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Quantum K-invariants and Gopakumar–Vafa invarints, II: Calabi–Yau threefolds at genus zero

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This is the second part of our ongoing project on the relations between *Gopakumar–Vafa BPS invariants* (GV) and *quantum K-theory* (QK) on the Calabi–Yau threefolds (CY3). We show that on CY3, a genus-zero quantum K-invariant can be written as a linear combination of a finite number of Gopakumar–Vafa invariants with coefficients from an explicit "multiple cover formula". Conversely, GV can be determined by QK in a similar manner. The technical heart is a proof of a remarkable conjecture by Hans Jockers and Peter Mayr.

This result is consistent with the "virtual Clemens conjecture" for the Calabi–Yau threefolds. A heuristic derivation of the relation between QK and GV via the virtual Clemens conjecture and the multiple cover formula is also given.

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0 Introduction

Main result This paper continues our study of relations between *Gopakumar–Vafa invariants* [1998a; 1998b] and *quantum K-theory* [Givental 2000; Lee 2004] on Calabi–Yau threefolds. The following is our main result:

Main theorem *A genus-zero quantum K-invariant (with "descendants") can be written as a linear combination of a finite number of Gopakumar–Vafa invariants with coefficients from an explicit multiple cover formula. Conversely, all genus-zero Gopakumar–Vafa invariants can be determined by a finite number of quantum K-invariants in a similar manner. Furthermore, the linear transformations between QK and GV preserve integrality.*

The multiple cover formula is stated in Proposition 4.4. The explicit formulation can be found in Theorem 1.2 together with the proof of Proposition 1.5. The quantum K-invariants of any genus are intrinsically integral invariants, while the integrality of the GV in our definition follows from the result in [Ionel and Parker 2018].

This theorem generalizes the results in [Chou and Lee 2022b] in two directions. First, the equivalence between genus-zero quantum K-invariants and Gopakumar–Vafa invariants now holds for all Calabi–Yau threefolds. Second, the relation extends to n-pointed quantum K-invariants thanks to the conjectural formula of H Jockers and P Mayr. The proof also follows a different approach, although both ultimately come from the virtual Hirzebruch–Riemann–Roch theorem for stacks [Tonita 2014].

Our techniques generalizes to higher dimensions. The Gopakumar–Vafa type invariants at genus zero have been defined to all semipositive (including Calabi–Yau and Fano) varieties with dimensions greater than or equal to 3. For $\gamma_1, \dots, \gamma_l \in H^*(X, \mathbb{Z})$, the numbers $\mathrm{GV}_{0,\beta}(\gamma_1, \dots, \gamma_l)$ are defined by

$$\sum_{\beta \neq 0} \mathrm{GW}_{0,\beta}(\gamma_1, \dots, \gamma_l) q^\beta =: \sum_{\beta \neq 0} \mathrm{GV}_{0,\beta}(\gamma_1, \dots, \gamma_l) \sum_{d=1}^{\infty} \frac{1}{d^{3-l}} q^{d\beta}$$

and conjectured to be integers in [Klemm and Pandharipande 2008]. The integrality has been proved by E-N Ionel and T H Parker [2018] using symplectic geometry. As an application of techniques developed in this series of papers, an *alternative proof of integrality* is given in [Chou 2024], by relating QK and GV.

Enumerative invariants on Calabi–Yau threefolds It is our belief that there should be only one set of numerical (virtual) enumerative invariants on Calabi–Yau threefolds, and Gopakumar–Vafa invariants should serve as the basic invariants to which all others can be reduced. While very explicit (conjectural, and partially verified) relations between Gromov–Witten, Donaldson–Thomas, Pandharipande–Thomas and Gopakumar–Vafa invariants have been available from the beginning, the relation with the quantum K-theory was not as clear at first. The situation changed when A Givental and his collaborators [Givental and Tonita 2014; Givental 2017a; 2015a; 2015b; 2015c; 2015d; 2015e; 2015f; 2015g; 2017b; 2020; 2017c] made a breakthrough in relating quantum K-theory with the Gromov–Witten theory. However, that relation is very general and is applicable to any smooth projective varieties (or DM stacks). Hence it is also quite complicated and its implications in this special case of Calabi–Yau threefolds are not as clear. This is the goal we set for ourselves in this series of papers.

As in part I [Chou and Lee 2022b], we use the ad hoc definition of Gopakumar–Vafa invariants in terms of Gromov–Witten invariants via the formula of R Gopakumar and C Vafa [1998a; 1998b]

$$\sum_{g=0}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \mathrm{GW}_{g,\beta} q^\beta \lambda^{2g-2} = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \mathrm{GV}_{g,\beta} \frac{1}{k} \left(2 \sin\left(\tfrac{1}{2} k\lambda\right)\right)^{2g-2} q^{k\beta}. \tag{0-1}$$

Therefore, relating QK and GV is in some sense a question of relating QK and GW. In this paper, we completely solve this problem and relate these two sets of invariants by an (invertible) integral linear transformation. This in particular implies the ad hoc definition of GV in (0-1) gives integral invariants in genus zero, as the quantum K-invariants are intrinsically *integral* by definition.

Perhaps it is worth pointing out that these relations do not render quantum K-invariants, or GW, DT and PT, obsolete. Quantum K-theory has *more direct* and *different* connections with finite-difference integrable systems, representation theory and theoretical physics. Its relation to 3-dimensional topological field theory can be found in the pioneering works of N Nekrasov, as well as Jockers and Mayr [2019; 2020]. Its connection to representation theory can be found in [Okounkov 2017]. Quantum K-theory of flag varieties is intimately related to finite difference integrable systems and quantum groups [Givental and Lee 2003]. It has also inspired much progress in geometric combinatorics through works like those of A Buch, P Chaput, C Li, L Mihalcea and N Perrin [Buch and Mihalcea 2011; Buch et al. 2013; 2020]. These works are developed from perspectives unique to the quantum K-theory.

We hope to extend these results to higher genus in future works.

Outline In Section 1 a conjecture by Jockers and Mayr [2019] is stated as Theorem 1.2 in an essentially identical form (our degree-zero invariants are different from theirs). This theorem gives a precise (finite) integral linear transformation from Gopakumar–Vafa invariants to an "essential subset" of genus-zero quantum K-invariants. In Proposition 1.5 we show that other genus-zero quantum K-invariants can be computed from these essential quantum K-invariants. Hence, the main theorem is reduced to Theorem 1.2.

Sections 2 and 3 are mainly devoted to the proof of Theorem 1.2. Our approach is to show that the right side of the formula $\tilde{J}$ in Theorem 1.2 belongs to a special class of (generating) functions to which J^K belongs. In the language of Givental, this is called "lying on the overruled Lagrangian cone". In Givental's framework, two functions both lying on the Lagrangian cone are related by a (generalized) mirror transformation, also known as Birkhoff factorization.

In the case of quantum K-theory, Givental developed a characterization of all functions lying on the overruled Lagrangian cone, called the *adelic characterization*. In Section 2, some necessary ingredients on Givental's adelic characterization are briefly summarized. Further details can be found in part I [Chou and Lee 2022b], or [Givental and Tonita 2014].

The adelic characterization is employed to show that $\tilde{J}$ lies on the Lagrangian cone in Section 3. Furthermore, by the uniqueness theorem, the mirror transformation is shown to be trivial, proving Theorem 1.2.

In the last section we study the multiple cover formula on the local model. Assuming the "virtual Clemens conjecture" on the Calabi–Yau threefold X, the Jockers–Mayr formula is derived from the "first principles". Of course, our proof as outlined above is completely different and admittedly much more involved, as the virtual Clemens conjecture remains a conjecture.

Indeed, this was how we first reached the formula (1-2), as at that time we were not able to understand the work of Jockers and Mayr [2019], written in language we were not used to. Thanks to their explanations, we now understand that their formula, dating back to 2019, is essentially the same.

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1 GW, GV, QK and the JM conjecture

1.1 GW, GV and QK

Let X be a smooth complex projective variety. (Cohomological) Gromov–Witten invariants of X are defined to be

$$\langle \tau_{d_1}(\phi_1)\cdots\tau_{d_n}(\phi_n)\rangle^{X,H}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \bigcup_{i=1}^{n} \mathrm{ev}_i^*(\phi_i) c_1(L_i)^{d_i} \in \mathbb{Q},$$

where $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ are the virtual fundamental classes, $\phi_1,\dots,\phi_n \in H(X)$, $d_1,\dots,d_n \in \mathbb{Z}_{\geq 0}$ and ev_i and L_i are the evaluation map and cotangent line bundle at the i^{th} marked point, respectively.

When X is a Calabi–Yau threefold, $\overline{\mathcal{M}}_{g,0}(X,\beta)$ has virtual dimension 0. We define

$$\mathrm{GW}_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1.$$

By string, dilaton and divisor equation, all Gromov–Witten invariants can be recovered from the 0-pointed one.

In theoretical physics, Gopakumar and Vafa [1998a; 1998b] introduced new *integral* topological invariants on *Calabi–Yau threefolds* (CY3) X, called *Gopakumar–Vafa invariants*. These invariants represent the counts of "numbers of BPS states" on X. There have been various attempts at defining Gopakumar–Vafa invariants mathematically. We refer the readers to [Maulik and Toda 2018].

A remarkable relation conjectured in [Gopakumar and Vafa 1998a; 1998b] between GV and GW can be expressed in terms of generating functions as in (0-1). In genus 0, the Gopakumar–Vafa relation can be written as

$$\text{(1-1)} \qquad \mathrm{GW}_{g=0,\beta} =: \mathrm{GW}_\beta = \sum_{k=1}^{\infty} \frac{1}{k^3}\mathrm{GV}_{\beta/k}, \quad \mathrm{GV}_{g=0,\beta} =: \mathrm{GV}_\beta := \sum_{k=1}^{\infty} \frac{\mu(k)}{k^3}\mathrm{GW}_{\beta/k},$$

where by definition

$$\mathrm{GW}_{\beta/k} = \mathrm{GV}_{\beta/k} = 0 \quad \text{if } \beta/k \notin H_2(X,\mathbb{Z}).$$

We have used the Möbius inversion and $\mu(e)$ is the Möbius function. For our purpose, we will *use* (1-1) *as the definition of* GV_β.

Quantum K-invariants are also *integral* invariants [Givental 2000; Lee 2004]. For any smooth projective variety X, quantum K-invariants are defined as

$$\langle \tau_{d_1}(\Phi_1)\cdots\tau_{d_n}(\Phi_n)\rangle^{X,K}_{g,n,\beta} := \chi\Big(\overline{\mathcal{M}}_{g,n}(X,\beta); \Big(\bigotimes_{i=1}^n \mathrm{ev}_i^*(\Phi_i)L_i^{d_i}\Big)\otimes \mathcal{O}^{\mathrm{vir}}\Big) \in \mathbb{Z}.$$

Here $\Phi_1,\dots,\Phi_n \in K^0(X)$, $d_1,\dots,d_n\in\mathbb{Z}$ and $\mathcal{O}^{\mathrm{vir}}$ is the virtual structure sheaf on $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

1.2 The Jockers–Mayr conjecture

Jockers and Mayr [2019] conjectured an explicit formula relating genus-zero quantum K-invariants (with divisor insertions t) and Gopakumar–Vafa invariants on CY3. S Garoufalidis and E Scheidegger [2022] computed the quintic case (with $t=0$) up to degree 6 and predicted the same formula. See Remark 1.3 for more discussion. We state a slightly revised version of their formula in Theorem 1.2.

