an $\mathcal{S}$-complex $\widetilde{C}$ over $R[U^{\pm 1}]$ with $\mathbb{Z}\times\mathbb{R}$-bigrading

$$\widetilde{C} = \bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{R}} \widetilde{C}_{i,j}$$

which satisfies

$$U\widetilde{C}_{i,j} \subset \widetilde{C}_{i+4,j+1}, \quad \tilde{d}\,\widetilde{C}_{i,j} \subset \bigcup_{k<j} C_{i-1,k} \quad \text{and} \quad \chi\widetilde{C}_{i,j} \subset \widetilde{C}_{i+1,j}.$$

We also impose that $\widetilde{C}$ is freely, finitely generated as an $R[U^{\pm 1}]$-module by homogeneously bigraded elements, and the canonical element 1 in the distinguished summand $R[U^{\pm 1}] \subset \widetilde{C}$ is in $\widetilde{C}_{0,0}$. Note that the ring $R[U^{\pm 1}]$ can be thought of as the trivial I-graded $\mathcal{S}$-complex, with U^j having $\mathbb{Z}\times\mathbb{R}$-grading $(4j, j)$. For any nonzero element $\zeta \in \widetilde{C}_{i,j}$, we denote the $\mathbb{R}$-grading (also called the *I-grading*) by $\deg_I(\zeta) = j$. This is extended to nonhomogeneous elements as follows: if $\zeta = \sum \zeta_i$ is a finite sum of homogeneous elements in distinct bigradings, then

$$\deg_I(\zeta) = \sup_i \{\deg_I(\zeta_i)\} \in \mathbb{R}\cup\{-\infty\} \tag{77}$$

with the convention that $\deg_I(0) = -\infty$. For an I-graded $\mathcal{S}$-complex $\widetilde{C}$, the dual I-graded $\mathcal{S}$-complex is formed by taking the underlying dual $\mathcal{S}$-complex $\widetilde{C}^\dagger$ and further defining

$$\deg_I(\zeta^\dagger) = \sup\{\deg_I(\zeta^\dagger(z)) - \deg_I(z) \mid 0 \neq z \in \widetilde{C}\}$$

for any $\zeta^\dagger \in C^\dagger = \mathrm{Hom}_{R[U^{\pm 1}]}(\widetilde{C}, R[U^{\pm 1}])$.

A height l morphism $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ *of level* δ between I-graded $\mathcal{S}$-complexes is a height l morphism of $\mathcal{S}$-complexes such that

$$\widetilde{\lambda}\widetilde{C}_{i,j} \subset \bigcup_{k \leqslant j+\delta} \widetilde{C}'_{i+2l,k}.$$

An $\mathcal{S}$-chain homotopy $\widetilde{K}$ of level δ between height l morphisms $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ (of any levels) is an R-module homomorphism and an $\mathcal{S}$-chain homotopy between $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ such that

$$\widetilde{K}\widetilde{C}_{i,j} \subset \bigcup_{k \leqslant j+\delta} \widetilde{C}'_{i+2l+1,k}.$$

We also have the analogous notions of strong height l morphisms and local morphisms in this context.

In the fortunate circumstance that no perturbations are needed in the construction of equivariant singular instanton Floer homology, the output is simply an I-graded $\mathcal{S}$-complex. In general, one must take a sequence of perturbations approaching zero. The type of algebraic structure that arises in this more general case is as follows:

Definition 5.1 An *enriched $\mathcal{S}$-complex* $\mathfrak{E}$ over $R[U^{\pm 1}]$ is a sequence $\{(\widetilde{C}^i, \tilde{d}^i, \chi^i)\}_{i \geq 1}$ of I-graded $\mathcal{S}$-complexes over $R[U^{\pm 1}]$, local morphisms $\widetilde{\phi}_i^j : \widetilde{C}^i \to \widetilde{C}^j$ of levels $\delta_{i,j}$, and a discrete subset $\mathfrak{K} \subset \mathbb{R}$ with no accumulation point satisfying the following:

(i) $\widetilde{\phi}_i^i = \mathrm{id}$ and $\widetilde{\phi}_j^k \circ \widetilde{\phi}_i^j$ is $\mathcal{S}$-chain homotopy equivalent to $\widetilde{\phi}_i^k$ by an $\mathcal{S}$-chain homotopy of level $\delta_{i,j,k}$.

(ii) For each $\delta > 0$, there exists an N such that $i, j, k > N$ implies $\delta_{i,j} \leqslant \delta$ and $\delta_{i,k,j} \leqslant \delta$.

(iii) For every $\delta > 0$, there exists $N \in \mathbb{Z}_{>0}$ such that, for any $n > N$ and any nonzero $\zeta \in \widetilde{C}^n$, we have

$$\deg_I(\zeta) \in B_\delta(\mathfrak{K}), \tag{78}$$

where $B_\delta(\mathfrak{K}) := \{r \in \mathbb{R} : |r - r'| < \delta \text{ for some } r' \in \mathfrak{K}\}$.

Remark 5.2 The I-graded $\mathcal{S}$-complexes defined above are I-graded $\mathcal{S}$-complexes of level 0 in the terminology of [Daemi and Scaduto 2024b]. Furthermore, in [Daemi and Scaduto 2024b], condition (iii) is not imposed in the definition of enriched $\mathcal{S}$-complexes. Condition (iii) is used in the course of defining the r_s-type invariants, following [Nozaki et al. 2024].

Proposition 5.3 *Let $\mathfrak{E}$ be an enriched $\mathcal{S}$-complex over $R[U^{\pm 1}]$ consisting of $\{\widetilde{C}^i\}$, $\{\widetilde{\phi}_i^j\}$, $\mathfrak{K}$. Define*

$$\widetilde{C}^{n,\leq s} := \{\zeta \in \widetilde{C}^n \mid \deg_I(\zeta) \leq s\}.$$

Then $\widetilde{C}^{n,\leq s}$ is a chain complex over R. For $s \notin \mathfrak{K}$ and sufficiently large $n, m \in \mathbb{Z}_{>0}$, the chain complexes $\widetilde{C}^{n,\leq s}$ and $\widetilde{C}^{m,\leq s}$ are canonically chain homotopy equivalent.

Proof By Definition 5.1(iii), for sufficiently large n and m, $\widetilde{\phi}_n^m$ induces a well-defined local morphism from $\widetilde{C}^{n,\leq s}$ to $\widetilde{C}^{m,\leq s}$. Moreover, conditions (i) and (ii) in Definition 5.1 imply that these maps provide canonical $\mathcal{S}$-chain homotopy equivalences between these $\mathcal{S}$-complexes. □

Remark 5.4 The chain complexes $\widetilde{C}^{n,\le s}$ over R in Proposition 5.3 have χ-actions and the result is compatible with these actions. However, these complexes are not $\mathcal{S}$-complexes over R, because they are not finitely generated, and the reducible summands do not have rank 1.

Definition 5.5 Fix a nonnegative integer l and nonnegative real number κ. A *morphism* $\mathfrak{L}\colon \mathfrak{E}(1) \to \mathfrak{E}(2)$ *of height l and level κ* between two enriched $\mathcal{S}$-complexes

$$\mathfrak{E}(r) = \big(\{\widetilde{C}^i(r)\}, \{\widetilde{\phi}_i^j(r)\}, \mathfrak{K}\big) \quad (r \in \{1,2\})$$

is a collection of height l, level $\delta_{i,j} + \kappa$ morphisms $\widetilde{\lambda}_i^j\colon \widetilde{C}^i(1) \to \widetilde{C}^j(2)$ of I-graded $\mathcal{S}$-complexes satisfying the following conditions:

(i) For each $\delta > 0$, there exists an N such that $i, j > N$ implies $\delta_{i,j} < \delta$.

(ii) The maps $\widetilde{\lambda}_j^k \circ \widetilde{\phi}_i^j(1)$ and $\widetilde{\phi}_j^k(2) \circ \widetilde{\lambda}_i^j$ are $\mathcal{S}$-chain homotopy equivalent to $\widetilde{\lambda}_i^k$ via an $\mathcal{S}$-chain homotopy of some level $\delta_{i,j,k}$. For every $\delta > 0$, there exists an N such that $\delta_{i,j,k} < \delta$ for $i, j, k > N$.

The morphism is *local* if each $\widetilde{\lambda}_i^j$ is a local morphism of I-graded $\mathcal{S}$-complexes. A morphism between enriched complexes is a *chain homotopy equivalence* if each $\widetilde{\lambda}_i^j$ is an $\mathcal{S}$-chain homotopy equivalence, where the involved $\mathcal{S}$-chain homotopy inverses and $\mathcal{S}$-chain homotopies have levels which converge to zero. A *weak* morphism is the above data, but without condition (ii) necessarily holding.

Let K be a knot in an integer homology 3-sphere Y. To a perturbation π of the Chern–Simons functional (along with other auxiliary choices) with nondegenerate critical set and unobstructed moduli spaces, one can associate an I-graded $\mathcal{S}$-complex $\widetilde{C}(Y, K, \pi)$ over $R[U^{\pm 1}]$. The generators of $C(Y, K, \pi)$ roughly correspond to homotopy classes $[A]$ of singular connections on $\mathbb{R} \times (Y, K)$, mod gauge, with limit at $-\infty$ some irreducible critical point α of the perturbed Chern–Simons functional, and limit at $+\infty$ the reducible θ. The connection $[A]$ determines an $\mathbb{R}$-valued lift of the perturbed Chern–Simons invariant of α, which determines the I-grading, and a $\mathbb{Z}$-valued lift of the $\mathbb{Z}/4$-grading of α. Setting $U = 1$ and forgetting the I-grading structure gives the $\mathbb{Z}/4$-graded $\mathcal{S}$-complex of (Y, K) considered in Section 2.1.

To define the enriched $\mathcal{S}$-complex $\mathfrak{E}(Y, K)$, we take a sequence of perturbations π_i as above (with the other necessary auxiliary choices) such that the norms $\|\pi_i\|$ converge to zero as $i \to \infty$. Then set

$$\mathfrak{E}(Y, K) := \big(\{\widetilde{C}^i = \widetilde{C}(Y, K, \pi_i)\}, \{\widetilde{\phi}_i^j\}, \mathfrak{K}\big).$$

The map $\widetilde{\phi}_i^j$ is induced by a 1-parameter family of perturbations (and other auxiliary data). Moreover, $\mathfrak{K} \subset \mathbb{R}$ is the set of critical values of the (unperturbed) Chern–Simons functional. By taking suitable 2-parameter families of perturbations and using the associated parametrized singular instanton moduli spaces, one can verify that $\widetilde{\phi}_j^k \circ \widetilde{\phi}_i^j$ is $\mathcal{S}$-chain homotopy equivalent to $\widetilde{\phi}_i^k$ via an $\mathcal{S}$-chain homotopy of some level $\delta_{i,k,j}$ as required. The enriched complex $\mathfrak{E}(Y, K)$ is an invariant of (Y, K) up to homotopy equivalence. See [Daemi and Scaduto 2024b, Section 7; 2024a, Section 2], although note that there the unperturbed Chern–Simons functional is used to define I-gradings.

For cobordism maps, we have the following. Recall the terminology of Definition 2.7.

Proposition 5.6 [Daemi and Scaduto 2024b, Theorem 7.18] *Suppose (W, S, c), with $c \in H^2(W;\mathbb{Z})$, is a cobordism $(Y, K) \to (Y', K')$ which is negative definite of height $l \geq 0$ over R. Then there is an induced height l morphism of level $2\kappa_{\min}(W, S, c)$ from $\mathfrak{E}(Y, K)$ to $\mathfrak{E}(Y', K')$ such that, for each morphism $\widetilde{\lambda}_i^j$ in the sequence, $c_l = \eta(W, S, c)$.*

Two enriched $\mathcal{S}$-complexes $\mathfrak{E}$ and $\mathfrak{E}'$ are *locally equivalent*, written $\mathfrak{E} \sim \mathfrak{E}'$, if there are local morphisms $\mathfrak{E} \to \mathfrak{E}'$ and $\mathfrak{E}' \to \mathfrak{E}$. Define the local equivalence group of enriched complexes

$$\Theta_R^{\mathfrak{E}} = \{\text{enriched } \mathcal{S}\text{-complexes over } R[U^{\pm 1}]\ \}/\sim.$$

Just as is the case for $\mathcal{S}$-complexes, this is an abelian group. The zero element is determined by $\{(\widetilde{C}^i, \tilde{d}^i, \chi^i) := (R[U^{\pm 1}], 0, 0)\}_{i\geq 1}$ with $\{\phi_i^j := \mathrm{id}_{R[U^{\pm 1}]}\}_{i,j\geq 1}$ and $\mathfrak{K} = \mathbb{Z} \subset \mathbb{R}$. Addition is defined by taking tensor products of enriched $\mathcal{S}$-complexes.

It is sometimes convenient to use a slightly weaker definition of local equivalence by replacing local morphisms with weak local morphisms. We call the resulting relation *weak local equivalence*.

Remark 5.7 The group $\Theta_R^{\mathfrak{E}}$ is uncountable. To see this, we construct for each $r \in \mathbb{R}_{>0}$ an enriched $\mathcal{S}$-complex as follows. First, define an I-graded $\mathcal{S}$-complex by

$$\widetilde{C}\{r\} = R[U^{\pm 1}]\langle\zeta\rangle \oplus R[U^{\pm 1}]\langle\chi(\zeta)\rangle \oplus R[U^{\pm 1}]_{(0)}.$$

The generator ζ has $\deg_I = r$ and $\mathbb{Z}$-grading 1. Further, $\tilde{d}(\zeta) = \delta_1(\zeta) = 1$. By taking the constant sequence of this I-graded $\mathcal{S}$-complex, we obtain an enriched $\mathcal{S}$-complex $\mathfrak{E}\{r\}$. For distinct positive real numbers r and r', one can show that $\mathfrak{E}\{r\}$ and $\mathfrak{E}\{r'\}$ are not locally equivalent.

Recall that $\Theta_{\mathbb{Z}}^{3,1}$ is the homology concordance group of knots in integer homology 3-spheres. We define a map Ω from $\Theta_{\mathbb{Z}}^{3,1}$ to the local equivalence group $\Theta_R^{\mathfrak{E}}$ by assigning to (Y, K) the class of the enriched complex $\mathfrak{E}(Y, K)$.

Theorem 5.8 [Daemi and Scaduto 2024b, Theorem 7.18] *The map $\Omega\colon \Theta_{\mathbb{Z}}^{3,1} \to \Theta_R^{\mathfrak{E}}$ is a homomorphism.*

5.2 Filtered special cycles

We next adapt some of the constructions involving special cycles from Section 3 to the setting of I-graded $\mathcal{S}$-complexes and enriched $\mathcal{S}$-complexes. As before, R is an integral domain algebra over $\mathbb{Z}[T^{\pm 1}]$.

We begin by discussing equivariant complexes in this context. Let $\widetilde{C}$ be an I-graded $\mathcal{S}$-complex over $R[U^{\pm 1}]$. Recall that in Section 2.2 we associated to the underlying $\mathcal{S}$-complex of $\widetilde{C}$ the large equivariant complexes $\widehat{\boldsymbol{C}}$, $\overline{\boldsymbol{C}}$, $\check{\boldsymbol{C}}$ and small equivariant complexes $\widehat{\mathfrak{C}}$, $\overline{\mathfrak{C}}$, $\check{\mathfrak{C}}$. These are $\mathbb{Z}$-graded complexes over $R[U^{\pm 1}][x]$. We extend $\deg_I$ in an obvious manner to each of these complexes. For example, given $\zeta = \sum_{i=0}^N \zeta_i x^i \in \widehat{\boldsymbol{C}}$ with $\zeta_i \in \widetilde{C}$, we have

$$\deg_I(\zeta) = \sup_i \{\deg_I(\zeta_i)\}.$$

Given this, it is immediate from the I-graded structure of $\widetilde{C}$ that $\deg_I(\hat{d}(\zeta)) \le \deg_I(\zeta)$ for each $\zeta \in \widehat{\boldsymbol{C}}$. The case for the other equivariant complexes is similar. Note that since $\overline{\boldsymbol{C}}$ and $\check{\boldsymbol{C}}$ contain elements having infinite Laurent power series in x^{-1}, the possible values of $\deg_I$ in these cases may include ∞.

The following result concerns the maps studied in the context of equivariant theories in Section 2.2:

Lemma 5.9 *The chain maps $\widehat{\Phi}$, $\overline{\Phi}$, $\check{\Phi}$ and $\widehat{\Psi}$, $\overline{\Psi}$, $\check{\Psi}$ are filtered maps (level 0): we have $\deg_I(\widehat{\Phi}(\zeta)) \le \deg_I(\zeta)$ for all $\zeta \in \widehat{\boldsymbol{C}}$, and similarly for the others. Also, the maps $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ and $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are filtered.*

Proof We prove $\overline{\Psi}$ is filtered. Recall that $\overline{\Psi}\colon \overline{\mathfrak{C}} \to \overline{\boldsymbol{C}}$ is defined by

$$\overline{\Psi}\Big(\sum_{j=-\infty}^{N} a_j x^i\Big) = \Big(\sum_{j=-\infty}^{N} \sum_{l=0}^{\infty} v^l \delta_2(a_j) x^{j-l-1}, 0, \sum_{j=-\infty}^{N} a_j x^j\Big).$$

Note that x does not affect the I-grading and both of δ_2 and v^j are decreasing in I-grading. This implies $\overline{\Psi}$ is also filtered. The proofs for the other maps are the same. □

To an I-graded $\mathcal{S}$-complex $\widetilde{C}$ over $R[U^{\pm 1}]$, each $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, define

(79) $$J_k^{\le s}(\widetilde{C}) := \{c_{-k} \in R \mid c_{-k}x^{-k} + c_{-k-1}x^{-k-1} + \cdots = \overline{\Phi}\boldsymbol{i}(z) \in R[U^{\pm 1}][\![x^{-1}, x]\!], z \in \widehat{\boldsymbol{C}}_{2k}, \deg_I(\hat{d}(z)) \le s\}.$$

This is an ideal of R. If $s \le s'$ then we have an inclusion $J_k^{\le s}(\widetilde{C}) \subset J_k^{\le s'}(\widetilde{C})$. We also have $J_k^{\le -\infty}(\widetilde{C}) = J_k(\widetilde{C})$, where $J_k(\widetilde{C})$ is the ideal defined in Section 3, regarding $\widetilde{C}$ as an $\mathcal{S}$-complex over R, having set $U = 1$. We will see below that for $s < 0$, these ideals give rise to local equivalence invariants.

Now we arrive at our definition of special cycles in this setting.

Definition 5.10 Let $f \in R$, $s \in \mathbb{R} \cup \{-\infty\}$, $k \in \mathbb{Z}$ and $\widetilde{C}$ be an I-graded $\mathcal{S}$-complex. We say $z \in \widehat{\boldsymbol{C}}$ is a *filtered special* (k, f, s)*-cycle* if there exists $\mathfrak{z} \in \widehat{\mathfrak{C}}_{2k}$ satisfying $\widehat{\Psi}(\mathfrak{z}) = z$ and also

$$\mathfrak{i}(\mathfrak{z}) = f x^{-k} + \sum_{i=-\infty}^{-k-1} b_i x^i, \quad \deg_I(\hat{\mathfrak{d}}\mathfrak{z}) \le s.$$

Remark 5.11 As in the case of unfiltered special cycles, the chain $\mathfrak{z}$ in Definition 5.10 is uniquely determined by the special cycle $z \in \widehat{\boldsymbol{C}}$. Note that if $s \le s'$ and z is a filtered special (k, f, s)-cycle, then z is also a filtered special (k, f, s')-cycle. Furthermore, z is a filtered special (k, f, s)-cycle, so $x \cdot z$ is a filtered special $(k-1, f, s)$-cycle. Consequently,

$$J_k^{\le s}(\widetilde{C}) \subset J_{k'}^{\le s'}(\widetilde{C})$$

if $k' \le k$ and $s' \le s$. Filtered special $(k, f, -\infty)$-cycles coincide with unfiltered special (k, f)-cycles. In general, filtered special cycles are *not* cycles in $\widehat{\boldsymbol{C}}$: they are cycles in $\widehat{\boldsymbol{C}}/\widehat{\boldsymbol{C}}^{\le s}$, where $\widehat{\boldsymbol{C}}^{\le s}$ consists of elements in $\widehat{\boldsymbol{C}}$ with $\deg_I \le s$. For any $k \in \mathbb{Z}$, $f \in R$ and $s \in \mathbb{R}$, there exists a special (k, f, s)-cycle in $\widehat{\boldsymbol{C}}$ if and only if $f \in J_k^{\le s}(\widetilde{C})$.

Remark 5.12 Let z be a special (k, f, s)-cycle with $f \neq 0$, where $z = \widehat{\Psi}(\mathfrak{z})$ and $\mathfrak{z} = \left(\alpha, \sum_{i=0}^{N} a_i x^i\right) \in \widehat{\mathfrak{C}}_{2k}$. Since $\widehat{\Phi}(z) = \mathfrak{z}$ and $\widehat{\Psi}$, $\widehat{\Phi}$ are filtered, we have $\deg_I(z) = \deg_I(\mathfrak{z})$. We further obtain

$$\deg_I(z) = \max\{\deg_I(\alpha), 0\}.$$

This follows from two observations: $\mathfrak{i}(\mathfrak{z})$ is equal to $f x^{-k}$ plus lower-order terms; and the $\mathbb{Z}$-grading of $\mathfrak{z}$ equals $2k$, implying that $a_i = f_i U^{(i+k)/2}$, where $f_i \in R$. In particular, $\deg_I(a_i) \leq 0$. Note that if $k > 0$, all the a_i are zero and $\deg_I(\alpha)$ is automatically positive.

The following describes the behavior of filtered special cycles under morphisms:

Lemma 5.13 *Let $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ be a height i morphism of level $\kappa \geq 0$ between I-graded $\mathcal{S}$-complexes. Suppose $s \in \mathbb{R}$ satisfies $s < -\kappa$. Then, for a special (k, f, s)-cycle $z \in \widehat{\boldsymbol{C}}_{2k}$, the chain defined by*

$$\widehat{\Psi}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z) \in \widehat{\boldsymbol{C}}'_{2k+2i}$$

is a special $(k+i, c_i f, s+\kappa)$-cycle, where c_i is defined as in (15). *Moreover,*

$$\deg_I(\widehat{\Psi}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z)) \leq \deg_I(z) + \kappa. \tag{80}$$

Proof This is a filtered version of Lemma 3.3. Note that $\widehat{\Psi}'$ and $\widehat{\Phi}'$ are filtered maps and $\widehat{\boldsymbol{\lambda}}$ is a map of level κ. Thus, we have (80). Let $z = \widehat{\Psi}(\mathfrak{z})$ and $\mathfrak{z}' = \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}}(z) = \widehat{\lambda}(\mathfrak{z})$. First note that $\mathfrak{d}'(\mathfrak{z}') = \mathfrak{d}'\widehat{\lambda}(\mathfrak{z}) = \widehat{\lambda}\mathfrak{d}(\mathfrak{z})$. Then $\deg_I(\mathfrak{d}\mathfrak{z}) \leq s$ and the assumption that $\widetilde{\lambda}$ has level κ imply $\deg_I(\mathfrak{d}'\mathfrak{z}') \leqslant s + \kappa$. Further,

$$\mathfrak{i}'(\mathfrak{z}') = \mathfrak{i}' \circ \widehat{\lambda}(\mathfrak{z}) = \mathfrak{i}' \circ \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}} \circ \widehat{\Psi}(\mathfrak{z}) = \overline{\Phi}' \circ \overline{\boldsymbol{\lambda}} \circ \boldsymbol{i} \circ \widehat{\Psi}(\mathfrak{z}) = \overline{\Phi}' \circ \overline{\boldsymbol{\lambda}} \circ \overline{\Psi} \circ \mathfrak{i}(\mathfrak{z}) - \overline{\Phi}' \circ \overline{\boldsymbol{\lambda}} \circ K_{\mathfrak{i}} \circ \hat{\mathfrak{d}}(\mathfrak{z}), \tag{81}$$

where we have used that

$$\overline{\Psi} \circ \mathfrak{i} - \boldsymbol{i} \circ \widehat{\Psi} = \bar{d} K_{\mathfrak{i}} + K_{\mathfrak{i}} \hat{\mathfrak{d}}, \tag{82}$$

with $K_{\mathfrak{i}}\colon \widehat{\mathfrak{C}} \to \overline{\boldsymbol{C}}$ being defined by

$$K_{\mathfrak{i}}\left(\alpha, \sum_{i=0}^{N} a_i x^i\right) = \left(-\sum_{i=0}^{\infty} v^i(\alpha) x^{-i-1}, 0, 0\right).$$

The first term in (81) has x-degree $-k-i$ with leading term $f c_i$ and I-grading equal to zero. The second term has I-grading at most $s + \kappa < 0$. Thus $\deg_I(\mathfrak{i}'(\mathfrak{z}')) < 0$. Furthermore, the second term in (81) can be written as follows, where each $a_j \in R$:

$$\sum_{j=-\infty}^{N} a_j U^{l_j} x^j.$$

Since the I-grading is negative, each $l_j < 0$. Since the $\mathbb{Z}$-grading is $2k + 2i$, we have $j = -2i - 2k - 4l_j$. Thus this power series has x-degree less than $-i - k$. Consequently, $\mathfrak{i}'(\mathfrak{z}')$ is dominated by the first term in (81), and has x-degree $-k-i$ with leading term $f c_i$. □

Corollary 5.14 *Let $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ be a height i morphism of level κ between I-graded $\mathcal{S}$-complexes. Then, for any $s < -\kappa$ and with c_i as defined in* (15), *we have*

$$c_i\, J_k^{\le s}(\widetilde{C}) \subseteq J_{k+i}^{\le s+\kappa}(\widetilde{C}').$$

Next, we consider tensor products of filtered special cycles.

