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Recently it has been proved that three-dimensional complete manifolds with nonnegatively pinched Ricci curvature must be flat or compact, thus confirming a conjecture of Hamilton. We generalise our work on the existence of Ricci flows from noncompact pinched three-manifolds in order to prove a higher-dimensional analogue. We construct a solution to Ricci flow, for all time, starting with an arbitrary complete noncompact manifold that is PIC1 pinched. As an application we prove that any complete manifold of nonnegative complex sectional curvature that is PIC1 pinched must be flat or compact.

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1 Introduction

Lee and Topping [35] proved the following result by lifting a remaining additional hypothesis of bounded curvature from the work of Deruelle, Schulze and Simon [22], which in turn appealed to work of Lott [36].

Theorem 1.1 (Hamilton’s pinching conjecture; cf Chow, Lu and Ni [20, Conjecture 3.39]) *Suppose (M^3, g_0) is a complete (connected) three-dimensional Riemannian manifold with $\text{Ric} \geq \varepsilon \text{scal} \geq 0$ for some $\varepsilon > 0$. Then (M^3, g_0) is either flat or compact.*

Here we develop a higher-dimensional version of the Ricci flow existence theory we established in [35], in order to prove a pinching result in general dimension. To state the result, we need to understand a little about the notion of isotropic curvature. See also Topping [50] for a different perspective.

Denote the space of algebraic curvature tensors on \mathbb{R}^n by $\mathcal{C}_B(\mathbb{R}^n)$. Given $R \in \mathcal{C}_B(\mathbb{R}^n)$, we can extend it by complex linearity to \mathbb{C}^n . Although we always have the symmetries of R in mind, we view it as a $(0, 4)$ tensor to avoid ambiguities of normalisation. To each two-complex-dimensional subspace Σ of \mathbb{C}^n we can then associate a complex sectional curvature. Concretely, if $v, w \in \mathbb{C}^n$ give an orthonormal basis of Σ , then the complex sectional curvature associated with Σ is $R(v, w, \bar{v}, \bar{w})$. We say that $R \in \mathcal{C}_B(\mathbb{R}^n)$ has nonnegative complex sectional curvature if all these curvatures are nonnegative, whichever Σ we choose, and we denote the cone of all such curvature tensors by $\mathcal{C}_{\text{PIC}2}$. Because this condition is $O(n)$ -invariant, we can talk of a manifold having nonnegative complex sectional curvature if this property holds for the curvature tensor at every tangent space.

We can weaken the notion of curvature positivity by restricting the sections Σ that we consider. In particular, one can ask for nonnegativity of complex sectional curvature only for PIC1 sections, defined to be those Σ that contain some nonzero vector v whose conjugate \bar{v} is orthogonal to the entire section Σ . The algebraic curvature tensors R with nonnegative complex sectional curvature for each such restricted Σ form a cone we denote by C_{PIC1} .

Asking that a curvature tensor lies in C_{PIC1} or C_{PIC2} is asking that certain natural curvature averages are nonnegative.

For $n \geq 4$, one can describe the cones C_{PIC1} and C_{PIC2} more explicitly as follows (see Brendle [5, Propositions 7.14 and 7.18]). The cone C_{PIC2} consists of curvature tensors R satisfying

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Similarly, C_{PIC1} is the cone of curvature tensors satisfying

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$.

Theorem 1.2 (main theorem) *Suppose (M^n, g_0) is a complete manifold of nonnegative complex sectional curvature with $n \geq 3$ that is pinched in the sense that*

$$(1-1) \quad R_{g_0} - \varepsilon_0 \text{scal}(R_{g_0}) \cdot I \in C_{\text{PIC1}}$$

for some $\varepsilon_0 > 0$. Then (M, g_0) is either flat or compact.

To clarify, we denote by R_g the curvature tensor of a Riemannian metric g , with (1-1) holding in every tangent space. Given $R \in \mathcal{C}_B(\mathbb{R}^n)$, we write $\text{Ric}(R)_{ik} := R_{ijkj}$ and $\text{scal}(R) := \text{Ric}(R)_{ii}$ in order to match the usual notions of Ricci and scalar curvature exactly. Meanwhile, I is the curvature tensor defined by

$$(1-2) \quad I_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Thus on the unit round sphere S^n we would have $R_g = I$.

Explicitly, our pinching condition (1-1) is saying exactly that when we compute the complex sectional curvature corresponding to a PIC1 section then not only should it be nonnegative, it should also be bounded below by $\varepsilon_0 \text{scal}(R_{g_0})$.

In arbitrary dimension, every $R \in C_{\text{PIC1}}$ automatically has nonnegative Ricci curvature, and for $n = 3$ the conditions are equivalent; see, for example, McLeod and Topping [37, Section 2]. For $n = 3$, the PIC1 pinching condition (1-1) considered in the theorem is equivalent to pinched Ricci curvature. This case is already handled by the theory we developed in [35] in that dimension, and thus we may focus here on the case that $n \geq 4$.

Our main theorem extends several earlier pinching results. Brendle and Schoen [9, Theorem 7.4] proved that if (M^n, g_0) is assumed, in addition to nonnegative complex sectional curvature, to satisfy a stronger PIC2 pinching condition, that is, one assumes

$$R_{g_0} - \varepsilon_0 \operatorname{scal}(R_{g_0}) \cdot I \in C_{\text{PIC2}},$$

for some $\varepsilon_0 > 0$, and additionally one assumes that the sectional curvature is uniformly bounded throughout, and that the scalar curvature is strictly positive, then (M, g_0) is compact. In turn, this generalises earlier work of Chen and Zhu [16] and of Ni and Wu [39].