From now on, let X be a CY3. For simplicity, $K(X)$ stands for the topological K-theory $K^0(X)$ of complex vector bundles.[1] Choose a basis of $K(X)$

$$\{\Phi_\alpha\}_{\alpha=1}^N = \bigsqcup_{i=0}^3 \{\Phi_{ij}\}_{j=1}^{n_i},$$

where $\{\mathrm{ch}(\Phi_{ij})\}_{j=1}^{n_i}$ forms a basis in $H^{2i}(X)$. In particular, $\{\Phi_{0j}\}_{j=1}^{n_0} = \{\Phi_{01}=\mathcal{O}\}$. Let $\{\Phi^{ij}\}$ be the dual basis of $\{\Phi_{ij}\}$ with respect to the K-theoretic Poincaré pairing

$$(\Phi_a,\Phi_b)^K := \chi(X,\Phi_a\otimes\Phi_b) = \int_X \mathrm{td}(TX)\,\mathrm{ch}(\Phi_a)\,\mathrm{ch}(\Phi_b).$$

The following fact will be used later:

Lemma 1.1 *For* $i=0$ *and* 1,

$$\mathrm{ch}(\Phi^{ij}) \in H^{2(3-i)}(X).$$

For $i=2$ *and* 3,

$$\mathrm{ch}(\Phi^{ij}) \in H^{\ge 2(3-i)}(X).$$

Proof Φ^{ij} can be written as

$$\Phi^{ij} := \mathrm{ch}^{-1}\big(\mathrm{td}(TX)^{-1}\,\mathrm{PD}(\mathrm{ch}(\Phi_{ij}))\big),$$

where PD denotes the dual under Poincaré pairing. The lemma follows from the definition of Φ_{ij} and that $\mathrm{td}(TX)^{-1}\in 1+H^{\ge 4}(X)$. ◻

[1]With suitable modifications, the results apply to the Grothendieck group of algebraic vector bundles.

Let us fix some notation. Let

$$t := \sum_{j=1}^{n_1} t_j \Phi_{1j}, \quad \beta_j := \int_\beta \mathrm{ch}(\Phi_{1j}), \quad \beta_t := \int_\beta \mathrm{ch}\, t := \sum_{j=1}^{n_1} t_j \beta_j.$$

Define a K-theoretic J-function with input t

$$J^K(t,q,Q) := (1-q) + t + \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{\alpha,n} \frac{Q^\beta}{n!} \Phi^\alpha \left\langle \frac{\Phi_\alpha}{1-qL}, t, \dots, t \right\rangle_{0,n+1,\beta}^{X,K}.$$

This is the "big J-function" in Definition 2.1 with the input specialized to $\boldsymbol{t} = t$. Define

$$\begin{aligned}(1\text{-}2) \quad \tilde{J} := (1-q) + t + \frac{\sum_{j=1}^{n_1} \Phi^{1j} \partial_{t_j} F(t)}{1-q} + \frac{q\Phi^{01} F(t)}{(1-q)^2} \\ + (1-q) \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} \sum_{r=1}^{\infty} \mathrm{GV}_\beta\, Q^{r\beta} \\ \times \left[\sum_{j=1}^{n_1} \Phi^{1j} \beta_j (a(r,q^r) + c(r,q,\beta_t)) + \Phi^{01}(b(r,q^r) + d(r,q,\beta_t)) \right],\end{aligned}$$

where

$$F(t) := \frac{1}{3!} \langle t,t,t \rangle_{0,3,0}^{X,K} = \frac{1}{6} \sum_{i,j,k=1}^{n_1} t_i t_j t_k \int_X \mathrm{ch}(\Phi_{1i})\, \mathrm{ch}(\Phi_{1j})\, \mathrm{ch}(\Phi_{1k}),$$

$$a(r,q^r) = \frac{(r-1)}{1-q^r} + \frac{1}{(1-q^r)^2},$$

$$b(r,q^r) = \frac{r^2-1}{1-q^r} + \frac{3}{(1-q^r)^2} - \frac{2}{(1-q^r)^3},$$

$$c(r,q,\beta_t) = \frac{\beta_t}{r(1-q)(1-q^r)} + \frac{e^{\beta_t} - 1 - \beta_t}{r^2(1-q)^2},$$

$$d(r,q,\beta_t) = \frac{\beta_t}{r(1-q)} \frac{r(1-q^r) - q^r}{(1-q^r)^2} + \frac{\beta_t^2}{2r^2(1-q)^2} + \frac{(1-3q)(e^{\beta_t} - 1 - \beta_t - \frac{\beta_t^2}{2}) + q\beta_t(e^{\beta_t} - 1 - \beta_t)}{r^3(1-q)^3}.$$

Theorem 1.2 (cf [Jockers and Mayr 2019]) *$J^K(t) = \tilde{J}$ for all Calabi–Yau threefolds.*

The proof of this theorem will occupy the entire Section 3.

Remark 1.3 (1) $\beta = 0$ terms in the definition of $J^K(t,q,Q)$ give nontrivial contribution

$$\frac{\sum_{j=1}^{n_1} \Phi^{1j} \partial_{t_j} F(t)}{1-q} + \frac{q\Phi^{01} F(t)}{(1-q)^2},$$

which comes from the Poincaré pairing on X. We point out that our $\beta = 0$ terms differ from those in [Jockers and Mayr 2019].

(2) When $t = 0$, we have $c(r, q, \beta_t) = d(r, q, \beta_t) = 0$. In this case, Theorem 1.2 specializes to the form of [Chou and Lee 2022b, Conjecture 1.1]. See also [Garoufalidis and Scheidegger 2022, Conjecture 1.1].

(3) One observes that $\tilde{J}$ is a rational function in q, with poles only at roots of unity, and the orders of poles are no greater than 3. This was explained in [Chou and Lee 2022b, Corollary 3.3].

(4) The dependence of $\tilde{J}$ on β_t is of the form e^{β_t}, except finitely many monomials in $\beta_t^{\le 2}$ and an overall factor β_t to some terms. This will be explained in geometric germs in Remark 3.14.

Remark 1.4 (multiple cover formula and virtual Clemens conjecture) Jockers and Mayr [2019] came up with the above formula based on arguments in string theory and evidences from local curve computations. Mathematically, the formula (1-2) is consistent with the (folklore) philosophy of the *virtual Clemens conjecture* and a genus-zero multiple cover formula in quantum K-theory. Namely, for the purpose of computing the (virtual) enumerative invariants on Calabi–Yau threefolds X, one may assume that there are only finitely many isolated "superrigid" smooth curves $\{C_i \subset X\}$ in various genera and degrees. The number of such curves of genus g in degree β is $\mathrm{GV}_{g,\beta}$. The enumerative invariants count these numbers combined with multiple-cover contributions.

In genus zero, C_i are all smooth $(-1,-1)$ curves. For the quantum K-theory in genus zero, the multiple-cover contributions can be summarized in the factor $[\cdot]$ in (1-2) involving functions a, b, c and d. This will be explained in more details in Section 4.

1.3 From the JM conjecture to the main theorem

In this subsection we show that one can obtain all genus-zero quantum K-invariants with cotangent lines L_i (descendants) as finite linear integral combinations of Gopakumar–Vafa invariants, assuming Theorem 1.2.

Proposition 1.5 *Theorem 1.2 implies the main theorem.*

Proof We note that the knowledge of $J^K(t)$ in Theorem 1.2 is equivalent to that of a collection of genus-zero quantum K-invariants with arbitrary insertions, including any powers of cotangent line bundles $L_1^{\otimes k}$ for $k \in \mathbb{Z}$ at the first marked point, and only $\mathrm{ch}^{-1}(H^2(X))$ insertions without L_i at all remaining marked points. In order to get all genus-zero quantum K-invariants, we will need to allow cotangent line bundles L_i and general K-classes at all marked points. This can be done as follows.

(1) **Inserting L_i at the i^{th} point** From [Lee and Pandharipande 2004, (2)], we have

$$L_i = L_1^{-1} \otimes \mathcal{O}_{D_{i\,|\,1}},$$

where $D_{i\,|\,1}$ is the virtual divisor whose general element is a map from a curve with a node separating the 1st and the i^{th} marked point to X. Inserting L_i can be *inductively* reduced to inserting additional L_1, which is already included in the formulation of Theorem 1.2.

(2) **Inserting K-classes in $\mathrm{ch}^{-1}(H^{\geq 4}(X))$** By (3-2) in Lemma 3.2, quantum K-invariants with insertions in $\mathrm{ch}^{-1}(H^{\geq 4}(X))$ vanish. Hence they can be omitted.

(3) **Inserting K-classes in $\mathrm{ch}^{-1}(H^0(X))$** For $\mathrm{ch}^{-1}(H^0(X))$, that is, $\mathcal{O}_X$, the K-theoretic *string equation* in [Lee 2004, (22)] reduce it to the "essential" invariants already covered in Theorem 1.2.

Therefore all quantum K-invariants can be reduced to the essential invariants by simple explicit formulas. □

2 Givental's adelic characterization

In this section, we briefly summarize results about adelic characterization in quantum K-theory of Givental and Tonita [2014]. Further details can be found in part I [Chou and Lee 2022b, Section 2] or the original papers by Givental [2016; 2015d; 2015g].

Let $\Lambda = \mathbb{Q}[[Q]]$ be the Novikov ring and

$$\boldsymbol{K} := K^0(X) \otimes \Lambda.$$

Givental's loop space for quantum K-theory is defined as

$$\mathcal{K} := K^0(X)(q) \otimes \Lambda.$$

$\mathcal{K}$ has a natural symplectic structure with the symplectic form Ω,

$$\mathcal{K} \ni f, g \mapsto \Omega(f,g) := (\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty})(f(q), g(q^{-1}))^K \frac{dq}{q}.$$

$\mathcal{K}$ admits the following Lagrangian polarization with respect to Ω:

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- := \boldsymbol{K}[q,q^{-1}] \oplus \{f(q) \in \mathcal{K} \mid f(0) \neq \infty,\ f(\infty) = 0\}.$$

Definition 2.1 The *big J-function* of X in the quantum K-theory is defined as a map $\mathcal{K}_+ \to \mathcal{K}$ given by

$$\boldsymbol{t} \mapsto J^K(\boldsymbol{t}) := (1-q) + \boldsymbol{t}(q) + \sum_{\alpha} \Phi^{\alpha} \sum_{n,\beta} \frac{Q^{\beta}}{n!} \left\langle \frac{\Phi_{\alpha}}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle_{0,n+1,\beta}^{X,K},$$

where $\{\Phi_\alpha\}$ and $\{\Phi^\alpha\}$ are Poincaré-dual bases of $K^0(X)$ with respect to $(\cdot,\cdot)^K$. The *overruled Lagrangian cone* $\mathcal{L}^K \subset \mathcal{K}$ is defined to be the range of $J^K(\mathcal{K}_+)$.

