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The chromatic colocalization of the spherical units is of height 1

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We show that for a large class of connective spectra, including the connective covers of L_n^f-local spectra for every integer n, the mapping space into the units of the p-complete sphere spectrum is insensitive to L_1-localization followed by taking connective cover.

55P42, 55Q50

1 Introduction

1.1 Background and motivation

Let $\mathbb{S}$ be the sphere spectrum and O be the space of infinite orthogonal matrices, that is, the colimit of the Lie groups O_n for $n \to \infty$. The classical J-homomorphism, introduced by Whitehead [1942], is a map $\pi_* O \to \pi_* \mathbb{S}$ constructed from the linear actions of orthogonal matrices on spheres. The J-homomorphism is almost a map of graded groups, except for π_0, where it is the map $\mathbb{Z}/2 \to \mathbb{Z}$ sending 1 to -1. This deficient behavior on π_0 may be explained by the fact that the J-homomorphism comes from a map of spectra $O \to \mathbb{S}^\times$, where $\mathbb{S}^\times$ is the connective spectrum of *units*[1] in the sphere spectrum, and where O is regarded as a connective spectrum via the operation of orthogonal sum of matrices. While the higher homotopy groups of $\mathbb{S}^\times$ are the same as that of $\mathbb{S}$, the lowest one is $\mathbb{Z}/2$, generated by -1 under multiplication, which is now compatible with the multiplication of $\pi_0 O \simeq \mathbb{Z}/2$ via the J-homomorphism.

[1] The spectrum of units of a commutative ring spectrum R is usually denoted by $\mathfrak{gl}_1(R)$ in the literature.

Starting with the work of Adams [1963; 1965a; 1965b], the image of the J-homomorphism has been extensively studied [Mahowald 1970; Bruner and Rognes 2022]. It constitutes the piece of the stable homotopy groups of spheres detected by the homotopy of the L_1-local sphere; see eg [Lurie 2010, Lecture 35, Theorem 7]. The fact that the image of J constitute classes of height 1 is dictated by the chromatic nature of the spectrum O: it is the connective cover of the L_1-local spectrum $\Sigma^{-1}KO$. As in the discussion in [Clausen 2011, Section 6.5], one could hope to similarly hit elements of higher chromatic height via higher-height analogs of the J-homomorphism. Focusing on the p-complete version of the sphere spectrum, one could try to map into $\mathbb{S}_p^\times$ the connective cover of Lubin–Tate theory E_n, or the K-theory spectrum of a ring spectrum of chromatic height $n \geq 1$, which by the chromatic redshift principle of Ausoni and Rognes detects height $n+1$ information; see [Rognes 2014; Land et al. 2024; Clausen et al. 2024; Burklund et al. 2022] for more precise formulations.

In this paper we show that there is no higher-height analog of the J-homomorphism. More precisely, under suitable assumptions on the chromatic behavior of the spectrum X, a map $J'\colon X \to \mathbb{S}_p^\times$ factors canonically through the connective cover of the L_1-localization $\tau_{\geq 0} L_1 X$ (and in fact, up to π_0, through the connective cover of its $K(1)$-localization). Hence, informally all the information obtained from "higher J-homomorphisms" on the stable homotopy groups of spheres is already seen at height 1.

1.2 Main result

For $n \geq 0$, let $T(n)$ be the telescope of a v_n-self map of a finite spectrum of type n, and set $T(\infty) := \mathbb{F}_p$. Let $\mathrm{T}[0,\infty] \subseteq \mathrm{Sp}$ be the full subcategory generated under colimits and desuspensions from the $T(n)$ for $n = 0, 1 \dots, \infty$.

Theorem A *Let $X \in \mathrm{T}[0,\infty]$ be a connective spectrum. Then,*

$$\mathrm{Map}(X, \mathbb{S}_p^\times) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\pi_0 X, \mathbb{F}_p^\times) \times \mathrm{Map}(\tau_{\geq 0} L_{K(1)} X, (\mathbb{S}_p^\times)_p^\wedge).$$

Remark 1.1 This result can be interpreted as a statement about the colocalization of $\mathbb{S}_p^\times$. Namely, let $\mathrm{Sp}^{\mathrm{cn}}$ be the ∞-category of connective (or (-1)-connected) spectra. The inclusion $\mathrm{T}[0,\infty] \cap \mathrm{Sp}^{\mathrm{cn}} \hookrightarrow \mathrm{Sp}^{\mathrm{cn}}$ admits a right adjoint

$$R\colon \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{T}[0,\infty] \cap \mathrm{Sp}^{\mathrm{cn}},$$

by the adjoint functor theorem. Comparing mapping spectra from a test connective spectrum, it follows from Theorem A that

$$R(\mathbb{S}_p^\times) \simeq \tau_{\geq 0} \hom(\tau_{\geq 0} L_1^f \mathbb{S}, \mathbb{S}_p^\times) \in \mathrm{T}[0,\infty] \cap \mathrm{Sp}^{\mathrm{cn}}.$$

Remark 1.2 The result in Theorem A could be improved. Conditionally on work in progress with Burklund and Ramzi, one can lift the (p-complete version of) the result from the units $\mathbb{S}_p^\times$ to the Picard spectrum $\mathrm{pic}(\mathbb{S}_p)$, which is arguably a more natural recipient for the J-homomorphism. We shall elaborate on this fact and its generalizations in the mentioned work in progress.

Remark 1.3 Miller and Priddy [1978] provide a conjectural description of the spectrum $\mathbb{S}^\times$, which in turn gives a description of the p-complete version $\mathbb{S}_p^\times$. In a private communication, Rezk explained to the author a reformulation of the p-complete version in terms of a pullback square

$$\begin{array}{ccc} (\mathbb{S}_p^\times)_p^\wedge & \longrightarrow & \mathbb{S}_p \\ \downarrow & & \downarrow \\ L_{K(1)}\mathbb{S} & \longrightarrow & L_{K(1)}\mathbb{S} \end{array}$$

where the bottom map is multiplication by the element $\eta \cdot \zeta \in \pi_0 L_{K(1)}\mathbb{S}$ (which vanishes at odd primes) and the upper map is a hypothetical spherical logarithm map.

Using the fact that $\mathrm{Map}(X, \mathbb{S}_p) = 0$ for every $X \in \mathrm{T}[0,\infty]$, one can easily show that the Miller–Priddy conjecture implies Theorem A. Thus, the main result of this paper can be seen as an evidence for their conjecture.

We conclude the discussion of our main result by pointing out that the class of spectra $\mathrm{T}[0,\infty]$ include several families of spectra of interest. First, it contains all the connective covers of L_n^f-local spectra, such as Lubin–Tate theories and the $K(n)$-local spheres. By the validity of the Quillen–Lichtenbaum conjectures for the integers [Kahn 1997; Rognes and Weibel 2000; Voevodsky 2011; Waldhausen 1984], $\mathrm{T}[0,\infty]$ contains also all the K-theory spectra of $\mathbb{Z}$-algebras; see Proposition 7.2. More generally, a higher-height generalization of the Quillen–Lichtenbaum conjecture, established by Hahn and Wilson [2022] for truncated Brown–Peterson spectra, shows that $\mathrm{T}[0,\infty]$ contains the K-theory of arbitrary $\mathrm{BP}\langle n\rangle$-algebras, where $\mathrm{BP}\langle n\rangle$ stands for an $\mathbb{E}_3$-form of the truncated Brown–Peterson spectra as in [Hahn and Wilson 2022]; see Proposition 7.3. One could optimistically hope that these examples are instances of a general phenomenon, closely related to chromatic redshift.

Conjecture 1.4 *Let R be an $\mathbb{E}_1$-ring spectrum. If $R \in \mathrm{T}[0,\infty]$ then $K(R) \in \mathrm{T}[0,\infty]$.*

Thus, if the conjecture holds, then our main theorem applies to the K-theory spectra of arbitrary $\mathbb{E}_1$-rings in $\mathrm{T}[0,\infty]$.