Lemma 5.15 *Let $z \in \widehat{\boldsymbol{C}}_{2k}$ (resp. $z' \in \widehat{\boldsymbol{C}}'_{2k'}$) be a special (k, f, s)-cycle (resp. (k', f', s')-cycle) such that*

$$\deg_I(z) + s', \deg_I(z') + s < 0. \tag{83}$$

Then the chain defined by

$$z^{\otimes} := \widehat{\Psi}^{\otimes} \circ \widehat{\Phi}^{\otimes} \circ \widehat{T}(z \otimes_{R[x]} z') \in \widehat{\boldsymbol{C}}^{\otimes}_{2k+2k'}$$

is a special $\bigl(k+k', ff', \max\{\deg_I(z)+s', \deg_I(z')+s\}\bigr)$-cycle. Moreover,

$$\deg_I(\widehat{\Psi}^{\otimes} \circ \widehat{\Phi}^{\otimes} \circ \widehat{T}(z \otimes_{R[x]} z')) \le \deg_I(z) + \deg_I(z'). \tag{84}$$

Proof This is a filtered version of Lemma 3.4. The inequality (84) follows since all the maps $\widehat{\Psi}^{\otimes}$, $\widehat{\Phi}^{\otimes}$ and $\widehat{T}$ are filtered. Since $z \in \widehat{\boldsymbol{C}}_{2k}$ (resp. $z' \in \widehat{\boldsymbol{C}}'_{2k'}$) is a special (k, f, s)-cycle (resp. (k', f', s')-cycle), we can take $\mathfrak{z} \in \widehat{\mathfrak{C}}_{2k}$ (resp. $\mathfrak{z}' \in \widehat{\mathfrak{C}}_{2k'}$) such that $\deg_I(\hat{\mathfrak{d}}(\mathfrak{z})) \le s$ (resp. $\deg_I(\hat{\mathfrak{d}}'(\mathfrak{z}')) \le s'$), $\widehat{\Psi}(\mathfrak{z}) = z$ (resp. $\widehat{\Psi}'(\mathfrak{z}') = z'$) and the leading term of $\mathfrak{i}(\mathfrak{z})$ (resp. $\mathfrak{i}'(\mathfrak{z}')$) is fx^{-k} (resp. $f'x^{-k'}$). Since $\widehat{\Psi}$ and $\widehat{\Psi}'$ are filtered chain maps, we also have $\deg_I(\hat{d}(z)) \le s$ (resp. $\deg_I(\hat{d}'(z')) \le s'$). Using this, we obtain

$$\begin{aligned}\deg_I(\hat{d}^{\otimes}(z \otimes_{R[x]} z')) &= \deg_I(\hat{d}(z) \otimes z' + z \otimes \hat{d}'(z')) \\ &\le \max\{\deg_I(\hat{d}(z)) + \deg(z'), \deg_I(\hat{d}'(z')) + \deg_I z\} \\ &\le \max\{\deg_I(z) + s', \deg_I(z') + s\}.\end{aligned} \tag{85}$$

Since $\widehat{\Psi}^{\otimes}$, $\widehat{\Phi}^{\otimes}$ and $\widehat{T}$ are filtered chain maps, we conclude that $\deg_I(\hat{d}^{\otimes}(z^{\otimes}))$ is bounded by the same term as in (85).

To complete the proof, we need to analyze $\mathfrak{i}^{\otimes}(\mathfrak{z}^{\otimes})$ with $\mathfrak{z}^{\otimes} = \widehat{\Phi}^{\otimes}\widehat{T}(z \otimes_{R[x]} z')$. Using Remark 2.12 and relations (31) and (82), we have

$$\begin{aligned}\mathfrak{i}^{\otimes}(\mathfrak{z}^{\otimes}) &= \overline{\Phi}^{\otimes}\boldsymbol{i}^{\otimes}\widehat{T}(z \otimes_{R[x]} z') = \overline{\Phi}^{\otimes}\overline{T}(\boldsymbol{i}\,\widehat{\Psi}(\mathfrak{z}) \otimes_{R[x]} \boldsymbol{i}'\widehat{\Psi}'(\mathfrak{z}')) \\ &= \overline{\Phi}^{\otimes}\overline{T}\bigl((\overline{\Psi} \circ \mathfrak{i} - \bar{d}K_{\mathfrak{i}} - K_{\mathfrak{i}}\hat{\mathfrak{d}})(\mathfrak{z}) \otimes_{R[x]} (\overline{\Psi}' \circ \mathfrak{i}' - \bar{d}'K'_{\mathfrak{i}} - K'_{\mathfrak{i}}\hat{\mathfrak{d}}')(\mathfrak{z}')\bigr).\end{aligned} \tag{86}$$

Among the nine terms obtained by expanding the above expression, first consider the term

$$\overline{\Phi}^{\otimes}\overline{T}(\overline{\Psi} \circ \mathfrak{i}(\mathfrak{z}) \otimes_{R[x]} \overline{\Psi}' \circ \mathfrak{i}'(\mathfrak{z}')).$$

Lemma 2.14 implies that the above expression is equal to $\mathfrak{i}(\mathfrak{z})\cdot\mathfrak{i}'(\mathfrak{z}')$, which has the leading term $ff'x^{-k-k'}$. Using the assumption in (83) and an argument as in the proof of Lemma 5.13, we can show the five terms in (86) involving either $\mathfrak{d}$ or $\mathfrak{d}'$ have x-degree less than $-k-k'$. It is also easy to see that the remaining three terms involving $\bar{d}$ or $\bar{d}'$ vanish. In summary, $\mathfrak{i}^{\otimes}(\mathfrak{z}^{\otimes})$ has the leading term $ff'x^{-k-k'}$. □

Next, we adapt the construction of the ideals in (79) to the case of enriched $\mathcal{S}$-complexes.

Lemma 5.16 *Let $\mathfrak{E}$ be an enriched $\mathcal{S}$-complex over $R[U^{\pm 1}]$ consisting of $\{\widetilde{C}^i\}$, $\{\widetilde{\phi}_i^j\}$, $\mathfrak{K}$. If $s \notin \mathfrak{K}$, then the ideal $J_k^{\le s}(\widetilde{C}^n) \subset R$ does not depend on n for sufficiently large n.*

Proof For an interval $[s_1, s_2] \subset \mathbb{R} \setminus \mathfrak{K}$, there is some $N > 0$ such that, for all $n > N$, the ideals $J_k^{\le s_1}(\widetilde{C}^n)$ and $J_k^{\le s_2}(\widetilde{C}^n)$ are equal. The proof follows from this observation and Corollary 5.14. □

From this lemma it follows that, for $s \notin \mathfrak{K}$, we can define $J_k^{\le s}(\mathfrak{E}) := J_k^{\le s}(\widetilde{C}^n)$ for a sufficiently large n. For $s \in \mathfrak{K}$, we define $J_k^{\le s}(\mathfrak{E}) := J_k^{\le s-\epsilon}(\mathfrak{E})$ for a sufficiently small $\epsilon > 0$. Moreover, from Corollary 5.14 we obtain:

Corollary 5.17 *Let $\mathfrak{L}\colon \mathfrak{E} \to \mathfrak{E}'$ be a height i morphism of level κ between enriched $\mathcal{S}$-complexes. Then, for any $s < -\kappa$ and with c_i as defined in (15) (which are the same for all $\widetilde{\lambda}_i^j$ involved in $\mathfrak{L}$), we have*

$$c_i J_k^{\le s}(\mathfrak{E}) \subseteq J_{k+i}^{\le s+\kappa}(\mathfrak{E}').$$

In particular, $J_k^{\le s}(\mathfrak{E})$ is a local equivalence invariant of the enriched $\mathcal{S}$-complex $\mathfrak{E}$.

Remark 5.18 In fact, the arguments show that $J_k^{\le s}(\mathfrak{E})$ is a weak local equivalence invariant.

Finally, we give conditions for when an enriched $\mathcal{S}$-complex is (weakly) locally equivalent to the trivial enriched complex. First we need the following:

Lemma 5.19 *Let $\widetilde{C}$ be a I-graded $\mathcal{S}$-complex. If there is a special $(0, 1, -\infty)$-cycle z such that*

$$\deg_I(z) = 0, \tag{87}$$

then there is a level 0 local map from the trivial I-graded $\mathcal{S}$-complex $R[U^{\pm 1}]_{(0)}$ to $\widetilde{C}$.

Proof A filtered special $(0, 1, -\infty)$-cycle $z = \widehat{\Psi}(\mathfrak{z})$, where $\mathfrak{z} = (\alpha, 1) \in \widehat{\mathfrak{C}}_{-2}$, satisfies $\deg_I(\mathfrak{d}\mathfrak{z}) \le -\infty$. This implies $d\alpha - \delta_2(1) = 0$. Define a local morphism of $\mathcal{S}$-complexes $\widetilde{\lambda}\colon R[U^{\pm 1}]_{(0)} \to \widetilde{C}$ by $\widetilde{\lambda}(0, 0, 1) = (0, \alpha, 1)$. If z is as in the statement of the lemma, by (87), $\deg_I(\alpha) = 0$ holds, which implies this is a level 0 morphism. □

Corollary 5.20 *An enriched complex $\mathfrak{E}$ consisting of $\{\widetilde{C}^i\}$, $\{\widetilde{\phi}_i^j\}$, $\mathfrak{K}$ is weakly local equivalent to the trivial enriched complex if and only if the following two conditions hold:*

(i) *There is a sequence of special $(0, 1, -\infty)$-cycles $\{z_j\}$ for $\{\widetilde{C}^j\}$ such that $\lim_{j\to\infty} \deg_I(z_j) \le 0$.*

(ii) *There is a sequence of special $(0, 1, -\infty)$-cycles $\{z_j\}$ for $\{(\widetilde{C}^j)^\dagger\}$ such that $\lim_{j\to\infty} \deg_I(z_j) \le 0$.*

Proof Using (i), as in the proof of Lemma 5.19, we define local morphisms $\widetilde{\lambda}_i\colon R[U^{\pm 1}]_{(0)} \to \widetilde{C}^i$ by $\widetilde{\lambda}_i(1) := (0, \alpha_i, 0)$, where α_i is an element in $\widetilde{C}^i$ such that $d\alpha_i = \delta_2(1)$ and $\lim_i \deg_I(\alpha_i) \le 0$. The sequence $\mathfrak{L} := \{\widetilde{\lambda}_i\}$ defines a weak local morphism from the trivial enriched $\mathcal{S}$-complex to $\mathfrak{E}$. Using (ii) and dualizing, we obtain a weak local morphism in the opposite direction. □

5.3 Numerical invariants for local equivalence classes of enriched complexes

Using filtered special cycles, we define a local equivalence invariant of enriched $\mathcal{S}$-complexes called $\mathcal{J}_{\mathfrak{E}}(k,s)$. This definition is motivated by a desire to generalize the Γ-invariants from [Daemi 2020; Daemi and Scaduto 2024b] and the r_s-type invariants from [Nozaki et al. 2024].

Definition 5.21 For an I-graded $\mathcal{S}$-complex $\widetilde{C}$, define

$$\mathcal{N}_{\widetilde{C}}(k,s) := \inf\{\deg_I(z) \mid z \text{ is a filtered special } (k,1,s)\text{-cycle}\} \in [0,\infty] \tag{88}$$

for $(k,s) \in \mathbb{Z} \times [-\infty,0)$.

Lemma 5.22 *For an I-graded $\mathcal{S}$-complex $\widetilde{C}$ and a positive integer k and $s \in [-\infty,0)$, we have*

$$\mathcal{N}_{\widetilde{C}}(k,s) = \inf\{\deg_I(\alpha) \mid \alpha \in C_{2k-1},\ \delta_1 v^j\alpha = 0 \text{ for } 0 \le j \le k-2,\ \delta_1 v^{k-1}\alpha = 1,\ \deg_I(d\alpha) \le s\} \tag{89}$$

and, if k is a nonpositive integer, we have

$$\mathcal{N}_{\widetilde{C}}(k,s) = \inf\left\{\max\{\deg_I(\alpha),0\} \;\middle|\; \alpha \in C_{2k-1},\ a_i \in R[U^{\pm 1}]\ (0 \le i \le -k),\ a_{-k} = 1,\ \deg_I\left(d\alpha - \sum_{i=0}^{-k} v^i\delta_2(a_i)\right) \le s\right\}. \tag{90}$$

In (90), we may assume $a_i = 0$ if $i \not\equiv k \pmod 2$, and otherwise $a_i = f_i U^{(k+i)/2}$ for some $f_i \in R$.

Proof This is a straightforward consequence of the definitions. For any α satisfying the condition in (89), we obtain the filtered special $(k,1,s)$-cycle $\widehat{\Psi}(\alpha,0)$ for a positive integer k, and, for any α and a_i satisfying (90), we obtain the filtered special $(k,1,s)$-cycle $\widehat{\Psi}\big(\alpha, \sum_{i=0}^{-k} a_i x^i\big)$ for a nonpositive integer k. For the last part, see Remark 5.12. □

Remark 5.23 Note that $\mathcal{N}_{\widetilde{C}}(k,s) = 0$ when $k \le 0$ and $s \in \big[\deg_I(\delta_2(1)), 0\big)$. To see this, one considers $\alpha = 0$ in the conditions appearing in (90).

The following monotonicity property is a consequence of Remark 5.11:

Lemma 5.24 *For any I-graded $\mathcal{S}$-complex $\widetilde{C}$, $\mathcal{N}_{\widetilde{C}}$ is increasing with respect to k and decreasing with respect to s. That is to say, for any $(k,s), (k',s') \in \mathbb{Z} \times [-\infty,0)$ with $k' \le k$ and $s' \ge s$, we have*

$$\mathcal{N}_{\widetilde{C}}(k',s') \le \mathcal{N}_{\widetilde{C}}(k,s).$$

Lemma 5.25 *Let $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ be a strong height i morphism of level $\kappa \ge 0$ between I-graded $\mathcal{S}$-complexes. Then, for $k \in \mathbb{Z}$ and $s \in [-\infty,0)$ satisfying $s+\kappa < 0$, we have*

$$\mathcal{N}_{\widetilde{C}'}(k+i, s+\kappa) \le \mathcal{N}_{\widetilde{C}}(k,s) + \kappa.$$

Proof If z is a filtered special $(k,1,s)$-cycle for $\widetilde{C}$, then Lemma 5.13 implies that

$$\widehat{\Psi}'\circ\widehat{\Phi}'\circ\widehat{\boldsymbol{\lambda}}(c_i^{-1}z) \tag{91}$$

is a filtered special $(k+i,1,s+\kappa)$-cycle for $\widetilde{C}'$, where c_i is defined as in (15). Moreover, $\deg_I$ of the special cycle in (91) is at most $\deg_I(z)+\kappa$. The claim follows from this observation. □

Lemma 5.26 *For an enriched complex $\mathfrak{E}=(\{\widetilde{C}^i\},\{\widetilde{\phi}_i^j\},\mathfrak{K})$, an integer k and $s\in[-\infty,0)\setminus\mathfrak{K}$, the limit of $\{\mathcal{N}_{\widetilde{C}^i}(k,s)\}_{i\in\mathbb{Z}_{>0}}$ exists in* $[0,\infty]$.

Proof If $[s_1,s_2]$ is an interval in $\mathbb{R}\setminus\mathfrak{K}$, then there is some $N>0$ such that, for all $n>N$, any special (k,f,s_2)-cycle for $\widetilde{C}^n$ is also a special (k,f,s_1)-cycle. In particular, for any such s_1, s_2 and N, we have

$$\mathcal{N}_{\widetilde{C}}(k,s_1)=\mathcal{N}_{\widetilde{C}}(k,s_2). \tag{92}$$

This observation together with Lemma 5.25 implies that $\{\mathcal{N}_{\widetilde{C}^i}(k,s)\}_{i\in\mathbb{Z}_{>0}}$ is a Cauchy sequence. Thus the limit exists. □

Lemma 5.26 allows us to extend Definition 5.21 to enriched complexes.

Definition 5.27 For an enriched $\mathcal{S}$-complex $\mathfrak{E}$ given by the data $(\{\widetilde{C}^i\},\{\widetilde{\phi}_i^j\},\mathfrak{K})$, define the invariant $\mathcal{N}_{\mathfrak{E}}\colon\mathbb{Z}\times\big([-\infty,0)\setminus\mathfrak{K}\big)\to[0,\infty]$ by

$$\mathcal{N}_{\mathfrak{E}}(k,s):=\lim_{i\to\infty}\mathcal{N}_{\widetilde{C}^i}(k,s). \tag{93}$$

If $s\in\mathfrak{K}\cap[-\infty,0)$, then we define $\mathcal{N}_{\mathfrak{E}}(k,s)=\lim_{s'\to s^-}\mathcal{N}_{\mathfrak{E}}(k,s')$.

Example 5.28 For the trivial enriched complex $\mathfrak{E}_0$, we have

$$\mathcal{N}_{\mathfrak{E}_0}(k,s)=\begin{cases}\infty & \text{if } k>0,\\ 0 & \text{if } k\le 0,\end{cases} \tag{94}$$

and, for the enriched complex $\mathfrak{E}\{r\}$ of Remark 5.7 and its dual $\mathfrak{E}\{r\}^\dagger$, we have

$$\mathcal{N}_{\mathfrak{E}\{r\}}(k,s)=\begin{cases}\infty & \text{if } k>1,\\ r & \text{if } k=1,\\ 0 & \text{if } k\le 0,\end{cases}\qquad \mathcal{N}_{\mathfrak{E}\{r\}^\dagger}(k,s)=\begin{cases}\infty & \text{if } k\ge 1,\\ \infty & \text{if } k=0 \text{ and } s<r,\\ 0 & \text{otherwise.}\end{cases} \tag{95}$$

Remark 5.29 For an enriched $\mathcal{S}$-complex $\mathfrak{E}$, it is not necessarily true — unlike for $\mathcal{S}$-complexes, in Remark 5.23 — that $\mathcal{N}_{\mathfrak{E}}(k,s)=0$ for $k\le 0$ and s close to zero. To see this, one can take $\mathfrak{E}$ to contain a sequence of $\mathcal{S}$-complexes $\widetilde{C}\{r_i\}^\dagger$ from Remark 5.7 where r_i are positive numbers approaching 0. See, however, Remark 5.44.

We may define a slight variation of $\mathcal{N}_{\mathfrak{E}}$ in the following way:

Definition 5.30 For an I-graded $\mathcal{S}$-complex $\widetilde{C}$ over $R[U^{\pm 1}]$, define

$$\underline{\mathcal{N}}_{\widetilde{C}}(k,s) := \inf\{\deg_I(z) \mid z \text{ is a filtered special } (k,f,s)\text{-cycle with } f \neq 0\} \in [0,\infty] \tag{96}$$

for $(k,s) \in \mathbb{Z} \times [-\infty, 0)$. For an enriched $\mathcal{S}$-complex $\mathfrak{E} = (\{\widetilde{C}^i\}, \{\widetilde{\phi}_i^j\}, \mathfrak{K})$ over $R[U^{\pm 1}]$, define

$$\underline{\mathcal{N}}_{\mathfrak{E}}(k,s) := \lim_{i\to\infty} \underline{\mathcal{N}}_{\widetilde{C}^i}(k,s) \tag{97}$$

if $(k,s) \in \mathbb{Z} \times \big([-\infty,0) \setminus \mathfrak{K}\big)$. Extend this definition to the case that $s \in \mathfrak{K}$ by requiring that $\underline{\mathcal{N}}_{\widetilde{C}}$ be continuous from the left with respect to the variable s.

Remark 5.31 We have an analogue of Lemma 5.22 for $\underline{\mathcal{N}}_{\widetilde{C}}(k,s)$ by replacing the condition $\delta_1 v^{k-1}\alpha = 1$ in (89) and the condition $a_{-k} = 1$ in (90), respectively, with $\delta_1 v^{k-1}\alpha \neq 0$ and $a_{-k} \neq 0$.

Remark 5.32 In the case that R is a field, the invariants $\mathcal{N}_{\mathfrak{E}}$ and $\underline{\mathcal{N}}_{\mathfrak{E}}$ agree. In general, we have $\mathcal{N}_{\mathfrak{E}}(k,s) \geq \underline{\mathcal{N}}_{\mathfrak{E}}(k,s)$. For instance, let $\mathfrak{E}'\{r\}$ be the enriched $\mathcal{S}$-complex with the same chain group as $\mathfrak{E}\{r\}$ and the differential that is a multiple of the differential of $\mathfrak{E}\{r\}$ by a nonunit element. Then $\mathcal{N}_{\mathfrak{E}'\{r\}}(1,s) = \infty$ and $\underline{\mathcal{N}}_{\mathfrak{E}'\{r\}}(1,s) = r$.

Next, we develop some basic properties for the invariants $\mathcal{N}_{\mathfrak{E}}$ and $\underline{\mathcal{N}}_{\mathfrak{E}}$.

Lemma 5.33 *For an enriched $\mathcal{S}$-complex $\mathfrak{E} = (\{\widetilde{C}^i\}, \{\widetilde{\phi}_i^j\}, \mathfrak{K})$, the map $\mathcal{N}_{\mathfrak{E}}$ is locally constant on the complement of $\mathbb{Z} \times \mathfrak{K}$ in $\mathbb{Z} \times [-\infty, 0)$. Moreover, $\mathcal{N}_{\mathfrak{E}}$ takes values in $\mathfrak{K} \cup \{\infty\}$. Similar properties hold for the map $\underline{\mathcal{N}}_{\mathfrak{E}}$.*

Proof The first claim follows from (92). The second part is straightforward, and similar proofs can be used to address the analogous claims for $\underline{\mathcal{N}}_{\mathfrak{E}}$. □

Lemma 5.34 *For any enriched $\mathcal{S}$-complex $\mathfrak{E}$, $\mathcal{N}_{\mathfrak{E}}$ and $\underline{\mathcal{N}}_{\mathfrak{E}}$ are increasing with respect to k and decreasing with respect to s. That is to say, for any $(k,s), (k',s') \in \mathbb{Z} \times [-\infty, 0)$ with $k' \leq k$ and $s' \geq s$, we have*

$$\mathcal{N}_{\mathfrak{E}}(k',s') \leq \mathcal{N}_{\mathfrak{E}}(k,s), \quad \underline{\mathcal{N}}_{\mathfrak{E}}(k',s') \leq \underline{\mathcal{N}}_{\mathfrak{E}}(k,s).$$

Proof In the case of $\mathcal{N}_{\mathfrak{E}}$, the claim can be verified using Lemma 5.24. The proof for $\underline{\mathcal{N}}_{\mathfrak{E}}$ is similar. □

Proposition 5.35 *Let $\mathfrak{L}\colon \mathfrak{E} \to \mathfrak{E}'$ be a strong height i morphism of level κ between enriched $\mathcal{S}$-complexes. Then, for $k \in \mathbb{Z}$ and $s \in [-\infty, 0)$, we have*

$$\mathcal{N}_{\mathfrak{E}'}(k+i, s) \leq \mathcal{N}_{\mathfrak{E}}(k, s-\kappa) + \kappa.$$

If $\mathfrak{L}\colon \mathfrak{E} \to \mathfrak{E}'$ is a height i morphism of level κ and c_i, defined as in (15), *is nonzero, then*

$$\underline{\mathcal{N}}_{\mathfrak{E}'}(k+i, s) \leq \underline{\mathcal{N}}_{\mathfrak{E}}(k, s-\kappa) + \kappa.$$

Proof The first part follows from Lemma 5.25, and the second part can be proved in a similar way. □

Remark 5.36 An analysis of the proof shows that Proposition 5.35 holds with the weaker assumption that the morphisms $\mathfrak{L}$ in the statement do not necessarily satisfy condition (ii) of Definition 5.5.

Corollary 5.37 *The invariants $\mathcal{N}_{\mathfrak{E}}$ and $\underline{\mathcal{N}}_{\mathfrak{E}}$ depend only on the weak local equivariance type of $\mathfrak{E}$.*

Proof Proposition 5.35 implies that $\mathcal{N}_{\mathfrak{E}}$ and $\underline{\mathcal{N}}_{\mathfrak{E}}$ depend only the local equivariance type of $\mathfrak{E}$. We can upgrade this fact to our desired claim using Remark 5.36. □

The following is a consequence of Lemma 5.15:

Theorem 5.38 *For two enriched complexes $\mathfrak{E}$ and $\mathfrak{E}'$, if $\mathcal{N}_{\mathfrak{E}}(k,s)$, $\mathcal{N}_{\mathfrak{E}'}(k',s')$ are finite and*

$$s^{\otimes} := \max\{\mathcal{N}_{\mathfrak{E}}(k,s)+s', \mathcal{N}_{\mathfrak{E}'}(k',s')+s\} < 0,$$

then

$$\mathcal{N}_{\mathfrak{E}\otimes\mathfrak{E}'}(k+k',s^{\otimes}) \le \mathcal{N}_{\mathfrak{E}}(k,s)+\mathcal{N}_{\mathfrak{E}'}(k',s'), \quad \underline{\mathcal{N}}_{\mathfrak{E}\otimes\mathfrak{E}'}(k+k',s^{\otimes}) \le \underline{\mathcal{N}}_{\mathfrak{E}}(k,s)+\underline{\mathcal{N}}_{\mathfrak{E}'}(k',s').$$

In [Daemi and Scaduto 2024b, Section 7.5], an invariant $\Gamma_{\mathfrak{E}}$ is defined for any enriched $\mathcal{S}$-complex $\Gamma_{\mathfrak{E}}$. The following is a straightforward consequence of the definition of Γ, Lemma 5.22 and Remark 5.31:

Lemma 5.39 *For any enriched $\mathcal{S}$-complex $\mathfrak{E}$ over $R[U^{\pm 1}]$, we have*

$$\Gamma_{\mathfrak{E}}(k) = \mathcal{N}_{\mathfrak{E}\otimes \operatorname{Frac} R}(k,-\infty) = \underline{\mathcal{N}}_{\mathfrak{E}}(k,-\infty), \tag{98}$$

where $\mathfrak{E}\otimes \operatorname{Frac} R$ is the enriched $\mathcal{S}$-complex obtained using the base change with respect to the field of fractions of R.