The strategy we use to prove Theorem 1.2 is to assume, contrary to the theorem, that our manifold is neither flat nor compact, and then to flow (M, g_0) under Ricci flow for all time and to analyse its asymptotic behaviour. The rough idea that echoes what has been done in previous work starting with Chen and Zhu [16] is that parabolic blow-downs of the Ricci flow would like to settle down to an expanding gradient soliton, but that this is incompatible with the (scale-invariant) pinching condition. The main challenge that we address is to establish the following existence theorem for Ricci flow, starting with a manifold of possibly unbounded curvature, that mirrors our earlier theory from [35].

Theorem 1.3 *For any $n \geq 4$ and $\varepsilon_0 \in (0, 1/(n(n-1)))$, there exist $a_0 > 0$ and $\varepsilon'_0 \in (0, 1/(n(n-1)))$ such that the following holds. Suppose (M^n, g_0) is a complete noncompact manifold such that*

$$(1-3) \quad R_{g_0} - \varepsilon_0 \operatorname{scal}(R_{g_0}) \cdot I \in C_{\text{PIC1}}$$

on M . Then there exists a smooth complete Ricci flow solution $g(t)$ on $M \times [0, +\infty)$ such that for all $t > 0$,

- (a) $R_{g(t)} - \varepsilon'_0 \operatorname{scal}(R_{g(t)}) \cdot I \in C_{\text{PIC1}}$,
- (b) $|R_{g(t)}| \leq a_0 t^{-1}$.

Note that the restriction $\varepsilon_0 < 1/(n(n-1))$, together with the pinching hypothesis, implies that $\operatorname{scal}(R_{g_0}) \geq 0$.

This existence theorem only requires PIC1 pinching, with no requirement for nonnegative complex sectional curvature. A theory for Ricci flow starting with general open manifolds with nonnegative complex sectional curvature was developed by Cabezas-Rivas and Wilking [10]. Although we do have that hypothesis at our disposal in the application, we must develop a new theory in order to obtain both long-time existence and the required curvature decay, both of which require the PIC1 pinching hypothesis. We emphasise that for the existence theory we do not make any noncollapsing assumption, we do not have any boundedness of curvature assumption, and we do not assume nonnegativity of the complex sectional curvature.

Remark 1.4 Our PIC1 pinching theorem raises the question of whether the cone in which the curvature of g_0 is assumed to lie can be weakened from C_{PIC2} to C_{PIC1} in Theorem 1.2. Our Ricci flow existence theory already works in this more general situation, but having some negative complex sectional curvature complicates the required blow-down argument in the proof of Theorem 4.4.

If we combine our main result with earlier work of Böhm and Wilking [2] and Brendle [5], we rapidly obtain the following consequence.

Corollary 1.5 *Suppose (M^n, g_0) is a complete manifold of nonnegative complex sectional curvature with $n \geq 3$ that is not everywhere flat, and is pinched in the sense of (1-1) for some $\varepsilon_0 > 0$. Then (M, g_0) is diffeomorphic to a spherical space form.*

We will give the proof of Corollary 1.5 towards the end of Section 4.

Our main existence theorem (Theorem 1.3) fits within several active research areas. The techniques we develop build on ideas introduced to construct complete Ricci flows starting with smooth manifolds with possibly unbounded curvature in other settings, necessarily on noncompact domains. This is achieved by the construction of local Ricci flows on an exhaustion of the domain, or alternative approximations, satisfying appropriate estimates that allow one to pass to a limit to give a global solution.

The first instance of this was the general theory of Ricci flow in two dimensions; see Giesen and Topping [24] and Topping [49]. In higher dimensions one can run the flow starting with manifolds of nonnegative complex sectional curvature; see Cabezas-Rivas and Wilking [10]. A common theme, following initial work of Simon [45], is to flow starting with a manifold that is globally noncollapsed and satisfies a lower bound on an appropriate notion of curvature. Examples of such curvature notions include Ricci curvature in three dimensions (see Hochard [26] and Simon and Topping [46]), complex sectional curvature (see Bamler, Cabezas-Rivas and Wilking [1]), the curvature associated with PIC1 (see Lai [30]) and, in the setting of Kähler manifolds, a combination of Ricci and orthogonal bisectional curvatures; see Lee and Tam [33]. Alternative interesting hypotheses include a Ricci lower bound and local almost-Euclidean isoperimetric inequality (see He [25]), and appropriate curvature lower bounds together with smallness of an L^1 Morrey-type norm of the curvature (ie smallness of rescaled averages of the curvature over balls of varying radius); see Chan and Lee [12]. See also Chan, Huang and Lee [11] and Cheng [17]. The case of $U(n)$ -invariant initial metrics on \mathbb{C}^n was handled by Chau, Li and Tam [14].

Note that many of the existence theorems above rely in an essential way on a noncollapsing assumption. Such an assumption is not permissible in our application, and the way in which we deal with this is one of the novelties of our work.

A related endeavour to the existence theory described above is the programme to use the Ricci flow to smooth out rough initial data. An early instance of this was Simon's work to smooth C^0 metrics [43]; cf later work of Chau and Lee in the Kähler setting [13]. Here our initial data is smooth, but we can draw on techniques from some of the papers that start the flow with some form of limit space (following Simon [45]) in the noncompact setting; see, for example, Hochard [26], Lai [30], Lee and Tam [32], McLeod and Topping [37] and Simon and Topping [46]. Although less related to this paper, a number of other Ricci flow papers handle rough initial data of various other types, including those of Chu and Lee [21], Koch and Lamm [29], Lamm and Simon [31] and Topping and Yin [51].

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2 Pinching cones

In this section, we briefly survey some general tools used throughout this paper that are largely taken from [6; 2]. For simplicity we work in dimension $n \geq 4$.