1.3 Outline of the proof

First, using the decomposition $\mathbb{S}_p^\times \simeq \mathbb{F}_p^\times \oplus (\mathbb{S}_p^\times)_p^\wedge$, the statement is equivalent to the isomorphism

$$(\star) \qquad \mathrm{Map}(X, (\mathbb{S}_p^\times)_p^\wedge) \simeq \mathrm{Map}(\tau_{\geq 0} L_{K(1)} X, (\mathbb{S}_p^\times)_p^\wedge).$$

The key features of $(\mathbb{S}_p^\times)_p^\wedge$ that come into the proof of this isomorphism are the following two:

(1) There are no maps from $\mathbb{Z}$ to its suspension:

$$\mathrm{Map}(\mathbb{Z}, \Sigma(\mathbb{S}_p^\times)_p^\wedge) \simeq \mathrm{pt}.$$

(2) There are no maps from the reduced suspension spectrum of B^2C_p to it. Equivalently, the pullback map
$$\pi^*\colon (\mathbb{S}_p^\times)_p^\wedge \to ((\mathbb{S}_p^\times)_p^\wedge)^{B^2C_p}$$
along the terminal map $\pi\colon B^2C_p \to \mathrm{pt}$ is an isomorphism.

The first property is an immediate consequence of the main result of [Carmeli 2023]. The second follows from a result of Lee [1992], giving the vanishing of the stable cohomotopy of the spaces B^mC_p for $m \geq 2$. Interestingly, these two properties both follow, in one way or another, from the Segal conjecture; see [Carlsson 1984; Lin 1980; Adams et al. 1985].

We proceed by showing that every p-complete connective spectrum satisfying (1) and (2) supports an isomorphism analogous to ($\star$). This is done in two steps. First, we show that condition (1) allows us to extrapolate uniquely $(\mathbb{S}_p^\times)_p^\wedge$ to negative homotopy degrees, obtaining a spectrum $\mathfrak{G}$ which receives no nonzero maps from any desuspension of $\mathbb{Z}$; see Proposition 3.3. Using condition (2), we then show that $\mathfrak{G}$ does not receive nonzero maps from any desuspension of $\Sigma^\infty B^2C_p$ as well.

The second step is to show that $\mathfrak{G}$, and in fact every p-complete spectrum which receives no maps from any desuspension of $\Sigma^\infty B^2C_p$, satisfies a stable version of the main result: for $X \in \mathrm{T}[0,\infty]$ we have
$$\mathrm{Map}(X,\mathfrak{G}) \simeq \mathrm{Map}(L_1X,\mathfrak{G}).$$
This is essentially a generation problem: we have to show that the localizing ideal $\langle \Sigma^\infty B^2C_p\rangle$ generated by $\Sigma^\infty B^2C_p$ contains the kernel of the functor $L_1\colon \mathrm{T}[0,\infty] \to L_1^f\mathrm{Sp}$.

More generally, we show in Proposition 5.3 that for all $n \geq 1$, the ideal $\langle \Sigma^\infty B^nC_p\rangle$ contains the kernel of the localization $L_{n-1}^f\colon \mathrm{T}[0,\infty] \to L_{n-1}^f\mathrm{Sp}$. As this kernel is generated by the spectra $T(m)$ for $m \geq n$ (Proposition 4.5), this reduces to the fact that $T(m) \in \langle \Sigma^\infty B^nC_p\rangle$ for $m \geq n$. We then reduce further to the case $m = n$, which we prove using the theory of chromatic cyclotomic extensions of Schlank, Yanovski and the author in [Carmeli et al. 2024]. Namely, the height-n cyclotomic extension of the $T(n)$-local sphere is a commutative $\mathbb{S}_{T(n)}$-algebra whose underlying spectrum is the $T(n)$-localization of $\mathbb{S}_{T(n)} \otimes \Sigma^\infty B^nC_p$. The fact that this extension is a finite $T(n)$-local Galois extension of $\mathbb{S}_{T(n)}$ allows us to write $T(n)$ as a retract of $T(n) \otimes \Sigma^\infty B^nC_p$, and hence show that $T(n) \in \langle \Sigma^\infty B^nC_p\rangle$.

Remark 1.5 Lee's theorem on the vanishing of the cohomotopy of B^2C_p can be interpreted as a statement about the *semiadditive height* of the p-complete sphere. Indeed, Yuan [2024] constructed a commutative ring spectrum A in the stable, 1-semiadditive ∞-category $\mathfrak{Z}_1$ of semiadditive height 1 from [Carmeli et al. 2021], whose underlying commutative ring spectrum is the p-complete sphere spectrum. The existence of A formally implies Lee's theorem. The fact that the vanishing of the reduced mapping spectrum from B^2C_p can be suitably interpreted as being "of height 1" is the main idea behind our proof of the main theorem.

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2 Ideals in symmetric monoidal stable categories

We recall the definition and basic properties of localizing ideals in symmetric monoidal stable ∞-categories. We warn the reader that we work in the presentable settings, which are different from various contexts for studying localizing ideals in the literature.

Definition 2.1 Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$. A *localizing ideal* in $\mathscr{C}$ is a full presentable subcategory $I \subseteq \mathscr{C}$ closed under colimits and tensoring with arbitrary objects of $\mathscr{C}$. We shall indicate that I is a localizing ideal by the notation $I \trianglelefteq \mathscr{C}$. The quotient $\mathscr{C}/I$ is the cofiber of the map $I \hookrightarrow \mathscr{C}$ in $\mathrm{Pr}_{\mathrm{st}}$.

A simple, yet useful consequence of being a localizing ideal is the following:

Proposition 2.2 *Let $I \trianglelefteq \mathscr{C}$ be a localizing ideal, and let $R \in I$ be an $\mathbb{E}_1$-ring in $\mathscr{C}$. Then, every R-module belongs to I.*

Proof The ∞-category $\mathrm{Mod}_R(\mathscr{C})$ of R-modules is generated under colimits, desuspensions, and tensoring with objects of $\mathscr{C}$ by the single object R. Since $R \in I$, and I is closed under these three operations, we deduce that $\mathrm{Mod}_R(\mathscr{C}) \subseteq I$. □

For a set of objects $S \subseteq \mathscr{C}$, we denote by $\langle S \rangle$ the localizing ideal generated by S, ie the full subcategory of $\mathscr{C}$ generated from S under colimits and tensoring with objects of $\mathscr{C}$.

Example 2.3 Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$. For $E \in \mathscr{C}$, the collection

$$\mathrm{Ann}(E) := \{X \in \mathscr{C} : X \otimes E = 0\}$$

is a localizing ideal, also known as the *Bousfield class* of E. The quotient $\mathscr{C}/\mathrm{Ann}(E)$ is equivalent to the Bousfield localization of $\mathscr{C}$ with respect to E and denoted by $\mathscr{C}_E$.

Warning 2.4 In various sources the notation $\langle E\rangle$ is reserved for the Bousfield class of E. Since Bousfield classes will play no role in this work, we hope that this notational conflict will not lead to a confusion.

For $I \trianglelefteq \mathscr{C}$, the quotient functor $\mathscr{C} \to \mathscr{C}/I$ is a symmetric monoidal localization. The right adjoint of this localization is a fully faithful embedding $\mathscr{C}/I \hookrightarrow \mathscr{C}$, whose essential image consists of those $Y \in \mathscr{C}$ for which $\mathrm{Map}(X, Y) = \mathrm{pt}$ for all $X \in I$. We denote the composition of the quotient functor $\mathscr{C} \to \mathscr{C}/I$ and its right adjoint $\mathscr{C}/I \hookrightarrow \mathscr{C}$ by

$$L_I : \mathscr{C} \to \mathscr{C},$$

and its image by $L_I\mathscr{C} \subseteq \mathscr{C}$. Thus, the composition

$$L_I\mathscr{C} \hookrightarrow \mathscr{C} \to \mathscr{C}/I$$

is an equivalence.

Warning 2.5 The Bousfield localization L_E and our localization $L_{\langle E\rangle}$ are different: the kernel of the first is the annihilator $\mathrm{Ann}(E)$, and the other is the localizing ideal $\langle E\rangle$ generated by it.