The *transpose* of $\mathcal{N}_{\mathfrak{E}}$ for an enriched $\mathcal{S}$-complex $\mathfrak{E}$ is the map $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}} : \mathbb{Z}\times[0,\infty] \to [-\infty,0]$ defined by

$$\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r) := \min\bigl\{\inf\{s\in[-\infty,0) \mid \mathcal{N}_{\mathfrak{E}}(k,s)\le r\}, 0\bigr\} \quad (r<\infty) \tag{99}$$

and $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,\infty) := \lim_{r\to\infty}\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)$. Note that $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)=0$ if and only if $\mathcal{N}_{\mathfrak{E}}(k,0)$ (and hence any $\mathcal{N}_{\mathfrak{E}}(k,s)$) is greater than r. It also follows immediately from the definition that

$$\mathcal{N}_{\mathfrak{E}}(k,-\infty)=0 \iff \mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,0)=-\infty. \tag{100}$$

This function is again increasing with respect to k and decreasing with respect to r. Lemma 5.33 implies that $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}$ for any enriched complex $\mathfrak{E}=(\{\widetilde{C}^i\},\{\widetilde{\phi}^j_i\},\mathfrak{K})$ takes values in $\mathfrak{K}\cup\{-\infty\}$, and it is locally constant on the complement of $\mathbb{Z}\times\mathfrak{K}$ in $\mathbb{Z}\times[0,\infty]$. Moreover, for any fixed value of k, the function $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,\cdot)$ is continuous from the right. We may recover $\mathcal{N}_{\mathfrak{E}}$ from $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}$ using the identity

$$\mathcal{N}_{\mathfrak{E}}(k,s) = \min\{r \mid \mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)\le s\} \quad \text{if } s\notin\mathfrak{K}.$$

The following is a consequence of the definition of $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)$ and Theorem 5.38:

$$\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}\otimes\mathfrak{E}'}(k+k',r+r') \le \max\{\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)+r', \mathcal{N}^{\mathsf{T}}_{\mathfrak{E}'}(k',r')+r\}. \tag{101}$$

Finally, we mention that the analogue of Proposition 5.35 holds for $\mathcal{N}^{\mathsf{T}}$: if there is a strong height i morphism $\mathfrak{E}\to\mathfrak{E}'$ of level $\kappa\ge 0$ between I-graded $\mathcal{S}$-complexes, and $k\in\mathbb{Z}$ and $r\in[0,\infty]$, we have

$$\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}'}(k+i,r+\kappa) \le \mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r)+\kappa. \tag{102}$$

The following is the analogue of Lemma 5.22, and the proof is straightforward:

Lemma 5.40 *For an I-graded $\mathcal{S}$-complex $\widetilde{C}$ and a positive integer k and $r \in [0,\infty]$, we have*

$$\mathcal{N}^{\mathsf{T}}_{\widetilde{C}}(k,r) = \min\bigl\{\inf\bigl\{\deg_I(d\alpha) \mid \alpha \in C_{2k-1},\ \delta_1 v^j \alpha = 0 \text{ for } 0 \le j \le k-2,\ \delta_1 v^{k-1}\alpha = 1,\ \deg_I(\alpha) \le r\bigr\}, 0\bigr\} \tag{103}$$

and, if k is a nonpositive integer, we have

$$\mathcal{N}^{\mathsf{T}}_{\widetilde{C}}(k,r) = \inf\left\{ \deg_I\left(d\alpha - \sum_{i=0}^{-k} v^i \delta_2(a_i)\right) \,\middle|\, \alpha \in C_{2k-1},\ a_i \in R[U^{\pm 1}]\ (0 \le i \le -k),\ a_{-k} = 1,\ \deg_I(\alpha) \le r \right\}. \tag{104}$$

In (104), *we may assume $a_i = 0$ if $i \not\equiv k \pmod 2$, and otherwise $a_i = f_i U^{(k+i)/2}$ for some $f_i \in R$.*

Motivated by [Nozaki et al. 2024], we define another numerical invariant for enriched complexes.

Definition 5.41 For any enriched $\mathcal{S}$-complex $\mathfrak{E}$ over $R[U^{\pm 1}]$ and any $s \in [-\infty, 0]$, we define

$$r_s(\mathfrak{E}) := -\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(0,-s) \in [0,\infty]. \tag{105}$$

The following is a corollary of Theorem 5.38 and (101):

Corollary 5.42 *The invariants $\Gamma_{\mathfrak{E}}$ and $r_s(\mathfrak{E})$ satisfy*

$$\Gamma_{\mathfrak{E}\otimes\mathfrak{E}'}(k+k') \le \Gamma_{\mathfrak{E}}(k) + \Gamma_{\mathfrak{E}'}(k'), \quad r_{s+s'}(\mathfrak{E}\otimes\mathfrak{E}') - s - s' \ge \min\{r_s(\mathfrak{E}) - s,\ r_{s'}(\mathfrak{E}') - s'\}.$$

Finally, the invariants that we have defined above can detect the (weak) local equivalence class of the trivial enriched $\mathcal{S}$-complex.

Corollary 5.43 *An enriched $\mathcal{S}$-complex $\mathfrak{E}$ is weakly locally equivalent to the trivial enriched $\mathcal{S}$-complex if and only if*

$$\mathcal{N}_{\mathfrak{E}}(0,-\infty) = 0 \quad \text{and} \quad \mathcal{N}_{\mathfrak{E}^\dagger}(0,-\infty) = 0. \tag{106}$$

These conditions are equivalent to $r_0(\mathfrak{E}) = \infty$ and $r_0(\mathfrak{E}^\dagger) = \infty$.

Proof The first statement follows from Corollary 5.20, and the last statement from (100). □

5.4 Homology concordance invariants

We now use the homomorphism $\Omega\colon \Theta^{3,1}_{\mathbb{Z}} \to \Theta^{\mathfrak{E}}_R$ constructed in Theorem 5.8, together with the numerical invariants of $\Theta^{\mathfrak{E}}_R$ defined above, to construct homology concordance invariants for knots in integer homology 3-spheres. Unless otherwise stated, R is an integral domain algebra over $\mathbb{Z}[T^{\pm 1}]$.

Let $K \subset Y$ be a knot in an integer homology 3-sphere, and write $\mathfrak{E}(Y,K)$ for the associated enriched $\mathcal{S}$-complex. Applying the construction of Definition 5.27, we obtain

$$\mathcal{N}_{(Y,K)}(k,s) := \mathcal{N}_{\mathfrak{E}(Y,K)}(k,s).$$

Here $k \in \mathbb{Z}$, $s \in [-\infty, 0)$ and $\mathcal{N}_{(Y,K)}(k,s) \in [0,\infty]$. We similarly define the homology concordance invariant $\underline{\mathcal{N}}_{(Y,K)}(k,s)$ using Definition 5.30. We also apply the construction of (99) to define

$$\mathcal{N}^{\mathsf{T}}_{(Y,K)}(k,r) := \mathcal{N}^{\mathsf{T}}_{\mathfrak{E}(Y,K)}(k,r),$$

where $k \in \mathbb{Z}$, $r \in [0,\infty]$, and $\mathcal{N}^{\mathsf{T}}_{(Y,K)}(k,r)$ takes values in $[-\infty, 0]$. By construction, all of these invariants factor through the homomorphism Ω and define homology concordance invariants. In the case that Y is the 3-sphere, we omit it from notation and write $\mathcal{N}_K$, and similarly for the other invariants.

Remark 5.44 For any sequence of perturbations π_i (with auxiliary choices) approaching zero, we have $\limsup \deg_I(\delta_2^i(1)) \le m < 0$, where $m := \max\{\mathfrak{K} \cap (-\infty,0)\}$, δ_2^i is the δ_2-map defined using π_i, and $\mathfrak{K}$ is the set of critical values of the unperturbed Chern–Simons functional. This follows from nondegeneracy of the reducible connection and standard compactness properties. Using this and Remark 5.23, we obtain that $\mathcal{N}_{\mathfrak{E}(Y,K)}(k,s) = 0$ for $k \le 0$ and $s \in [m, 0)$. In particular, $\mathcal{N}^{\mathsf{T}}_{\mathfrak{E}}(k,r) \le m$ when $k \le 0$.

As special values, we have the homology concordance invariants

$$\Gamma_{(Y,K)}(k) := \underline{\mathcal{N}}_{(Y,K)}(k,-\infty), \quad r_s(Y,K) := -\mathcal{N}^{\mathsf{T}}_{(Y,K)}(0,-s).$$

The invariant $\Gamma_{(Y,K)}$ is the same as the invariant $\Gamma^R_{(Y,K)}$ studied in [Daemi and Scaduto 2024b; 2024a]. The invariant $r_s(Y,K)$ is an analogue of the integer homology 3-sphere invariant defined in [Nozaki et al. 2024]. Later in this section, we describe an explicit relationship between this latter invariant and $r_s(Y,U_1)$, where U_1 is an unknot in a small ball inside Y; see (120).

The following connected sum inequalities follow from the inequalities of Theorem 5.38 and (101), and the fact that $\Omega\colon \Theta^{3,1}_{\mathbb{Z}} \to \Theta^{\mathfrak{E}}_{R}$ is a homomorphism:

Theorem 5.45 *Given knots in integer homology* 3*-spheres* (Y,K) *and* (Y',K'), *suppose* $s^{\otimes} < 0$, *where* $s^{\otimes} := \max\{\mathcal{N}_{(Y,K)}(k,s) + s', \mathcal{N}_{(Y',K')}(k',s') + s\}$. *Then*

$$\mathcal{N}_{(Y\#Y',K\#K')}(k+k',s^{\otimes}) \le \mathcal{N}_{(Y,K)}(k,s) + \mathcal{N}_{(Y',K')}(k',s'),$$

with the same inequality holding for the $\underline{\mathcal{N}}$*-invariants. Furthermore,*

$$\mathcal{N}^{\mathsf{T}}_{(Y\#Y',K\#K')}(k+k',r+r') \le \max\{\mathcal{N}^{\mathsf{T}}_{(Y,K)}(k,r) + r', \mathcal{N}^{\mathsf{T}}_{(Y,'K')}(k',r') + r\}.$$

Specializing to the case of the invariants $\Gamma_{(Y,K)}$ and $r_s(Y,K)$, we obtain the following:

Corollary 5.46 *The invariants* Γ *and* r_s *satisfy the inequalities*

$$\Gamma_{(Y\#Y',K\#K')}(k+k') \le \Gamma_{(Y,K)}(k) + \Gamma_{(Y',K')}(k'),$$
$$r_{s+s'}(Y\#Y',K\#K') - s - s' \ge \min\{r_s(Y,K)-s,\, r_{s'}(Y',K')-s'\}.$$

Note that Theorem 5.45 and Corollary 5.46 prove Theorem 1.19.

We next consider the behavior of these invariants under cobordisms. What follows is a straightforward generalization of some material from [Daemi and Scaduto 2024b, Section 4.4]. Recall from the discussion surrounding Definition 2.7 that associated to cobordism data (W,S,c) with $b_1(W)=b^+(W)=0$ there is an integer

$$i := 4\kappa_{\min}(W,S,c) + \tfrac{1}{4}S\cdot S + \tfrac{1}{2}\chi(S) + \tfrac{1}{2}\sigma(Y,K) - \tfrac{1}{2}\sigma(Y',K') \tag{107}$$

such that, if $i \ge 0$, then there is an associated height i morphism of $\mathcal{S}$-complexes. This construction can be used to construct a height i morphism of the associated enriched $\mathcal{S}$-complexes.

Theorem 5.47 *Let* $(W,S)\colon (Y,K)\to(Y',K')$ *be a cobordism, as in Definition 2.7, which is negative definite of strong height* $i\ge 0$ *over* R, *where* i *can be computed from* (107). *Then*

$$\mathcal{N}_{(Y',K')}(k+i,s) \le \mathcal{N}_{(Y,K)}(k,s-2\kappa_{\min}(W,S)) + 2\kappa_{\min}(W,S). \tag{108}$$

Moreover, if equality is achieved in (108) *for some* $k\in\mathbb{Z}$ *and* $s\in[-\infty,0)$ *with both sides finite and positive, then there exists an irreducible traceless* $\mathrm{SU}(2)$*-representation of* $\pi_1(W\setminus S)$. *The same conclusion holds for the* $\underline{\mathcal{N}}$*-invariants, under the weaker assumption that the cobordism is not necessarily strong, but satisfies* $\eta(W,S)\ne 0$.

Proof Inequality (108) follows from Propositions 5.6 and 5.35.

Now we prove the second statement, regarding when equality is achieved in (108). For now, assume our enriched $\mathcal{S}$-complexes are simply I-graded $\mathcal{S}$-complexes, and morphisms are level 0. In particular, the singular instanton $\mathcal{S}$-complexes, and the relevant cobordism maps below, can be defined without perturbations, and the I-gradings are determined by the unperturbed Chern–Simons functional.

Define $\kappa := \kappa_{\min}(W,S)$. Let $z=\widehat{\Psi}(\mathfrak{z})$ be a filtered special $(k,1,s-2\kappa)$-cycle with $\mathfrak{z}=\bigl(\alpha,\sum_{i=0}^N a_i x^i\bigr)$ in $\widehat{\mathfrak{C}}_{2k}(Y,K)$. Recall from Remark 5.12 that

$$\deg_I(z)=\deg_I(\mathfrak{z})=\max\{\deg_I(\alpha),0\},$$

and $\deg_I(a_i)\le 0$. Assume z is a filtered special cycle that realizes the value of $\mathcal{N}_{(Y,K)}(k,s-2\kappa)$. This is possible because the I-gradings take values in the image of the Chern–Simons functional, a discrete subset of $\mathbb{R}$. Thus,

$$\mathcal{N}_{(Y,K)}(k,s-2\kappa)=\max\{\deg_I(\alpha),0\}. \tag{109}$$

Let $\widehat{\lambda}\colon \widehat{\mathfrak{C}}(Y,K) \to \widehat{\mathfrak{C}}(Y',K')$ be induced by (W,S,c) on small equivariant (filtered) complexes. By Lemma 5.13, $z' = \widehat{\Psi}'(\mathfrak{z}')$, where $\mathfrak{z}' = \widehat{\lambda}(\mathfrak{z})$, is a filtered special $(k+i,1,s)$-cycle. We obtain

$$\mathcal{N}_{(Y',K')}(k+i,s) \le \deg_I(\widehat{\lambda}(\mathfrak{z})) \le \deg_I(\mathfrak{z}) + 2\kappa = \mathcal{N}_{(Y,K)}(k,s-2\kappa) + 2\kappa, \tag{110}$$

where we have used in the second inequality that $\widehat{\lambda}$ has level 2κ, and we have also used $\deg_I(\mathfrak{z}') = \deg_I(z')$. By our assumption that (108) is an equality, the inequalities in (110) are equalities. Next, $\mathfrak{z}' = \left(\alpha', \sum_{i=0}^{N'} a_i' x^i\right)$, where

$$\alpha' = \lambda(\alpha) + \sum_{i=1}^{N}\sum_{j=0}^{i-1} \mu v^{i-j-1}\delta_2(a_i) + \sum_{i=0}^{N}(v')^i\,\Delta_2(a_i). \tag{111}$$

This follows from direct computation of $\mathfrak{z}' = \widehat{\lambda}(\mathfrak{z}) = \widehat{\Phi}' \circ \widehat{\boldsymbol{\lambda}} \circ \widehat{\Psi}'(\mathfrak{z})$ using the definitions of the maps from Section 2.2. As $\mathcal{N}_{(Y',K')}(k+i,s) > 0$, we have by Remark 5.12 that

$$\mathcal{N}_{(Y',K')}(k+i,s) = \deg_I(\mathfrak{z}') = \deg_I(\alpha').$$

Now $\deg_I(\alpha')$ is the maximum of the I-degrees of the three terms appearing in (111).

First, consider the case in which this maximum is realized by the first term:

$$\deg_I(\alpha') = \deg_I(\lambda(\alpha)).$$

From (109), and the string of equalities achieved in (110), we obtain

$$\deg_I(\alpha') \ge \deg_I(\alpha) + 2\kappa. \tag{112}$$

Furthermore, $\deg_I(\alpha') \le \deg_I(\alpha) + 2\kappa$ because $\widetilde{\lambda}$ has level 2κ. Thus (112) is an equality. Recall from [Daemi and Scaduto 2024a] that $\alpha = \sum r_i \widetilde{\alpha}_i$ and $\alpha' = \sum r_i' \widetilde{\alpha}_i'$ are linear combinations over R of irreducible flat connections (in fact, paths of flat connections to the reducible), and $\lambda(\widetilde{\alpha}_i) = \sum d_{ij}\widetilde{\alpha}_j'$, where $d_{ij} \in \mathbb{Z}$ is the signed count of elements in a moduli space of singular instantons on the cobordism (W,S) with cylindrical ends attached. For such an instanton A in this count, we have

$$\deg_I(\widetilde{\alpha}_i) + 2\kappa = \deg_I(\widetilde{\alpha}_j') + 2\kappa(A).$$

Suppose $\widetilde{\alpha}_j'$ is chosen such that $\deg_I(\alpha') = \deg_I(\widetilde{\alpha}_j')$ and $r_j' \ne 0$. We obtain

$$\deg_I(\alpha) + 2\kappa \ge \deg_I(\widetilde{\alpha}_i) + 2\kappa = \deg_I(\widetilde{\alpha}_j') + 2\kappa(A) = \deg_I(\alpha') + 2\kappa(A).$$

Using (112), we obtain $\kappa(A) \le 0$. The energy of an instanton is necessarily nonnegative, and thus $\kappa(A) = 0$, and A is flat. This flat singular connection corresponds to an SU(2)-representation of $\pi_1(W \setminus S)$ that extends irreducible representations of $\pi_1(Y \setminus K)$ and $\pi_1(Y' \setminus K')$ which send meridians to traceless elements in SU(2).

Now suppose $\deg_I(\alpha')$ is realized by the second term in (111). Here, $\deg_I(\alpha') = \deg_I(\mu(\beta))$, where

$$\beta := \sum_{i=1}^{N}\sum_{j=0}^{i-1} v^{i-j-1}\delta_2(a_i).$$

Since each $\deg_I(a_i) \le 0$, we have $\deg_I(\beta) \le 0$. Since $\widetilde{\lambda}$ is of level 2κ, $\deg_I(\alpha') \ge \deg_I(\beta) + 2\kappa$, analogous to (112). The argument now proceeds as in the previous case, and we obtain a representation of $\pi_1(W \setminus S)$ that extends irreducible representations at (Y, K) and (Y', K'), just as before. In the last case, in which $\deg_I(\alpha')$ is realized by the third term in (111), we have some $r \in R[U^{\pm 1}]$ such that $\deg_I(r) \le 0$ and $\deg_I(\Delta_2(r)) \ge \deg_I(r) + 2\kappa$. The argument gives a representation of $\pi_1(W \setminus S)$ as before, except that it is only irreducible at the end of (Y', K').

The case involving nontrivial perturbations is similar, but also uses limiting arguments similar to those in [Daemi 2020, Section 3.2; Nozaki et al. 2024, Section 3]. □

Substituting $s = -\infty$, Theorem 5.47 recovers one of the inequalities in [Daemi and Scaduto 2024a, Proposition 4.33].

Remark 5.48 The proof of Theorem 5.47 given above shows the following. If equality occurs in (108) and $\mathcal{N}_{(Y,K)}(k, s - 2\kappa_{\min}(W, S))$ is finite and positive, then there exists an irreducible SU(2)-representation of $\pi_1(W \setminus S)$ extending irreducible traceless representations of $\pi_1(Y \setminus K)$ and $\pi_1(Y' \setminus K')$. However, if $\mathcal{N}_{(Y,K)}(k, s - 2\kappa_{\min}(W, S)) = 0$ and $\mathcal{N}_{(Y',K')}(k + i, s) > 0$, then the representation obtained is irreducible at (Y', K'), but possibly reducible at (Y, K).

Remark 5.49 A version of Theorem 5.47 also holds for nontrivial bundles. If (W, S, c) is cobordism data as in Definition 2.7, which is negative definite of strong height $i \ge 0$ over R, then

$$\mathcal{N}_{(Y',K')}(k + i, s) \le \mathcal{N}_{(Y,K)}(k, s - 2\kappa_{\min}(W, S, c)) + 2\kappa_{\min}(W, S, c).$$

In this case, if both sides are finite and positive, there exists an irreducible SU(2)-representation of $\pi_1(W \setminus (S \cup F))$ which is traceless around meridians of S and equal to -1 around meridians of F, where F is an embedded surface in W such that $[F]$ is Poincaré dual to $c \in H^2(W; \mathbb{Z})$.

The following is an analogue of Theorem 5.47 for $\mathcal{N}^{\mathsf{T}}$, and the proof is similar (see also (102)):

Theorem 5.50 *Let $(W, S)\colon (Y, K) \to (Y', K')$ be a cobordism, as in Definition 2.7, which is negative definite of strong height $i \ge 0$ over R, where i can be computed from (107). Then*

$$\mathcal{N}^{\mathsf{T}}_{(Y',K')}(k + i, r + 2\kappa_{\min}(W, S)) \le \mathcal{N}^{\mathsf{T}}_{(Y,K)}(k, r) + 2\kappa_{\min}(W, S) \tag{113}$$

Moreover, if equality is achieved in (113) for some $k \in \mathbb{Z}$ and $r \in [0, \infty]$ with both sides finite and negative, then there exists an irreducible traceless SU(2)-representation of $\pi_1(W \setminus S)$.

Remark 5.51 If $k + i \le 0$, then both sides of (108) are automatically negative by Remark 5.44.

We now apply Theorem 5.47 to relate our invariants to knot surgeries.

Corollary 5.52 *Let K be a knot in an integer homology 3-sphere Y satisfying $\sigma(Y,K)\le 0$. Then we have the following inequality, where $Y_1(K)$ denotes 1-surgery on K:*

$$\mathcal{N}_{(Y,K)}\big(k-\tfrac{1}{2}\sigma(Y,K),s\big)\le \mathcal{N}_{(Y_1(K),U_1)}\big(k,s-\tfrac{1}{8}\big)+\tfrac{1}{8}.$$

A similar inequality holds for the $\underline{\mathcal{N}}$-invariants.

Proof Consider the 4-manifold with boundary W obtained by adding a 1-framed 2-handle to $[0,1]\times Y$ along $\{1\}\times K$, and reversing orientation. We view W as a cobordism $Y_1(K)\to Y$. In this cobordism there is an embedded annulus S formed by taking the core of the 2-handle and connect-summing with a small 2-disk whose boundary is an unknot $U_1\subset Y_1(K)$. We obtain a cobordism of pairs $(W,S)\colon (Y_1(K),U_1)\to (Y,K)$. Note $H^2(W;\mathbb{Z})$ is generated by $[S]$, $S\cdot S=-1$, and S has genus zero, and W is simply connected. Choosing bundle data $c=0$, in this case (16) is minimized by the unique choice $c_1(L)=0$, and we compute

$$\kappa_{\min}(W,S,c)=-\big(\tfrac{1}{4}S\big)^2=\tfrac{1}{16}.$$

Furthermore, (107) is computed to be $i=-\frac{1}{2}\sigma(Y,K)$. Note also $\eta(W,S,c)=1$, so the induced morphism is strong. The result now follows from Theorem 5.47. □

Substituting $s=-\infty$, we obtain the following:

Corollary 5.53 *Let K be a knot in an integer homology 3-sphere Y satisfying $\sigma(Y,K)\le 0$. Then*

$$\Gamma_{(Y,K)}\big(k-\tfrac{1}{2}\sigma(Y,K)\big)\le \Gamma_{(Y_1(K),U_1)}(k)+\tfrac{1}{8}.$$

A consequence of Theorem 5.50 and Remark 5.51 is the following:

Corollary 5.54 *Let $(W,S)\colon (Y,K)\to(Y',K')$ be a cobordism of pairs, as in Definition 2.7, which is negative definite of strong height 0 over R. Let $\kappa:=\kappa_{\min}(W,S)$. Then, for all $s\in[-\infty,0]$, we have*

$$r_{s-2\kappa}(Y,K)\le r_s(Y',K')+2\kappa. \tag{114}$$

Furthermore, if $r_{s-2\kappa}(Y,K)$ is finite and equality is achieved in (114), then there exists an irreducible traceless SU(2)-representation of $\pi_1(W\setminus S)$.

In the sequel, the above is typically applied to the case in which $(W,S)\colon (Y,K)\to(Y',K')$ is a cobordism of pairs satisfying $b_1(W)=b^+(W)=0$, with $S\subset W$ a null-homologous annulus and $\sigma(Y,K)=\sigma(Y',K')$. In this case we have, for any $s\in[-\infty,0]$,

$$r_s(Y,K)\le r_s(Y',K'). \tag{115}$$

If furthermore $\pi_1(W\setminus S)\cong\mathbb{Z}$, then there cannot be any irreducible traceless SU(2)-representation of $\pi_1(W\setminus S)$. In this case, for $s\in(-\infty,0]$, (115) is a strict inequality.