If we apply the virtual Kawasaki–Hirzebruch–Riemann–Roch formula for Deligne–Mumford stacks (VKHRR) [Tonita 2014] on $\overline{\mathcal{M}}_{g,n}(X,\beta)$, we get

$$\begin{aligned} J^K(\boldsymbol{t}) &= (1-q) + \boldsymbol{t}(q) + \sum_{n,\beta,\alpha} \frac{Q^{\beta}\Phi^{\alpha}}{n!} \left\langle \frac{\Phi_{\alpha}}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle_{0,n+1,\beta}^{X,K} \\ &= (1-q) + \boldsymbol{t}(q) + \sum_{n,\beta,\alpha} \frac{Q^{\beta}\Phi^{\alpha}}{n!} \sum_{\zeta} \left\langle \frac{\Phi_{\alpha}}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle_{0,n+1,\beta}^{X_\zeta}, \end{aligned}$$

where in $\sum_\zeta$ ζ runs through all roots of unity (including 1), X_ζ stand for the collection of inertia stacks ("Kawasaki strata") where g acts on L_1 with eigenvalue ζ, and $\langle\cdot\rangle^{X_\zeta}$ denotes the contributions of X_ζ in the Riemann–Roch formula. In other words, $\langle\cdot\rangle^{X,K}$ represents (true) quantum K-invariants while $\langle\cdot\rangle^{X_\zeta}$ stands for (collections of) *cohomological* invariants. Symbolically, we have

$$\langle\cdot\rangle^{X,K} = \sum_\zeta \langle\cdot\rangle^{X_\zeta}.$$

Let ζ be a primitive r^{th} root of unity. Recall in part I [Chou and Lee 2022b, Section 2.3] (taken from [Givental and Tonita 2014, Section 7]) that we use $\operatorname{arm}(L)$, $\operatorname{leg}_r(L)$ and $\operatorname{tail}_\zeta(L)$ to denote certain specific contributions from Kawasaki strata in the Riemann–Roch formula for stacks:

$$\begin{aligned}
&\operatorname{arm}(q) = \sum_{n,\beta,\alpha} \frac{Q^\beta \Phi^\alpha}{n!} \sum_{\zeta'\neq 1} \left\langle \frac{\Phi_\alpha}{1-qL}, \boldsymbol{t}(L),\dots,\boldsymbol{t}(L)\right\rangle_{0,n+1,\beta}^{X_{\zeta'}}, \\
&\operatorname{leg}_r(q) = \Psi^r(\operatorname{arm}(q)|_{\boldsymbol{t}=0}), \qquad\qquad (2\text{-}1)\\
&\operatorname{tail}_\zeta(q) = \sum_{n,\beta,\alpha} \frac{Q^\beta \Phi^\alpha}{n!} \sum_{\zeta'\neq \zeta} \left\langle \frac{\Phi_\alpha}{1-qL}, \boldsymbol{t}(L),\dots,\boldsymbol{t}(L)\right\rangle_{0,n+1,\beta}^{X_{\zeta'}}.
\end{aligned}$$

Here ζ' is an arbitrary root of unity and Ψ^r are the *Adams operations*. Recall that Adams operations are additive and multiplicative endomorphisms of K-theory, or more generally λ-rings, acting on line bundles by $\Psi^r(L) = L^{\otimes r}$. Here Ψ^r also act on the Novikov variables by $\Psi^r(Q^\beta) = Q^{r\beta}$ and on q by $\Psi^r(q) = q^r$.

Another necessary ingredient in the proof is the *fake quantum K-theory*, defined by "naively" applying the virtual Riemann–Roch for schemes to stacks

$$\langle \tau_{d_1}(\Phi_1)\cdots\tau_{d_n}(\Phi_n)\rangle_{g,n,\beta}^{X,\text{fake}} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}} \operatorname{td}(T^{\text{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}) \operatorname{ch}\left(\bigotimes_{i=1}^n \operatorname{ev}_i^*(\Phi_i) L_i^{d_i}\right),$$

where $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{vir}}$ is the (cohomological) virtual fundamental class and $[T^{\text{vir}}]$ the virtual tangent bundle. There is a similar Givental formalism for the fake quantum K-theory. Givental's loop space is changed to the formal series

$$\mathcal{K}^{\text{fake}} := \boldsymbol{K}[\![1-q,(1-q)^{-1}]\!]$$

with the symplectic form Ω^{fake}

$$\Omega^{\text{fake}}(f,g) := -\operatorname{Res}_{q=1}(f(q), g(q^{-1}))^K \frac{dq}{q},$$

and a corresponding Lagrangian polarization:

$$\mathcal{K}^{\text{fake}} = \mathcal{K}_+^{\text{fake}} \oplus \mathcal{K}_-^{\text{fake}} := \boldsymbol{K}[\![q-1]\!] \oplus \operatorname{span}_{\boldsymbol{K}} \left\{ \frac{q^k}{(1-q)^{k+1}} \right\}_{k\geq 0}.$$

The big J-function of X in the fake K-theory is defined as a map $\mathcal{K}_+^{\text{fake}} \to \mathcal{K}^{\text{fake}}$:

$$\boldsymbol{t} \mapsto J^{\text{fake}}(\boldsymbol{t}) := (1-q) + \boldsymbol{t}(q) + \sum_\alpha \Phi^\alpha \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\Phi_\alpha}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle_{0,n+1,d}^{X,\text{fake}}.$$

The definitions in (2-1) can be interpreted as formal power series as in part I. For this purpose a more general version of arm and tail, which allows "nongeometric" input $f \in \mathcal{K}$, will be defined. Let $f \in \mathcal{K}$ and ζ be any primitive r^{th} root of unity with $r \geq 2$. Define

$$\begin{aligned} \operatorname{arm}(f;q) &:= \pi_+^{\text{fake}}(f|_{q=1}) - \pi_+(f)|_{q=1} \in \boldsymbol{K}[\![1-q]\!], \\ \operatorname{leg}_r(q) &:= \Psi^r(\pi_+^{\text{fake}}(J^K(0)|_{q=1})) \in \boldsymbol{K}[\![1-q]\!], \\ \operatorname{tail}_\zeta(f;q) &:= \pi_+^{1-\zeta q}(f|_{q=\zeta^{-1}}) - \pi_+(f)|_{q=\zeta^{-1}} \in \boldsymbol{K}[\![1-\zeta q]\!], \\ \delta_\zeta(f;q) &:= [1-\zeta^{-1}q + \pi_+(f)(\zeta^{-1}q) + \operatorname{tail}_\zeta(f;\zeta^{-1}q)]_{q=1} \in \boldsymbol{K}[\![1-q]\!], \end{aligned} \tag{2-2}$$

where

$$\pi_+ \colon \mathcal{K} \to \mathcal{K}_+, \quad \pi_+^{\text{fake}} \colon \mathcal{K}^{\text{fake}} \to \mathcal{K}_+^{\text{fake}}, \quad \pi_+^{1-\zeta q} \colon \boldsymbol{K}[\![1-\zeta q, (1-\zeta q)^{-1}]\!] \to \boldsymbol{K}[\![1-\zeta q]\!]$$

are projections and $[\cdot]|_{q=\zeta}$ *stands for series expansion at* $q=\zeta$.

Remark 2.2 Suitably interpreted, the generalized definitions above are actually the same as those in (2-1) when $f = J^K(\boldsymbol{t})$. For example, we will see in Proposition 2.3 that

$$\pi_+^{\text{fake}}(J^K(\boldsymbol{t})|_{q=1}) - \pi_+(J^K(\boldsymbol{t}))|_{q=1} = \pi_+^{\text{fake}}(J^K(\boldsymbol{t})|_{q=1}) - [(1-q)+\boldsymbol{t}] = \operatorname{arm}(q),$$

and hence the two definitions of $\operatorname{arm}(q)$ agree. The $\operatorname{leg}_r(q)$ in (2-2) is consistent with $\operatorname{leg}_r(q)|_{q=1}$ in (2-1). With this understanding, when f takes the form

$$1 - q + \boldsymbol{t} + f_-(\boldsymbol{t}, q)$$

with $f_-(\boldsymbol{t}, q) \in \mathcal{K}_-$, we *abuse the notation* and do not distinguish these two.

Propositions 2.3 and 2.4 justify the definition in (2-2).

Proposition 2.3 (essentially [Givental and Tonita 2014, Proposition 1]) *We have*

$$J^K(\boldsymbol{t})|_{q=1} = J^{\text{fake}}(\boldsymbol{t} + \operatorname{arm}),$$

where $\operatorname{arm}(q) := \operatorname{arm}(J^K; q)$.

For $\zeta \neq 1$ a primitive r^{th} root of unity, another "Kawasaki strata" will also be used, namely, the *stem space*, which is isomorphic to

$$\overline{\mathcal{M}}_{0,n+2,\beta}^X(\zeta) := \overline{\mathcal{M}}_{0,n+2}([X/\mathbb{Z}_r], \beta; (g, 1, \dots, 1, g^{-1})).$$

Here the group elements g, 1 and g^{-1} signal the twisted sectors in which the marked points lie. As in part I, the notation $[\cdot]^{X_\zeta}$ will be reserved for stem contributions

(2-3) $$[T_1(L), T(L), \dots, T(L), T_{n+2}(L)]^{X_\zeta}_{0,n+2,\beta} := \int_{[\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta)]^{\mathrm{vir}}} \mathrm{td}(T_{\overline{\mathcal{M}}})\,\mathrm{ch}\biggl(\frac{\mathrm{ev}_1^*(T_1(L))\,\mathrm{ev}_{n+2}^*(T_{n+2}(L))\prod_{i=2}^{n+1}\mathrm{ev}_i^*\,T(L)}{\mathrm{Tr}(\Lambda^* N^*_{\overline{\mathcal{M}}})}\biggr),$$

where $[\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta)]^{\mathrm{vir}}$ is the virtual fundamental class, $T_{\overline{\mathcal{M}}}$ is the (virtual) tangent bundle to $\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta)$ and $N_{\overline{\mathcal{M}}}$ is the (virtual) normal bundle of $\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta) \subset \overline{\mathcal{M}}_{0,nr+2}(X, r\beta)$. We define the trace bundle $\mathrm{Tr}(F)$ to be the virtual bundle

$$\mathrm{Tr}(F) := \sum_\lambda \lambda F_\lambda,$$

where the sum is over the eigenvalue of g-action with the corresponding eigenspace.

Proposition 2.4 [Givental and Tonita 2014, Section 8] *Let ζ be a primitive r^{th} root of unity. We have*

$$\sum_{n,\beta,\alpha} \frac{Q^\beta \Phi^\alpha}{n!}\Bigl\langle \frac{\Phi_\alpha}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\Bigr\rangle^{X_\zeta}_{0,n+1,\beta} = \sum_{n,\beta,\alpha} \frac{Q^{r\beta}\Phi^\alpha}{n!}\Bigl[\frac{\Phi_\alpha}{1-q\zeta L^{1/r}}, \mathrm{leg}_r(L), \dots, \mathrm{leg}_r(L), \delta_\zeta(L^{1/r})\Bigr]^{X_\zeta}_{0,n+2,\beta},$$

where $\delta_\zeta(q) = \delta_\zeta(J^K; q)$ in (2-2).

Remark 2.5 Since $L^{1/r}$ is unipotent on a fixed moduli, the expansion

$$\frac{1}{1-q\zeta L^{1/r}} = \sum_{i\geq 0} \frac{(\zeta q)^i (L^{1/r}-1)^i}{(1-\zeta q)^{i+1}}$$

is a finite sum. When combined with the VKHRR for $\overline{\mathcal{M}}_{0,n}(X,\beta)$, this expresses $J^K(\boldsymbol{t}, q) - (1-q) - \boldsymbol{t}(q)$ as a sum of rational functions with poles only at roots of unity.

Theorem 2.6 (adelic characterization, [Givental and Tonita 2014, Section 6]) *We have $f \in \mathcal{L}^K$ if and only if*:

(i) *$f|_{\mathcal{K}_-}$ has poles only at roots of unity.*

(ii) $f|_{q=1} = J^{\mathrm{fake}}(-(1-q) + \pi_+(f)|_{q=1} + \mathrm{arm}(f))$.

(iii) *Let ζ be any primitive r^{th} roots of unity with $r \geq 2$. We have*

$$f|_{q=\zeta^{-1}} = \delta_\zeta(f; \zeta q) + \sum_{n,\beta,\alpha} \frac{Q^{r\beta}\Phi^\alpha}{n!}\Bigl[\frac{\Phi_\alpha}{1-\zeta q L^{1/r}}, \mathrm{leg}_r(L), \dots, \mathrm{leg}_r(L), \delta_\zeta(f; L^{1/r})\Bigr]^{X_\zeta}_{0,n+2,\beta}.$$

Remark 2.7 Our formulation of criteria (ii) and (iii) in the adelic characterization might appear different from the ones in [loc. cit., Section 6]. They are actually equivalent.

The criterion (ii), $f|_{q=1} \in \mathcal{L}^{\text{fake}}$, in [loc. cit., Section 6] is equivalent to our formulation since if $f \in \mathcal{L}^K$, then

$$f|_{q=1} = J^{\text{fake}}(-(1-q) + \pi_+^{\text{fake}}(f)) = J^{\text{fake}}(-(1-q) + \pi_+(f) + \operatorname{arm}(f)).$$

Conversely, $J^{\text{fake}}(-(1-q) + \pi_+(f) + \operatorname{arm}(f)) \in \mathcal{L}^{\text{fake}}$ is clear.