Given $R \in \mathcal{C}_B(\mathbb{R}^n)$, define a new algebraic curvature tensor $Q(R)$ by

$$(2-1) \quad Q(R)_{ijkl} = R_{ijpq}R_{klpq} + 2R_{ipkq}R_{jplq} - 2R_{iplq}R_{jpkq},$$

and consider the ‘‘Hamilton ODE’’ $(d/dt)R = Q(R)$. For the significance of this ODE, and the geometry behind the quantity $Q(R)$, one can refer to [2].

We are interested in cones $C \subset \mathcal{C}_B(\mathbb{R}^n)$ that are invariant under the Hamilton ODE, such as C_{PIC1} and C_{PIC2} [8; 52]. We will consider cones C with the following properties:

- (I) C is closed, convex and $O(n)$ -invariant.
- (II) C is transversally invariant under the Hamilton ODE.
- (III) Every $R \in C \setminus \{0\}$ has positive scalar curvature.
- (IV) The curvature tensor I defined in (1-2) lies in the interior of C .

In (II), transversally invariant is equivalent to saying that $Q(R)$ lies in the interior of the tangent cone $T_R C$ for all $R \in C \setminus \{0\}$.

In the following, we will recall two families of cones that satisfy (I)–(IV), adopting notation from [2; 5] where appropriate. We start by recalling the endomorphism $\ell_{a,b}$ on $\mathcal{C}_B(\mathbb{R}^n)$ introduced by Böhm and Wilking [2]. For any $a, b \in \mathbb{R}$ and $R \in \mathcal{C}_B(\mathbb{R}^n)$, we define

$$\ell_{a,b}(R) = R + b \cdot \left(\text{Ric}(R) - \frac{\text{scal}(R)}{n} g \right) \otimes g + \frac{a}{n} \text{scal}(R) \cdot g \otimes g = R + b \cdot \text{Ric}(R) \otimes g + \frac{a-b}{n} \text{scal}(R) \cdot g \otimes g,$$

where g is the metric ($g_{ij} = \delta_{ij}$ on \mathbb{R}^n) and \otimes denotes the Kulkarni–Nomizu product:

$$(A \otimes B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}.$$

We will occasionally use that $g \otimes g = 2I$.

In [8], the following family of cones $\hat{C}(s)$ was introduced by Brendle and Schoen in order to prove the differentiable sphere theorem, in the spirit of the work of Böhm and Wilking [2].

Definition 2.1 Define a family of closed convex $O(n)$ -invariant cones $\hat{C}(s)$ for $s \in (0, +\infty)$ as follows:

- (a) For $s \in (0, \frac{1}{2}]$,

$$\hat{C}(s) = \left\{ \ell_{a,s}(R) : R \in C_{\text{PIC2}}, \text{Ric}(R) \geq \frac{p}{n} \text{scal}(R) \right\},$$

where

$$2a = \frac{2s + (n-2)s^2}{1 + (n-2)s^2}, \quad p = 1 - \frac{1}{1 + (n-2)s^2}.$$

(b) For $s \in (\frac{1}{2}, +\infty)$,

$$\widehat{C}(s) = \left\{ \ell_{s,1/2}(R) : R \in C_{\text{PIC2}}, \text{Ric}(R) \geq \frac{p}{n} \text{scal}(R) \right\},$$

where

$$p = 1 - \frac{4}{n-2+8s}.$$

The importance of the cones $\widehat{C}(s)$ stems from the following proposition, which can be used to show that certain Ricci flows become round.

Proposition 2.2 (Propositions 13–15 in [8]; cf [2]) *For every $s > 0$, the cone $\widehat{C}(s)$ satisfies the properties (I)–(IV). Moreover, $\widehat{C}(s)$ varies continuously in s .*

The continuity here is with respect to the Hausdorff topology once we intersect with the closed unit ball $\{R \in C_B(\mathbb{R}^n) : |R| \leq 1\}$. As we are dealing with *convex* cones, the complements of the cones also vary continuously.

As $s \rightarrow +\infty$, the cone $\widehat{C}(s)$ converges to $\mathbb{R}_{\geq 0}I$. In particular, knowing that there exists $s_0 > 0$ such that $R \in \widehat{C}(s)$ for all $s \geq s_0$ is equivalent to knowing that R is the curvature tensor of a space-form with nonnegative curvature. It was shown by Brendle, Huisken and Sinestrari [6] that compact ancient solutions of Ricci flow for $t \in (-\infty, 0)$ with $R_{g(t)} \in \widehat{C}(s_0)$ for some $s_0 > 0$ and all $t < 0$ (on each tangent space) must have constant sectional curvature for each $t < 0$. This was later generalised by Yokota [54] to the case of complete ancient Ricci flows with possibly unbounded curvature.

We follow the idea of [6] and define, for each $s > 0$, $\check{C}(s)$ to be the cone of all algebraic curvature tensors $R \in C_B(\mathbb{R}^n)$ satisfying

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} + \frac{1}{s}(1 - \lambda^2)(1 - \mu^2) \cdot \text{scal}(R) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and $\lambda, \mu \in [0, 1]$. Clearly $C_{\text{PIC2}} \subset \check{C}(s) \subset C_{\text{PIC1}}$ for all $s > 0$. See the appendix for a geometric interpretation of $\check{C}(s)$, which also implies these inclusions. The cones $\check{C}(s)$ are nested, getting smaller as s increases, and their intersection is C_{PIC2} . Moreover, $\check{C}(s)$ converges to C_{PIC2} as $s \rightarrow \infty$. Following the idea in the proof of [6, Theorem 12], in the spirit of [2], we define

$$\tilde{C}(b, s) := \ell_b(\check{C}(s)),$$

where we define

$$(2-2) \quad \ell_b(C) := \ell_{a,b}(C) \quad \text{for } a := b + \frac{1}{2}(n-2)b^2.$$

Proposition 2.3 For any $n \geq 4$, $s > 0$ and $b \in (0, \Upsilon_n)$, where

$$(2-3) \quad \Upsilon_n := \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)},$$

the cone $\tilde{C}(b, s)$ satisfies (I)–(IV). Moreover, for each such b , the cones $\tilde{C}(b, s)$ vary continuously in s .