In proving linear independence of a given sequence of elements in $\Theta^{3,1}_{\mathbb{Z}}$, we will use the following:

Proposition 5.55 *Let $\{(Y_i, K_i)\}_{i\in\mathbb{Z}_{>0}}$ be a sequence of representatives in $\Theta^{3,1}_{\mathbb{Z}}$ satisfying*:

(i) $\infty > r_0(Y_1, K_1) > r_0(Y_2, K_2) > \cdots$.

(ii) $r_0(-Y_i, -K_i) = \infty$.

Then, for any linear combination $[(Y, K)] := \sum_{i=0}^N m_i[(Y_i, K_i)]$ in $\Theta^{3,1}_{\mathbb{Z}}$ with $m_N > 0$, we have $r_0(Y, K) < \infty$. In particular, $\{(Y_i, K_i)\}_{i\in\mathbb{Z}_{>0}}$ is a linearly independent subset of $\Theta^{3,1}_{\mathbb{Z}}$.

Proof We first claim that, for any positive integer m, we have

$$r_0\big(\#_m(Y_i, K_i)\big) = r_0(Y_i, K_i), \quad r_0\big(\#_m(-Y_i, -K_i)\big) = \infty. \tag{116}$$

Applying Corollary 5.46 to $(Y_i, K_i) \# (Y_i, K_i)$ yields the inequality $r_0\big(\#_2(Y_i, K_i)\big) \geq r_0(Y_i, K_i)$. On the other hand, we have $r_0(Y_i, K_i) \geq r_0\big(\#_2(Y_i, K_i)\big)$ by applying Corollary 5.46 to the decomposition $[(Y_i, K_i)] = 2[(Y_i, K_i)] - [(Y_i, K_i)]$ in $\Theta^{3,1}_{\mathbb{Z}}$. This yields the first part of (116) for $m = 2$. The case for larger values of m follows inductively. The second claim in (116) also follows from Corollary 5.46.

Now let $[(Y, K)] := \sum_{i=0}^N m_i[(Y_i, K_i)]$ be as in the statement of the proposition. A similar argument as above using Corollary 5.46 gives

$$r_0(Y, K) = \min\{r_0(Y_i, K_i) \mid 1 \leq i \leq N, m_i > 0\} = r_0(Y_N, K_N) < \infty. \tag{117}$$

To obtain the last part of the proposition, note that, if we have a linear combination $\sum_{i=0}^N m_i[(Y_i, K_i)]$ which is zero in $\Theta^{3,1}_{\mathbb{Z}}$, we may assume that $m_N > 0$ without loss of generality, which is a contradiction by the first part of the proposition. □

The following is a relation between r_0 and $\tilde{s}$:

Theorem 5.56 *For a knot K in S^3 with $\tilde{s}(K) \neq 0$, we have $\min\{r_0(K), r_0(K^*)\} < \infty$.*

Proof By construction, $\tilde{s}(K)$ depends only on the weak local equivalence class of the enriched $\mathcal{S}$-complex $\mathfrak{E}(Y, K)$. Thus if $\tilde{s}(K)$ is nonzero, $\mathfrak{E}(Y, K)$ is not weakly locally equivalent to the trivial enriched $\mathcal{S}$-complex. By Corollary 5.43, we obtain that one of $r_0(K)$ or $r_0(K^*)$ is finite. □

We next make some remarks on the relationship between invariants for unknots and the 3-manifold invariants of [Daemi 2020; Nozaki et al. 2024]. First we give another expression for $r_s(Y, K)$. Assume for simplicity that no perturbations are required, and thus the enriched $\mathcal{S}$-complex $\mathfrak{E}(Y, K)$ is an I-graded $\mathcal{S}$-complex $\widetilde{C} = \widetilde{C}(Y, K; \Delta_R)$. Then we compute

$$\begin{aligned} r_s(Y, K) &= -\min\big\{\inf\{\deg_I(d\alpha - \delta_2(1)) \mid \alpha \in C_{-1}, \deg_I(\alpha) \leq -s\}, 0\big\} \\ &= \max\big\{\sup\{\deg_I(f) \mid f \in C_1^\dagger, \deg_I(\alpha) \leq -s, \alpha \in C_{-1}, (d^\dagger f)(\alpha) - \delta_1^\dagger(f) \in R \setminus 0\}, 0\big\} \\ &= \inf\{r \in (0, \infty] \mid [\delta_1^\dagger] \neq 0 \in H(C^{\leq -s}/C^{\leq r})\}, \end{aligned} \tag{118}$$

where $C^{\leq r}$ denotes the R-chain complex consisting of elements with $\deg_I \leq r$. The invariant $r_s^R(Y)$ of [Nozaki et al. 2024] is defined similarly as in the last line, but using the complex $C_*(Y;R)$ of the integer homology sphere Y, with Chern–Simons filtration, and the map $D_1\colon C_*(Y;R) \to R$ (called θ_Y in [loc. cit.]) in place of δ_1. (There are also some convention differences, reflected in the discussion below; see also [ibid., Section 4.1].) The chain complex $C_*(Y;R)$ is defined with respect to the coefficient ring R, forgetting the $\mathbb{Z}[T^{\pm 1}]$-algebra structure.

There is a chain map $\phi\colon C_*(Y,U_1;\Delta_R) \to C_*(Y;R)$ and a map $\mathfrak{D}\colon C_1(Y,U_1;\Delta_R) \to R$ satisfying

$$\mathfrak{D}d + \delta_1 + D_1\phi = 0. \tag{119}$$

For details see [Daemi and Scaduto 2024a, Proposition 5.3]. The map ϕ preserves the Chern–Simons filtrations, and we obtain, much like in the argument that proves Theorem 5.47, using the dual version of (119), an inequality

$$r_s(Y,U_1) \leq 2r_s^R(Y). \tag{120}$$

The factor of 2 that appears is because of the difference of conventions for the Chern–Simons functional. A similar argument using (119) and also the dual version yields the inequalities

$$\Gamma_Y(1) \leq \Gamma_{(Y,U_1)}(1), \tag{121}$$

$$\Gamma_Y(0) \geq \Gamma_{(Y,U_1)}(0), \tag{122}$$

where Γ_Y is as defined in [Daemi 2020] over $\mathbb{Q}$, and $\Gamma_{(Y,U_1)}$ is defined with the coefficient ring R being any integral domain algebra over $\mathbb{Q}[T^{\pm 1}]$.

In fact, the inequalities (120)–(122) are equalities. A proof follows along the same lines as the one sketched in [Daemi and Scaduto 2024a, Section 5.3], where it is explained that $C_*(Y,U_1;R)$ is chain homotopy equivalent to the $\mathbb{Z}/4$-graded chain complex $(C_*(Y;R),d_Y) \oplus (C_{*-2}(Y;R),d_Y)$ and that, under this equivalence, δ_1 corresponds to $D_1 \oplus 0$. The equivalence, which is roughly induced by certain cobordism maps, is filtration-preserving (adjusting for differences in convention). Given this explicit identification, equality in (120)–(122) follows. As we do not use this result elsewhere in the paper, we omit the details.

We have the following result regarding homology cobordisms of integer homology 3-spheres:

Corollary 5.57 *If $r_0(Y,U_1)$ is finite, then, for any homology cobordism W from Y to itself, there is an irreducible $\mathrm{SU}(2)$-representation of $\pi_1(W)$.*

Proof Let $W\colon Y \to Y$ be a homology cobordism. Choose a submanifold of W, diffeomorphic to $[0,1] \times D^3$, which is a regular neighborhood of a path from the incoming copy of Y to the outgoing copy of Y. We obtain a cobordism of pairs $(W,S)\colon (Y,U_1) \to (Y,U_1)$, where S is an unknotted annulus inserted in the submanifold identified with $[0,1] \times D^3$. By the Seifert–Van Kampen theorem, we have

$$\pi_1(W \setminus S) \cong \pi_1(W) * \mathbb{Z},$$

where the copy of $\mathbb{Z}$ is generated by any circle fiber of the normal bundle of S. Now our assumption on $r_0(W,S)$, together with Corollary 5.54, implies that there is an irreducible traceless representation $\rho\colon \pi_1(W) * \mathbb{Z} \to \mathrm{SU}(2)$. The traceless condition means that, after possibly conjugating, $\rho(1) = \boldsymbol{i} = \mathrm{diag}(i,-i) \in \mathrm{SU}(2)$, where $1 \in \mathbb{Z}$. Consider the representation $\rho' := \rho \circ \iota$, where $\iota\colon \pi_1(W) \to \pi_1(W) * \mathbb{Z}$ is the inclusion. If ρ' is reducible, then, because $\mathrm{Hom}(\pi_1(W),\mathbb{Z}) = H^1(W;\mathbb{Z}) = 0$, it is trivial. But then ρ has image $\boldsymbol{i}$ and is reducible, a contradiction. Thus ρ' is the desired irreducible representation. □

Remark 5.58 An alternative proof of Corollary 5.57 may be obtained once equality in (120) is established. For then finiteness of $r_0(Y,U_1)$ implies finiteness of $r_0^R(Y)$, and the result then follows from [Nozaki et al. 2024, Theorem 1.1(1)].

We end this section with some results involving the invariant $s^\sharp(Y,K)$. In what follows, for an integer homology 3-sphere Y, we write $h(Y) \in \mathbb{Z}$ for Frøyshov's instanton invariant [2002], which is constructed in the setting of SO(3)-equivariant instanton Floer theory for Y with rational coefficients. We study the relationship between $\tilde{s}(Y,K)$ and $h(Y)$. For the remainder of this section, $R = \mathbb{Q}[\![\Lambda]\!]$.

Proposition 5.59 *If K is a knot in an integer homology 3-sphere Y satisfying $h(Y) = h(Y_1(K))$, then*

$$\tilde{s}(Y,K) \le \tilde{s}(Y_1(K),U_1).$$

Proof Let $i := -\frac{1}{2}\sigma(Y,K)$. First assume $i \ge 0$. Consider the cobordism $(W,S)\colon (Y_1(K),U_1) \to (Y,K)$ from the proof of Corollary 5.52. There, it was shown that (W,S) is strong and negative definite over R of height i. Then

$$i = -\tfrac{1}{2}\sigma(Y,K) = h(Y,K) - 4h(Y) = h(Y,K) - 4h(Y_1(K)) = h(Y,K) - h(Y_1(K),U_1),$$

where we have used Theorem 2.9 in the second and fourth equalities, and the assumption $h(Y) = h(Y_1(K))$ in the third. The result now follows from an application of Proposition 4.11. If $i \le 0$, using the trick employed in Section 4.4, we form a cobordism $(W',S')\colon \big(Y_1(K), \#_{-i}\, T^*_{2,3}\big) \to (Y,K)$ by splicing on $-i$ copies of the blown-up version of the cobordism from Lemma 4.39 onto (W,S). Then (W',S') has height 0 with $c_0 = \eta(W',S')$ equal to Λ^{-i} up to a unit. An application of Proposition 4.2 gives the result. □

Proposition 5.60 *If $h(Y) = 0$, then $\tilde{s}(Y,U_1) = 0$.*

Proof Consider the map $\psi\colon C(Y;R) \to C(Y,U_1;\Delta_R)$ which is the dual of $\phi\colon C(-Y,-U_1;\Delta_R) \to C(-Y;R)$ from (119). Then (119) gives a dual relation of the form

$$d\mathfrak{D}' + \delta_2 + \psi D_2 = 0.$$

Under the standing assumption that $R = \mathbb{Q}[\![\Lambda]\!]$, we have $C(Y;R) = C(Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[\![\Lambda]\!]$. Further, $h(Y) = 0$ implies that $D_2(1) = d_Y\alpha$ for some $\alpha \in C(Y;\mathbb{Q})$. Now

$$\delta_2(1) = -\psi D_2(1) - d\mathfrak{D}'(1) = -\psi d_Y(\alpha) - d\mathfrak{D}'(1) = d\beta,$$

where $\beta = -\psi(\alpha) - \mathfrak{D}'(1)$. Then $(0, \beta, 1) \in \widehat{\boldsymbol{C}}$ defines a special $(0,1)$-cycle for $\widetilde{C}(Y, U_1; \Delta_R)$. From the definition of $\tilde{s}$ we obtain $\tilde{s}(Y, U_1) \leq 0$. Applying the same argument to the orientation reversal yields the inequality $\tilde{s}(-Y, -U_1) \leq 0$. We then have

$$0 \leq -\tilde{s}(-Y, -U_1) \leq \tilde{s}(Y, U_1)$$

by (52). □

The following result implies Theorem 1.5:

Proposition 5.61 *For a knot K in Y with $h(Y) = 0$, if $\tilde{s}(Y, K) > 0$, then $h(Y_1(K)) < 0$.*

Proof Note that we always have $h(Y_1(K)) \leq h(Y) = 0$; see [Frøyshov 2002]. Assume $h(Y_1(K)) = 0$. Then

$$0 = \tilde{s}(Y_1(K), U_1) \geq \tilde{s}(Y, K)$$

from Propositions 5.59 and 5.60. This contradicts $\tilde{s}(Y, K) > 0$. □

Remark 5.62 Following the outline given in [Daemi and Scaduto 2024a, Section 5.3], we expect that $\widetilde{C}(Y, U_1; \Delta_{\mathbb{Q}[\![\Lambda]\!]})$ is homotopy equivalent to $\widetilde{C} \otimes_{\mathbb{Q}} \mathbb{Q}[\![\Lambda]\!]$, where $\widetilde{C}$ is an $\mathcal{S}$-complex over $\mathbb{Q}$. This would imply $\tilde{s}(Y, U_1) = 0$ for any integer homology 3-sphere Y. We would then obtain that $\tilde{s}(Y, K) > 0$ implies $h(Y_1(K)) < h(Y)$.

6 Applications

We now use the invariants introduced in this paper to prove various topological applications. In the first section below, we provide slice genus bounds for the invariants $s^\sharp_\pm(K)$. We then turn to prove the applications described in the introduction involving knot concordance, homology cobordism, and nonabelian traceless SU(2)-representations on concordance complements.

6.1 Genus bounds from $s^\sharp_\pm$

Kronheimer and Mrowka [2013] prove that $s^\sharp(K)$ gives a lower bound for twice the slice genus of K. From the definitions given by Gong [2021], one obtains slice genus bounds for $s^\sharp_\pm(K)$ as well. Using our description of these invariants, we improve these bounds:

Theorem 6.1 *For any knot K, we have the following genus bounds*:

(i) $|s^\sharp_+(K)|, |s^\sharp_-(K) + 1| \leq g_4(K)$ *hold if* $\sigma(K) < 0$.

(ii) $|s^\sharp_\pm(K)| \leq g_4(K)$ *hold if* $\sigma(K) = 0$.

(iii) $|s^\sharp_+(K) - 1|, |s^\sharp_-(K)| \leq g_4(K)$ *hold if* $\sigma(K) > 0$.

Before proving this theorem, we observe that the above genus bounds can be used to improve a result from [Gong 2021] about the values of $s^\sharp_\pm$ for quasipositive knots:

Corollary 6.2 *Let K be a quasipositive knot. If $\sigma(K) \le 0$, then $s^\sharp_+(K) = g_4(K)$. If $\sigma(K) < 0$, then $s^\sharp_-(K) = g_4(K) - 1$.*

This corollary determines $s^\sharp_\pm$ for any algebraic knot, or more generally any positive knot, because any such nontrivial knot has negative signature [Przytycki 1989].

Proof Gong [2021] shows that, for any quasipositive knot K, if $g_4(K)$ is even, then

$$s^\sharp_+(K) = g_4(K), \quad g_4(K) - 1 \le s^\sharp_-(K) \le g_4(K),$$

and, if $g_4(K)$ is odd, then

$$g_4(K) \le s^\sharp_+(K) \le g_4(K) + 1, \quad s^\sharp_-(K) = g_4(K) - 1.$$

Now the claim follows from combining the above bounds with Theorem 6.1. □

In order to prove Theorem 6.1, we use the following lemma:

Lemma 6.3 [Sato 2023, Section 3] *For any knot K, there exist a nonnegative integer k, positive integers m_i, n_i for any $1 \le i \le g_4(K)$, and a cobordism of pairs*

$$(W, S)\colon (S^3, K) \to \big(S^3, \#_{i=1}^{g_4(K)} K^*_{m_i,n_i}\big)$$

such that $K_{m,n}$ is the double twist knot with parameters $m, n \in \mathbb{Z}_{>0}$, W is the connected sum of the product cobordism with k copies of $\overline{\mathbb{C}P}^2$, $g(S) = 0$, and the homology class of S is equal to $e_1 + \cdots + e_l$, where $0 \le l \le k$ and e_i is a generator of H_2 of the i^{th} copy of $\overline{\mathbb{C}P}^2$ in W.

Proof of Theorem 6.1 For the knot K, take the cobordism (W, S) provided by Lemma 6.3. There is a unique minimal reducible on (W, S) of index $2i - 1$ with $i = \frac{1}{2}\sigma(K) - g_4(K)$. In particular, $\eta(W, S) = 1$. Since $\sigma\big(\#_{i=1}^{g_4(K)} K^*_{m_i,n_i}\big) = 2g_4(K) \ge \sigma(K)$, the integer i is nonpositive, and if it is negative then (W, S) is not a negative definite cobordism of pairs with a nonnegative height. To remedy this, take the blown-up version of the cobordism from Lemma 4.39 and view it as a cobordism from the unknot to $T_{2,3}$. Then splice $-i$ copies of this cobordism along (W, S) to obtain a cobordism

$$(W', S')\colon (S^3, K) \to \big(S^3, \#_{i=1}^{g_4(K)} K^*_{m_i,n_i} \# \#_{-i}\, T_{2,3}\big).$$

It is straightforward to check that (W', S') is a negative definite cobordism of pairs with vanishing height and $\eta(W', S') = (T^2 - T^{-2})^{-i}$ up to a unit. Thus, from Propositions 2.8 and 4.11, we have

$$s^\sharp_\pm\big(\#_{i=1}^{g_4(K)} K^*_{m_i,n_i} \# \#_{-i}\, T_{2,3}\big) \le s^\sharp_\pm(K) - i. \tag{123}$$

Proposition 4.50 implies that the $\mathcal{S}$-complex of $\#_{i=1}^{g_4(K)} K^*_{m_i,n_i} \# \#_{-i}\, T_{2,3}$ is locally equivalent to the connected sum of $-g_4(K) - i = -\frac{1}{2}\sigma(K)$ copies of $T_{2,3}$ over $\mathbb{Q}[\![\Lambda]\!]$. Therefore,

$$s^\sharp_\pm\big(\#_{i=1}^{g_4(K)} K^*_{m_i,n_i} \# \#_{-i}\, T^*_{2,3}\big) = s^\sharp_\pm\big(\#_{-\sigma(K)/2}\, T_{2,3}\big).$$

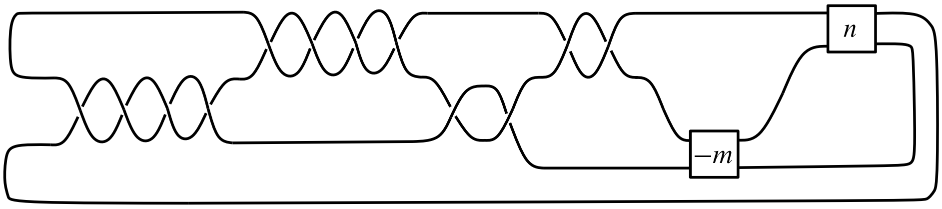


Figure 3: Two-bridge knot $K_{m,n} = K(212mn - 68n + 53, 106m - 34)$. In the box labeled "$-m$" there are m full negative twists, and in the box labeled "n" there are n full positive twists.

This together with (123) gives

$$s^{\sharp}_{\pm}\big(\#_{-\sigma(K)/2}\, T_{2,3}\big) + \tfrac{1}{2}\sigma(K) - g_4(K) \le s^{\sharp}_{\pm}(K).$$

By substituting K^* for K, we obtain

$$-s^{\sharp}_{\mp}\big(\#_{-\sigma(K)/2}\, T_{2,3}\big) - \tfrac{1}{2}\sigma(K) - g_4(K) \le -s^{\sharp}_{\mp}(K).$$

Combining the last two inequalities, we obtain the desired claim

$$\big|s^{\sharp}_{\pm}(K) - \tfrac{1}{2}\sigma(K) - s^{\sharp}_{\pm}\big(\#_{-\sigma(K)/2}\, T_{2,3}\big)\big| \le g_4(K). \qquad \square$$

6.2 Applications to knot concordance and H-sliceness

Here we prove the following, which is a restatement of Theorem 1.15:

Theorem 6.4 *Let m be a positive integer and $\{K_{m,n}\}_{n\in\mathbb{Z}_{\ge 0}}$ be the sequence of two-bridge knots defined by $K_{m,n} := K(212mn - 68n + 53, 106m - 34)$. Let K be a knot whose concordance class is of the form*

$$[K] = a_1[K_{m,0}] + a_2[K_{m,1}] + \cdots + a_N[K_{m,N}],$$

where $a_i \in \mathbb{Z}$ and a_N is positive. Then the knot K is not smoothly H-slice in any positive definite closed 4-manifold with $b_1 = 0$.

Note that $K_{m,0} = 10_{28}$ for any $m \in \mathbb{Z}_{>0}$. The diagram of $K_{m,n}$ is depicted in Figure 3. Firstly, we observe that almost all $K_{m,n}$ are algebraically slice. Recall the Tristram–Levine signature function

$$\sigma_\omega(Y, K) := \operatorname{sgn}[(1 - e^{4\pi i\omega})A_K + (1 - e^{-4\pi i\omega})A_K^{\mathsf{T}}]. \tag{124}$$

Here A_K is a Seifert matrix for the knot $K \subset Y$, $\omega \in \big(0, \frac{1}{2}\big)$, and $\operatorname{sgn}(B)$ is the signature of the Hermitian matrix B, which is the number of positive eigenvalues minus the number of negative eigenvalues of B.

Lemma 6.5 *We have $\sigma(K_{m,n}) = \sigma_{1/4}(K_{m,n}) = 0$ for $m \ge 1$ and $n \ge 0$. Moreover, if $m \ge 7$, then $\sigma_\omega(K_{m,n}) = 0$ for $\omega \in \big(0, \frac{1}{2}\big)$. Thus $K_{m,n}$ is torsion in the algebraic concordance group if $m \ge 7$, $n \ge 0$.*

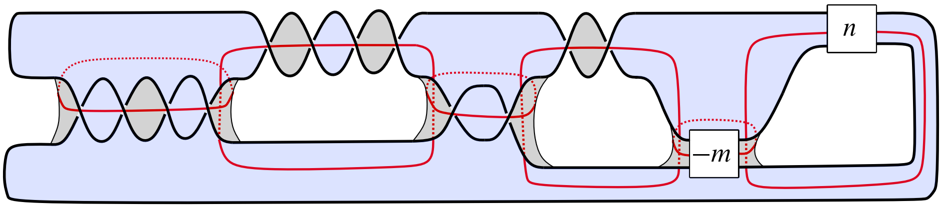


Figure 4: A Seifert surface for $K_{m,n}$ and its symplectic basis.

Proof Take a Seifert surface for the knot $K_{m,n}$, together with a choice of symplectic basis, as shown in Figure 4. Then the corresponding Seifert matrix $A_{m,n}$ is

$$A_{m,n} = \begin{bmatrix} 2 & & & & & \\ -1 & 2 & & & & \\ & -1 & -1 & & & \\ & & -1 & -1 & & \\ & & & -1 & -m & \\ & & & & -1 & n \end{bmatrix}.$$

Now $\sigma_\omega(K_{m,n})$ is the signature of the matrix $A_{m,n,\omega} := (1-e^{4\pi i\omega})A_{m,n} + (1-e^{-4\pi i\omega})A_{m,n}^{\mathsf{T}}$. We first show that the number of positive and negative eigenvalues of this Hermitian matrix does not change in the case that $\omega = \frac{1}{4}$ and $m \geq 1$, $n \geq 0$, and also in the case that ω is arbitrary and $m \geq 7$, $n \geq 0$.