Ideals in symmetric monoidal stable ∞-categories behave much like ordinary ideals of commutative rings. For example, if $I, J \trianglelefteq \mathscr{C}$, their intersection $I \cap J$ is again a localizing ideal of $\mathscr{C}$. Similarly, we define the sum $I + J$ to be the localizing ideal generated from I and J under colimits and tensoring with objects of $\mathscr{C}$. The product $I \cdot J$ is the localizing ideal generated by the objects $X \otimes Y$ for $X \in I$ and $Y \in J$.

Proposition 2.6 *Let $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$, let $M \in \mathscr{C}$ and let $x : \Sigma^k M \to M$ be a self-map of some degree k. Then*

$$\langle M/x\rangle + \langle x^{-1}M\rangle = \langle M\rangle.$$

Proof On the one hand, M/x and $x^{-1}M$ are colimits of copies of M and therefore belong to $\langle M\rangle$. On the other hand, the cofiber of the map $M \to x^{-1}M$ can be written as a colimit of the form

$$\mathrm{Cofib}(M \to x^{-1}M) \simeq \varinjlim_{\ell\to\infty} (\Sigma^{-\ell k} M/x^{\ell}).$$

Since each of the objects $\Sigma^{-\ell k} M/x^{\ell}$ belong to the thick subcategory generated by M/x, we deduce that $M \in \langle x^{-1}M\rangle + \langle M/x\rangle$, and the result follows. □

3 Ideals of discrete spectra

We now consider ideals of spectra concentrated in degree 0. The two cases of interest are $\langle\mathbb{Z}\rangle$ and $\langle\mathbb{F}_p\rangle$. Note that by Proposition 2.6 we have

(1) $$\langle\mathbb{Z}\rangle = \left\langle\mathbb{Z}\left[\frac{1}{p}\right]\right\rangle + \langle\mathbb{F}_p\rangle.$$

Proposition 3.1 *The ideal $\langle\mathbb{Z}\rangle \subseteq \mathrm{Sp}$ contains all the bounded above spectra, while the ideal $\langle\mathbb{F}_p\rangle$ contains all bounded above spectra with homotopy groups which are p-power torsion.*

Proof Using the Whitehead tower, we can present a bounded above spectrum as a filtered colimit of bounded spectra. Then, every bounded spectrum is a finite extension of Eilenberg–Mac Lane spectra, which admit $\mathbb{Z}$-module structures and therefore belong to $\langle \mathbb{Z} \rangle$; see Proposition 2.2. If all the homotopy groups are p-power torsion, then the Eilenberg–Mac Lane spectra that appear in this process are all built from extensions out of $\mathbb{F}_p$-modules so we further get $X \in \langle \mathbb{F}_p \rangle$. $\square$

Proposition 3.1 above shows that $L_{\langle \mathbb{Z} \rangle}$ depends only on arbitrary connected covers.

Proposition 3.2 *Let X be a spectrum. The m^{th} connective cover map $\tau_{\geq m} X \to X$ induces an isomorphism*

$$L_{\langle \mathbb{Z} \rangle}(\tau_{\geq m} X) \xrightarrow{\sim} L_{\langle \mathbb{Z} \rangle}(X).$$

If $\tau_{\leq m-1} X$ has p-power torsion homotopy groups, then we also have

$$L_{\langle \mathbb{F}_p \rangle}(\tau_{\geq m} X) \xrightarrow{\sim} L_{\langle \mathbb{F}_p \rangle}(X).$$

Proof The fiber of the map $\tau_{\geq m} X \to X$ is bounded above and hence belongs to the kernel of $L_{\langle \mathbb{Z} \rangle}$ by the first part of Proposition 3.1. If $\tau_{\leq m-1} X$ has only p-power torsion homotopy groups, then this fiber is in the kernel of $L_{\langle \mathbb{F}_p \rangle}$ as well by the second part of Proposition 3.1. $\square$

Let $\mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$ be the full subcategory spanned by the connective spectra. From the above, we deduce that $L_{\langle \mathbb{Z} \rangle}\mathrm{Sp}$ exhibits "connective behavior" in the following sense.

Proposition 3.3 *The connective cover functor $\tau_{\geq 0} \colon L_{\langle \mathbb{Z} \rangle}\mathrm{Sp} \to \mathrm{Sp}^{\mathrm{cn}}$ is a fully faithful right adjoint, with left adjoint given by the restriction of $L_{\langle \mathbb{Z} \rangle}$ to connective spectra.*

Proof The fact that it is a right adjoint with the claimed left adjoint follows by composing the adjunctions

$$L_{\langle \mathbb{Z} \rangle}\mathrm{Sp} \simeq \mathrm{Sp}/\langle \mathbb{Z} \rangle \leftrightarrows \mathrm{Sp} \leftrightarrows \mathrm{Sp}^{\mathrm{cn}}.$$

To show that $\tau_{\geq 0}$ is fully faithful on $L_{\langle \mathbb{Z} \rangle}\mathrm{Sp}$, note that for $X, Y \in L_{\langle \mathbb{Z} \rangle}\mathrm{Sp}$, by Proposition 3.2,

$$\mathrm{Map}(X, Y) \simeq \mathrm{Map}(L_{\langle \mathbb{Z} \rangle}\tau_{\geq 0} X, Y) \simeq \mathrm{Map}(\tau_{\geq 0} X, Y) \simeq \mathrm{Map}(\tau_{\geq 0} X, \tau_{\geq 0} Y). \quad \square$$

As a consequence of the proposition above, the essential image of the functor $\tau_{\geq 0} L_{\langle \mathbb{Z} \rangle} \colon \mathrm{Sp} \hookrightarrow \mathrm{Sp}^{\mathrm{cn}}$ may be described as those connective spectra X for which $X \xrightarrow{\sim} \tau_{\geq 0} L_{\langle \mathbb{Z} \rangle} X$. There is a more explicit description of these connective spectra:

Proposition 3.4 *The following conditions for a connective spectrum X are equivalent.*

1. $X \xrightarrow{\sim} \tau_{\geq 0} L_{\langle \mathbb{Z} \rangle} X$.
2. $\tau_{\geq -1} \hom(\mathbb{Z}, X) = 0$.
3. *There is a connective spectrum Y such that* $\mathrm{Map}(\mathbb{Z}, Y) = \mathrm{pt}$ *and* $\tau_{\geq 0}\Sigma^{-1} Y \simeq X$.

Proof (2) $\iff$ (3) If $\tau_{\geq -1}\hom(\mathbb{Z}, X) = 0$, we can choose $Y = \Sigma X$. Conversely, if there is such a Y then we have a fiber sequence

$$\Sigma^{-2}\pi_0 Y \to X \to \Sigma^{-1} Y.$$

Applying $\hom(\mathbb{Z}, -)$ we obtain a fiber sequence

$$\Sigma^{-2}\hom(\mathbb{Z}, \pi_0 Y) \to \hom(\mathbb{Z}, X) \to \Sigma^{-1}\hom(\mathbb{Z}, Y).$$

Now, $\tau_{\geq -1}\Sigma^{-1}\hom(\mathbb{Z}, Y) \simeq 0$ by our assumption on Y, and $\tau_{\geq -1}\Sigma^{-2}\hom(\mathbb{Z}, \pi_0 Y) = 0$ since $\pi_0 Y$ is discrete. Together, these imply that $\tau_{\geq -1}\hom(\mathbb{Z}, X) = 0$ as well.

(1) $\implies$ (3) If $X \simeq \tau_{\geq 0} L_{\langle \mathbb{Z}\rangle} X$ then we can choose $Y = \tau_{\geq 0}\Sigma L_{\langle \mathbb{Z}\rangle} X$. Indeed, this spectrum has a contractible space of maps from $\mathbb{Z}$ since its the connective cover of an object of $L_{\langle \mathbb{Z}\rangle}\mathrm{Sp}$.