Proof This is contained in the proof of [6, Theorem 12], which in turn appeals to [6, Proposition 10; 2, Proposition 3.2]. For property (III) see also Lemma A.2. □

3 A priori estimates under pinching conditions

In this section, we will establish local a priori estimates along the Ricci flow. But first we give a result that says that $\ell_{a,b}$ turns positive curvatures into pinched curvatures.

Lemma 3.1 Suppose, for $n \geq 3$, that $C \subset \mathcal{C}_B(\mathbb{R}^n)$ is a convex cone with the properties that $\text{Ric}(R) \geq 0$ for every $R \in C$ and that every $R \in \mathcal{C}_B(\mathbb{R}^n)$ of nonnegative curvature operator lies in C . Suppose further that $a > b \geq 0$. Then there exists $\delta > 0$ depending on a, b, n and the cone C such that if $S \in C$ and $R = \ell_{a,b}(S)$, then

$$R - \delta \text{scal}(R) \cdot I \in C.$$

Proof By assumption, $S \in C$ and hence $\text{Ric}(S) \geq 0$. We claim that this implies that $\text{Ric}(S) \otimes g$ has nonnegative curvature operator, in which case it must be in the cone C by assumption. Indeed, a computation tells us that for a general element $A = \alpha_{ij}e_i \otimes e_j \in \Lambda^2\mathbb{R}^n$, we have

$$\text{Ric}(S) \otimes g(A, A) = 4\alpha_{ij}\alpha_{kj} \text{Ric}(S)_{ik},$$

and the right-hand side is nonnegative for each j separately.

Given any symmetric bilinear form h on \mathbb{R}^n , we can compute that

$$\text{scal}(h \otimes g) = 2(n-1) \text{tr}(h),$$

and hence

$$\text{scal}(\text{Ric}(S) \otimes g) = 2(n-1) \text{scal}(S)$$

and

$$\text{scal}(I) = n(n-1).$$

Unravelling the definition of R gives

$$R = S + b \text{Ric}(S) \otimes g + \frac{2(a-b)}{n} \text{scal}(S) \cdot I,$$

so

$$\text{scal}(R) = (2a(n-1) + 1) \text{scal}(S).$$

Therefore for $\delta \in (0, 2(a - b)/(n(2a(n - 1) + 1))]$, we obtain

$$\begin{aligned} R - \delta \operatorname{scal}(R) \cdot I &= S + b \operatorname{Ric}(S) \otimes g + \frac{2(a - b)}{n} \operatorname{scal}(S) \cdot I - \delta \operatorname{scal}(R) \cdot I \\ &= S + b \operatorname{Ric}(S) \otimes g + \left(\frac{2(a - b)}{n} - \delta(2a(n - 1) + 1) \right) \operatorname{scal}(S) \cdot I \in C. \quad \square \end{aligned}$$

We assemble some specific instances of the previous lemma:

Lemma 3.2 *Suppose $R = \ell_{a,b}(S)$ for some $a > b \geq 0$ and $S \in \mathcal{C}_B(\mathbb{R}^n)$, for $n \geq 3$. Then there exists $\delta(n, a, b) > 0$ such that the following hold:*

- (i) $\operatorname{Ric}(R) \geq \delta \operatorname{scal}(R)g$ if $\operatorname{Ric}(S) \geq 0$.
- (ii) $R - \delta \operatorname{scal}(R) \cdot I \in C_{\text{PIC1}}$ if $S \in C_{\text{PIC1}}$.
- (iii) $R - \delta \operatorname{scal}(R) \cdot I \in C_{\text{PIC2}}$ if $S \in C_{\text{PIC2}}$.

It is also convenient to record a converse statement:

Lemma 3.3 *Suppose for $n \geq 3$ that C is a closed convex cone in $\mathcal{C}_B(\mathbb{R}^n)$ such that*

- (1) I is in the interior of the cone C , and
- (2) every $R \in C \setminus \{0\}$ has positive scalar curvature.

Then given $\varepsilon \in (0, 1/(n(n - 1)))$, for small enough $b > 0$ depending on ε, n and C , every $R \in \mathcal{C}_B(\mathbb{R}^n)$ with $R - \varepsilon \operatorname{scal}(R) \cdot I \in C$ satisfies $R \in \ell_b(C)$.

Recall that ℓ_b is defined in (2-2).