The criterion (iii) in [loc. cit., Section 6] is formulated as $f|_{q=\zeta^{-1}} \in T^r$. Here T^r is some tangent space of $\mathcal{L}^{\text{fake}}$ twisted by some characteristic classes and with suitable change of variable. This criterion is exactly the reformulation of the stem theory computation; see the corollary in [loc. cit., Section 8] for detail. Our formalism is exactly the corresponding stem theory equality.

3 Proof of $\tilde{J} = J^K$

In this section, we use adelic characterization (Theorem 2.6) to show that $\tilde{J}$ defined in (1-2) is in $\mathcal{L}^K$. It follows from a uniqueness theorem that the mirror transformation is trivial and $\tilde{J} = J^K$. Combined with Proposition 1.5, this concludes the proof of the main theorem.

3.1 Outline of the proof

(1) In Section 3.2 we recall basic facts about (twisted) Gromov–Witten invariants on Calabi–Yau threefolds. In particular, a vanishing theorem which was used in Proposition 1.5 is proved.

(2) To check the adelic characterization, $\tilde{J}$ is expanded in partial fractions with poles at $1-\zeta q$ for all roots of unity ζ. Note that the orders of poles are ≤ 3 by definition of $\tilde{J}$. The computation is divided into four parts in Lemmas 3.3–3.6 for the $a(r, q^r)$, $b(r, q^r)$, $c(r, q, \beta_t)$ and $d(r, q, \beta_t)$ terms, respectively.

(3) Motivated by Propositions 2.3 and 2.4, we emulate the geometric situation to formulate the following functions for the $\tilde{J}$ in Section 3.4:

$$\widetilde{\operatorname{arm}}(q) := \operatorname{arm}(\tilde{J}; q) \in \boldsymbol{K}[\![1-q]\!], \qquad \widetilde{\operatorname{leg}}_r(q) := \Psi^r(\widetilde{\operatorname{arm}}(q)|_{t=0}) \in \boldsymbol{K}[\![1-q]\!],$$
$$\widetilde{\operatorname{tail}}_\zeta(q) := \operatorname{tail}_\zeta(\tilde{J}; q) \in \boldsymbol{K}[\![1-\zeta q]\!], \qquad \tilde{\delta}_\zeta(q) := \delta_\zeta(\tilde{J}; q) \in \boldsymbol{K}[\![1-q]\!].$$

They serve as bridges for comparison with the corresponding functions associated to J^K from the Kawasaki strata.

(4) In Propositions 3.10 and 3.12, we show that $\widetilde{\operatorname{arm}}$, $\widetilde{\operatorname{leg}}$ and $\widetilde{\operatorname{tail}}$ satisfy the properties of their geometric counterparts in Propositions 2.3 and 2.4. We then apply the adelic characterization (Theorem 2.6) to conclude $\tilde{J} \in \mathcal{L}^K$, and moreover $\tilde{J} = J^K(t)$.

3.2 Twisted Gromov–Witten theory on CY3

Recall the notation of the *stem invariants*, which are important organizing ingredients in the virtual Hirzebruch–Riemann–Roch for stacks [Givental and Tonita 2014; Chou and Lee 2022b]:

$$[T_1(L), T(L), \dots, T(L), T_{n+2}(L)]^{X_\zeta}_{0,n+2,\beta}$$
$$:= \int_{[\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta)]^{\mathrm{vir}}} \mathrm{td}(T_{\overline{\mathcal{M}}})\,\mathrm{ch}\biggl(\frac{\mathrm{ev}_1^*(T_1(L))\,\mathrm{ev}_{n+2}^*(T_{n+2}(L))\prod_{i=2}^{n+1}\mathrm{ev}_i^*\,T(L)}{\mathrm{Tr}(\Lambda^* N^*_{\overline{\mathcal{M}}})}\biggr).$$

These can be interpreted as *twisted* (cohomological) GW invariants with the twisting class

$$\mathrm{td}(T_{\overline{\mathcal{M}}})\,\mathrm{ch}\biggl(\frac{1}{\mathrm{Tr}(\Lambda^* N^*_{\overline{\mathcal{M}}})}\biggr).$$

Such twisting classes come from the deformation theory of the moduli of stable maps and consist of three parts; see eg [Givental and Tonita 2014, Section 8].

Type *A* $\mathrm{td}(\pi^K_*\,\mathrm{ev}^*(TX))\prod_{k=1}^{r-1}\mathrm{td}_{\zeta^k}(\pi^K_*\,\mathrm{ev}^*(T_X\otimes\mathbb{C}_{\zeta^k}))$ Here $\pi:\mathcal{C}\to\overline{\mathcal{M}}$ and $\mathrm{ev}:\mathcal{C}\to X/\mathbb{Z}_r$ form the universal (orbifold) stable map diagram, π^K_* is the K-theoretic pushforward of π, $\mathbb{C}_{\zeta^k}$ is the line bundle over $B\mathbb{Z}_r$ with g acting by ζ^k, and for any line bundle l, define the invertible multiplicative characteristic classes

$$\mathrm{td}(l) := \frac{c_1(l)}{1-e^{-c_1(l)}},\qquad \mathrm{td}_\lambda(l) := \frac{1}{1-\lambda e^{-c_1(l)}}.$$

Type *B* $\mathrm{td}(\pi^K_*(-L^{-1}))\prod_{k=1}^{r-1}\mathrm{td}_{\zeta^k}(\pi^K_*(-L^{-1}\otimes\mathrm{ev}^*(\mathbb{C}_{\zeta^k})))$ Here $L=L_{n+3}$ is the universal cotangent line bundle of

$$\mathcal{C}\cong\overline{\mathcal{M}}^{X/\mathbb{Z}_r,\beta}_{0,n+3}(g,1,\dots,1,g^{-1},1).$$

Type *C* $\mathrm{td}^\vee(-\pi^K_* i_*\mathcal{O}_{Z_g})\,\mathrm{td}^\vee(-\pi^K_* i_*\mathcal{O}_{Z_1})\prod_{i=1}^{k-1}\mathrm{td}^\vee_{\zeta^k}(-\pi^K_* i_*\mathcal{O}_{Z_1})$ Here Z_1 stands for the unramified nodal locus, and Z_g stands for ramified one with $i:Z\to\mathcal{C}$ the embedding of nodal locus. For any line bundle l,

$$\mathrm{td}^\vee(l) = \frac{-c_1(l)}{1-e^{c_1(l)}},\qquad \mathrm{td}^\vee_\lambda(l) = \frac{1}{1-\lambda e^{c_1(l)}}.$$

We start with the following two observations:

Lemma 3.1 *We have*

$$\mathrm{td}(T_{\overline{\mathcal{M}}})\,\mathrm{ch}\biggl(\frac{1}{\mathrm{Tr}(\Lambda^* N^*_{\overline{\mathcal{M}}})}\biggr) =: r^{-n} + T_{0,n+2,\beta}(\zeta),$$

where $T_{0,n+2,\beta}(\zeta)\in H^{>0}(\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta))$, *with* 0, n, β *and* ζ *inherited from* $\overline{\mathcal{M}}=\overline{\mathcal{M}}^X_{0,n+2,\beta}(\zeta)$.

Proof Note that $N_{\overline{\mathcal{M}}}$ has virtual dimension $(r-1)n$, since $\overline{\mathcal{M}}$ is considered as Kawasaki strata in $\overline{\mathcal{M}}^X_{0,nr+2,\beta}$.

Only the twisting classes on the normal bundle give a nontrivial constant. It is of the form

$$\prod_{k=1}^{r-1}\mathrm{td}_{\zeta^k}\bigl(\pi^K_*(E\otimes\mathrm{ev}^*(\mathbb{C}_{\zeta^k}))\bigr),$$

where E is a virtual bundle of rank n. The constant term is given by

$$\left(\prod_{k=1}^{r-1}\frac{1}{1-\zeta^k}\right)^n = r^{-n}. \qquad \square$$

Lemma 3.2 *Let X be a Calabi–Yau threefold. If $\deg_{\mathbb{C}}\phi_1 \geq 2$ then*

$$\langle\tau_{k_1}(\phi_1),\dots,\tau_{k_n}(\phi_n)\rangle^{\#}_{g,n,\beta\neq 0} = 0, \tag{3-1}$$

where # denotes cohomological GW invariants with any twisting.

In K-theory let $\Phi \in K(X)$. If $\deg_{\mathbb{C}}\operatorname{ch}(\Phi_1)\geq 2$, we have

$$\langle\tau_{k_1}(\Phi_1),\dots,\tau_{k_n}(\Phi_n)\rangle^{K}_{g,n,\beta\neq 0} = 0. \tag{3-2}$$

Proof We start with (3-1). Let

$$\pi_1\colon \overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,1}(X,\beta)$$

be the forgetful map forgetting the last $n-1$ marked points and $T\in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ be the twisting class. By the projection formula

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} T\prod_{i=1}^{n}(\psi_i^{k_i}\operatorname{ev}_i^*\phi_i) = \int_{[\overline{\mathcal{M}}_{g,1}(X,\beta)]^{\mathrm{vir}}}(\operatorname{ev}_1^*\phi_1)(\pi_1)_*\left(T\psi_1^{k_1}\prod_{i=2}^{n}\psi_i^{k_i}\operatorname{ev}_i^*\phi_i\right).$$

Since $[\overline{\mathcal{M}}_{g,1}(X,\beta)]^{\mathrm{vir}}$ has virtual dimension 1, while $\deg_{\mathbb{C}}(\phi_i)\geq 2$, the integral in the last equation must vanish. This shows (3-1).

Arguing similarly for K-theory, let $F\in K(\overline{M}_{g,n}(X,\beta))$ be the twisting bundle. By the projection formula

$$\chi\left(\overline{M}_{g,n}(X,\beta);\mathcal{O}^{\mathrm{vir}}\otimes F\otimes\left(\bigotimes_{i=1}^{n}L_i^{d_i}\operatorname{ev}_i^*(\Phi_i)\right)\right)$$
$$= \chi\left(\overline{M}_{g,1}(X,\beta);\mathcal{O}^{\mathrm{vir}}\otimes\operatorname{ev}_1^*\Phi_1\otimes(\pi_1^K)_*\left(F\otimes L_1^{d_1}\otimes\bigotimes_{i=2}^{n}L_i^{k_i}\operatorname{ev}_i^*\Phi_1\right)\right) = 0,$$

where the vanishing holds for the same dimensional reason. $\square$

3.3 Expansion of $\tilde{J}$

In this subsection, we rewrite $\tilde{J}$ in terms of the sum of a polynomial and fractions of the form $(1-\zeta q)^{-i}$ for ζ any root of unity and $i\in\mathbb{Z}>0$. The computation will be long and tedious. We separate it into four parts.