(2) $\implies$ (1) Define a sequence of spectra iteratively as follows: Set $X_0 = X$ and X_{k+1} to be the cofiber

$$\hom(\mathbb{Z}, X_k) \xrightarrow{\mathrm{ev}} X_k \to X_{k+1}.$$

We shall show inductively that $\tau_{\geq -k-1}\hom(\mathbb{Z}, X_k) = 0$. Indeed, let C be the cofiber of the left unit map $\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}$. Since $\pi_1 \mathbb{Z} \otimes \mathbb{Z} = 0$ and $\pi_0 \mathbb{Z} \to \pi_0 \mathbb{Z} \otimes \mathbb{Z}$ is injective, the $\mathbb{Z}$-module spectrum C is 2-connective. Applying $\hom(\mathbb{Z}, -)$ to the cofiber sequence defining X_{k+1}, and using the canonical identification $\hom(\mathbb{Z}, \hom(\mathbb{Z}, -)) \simeq \hom(\mathbb{Z} \otimes \mathbb{Z}, -)$ we get a fiber sequence

$$\hom(\mathbb{Z} \otimes \mathbb{Z}, X_k) \to \hom(\mathbb{Z}, X_k) \to \hom(\mathbb{Z}, X_{k+1}),$$

from which we deduce that

$$\hom(\mathbb{Z}, X_{k+1}) \simeq \Sigma\hom(C, X_k) \simeq \Sigma\hom_{\mathbb{Z}}(C, \hom(\mathbb{Z}, X_k)) \simeq \hom_{\mathbb{Z}}(C, \Sigma\hom(\mathbb{Z}, X_k))$$

Since, by the inductive hypothesis, $\tau_{\geq -k}\Sigma\hom(\mathbb{Z}, X_k) = 0$ and $\tau_{\leq 1} C = 0$, we deduce that

$$\tau_{\geq -k-2}\hom(\mathbb{Z}, X_{k+1}) \simeq \tau_{\geq -k-2}\hom_{\mathbb{Z}}(C, \Sigma\hom(\mathbb{Z}, X_k)) = 0,$$

hence verifying the inductive step.

Let $X_\infty := \varinjlim_k X_k$. We now use the above estimate to show that $\tau_{\geq 0} X_\infty = X$ and that $X_\infty \in L_{\langle \mathbb{Z}\rangle}\mathrm{Sp}$. These imply that $X_\infty = L_{\langle \mathbb{Z}\rangle} X$ by Proposition 3.3. For the first claim, we have $\tau_{\geq 0} X_k \xrightarrow{\sim} \tau_{\geq 0} X_{k+1}$ for all $k \in \mathbb{N}$. Indeed, this follows from the fact that the fiber of the map $X_k \to X_{k+1}$ has vanishing homotopy groups in degrees -1 and above. Finally, we show that $X_\infty \in L_{\langle \mathbb{Z}\rangle}\mathrm{Sp}$, or equivalently that $\hom(\mathbb{Z}, X_\infty) = 0$. By the fact that $\tau_{\geq -k-1}\hom(\mathbb{Z}, X_k) = 0$, it would suffice to show that the assembly map

$$\varinjlim \hom(\mathbb{Z}, X_k) \to \hom(\mathbb{Z}, \varinjlim X_k)$$

is an isomorphism. This assembly map participates in a cofiber sequence of assembly maps

$$\begin{array}{ccccc}
\varinjlim \hom(\mathbb{Z}, \tau_{\geq 0} X_k) & \longrightarrow & \varinjlim \hom(\mathbb{Z}, X_k) & \longrightarrow & \varinjlim \hom(\mathbb{Z}, \tau_{\leq -1} X_k) \\
\downarrow & & \downarrow & & \downarrow \\
\hom(\mathbb{Z}, \varinjlim \tau_{\geq 0} X_k) & \longrightarrow & \hom(\mathbb{Z}, \varinjlim X_k) & \longrightarrow & \hom(\mathbb{Z}, \varinjlim \tau_{\leq -1} X_k)
\end{array}$$

The left vertical map is an isomorphism since the sequence $\tau_{\geq 0}X_0 \to \tau_{\geq 0}X_1 \to \cdots$ is constant, by our proof of the first claim. The right vertical map is an isomorphism since $\mathbb{Z}$ is a spectrum of finite type, and hence $\hom(\mathbb{Z}, -)$ preserves filtered colimits of coconnective spectra. This shows that the middle vertical map is an isomorphism, concluding the proof. □

4 Chromatic ideals

For $n \geq 1$ let[2] $F(n)$ be a finite spectrum of type n at an (implicit) prime p. We also set $F(0) := \mathbb{S}$. Then, the quotient

$$\mathrm{Sp}/\langle F(n)\rangle \simeq L_{n-1}^f \mathrm{Sp}$$

is the $(n-1)^{\mathrm{st}}$ finite localization, and the corresponding localization functor is denoted by L_{n-1}^f. By the thick subcategory theorem of Hopkins and Smith [1998], after localizing at the prime p, the chains of ideals

$$\mathrm{Sp} = \langle F(0)\rangle \supseteq \langle F(1)\rangle \supseteq \langle F(2)\rangle \supseteq \cdots \supseteq \{0\}$$

contain all the compactly generated localizing ideals in $\mathrm{Sp}_{(p)}$.

Let $v_n\colon \Sigma^k F(n) \to F(n)$ be a v_n-self map, and let $T(n) := v_n^{-1} F(n)$ be the associated telescope. We define also $T(\infty) := \mathbb{F}_p$. Using Proposition 2.6 iteratively we get, for $n \geq m$,

$$\langle F(m)\rangle = \langle T(m), \dots, T(n), F(n+1)\rangle. \tag{2}$$

Taking $m = 0$ and applying L_n^f-localization, which precisely annihilates $F(n+1)$, we get an equality

$$\langle T(0), \dots, T(n)\rangle = L_n^f \mathrm{Sp} \tag{3}$$

of ideals in $L_n^f \mathrm{Sp}$. Motivated by this example, we consider the following localizing ideal.

Definition 4.1 We define the localizing ideal

$$\mathrm{T}[0,\infty] := \langle T(0), T(1), \dots, T(\infty)\rangle \trianglelefteq \mathrm{Sp}.$$

Informally, $\mathrm{T}[0,\infty]$ consists of those spectra which can be built from the "chromatic cells" $T(i)$ for $i = 0, \dots .\infty$. Note that since the ideal $\langle T(n)\rangle$ is independent of the specific choice of finite complex $F(n)$ and v_n-self map $v_n\colon \Sigma^k F(n) \to F(n)$, the ideal $\mathrm{T}[0,\infty]$ is also independent of these choices. Furthermore, since $T(\infty) = \mathbb{F}_p \in \mathrm{T}[0,\infty]$, this ideal contains all bounded above spectra with p-torsion homotopy by Proposition 3.1. Thus, we may informally think of $\mathrm{T}[0,\infty]$ as those spectra whose p-torsion part is a mixture of bounded above and periodic pieces of various chromatic heights.

We have the following source of examples of objects in $\mathrm{T}[0,\infty]$.

Definition 4.2 [Barthel et al. 2024, Definition 7.13] A spectrum X is called *almost L_n^f-local* if $X \otimes F(n+1)$ is bounded above.

[2]The notation $F(n)$ for a finite spectrum of type n is nonstandard. Note, for example, that if a Smith–Toda complex $V(n)$ exists then we can choose $F(n+1) := V(n)$.

Proposition 4.3 *If X is almost L_n^f-local then $X \in \mathrm{T}[0,\infty]$.*

Proof Tensoring equation (2) for $m = 0$ with X, we get

$$\langle X\rangle \subseteq \sum_{k=0}^{n} \langle X \otimes T(k)\rangle + \langle X \otimes F(n+1)\rangle \subseteq \sum_{k=0}^{n} \langle T(k)\rangle + \langle X \otimes F(n+1)\rangle.$$

Since $\langle T(k)\rangle \subseteq \mathrm{T}[0,\infty]$, it remains to show that the same holds for $\langle X \otimes F(n+1)\rangle$. But $X \otimes F(n+1)$ is bounded above by assumption, and all its homotopy groups are p-power torsion since p is nilpotent on $F(n+1)$. By Proposition 3.1, these imply that

$$X \otimes F(n+1) \in \langle \mathbb{F}_p\rangle \subseteq \mathrm{T}[0,\infty]. \qquad \square$$

On the other hand, there are many spectra of interest which are not in $\mathrm{T}[0,\infty]$.