To this end, a direct computation gives the following expression for the determinant:

$$\begin{aligned} f(m,n,\omega) &:= \det(A_{m,n,\omega}) \\ &= 64\sin^6(2\pi\omega)\big(8mn\cos(12\pi\omega) + \cos(4\pi\omega)((98m-34)n+26) \\ &\qquad + \cos(8\pi\omega)(-42mn+8n-8) - 64mn + 26n - 19\big). \end{aligned}$$

Now we consider m, n as continuous variables. Setting $x = \cos(4\pi\omega)$, we compute

$$\begin{aligned} \frac{df}{dm} &= -256n\sin^8(2\pi\omega)p(x), \quad & p(x) &= 16x^2 - 26x + 11, \\ \frac{df}{dn} &= -256\sin^8(2\pi\omega)q(x), \quad & q(x) &= 16mx^2 - 26mx + 11m + 8x - 9. \end{aligned}$$

The quadratic $p(x)$ is always positive, and thus $df/dm \leq 0$. On the other hand, $q(x)$ is always positive when $m \geq 7$, and in this case we thus have $df/dn < 0$. In summary, the function $f(m,n,\omega)$ is nonincreasing in n, m when $n \geq 0$ and $m \geq 7$. We also have, with x as before,

$$f(m,0,\omega) = -64\sin^6(2\pi\omega)p(x) < 0.$$

We conclude that $f(m,n,\omega)$ is negative for all $m \geq 7$, $n \geq 0$, $\omega \in \big(0, \frac{1}{2}\big)$. Also

$$f\big(m,n,\tfrac{1}{4}\big) = -64(53 + 4(-17+53m)n),$$

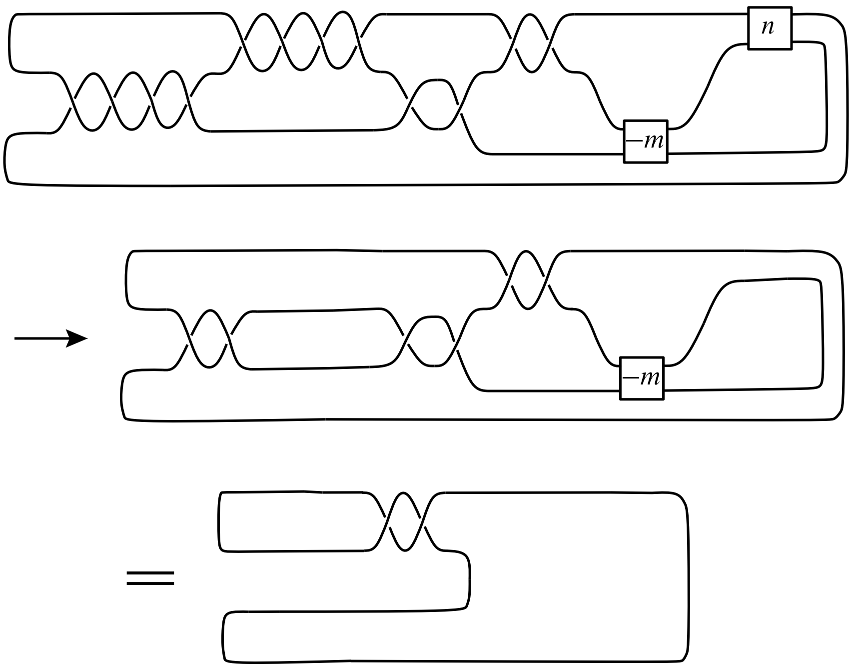


Figure 5: Changing $n+3$ negative crossings in the diagram for $K_{m,n}$ produces an unknot.

which is negative for all $m \geq 1$, $n \geq 0$. Thus the number of positive and negative eigenvalues of $A_{m,n,\omega}$ does not change under the stated conditions. Thus $\sigma_\omega(K_{m,n}) = \sigma(K_{1,0})$ for m, n, ω as in the statement of the lemma. The proof is completed by showing that $\sigma(K_{1,0}) = \sigma(10_{28}) = 0$, which is straightforward. □

In order to prove Theorem 6.4, we use the following property of the invariant $r_0(K)$:

Proposition 6.6 *For a knot K in S^3, if the invariant $r_0(K)$ is finite and $\sigma(K) = 0$, then K is not smoothly H-slice in any negative definite closed 4-manifold with $b_1 = 0$.*

Proof If K is H-slice in a negative definite 4-manifold with $b_1 = 0$, then we obtain a negative definite cobordism of pairs $(W, S)\colon (S^3, U_1) \to (S^3, K)$ of height 0 with $\eta(W, S) = 1$ and $\kappa_{\min}(W, S) = 0$. Now applying Corollary 5.54 gives a contradiction. □

Proof of Theorem 6.4 It suffices to verify the hypotheses (i) and (ii) of Proposition 5.55 for the family of knots $\{K^*_{m,n}\}_{n\geq 0}$, where $m > 0$ is fixed. Indeed, the result then follows from Propositions 5.55 and 6.6.

Consider the negative to positive crossing changes from $K_{m,n}$ to the unknot shown in Figure 5. These give rise to an immersed cobordism from the unknot to $K_{m,n}$ with genus 0, no positive double points and $n+3$ negative double points. By blowing up and capping off the unknot, we obtain a null-homologous disk with boundary $K_{m,n}$ in $\left(\#_{n+3}\,\overline{\mathbb{C}P^2}\right) \setminus B^4$. Now Lemma 6.5 and Proposition 6.6 imply

$$r_0(K_{m,n}) = \infty. \tag{125}$$

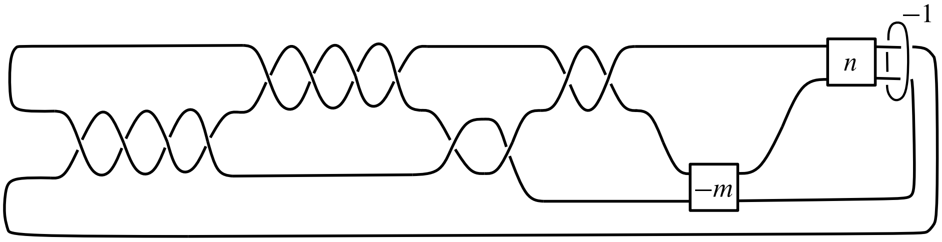


Figure 6: A Kirby diagram for a cobordism $(W, S_{m,n})\colon (S^3, K_{m,n}) \to (S^3, K_{m,n+1})$. The cobordism is obtained by attaching the (-1)-framed 2-handle to $(S^3, K_{m,n})$.

Similar to the previous construction, a positive to negative crossing change from $K_{m,n}$ to $K_{m,n+1}$ induces, after blowing up, a cobordism $(W_{m,n}, S_{m,n})\colon (S^3, K_{m,n}) \to (S^3, K_{m,n+1})$, where $S_{m,n}$ is a null-homologous annulus and $W_{m,n}$ is a twice-punctured $\overline{\mathbb{C}P}^2$. Then Corollary 5.54 gives

$$r_0(K^*_{m,n+1}) \le r_0(K^*_{m,n}). \tag{126}$$

Here we have used that $\sigma(K_{m,n}) = \sigma(K_{m,n+1}) = 0$, which implies that the cobordism $(W_{m,n}, S_{m,n})$ is negative definite over R of strong height 0.

The cobordism $(W_{m,n}, S_{m,n})$ is given by the Kirby diagram in Figure 6. Note that the two-bridge presentation of $K_{m,n}$ implies that $\pi_1(S^3 \setminus K_{m,n})$ is generated by two meridional loops, one for each of the two strands that pass through the (-1)-framed 2-handle. This added 2-handle induces a relation on $\pi_1(W_{m,n} \setminus S_{m,n})$ that equates these two elements. Thus $\pi_1(W_{m,n} \setminus S_{m,n}) \cong \mathbb{Z}$. As such a cobordism does not admit any irreducible SU(2)-representations, Corollary 5.54 implies that (126) is a strict inequality

$$r_0(K^*_{m,n+1}) < r_0(K^*_{m,n}). \tag{127}$$

Now consider the case $n = 0$. Figure 7 describes a positive to negative crossing change from $K_{m,0} = 10_{28}$ to 7^*_4. This gives an immersed cobordism from 10^*_{28} to 7_4 with genus 0 and one positive double point. Therefore, it follows from [Daemi and Scaduto 2024a, Corollary 3.24 and Proposition 4.33] that

$$\tfrac{3}{5} = \Gamma_{7_4}(1) \le \tfrac{1}{2} + \Gamma_{10^*_{28}}(0).$$

This shows $\Gamma_{10^*_{28}}(0) > 0$, and property (100) implies $r_0(K^*_{m,0}) = r_0(10^*_{28}) < \infty$. Together with (125) and (127), this verifies the hypotheses in Proposition 5.55. □

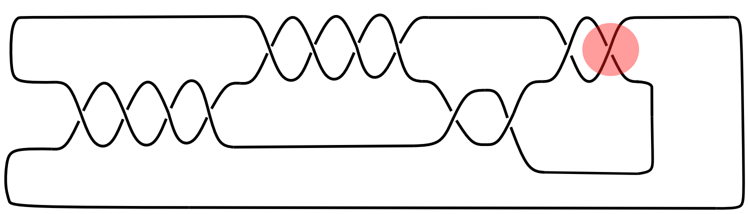


Figure 7: Changing a positive crossing in 10_{28} gives 7^*_4. (The knot 7_4 is depicted in Figure 10.)

6.2.1 The result on satellite operations We next prove Theorem 1.11 and Corollary 1.12, the results on satellite operations described in the introduction. The following implies Theorem 1.11, as we will see below:

Theorem 6.7 *Suppose that a pattern $P \subset S^1 \times D^2$ satisfies:*

(i) *There is some knot K such that $r_0(P(K)) < \infty$ and $r_0(P(K)^*) = \infty$.*

(ii) *$P(U_1)$ is the unknot.*

Then the image of the induced map $P: \mathcal{C} \to \mathcal{C}$ generates an infinite rank subgroup of $\mathcal{C}$.

Suppose that $(W, S): (S^3, K) \to (S^3, K')$ is a cobordism of pairs satisfying $b_1(W) = b^+(W) = 0$, with S a null-homologous annulus. Cut out a regular neighborhood of S and glue in the pair $(S^1 \times D^2 \times I, P \times I)$ along its boundary in the standard way to obtain a cobordism of pairs

$$(W, S_P): (S^3, P(K)) \to (S^3, P(K')), \tag{128}$$

where S_P is the image of $P \times I$ after gluing. Note that S_P is null-homologous in W.

Lemma 6.8 *Suppose $P(U_1)$ is the unknot. If $\pi_1(W \setminus S) \cong \mathbb{Z}$, then $\pi_1(W \setminus S_P) \cong \mathbb{Z}$.*

Proof Write $W \setminus P_S = W_S \cup X_P$, where $W_S = W \setminus N(S)$ is the complement of a regular neighborhood of S in W and $X_P = (S^1 \times D^2 \setminus P) \times I$. Thus $W \setminus S_P$ is the gluing of W_S and X_P along their common boundary $Z \cong T^2 \times I$. By the Seifert–Van Kampen theorem,

$$\pi_1(W \setminus S_P) \cong \pi_1(W_S) *_{\pi_1(Z)} \pi_1(X_P). \tag{129}$$

Note $\pi(Z) \cong \mathbb{Z} \oplus \mathbb{Z}$ and, by assumption, $\pi_1(W_S) \cong \mathbb{Z}$. In particular, the amalgamated product in (129) is unchanged if $\pi_1(W_S)$ is replaced by $\pi_1((S^3 \setminus U_1) \times I) \cong \mathbb{Z}$. Another application of Seifert–Van Kampen says that this latter amalgamated product is isomorphic to $\pi_1\big((S^3 \setminus P(U_1)) \times I\big)$. By our assumption that $P(U_1) = U_1$, this group is isomorphic to $\mathbb{Z}$, and the result follows. □

Lemma 6.9 *Let $K \subset S^3$ be a knot. Then there is a knot $K' \subset S^3$ and a cobordism $(W, S): (S^3, K') \to (S^3, K)$ such that $b_1(W) = b^+(W) = 0$, S is null-homologous in W, and $\pi_1(W \setminus S) \cong \mathbb{Z}$.*

Proof Let K' be obtained from K by changing n negative crossings to positive crossings in some diagram. Then, in the standard manner using blow-ups, there is an associated cobordism $(W, S): (S^3, K') \to (S^3, K)$ such that W is the cylinder $S^3 \times I$ blown up n times. To arrange that $\pi_1(W \setminus S) \cong \mathbb{Z}$, we proceed as follows. Let K have a diagram which includes the crossings that are altered to obtain K'. The group $\pi_1(W \setminus S)$ has presentation given by a Wirtinger presentation for $\pi_1(S^3 \setminus K)$ using this diagram, but with additional relations: for each Wirtinger relation $x_j x_i = x_k x_j$ corresponding to a crossing that is changed, we obtain relations $x_j = x_i$ and $x_j = x_k$. To guarantee $\pi_1(W \setminus S) \cong \mathbb{Z}$, one must simply have enough crossing changes that these additional relations identify all the Wirtinger generators.

If K is given as the closure of a braid β with p strands, K' can be chosen explicitly as follows. This is similar to what is done in the proof of [Nozaki et al. 2024, Theorem 5.17]. Let

$$\Delta_H = (\sigma_1\sigma_2\cdots\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2})\cdots(\sigma_1\sigma_2)\sigma_1,$$

where σ_i are standard generators of the braid group on p strands. Then let K' be the closure of $\Delta_H^2\beta$. Note that K' is obtained from K by viewing K as the closure of $\Delta_H^{-1}\Delta_H\beta$ and changing all of the negative crossings in Δ_H^{-1} to positive crossings. As $\pi_1(S^3\setminus K)$ is generated by a collection of p meridians around the braid strands at the top or bottom of the braid closure, and the crossings of Δ_H relate all of these strands, these p generators are identified in $\pi_1(W\setminus S)$, as desired. □

In what follows, we use the signature formula for satellites due to Litherland [1979]. With our conventions, this reads as follows. For a pattern P with winding number p, and a knot K in S^3, we have

$$\sigma_\omega(P(K)) = \sigma_{p\omega}(K) + \sigma_\omega(P(U_1)), \tag{130}$$

where $p\omega$ is taken modulo $\frac{1}{2}\mathbb{Z}$ to lie in $\left[0,\frac{1}{2}\right)$. Here we assume $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of the satellite knot $P(K)$.

Proof of Theorem 6.7 The result is known for nonzero winding number patterns by [Hedden and Pinzón-Caicedo 2021, Proposition 8]. Thus we assume the winding number of P is zero. By assumption, we have a knot $K_1 := K$ that satisfies $r_0(P(K_1)) < \infty$ and $r_0(P(K_1)^*) = \infty$. Let $K_2 := K'$ be a knot as given by Lemma 6.9. Then, since $b_1(W) = b^+(W) = 0$, S is null-homologous and $\pi_1(W\setminus S)\cong\mathbb{Z}$, we may form the associated cobordism as in (128) to obtain $(W,S_P)\colon (S^3,P(K_2))\to(S^3,P(K_1))$. By (130) with $\omega=\frac{1}{4}$, and the assumption that P has winding number zero, we have

$$\sigma(P(K_1)) = \sigma(P(U_1)) = 0,$$

and similarly $\sigma(P(K_2)) = 0$. Thus (W,S_P) is a cobordism which is negative definite of strong height 0. Then Corollary 5.54 and the discussion following it applied to the cobordism (W,S_P) gives us

$$r_0(P(K_2)) < r_0(P(K_1)).$$

That this inequality is strict uses $\pi_1(W\setminus S_P)\cong\mathbb{Z}$, which follows from Lemma 6.8, $\pi_1(W\setminus S)\cong\mathbb{Z}$ and our assumption $P(U_1) = U_1$. Similarly, viewing (W,S) as a cobordism $(S^3,P(K_1)^*)\to(S^3,P(K_2)^*)$, Corollary 5.54 yields $\infty = r_0(P(K_1)^*) = r_0(P(K_2)^*)$. We continue in this fashion, inductively defining K_i from K_{i-1} using Lemma 6.9. Then $\{K_i\}_{i=1}^{\infty}$ satisfies the hypotheses of Proposition 5.55, and thus gives a linearly independent set in the concordance group. □

Proof of Theorem 1.11 Let K be a knot that can be unknotted by a sequence a positive to negative crossing changes. As in the proof of Theorem 6.7, we may assume the winding number of P is zero, so $\sigma(P(K)) = \sigma(P(U_1)) = 0$.

We obtain from the crossing changes of K a cobordism $(W,S)\colon (S^3,K)\to(S^3,U_1)$ with $b_1(W)=b^+(W)=0$ such that S is a null-homologous annulus. Applying the construction of (128), we obtain $(W,S_P)\colon(S^3,P(K))\to(S^3,P(U_1))$. As $P(U_1)=U_1$, we can cap this off, and conclude that $P(K)^*$ is H-slice in a negative definite 4-manifold with $b_1=0$. Then Proposition 6.6 implies $r_0(P(K)^*)=\infty$.

Next, since $P(K)$ is by assumption quasipositive and nonslice, Corollary 1.2 implies $\tilde{s}(P(K))=g_4(P(K))>0$. In particular, the enriched $\mathcal{S}$-complex associated to $P(K)$ is not weakly locally equivalent to the trivial enriched $\mathcal{S}$-complex. By Corollary 5.43, we then have either $r_0(P(K))<\infty$ or $r_0(P(K)^*)<\infty$. Having shown $r_0(P(K)^*)=\infty$ above, it must be that $r_0(P(K))<\infty$. As the pattern P and the knot K satisfy the hypotheses of Theorem 6.7, the result follows. □

Proposition 6.10 *If P satisfies the conditions of Theorem 6.7 with $K=T_{p,q}$ in condition* (i), *then*

$$\{P(T_{p,q+pn})\}_{n=0}^{\infty}$$

is a linearly independent set in the homology concordance group.

Proof This follows from the proof of Lemma 6.9, which gives the construction for the sequence of knots in the proof of Theorem 6.7, and the following observation: if $T_{p,q+np}$ is obtained in a standard way by taking the closure of a p-strand braid β, then $T_{p,q+(n+1)p}$ is the braid closure of $\Delta_H^2\beta$. □

Proof of Corollary 1.12 This follows from Proposition 6.10 with $P=\mathrm{Wh}^r$. □

We next show that the patterns of Figure 1 satisfy the hypotheses of Theorem 1.11. In what follows, we allow m or n to be zero, in which case the corresponding sequences are empty.

Proposition 6.11 *Let P be a pattern as in Figure 1 for $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^n$ sequences of negative integers with $m\geq n-1$ and $\max\{m,n\}>0$. Then the image of the induced concordance map $P\colon\mathcal{C}\to\mathcal{C}$ generates an infinite rank subgroup of $\mathcal{C}$.*

Write $A:=\{a_i\}_{i=1}^m$ and $B:=\{b_i\}_{i=1}^n$. Note that when $A=\varnothing$ and $B=\{-1\}$, the pattern P is the Whitehead double. When $A=\{-1,\dots,-1\}$ and $B=\varnothing$, we obtain the $(m+1,1)$-cable. The case of $A=\{-k-1\}$ and $B=\{-1\}$ is Yasui's pattern $P_{0,k}$ [2015, Figure 10].

Lemma 6.12 *A pattern P as in Proposition 6.11 preserves strong quasipositivity, in the following sense: if K is a nontrivial strongly quasipositive knot, then so too is $P(K)$.*

It is proved in [Rudolph 1993, Section 2] that the Whitehead double of strongly quasipositive knot is also strongly quasipositive. Lemma 6.12 gives a generalization of this fact. We follow the method in [loc. cit.] to prove Lemma 6.12. In what follows we write $g(K)$ for the Seifert genus of a knot, and use that quasipositive Seifert surfaces realize $g(K)$; see [loc. cit.].

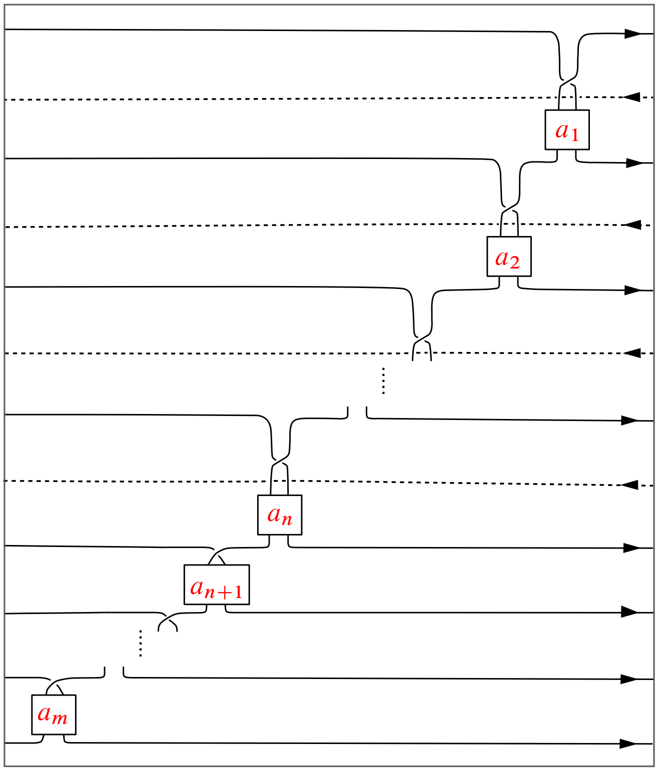


Figure 8: The pattern P' (omit horizontal dotted lines) and P'' (include horizontal dotted lines).

Proof We first show that the pattern P' shown in Figure 8, the case where $B = \varnothing$, preserves strong quasipositivity in this sense. Let S be a quasipositive Seifert surface for a nontrivial knot K. Then the union of $m+1$ parallel copies of S is also quasipositive, which we denote by $\widetilde{S}$. Consider a surface $\overline{S}$ with $m+1$ annuli banded together as in Figure 9. Recall that $P'(K)$ is formed by cutting out from (S^3, K) a regular neighborhood pair $(S^2 \times D^2, S^1)$ of the knot K, and gluing in the pair $(S^1 \times D^2, P')$.

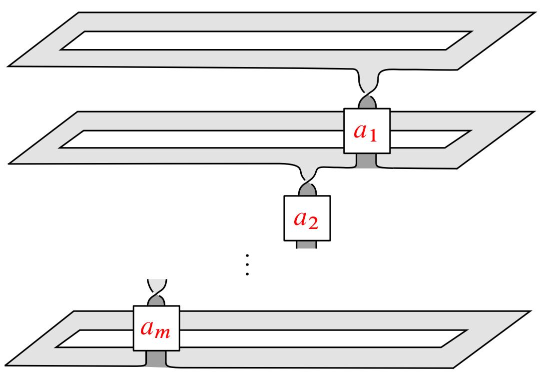


Figure 9: The surface $\overline{S}$.

Upon forming $P'(K)$, the surface $\overline{S}$ is transferred to $S^1 \times D^2$ in such a way that the interior boundaries of the annuli in $\overline{S}$ glue to the parallel copies of S in $\widetilde{S}$. The resulting surface S' is a quasipositive Seifert surface for $P'(K)$, and thus P' preserves quasipositivity. Note that the Euler characteristic of S' is

$$\chi(S') = (m+1)\chi(S) - m.$$

We next use Rudolph's result [1992] that any full subsurface of a quasipositive surface is also quasipositive. (Recall that a *full subsurface* is a subsurface whose inclusion induces an injection on fundamental groups.) Let $\widetilde{S}'$ be the union of n parallel copies of a regular neighborhood of ∂S in S and $m-n+1$ parallel copies of S. Then $\widetilde{S}'$ is regarded as a subsurface of $\widetilde{S}$. Fitting this into the above construction, where we glue $\widetilde{S}'$ and $\overline{S}$, we obtain a corresponding subsurface S'' of S'. Since $g(S) > 0$, the boundary of S is not null-homotopic in S, and hence S'' is a full subsurface of S'. Consequently, S'' is a quasipositive Seifert surface for $P''(K)$ with Euler characteristic

$$\chi(S'') = (m-n+1)\chi(S) - m,$$

where P'' is the pattern depicted in Figure 8, with dotted lines included.

Next, we consider plumbings of S'' with twisted annuli B_i $(1 \le i \le n)$, each of which has b_i full twists, such that the resulting surface S''' has boundary $P(K)$. Then, by [Rudolph 1998, Theorem], the quasipositivity of S'' and B_i implies that S''' is quasipositive. Moreover,

$$\chi(S''') = \chi(S'') - n = (m-n+1)\chi(S) - m - n.$$

This implies $g(P(K)) = g(S''') = (m-n+1)g(K) + n$. Under our assumptions on m and n, we obtain $g(P(K)) > 0$. In particular, $P(K)$ is nontrivial. □

Proof of Proposition 6.11 Let K be any nontrivial positive knot, such as a positive torus knot $T_{p,q}$. Then, by Lemma 6.12, $P(K)$ is a nontrivial strongly quasipositive knot, and hence P satisfies condition (i) of Theorem 1.11. Condition (ii) — that $P(U_1) = U_1$ — is straightforward. □

The property of preserving strongly quasipositive knots is closed under composition of patterns, and so too is the property $P(U_1) = U_1$. Thus, if $P_1, \dots, P_l$ are patterns as in Proposition 6.11, then $P_l \circ \cdots \circ P_1$ also satisfies Theorem 1.11. This remark, together with Proposition 6.10, gives the following:

Corollary 6.13 *Let $P_1, \dots, P_l$ be patterns as in Proposition 6.11, and $P := P_l \circ \cdots \circ P_1$. Then*

$$\{P(T_{p,q+pn})\}_{n=0}^{\infty}$$

is a linearly independent set in the homology concordance group.

6.3 Applications to the homology cobordism group

We now give several applications to the homology cobordism group of homology 3-spheres. We begin by proving Theorem 1.14. To this end, we have:

Theorem 6.14 *For a knot K in S^3 with $\tilde{s}(K) > 0$, the set $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ is linearly independent in the homology cobordism group.*

Proof Proposition 5.61 implies $h(S_1(K)) < 0$. Now [Nozaki et al. 2024, Theorem 1.8] completes the proof. □

Proof of Theorem 1.14 By Theorem 6.14, we only need to check $\tilde{s}(K) > 0$ for knots listed in (i) and (ii), which follows from Theorem 4.24 and Corollary 4.25. □

The following is a restatement of Theorem 1.20:

Theorem 6.15 *Let K be a knot in an integer homology 3-sphere Y satisfying $\sigma(Y, K) \le 0$. Suppose $\frac{1}{8} < \Gamma_{(Y,K)}\big(-\frac{1}{2}\sigma(Y, K)\big)$. Then $\Gamma_{Y_1(K)}(0) > 0$ and $r_0(Y_1(K)) < \infty$.*

Proof By Corollary 5.53 with $k = 0$, we have

$$0 < \Gamma_{(Y,K)}\big(-\tfrac{1}{2}\sigma(Y, K)\big) - \tfrac{1}{8} \le \Gamma_{(Y_1(K),U_1)}(0).$$

Using (122), we conclude that $\Gamma_{Y_1(K)}(0) > 0$. The second claim follows from the analogue of (100) for integer homology spheres. □

The following is a useful criterion for determining when the surgeries of a knot determine a linearly independent set in homology cobordism. It will be used to prove Theorem 1.13.