Proof Because $\varepsilon \in (0, 1/(n(n - 1)))$, by considering the scalar curvature of $R - \varepsilon \operatorname{scal}(R) \cdot I \in C$ and using that each element of C has nonnegative scalar curvature, we deduce that $\operatorname{scal}(R) \geq 0$. Using that every $\tilde{R} \in C \setminus \{0\}$ has positive scalar curvature, and also that the cone is closed, we deduce that $|\tilde{R}| \leq c \operatorname{scal}(\tilde{R})$ for every $\tilde{R} \in C$ and some $c > 0$ depending only on C . This ensures that if R is scaled to have $|R| = 1$ then not only is the term $\varepsilon \operatorname{scal}(R) \cdot I$ in the cone, but so is a neighbourhood of radius $r_0 > 0$ depending on ε and the cone, including on c , but not on the specific R chosen with $|R| = 1$. Thus by convexity of the cone we can add an error S of magnitude $|S| < r_0$ to R and still have

$$R + S = [R - \varepsilon \operatorname{scal}(R) \cdot I] + [S + \varepsilon \operatorname{scal}(R) \cdot I] \in C.$$

In particular, since $\ell_b(C)$ varies continuously in b , we conclude that for small enough $b > 0$ we have $R \in \ell_b(C)$. □

The following lemma translates the pinching condition into a cone condition:

Lemma 3.4 *Suppose $R \in \mathcal{C}_B(\mathbb{R}^n)$, $\varepsilon_0 \in (0, 1/(n(n - 1)))$ and*

$$(3-1) \quad R - \varepsilon_0 \operatorname{scal}(R) \cdot I \in C_{\text{PIC1}}.$$

Then $R \in \tilde{C}(b, s_0)$ for some $s_0(n, \varepsilon_0) > 0$ and $0 < b(n, \varepsilon_0) < \Upsilon_n$.

Note that by tracing the hypothesis (3-1) and using that $\varepsilon_0 \in (0, 1/(n(n-1)))$, we see that $\text{scal}(R) \geq 0$ automatically in the lemma.

Proof We apply Corollary A.3 in the appendix to $S := R - \varepsilon_0 \text{scal}(R) \cdot I \in C_{\text{PIC1}}$ in place of R , and with $\varepsilon = \frac{1}{2}\varepsilon_0$. We deduce that there exists $s_0 > 0$ depending only on n and ε_0 such that

$$S + \frac{1}{2}\varepsilon_0 \text{scal}(S) \cdot I \in \check{C}(s_0).$$

Because $\text{scal}(S) = (1 - \varepsilon_0 n(n-1)) \text{scal}(R) \leq \text{scal}(R)$, we can expand S to give

$$R - \frac{1}{2}\varepsilon_0 \text{scal}(R) \cdot I \in \check{C}(s_0),$$

where we are using that I lies in the convex cone $\check{C}(s_0)$. In fact, I lies in the interior of the cone $\check{C}(s_0)$ (even of the smaller cone C_{PIC2}) and every $R \in \check{C}(s_0) \setminus \{0\}$ has positive scalar curvature. This latter fact follows because $\check{C}(s_0) \subset C_{\text{PIC1}}$ and the same is true for C_{PIC1} , eg by Lemma A.2. We can then invoke Lemma 3.3 to conclude that for some $b \in (0, \Upsilon_n)$ we have $R \in \ell_b(\check{C}(s_0)) = \tilde{C}(b, s_0)$. \square

We now prove a local version of Hamilton’s celebrated ODE–PDE theorem.

Theorem 3.5 (local ODE–PDE theorem) *Suppose for $n \geq 3$ that C is a closed convex $O(n)$ -invariant cone in $C_B(\mathbb{R}^n)$ with the properties*

- (1) *for some $\varepsilon > 0$, if $S \in C_B(\mathbb{R}^n)$ satisfies $|S - I| \leq \varepsilon$, then $S \in C$,*
- (2) *C is invariant under the Hamilton ODE.*

Suppose further that $(M^n, g(t))$ for $t \in [0, T]$ is a smooth solution to the Ricci flow with $g(0) = g_0$ such that for some $x_0 \in M$, we have

- (i) $B_{g_0}(x_0, 1) \Subset M$,
- (ii) $R_{g_0} \in C$ on $B_{g_0}(x_0, 1)$,
- (iii) $|R_{g(t)}| \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$ for some $c_0 < \infty$.

Then for every $l \geq 0$, there exists $S_0(n, c_0, \varepsilon, l) > 0$ such that for all $t \in [0, T \wedge S_0]$, we have

$$R_{g(t)}(x_0) + t^l \cdot I \in C.$$

Here we use the notation $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$.

In this work we will apply the local ODE–PDE theorem in the case that $l = 0$ and $C = \tilde{C}(b, s_0)$ for some $s_0 > 0$ and $b \in (0, \Upsilon_n)$, where Υ_n was defined in (2-3), in order to show that the pinching condition is almost preserved locally. In that case it can be viewed as being analogous to [35, Lemma 3.1] in the three-dimensional theory. For ease of reference we record the consequence we require in the following lemma:

Lemma 3.6 Suppose for $n \geq 4$ that $(M^n, g(t))$ for $t \in [0, T]$ is a smooth solution to the Ricci flow with $g(0) = g_0$ such that for some $x_0 \in M$, we have

- (i) $B_{g_0}(x_0, 1) \Subset M$,
- (ii) $R_{g_0} \in \tilde{C}(b, s_0)$ on $B_{g_0}(x_0, 1)$ for some $s_0 > 0$ and $b \in (0, \Upsilon_n)$,
- (iii) $|R_{g(t)}| \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$ for some $c_0 < \infty$.

Then there exists $S_0(n, c_0, b, s_0) > 0$ such that for all $t \in [0, T \wedge S_0]$,

$$R_{g(t)}(x_0) + I \in \tilde{C}(b, s_0).$$

Proof of Theorem 3.5 We denote by d the distance $d(R, S) = |R - S|$ on $\mathcal{C}_B(\mathbb{R}^n)$, where the norm is the standard extension of the Euclidean norm on \mathbb{R}^n . The corresponding inner product could be written $\langle R, S \rangle = R_{ijkl}S_{ijkl}$.