Example 4.4 The p-complete sphere spectrum $\mathbb{S}_p$ does not belong to $\mathrm{T}[0,\infty]$, and in fact, it belongs to the localization $L_{\mathrm{T}[0,\infty]}\mathrm{Sp}$. Indeed, it suffices to check that $\hom(T(m), \mathbb{S}_p) = 0$ for all $m = 0, \dots, \infty$.

- For $m = 0$ this follows from the fact that $\mathbb{S}_p$ is p-complete.
- For $m = \infty$ this follows from the chromatic convergence theorem. Namely,

$$\hom(\mathbb{F}_p, \mathbb{S}_p) \simeq \varprojlim \hom(\mathbb{F}_p, L_n^f \mathbb{S}_p^\wedge),$$

  and $\hom(\mathbb{F}_p, X) = 0$ for every L_n^f-local spectrum X.
- For $1 \le m < \infty$, this follows from the periodicity of $T(m)$. Namely, choosing $T(m)$ to be an $\mathbb{E}_1$-ring, and considering $T(m) \otimes F(m)^\vee$, which generates the same thick subcategory, it suffices to show that

$$\hom(T(m) \otimes F(m)^\vee, \mathbb{S}_p) \simeq \hom(T(m), F(m)) = 0.$$

  Now, a v_m-self map of $F(m)$ acts invertibly on this spectrum because it is a $T(m)$-module. But $F(m)$ is bounded below, so

$$\begin{aligned}\hom(T(m), F(m)) &\simeq \varprojlim\big(\cdots \xrightarrow{v_m} \hom(T(m), \Sigma^d F(m)) \xrightarrow{v_m} \hom(T(m), F(m))\big)\\ &\simeq \hom\Big(T(m), \varprojlim\big(\cdots \xrightarrow{v_m} \Sigma^d F(m) \xrightarrow{v_m} F(m)\big)\Big) \simeq \hom(T(m), 0) = 0.\end{aligned}$$

By construction, the inclusion $L_n^f\mathrm{Sp} \hookrightarrow \mathrm{Sp}$ lands in $\mathrm{T}[0,\infty]$, and we can restrict L_n^f to a localization $L_n^f : \mathrm{T}[0,\infty] \to L_n^f\mathrm{Sp}$. Moreover, the kernel of the localization L_n^f simplifies after this restriction of its source.

Proposition 4.5 *The kernel of $L_n^f : \mathrm{T}[0,\infty] \to L_n^f\mathrm{Sp}$ is the localizing ideal $\langle T(n+1), \dots, T(\infty)\rangle$ of* Sp. *Namely,*

$$\mathrm{T}[0,\infty] \cap \langle F(n+1)\rangle = \langle T(n+1), \dots, T(\infty)\rangle.$$

Proof Clearly, the right-hand side is contained in the left-hand side. On the other hand, note that

$$\mathrm{T}[0,\infty] + \langle F(n+1)\rangle = \mathrm{Sp},$$

and hence

$$\begin{aligned}\langle F(n+1)\rangle \cap \mathrm{T}[0,\infty] &= (\langle F(n+1)\rangle \cap \mathrm{T}[0,\infty]) \cdot (\mathrm{T}[0,\infty] + \langle F(n+1)\rangle) \\ &\subseteq \langle F(n+1)\rangle \cdot \mathrm{T}[0,\infty] = \sum_{m=0}^{\infty} \langle F(n+1)\rangle \cdot \langle T(m)\rangle = \sum_{m=0}^{\infty} \langle F(n+1) \otimes T(m)\rangle.\end{aligned}$$

Finally, if $m < n+1$ then $F(n+1) \otimes T(m) = 0$, while if $m \geq n+1$ then $T(m)$ and $T(m) \otimes F(n+1)$ are both telescopes on v_m-self maps of spectra of type m, and hence generate the same thick subcategory, by the thick subcategory theorem. Hence,

$$\langle F(n+1) \otimes T(m)\rangle = \begin{cases} \langle T(m)\rangle & \text{if } m \geq n+1, \\ 0 & \text{if } m \leq n, \end{cases}$$

and thus

$$\langle F(n+1)\rangle \cap \mathrm{T}[0,\infty] \subseteq \sum_{m=0}^{\infty} \langle F(n+1) \otimes T(m)\rangle = \sum_{m=n+1}^{\infty} \langle T(m)\rangle. \qquad \square$$

5 Ideals of Eilenberg–Mac Lane spaces

For a pointed space A, let $\overline{\mathbb{S}}[A] := \Sigma^{\infty} A$ be the reduced suspension spectrum of A. Write $\langle A\rangle := \langle \overline{\mathbb{S}}[A]\rangle$. Let $B^n C_p$ be the Eilenberg–Mac Lane space with

$$\pi_i B^n C_p = \begin{cases} C_p & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

For $n \geq 1$ the spectrum $\overline{\mathbb{S}}[B^n C_p]$ is p-local.

Our goal from now on is to compare the ideals $\langle B^n C_p\rangle$ with the compactly generated ideals $\langle F(n)\rangle$. First, we show that like the latter, the former family form a descending chain.

Proposition 5.1 *The ideals $\langle B^n C_p\rangle$ satisfy*

$$\mathrm{Sp} = \langle B^0 C_p\rangle \supseteq \langle B^1 C_p\rangle \supseteq \langle B^2 C_p\rangle \cdots \supseteq \langle B^n C_p\rangle \supseteq \cdots. \tag{4}$$

Proof First, recall that the reduced suspension spectrum functor $\overline{\mathbb{S}}[-]\colon \mathcal{S}_* \to \mathrm{Sp}$ is colimit-preserving. Now, the bar construction gives a colimit presentation

$$B^n C_p \simeq \varinjlim_{[k]\in\Delta} (B^{n-1} C_p)^k \in \mathcal{S}_*,$$

and hence

$$\overline{\mathbb{S}}[B^n C_p] \simeq \varinjlim_{[k]\in\Delta} \overline{\mathbb{S}}[(B^{n-1} C_p)^k].$$

It will therefore suffice to show that $\overline{\mathbb{S}}[(B^{n-1} C_p)^k] \in \langle B^{n-1} C_p\rangle$ for all $k \in \mathbb{N}$. For $k = 1$ this is obvious, and for $k > 1$ we can argue inductively as follows. For pointed spaces A and B we have a cofiber sequence

$$A \vee B \to A \times B \to A \wedge B.$$

Using this for $A = B^n C_p$ and $B = (B^n C_p)^{k-1}$ and applying the functor $\overline{\mathbb{S}}[-]$, we obtain a cofiber sequence

$$\overline{\mathbb{S}}[B^n C_p] \oplus \overline{\mathbb{S}}[(B^n C_p)^{k-1}] \to \overline{\mathbb{S}}[(B^n C_p)^k] \to \overline{\mathbb{S}}[B^n C_p] \otimes \overline{\mathbb{S}}[(B^n C_p)^{k-1}].$$

By the inductive hypothesis, the left and right most objects in the sequence belong to $\langle B^n C_p \rangle$, and hence so does $\overline{\mathbb{S}}[(B^n C_p)^k]$. □

Next, we show that the chain of Eilenberg–Mac Lane ideals is majorized by the chain of ideals of finite spectra.

Proposition 5.2 *For all $n \in \mathbb{N}$ there is an inclusion of ideals $\langle B^n C_p \rangle \subseteq \langle F(n) \rangle$.*

Proof Since $\langle F(n) \rangle$ is the kernel of the localization L_{n-1}^f, the statement of the proposition is equivalent to the vanishing of $L_{n-1}^f \overline{\mathbb{S}}[B^n C_p]$. This, in turn, follows from [Carmeli et al. 2022, Corollary 5.3.7]. □

However, the gap between $\langle B^n C_p \rangle$ and $\langle F(n) \rangle$ is unseen by chromatic homotopy theory. Concretely, the following result holds.