Theorem 6.16 *Let K be a knot in S^3 satisfying $\sigma(K) \le 0$. Suppose $\frac{1}{8} < \Gamma_K\big(-\frac{1}{2}\sigma(K)\big)$. Then the set $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ is linearly independent in the homology cobordism group.*

Proof By Theorem 6.15, we have $r_0(S_1(K)) < \infty$. Then the proof of [Nozaki et al. 2024, Theorem 5.12] implies

$$\infty > r_0(S_1(K)) > r_0(S_{1/2}(K)) > \cdots \quad \text{and} \quad \infty = r_0(-S_1(K)) = r_0(-S_{1/2}(K)) = \cdots.$$

The proof now follows from [ibid., Corollary 5.6], the 3-manifold analogue of Proposition 5.55. □

The following result, with Lemma 6.5, implies Theorem 1.13:

Theorem 6.17 *For any of the two-bridge knots $K_{m,n}$ defined in (6) with $m \ge 1$ and $n \ge 0$, the set $\{S^3_{1/k}(K_{m,n})\}_{k=1}^{\infty}$ is linearly independent in the homology cobordism group.*

Proof For $K_{m,0} = 10_{28}$ (where m is arbitrary), we show below in Corollary 6.22 that

$$\Gamma_{10^*_{28}}(0) = \tfrac{8}{53} > \tfrac{1}{8}.$$

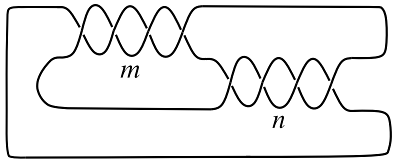


Figure 10: The double twist knot $D_{m,n}$ is a two-bridge knot involving a strand of m full twists and one of n full twists. The particular example shown is $D_{2,2}$, which is the knot 7_4 in Rolfsen notation.

Next, suppose $m \geq 1$ and $n \geq 0$. Recall the crossing change cobordism $(S^3, K^*_{m,n+1}) \to (S^3, K^*_{m,n})$ from the proof of Theorem 6.4, which is negative definite of strong height zero (and level zero), since $\sigma(K^*_{m,n}) = (K^*_{m,n+1}) = 0$. From Theorem 5.47 (see also [Daemi and Scaduto 2024a, Proposition 4.33]), we obtain

$$\Gamma_{K^*_{m,n}}(0) \leq \Gamma_{K^*_{m,n+1}}(0).$$

We inductively obtain $\Gamma_{K^*_{m,n+1}}(0) > \frac{1}{8}$. The result now follows from Theorem 6.16. □

We provide some more examples using Theorem 6.16, where the knot is a two-bridge knot. Note that for a two-bridge knot K with nonzero signature, Theorem 1.14 provides a linearly independent set of the surgeries on K. For this reason we focus on examples with zero signature. We will use the following:

Lemma 6.18 *For $m, n \geq 2$ with $\max\{m, n\} \geq 3$, let $D_{m,n}$ be a double twist knot as described in Figure 10. Suppose K is a knot with $\sigma(K) = 0$ obtained from $D_{m,n}$ by changing a positive crossing to a negative crossing. Then $\{S^3_{1/k}(K)\}_{k=1}^{\infty}$ is linearly independent in the homology cobordism group.*

Proof Since a single negative crossing change gives rise to a negative definite cobordism of height 1 and $\kappa_{\min} = \frac{1}{4}$, Proposition 4.33 of [Daemi and Scaduto 2024a] implies

$$\Gamma_{D_{m,n}}(1) - \tfrac{1}{2} \leq \Gamma_K(0).$$

On the other hand, Corollary 3.24 of [Daemi and Scaduto 2024a] computes

$$\Gamma_{D_{m,n}}(1) = \frac{(2m-1)(2n-1)}{4mn-1}.$$

It is then straightforward to verify, using these two relations, that $\Gamma_K(0) > \frac{1}{8}$ for $m \geq 2$, $n \geq 3$. Now the assertion follows from Theorem 6.16. □

With our conventions, the double twist knot $D_{m,n}$ has the two-bridge knot description

$$D_{m,n} = K(4mn-1, -2m).$$

Given an integer $l \geq 1$ we consider the two-bridge knot

$$D_{l,m,n} := K(16lmn - 4l - 4mn + 4m + 1, -8ln + 2n - 2).$$

This has Hirzebruch–Jung continued fraction $(-2m, -2n, 2l, 2)$, and, by changing a negative crossing in the twist corresponding to "2", we obtain $D_{m,n}$. The mirror of the case of $m = n = 2$ and $l = 1$ is depicted in Figure 7. Indeed, in this case, $D_{1,2,2} = K(53, -14) = 10^*_{28}$.

Theorem 6.19 *For integers $l \geq 1$, $m, n \geq 2$, with $\max\{m, n\} \geq 3$, the two-bridge knot $D_{l,m,n}$ has $\sigma(D_{l,m,n}) = 0$, and $\{S^3_{1/k}(D_{l,m,n})\}_{k=1}^{\infty}$ is linearly independent in the homology cobordism group.*

Proof By the above description of $D_{l,m,n}$ and Lemma 6.18, it suffices to show $\sigma(D_{l,m,n}) = 0$. Similar to the setup in the proof of Lemma 6.5, $\sigma(D_{l,m,n})$ is given by the signature of the matrix

$$\begin{bmatrix} -2m & -1 & & \\ -1 & -2n & -1 & \\ & -1 & 2l & -1 \\ & & -1 & 2 \end{bmatrix},$$

which, given that l, m, n are positive, is clearly zero. □

Note that $D_{1,2,2} = 10^*_{28}$ is not included in this statement, although the result still holds (in fact, it is contained in Theorem 6.17). Special cases of Theorem 6.19 include

$$\begin{aligned} D_{1,2,3} &= K(77, 50) = 12a^*_{380}, & D_{1,3,2} &= K(81, 52) = 12a^*_{596}, \\ D_{1,4,2} &= K(109, 70), & D_{1,2,4} &= K(101, 66). \end{aligned} \tag{131}$$

In the next section, a more explicit way of proving these cases is explained. We remark that $D_{l,m,n}$ is not always algebraically slice; for example, $D_{1,2,3}$ has nontrivial Tristram–Levine signature function.

6.4 More two-bridge examples using the ADHM construction

Daemi and Scaduto [2024a] study the $\mathcal{S}$-complex of a two-bridge knot $K(p, q)$ using information derived from an equivariant ADHM correspondence. The data of d, δ_1, δ_2 are completely determined, and constraints are given for which components of the υ-map can possibly be nonzero. All of this information is represented entirely by arithmetic conditions in terms of (p, q). Here, we use this information to compute the local equivalence class of the enriched $\mathcal{S}$-complex for 10_{28}. We also provide some further examples. We refer to [loc. cit.] for the details on the structure of $\mathcal{S}$-complexes used below.

Remark 6.20 The conventions of this paper differ from those in [Daemi and Scaduto 2024b; 2024a]: the two-bridge knot $K(p, q)$ in this paper corresponds to the two-bridge knot $K(p, -q)$ in those references.

For two-bridge knots, one can avoid perturbations and the generality of enriched $\mathcal{S}$-complexes, and work exclusively with I-graded $\mathcal{S}$-complexes. Throughout this section, we work with I-graded $\mathcal{S}$-complexes over $R[U^{\pm 1}]$, where $R = \mathbb{Z}[T^{\pm 1}]$. We begin by giving a simple expression for the local equivalence class of the I-graded $\mathcal{S}$-complex for the two-bridge knot 10^*_{28}. Define an I-graded $\mathcal{S}$-complex

$$\widetilde{C}'_* = C'_* \oplus C'_{*-1} \oplus R[U^{\pm 1}], \tag{132}$$

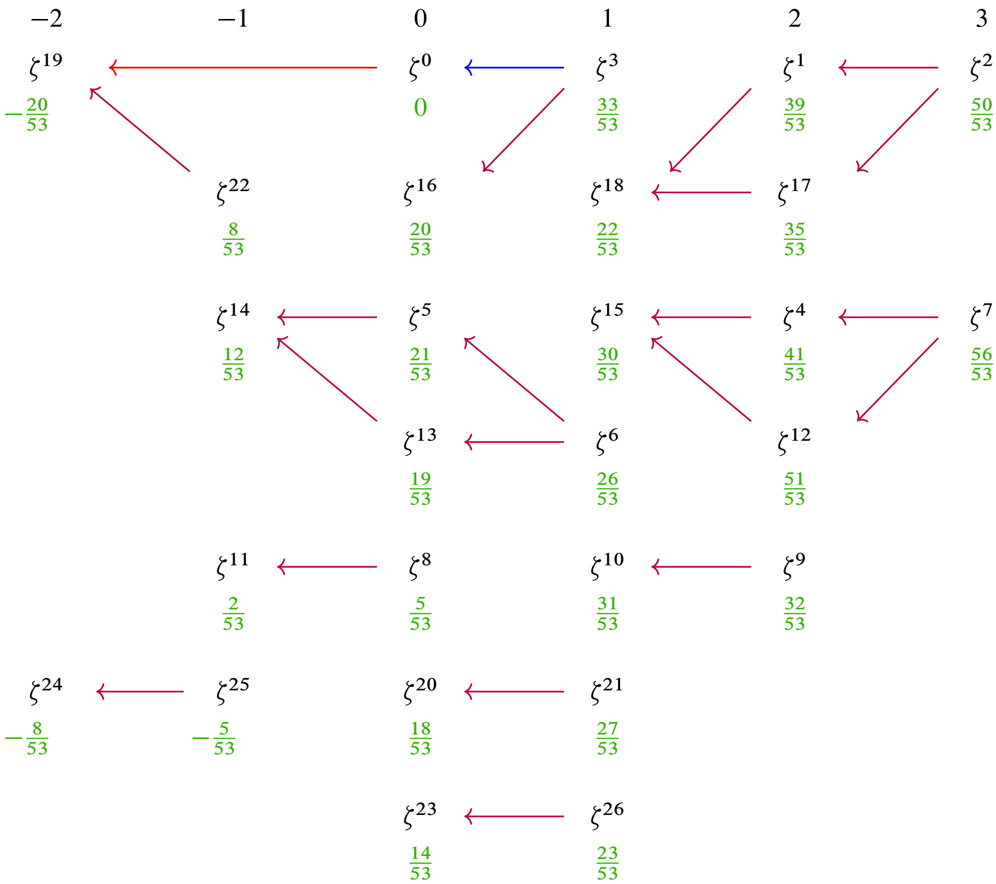


Figure 11: This diagram shows generators for C_* and the reducible ζ^0 for the knot $10^*_{28} = K(53,34)$. All arrows are multiplication by $\pm(T^2 - T^{-2})$. The displayed arrows represent d except for the two which are incident to ζ^0, which are δ_1 and δ_2. The v-map might have nonzero components (not displayed).

where C'_* is freely generated by α, β. The differential $\tilde{d}'$ satisfies $d'(\beta) = (T^2 - T^{-2})\alpha$ and $\delta'_2(1) = (T^2 - T^{-2})\alpha$, and is otherwise zero. The $\mathbb{Z}\times\mathbb{R}$-gradings for the generators α and β are respectively defined to be $\left(-2, -\frac{20}{53}\right)$ and $\left(-1, \frac{8}{53}\right)$.

Proposition 6.21 *The I-graded $\mathcal{S}$-complex of the knot 10^*_{28} over $R[U^{\pm 1}]$, where $R = \mathbb{Z}[T^{\pm 1}]$, is locally equivalent to the I-graded $\mathcal{S}$-complex $\widetilde{C}'$ which is defined above.*

Proof Recall that 10^*_{28} is the two-bridge knot $K(53,34)$. Let $\widetilde{C} = \widetilde{C}(10^*_{28}; \Delta_R)$ be the associated I-graded $\mathcal{S}$-complex over $R[U^{\pm 1}]$. We define morphisms $\widetilde{\lambda}\colon \widetilde{C} \to \widetilde{C}'$ and $\widetilde{\lambda}'\colon \widetilde{C}' \to \widetilde{C}$. First we define $\widetilde{\lambda}'$. We write the components of this morphism in the usual way:

$$\widetilde{\lambda}' = \begin{bmatrix} \lambda' & 0 & 0 \\ \mu' & \lambda' & \Delta'_2 \\ \Delta'_1 & 0 & 1 \end{bmatrix}.$$

We declare that the components of $\widetilde{\lambda}'$ as indicated above are given by

$$\lambda'(\alpha)=\zeta^{19},\quad \lambda'(\beta)=\zeta^{22},\quad \mu'(\alpha)=0,\quad \mu'(\beta)=cU^{-1}\zeta^{1}$$

and $\Delta_1'=\Delta_2'=0$. Here c is an integer such that $v(\zeta^{22})=cU^{-1}(T^2-T^{-2})\zeta^{18}$. We are using the constraint from [Daemi and Scaduto 2024a] that the only possible nonzero component of the v-map acting on ζ^{22} is some multiple of ζ^{18}. It is straightforward to check that $\widetilde{\lambda}'$ is a local map of $\mathcal{S}$-complexes. Furthermore, because $\deg_I(\beta)=\frac{8}{53}>-\frac{14}{53}=\deg_I(U^{-1}\zeta^1)$, it is a level 0 local morphism of I-graded $\mathcal{S}$-complexes. Now we turn to the construction of $\widetilde{\lambda}$. For this, we declare the nonzero components:

$$\lambda(\zeta^{19})=\alpha,\quad \lambda(\zeta^{22})=\beta,\quad \mu(\zeta^{16})=-c_1\beta,\quad \mu(\zeta^{20})=-c_2\beta,\quad \mu(\zeta^{23})=-c_3\beta.$$

Here c_1, c_2, c_3 are constants defined by the relations

$$v(\zeta^{3})=c_1(T^2-T^{-2})\zeta^{22},\quad v(\zeta^{21})=c_2(T^2-T^{-2})\zeta^{22},\quad v(\zeta^{26})=c_3(T^2-T^{-2})\zeta^{22}.$$

In writing these expressions, we are again using the constraint on the form of the v-map from [loc. cit.], which is a numerical computation. We also set $\Delta_1=\Delta_2=0$. It is straightforward to verify that $\widetilde{\lambda}$ as defined is a local map of $\mathcal{S}$-complexes. Furthermore, because

$$\deg_I(\zeta^{16})=\tfrac{20}{53},\quad \deg_I(\zeta^{20})=\tfrac{18}{53},\quad \deg_I(\zeta^{23})=\tfrac{14}{53}$$

are all greater than $\deg_I(\beta)=\frac{8}{53}$, this is a level 0 local morphism of I-graded $\mathcal{S}$-complexes.

Having constructed level 0 local morphisms in each direction, we have proved the result. □

In what follows, we denote by

$$r_s'(K)=-\inf\{r\in[-\infty,0)\mid \underline{\mathcal{N}}_K(0,r)\le -s\}$$

the version of the r_s-invariant for a knot K which only depends on the enriched $\mathcal{S}$-complex defined over the field of fractions of $R=\mathbb{Z}[T^{\pm 1}]$.

Corollary 6.22 *The nontrivial Γ- and r_s'-invariants for 10_{28} and its mirror are given by*

$$\Gamma_{10_{28}^*}(i)=\begin{cases}0 & \text{if } i<0,\\ \frac{8}{53} & \text{if } i=0,\\ \infty & \text{if } i>0,\end{cases}\qquad r_s'(10_{28})=\begin{cases}\infty & \text{if } s\in\left(-\infty,-\frac{8}{53}\right),\\ \frac{19}{53} & \text{if } s\in\left(-\frac{8}{53},0\right].\end{cases}$$

Similar computations can be carried out for other two-bridge knots. In Table 1, we list some such examples, focusing on the case of two-bridge knots with zero signature. The computations were partially done by computer, following the algorithm given in [Daemi and Scaduto 2024a] for $\mathcal{S}$-complexes of two-bridge knots. Note that the examples (131) appear in this table.

(p, q)	other name	$\Gamma_K(0)$	$\Gamma_K(0) > \frac{1}{8}$?	$r'_{s^\star}(K^*)$
$(37, 17)$	10^*_{34}	$\frac{2}{37}$	N	$\frac{13}{37}$
$(49, 23)$	$12a_{169}$	$\frac{4}{49}$	N	$\frac{17}{49}$
$(53, 34)$	10^*_{28}	$\frac{8}{53}$	Y	$\frac{19}{53}$
$(61, 29)$		$\frac{6}{61}$	N	$\frac{21}{61}$
$(65, 51)$	$11a^*_{333}$	$\frac{14}{65}$	Y	$\frac{25}{65}$
$(69, 29)$	10_{32}	$\frac{6}{69}$	N	$\frac{19}{69}$
$(73, 23)$	$12a_{1148}$	$\frac{2}{73}$	N	$\frac{19}{73}$
$(73, 35)$		$\frac{8}{73}$	N	$\frac{25}{73}$
$(77, 50)$	$12a^*_{380}$	$\frac{18}{77}$	Y	$\frac{27}{77}$
$(81, 52)$	$12a^*_{596}$	$\frac{16}{81}$	Y	$\frac{29}{81}$
$(85, 41)$		$\frac{10}{85}$	N	$\frac{29}{85}$
$(93, 73)$		$\frac{22}{93}$	Y	$\frac{35}{93}$
$(93, 41)$	$11a^*_{93}$	$\frac{8}{93}$	N	$\frac{27}{93}$
$(97, 31)$		$\frac{4}{97}$	N	$\frac{25}{97}$
$(97, 47)$		$\frac{12}{97}$	N	$\frac{33}{97}$
$(101, 66)$		$\frac{28}{101}$	Y	$\frac{35}{101}$
$(105, 41)$	$11a_{175}$	$\frac{6}{105}$	N	$\frac{25}{105}$
$(109, 70)$		$\frac{24}{109}$	Y	$\frac{39}{109}$
$(109, 53)$		$\frac{14}{109}$	Y	$\frac{37}{109}$
$(109, 51)$		$\frac{2}{109}$	N	$\frac{47}{109}$

Table 1: Every two-bridge knot $K = K(p,q)$ with (i) $|p| \leqslant 109$, (ii) $\sigma(K) = 0$, and (iii) $\Gamma_K(0), \Gamma_{K^*}(0)$ not both zero is equivalent to one of the above, after possibly taking the mirror. The identifiers for the knots in the second column are from KnotInfo [Livingston and Moore 2004]. In each case, $\Gamma_{K^*} = \Gamma_{U_1}$ and $r'_s(K) = r'_s(U_1)$, while for the mirror, $r'_s(K^*) = \infty$ for $s < -\Gamma_K(0)$ and $r'_s(K^*)$ is finite for $s \in (-\Gamma_K(0), 0]$. In the last column above, $s^\star$ is any real number in $(-\Gamma_K(0), 0]$. The knots in the shaded rows are those to which Theorem 6.16 applies.

6.5 Irreducible SU(2)-representations

We now consider results on the existence of nonabelian traceless SU(2)-representations for the fundamental groups of homology concordance complements, and a related result for homology cobordisms. (Note that Theorem 1.16, which involves arbitrary holonomy parameters, is proved in the next section.)

Theorem 6.23 *Let K be a knot in an integer homology 3-sphere Y. If the enriched $\mathcal{S}$-complex of (Y, K) is not weakly locally equivalent to the trivial enriched $\mathcal{S}$-complex, then, for any homology concordance $(W, S)\colon (Y, K) \to (Y', K')$, there exists an irreducible traceless representation $\pi_1(W \setminus S) \to \mathrm{SU}(2)$.*

A similar result involving a nonvanishing condition on the signature was proven in [Daemi and Scaduto 2024b].

Proof If $\mathfrak{E}(Y,K)$ is not locally equivalent to the trivial enriched $\mathcal{S}$-complex, Corollary 5.43 says that either $r_0(K)<\infty$ or $r_0(K^*)<\infty$. Then Corollary 5.54 completes the proof. □

As a corollary, we have the following, which implies Theorem 1.17:

Corollary 6.24 *Let K be a knot in S^3. Suppose one of the following invariants is nontrivial* (*ie not equal to the value of an invariant for the unknot*):

- (i) $\sigma(K)$, $s^\sharp(K)$, $s^\sharp_\pm(K)$, $\tilde{s}(K)$ *or* $\tilde{\varepsilon}(K)$;
- (ii) *one of Kronheimer and Mrowka's invariants* [2021b], *such as* $f_\sigma(K)$ *or* $z^\natural(K)$.

Then, for any knot concordance $S\subset I\times S^3$ from K to K', there exists an irreducible traceless representation $\pi_1(I\times S^3\setminus S)\to \mathrm{SU}(2)$.

Proof If one of the above invariants is nontrivial, Theorem 4.49 implies that the enriched $\mathcal{S}$-complex of K is not weakly locally equivalent to that of the unknot. Thus, the result follows from Theorem 6.23. □

We also provide a result on the existence of irreducible representations on homology cobordisms.

Theorem 6.25 *Let K be a knot in an integer homology 3-sphere Y satisfying $\sigma(Y,K)\le 0$. Suppose $\frac{1}{8}<\Gamma_{(Y,K)}\bigl(-\frac{1}{2}\sigma(Y,K)\bigr)$. Fix a positive integer n. Then, for any homology cobordism W from $Y_{1/n}(K)$ to itself, there exists an irreducible* $\mathrm{SU}(2)$*-representation on* $\pi_1(W)$.

Proof Theorem 6.15 and (120) imply $r_0(Y_1(K),U_1)<\infty$. There is a standard negative definite cobordism with $b_1=0$ from $Y_{1/n}(K)$ to $Y_1(K)$ by attaching a chain of (-2)-framed two-handles of length $n-1$ to a meridian of $K\subset Y_{1/n}(K)$, and this gives $r_0(Y_{1/n}(K),U_1)\le r_0(Y_1(K),U_1)<\infty$. The result then follows from Corollary 5.57. □

7 General holonomy parameters

Singular connections are the key geometrical player in the definition of the $\mathcal{S}$-complex of a knot. Up to this point, we have only considered $\mathcal{S}$-complexes defined by singular connections whose holonomies along meridians of a codimension two submanifold are asymptotic to a conjugate of the element

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\in \mathrm{SU}(2). \tag{133}$$

One can consider more generally singular connections where (133) is replaced with

$$\begin{bmatrix} e^{2\pi i\omega} & 0 \\ 0 & e^{-2\pi i\omega} \end{bmatrix}\in \mathrm{SU}(2) \tag{134}$$

for some fixed $\omega \in \left(0, \frac{1}{2}\right)$ [Kronheimer and Mrowka 1993; 2011b]. In Section 7.1, we review how such singular connections can also be used to produce $\mathcal{S}$-complexes for knots. In this section, we mainly follow [Imori 2024]; see also [Echeverria 2019]. However, there are some minor differences in our conventions that we discuss in detail below. We also study functoriality of $\mathcal{S}$-complexes of knots with holonomy parameter ω with respect to cobordisms of pairs. In Section 7.2, we study filtered special cycles with holonomy parameter ω and generalize the construction of the numerical invariants of Section 5 to the case of arbitrary holonomy parameter ω. In particular, the proof of Theorem 1.16 will be given in this section.

7.1 $\mathcal{S}$-complexes from singular instantons with general holonomy parameters

A singular connection with holonomy parameter ω for a pair (Y, K) of a knot in an integer homology 3-sphere is, roughly speaking, a connection on the trivial $\mathrm{SU}(2)$-bundle over Y that is singular along K and whose holonomy along any family of shrinking meridians of K is asymptotic to a conjugate of the matrix (134). This definition can be made precise by picking a model singular connection A_0^ω and then considering connections of the form $A_0^\omega + a$, where a is a 1-form with coefficients in $\mathfrak{su}(2)$ that lives in a weighted Sobolev space and has appropriate decay behavior in a neighborhood of the knot K. We refer the reader to [Kronheimer and Mrowka 1993; 2011b] for the details of the definition of singular connections, and we only review some of the features of singular connections that are essential for our purposes. We write $\mathcal{A}^\omega(Y, K)$ for the space of all singular connections for the pair (Y, K). There is an action of a gauge group $\mathcal{G}^\omega(Y, K)$ on $\mathcal{A}^\omega(Y, K)$ and the quotient space is denoted by $\mathcal{B}^\omega(Y, K)$.

The definition of singular connections can be adapted to pairs (W, S) of an embedded surface S in a 4-manifold W (or more generally submanifolds of codimension two in a smooth manifold). We will be interested in the case that $(W, S)\colon (Y, K) \to (Y', K')$ is a cobordism of pairs with cylindrical ends, and consider singular $U(2)$ connections with holonomy parameter ω that are asymptotic to fixed representatives of $\alpha \in \mathcal{B}^\omega(Y, K)$ and $\alpha' \in \mathcal{B}^\omega(Y', K')$ such that the induced connection on the determinant bundle is some fixed (nonsingular) $U(1)$ connection. The configuration spaces of such connections modulo the gauge group is denoted by $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$, where $c \in H^2(W; \mathbb{Z})$ denotes the first Chern class of the determinant bundle. The set of connected components of $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$ can be characterized as a torsor over $\mathbb{Z} \oplus \mathbb{Z}$, where $(1, 0) \in \mathbb{Z} \oplus \mathbb{Z}$ acts by *adding an instanton* and $(0, 1) \in \mathbb{Z} \oplus \mathbb{Z}$ acts by *adding a monopole*. Any element z of this torsor is called a *path from α to α' along* (W, S) and the corresponding connected component of $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$ is denoted by $\mathcal{B}_z^\omega(W, S, c; \alpha, \alpha')$.