Define a Lipschitz function $\varphi: B_{g_0}(x_0, 1) \times [0, T] \rightarrow \mathbb{R}$ by $\varphi(x, t) = d(R_{g(t)}(x), C)$. We first show that φ satisfies

$$(3-2) \quad \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\varphi \leq C(n)|R_{g(t)}|\varphi$$

in the (strong) barrier sense whenever $\varphi > 0$. By this we mean that for each point

$$(x_1, t_1) \in B_{g_0}(x_0, 1) \times (0, T) \quad \text{where } \varphi(x_1, t_1) > 0,$$

we can find a smooth function $\underline{\varphi}(x, t)$ defined in a space-time neighbourhood of (x_1, t_1) such that $\varphi \geq \underline{\varphi}$ where defined, while at (x_1, t_1) we have both $\varphi(x_1, t_1) = \underline{\varphi}(x_1, t_1)$ and

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t_1)}\right)\underline{\varphi} \leq C(n)|R_{g(t_1)}|\underline{\varphi}.$$

(This notion of barrier sense is particularly well adapted to applications of the maximum principle.)

If this is the case, then it follows from [34, Theorem 1.1] that for any $l \geq 0$, there exists $S_0 > 0$ depending only on n, c_0 and l such that

$$\varphi(x_0, t) \leq t^{c_0+2+l} \quad \text{if } t \in (0, S_0]$$

since $\varphi(\cdot, 0) = 0$ on $B_{g_0}(x_0, 1)$ and $\varphi \leq d(R_{g(t)}(x), 0) \leq c_0 t^{-1}$ on $B_{g_0}(x_0, 1) \times (0, T]$. By shrinking $S_0 > 0$, depending now additionally on ε , we may assume $t^{c_0+2} < \varepsilon$, and hence

$$(3-3) \quad \varphi(x_0, t) \leq \varepsilon t^l \quad \text{if } t \in (0, S_0].$$

By definition of φ we have

$$R_{g(t)}(x_0) - \varphi(x_0, t)\xi(R_{g(t)}(x_0)) \in C$$

whenever $R_{g(t)}(x_0) \notin C$, where for $S \notin C$ we write

$$(3-4) \quad \xi(S) = (S - \pi(S))/|S - \pi(S)|,$$

where π is the projection to the cone C . In particular, convexity of the cone implies that if $R_{g(t)}(x_0) \notin C$ then for $t \in (0, S_0]$ we have

$$R_{g(t)}(x_0) + t^l I = [R_{g(t)}(x_0) - \varphi(x_0, t)\xi(R_{g(t)}(x_0))] + [t^l I + \varphi(x_0, t)\xi(R_{g(t)}(x_0))] \in C,$$

by (3-3) and the definition of ε in the theorem. This will complete the proof.

In order to prove (3-2) at an arbitrary point (x_1, t_1) where $\varphi(x_1, t_1) > 0$, we must locally trivialise the tangent bundle, and hence the bundle in which $R_{g(t)}$ lives, near (x_1, t_1) , so that we can compare $R_{g(t)}(x)$ and $R_{g(t_1)}(x_1)$ for (x, t) near (x_1, t_1) . At the fixed time t_1 we do this by radial parallel transport centred at x_1 . In the time direction we adopt the time-dependent gauge transformation often referred to as the Uhlenbeck trick [19]. This allows us to compare tensors at different points. It makes the fibre metric (and derived cones, and π and ξ) constant in a neighbourhood of (x_1, t_1) , and leads to a simpler evolution equation

$$(3-5) \quad \left(\frac{\partial}{\partial t} - \Delta\right)R = Q(R)$$

for the curvature tensor, for Q as in (2-1).

To show the evolution equation (3-2) in the barrier sense, we work near our point (x_1, t_1) where $\varphi(x_1, t_1) > 0$, or equivalently $d(R_{g(t_1)}(x_1), C) > 0$. Since $\varphi(x, t)$ is in general only continuous, we construct a barrier as follows. Let

$$\underline{\varphi}(x, t) = \langle \xi(R_{g(t_1)}(x_1)), R_{g(t)}(x) \rangle$$

for (x, t) sufficiently close to (x_1, t_1) so that the bundle trivialisation above is valid. Then $\varphi(x_1, t_1) = \underline{\varphi}(x_1, t_1)$ and convexity of the cone implies $\underline{\varphi}(x, t) \leq \varphi(x, t)$ for (x, t) in the neighbourhood of (x_1, t_1) where $\underline{\varphi}$ is defined. Hence $\underline{\varphi}$ serves as a lower barrier for φ in the strongest possible sense.

In order to compute the evolution equation for $\underline{\varphi}$ at (x_1, t_1) , we first note that we always have $|\pi(R)| \leq |R|$, and so writing L_0 for the Lipschitz constant of Q on the unit ball in $\mathcal{C}_B(\mathbb{R}^n)$ (which depends only on n) we can compute

$$|Q(R) - Q(\pi(R))| \leq |R|^2 \cdot \left| Q\left(\frac{R}{|R|}\right) - Q\left(\frac{\pi(R)}{|\pi(R)|}\right) \right| \leq |R|^2 L_0 \left| \frac{R}{|R|} - \frac{\pi(R)}{|\pi(R)|} \right| = L_0 |R| \cdot |R - \pi(R)|.$$

At (x_1, t_1) we then have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{g(t_1)}\right)\underline{\varphi} &= \langle \xi(R_{g(t_1)}(x_1)), Q(R_{g(t_1)}(x_1)) \rangle \\ &\leq \langle \xi(R_{g(t_1)}(x_1)), Q(R_{g(t_1)}(x_1)) - Q(\pi(R_{g(t_1)}(x_1))) \rangle \leq L_0 |R_{g(t_1)}| \varphi. \end{aligned}$$

Here we have used the fact that $\pi(R_{g(t_1)}(x_1)) \in \partial C$, so that

$$\langle \xi(R_{g(t_1)}(x_1)), Q(\pi(R_{g(t_1)}(x_1))) \rangle \leq 0$$

because C is invariant under the Hamilton ODE. We have also used that although $\xi(R_{g(t_1)}(x_1))$ is not parallel, its Laplacian vanishes at (x_1, t_1) by construction. □

Next we will show that for $R_{g(t)}$ inside the pinching cones $\widehat{C}(s)$ or $\widetilde{C}(b, s)$ of large magnitude, the pinching improves with time. We use an approach of Brendle, Huisken and Sinestrari [6] developed in the global setting.