Proposition 5.3 *For all $n \in \mathbb{N}$,*

$$\langle B^n C_p \rangle \cap \mathrm{T}[0, \infty] = \langle F(n) \rangle \cap \mathrm{T}[0, \infty].$$

Proof First, by Proposition 5.2, $\langle B^n C_p \rangle \subseteq \langle F(n) \rangle$. Thus, it remains to show that

$$\langle F(n) \rangle \cap \mathrm{T}[0, \infty] \subseteq \langle B^n C_p \rangle \cap \mathrm{T}[0, \infty].$$

Second, by Proposition 4.5,

$$\langle F(n) \rangle \cap \mathrm{T}[0, \infty] = \langle T(n), \dots, T(\infty) \rangle,$$

so the above inclusion is equivalent to the fact that $T(m) \in \langle B^n C_p \rangle$ for $m \geq n$, including $m = \infty$. We begin with the case of finite m. Since by Proposition 5.1 there is a containment $\langle B^n C_p \rangle \supseteq \langle B^m C_p \rangle$, it would suffice to check this for $n = m$, that is, we only have to show that $T(n) \in \langle B^n C_p \rangle$ for all $n \in \mathbb{N}$. The tensor product $T(n) \otimes \overline{\mathbb{S}}[B^n C_p]$ belongs to $\langle B^n C_p \rangle$, so it would suffice to exhibit $T(n)$ as a retract of this tensor product. By definition, $L_{T(n)} \overline{\mathbb{S}}[B^n C_p] = \mathbb{S}_{T(n)}[\omega_p^{(n)}]$ is the p^{th} cyclotomic extension of the $T(n)$-local sphere from [Carmeli et al. 2024]. By [Carmeli et al. 2024, Proposition 5.2], this cyclotomic extensions is an $\mathbb{F}_p^\times$-Galois extension of $\mathbb{S}_{T(n)}$ and in particular

$$\mathbb{S}_{T(n)}[\omega_p^{(n)}]^{h\mathbb{F}_p^\times} \simeq \mathbb{S}_{T(n)}.$$

Tensoring this isomorphism with $T(n)$ we get

$$(T(n) \otimes \overline{\mathbb{S}}[B^n C_p])^{h\mathbb{F}_p^\times} \simeq T(n) \otimes \overline{\mathbb{S}}[B^n C_p])^{h\mathbb{F}_p^\times} \simeq T(n) \otimes L_{T(n)} \overline{\mathbb{S}}[B^n C_p])^{h\mathbb{F}_p^\times} \simeq T(n),$$

where we used the facts that tensoring with $T(n)$ preserves $\mathbb{F}_p^\times$-fixed points and that it is invariant under $T(n)$-localization. Finally, since $T(n) \otimes \overline{\mathbb{S}}[B^n C_p]$ is p-complete and $\mathbb{F}_p^\times$ is of order prime to p, the fixed points for the action are a retract of the underlying spectrum, that is, $T(n)$ is a retract of $T(n) \otimes \overline{\mathbb{S}}[B^n C_p]$.

It remains to show that $T(\infty) := \mathbb{F}_p \in \langle B^n C_p \rangle$. Similarly to the finite case, it would suffice to show that $\mathbb{F}_p$ is, up to suspension, a retract of $\overline{\mathbb{S}}[B^n C_p] \otimes \mathbb{F}_p$. But $\overline{\mathbb{S}}[B^n C_p] \otimes \mathbb{F}_p$ splits into a nonempty sum of suspensions of copies of $\mathbb{F}_p$, because it is a nonzero $\mathbb{F}_p$-module spectrum. □

Remark 5.4 Robert Burklund explained to the author that a variant of an unpublished argument of Hopkins and Smith for the equality of the Bousfield classes of $\overline{\mathbb{S}}[BC_p]$ and $F(1)$ gives the equality of ideals $\langle BC_p \rangle = \langle F(1) \rangle$. Consequently, for $n = 1$ the localizing subcategory $L_{\langle BC_p \rangle}\mathrm{Sp} \subseteq \mathrm{Sp}$ is the ∞-category of $\mathbb{S}[1/p]$-module spectra. Such a simple description is impossible for higher values of m, though, as the following example shows.

Example 5.5 The ideals $\langle B^2 C_p \rangle$ and $\langle F(2) \rangle$ are different. In fact, the p-complete sphere spectrum $\mathbb{S}_p$ belongs to $L_{\langle B^2 C_p \rangle}\mathrm{Sp}$. Indeed, this statement is equivalent to the claim that the pullback morphism $\pi^* \colon \mathbb{S}_p \to \mathbb{S}_p^{B^2 C_p}$ along the terminal map $\pi \colon B^2 C_p \to \mathrm{pt}$ is an isomorphism, which is a special case of [Lee 1992, Theorem 2.5].

It will be convenient to reformulate Proposition 5.3 in terms of localization functors and mapping spectra.

Corollary 5.6 *For all $n \geq 1$, the localization L_{n-1}^f is a further localization of $L_{\langle B^n C_p \rangle}$. On the other hand, they agree when restricted to* $\mathrm{T}[0,\infty]$. *Namely, for $X \in \mathrm{T}[0,\infty]$ we have*

$$L_{\langle B^n C_p \rangle} X \simeq L_{n-1}^f X.$$

Proof Recall that a localization L is a further localization of L' if and only if the kernel of L contains that of L'. Thus, the first part follows from Proposition 5.2 applied to the localizations $L_{\langle B^n C_p \rangle}$ and L_{n-1}^f of Sp.

For the second part, we have to show that for $X \in \mathrm{T}[0,\infty]$, the spectrum $L_{\langle B^n C_p \rangle} X$ is L_{n-1}^f-local. In other words, we have to show that the (colimit preserving) functor $L_{\langle B^n C_p \rangle}|_{\mathrm{T}[0,\infty]} \colon \mathrm{T}[0,\infty] \to L_{\langle B^n C_p \rangle}\mathrm{Sp}$ factors through $L_{n-1}^f\mathrm{Sp}$. Since $L_{n-1}^f\mathrm{Sp} \simeq \mathrm{T}[0,\infty]/\mathrm{T}[0,\infty] \cap \langle F(n) \rangle$, this follows from the fact that the kernel of $L_{\langle B^n C_p \rangle}|_{\mathrm{T}[0,\infty]}$ is $\mathrm{T}[0,\infty] \cap \langle B^n C_p \rangle$, which equals $\mathrm{T}[0,\infty] \cap \langle F(n) \rangle$ by Proposition 5.3. □

Corollary 5.7 *Let X be an almost L_n^f-local spectrum for some $n \in \mathbb{N}$. Then*

$$L_{\langle B^n C_p \rangle} X \simeq L_{n-1}^f X.$$

Proof By Proposition 4.3, $X \in \mathrm{T}[0,\infty]$, so the result follows from Corollary 5.6. □

Corollary 5.8 *Let $Y \in L_{\langle B^n C_p \rangle}\mathrm{Sp}$ and let $X \in \mathrm{T}[0,\infty]$ (eg an almost L_m^f-local spectrum for some $m \in \mathbb{N}$). Then,*

$$\hom(X, Y) \simeq \hom(L_{n-1}^f X, Y).$$

Proof Since $Y \in L_{\langle B^n C_p \rangle}\mathrm{Sp}$, we have

$$\hom(X, Y) \simeq \hom(L_{\langle B^n C_p \rangle} X, Y).$$

But $L_{\langle B^n C_p \rangle} X \simeq L_{n-1}^f X$ by Corollary 5.6 and the result follows. □

6 Maps into the units of the sphere spectrum

We shall now consider the implications of the theory developed in the previous section to the spectrum $\mathbb{S}_p^\times$. Note that

$$\mathbb{S}_p^\times \simeq (\mathbb{S}_p^\times)_p^\wedge \oplus \mathbb{F}_p^\times, \tag{5}$$

so studying maps of connective spectra into $\mathbb{S}_p^\times$ and its p-completion are essentially the same. We first show that it is safe to replace the p-completion with a suitable delooping, so that we can work stably.