Associated to any singular connection A representing an element of $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$, there are two quantities that are called *topological energy* and *monopole number*. The topological energy of A is defined with the same Chern–Weil integral as before,

$$\kappa(A) = \frac{1}{8\pi^2} \int_W \mathrm{tr}(F_{\mathrm{ad}(A)} \wedge F_{\mathrm{ad}(A)}),$$

where $F_{\mathrm{ad}(A)}$ denotes the trace-free part of the curvature of A corresponding to the splitting $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathbb{R}$ of the Lie algebra of $U(2)$. The curvature of A extends over the singular locus and has the form

$$\begin{bmatrix} \Omega & 0 \\ 0 & \Omega' \end{bmatrix}$$

with respect to the splitting that gives the presentation in (134). The monopole number of A is defined as

$$\nu(A) = -\frac{i}{2\pi} \int_S (\Omega - \Omega') + 2\omega S \cdot S.$$

The topological energy and the monopole number of A are locally constant and depend only on the path z represented by A. Adding an instanton to A changes $(\kappa(A), \nu(A))$ to $(\kappa(A) + 1, \nu(A))$, and adding a monopole changes $(\kappa(A), \nu(A))$ to $(\kappa(A) + 2\omega, \nu(A) + 2)$. Moreover, the connected components of $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$ are determined by their topological energies and monopole numbers.

The subspace of $\mathcal{B}^\omega(Y, K)$ consisting of connections with trivial curvature has a topological description. By taking holonomy along closed curves, the space of such flat connections, denoted by $\mathfrak{C}(Y, K, \omega)$, can be identified with the character variety

$$\{\varphi \in \mathrm{Hom}(\pi_1(Y \setminus K), \mathrm{SU}(2)) \mid \varphi(\mu) \text{ is conjugate to (134)}\}/\mathrm{SU}(2), \tag{135}$$

where μ is any meridian of K. In particular, there is a distinguished flat singular connection θ^ω in $\mathcal{B}^\omega(Y, K)$ that is reducible and it corresponds to the class of the abelian representation in (135). A flat singular connection is nondegenerate if it is cut down transversely by the flat curvature equation. The reducible connection θ^ω is nondegenerate if and only if $e^{4\pi i \omega}$ is not a root of the Alexander polynomial [Echeverria 2019, Lemma 15]. (See also [Kronheimer and Mrowka 2011b].)

The ASD (anti-self-dual) equation is naturally defined for singular connections on a 4-manifold, and we write $M_z^\omega(W, S, c; \alpha, \alpha')$ for the subspace of $\mathcal{B}_z^\omega(W, S, c; \alpha, \alpha')$ given by the solutions of the ASD equation. The local behavior of this moduli space around the class of a singular connection is controlled by an ASD operator D_A. Assuming that α and α' are nondegenerate, D_A is a Fredholm operator defined on appropriate weighted Sobolev spaces that in particular give exponential decay at the ends. In this case, the moduli space $M_z^\omega(W, S, c; \alpha, \alpha')$ is a smooth manifold of dimension $\mathrm{ind}(D_A)$ in a neighborhood of the class of A if D_A is surjective. Analogous to the topological energy and monopole number, the index of D_A depends only on z. In fact, the dimension formulas of [Kronheimer and Mrowka 1993] together with excision imply that, for any singular connection A representing an element of $\mathcal{B}^\omega(W, S, c; \alpha, \alpha')$, the expression

$$\mathrm{ind}(D_A) - \left(8\kappa(A) + 2(1 - 4\omega)\nu(A) - \tfrac{3}{2}(\chi(W) + \sigma(W)) + \chi(S) + 8\omega^2 S \cdot S\right)$$

depends only on α and α'. Motivated by this discussion, we write $\mathrm{ind}(z)$ for the index of the ASD operator of any element (and hence all elements) of $\mathcal{B}_z^\omega(W, S, c; \alpha, \alpha')$.

Example 7.1 Suppose $(W,S)\colon (Y,K)\to(Y',K')$ is a cobordism of pairs such that $b^1(W)=b^+(W)=0$. We also assume that $e^{4\pi i\omega}$ is not a root of the Alexander polynomials of (Y,K) and (Y',K') and thus the reducible singular flat connections θ^ω and θ'^ω for (Y,K) and (Y',K') are nondegenerate. Also let $c\in H^2(W;\mathbb{Z})$ and B_0 be a $U(1)$ connection on the line bundle with $c_1=c$. If L is a $U(1)$-bundle on W, then there is a reducible instanton A_L^ω that has the form $B\oplus B^*\otimes B_0$ with B a singular $U(1)$ connection such that $\frac{i}{2\pi}F_B$ represents the cohomology class $c_1(L)+\omega S$. Then

$$\kappa(A_L^\omega)=-\bigl(c_1(L)+\omega S-\tfrac{1}{2}c\bigr)^2,\quad \nu(A_L^\omega)=(c-2c_1(L))\cdot S$$

and we have the index formula

$$\operatorname{ind}(D_{A_L^\omega})=8\kappa(A_L^\omega)+2(1-4\omega)\nu(A_L^\omega)-\tfrac{3}{2}(\chi(W)+\sigma(W))+\chi(S)+8\omega^2S\cdot S \\ +\sigma_\omega(Y,K)-\sigma_\omega(Y',K')-1, \tag{136}$$

where $\sigma_\omega(Y,K)$, $\sigma_\omega(Y',K')$ are the Tristram–Levine signatures of (Y,K), (Y',K') as given in (124). This formula, in the case that ω is a rational number, is proved in [Imori 2024]. To extend the formula to the irrational case, note that if $[\omega_0-\epsilon,\omega_0+\epsilon]$ is an interval such that $e^{4\pi i\omega}$ is not a root of Alexander polynomial for any ω in this interval, then the expression on the left-hand side of (136) is constant for all values of $\omega\in[\omega_0-\epsilon,\omega_0+\epsilon]$. The left-hand side is an index of an elliptic operator, and it is clear from the Fredholm theory developed in [Kronheimer and Mrowka 1993; 2011b] that the ASD operators $D_{A_L^\omega}$ for $\omega\in[\omega_0-\epsilon,\omega_0+\epsilon]$ form a continuous family of Fredholm operators with a continuously varying domains and codomains. In particular, standard properties of the indices of Fredholm operators imply that the left-hand side of (136) is also constant. Now we can use this observation to obtain (136) in the case that ω is irrational.

A *lifted singular connection* $\widetilde{\alpha}$ on (Y,K) with holonomy parameter ω is an element α of $\mathcal{B}^\omega(Y,K)$ together with a choice of a homotopy class of a path from α to θ^ω in $\mathcal{B}^\omega(Y,K)$. The path from α to θ^ω determines a connected component z of $\mathcal{B}^\omega(\mathbb{R}\times Y,\mathbb{R}\times K;\alpha,\theta^\omega)$. Using this path z, we can define[1]

$$\operatorname{cs}(\widetilde{\alpha})=2\kappa(z),\quad \operatorname{hol}_K(\widetilde{\alpha})=\nu(z),\quad \deg_{\mathbb{Z}}(\widetilde{\alpha})=\operatorname{ind}(z).$$

There is a forgetful map that sends the lifted singular connection $\widetilde{\alpha}$ to the underlying singular connection α. This map identifies lifted singular connections as the universal cover of $\mathcal{B}^\omega(Y,K)$. Then cs and hol_K define two real-valued functions on the universal cover. The critical points of the Chern–Simons functional cs are given by lifted flat singular connections. For a pair of integers k, l, we can add k instantons and l monopoles to the path component of a lifted connected $\widetilde{\alpha}$ to obtain a new lifted connection with the same underlying singular connection. If we denote this lifted connection by $U^{2k+l}T^{2l}\widetilde{\alpha}$, then

$$\operatorname{cs}(U^iT^j\widetilde{\alpha})=\operatorname{cs}(\widetilde{\alpha})+i+j\bigl(2\omega-\tfrac{1}{2}\bigr),\quad \operatorname{hol}_K(U^iT^j\widetilde{\alpha})=\operatorname{hol}_K(\widetilde{\alpha})+j, \tag{137}$$

$$\deg_{\mathbb{Z}}(U^iT^j\widetilde{\alpha})=4i+\deg_{\mathbb{Z}}(\widetilde{\alpha}) \tag{138}$$

for any pair of integers i and j satisfying the property that j is an even integer $2i-j$ is divisible by 4.

[1] Our convention of the Chern–Simons functional cs differs from the convention of [Imori 2024] by a factor of 2.

We formally extend the definition of lifted flat connections to include the case that i, j are arbitrary integers by requiring (137).

The computation in (137) motivates the definition of a ring with two gradings. From now on we assume that $\omega \in \left(0, \frac{1}{4}\right)$. Let R be a ring, and consider the ring $R[T^{-1}, T]][U^{\pm 1}]$ of Laurent polynomials in the variable U with coefficients in the ring of Laurent power series in the variable T. Thus, for any element q of this ring, there is a finite positive integer N such that

$$q = \sum_{i=-N}^{N} r_i(T) U^i$$

with $r_i(T) \in R[T^{-1}, T]]$. We may regard $R[T^{-1}, T]][U^{\pm 1}]$ as an algebra over $R[U^{\pm 1}, T^{\pm 1}]$ in the obvious way. We define the I-grading of a monomial rU^iT^j for a nonzero $r \in R$ to be $i + j\left(2\omega - \frac{1}{2}\right)$. Now the I-grading of an arbitrary nonzero element q of $R[T^{-1}, T]][U^{\pm 1}]$ is defined to be the maximum of the I-gradings of all monomials in q. In particular, the assumption that $\omega < \frac{1}{4}$ implies that the I-grading of any nonzero element is a finite number. We extend the I-grading to the zero element of $R[T^{-1}, T]][U^{\pm 1}]$ by $-\infty$. We also define a $\mathbb{Z}$-grading $\deg_{\mathbb{Z}}$ on $R[T^{-1}, T]][U^{\pm 1}]$ by setting elements of the form $r(T)U^i$ to be homogenous of degree $4i$.

Next, we recall the definition of the $\mathcal{S}$-complex of (Y, K) for a holonomy parameter ω such that $e^{4\pi i \omega}$ is not a root of the Alexander polynomial of (Y, K). The assumption on ω implies that the reducible θ^ω for the pair (Y, K) is nondegenerate. After a small perturbation of cs, we may assume that its critical points in the space of lifted singular connections are nondegenerate. (We assume that the perturbation is invariant with respect to the action of U and T on the space of lifted singular connections.) Let $C^\omega(Y, K)^\circ$ be the R-module freely generated by the critical points of the perturbed cs. Then there is an action of $R[U^{\pm 1}, T^{\pm 1}]$ on this module, and $C^\omega(Y, K)^\circ$ is in fact a finitely generated free module over $R[U^{\pm 1}, T^{\pm 1}]$. For a reason that becomes clear in a moment, we take the completion

$$C^\omega(Y, K) := C^\omega(Y, K)^\circ \otimes_{R[U^{\pm 1}, T^{\pm 1}]} R[T^{-1}, T]][U^{\pm 1}] \tag{139}$$

to obtain a finitely generated free module over $R[T^{-1}, T]][U^{\pm 1}]$. We use the value of the perturbed cs to define $\deg_I$ for any element $r\widetilde{\alpha} \in C^\omega(Y, K)$ where $\widetilde{\alpha}$ is a lifted connection and $r \in R$ is nonzero. We extend $\deg_I$ to all elements of $C^\omega(Y, K)$ using (77). This grading, called I-grading, has the property that

$$\deg_I(q \cdot \zeta) = \deg_I(q) + \deg_I(\zeta) \tag{140}$$

for any $\zeta \in C^\omega(Y, K)$ and $q \in R[T^{-1}, T]][U^{\pm 1}]$. We also use $\deg_{\mathbb{Z}}$ to define a $\mathbb{Z}$-grading on $C^\omega(Y, K)$.

The moduli spaces of singular ASD connections determine a differential d on $C^\omega(Y, K)$. To be more specific, let $\widetilde{\alpha}$ be a lifted (perturbed) flat connection with the underlying singular flat connection α. Suppose z is a path from α to another irreducible perturbed flat connection α' such that $\mathrm{ind}(z) = 1$. The path z determines a lift of $\widetilde{\alpha}'$ of α' by requiring that

$$\mathrm{cs}(\widetilde{\alpha}) = 2\kappa(z) + \mathrm{cs}(\widetilde{\alpha}'), \quad \mathrm{hol}_K(\widetilde{\alpha}) = \nu(z) + \mathrm{hol}_K(\widetilde{\alpha}').$$

In particular, $\deg_{\mathbb{Z}}(\widetilde{\alpha})$ is equal to $\mathrm{ind}(z) + \deg_{\mathbb{Z}}(\widetilde{\alpha}')$, and hence the difference between the $\mathbb{Z}$-gradings of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ is 1. The downward gradient flowline equation for the perturbed cs can be identified as a perturbation of the ASD equation over $\mathbb{R} \times Y$, and with a slight abuse of notation, we still write $M_z^{\omega}(\mathbb{R} \times Y, \mathbb{R} \times K; \alpha, \alpha')$ for the solutions of this perturbed ASD equation in $\mathcal{B}_z^{\omega}(\mathbb{R} \times Y, \mathbb{R} \times K; \alpha, \alpha')$. By choosing the perturbation of cs generically, we may assume that $M_z^{\omega}(\mathbb{R} \times Y, \mathbb{R} \times K; \alpha, \alpha')$ is cut down transversely and hence it is a smooth 1-dimensional manifold. This moduli space admits a natural orientation and a free $\mathbb{R}$-action given by translation. Therefore, we obtain a well-defined integer m_z by the signed count of the elements of the quotient space $M_z^{\omega}(\mathbb{R} \times Y, \mathbb{R} \times K; \alpha, \alpha')/\mathbb{R}$. Now we define

$$d(\widetilde{\alpha}) = \sum_z m_z \cdot \widetilde{\alpha}', \tag{141}$$

where the sum is over paths z as above. It is clear from the construction that $\deg_I(\widetilde{\alpha}')$ is smaller than $\deg_I(\widetilde{\alpha})$. However, we cannot guarantee that there are finitely many terms in (141) and this is the reason that we need to work with the completion in (139).

The map d extends to an $R[T^{-1}, T]][U^{\pm 1}]$-module homomorphism. This defines a differential on $C^{\omega}(Y, K)$. This differential decreases $\deg_{\mathbb{Z}}$ by 1 and decreases the I-grading. We similarly define maps

$$\delta_1 \colon C^{\omega}(Y, K) \to R[T^{-1}, T]][U^{\pm 1}], \quad \delta_2 \colon R[T^{-1}, T]][U^{\pm 1}] \to C^{\omega}(Y, K)$$

using the singular ASD connections that are asymptotic to θ^{ω}, and a homomorphism

$$v \colon C^{\omega}(Y, K) \to C^{\omega}(Y, K)$$

by cutting down the moduli spaces by holonomy along the path $\mathbb{R} \times \{p\}$, where p is a basepoint on K.

Using these homomorphisms, we can form $\widetilde{C}^{\omega}(Y, K)$ together with a differential $\tilde{d}$ and a map χ as in Section 2.1. The $\mathbb{Z}$-grading $\deg_{\mathbb{Z}}$ and the I-grading $\deg_I$ define the structure of a variant of I-graded $\mathcal{S}$-complexes: the differential $\tilde{d}$ and the homomorphism χ respectively decrease and increase $\mathbb{Z}$-grading by 1. The map $\tilde{d}$ decreases the I-grading and χ preserves the I-grading. Multiplication of an element of $\widetilde{C}^{\omega}(Y, K)$ by an element of $R[T^{-1}, T]][U^{\pm 1}]$ changes the I-grading as in (140). Moreover, multiplying a homogenous element (with respect to $\deg_{\mathbb{Z}}$) in $\widetilde{C}^{\omega}(Y, K)$ by $r(T)U^i$ increases $\deg_{\mathbb{Z}}$ by $4i$. This algebraic structure of $\widetilde{C}^{\omega}(Y, K)$ motivates a generalization of the notion of I-graded $\mathcal{S}$-complexes. We continue to call any such algebraic structure an I-graded $\mathcal{S}$-complex; if we want to specify the role of ω, we say an I-graded $\mathcal{S}$-complex *with holonomy parameter* ω.

Most of the algebraic notions introduced in Section 5.1 can be adapted to the present setup. We define, just as before, height i morphisms of I-graded $\mathcal{S}$-complexes with holonomy parameter ω and with level δ. There is a small modification that we need to make in the definition of strong morphisms because the variable T has nontrivial I-grading. A height i morphism of I-graded $\mathcal{S}$-complexes with level δ is strong if the element c_i, defined as in (15), has the form

$$c_i = r_0 + r_1 T + r_2 T^2 + \cdots \in R[T^{-1}, T]],$$

where r_0 is an invertible element of the ring R. This is equivalent to saying that c_i is invertible and $\deg_I(c_i) = 0$. We define enriched complexes with holonomy parameter ω as in Definition 5.1 with the change that instead of one discrete subset $\mathfrak{K}$ of $\mathbb{R}$, we have one such discrete set $\mathfrak{K}_m$ for any integer m, and we update item (iii) by replacing (78) with

$$\deg_{\mathbb{Z}}(\zeta) = m \implies \deg_I(\zeta) \in B_\delta(\mathfrak{K}_m). \tag{142}$$

Finally, we follow Definition 5.5 without any change to define (weak) local equivalence of enriched complexes with holonomy parameter ω.

Now we return to our topological setup. The following lemma justifies our requirement in (142) on the algebraic side:

Lemma 7.2 *For any (Y, K) and any ω, there is a discrete subset $\mathfrak{K}^0_m \subset \mathbb{R}$ for any integer m such that, for any lifted singular flat connection $\widetilde{\alpha}$ on (Y, K) with holonomy parameter ω and $\deg_{\mathbb{Z}}(\alpha) = m$, we have $\mathrm{cs}(\widetilde{\alpha}) \in \mathfrak{K}^0_m$. Moreover, there is a constant M_0 such that the following holds: for any positive real number δ, there is a positive constant ϵ such that, if we perturb* cs *by a perturbation term whose norm is less than ϵ and $\widetilde{\alpha}$ is a lift of a critical point of the perturbed* cs *with $\deg_{\mathbb{Z}}(\alpha) = m$, then*

$$\mathrm{cs}(\widetilde{\alpha}) \in \bigcup_{i=m-M_0}^{m+M_0} B_\delta(\mathfrak{K}^0_i). \tag{143}$$

In the statement of this lemma, we do not assume that perturbed singular flat connections on (Y, K) are necessarily nondegenerate. In this case, the ASD operator associated to a connection A from some singular flat connection α to θ^ω is not necessarily Fredholm. To define $\deg_{\mathbb{Z}}$ in this case, we modify the function spaces in the definition of D_A to weighted Sobolev spaces. That is to say, we replace L^2_k function spaces with $L^2_{k,\delta}$ for some sufficiently small positive value of δ. An element f of some function space $L^2_{k,\delta}$ over the cylinder $\mathbb{R} \times Y$ is characterized by the property that $e^{\delta|t|} f$ is an element of L^2_k, where t denotes the parameter on the $\mathbb{R}$ factor.

Proof The moduli space of singular flat connections $\mathfrak{C}(Y, K, \omega)$ is compact by the Uhlenbeck compactness theorem. Thus $\mathfrak{C}(Y, K, \omega)$ has finitely many path connected components, and we pick a basepoint from each path connected component to obtain a finite set of singular flat connections $\{\alpha_i\}_{i=1}^N$. Any lifted flat connection $\widetilde{\alpha}$ can be written as the concatenation of a lift $\widetilde{\alpha}_i$ of a basepoint singular flat connection α_i and a path z within one of the connected components of $\mathfrak{C}(Y, K, \omega)$. The path z is represented by a singular flat connection A on $\mathbb{R} \times Y$ with holonomy parameter ω such that the restriction of A to $\{t\} \times Y$ for any $t \in \mathbb{R}$ is flat. In particular, the topological energy of A vanishes, and $\mathrm{cs}(\widetilde{\alpha}) = \mathrm{cs}(\widetilde{\alpha}_i)$.

Although $\deg_{\mathbb{Z}}(\widetilde{\alpha})$ is not necessarily equal to $\deg_{\mathbb{Z}}(\widetilde{\alpha}_i)$, we claim that the difference between these quantities is uniformly bounded. First note that $h^1(\alpha)$, the dimension of the Zariski tangent space of $\mathfrak{C}(Y, K, \omega)$ at $\alpha \in \mathfrak{C}(Y, K, \omega)$ (or equivalently the kernel of the gradient of cs at α), is bounded by some finite constant M. This follows from the compactness of $\mathfrak{C}(Y, K, \omega)$ and the fact that $h^1(\alpha)$ defines a lower semicontinuous function on $\mathfrak{C}(Y, K, \omega)$. For any $\alpha \in \mathfrak{C}(Y, K, \omega)$, there is a small neighborhood of α in $\mathcal{B}^\omega(Y,K)$ such that, for any path z of singular flat connections within this neighborhood, the ASD index of the path is at most $h^1(\alpha)$. This can be seen from the spectral flow interpretation of the ASD index over a cylinder. From this observation, we see that the difference between $\deg_{\mathbb{Z}}(\widetilde{\alpha})$ and $\deg_{\mathbb{Z}}(\widetilde{\alpha}_i)$ is bounded by a constant M_0 that is independent of $\widetilde{\alpha}$. As a consequence of the relationship between $(\mathrm{cs}(\widetilde{\alpha}), \deg_{\mathbb{Z}}(\widetilde{\alpha}))$ and $(\mathrm{cs}(\widetilde{\alpha}_i), \deg_{\mathbb{Z}}(\widetilde{\alpha}_i))$, it suffices to prove the first claim in the case that $\widetilde{\alpha}$ is a lift of one of the basepoints. Now the claim follows because there are finitely many basepoints and the topological energy and the indices of the lifts of these basepoints satisfy (137) and (138).

For the second claim of the lemma, the key point is that any critical point α of the perturbed cs for a small enough perturbation belongs to a small neighborhood of $\alpha_0 \in \mathfrak{C}(Y,K,\omega)$. We may assume that this neighborhood is small enough that a path z from α_0 to α in this neighborhood has the property that

$$\kappa(z) < \tfrac{1}{2}\delta, \quad \mathrm{ind}(z) < M_0,$$

where M_0 is the constant obtained in the previous paragraph. Since any lift $\widetilde{\alpha}$ of α is obtained from concatenating a lift of $\widetilde{\alpha}_0$ and a path z as above, the property in (143) holds. □

For two perturbations of the Chern–Simons functional in the definition of $\widetilde{C}^\omega(Y,K)$, we obtain morphisms of some level δ between the corresponding I-graded $\mathcal{S}$-complexes. Therefore, taking a sequence of perturbations going to zero gives an enriched complex $\mathfrak{E}^\omega(Y,K)$ in the above generalized sense. Although (78) does not hold in the case that ω is irrational, we have (142) for arbitrary ω, as a consequence of Lemma 7.2, by taking $\mathfrak{K}_m$ to be the union of the discrete sets $\mathfrak{K}_i^0$ for i between $m - M_0$ and $m + M_0$. The homotopy type of this enriched complex is a topological invariant of (Y,K).

Remark 7.3 Suppose l is the $\mathbb{Z}/2$-bundle over $Y \setminus K$ that restricts to a nontrivial bundle on any meridian of K. Since the structure group of l is discrete, it comes with a canonical connection ι. Given a singular connection A on (Y,K) with holonomy parameter ω, the connection $A \otimes \iota$ determines a singular connection on (Y,K) with holonomy parameter $\frac{1}{2} - \omega$. The above construction can be modified in a straightforward way to define an enriched complex $\mathfrak{E}^\omega(Y,K)$ for $\omega \in \left(\frac{1}{4}, \frac{1}{2}\right)$, which is now defined over the ring $R[\![T^{-1}, T]\!][U^{\pm 1}]$. The operation of taking tensor product with ι allows us to obtain this enriched complex from $\mathfrak{E}^{1/2-\omega}(Y,K)$. In particular, we do not get any new information by considering the values of ω that are greater than $\frac{1}{4}$.

This construction of enriched complexes of knots with arbitrary holonomy parameter ω is functorial with respect to cobordisms of pairs. First we need the following generalization of Definition 2.7:

Definition 7.4 Let $(W,S)\colon (Y,K) \to (Y',K')$ be a cobordism of pairs and $c \in H^2(W;\mathbb{Z})$. For a nonnegative integer i, we say (W,S,c) is a *height i negative definite cobordism* with respect to the holonomy parameter ω if it satisfies the following properties:

(i) $b^1(W) = b^+(W) = 0$ and $e^{4\pi i\omega}$ is not the root of the Alexander polynomials of (Y,K), (Y',K').

(ii) For any L, the index of the reducible ASD connection A_L^ω from Example 7.1 is at least $2i - 1$.

We say (W,S,c) is a *strong height i negative definite cobordism* (*over $R[\![T]\!]$*) if

$$\eta^{\omega}(W,S,c) := \sum_{\substack{c_1(L)\in H^2(W;\mathbb{Z})\\ \mathrm{ind}(A_L)=2i-1}} (-1)^{c_1(L)^2} T^{\nu(A_0)-\nu(A)} \in R[\![T]\!] \tag{144}$$

is invertible and has vanishing $\deg_I$. Here A_0 is a reducible ASD connection of index $2i-1$ on (W,S,c) whose topological energy is equal to

$$\kappa^{\omega}_{\min}(W,S,c) := \min\{\kappa(A^{\omega}_L) \mid \mathrm{ind}(A^{\omega}_L) = 2i-1\}. \tag{145}$$

In what follows, if c is trivial, then we drop it from the notation in (144) and (145).