We first fix some notation/conventions. Let $\sum_{\zeta^r=1}$ denote the sum over primitive r^{th} roots of unity. For $\beta\in H_2(X,\mathbb{Z})\setminus\{0\}$,

$$\operatorname{ind}(\beta) := \max\{k\in\mathbb{N}\mid \beta/k\in H_2(X,\mathbb{Z})\}.$$

Finally for $\operatorname{ind}(\beta)=1$, we define

$$\mathrm{GV}^{(\gamma)}_{d\beta} := \sum_{k\mid d}k^{\gamma}\,\mathrm{GV}_{k\beta}.$$

Lemma 3.3 *We have*

$$(1-q)\sum_{j=1}^{n_1}\Phi^{1j}\sum_{0\neq\beta\in H_2(X,\mathbb{Z})}\sum_{r=1}^{\infty}\mathrm{GV}_\beta\, Q^{r\beta}\beta_j a(r,q^r)$$

$$= (1-q)\sum_{j=1}^{n_1}\Phi^{1j}\sum_{\mathrm{ind}(\beta)=1}\sum_{r=1}^{\infty}\sum_{\zeta^r=1}\sum_{d=1}^{\infty}Q^{rd\beta}\beta_j\biggl(\frac{\mathrm{GV}^{(1)}_{d\beta}-\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2)}{1-\zeta q}+\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2)}{(1-\zeta q)^2}\biggr)$$

$$= \sum_{j=1}^{n_1}\Phi^{1j}\sum_{\mathrm{ind}(\beta)=1}\sum_{r=1}^{\infty}\sum_{\zeta^r=1}\sum_{d=1}^{\infty}Q^{rd\beta}\beta_j$$

$$\times\biggl(\zeta^{-1}\Bigl(\mathrm{GV}^{(1)}_{d\beta}-\frac{1}{r^2d^2}\mathrm{GV}^{(3)}_{d\beta}\Bigr)+\frac{(1-\zeta^{-1})(\mathrm{GV}^{(1)}_{d\beta}-\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))+\zeta^{-1}(\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))}{1-\zeta q}$$

$$+\frac{(1-\zeta^{-1})(\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))}{(1-\zeta q)^2}\biggr),$$

Proof For the first equality, we use the following expansions:

$$\frac{1}{1-q^r}=\sum_{k\mid r}\sum_{\zeta^k=1}\frac{1}{r(1-\zeta q)},\qquad \frac{1}{(1-q^r)^2}=\sum_{k\mid r}\sum_{\zeta^k=1}\Bigl(\frac{1}{r^2(1-\zeta q)^2}+\frac{r-1}{r^2(1-\zeta q)}\Bigr).$$

Let ζ be any primitive r^{th} root of unity with $r\mid M$ and $\mathrm{ind}(\beta')=1$. We compute

$$\mathrm{Coeff}\Bigl(\sum_{0\neq\beta\in H_2(X,\mathbb{Z})}\sum_{r=1}^{\infty}\mathrm{GV}_\beta\, Q^{r\beta}a(r,q^r),\frac{Q^{M\beta'}}{1-\zeta q}\Bigr)$$

$$=\mathrm{Coeff}\Bigl(\sum_{k\mid(M/r)}\Bigl(\frac{(M/(rk))(rk-1)}{1-q^{rk}}+\frac{M/(rk)}{(1-q^{rk})^2}\Bigr)\mathrm{GV}_{M\beta'/rk},\frac{1}{1-\zeta q}\Bigr)$$

$$=\mathrm{Coeff}\Bigl(\sum_{k\mid(M/r)}\Bigl(\frac{(M/(rk))(rk-1)}{rk(1-\zeta q)}+\frac{(M/(rk))(rk-1)}{(rk)^2(1-\zeta q)}\Bigr)\mathrm{GV}_{M\beta'/rk},\frac{1}{1-\zeta q}\Bigr)$$

$$=\mathrm{GV}^{(1)}_{M\beta'/r}-\frac{1}{M^2}\mathrm{GV}^{(3)}_{M\beta'/r}.$$

The computation for the coefficient of $1/(1-\zeta q)^2$ is similar.

For the second equality, we write $1-q=(1-\zeta^{-1})+\zeta^{-1}(1-\zeta q)$. □

Lemma 3.4 *We have*

$$(1-q)\Phi^{01}\sum_{0\neq\beta\in H_2(X,\mathbb{Z})}\sum_{r=1}^{\infty}\mathrm{GV}_\beta\, Q^{r\beta}b(r,q^r)$$

$$=(1-q)\Phi^{01}\sum_{\mathrm{ind}(\beta)=1}\sum_{r=1}^{\infty}\sum_{\zeta^r=1}\sum_{d\geq 1}Q^{rd\beta}$$

$$\times\Bigl(\frac{rd\,\mathrm{GV}^{(-1)}_{d\beta}-\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{1-\zeta q}+\frac{3\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{(1-\zeta q)^2}+\frac{-2\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{(1-\zeta q)^3}\Bigr)$$

$$= \Phi^{01} \sum_{\operatorname{ind}(\beta)=1} \sum_{r=1}^{\infty} \sum_{\zeta^r=1} \sum_{d\ge 1} Q^{rd\beta} \Biggl(\zeta^{-1} \Bigl(rd\,\mathrm{GV}^{(-1)}_{d\beta} - \frac{1}{r^3 d^3} \mathrm{GV}^{(3)}_{d\beta} \Bigr) + \frac{(1-\zeta^{-1})(rd\,\mathrm{GV}^{(-1)}_{d\beta} - \mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)) + \zeta^{-1}(3\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{1-\zeta q} + \frac{(1-\zeta^{-1})(3\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)) + \zeta^{-1}(-2\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{(1-\zeta q)^2} + \frac{(1-\zeta^{-1})(-2\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{(1-\zeta q)^3} \Biggr).$$

Proof The computation is similar to Lemma 3.3 with the only difference that one more expansion is used:

$$\frac{1}{(1-q^r)^3} = \sum_{k \mid \frac{M}{r}} \sum_{\zeta^k=1} \Bigl(\frac{1}{r^3(1-\zeta q)^3} + \frac{3(r-1)}{2r^3(1-\zeta q)^2} + \frac{2r^2-3r+1}{2r^3(1-\zeta q)} \Bigr). \qquad \square$$

Expansions for $c(r,q,\beta_t)$ and $d(r,q,\beta_t)$ can be computed using the same method.

Lemma 3.5 *We have*

$$(1-q) \sum_{j=1}^{n_1} \Phi^{1j} \sum_{0\ne\beta\in H_2(X,\mathbb{Z})} \sum_{r=1}^{\infty} \mathrm{GV}_\beta\, Q^{r\beta} \beta_j c(r,q,\beta_t)$$
$$= \sum_{j=1}^{n_1} \Phi^{1j} \sum_{\operatorname{ind}(\beta)=1} \sum_{d=1}^{\infty} \beta_j \beta_t \times \Biggl[\sum_{r\ge 2} \sum_{\zeta^r=1} Q^{rd\beta} \Bigl(\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2)}{1-\zeta q} \Bigr) + Q^{d\beta} \Bigl(\frac{\mathrm{GV}^{(2)}_{d\beta}}{2d} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{2d^2} + \frac{((e^{\beta_t}-1)/\beta_t)(\mathrm{GV}^{(3)}_{d\beta}/d^2)}{1-q} \Bigr) \Biggr].$$

Lemma 3.6 *We have*

$$(1-q)\Phi^{01} \sum_{0\ne\beta\in H_2(X,\mathbb{Z})} \sum_{r=1}^{\infty} \mathrm{GV}_\beta\, Q^{r\beta} d(r,q,\beta_t)$$
$$= \Phi^{01} \sum_{\operatorname{ind}(\beta)=1} \sum_{d=1}^{\infty} \Biggl[\sum_{r\ge 2} \sum_{\zeta^r=1} Q^{rd\beta} \beta_t \Bigl(\frac{\mathrm{GV}^{(1)}_{d\beta}/(rd) + \mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{1-\zeta q} + \frac{-\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{(1-\zeta q)^2} \Bigr) + Q^{d\beta} \Bigl(f_0(d,\beta_t) + \frac{f_1(d,\beta_t)}{1-q} + \frac{f_2(d,\beta_t)}{(1-q)^2} \Bigr) \Biggr].$$

where

$$f_0(d,\beta_t) = \beta_t \Bigl(\tfrac{1}{2} \mathrm{GV}^{(0)}_{d\beta} - \frac{5\,\mathrm{GV}^{(1)}_{d\beta}}{12d} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{12d^3} \Bigr),$$
$$f_1(d,\beta_t) = \frac{\beta_t\,\mathrm{GV}^{(1)}_{d\beta}}{d} + \frac{\beta_t^2\,\mathrm{GV}^{(2)}_{d\beta}}{2d^2} + \bigl((3-\beta_t)e^{\beta_t} - 3 - \beta_t - \tfrac{1}{2}\beta_t^2\bigr) \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3},$$
$$f_2(d,\beta_t) = (e^{\beta_t}(\beta_t-2)+2) \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3}.$$

3.4 Arm, leg and tail

Lemma 3.7 *We have*

$$\widetilde{\mathrm{arm}}(q) = \sum_{\mathrm{ind}(\beta)=1} \sum_{d=1}^{\infty} Q^{d\beta} \Bigg[\sum_{j=1}^{n_1} \Phi^{1j} \beta_j \left(\mathrm{GV}^{(1)}_{d\beta} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^2} + \frac{\beta_t \, \mathrm{GV}^{(2)}_{d\beta}}{2d} - \frac{\beta_t \, \mathrm{GV}^{(3)}_{d\beta}}{2d^2} \right) + \Phi^{01} \left(d \, \mathrm{GV}^{(-1)}_{d\beta} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3} + f_0(d, \beta_t) \right) \Bigg] + O(1-q).$$

In particular, $\mathrm{ch}(\widetilde{\mathrm{arm}}(q)) \in H^{\geq 4}(X)[\![1-q, Q]\!]$.

Proof Note that

$$\pi_+^{\mathrm{fake}} \left(\zeta^{-1}\alpha + \frac{(1-\zeta^{-1}\alpha + \zeta^{-1}\beta)}{1-\zeta q} + \frac{(1-\zeta^{-1})\beta + \zeta^{-1}\gamma}{(1-\zeta q)^2} + \frac{(1-\zeta^{-1})\gamma}{(1-\zeta q)^3} \right) = \pi_+^{\mathrm{fake}} \left((1-q) \left(\frac{\alpha}{1-\zeta q} + \frac{\beta}{(1-\zeta q)^2} + \frac{\gamma}{(1-\zeta q)^3} \right) \right) = \alpha \delta_{1,\zeta} + O(1-q).$$

The lemma follows from Lemmas 3.3–3.6 by taking only $\zeta = 1$ terms. □

Lemma 3.8 *We have*

$$\widetilde{\mathrm{leg}}_r(q) := \Psi^r (\widetilde{\mathrm{arm}}(q)|_{t=0}) = \sum_{\mathrm{ind}(\beta)=1} \sum_{d=1}^{\infty} Q^{rd\beta} \Bigg[\sum_{j=1}^{n_1} r^2 \Phi^{1j} \beta_j \left(\mathrm{GV}^{(1)}_{d\beta} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^2} \right) + r^3 \Phi^{01} \left(d \, \mathrm{GV}^{(-1)}_{d\beta} - \frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3} \right) \Bigg] + O(1-q).$$

In particular, $\mathrm{ch}(\widetilde{\mathrm{leg}}_r(q)) \in H^{\geq 4}(X)[\![1-q, Q]\!]$.

Proof This follows directly from

$$\Psi^r(\Phi^{ij}) := \mathrm{ch}^{-1} \Psi^r \, \mathrm{ch}(\Phi^{ij}) = r^{3-i} \Phi^{ij}$$

for $i \leq 2$, since $\mathrm{ch}(\Phi^{ij}) \in H^{2(3-i)}(X)$ for $i \leq 2$ by Lemma 1.1. □

We won't need any explicit expression for $\widetilde{\mathrm{tail}}$ except for the fact that it has $\deg_{\mathbb{C}} \geq 2$.

Lemma 3.9 $$\mathrm{ch}(\widetilde{\mathrm{tail}}_\zeta(q)) \in H^{\geq 4}(X)[\![1-\zeta q, Q]\!].$$

Proof This follows from the definition of $\widetilde{\mathrm{tail}}$ and the observation that

$$\mathrm{ch}(\pi_+^{1-\zeta q}(\tilde{J} - (1-q) - t)) \in H^{\geq 4}(X)[\![1-\zeta q, Q]\!].$$ □

3.5 Verification of the adelic characterization

We show that $\tilde{J}$ satisfies Theorem 2.6 and hence lies on the K-theoretic Lagrangian cone.