Lemma 3.7 *Suppose C is either $\widehat{C}(s)$ or $\widetilde{C}(b, s)$, for $s > 0$ and $b \in (0, \Upsilon_n)$, with $n \geq 4$. Then there exists $\widehat{T}(n, C) > 0$ such that the following holds: Suppose $(M, g(t))$ for $t \in [0, T]$ is a smooth solution to the Ricci flow such that for some $x_0 \in M$ and $r_0 \in (0, 1)$, we have*

- (i) $B_{g(t)}(x_0, 8(n-1)r_0^{-1}) \Subset M$,
- (ii) $R_{g(t)} + I \in C$ on $B_{g(t)}(x_0, 8(n-1)r_0^{-1})$,
- (iii) $\text{Ric}_{g(t)} \leq (n-1)r_0^{-2}$ on $B_{g(t)}(x_0, r_0)$,

for all $t \in [0, T]$. Then for all $t \in [0, T \wedge \widehat{T}]$,

$$R_{g(t)}(x_0) + (4 - t \cdot \text{scal}(R_{g(t)}(x_0)))I \in C.$$

Proof For each $t \in [0, T]$ and $x \in B_{g(t)}(x_0, 8(n-1)r_0^{-1})$, we write

$$\begin{aligned} S &= S_{g(t)}(x) := R_{g(t)}(x) + (2 - tn(n-1) - t \text{scal}(R_{g(t)}(x))) \cdot I \\ &= (R_{g(t)}(x) + I) + (1 - t \text{scal}(R_{g(t)}(x) + I)) \cdot I = \widehat{R} + (1 - t \text{scal}(\widehat{R})) \cdot I, \end{aligned}$$

where we abbreviate $\widehat{R} := R_{g(t)}(x) + I$, and define

$$\varphi(x, t) = d(C, S_{g(t)}(x)).$$

As we shall see at the end, constraining φ not to grow too much will imply the conclusion of the lemma.

Observe that by convexity of the cone C , and the facts that $R_{g(0)} + I \in C$ on $B_{g(0)}(x_0, 8(n-1)r_0^{-1})$ and $I \in C$, we have $S_{g(0)} = (R_{g(0)} + I) + I \in C$, and hence $\varphi(\cdot, 0) \equiv 0$, throughout $B_{g(0)}(x_0, 8(n-1)r_0^{-1})$.

We first show that there exist $\widehat{T}, \delta > 0$ depending only on n and C such that φ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi \leq -\delta t^{-2} \varphi^2$$

in the barrier sense at every point (x_1, t_1) for which $t_1 \in (0, T \wedge \widehat{T}]$, $x_1 \in B_{g(t_1)}(x_0, 8(n-1)r_0^{-1})$ and $\varphi(x_1, t_1) > 0$. We want to construct a lower barrier for φ at (x_1, t_1) satisfying the desired evolution inequality.

Borrowing ideas and notation from the proof of Theorem 3.5, including the bundle trivialisation near to (x_1, t_1) , we define

$$\underline{\varphi}(x, t) = \langle \xi(S_{g(t_1)}(x_1)), S_{g(t)}(x) \rangle,$$

where ξ is given by (3-4). This satisfies $\underline{\varphi}(x, t) \leq \varphi(x, t)$ for all (x, t) in the neighbourhood of (x_1, t_1) where the bundle trivialisation holds, with equality at (x_1, t_1) .

Recall the evolution equation (3-5) for $R_{g(t)}(x)$, and the evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \text{scal}(R_{g(t)}) = 2|\text{Ric}(R_{g(t)})|^2$$

for $\text{scal}(R_{g(t)})$ [48, Proposition 2.5.4], which give us

$$\left(\frac{\partial}{\partial t} - \Delta\right)S = Q(R) - \text{scal}(R)I - 2t|\text{Ric}(R)|^2I - n(n-1)I.$$

At (x_1, t_1) we thus obtain

$$(3-6) \quad \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\varphi = \langle \xi(S), Q(R) - \text{scal}(R)I - 2t|\text{Ric}(R)|^2I - n(n-1)I \rangle.$$

We will estimate the right-hand side by adapting the approach of [6, Lemma 5; 54, Lemma 10]. A useful heuristic for the proof is that by constraining $\hat{T} > 0$ (and hence $t = t_1$) to be small, we will show that $|R|$ is large, and hence that the dominant term on the right-hand side of (3-6) is the quadratic term $\langle \xi(S), Q(R) \rangle$. This term will be comparable to $\langle \xi(S), Q(\pi(S)) \rangle$ because we will show that R and $\pi(S)$ are close, relative to their magnitude. To control this we follow [54] (cf [6]) and use property (IV) to obtain the estimate

$$(3-7) \quad \langle \xi(S), Q(\pi(S)) \rangle \leq -3\mu|\pi(S)|^2$$

for some $\mu > 0$ depending on n and the cone. The right-hand side will be negative enough to absorb all the error terms and conclude the lemma. We now give the details.