Note that the index formula (136) and the assumption on A_0 imply that the terms in (144) have nonnegative powers of T and hence $\eta^{\omega}(W,S,c)$ belongs to $R[\![T]\!]$. Moreover, $\eta^{\omega}(W,S,c)$ is independent of the choice of A_0 because $\nu(A_0)$ is determined by $\kappa^{\omega}_{\min}(W,S,c)$. We also remark that the subset of $H^2(W;\mathbb{Z})$ given by c_1 of connections A^{ω}_L with minimal index is independent of ω. In particular, $\eta^{\omega}(W,S,c)$ (defined using the value of i that minimizes the index of the ASD operator) is independent of ω.

Example 7.5 Let K' be obtained from a knot $K \subset Y$ by changing a negative crossing to a positive crossing, and $(W,S)\colon (Y,K) \to (Y,K')$ the blow-up of the crossing change cobordism. There are two reducibles A_0 and A_1, corresponding to the trivial cohomology class and the exceptional class, that minimize the index. The index of these reducibles is $\sigma_\omega(Y,K) - \sigma_\omega(Y',K) - 1 \in \{\pm 1\}$. We have

$$(\kappa(A_0), \nu(A_0)) = (4\omega^2, 0), \quad (\kappa(A_1), \nu(A_1)) = (1 - 4\omega + 4\omega^2, -4),$$

which implies that

$$\eta^{\omega}(W,S) = 1 - T^4.$$

This is an invertible element of $\mathbb{Z}[\![T]\!]$ with trivial $\deg_I$. Therefore, (W,S) gives a strong height 0 or 1 negative definite cobordism. In the case that K' is obtained from K by changing a positive crossing to a negative crossing, the cobordism (W,S) has a unique reducible with minimal index $\sigma_\omega(Y,K) - \sigma_\omega(Y',K) - 1 \in \{-3,-1\}$ and vanishing values of κ, ν. In particular, $\eta^{\omega}(W,S) = 1$. Thus (W,S) is a strong height 0 negative definite cobordism if $\sigma_\omega(Y,K) = \sigma_\omega(Y',K)$.

Example 7.6 As a generalization of the previous example, let $S\colon K \to K'$ be an immersed knot cobordism in $[0,1]\times Y$ with transverse double points. We write s_+ and s_- for the number of positive and negative double points of S. Then blowing up the double points of S determines a cobordism $(W,\bar{S})\colon (Y,K) \to (Y,K')$. If $i = \frac{1}{2}(\sigma_\omega(K) - \sigma_\omega(K')) - g(S) \geq 0$, then $(W,\bar{S})$ gives a height i negative definite cobordism with $\eta^{\omega}(W,S) = (1-T^4)^{s_+}$ and $\kappa^{\omega}_{\min}(W,S,c) = 4s_+\omega^2$.

For a negative definite cobordism $(W,S,c)\colon (Y,K) \to (Y',K')$ of height i, we define maps

$$\begin{gathered} \lambda,\mu\colon C^{\omega}(Y,K) \to C^{\omega}(Y',K'), \\ \Delta_1\colon C^{\omega}(Y,K) \to R[T^{-1},T]\!][U^{\pm 1}], \\ \Delta_2\colon R[T^{-1},T]\!][U^{\pm 1}] \to C^{\omega}(Y',K'), \end{gathered} \tag{146}$$

essentially in the same way as in the case $\omega = \frac{1}{4}$, using moduli spaces $M_z^\omega(W,S,c;\alpha,\alpha')$. We need only clarify our convention on relating lifted flat connections. Fix a reducible connection A_0 as in Definition 7.4. Let $\widetilde{\alpha}$ be a lifted singular connection on (Y,K) with the underlying singular connection α and z be a path from α to α' along (W,S,c). (In practice, and for the definition of the cobordism maps above, the path z is determined by a singular instanton.) Then we fix a lift $\widetilde{\alpha}'$ of α' by requiring

$$\operatorname{cs}(\widetilde{\alpha}') + 2\kappa(z) = \operatorname{cs}(\widetilde{\alpha}) + 2\kappa^\omega_{\min}(W,S,c), \quad \operatorname{hol}_K(\widetilde{\alpha}') + \nu(z) = \operatorname{hol}_K(\widetilde{\alpha}) + \nu(A_0). \tag{147}$$

As mentioned above, $\nu(A_0)$ is independent of A_0 as long as its energy is equal to $\kappa^\omega_{\min}(W,S,c)$.

The maps in (146) combine, in the way described in Section 2.1, to define a height i morphism

$$\widetilde{\lambda}^\omega_{(W,S,c)} \colon \widetilde{C}^\omega(Y,K) \to \widetilde{C}^\omega(Y',K').$$

That is to say, $\widetilde{\lambda}_{(W,S,c)}$ is a chain map, and if we define c_j as in Definition 2.6, then $c_i = \eta^\omega(W,S,c)$ and $c_j = 0$ for $j < i$. For $i = 0$ this is proved in [Imori 2024] and for higher values of i the argument of [Daemi and Scaduto 2024a] adapts without any essential change. In the absence of perturbation terms, the first identity in (147) and nonnegativity of the topological energy of instantons imply that $\widetilde{\lambda}_{(W,S,c)}$ is a morphism of I-graded complexes with level $2\kappa^\omega_{\min}(W,S,c)$. In general, we obtain a morphism of enriched complexes

$$\mathfrak{L}^\omega_{(W,S,c)} \colon \mathfrak{E}^\omega(Y,K) \to \mathfrak{E}^\omega(Y',K')$$

with height i and level $2\kappa^\omega_{\min}(W,S,c)$ using sequences of perturbation terms that go to zero.

We may forget the I-grading in the above construction and obtain an $\mathcal{S}$-complex in the sense of Definition 2.1 over the ring $R[T^{-1},T]\!]$ with a $\mathbb{Z}/4$-grading. Here we use the isomorphism of degree 4 provided by U to roll the $\mathbb{Z}$-graded complex $\widetilde{C}^\omega(Y,K)$ into a $\mathbb{Z}/4$-graded complex, which retains the same notation. Unlike the case of holonomy parameter $\omega = \frac{1}{4}$, the $\mathcal{S}$-complex $\widetilde{C}^\omega(Y,K)$ does not see much about the knottedness of K in Y. To be more precise, we have the following:

Proposition 7.7 [Imori 2024] *If K and K' are two homotopic knots in an integer homology sphere Y with $\sigma_\omega(Y,K) = \sigma_\omega(Y,K')$ and nonvanishing values of the Alexander polynomial at $e^{4\pi i\omega}$, then the $\mathcal{S}$-complexes $\widetilde{C}^\omega(Y,K)$ and $\widetilde{C}^\omega(Y,K')$ are chain homotopy equivalent.*

Proof We may obtain K' from K by a sequence of crossing changes. This gives a cylinder cobordism $S \colon K \to K'$ immersed into $[0,1]\times Y$. Use the construction of $(W,\bar{S}) \colon (Y,K) \to (Y',K')$ in Example 7.6 to obtain a height 0 negative definite cobordism. The above functoriality discussion gives a morphism $\widetilde{\lambda} \colon \widetilde{C}^\omega(Y,K) \to \widetilde{C}^\omega(Y,K')$. We obtain a morphism $\widetilde{\lambda}'$ of $\mathcal{S}$-complexes in the reverse direction by flipping $(W,\bar{S})$. The composition of $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ in either order is the identity multiplied by a term of the form $(1-T^4)^n$ [loc. cit.]. This gives the desired claim, because $1-T^4$ is a unit in $R[T^{-1},T]\!]$. □

In the case of a knot K in S^3, we can completely characterize $\widetilde{C}^\omega(K)$. For any ω, there is a two-bridge torus knot $T_{2,2n+1}$ such that $\sigma_\omega(T_{2,2n+1}) = -2$ and $\Delta_{T_{2,2n+1}}(e^{4\pi i\omega}) \neq 0$ [loc. cit.]. For any such knot, the character variety (135) is nondegenerate and has one reducible element θ^ω and one irreducible element α. In particular, we can use a perturbation in the definition of $\widetilde{C}^\omega(T_{2,2n+1})$ such that $C^\omega(T_{2,2n+1}) = R[T^{-1}, T]\!]$. There is an iterated crossing change cobordism from the unknot to $T_{2,2n+1}$ which gives rise to a strong height 1 negative definite cobordism of pairs by following the construction of Example 7.6. In particular, this implies that the map δ_1 is multiplication by a unit. Now degree consideration shows that the remaining maps involved in the $\mathcal{S}$-complex $\widetilde{C}^\omega(T_{2,2n+1})$ have to be trivial. After a change of basis, we may also assume that $\delta_1 \colon R[T^{-1}, T]\!] \to R[T^{-1}, T]\!]$ is the identity map. We denote this $\mathcal{S}$-complex by $\widetilde{C}\langle 1\rangle$. More generally, for any integer k, let $\widetilde{C}\langle k\rangle$ be given by tensoring k copies of $\widetilde{C}\langle 1\rangle$. (For negative k, this means that $\widetilde{C}\langle k\rangle$ is the tensor product of $-k$ copies of the dual of $\widetilde{C}\langle 1\rangle$.) The connected sum theorem of [loc. cit.] implies that $\widetilde{C}^\omega(\#_k T_{2,2n+1})$ is chain homotopy equivalent to $\widetilde{C}\langle k\rangle$. Therefore, the following is a corollary of Proposition 7.7:

Corollary 7.8 [Imori 2024] *For any knot K in S^3 and $\omega \in \left(0, \frac{1}{4}\right)$ that $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of K, the $\mathcal{S}$-complex $\widetilde{C}^\omega(K)$ is chain homotopy equivalent to $\widetilde{C}\left\langle -\frac{1}{2}\sigma_\omega(K)\right\rangle$.*

In the more general case of arbitrary integer homology spheres Y, it is reasonable to expect that $\widetilde{C}^\omega(Y, K)$ is determined by Y and the integer $\sigma_\omega(Y, K)$. At least in the case that K is null-homotopic, we can verify this by following the argument as in the proof of Corollary 7.8. Applying the connected sum theorem of [loc. cit.], we conclude that $\widetilde{C}^\omega(Y, K)$ is chain homotopy equivalent to

$$\widetilde{C}^\omega(Y, U_1) \otimes \widetilde{C}\left\langle -\tfrac{1}{2}\sigma_\omega(Y, K)\right\rangle. \tag{148}$$

Now, if R is an integral domain, (148) implies that the following identity holds for the Frøyshov invariant $h^\omega(Y, K) := h(\widetilde{C}^\omega(Y, K))$ of the $\mathcal{S}$-complex $\widetilde{C}^\omega(Y, K)$:

$$h^\omega(Y, K) := h^\omega(Y, U_1) - \tfrac{1}{2}\sigma_\omega(Y, K).$$

Following the same argument as in the proof of [Daemi and Scaduto 2024a, Proposition 5.3], we can see that $h^\omega(Y, U_1) = 4h(Y)$ if $2 \in R$ is nonzero. Here $h(Y)$ is the Frøyshov invariant [2002] of the integral homology 3-sphere Y defined using a coefficient ring R. In summary, we have the following:

Proposition 7.9 *Let R be an integral domain with $2 \in R$ nonzero. Let K be a null-homotopic knot in an integer homology sphere Y such that $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of K. Then*

$$h^\omega(Y, K) = 4h(Y) - \tfrac{1}{2}\sigma_\omega(Y, K).$$

7.2 Filtered special cycles for general holonomy parameters

In the previous section, we learned that $\mathcal{S}$-complexes of knots in the case $\omega \in \left(0, \frac{1}{4}\right)$, after forgetting the I-gradings, do not contain any interesting information. This is in contrast to the case $\omega = \frac{1}{4}$, where we have obtained nontrivial topological information from the $\mathcal{S}$-complex of a knot K, such as the invariant $\tilde{s}(K)$. Nevertheless, the I-grading of $\mathfrak{C}^\omega(Y, K)$ is expected to say more about the topological type of K. This I-grading was already used in [Imori 2024] to study concordances between torus knots. In this section, we initiate the study of some of the concordance invariants that one can obtain by applying the filtered constructions of the previous sections.

For any ω, our concordance invariant in a raw form is the local equivalence class of the enriched complex $\mathfrak{E}^\omega(Y,K)$. For any ring R, we define $\Theta_R^{\mathfrak{E},\omega}$ to be the set of local equivalence classes of enriched $\mathcal{S}$-complexes over the ring $R[T^{-1},T]][U^{\pm 1}]$ with holonomy parameter ω. As in the previous instances, tensor product gives an additive structure on this set and we would like to define a homomorphism

$$\Omega^{\mathfrak{E},\omega}\colon \Theta_{\mathbb{Z}}^{3,1}\to\Theta_R^{\mathfrak{E},\omega}$$

by mapping $[(Y,K)]$ to the local equivalence class of $\mathfrak{E}^\omega(Y,K)$. There is an issue with this definition because $e^{4\pi i\omega}$ might be a root of Alexander polynomial of K, and hence $\mathfrak{E}^\omega(Y,K)$ might not be well defined. One obvious fix for this issue is to limit to the values of ω such that $e^{4\pi i\omega}$ is not the root of Alexander polynomial of any K in an integer homology sphere. For instance, we can pick ω to be any rational number if the form $m/4p^k$, where p is a prime number. For any such ω, $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of any knot (See [Livingston 2002, Corollary 3.2], for example.)

The construction of filtered special cycles in Section 5 can be adapted to enriched complexes of knots with general holonomy parameters. If $\widetilde{C}^\omega$ is an $\mathcal{S}$-complex over the ring $R[T^{-1},T]][U^{\pm 1}]$ with parameter ω, then we can form the equivariant complexes and the maps between them as in Section 2.2. For any $f\in R$, $s\in\mathbb{R}\cup\{-\infty\}$, $k\in\mathbb{Z}$, we define a filtered special (k,f,s)-cycle to be an element $z\in\widehat{\boldsymbol{C}}_{2k}^\omega$ such that $z=\widehat{\Psi}(\mathfrak{z})$ for some $\mathfrak{z}\in\widehat{\mathfrak{C}}_{2k}^\omega$ and

$$\mathfrak{i}(\mathfrak{z})=\sum_{i=-\infty}^{-k} b_i x^i,\quad \deg_I(\hat{\mathfrak{d}}\mathfrak{z})\le s,$$

where $b_{-k}=f+\sum_{i=1}^\infty r_iT^i\in R[\![T]\!]$ with $r_i\in R$. In particular, if $f\neq 0$, then $\deg_I(b_{-k})=0$. Using filtered special cycles, we have the following counterparts of Definitions 5.21 and 5.30:

$$\mathcal{N}_{\widetilde{C}^\omega}(k,s):=\inf\{\deg_I(z)\mid z \text{ is a filtered special } (k,1,s)\text{-cycle}\}\in[0,\infty],$$
$$\underline{\mathcal{N}}_{\widetilde{C}^\omega}(k,s):=\inf\{\deg_I(z)\mid z \text{ is a filtered special } (k,f,s)\text{-cycle with } f\neq 0\}\in[0,\infty].$$

Property (142) implies that the above definition extends to enriched complexes of holonomy parameter ω by a limiting process as in Section 5.2. For an enriched $\mathcal{S}$-complex $\mathfrak{E}^\omega$, the transpose $\mathcal{N}_{\mathfrak{E}^\omega}^{\mathsf{T}}$ of $\mathcal{N}_{\mathfrak{E}^\omega}$ is defined as in (99). The value of $\mathcal{N}_{\mathfrak{E}^\omega}(k,s)$ belongs to $\mathfrak{K}_{2k-1}$, and similarly for $\underline{\mathcal{N}}_{\mathfrak{E}^\omega}$ and $\mathcal{N}_{\mathfrak{E}^\omega}^{\mathsf{T}}$. As a counterpart of Lemma 5.24, $\mathcal{N}_{\mathfrak{E}^\omega}(k,s)$ and $\underline{\mathcal{N}}_{\mathfrak{E}^\omega}(k,s)$ are increasing with respect to k and decreasing with respect to s. Thus, $\underline{\mathcal{N}}_{\mathfrak{E}^\omega}(k,r)$ is increasing with respect to k and decreasing with respect to r. Define

$$\Gamma_{\mathfrak{E}^\omega}(k):=\underline{\mathcal{N}}_{\mathfrak{E}^\omega}(k,-\infty),\quad r_s(\mathfrak{E}^\omega):=-\mathcal{N}_{\mathfrak{E}^\omega}^{\mathsf{T}}(0,-s).$$

Applying these to the enriched complex $\mathfrak{E}^\omega(Y,K)$ of a knot K gives the topological invariants $\mathcal{N}_{(Y,K)}^\omega$, $\underline{\mathcal{N}}_{(Y,K)}^\omega$, $\mathcal{N}_{(Y,K)}^{\omega,\mathsf{T}}$, $\Gamma_{(Y,K)}^\omega$ and $r_s^\omega(Y,K)$.

The connected sum theorem of [Imori 2024] gives rise to the analogues of the connected sum inequalities in Theorem 5.45. For the pairs (Y,K) and (Y',K'), let k,k' be integers and s,s' be negative real numbers such that $s^\otimes:=\max\{\mathcal{N}_{(Y,K)}^\omega(k,s)+s',\mathcal{N}_{(Y',K')}^\omega(k',s')+s\}$ is negative. Then

$$\mathcal{N}_{(Y\#Y',K\#K')}^\omega(k+k',s^\otimes)\le\mathcal{N}_{(Y,K)}^\omega(k,s)+\mathcal{N}_{(Y',K')}^\omega(k',s').$$

A similar inequality holds for $\underline{\mathcal{N}}^{\omega}_{(Y,K)}$. If r and r' are positive real numbers, then

$$\mathcal{N}^{\omega,\mathsf{T}}_{(Y\#Y',K\#K')}(k+k',r+r') \le \max\{\mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}(k,r)+r', \mathcal{N}^{\omega,\mathsf{T}}_{(Y,'K')}(k',r')+r\}.$$

The following result generalizes Theorems 5.47 and 5.50 to other holonomy parameters:

Theorem 7.10 *Let (Y,K) and (Y',K') be given such that $e^{4\pi i\omega}$ is not a root of the Alexander polynomials of these knots. Let $(W,S)\colon (Y,K)\to(Y',K')$ be a cobordism which is negative definite of strong height $i\ge 0$ with respect to the holonomy parameter ω. Then*

$$\mathcal{N}^{\omega}_{(Y',K')}(k+i,s) \le \mathcal{N}^{\omega}_{(Y,K)}(k,s-2\kappa^{\omega}_{\min}(W,S))+2\kappa^{\omega}_{\min}(W,S), \tag{149}$$

$$\mathcal{N}^{\omega,\mathsf{T}}_{(Y',K')}(k+i,r+2\kappa^{\omega}_{\min}(W,S)) \le \mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}(k,r)+2\kappa^{\omega}_{\min}(W,S). \tag{150}$$

Moreover, if equality is achieved in (149) (resp. (150)) for some $k\in\mathbb{Z}$ and $s\in[-\infty,0)$ (resp. $r\in[0,\infty]$), with both sides finite and positive (resp. negative), then there exists an irreducible $\mathrm{SU}(2)$-representation of $\pi_1(W\setminus S)$ that maps a meridian of S to (134). The same conclusion holds for $\underline{\mathcal{N}}^{\omega}_{(Y,K)}$, under the weaker assumption that the cobordism is not necessarily strong, but satisfies $\deg_I(\eta^{\omega}(W,S))=0$.

Example 7.11 The roots of the Alexander polynomial of the trefoil $T_{2,3}$ are $e^{\pi i/3}$ and $e^{5\pi i/3}$, and $\sigma_\omega(T_{2,3})$ is equal to -2 for $\frac{1}{12}<\omega<\frac{5}{12}$ and is equal to 0 for $0<\omega<\frac{1}{12}$ or $\frac{5}{12}<\omega<\frac{1}{2}$. This computation of the Tristram–Levine signature is reflected in the fact that the moduli space of flat connections $\mathfrak{C}(T_{2,3},\omega)$ for $\frac{1}{12}<\omega<\frac{5}{12}$ has a unique nondegenerate irreducible α_ω and a nondegenerate reducible. For any such ω that has the form of a rational number $m/2p$ with $p=6n-1$, we have

$$\mathrm{cs}(\widetilde{\alpha}_\omega) \in \left\{\frac{(5p+6m)^2}{12p^2}+\frac{k}{p}\right\}_{k\in\mathbb{Z}}, \tag{151}$$

where $\widetilde{\alpha}_\omega$ is a lift of α_ω. To see this, note that α_ω can be lifted to a flat connection $\widehat{\alpha}_\omega$ on the Brieskorn homology sphere $\Sigma(2,3,p)$. Following the arguments in [Collin and Saveliev 1999; Imori 2024], one first determines the flat connection $\widehat{\alpha}_\omega$ on $\Sigma(2,3,p)$ and then uses [Fintushel and Stern 1990] to show that the (ordinary) Chern–Simons functional of $\widehat{\alpha}_\omega$ is equal to $(5p+6m)^2/24p\in\mathbb{R}/\mathbb{Z}$. Now (151) follows from this computation.

For the rest of this example, assume that the above value of ω is less than $\frac{1}{4}$. Since α_ω is nondegenerate, we may form the enriched complex $\mathfrak{E}^{\omega}(T_{2,3})$ of $T_{2,3}$ using a sequence of perturbations of the Chern–Simons functional that are trivial in a neighborhood of the elements of $\mathfrak{C}(T_{2,3},\omega)$. In particular, $\Gamma^{\omega}_{T_{2,3}}(1)$ is a positive real number in the set in (151). Moreover, applying Example 7.5 to the crossing change cobordism from the unknot to the trefoil and then applying Theorem 7.10 implies that

$$\Gamma^{\omega}_{T_{2,3}}(1) \le \frac{2m^2}{p^2}.$$

In particular, if $2m^2<p$, then the above two constraints uniquely determine $\Gamma^{\omega}_{T_{2,3}}(1)$. We can use this to show the identities

$$\Gamma^{1/10}_{T_{2,3}}(1)=\tfrac{1}{300}, \quad \Gamma^{1/11}_{T_{2,3}}(1)=\tfrac{1}{1452}.$$

In general, we expect that

$$\Gamma^{\omega}_{T_{2,3}}(1) = 12\big(\omega - \tfrac{1}{12}\big)^2$$

for $\omega \in \big(\frac{1}{12}, \frac{1}{4}\big]$.

Corollary 7.12 *Suppose K is a knot in integer homology sphere Y such that $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of K and the weak local equivalence class of $\mathfrak{E}^{\omega}(Y,K)$ is nontrivial. Then, for any homology concordance $(W,S)\colon (Y,K) \to (Y',K')$, there is a representation of $\pi_1(W \setminus S)$ into* SU(2) *that restricts to irreducible representations of $Y \setminus K$ and $Y' \setminus K'$, and the conjugacy class of a meridian of S is mapped to the conjugacy class of* (134) *in* SU(2).

This corollary follows from Theorem 7.10. A homology concordance (W,S) determines a negative definite cobordism of strong height 0 with $\kappa^{\omega}_{\min}(W,S) = 0$ and $\eta^{\omega}(W,S) = 1$. The cobordism from (Y',K') to (Y,K) obtained by flipping (W,S) satisfies a similar property. Thus (149) and (150) imply

$$\mathcal{N}^{\omega}_{(Y,K)}(k,s) = \mathcal{N}^{\omega}_{(Y',K')}(k,s), \quad \mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}(k,r) = \mathcal{N}^{\omega,\mathsf{T}}_{(Y',K')}(k,r)$$

for all values of k, s and r. In particular, we obtain the claim in Corollary 7.12 if we can show that either $\mathcal{N}^{\omega}_{(Y,K)}$ or $\mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}$ is finite and nonzero. Arguing as in Remark 5.44, the value of $\mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}(k,s)$ is nonzero for $k \le 0$. Since $\mathfrak{E}^{\omega}(Y,K)$ is not weakly locally equivalent to the trivial enriched complex, the following generalization of Corollary 5.43 shows that $\mathcal{N}^{\omega,\mathsf{T}}_{(Y,K)}(0,0)$ is finite:

Proposition 7.13 *An enriched $\mathcal{S}$-complex $\mathfrak{E}^{\omega}$ with holonomy parameter ω is weakly locally equivalent to the trivial enriched $\mathcal{S}$-complex if and only if*

$$\mathcal{N}_{\mathfrak{E}^{\omega}}(0,-\infty) = 0 \quad \text{and} \quad \mathcal{N}_{(\mathfrak{E}^{\omega})^{\dagger}}(0,-\infty) = 0. \tag{152}$$

These conditions are equivalent to $r_0(\mathfrak{E}^{\omega}) = \infty$ and $r_0((\mathfrak{E}^{\omega})^{\dagger}) = \infty$.

Corollary 7.14 *Suppose K is a knot in an integer homology sphere Y such that $e^{4\pi i\omega}$ is not a root of the Alexander polynomial of K and it satisfies one of the following assumptions*:

(i) $4h(Y) + \sigma_{\omega}(Y,K) \neq 0$.

(ii) $\mathcal{N}^{\omega}_{(Y,K)} \neq \mathcal{N}^{\omega}_{(S^3,U_1)}$.

(iii) $\underline{\mathcal{N}}^{\omega}_{(Y,K)} \neq \underline{\mathcal{N}}^{\omega}_{(S^3,U_1)}$.

Then the same claim as in Corollary 7.12 holds.

Proof All of the invariants $\mathcal{N}^{\omega}_{(Y,K)}$, $\underline{\mathcal{N}}^{\omega}_{(Y,K)}$ and $4h(Y) + \sigma_{\omega}(Y,K)$ depend on the weak local equivalence class of (Y,K). So, if any one of them is nontrivial, then the weak local equivalence class of $\mathfrak{E}^{\omega}(Y,K)$ is nontrivial, too. Now the claim follows from Corollary 7.12. □

Proof of Theorem 1.16 This follows from Corollary 7.14(i) applied to knots in the 3-sphere. □

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