Proposition 3.10 $$\tilde{J}|_{q=1} = J^{\text{fake}}(t+\widetilde{\text{arm}}).$$

Proof Note that

$$\begin{aligned}
\tilde{J}|_{q=1} - \left((1-q)+t+\frac{\sum_{j=1}^{n_1}\Phi^{1j}\partial_{t_j}F(t)}{1-q}+\frac{q\Phi^{01}F(t)}{(1-q)^2}\right) \\
= \widetilde{\text{arm}}(q) + \sum_{\text{ind}(\beta)=1}\sum_{d=1}^{\infty} Q^{d\beta}\Bigg[\sum_{j=1}^{n_1}\Phi^{1j}\beta_j\left(\frac{e^{\beta_t}\,\text{GV}^{(3)}_{d\beta}/d^2}{1-q}\right) \\
+\Phi^{01}\left(\frac{f_1(d,\beta_t)+3\,\text{GV}^{(3)}_{d\beta}/d^3}{1-q}+\frac{f_2(d,\beta_t)-2\,\text{GV}^{(3)}_{d\beta}/d^3}{(1-q)^2}\right)\Bigg],
\end{aligned}$$

and that

$$\begin{aligned}
J^{\text{fake}}(t) - \left((1-q)+t+\frac{\sum_{j=1}^{n_1}\Phi^{1j}\partial_{t_j}F(t)}{1-q}+\frac{q\Phi^{01}F(t)}{(1-q)^2}\right) \\
= \sum_{0\neq\beta\in H_2(X,\mathbb{Z})}\text{GW}_\beta\, Q^\beta e^{\beta_t}\left[\sum_{j=1}^{n_1}\beta_j\Phi^{1j}\left(\frac{1}{1-q}\right)+\Phi^{01}\left(\frac{3-\beta_t}{1-q}+\frac{\beta_t-2}{(1-q)^2}\right)\right] \\
= \sum_{\text{ind}(\beta)=1}\sum_{d\geq 1}\frac{\text{GV}^{(3)}_{d\beta}}{d^3}Q^{d\beta}e^{\beta_t}\left[\sum_{j=1}^{n_1}\beta_j\Phi^{1j}\left(\frac{d}{1-q}\right)+\Phi^{01}\left(\frac{3-\beta_t}{1-q}+\frac{\beta_t-2}{(1-q)^2}\right)\right].
\end{aligned}$$

Invariants of fake theory can be computed by quantum Riemann–Roch [Coates and Givental 2006]. The first equality can be derived by the (cohomological) divisor equation and the fact that $\langle 1\rangle^{\text{fake}}_{0,1,\beta} = \text{GW}_\beta$. The second equality uses the definition (1-1) of GV.

Their difference $\tilde{J} - J^{\text{fake}}$ comes from the $\widetilde{\text{arm}}$ contribution:

$$\tilde{J}|_{q=1} - J^{\text{fake}}(t) = \widetilde{\text{arm}}(q) + \sum_{\text{ind}(\beta)=1}\sum_{d=1}^{\infty}\Phi^{01}\left(\frac{f_1(d,\beta_t)+3\,\text{GV}^{(3)}_{d\beta}/d^3}{1-q}\right) = J^{\text{fake}}(t+\widetilde{\text{arm}}) - J^{\text{fake}}(t).$$

The last equality follows from the Poincaré pairing

$$\left\langle \frac{1}{1-qL}, \widetilde{\text{arm}}(L), t\right\rangle_{0,3,\beta'=0} = \sum_{\text{ind}(\beta)=1}\sum_{d=1}^{\infty}\left(\frac{f_1(d,\beta_t)+3\,\text{GV}^{(3)}_{d\beta}/d^3}{1-q}\right),$$

since $\text{ch}(\widetilde{\text{arm}}(q)) \in H^{\geq 4}(X)[[1-q, Q]]$ by Lemma 3.7, and hence all terms with $\beta' \neq 0$ vanish by Lemma 3.2. □

We divide the computation of $\tilde{J}|_{q=\zeta^{-1}}$ into two parts, $t=0$ and $t\neq 0$.

Proposition 3.11 [Chou and Lee 2022b, Section 4] *Let $t=0$ and ζ be a primitive r^{th} root of unity. Then*

$$\tilde{J}|_{q=\zeta^{-1}} = \tilde{\delta}_\zeta(\zeta q) + \sum_{n,\beta,\alpha}\frac{Q^{r\beta}\Phi_\alpha}{n!}\left[\frac{\Phi^\alpha}{1-\zeta q L^{1/r}}, \text{leg}_r(L),\ldots,\text{leg}_r(L),\tilde{\delta}_\zeta(L^{1/r})\right]^{X_\zeta}_{0,n+2,\beta}.$$

Proof The computation is similar to [Chou and Lee 2022b, Section 4] for X the quintic threefold. We sketch the proof here.

From the definition of $\widetilde{\mathrm{tail}}_\zeta$, the polynomial term of $\tilde{J}|_{q=\zeta^{-1}}$ equals

$$1-q+\widetilde{\mathrm{tail}}_\zeta(q)=(1-\zeta^{-1})+\zeta^{-1}(1-\zeta q)+\widetilde{\mathrm{tail}}_\zeta(q)=\tilde{\delta}_\zeta(\zeta q)\in \boldsymbol{K}[\![1-\zeta q]\!].$$

We use Lemmas 3.3–3.6 for the principal part of the left side. The principal part on the right equals

$$\sum_\alpha \Phi_\alpha\Biggl(\biggl[\frac{\Phi^\alpha}{1-\zeta q L^{1/r}},\mathrm{leg}_r(L),1-\zeta^{-1}L^{1/r}+\widetilde{\mathrm{tail}}_\zeta(\zeta^{-1}L^{1/r})\biggr]^{X_\zeta}_{0,3,\beta=0} \\ +\sum_\beta Q^{r\beta}\biggl[\frac{\Phi^\alpha}{1-\zeta q L^{1/r}},1-\zeta^{-1}L^{1/r}+\widetilde{\mathrm{tail}}_\zeta(\zeta^{-1}L^{1/r})\biggr]^{X_\zeta}_{0,2,\beta}\Biggr).$$

Since $\mathrm{ch}(\mathrm{leg}_r(L))\in H^{\geq 4}(X)[\![1-q,Q]\!]$ by Lemma 3.8, only the $\beta=0$ and $n=3$ term for the first expression has nonzero contribution. It can be computed directly:

$$\text{(3-3)}\quad \sum_\alpha \Phi_\alpha\biggl[\frac{\Phi^\alpha}{1-\zeta q L^{1/r}},\mathrm{leg}_r(L),1-\zeta^{-1}L^{1/r}+\widetilde{\mathrm{tail}}_\zeta(\zeta^{-1}L^{1/r})\biggr]^{X_\zeta}_{0,3,0} \\ =\frac{1-\zeta^{-1}}{r^2(1-\zeta q)}\sum_\alpha \Phi_\alpha\langle \Phi^\alpha,\mathrm{leg}_r(L),1\rangle^X_{0,3,0} \\ =\frac{1-\zeta^{-1}}{1-\zeta q}\sum_{\mathrm{ind}(\beta)=1}\sum_{d=1}^\infty Q^{rd\beta}\biggl[\sum_{j=1}^{n_1}\Phi^{1j}\beta_j\biggl(\mathrm{GV}^{(1)}_{d\beta}-\frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3}\biggr)+r\Phi^{01}\biggl(d\,\mathrm{GV}^{(-1)}_{d\beta}-\frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3}\biggr)\biggr].$$

In the above computation, all terms involving $\widetilde{\mathrm{tail}}$ vanish by Lemma 3.2. The r^{-2} after the first equality follows from the constant term of the twisting class by Lemma 3.1 and the Poincaré pairing on $X/\mathbb{Z}_r$.

We compute the second expression as follows:

$$\sum_\beta Q^{r\beta}\biggl[\frac{\Phi^\alpha}{1-\zeta q L^{1/r}},1-\zeta^{-1}L^{1/r}+\widetilde{\mathrm{tail}}_\zeta(\zeta^{-1}L^{1/r})\biggr]^{X_\zeta}_{0,2,\beta} \\ =\tfrac{1}{r}\sum_\alpha \Phi^\alpha\biggl\langle\frac{\Phi_\alpha}{1-\zeta q L^{1/r}},\Psi^r\Bigl(\sum_\alpha \Phi^\alpha\langle\Phi_\alpha\rangle^{\mathrm{fake}}\Bigr),1-\zeta^{-1}L^{1/r}\biggr\rangle^{X/\mathbb{Z}_r,K}_{0,3,0} \\ +\sum_\alpha\sum_\beta Q^{r\beta}\Phi_\alpha\biggl\langle\frac{\Phi^\alpha}{1-\zeta q L^{1/r}},1-\zeta^{-1}L^{1/r}\biggr\rangle^{X/\mathbb{Z}_r,K}_{0,2,r\beta}.$$

All terms involving $\widetilde{\mathrm{tail}}$ vanish by Lemma 3.2. The term involving $\langle\,\rangle^{\mathrm{fake}}$ comes from the twisting of type C. Two expressions can be computed directly:

$$\text{(3-4)}\quad \frac{1}{r}\sum_\alpha \Phi^\alpha\biggl\langle\frac{\Phi_\alpha}{1-\zeta q},\Psi^r\Bigl(\sum_{\alpha'}\Phi^{\alpha'}\langle\Phi_{\alpha'}\rangle^{\mathrm{fake}}\Bigr),1\biggr\rangle^{X/\mathbb{Z}_r,K}_{0,3,0} \\ =\frac{1-\zeta^{-1}}{r^2(1-\zeta q)}\sum_\alpha \Phi^\alpha\biggl\langle\Phi_\alpha,\sum_{\beta>0}Q^{r\beta}\Bigl(\sum_{j=1}^{n_1}r^2\Phi^{1j}\beta_j\,\mathrm{GW}_\beta+r^3\Phi^{01}\,\mathrm{GW}_\beta\Bigr),1\biggr\rangle^{X,K}_{0,3,0} \\ =\frac{1-\zeta^{-1}}{1-\zeta q}\sum_{\mathrm{ind}(\beta)=1}\sum_{d=1}^\infty\biggl[\sum_{j=1}^{n_1}\Phi^{1j}\beta_j\biggl(\frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3}\biggr)+r\Phi^{01}\biggl(\frac{\mathrm{GV}^{(3)}_{d\beta}}{d^3}\biggr)\biggr].$$

$$\sum_\alpha\sum_\beta Q^{r\beta}\Phi^\alpha\Big\langle\frac{\Phi_\alpha}{1-\zeta qL^{1/r}},1-\zeta^{-1}L^{1/r}\Big\rangle^{X/\mathbb{Z}_r,K}_{0,2,r\beta} \tag{3-5}$$
$$\begin{aligned}=\sum_{\operatorname{ind}(\beta)=1}\sum_{d=1}^{\infty}Q^{rd\beta}\Bigg[\sum_{j=1}^{n_1}\Phi^{1j}\beta_j\Bigg(&\frac{(1-\zeta^{-1})(-\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))+\zeta^{-1}(\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))}{1-\zeta q}\\&+\frac{(1-\zeta^{-1})(\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2))}{(1-\zeta q)^2}\Bigg)\\+\Phi^{01}\Bigg(&\frac{(1-\zeta^{-1})(-\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))+\zeta^{-1}(3\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{1-\zeta q}\\&+\frac{(1-\zeta^{-1})(\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))+\zeta^{-1}(-2\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{(1-\zeta q)^2}\\&+\frac{(1-\zeta^{-1})(-2\,\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3))}{(1-\zeta q)^3}\Bigg)\Bigg].\end{aligned}$$

Combining (3-3)–(3-5) gives the principal part of the right side, and it coincides with the principal part of the left side. □

Proposition 3.12 *Let $t=\sum_{j=1}^{n_1}t_j\Phi_{1j}$ and ζ be a primitive r^{th} root of unity. We have*

$$\tilde{J}_{q=\zeta^{-1}}=\tilde{\delta}_\zeta(\zeta q)+\sum_{n,\beta,\alpha}\frac{Q^{r\beta}\Phi_\alpha}{n!}\Big[\frac{\Phi^\alpha}{1-\zeta qL^{1/r}},\operatorname{leg}_r(L),\dots,\operatorname{leg}_r(L),\tilde{\delta}_\zeta(L^{1/r})\Big]^{X_\zeta}_{0,n+2,\beta}.$$