Proposition 6.1 *The canonical map $(\mathbb{S}_p^\times)_p^\wedge \to \tau_{\geq 0} L_{\langle\mathbb{Z}\rangle}(\mathbb{S}_p^\times)_p^\wedge$ is an isomorphism.*

Proof By Proposition 3.4, it would suffice to show that $\mathrm{Map}(\mathbb{Z}, \Sigma(\mathbb{S}_p^\times)_p^\wedge) = \mathrm{pt}$. By (5), this is the case if and only if the map $\mathbb{S}_p^\times \to \mathbb{F}_p^\times$ induces an isomorphism

$$\tau_{\geq 0}\,\mathrm{hom}(\mathbb{Z}, \Sigma\mathbb{S}_p^\times) \xrightarrow{\sim} \tau_{\geq 0}\,\mathrm{hom}(\mathbb{Z}, \Sigma\mathbb{F}_p^\times).$$

By [Carmeli 2023, Proposition 3.8], the target of this map identifies with $\Sigma\mathbb{F}_p^\times$. The claim now follows from the combination of [Carmeli 2023, Corollary 1.2] and [Carmeli 2023, Lemma 4.17], which together show that our map is the composition of isomorphisms

$$\tau_{\geq 0}\,\mathrm{hom}(\mathbb{Z}, \Sigma\mathbb{S}_p^\times) \xrightarrow{\sim} \tau_{\geq 0}\,\mathrm{hom}(\mathbb{Z}, \mathrm{pic}(\mathbb{S}_p)) \simeq \Sigma\mathbb{F}_p^\times. \qquad \square$$

From now on, we shall work with this nonconnective spectrum, which we abbreviate as follows:

Definition 6.2 We define the spectrum $\mathfrak{G} := L_{\langle\mathbb{Z}\rangle}(\mathbb{S}_p^\times)_p^\wedge$.

Hence,

$$\tau_{\geq 0}\mathfrak{G} \simeq (\mathbb{S}_p^\times)_p^\wedge,$$

and by definition, $\mathfrak{G} \in L_{\langle\mathbb{Z}\rangle}\mathrm{Sp}$. We shall need two other such locality properties of $\mathfrak{G}$.

Proposition 6.3 *The spectrum $\mathfrak{G}$ is p-complete.*

Proof We have to show that $\mathrm{Map}(X, \mathfrak{G}) \simeq \mathrm{pt}$ for every spectrum X on which p is invertible. Since $\mathfrak{G} \in L_{\langle\mathbb{Z}\rangle}\mathrm{Sp}$, by Proposition 3.3 we have the isomorphisms

$$\mathrm{Map}(X, \mathfrak{G}) \simeq \mathrm{Map}(\tau_{\geq 0}X, \tau_{\geq 0}\mathfrak{G}) \simeq \mathrm{Map}(\tau_{\geq 0}X, (\mathbb{S}_p^\times)_p^\wedge).$$

Since p is invertible also on $\tau_{\geq 0}X$ and $(\mathbb{S}_p^\times)_p^\wedge$ is p-complete, the latter mapping space is contractible and the result follows. $\square$

By Lee's theorem, no nonzero maps of connective spectra $\overline{\mathbb{S}}[B^2C_p] \to \mathbb{S}_p^\times$ exist. This formally implies an analogous property for our spectrum $\mathfrak{G}$.

Proposition 6.4 *The spectrum* $\mathfrak{G}$ *satisfies* $\mathfrak{G} \in L_{\langle B^2C_p\rangle}\mathrm{Sp}$.

Proof We have to show that $\hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) = 0$. Now,

$$\begin{aligned}\Omega^\infty \hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) &\simeq \mathrm{Map}(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) \simeq \mathrm{Map}(\overline{\mathbb{S}}[B^2C_p], \tau_{\geq 0}\mathfrak{G}) \\ &\simeq \mathrm{Map}(\overline{\mathbb{S}}[B^2C_p], (\mathbb{S}_p^\times)^\wedge_p) \simeq \mathrm{Map}(\overline{\mathbb{S}}[B^2C_p], \mathbb{S}_p^\times) \\ &\simeq \mathrm{Fib}((\mathbb{S}_p^\times)^{B^2C_p} \to \mathbb{S}_p^\times) \simeq \mathrm{Fib}((\mathbb{S}_p^{B^2C_p})^\times \to \mathbb{S}_p^\times),\end{aligned}$$

where the second isomorphism is since $\overline{\mathbb{S}}[B^2C_p]$ is connective, the third by Proposition 3.1, and the fourth since $\overline{\mathbb{S}}[B^2C_p] \in \langle F(1)\rangle$, so maps from it to a spectrum and to its p-completion are the same.

We have that $\mathbb{S}_p \xrightarrow{\sim} \mathbb{S}_p^{B^2C_p}$ by Lee's theorem (see Example 5.5), so the fiber of the induced map on units $(\mathbb{S}_p^{B^2C_p})^\times \to \mathbb{S}_p^\times$ is contractible. We deduce that $\Omega^\infty \hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) = \mathrm{pt}$, so that in particular $\hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G})$ is bounded above. However, by definition, $\mathfrak{G} \in L_{\langle\mathbb{Z}\rangle}\mathrm{Sp}$. Since $\langle\mathbb{Z}\rangle$ is closed under tensoring with $\overline{\mathbb{S}}[B^2C_p]$, the localizing subcategory $L_{\langle\mathbb{Z}\rangle}\mathrm{Sp}$ is closed under applying $\hom(\overline{\mathbb{S}}[B^2C_p], -)$ and we deduce that $\hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) \in L_{\langle\mathbb{Z}\rangle}\mathrm{Sp}$. Together, by Proposition 3.1 these imply that $\hom(\overline{\mathbb{S}}[B^2C_p], \mathfrak{G}) = 0$. □

From the relationship between the ideal of B^2C_p and the chromatic localizations of the previous section, we deduce the following:

Corollary 6.5 *With the notation as above, for* $X \in \mathrm{T}[0,\infty]$ *we have*

$$\mathrm{Map}(X, \mathfrak{G}) \simeq \mathrm{Map}(L_{K(1)}X, \mathfrak{G}).$$

Proof Since $\mathfrak{G} \in L_{\langle B^2C_p\rangle}\mathrm{Sp}$, we deduce from Corollary 5.8 that

$$\mathrm{Map}(X, \mathfrak{G}) \simeq \mathrm{Map}(L_1^f X, \mathfrak{G}).$$

Since by Proposition 6.3, $\mathfrak{G}$ is also p-complete and $L_{K(1)}X$ is the p-completion of $L_1^f X$, we obtain

$$\mathrm{Map}(L_1^f X, \mathfrak{G}) \simeq \mathrm{Map}((L_1^f X)^\wedge_p, \mathfrak{G}) \simeq \mathrm{Map}(L_{K(1)}X, \mathfrak{G}). \qquad \square$$

For connective X, this immediately translates to our main result regarding $\mathbb{S}_p^\times$.