Proof The computation for the $\beta_t=0$ part was covered by the previous lemma. We compare the $\beta_t\neq 0$ part. On the left, we consider only $c(r,q,\beta_t)$ and $d(r,q,\beta_t)$ terms. On the right, we only need to consider

$$\sum_{n,\beta,\alpha}\frac{Q^{r\beta}\Phi_\alpha}{n!}\Big[\frac{\Phi^\alpha}{1-\zeta qL^{1/r}},\operatorname{leg}_r(L),\dots,\operatorname{leg}_r(L),t\Big]^{X_\zeta}_{0,n+2,\beta},$$

since $\operatorname{leg}_\zeta(L)$ is independent of t. With the same argument as in Proposition 3.11, we compute the following three expressions:

$$\sum_\alpha\Phi_\alpha\Big[\frac{\Phi^\alpha}{1-\zeta qL^{1/r}},\operatorname{leg}_r(L),t\Big]^{X_\zeta}_{0,3,0}=\Phi^{01}\beta_t\Bigg(\frac{\mathrm{GV}^{(1)}_{d\beta}-\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{1-\zeta q}\Bigg), \tag{3-6}$$

$$\frac{1}{r}\sum_\alpha\Phi^\alpha\Big\langle\frac{\Phi_\alpha}{1-\zeta q},\Phi^r\Big(\sum_\alpha\Phi^\alpha\langle\Phi_\alpha\rangle^{\mathrm{fake}}\Big),t\Big\rangle^{X/\mathbb{Z}_r,K}_{0,3,0}=\Phi^{01}\beta_t\Bigg(\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{1-\zeta q}\Bigg), \tag{3-7}$$

$$\sum_\alpha\sum_\beta Q^{r\beta}\Phi_\alpha\Big\langle\frac{\Phi^\alpha}{1-\zeta qL^{1/r}},t\Big\rangle^{X/\mathbb{Z}_r,K}_{0,2,r\beta} \tag{3-8}$$
$$=\sum_{j=1}^{n_1}\beta_j\Phi^{1j}\beta_t\Bigg(\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^2d^2)}{1-\zeta q}\Bigg)+\Phi^{01}\beta_t\Bigg(\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{1-\zeta q}-\frac{\mathrm{GV}^{(3)}_{d\beta}/(r^3d^3)}{(1-\zeta q)^2}\Bigg).$$

Combining (3-6)–(3-8) gives the right side, and it coincides with left side. □

Theorem 3.13 (Theorem 1.2) *$\tilde{J} \in \mathcal{L}^K$. Moreover, $\tilde{J} = J^K(t)$.*

Proof For the first statement $\tilde{J} \in \mathcal{L}^K$, we use the adelic characterization theorem, Theorem 2.6. It suffices to check the following three criteria:

(i) $\tilde{J}$ only has poles at roots of unity. This criterion follows immediately from the expression of $\tilde{J}$.

(ii) The fake theory computation follows from Proposition 3.10.

(iii) The stem theory computation follows from Proposition 3.12.

This proves the first statement. The second statement follows from Givental's uniqueness theorem. Since every function f whose range lies on the Lagrangian cone is completely determined by $\pi_+(f)$ and

$$\pi_+(\tilde{J}) = 1 - q + t = \pi_+(J^K(t)),$$

we conclude that $\tilde{J} = J^K(t)$. □

Remark 3.14 The dependence of J^K on β_t is of the form e^{β_t}, except finitely many monomials in $\beta_t^{\leq 2}$ and an overall linear factor β_t to some terms. This can be explained in geometric terms.

Recall that for GW on CY3, invariants with (even) classes of $\deg_{\mathbb{C}} \geq 2$ vanish. The divisor and string equations "remove" $\deg_{\mathbb{C}} = 1$ and 0 classes in the insertions, and reduce all invariants to 0-pointed $\langle \cdot \rangle_{g,n=0,\beta}$. Furthermore, the dependence on the $\deg_{\mathbb{C}} = 1$ classes D is of the form $e^{\int_\beta D}$. A closed formula of this reduction can be found in [Fan and Lee 2019].

If the moduli of stable maps at genus zero were virtually smooth *schemes*, the virtual HRR would imply that quantum K-invariants behave almost like GW invariants. As we are restricting the insertions in J^K to $t \in \mathrm{ch}^{-1} H^2(X)$ and since the n-pointed moduli has virtual dimension n, reduction from n-point to 1-point would be simply applying the divisor axiom in GW, and the result should depend on e^{β_t}. The extra linear overall factor to some terms also comes from the divisor axiom.

This is however not true, but the stacky contributions [Edidin 2013, Theorem 5.1] in this case are limited to finitely many graphs. This implies the simple dependence of $J^K(t)$ on β_t. For completeness, we list the possible "Kawasaki strata" below.

The constant term ($\beta_t = 0$) comes from the graph sum of the Kawasaki strata in Figure 1. The linear terms of β_t and higher-order terms are given by Figures 2 and 3, respectively.

Notation is explained below; see [Chou and Lee 2022b] or [Givental and Tonita 2014] for more detail.

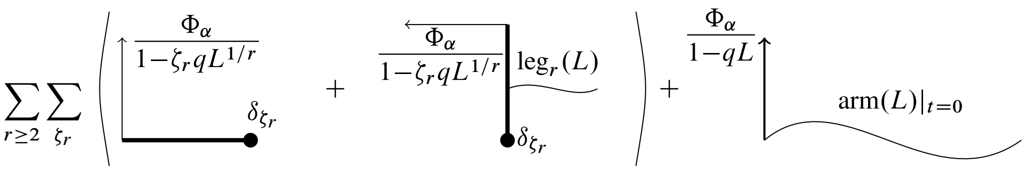

Figure 1: Kawasaki strata, β_t^0.

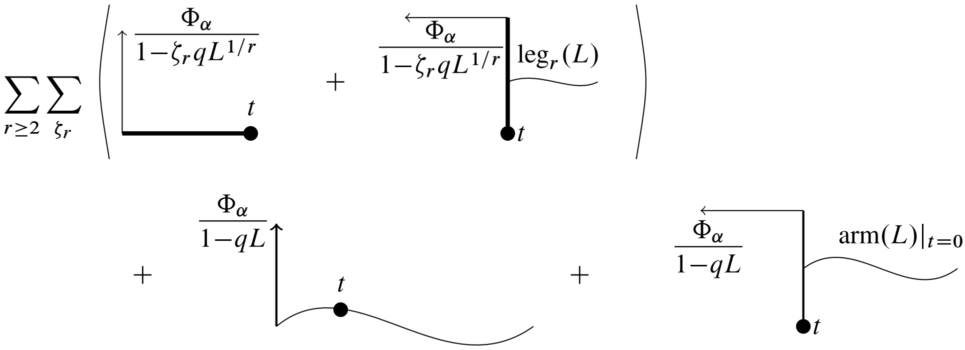

Figure 2: Kawasaki strata, β_t^1.

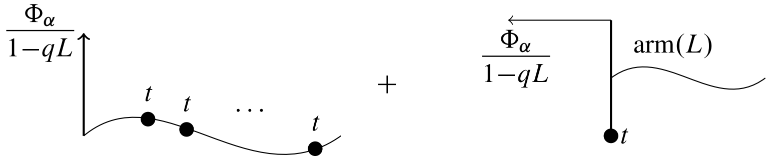

Figure 3: Kawasaki strata, $\beta_t^{\geq 2}$.

- The black points denote the marked points.
- $\delta_\zeta := \delta_\zeta(J^K(0);q)$ and $\mathrm{arm}(q) := \mathrm{arm}(J^K(t);q)$.
- Given a stable map with symmetry g, the action of g fixes the marked points and acts on L_1 with an eigenvalue, which is denoted by ζ_r. Here $\sum_{\zeta_r}$ denotes the sum over all primitive r^{th} roots of unity.
- The first marked point is denoted by an arrow with input $\Phi_\alpha/(1-\zeta_r qL^{1/r})$ for the stem theory and $\Phi_\alpha/(1-qL)$ for the fake theory.
- The vertical line denotes the curve that contracts to a point under the stable map.
- The stem curve is emphasized with a thick line.

Note that $\mathrm{arm}(L)$ only has monomials in $\beta_t^{\leq 1}$ since stem theory only has those. Hence the right graph in Figure 3 contributes terms with $\beta_t^{\leq 2}$. All $\beta_t^{\geq 3}$ terms come only from the left graph in Figure 3 and the virtual Hirzebruch–Riemann–Roch for *schemes* applies. For those, the usual divisor axiom in Gromov–Witten theory gives e^{β_t}, as explained above.

4 The multiple cover formula and virtual Clemens conjecture

In this section, we study the K-theoretic multiple cover formula at genus zero and give a heuristic derivation of Theorem 1.2 based on the "virtual Clemens conjecture", a folklore ansatz for enumerative geometry on Calabi–Yau threefolds.

We note that Jockers and Mayr [2019, Section 2.3] have already performed a similar computation for the "resolved conifold" where the equivariant quantum K-theory was employed.

4.1 The multiple cover formula

Lemma 4.1 *For the total space $X_{-1,-1}$ of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, the Gopakumar–Vafa invariants satisfy* $\mathrm{GV}_{0,1} = 1$ (*genus zero and degree* 1) *and* $\mathrm{GV}_{g,d} = 0$ (*otherwise*).

Proof This follows from the Gopakumar–Vafa equation (1-1), and the results in Gromov–Witten theory: for $g = 0$ the Voisin–Aspinwall–Morrison formula [Voisin 1996; Aspinwall and Morrison 1993]

$$\mathrm{GW}_{0,0,d} = \frac{1}{d^3},$$

for $g = 1$ the BCOV and Graber–Pandharipande formula [1999]

$$\mathrm{GW}_{1,0,d} = \frac{1}{12d},$$

and for $g \geq 2$ the Faber–Pandharipande formula [2000]

$$\mathrm{GW}_{g,0,d} = \frac{|B_{2g}| d^{2g-3}}{2g(2g-2)!}. \qquad \square$$

In order to consider the small J-function of $X_{-1,-1}$ in quantum K-theory, we consider its compactification by the "infinity divisor". Let

$$Y_{-1,-1} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}).$$

Let $P = \pi^*\mathcal{O}(-1)$ with $\pi : Y_{-1,-1} \to \mathbb{P}^1$, and $T = \mathcal{O}(D_\infty)$ with $D_\infty \subset Y_{-1,-1}$, the infinity divisor. We define $Q^r := Q^{r\ell}$, where

$$\ell := [\mathbb{P}^1] \overset{0}{\hookrightarrow} Y_{-1,-1}$$

is the line class in the "zero section" of the projective bundle. In the following, we consider the *specialized* small permutation-equivariant I-function and J-function of $Y_{-1,-1}$ only for the curve classes in the zero section, ie multiples of ℓ.

Lemma 4.2 *The small I-function for $Y_{-1,-1}$, with curve classes in the zero section, is*

$$I^{Y_{-1,-1}}(q, Q) = (1-q)\left[1 + \sum_{r=1}^{\infty} Q^r \frac{(1-PT)^2 \prod_{m=1}^{r-1}(1-PTq^m)^2}{(PT)^{2r} q^{r(r-1)} \prod_{m=1}^{r}(1-Pq^m)^2}\right].$$

Proof This lemma follows from the computations of the I-functions for toric manifolds via fixed-point localization by Givental and collaborators; see [Givental and Tonita 2014; Givental 2015d]. Their formula for small I-functions gives

$$I^{Y_{-1,-1}}(q, Q) = (1-q)\left[1 + \sum_{r=1}^{\infty} Q^r \frac{\prod_{m=-r+1}^{0}(1-P^{-1}T^{-1}q^m)^2}{\prod_{m=1}^{r}(1-Pq^m)^2}\right].$$

A simple manipulation gives the above presentation. $\square$

We note that in this case the small I-function includes a factor of $(1-PT)^2$, the K-theoretic normal bundle of $\mathbb{P}^1$ embedded in $Y_{-1,-1}$. One may think of this as an I-function of a toric completion

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O})$, with curve class in the base $\mathbb{P}^1$. This small I-function is different from the small J-function, as it has poles at $q=0$. (However, it satisfies $\lim_{q\to\infty} I^{X_{-1,-1}}(q)=0$.) Using this I-function one can obtain the small J-function via the "*generalized* mirror transform" (also known as the explicit reconstruction, or Birkhoff factorization).