Theorem 6.6 *Let* X *be a connective spectrum in* $\mathrm{T}[0,\infty]$. *Then*,

$$\mathrm{Map}(X, \mathbb{S}_p^\times) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\pi_0 X, \mathbb{F}_p^\times) \times \mathrm{Map}(\tau_{\geq 0} L_{K(1)}X, (\mathbb{S}_p^\times)^\wedge_p).$$

Proof Using the decomposition

$$\mathbb{S}_p^\times \simeq (\mathbb{S}_p^\times)^\wedge_p \oplus \mathbb{F}_p^\times$$

we immediately reduce the statement to

$$\mathrm{Map}(X, (\mathbb{S}_p^\times)^\wedge_p) \simeq \mathrm{Map}(\tau_{\geq 0} L_{K(1)}X, (\mathbb{S}_p^\times)^\wedge_p).$$

This, in turn, follows from the chain of isomorphisms

$$\begin{aligned}\operatorname{Map}(X,(\mathbb{S}_p^{\times})_p^{\wedge}) \simeq \operatorname{Map}(X,\mathfrak{G}) \simeq \operatorname{Map}(L_{K(1)}X,\mathfrak{G}) &\simeq \operatorname{Map}(\tau_{\geq 0}L_{K(1)}X,\tau_{\geq 0}\mathfrak{G})\\ &\simeq \operatorname{Map}(\tau_{\geq 0}L_{K(1)}X,(\mathbb{S}_p^{\times})_p^{\wedge}),\end{aligned}$$

where the first isomorphism is since X is connective and $\tau_{\geq 0}\mathfrak{G} \simeq (\mathbb{S}_p^{\times})_p^{\wedge}$, the second is Corollary 6.5, the third follows from Proposition 3.3 and the fourth is again by $\tau_{\geq 0}\mathfrak{G} \simeq (\mathbb{S}_p^{\times})_p^{\wedge}$. □

Remark 6.7 Expanding Remark 1.2, it is natural to ask if one can obtain a similar result for $\operatorname{pic}(\mathbb{S}_p)$. Note that our result for $\mathbb{S}_p^{\times}$ depends on a vanishing result for $\operatorname{pic}(\mathbb{S}_p)_p^{\wedge}$, namely, that it receives no maps from $\mathbb{Z}$. Similarly, we could lift the result from $\mathbb{S}_p^{\times}$ to $\operatorname{pic}(\mathbb{S}_p)$ if we knew that there were no maps from $\mathbb{Z}$ to the p-completion of the Brauer spectrum of $\mathbb{S}_p$. This is a result of a work in progress with Burklund and Ramzi.

7 Applications

To demonstrate the utility of Theorem 6.6, we list some examples of familiar spectra belonging to $\mathrm{T}[0,\infty]$, for which Theorem A applies. First, observe that the connective cover of every L_n^f-local spectrum is in $\mathrm{T}[0,\infty]$. For example, this applies to Lubin–Tate theories. Let E_n be a Lubin–Tate theory associated with a formal group law of height n over a perfect ring κ of characteristic p, so that $\pi_0 E_n \simeq \mathbb{W}(\kappa)[\![u_1,\dots,u_n]\!]_p^{\wedge}$.

Proposition 7.1 *Every map $\tau_{\geq 0}E_n \to \mathbb{S}_p^{\times}$ factors uniquely through the map $\tau_{\geq 0}E_n \to \tau_{\geq 0}E_n[u_1^{-1}]_p^{\wedge}$:*

$$\begin{array}{ccc} \tau_{\geq 0}E_n & \longrightarrow & \mathbb{S}_p^{\times} \\ \downarrow & \nearrow & \\ \tau_{\geq 0}E_n[u_1^{-1}]_p^{\wedge} & & \end{array}$$

Proof This follows from Theorem 6.6 by observing that $\pi_0 E_n/(p-1) = 0$ and hence it has no nonzero maps to $\mathbb{F}_p^{\times}$, and that $L_{K(1)}E_n \simeq E_n[u_1^{-1}]_p^{\wedge}$. □

Next, we discuss maps from K-theory spectra (and their suspensions) into $\mathbb{S}_p^{\times}$. The following result is an immediate consequence of Waldhausen's interpretation of the Quillen–Lichtenbaum conjecture.

Proposition 7.2 *Let R be a $\mathbb{Z}$-algebra in* Sp *(eg an ordinary associative ring). Then $K(R) \in \mathrm{T}[0,\infty]$. Consequently, for all $t \in \mathbb{N}$,*

$$\operatorname{Map}(\tau_{\geq 0}\Sigma^{-t}K(R),\mathbb{S}_p^{\times}) \simeq \operatorname{Hom}(\pi_t K(R),\mathbb{F}_p^{\times}) \times \operatorname{Map}(\tau_{\geq 0}\Sigma^{-t}L_{K(1)}K(R),\mathbb{S}_p^{\times}).$$

Proof First, observe that since bounded above spectra are in $\mathrm{T}[0,\infty]$, if $K(R) \in \mathrm{T}[0,\infty]$ so does $\tau_{\geq 0}\Sigma^{-t}K(R)$. Hence, the "consequently" part indeed follows from the first claim. Next, since $K(R)$ is a module over $K(\mathbb{Z})$, by Proposition 2.2 it suffices to show that $K(\mathbb{Z}) \in \mathrm{T}[0,\infty]$. As shown in [Waldhausen 1984, Appendix 4], the validity of the Quillen–Lichtenbaum conjecture for the ring $\mathbb{Z}$ (see [Rognes and Weibel 2000; Kahn 1997; Voevodsky 2011]) implies that the map $K(\mathbb{Z}) \to L_{K(1)}K(\mathbb{Z})$ has bounded above fiber, which in turn implies that $K(\mathbb{Z}) \in \mathrm{T}[0,\infty]$ by Proposition 3.1 (note that $\langle \mathbb{Z} \rangle \subseteq \mathrm{T}[0,\infty]$). □

Finally, Hahn and Wilson [2022] constructed $\mathbb{E}_3$-MU-algebra structures on the truncated Brown–Peterson spectra $\mathrm{BP}\langle n \rangle$. For these $\mathbb{E}_3$-algebras, they established a higher-height analog of the Quillen–Lichtenbaum conjecture, and hence we have an analog of Proposition 7.2.

Proposition 7.3 *Let* $\mathrm{BP}\langle n \rangle$ *be the* $\mathbb{E}_3$*-algebra as in* [Hahn and Wilson 2022]. *For every* $\mathrm{BP}\langle n \rangle$*-algebra* R, *the spectrum* $K(R)$ *belongs to* $\mathrm{T}[0,\infty]$. *Consequently, for all* $t \in \mathbb{N}$,

$$\mathrm{Map}(\tau_{\geq 0}\Sigma^{-t}K(R), \mathbb{S}_p^{\times}) \simeq \mathrm{Hom}(\pi_t K(R), \mathbb{F}_p^{\times}) \times \mathrm{Map}(\tau_{\geq 0}\Sigma^{-t}L_{K(1)}K(R), \mathbb{S}_p^{\times}).$$

Proof As in the proof of Proposition 7.2, we can reduce to showing that $K(\mathrm{BP}\langle n \rangle)$ belongs to $\mathrm{T}[0,\infty]$. By [Hahn and Wilson 2022, Corollary 3.4.4] the fiber of the map $K(R)_{(p)} \to L_{n+1}^f K(R)_{(p)}$ is bounded above. Since the target is in $\mathrm{T}[0,\infty]$, so is the source. Thus, $K(R) \otimes F(1) \simeq K(R)_{(p)} \otimes F(1) \in \mathrm{T}[0,\infty]$, which implies that $K(R)$ belongs to $\mathrm{T}[0,\infty]$ as well, since $\langle K(\mathbb{Z}) \rangle \subseteq \langle K(\mathbb{Z})[\frac{1}{p}] \rangle + \langle K(\mathbb{Z}) \otimes F(1) \rangle$; see Proposition 2.6. □

By the purity result of [Land et al. 2024, Theorem A], $L_{K(1)}K(R) \simeq L_{K(1)}K(L_1^f R)$, and similarly $\pi_t K(R)/(p-1) \simeq \pi_t K(L_1^f R)/(p-1)$. Combining with Proposition 7.3, we deduce the following:

Corollary 7.4 *With the settings of Proposition 7.3, the map* $R \to L_1^f R$ *induces isomorphisms*

$$\mathrm{Map}(\tau_{\geq 0}\Sigma^{-t}K(R), \mathbb{S}_p^{\times}) \simeq \mathrm{Map}(\tau_{\geq 0}\Sigma^{-t}K(L_1^f R), \mathbb{S}_p^{\times})$$

for all $t \in \mathbb{N}$.

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