Proposition 4.3 *The small J-function for $Y_{-1,-1}$, with curve classes in the zero section, is*

$$\frac{1}{1-q}J^{Y_{-1,-1}}(q,Q) = 1+(1-PT)^2(1+(1-P))\sum_{r\geq 1}Q^r a(r,q^r)+(1-PT)^2(1-P)\sum_{r\geq 1}Q^r b(r,q^r). \tag{4-1}$$

Proof The K-ring of $Y_{-1,-1}$ has the following presentation:

$$K(Y_{-1,-1})=\frac{\mathbb{Z}[P,T]}{((1-P)^2,(1-PT)^2(1-T))}.$$

We use the relations of $K(Y_{-1,-1})$ to rewrite $I^{Y_{-1,-1}}$ as follows. Since $(1-PT)^2(1-T)=0$, we have

$$(1-PT)^2(1-Pq^mT)=(1-PT)^2(1-Pq^m[1-(1-T)])=(1-PT)^2(1-Pq^m). \tag{4-2}$$

Therefore,

$$\begin{aligned} I^{Y_{-1,-1}}-(1-q) &= (1-q)\sum_{r=1}^{\infty}Q^r\frac{(1-PT)^2}{(PT)^{2r}q^{r(r-1)}(1-Pq^r)^2} \\ &= (1-q)\sum_{r\geq 1}Q^r\Big[\Big(\sum_{i=1}^{r-1}\frac{i}{q^{r(r-i)}}\Big)(1-PT)^2+\Big(\sum_{i=1}^{r-1}\frac{i(2r-i+1)}{q^{r(r-i)}}\Big)(1-PT)^3\Big] \\ &\quad +(1-q)((1-PT)^2+(1-PT)^3)\sum_{r\geq 1}Q^r\Big(\frac{r-1}{1-q^r}+\frac{1}{(1-q^r)^2}\Big) \\ &\quad +(1-q)(1-PT)^3\sum_{r\geq 1}Q^r\Big(\frac{r^2-1}{1-q^r}+\frac{3}{(1-q^r)^2}-\frac{2}{(1-q^r)^3}\Big). \end{aligned}$$

In the first equality, the factor $\prod_{m=1}^{r-1}(1-PTq^m)^2$ in the numerator and $\prod_{m=1}^{r-1}(1-Pq^m)^2$ in the denominator cancel each other due to the presence of $(1-PT)^2$ and (4-2). The second equality follows from an explicit computation.

The first line after the second equal sign lies in $\mathcal{K}_+$ and the rest lies in $\mathcal{K}_-$. Consider the reconstruction theorem for permutation-equivariant K-theory [Givental 2015g, Theorem 2]

$$\begin{aligned} J^{Y_{-1,-1}}(q,Q) &= \sum_{r\geq 0} I_r^{Y_{-1,-1}}Q^r \\ &\quad \times\exp\Big(\Big(\sum_{k>0}\frac{\Psi^k}{k}\Big)\Big(\frac{\sum_{i=0}^{1}\delta_i(Q)(1-Pq^r)(1-PTq^r)^i+\sum_{i=0}^{3}\epsilon_i(Q)(1-PTq^r)^i}{(1-q)}\Big)\Big) \\ &\quad \times\Big(\sum_{i=0}^{1}s_i(q,Q)(1-Pq^r)(1-PTq^r)^i+\sum_{i=0}^{3}u_i(q,Q)(1-PTq^r)^i\Big), \end{aligned}$$

for some uniquely determined $\epsilon_i(Q)$, $\delta_i(Q)$, $s_i(q,Q)$ and $u_i(q,Q)$, where

$$\epsilon_i(Q)=\sum_{j\geq 1}\epsilon_{ij}Q^j\in\mathbb{Q}[\![Q]\!], \qquad \delta_i(Q)=\sum_{j\geq 1}\delta_{ij}Q^j\in\mathbb{Q}[\![Q]\!],$$

$$u_i(q,Q)=\sum_{j\geq 0}u_{ij}(q)Q^j\in\mathbb{Q}[q,q^{-1}][\![Q]\!], \quad s_i(q,Q)=\sum_{j\geq 0}s_{ij}(q)Q^j\in\mathbb{Q}[q,q^{-1}][\![Q]\!].$$

A direct computation by induction on the degree of the Novikov variable shows that

$$u_0(q,Q)=1, \quad u_1(q,Q)=\epsilon_i(Q)=\delta_j(Q)=s_j(q,Q)=0,$$

where $0\leq i\leq 3$ and $0\leq j\leq 1$. Note that $u_2(q,Q)$ and $u_3(q,Q)$ will not change the $\mathcal{K}_-$ part and that the $\mathcal{K}_-$ part coincides with the definition of $a(r,q^r)$ and $b(r,q^r)$. □

We go through the same argument with input $t=t_1(1-P)$.

Proposition 4.4 *The J-function for $Y_{-1,-1}$, with curve classes in the zero section, is*

$$\frac{1}{1-q}J^{Y_{-1,-1}}(t,q,Q)=1+\frac{t_1(1-P)}{1-q}+(1-PT)^2(1+(1-P))\sum_{r\geq 1}Q^r(a(r,q^r)+c(r,q,t_1)) + (1-PT)^2(1-P)\sum_{r\geq 1}Q^r(b(r,q^r)+d(r,q,t_1)). \tag{4-3}$$

Proof We compute $J^{Y_{-1,-1}}$ in two steps. Consider the flow

$$J^{Y_{-1,-1}}(t^*,q,Q)=\sum_{r\geq 0}Q^r\exp\biggl(t_1\frac{1-Pq^r}{1-q}\biggr)J^{Y_{-1,-1}}(0,q,Q),$$

where

$$t^*=t+(1-PT)^2(E_1(t_1,q,Q))+(1-P)(1-PT)^2(E_2(t_1,q,Q))$$

with

$$E_1(t_1,q,Q)\in F_1(t_1,Q)+(1-q)\mathbb{Q}[t_1,q^{-1},q][\![Q]\!], \quad E_2(t_1,q,Q)\in\mathbb{Q}[t_1,q^{-1},q][\![Q]\!].$$

Then we consider the explicit reconstruction for ordinary K-theory

$$\begin{aligned}J^{Y_{-1,-1}}(t,q,Q)=\sum_{r\geq 0}&J_r^{Y_{-1,-1}}(t^*,q,Q)Q^r\\ &\times\exp\biggl(\frac{\sum_{i=0}^{1}\delta_i(t_1,Q)(1-Pq^r)(1-PTq^r)^i+\sum_{i=0}^{3}\epsilon_i(t_1,Q)(1-PTq^r)^i}{(1-q)}\biggr)\\ &\times\biggl(\sum_{i=0}^{1}s_i(t_1,q,Q)(1-Pq^r)(1-PTq^r)^i+\sum_{i=0}^{3}u_i(t_1,q,Q)(1-PTq^r)^i\biggr),\end{aligned}$$

for some uniquely determined $\delta_i(t_1,Q)$, $\epsilon_i(t_1,Q)$, $s_i(t_1,q,Q)$ and $u_i(t_1,q,Q)$, where

$$\epsilon_2(t_1,Q)=-F_1(t_1,Q), \quad u_0(t_1,q,Q)=1, \quad u_1(t_1,q,Q)=\epsilon_i(t_1,Q)=\delta_i(t_1,Q)=s_i(t_1,q,Q)=0,$$

for all $i=0,1$. Note that any choice of $\epsilon_3(t_1,Q)$, $u_2(t_1,q,Q)$ and $u_3(t_1,q,Q)$ will not change the $\mathcal{K}_-$ part. The proposition follows from a long and direct computation. □

Remark 4.5 The invariance of the $\mathcal{K}_-$ part in the proof of Proposition 4.3 is not surprising. Any choice of $u_2(q,Q)$ and $u_3(q,Q)$ corresponds to the input lying in $(1-q)H^{\geq 4}(Y_{-1,-1})[\![q,Q]\!]$, and hence it gives no contribution by Lemma 3.2.

The same argument works for Proposition 4.4 with one addition. Let

$$\boldsymbol{t} \in t_1(1-P) + \sum_{i=2}^{3} \epsilon_i(t_1,Q)(1-PT)^i + (1-q)H^{\geq 4}[t_1,q^{-1},q][\![Q]\!].$$

We have

$$\sum_{n\geq 0}\sum_{r\geq 0} \frac{Q^r}{n!}\left\langle \frac{1}{1-qL}, \boldsymbol{t}, \dots, \boldsymbol{t} \right\rangle_{0,n+1,r}^{Y_{-1,-1}} = \frac{t_1\epsilon_2(t_1,Q)}{1-q}$$

by Lemma 3.2. Note that all terms equal zero except when $n=2$ and $r=0$. In that case, the contribution only involves $\epsilon_2(t_1,Q)$.

The following corollary was first obtained by Jockers and Mayr.

Corollary 4.6 (multiple cover formula [Jockers and Mayr 2019, Section 2.3]) *For $t=t_1(1-P)$, the small J-function for $X_{-1,-1}$ with input* 0 *and t are*

$$\frac{1}{1-q}J^{X_{-1,-1}}(0,q,Q) = 1 + (1+(1-P))\sum_{r\geq 1} Q^r a(r,q^r) + (1-P)\sum_{r\geq 1} Q^r b(r,q^r),$$

$$\frac{1}{1-q}J^{X_{-1,-1}}(t,q,Q)$$

$$= 1 + \frac{t_1(1-P)}{1-q} + (1+(1-P))\sum_{r\geq 1} Q^r(a(r,q^r)+c(r,q,t_1)) + (1-P)\sum_{r\geq 1} Q^r(b(r,q^r)+d(r,q,t_1)).$$

Proof Since the zero section $\mathbb{P}^1$ has normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, the quantum K-invariants of $r\ell$ in $X_{-1,-1}$ are exactly the same as those in $Y_{-1,-1}$. The only difference in J-functions comes from different bases of the K-groups and the Poincaré pairings. The net result is the removal of the factor $(1-PT)^2$ from the specialized $J^{Y_{-1,-1}}(q,Q)$ for nonzero-degree terms in Propositions 4.3 and 4.4. □

4.2 The virtual Clemens conjecture

We now give a heuristic derivation and an interpretation of the relationship between Gopakumar–Vafa invariants and quantum K-invariants at genus zero by a multiple cover formula. This has served to guide us in our search for the current formulation of Theorem 1.2, even though the actual proof follows a completely different approach. Of course, the original formulations of Jockers and Mayr [2019] and Garoufalidis and Scheidegger [2022] have been helpful too.

The following "folklore conjecture" in enumerative geometry is our starting point.

Conjecture 4.7 (virtual Clemens conjecture) *For the purpose of computing genus-zero enumerative invariants on Calabi–Yau threefolds X, one may assume that there are only finitely many isolated rational curves $\{C_i \subset X\}$. Furthermore, C_i are all smooth $(-1,-1)$ curves.*

Assume that we are given an "ideal" Calabi–Yau threefold X satisfying the above conjecture; by Lemma 4.1, each isolated $(-1,-1)$-curve (in any degree β) contributes 1 to the Gopakumar–Vafa invariants, independently of β. Therefore there are $\mathrm{GV}_{0,\beta}$ isolated $(-1,-1)$-curves in degree β. For each of these isolated curves, quantum K-theory allows multiple r-covers of the isolated $(-1,-1)$-curve. The coefficients $a(r,q^r)$, $b(r,q^r)$, $c(r,q,t_1)$ and $d(r,q,t_1)$ of the r-covers come from the J-function of $X_{-1,-1}$. The only addition is the factor of $\beta_j := \int_\beta D_j$ with $D_j = \mathrm{ch}(\Phi^{1j})$, which comes from the divisor axiom.

In summary, the virtual Clemens conjecture and the multiple cover formula imply that GV is equivalent to QK for all Calabi–Yau threefolds in genus zero.

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