Because $\hat{R} \in \mathcal{C}$ and $I \in \mathcal{C}$ but $S = \hat{R} + (1 - t \text{scal}(\hat{R})) \cdot I \notin \mathcal{C}$, we must have $t \text{scal}(\hat{R}) > 1$ at (x_1, t_1) . But $\text{scal}(\hat{R}) \leq \alpha|\hat{R}|$, where we will use α to represent a n -dependent constant that is allowed to increase with each use. In particular, $|\hat{R}| \geq 1/(\alpha t)$. By reducing $\hat{T} > 0$ if necessary, this forces $|\hat{R}| \geq 2|I|$, which implies that $R = \hat{R} - I$ has comparable magnitude in that

$$\frac{1}{2}|R| \leq |\hat{R}| \leq 2|R|.$$

In particular,

$$(3-8) \quad |I| \leq |R|.$$

We compare R and $\pi(S)$ by estimating

$$(3-9) \quad |R - \pi(S)| = |\hat{R} - \pi(S) - I| \leq |\hat{R} - \pi(S)| + |I| \leq |\hat{R} - S| + |I| = (t \text{scal}(\hat{R}) - 1)|I| + |I| = t \text{scal}(\hat{R})|I|.$$

A first consequence is that $|R - \pi(S)| \leq \alpha t |\hat{R}| \leq 2\alpha t |R|$, so by reducing $\hat{T} > 0$ if necessary, we may assume that

$$|\pi(S)| \geq |R| - |R - \pi(S)| \geq (1 - 2\alpha t)|R| \geq \frac{1}{2}|R|.$$

Therefore $\text{scal}(\hat{R}) \leq \alpha|\pi(S)|$ and $|\text{scal}(R)| \leq \alpha|\pi(S)|$. In particular, (3-9) develops to

$$(3-10) \quad |R - \pi(S)| \leq \alpha t |\pi(S)|.$$

We can use this estimate to control the first term on the right-hand side of (3-6). By the quadratic nature of Q ,

$$Q(R) = Q(\pi(S) + (R - \pi(S))) = Q(\pi(S)) + Q(R - \pi(S)) + \pi(S) * (R - \pi(S)),$$

where the $*$ notation is explained in [48, Section 2.1]. In particular,

$$\begin{aligned} \langle \xi(S), Q(R) \rangle &\leq \langle \xi(S), Q(\pi(S)) \rangle + \alpha |R - \pi(S)|^2 + \alpha |\pi(S)| \cdot |R - \pi(S)| \\ &\leq -3\mu |\pi(S)|^2 + \alpha (t^2 + t) |\pi(S)|^2 \leq -2\mu |\pi(S)|^2, \end{aligned}$$

by (3-7) and (3-10), where (as usual) α is a n -dependent constant that can increase with each use, and we may have had to reduce $\hat{T} > 0$. The second term on the right-hand side of (3-6) can be controlled by

$$\langle \xi(S), -\text{scal}(R)I \rangle \leq |\text{scal}(R)| \cdot |I| \leq \alpha |\pi(S)| \leq (t \text{scal}(\hat{R}))\alpha |\pi(S)| \leq t\alpha |\pi(S)|^2.$$

Similarly, the fourth term on the right-hand side of (3-6) can be controlled by

$$\langle \xi(S), -n(n-1)I \rangle \leq \alpha |I| \leq \alpha |R| \leq (t \text{scal}(\hat{R}))\alpha |\pi(S)| \leq t\alpha |\pi(S)|^2,$$

where we have used (3-8). Finally, the third term on the right-hand side of (3-6) can be controlled by

$$\langle \xi(S), -2t|\text{Ric}(R)|^2 I \rangle \leq \alpha t |R|^2 \leq t\alpha |\pi(S)|^2.$$

Combining, (3-6) gives

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\underline{\varphi} \leq -2\mu |\pi(S)|^2 + t\alpha |\pi(S)|^2 \leq -\mu |\pi(S)|^2,$$

after possibly reducing $\hat{T} > 0$ again. Meanwhile, because $\hat{R} \in C$, we compute

$$\varphi(x_1, t_1) \leq |\hat{R} - S| = (t \text{scal}(\hat{R}) - 1)|I| \leq t \text{scal}(\hat{R})|I| \leq \alpha t |\pi(S)|.$$

Thus there exists $\delta > 0$ depending only on n and C such that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\underline{\varphi} \leq -\delta t^{-2} \varphi^2$$

at (x_1, t_1) , as claimed.

Now we are ready to set up a maximum principle argument. By hypothesis (iii) and [41, Lemma 8.3(a)], the function $\eta(x, t) = d_{g(t)}(x, x_0) + \frac{5}{3}(n-1)r_0^{-1}t$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\eta \geq 0$$

in the barrier sense whenever $d_{g(t)}(x, x_0) \geq r_0$. Let ϕ be a smooth nonincreasing function on \mathbb{R} such that $\phi \equiv 1$ on $(-\infty, 1]$, ϕ vanishes outside $(-\infty, 2]$ and satisfies $\phi'' \geq -10^2\phi$, and $|\phi'| \leq 10^2\phi^{1/2}$. Define

$$\Phi(x, t) = e^{-10^2 t A^{-2} r_0^2} \phi\left(\frac{\eta(x, t)}{A r_0^{-1}}\right), \quad A = 4(n-1)$$

so that $(\partial_t - \Delta_{g(t)})\Phi \leq 0$ in the sense of barriers. Moreover, due to the choice of A , $\Phi(x_0, t) = e^{-10^2 t A^{-2} r_0^2}$ for $t \in [0, T \wedge \hat{T}]$ and Φ vanishes outside $B_{g(t)}(x_0, 2A r_0^{-1})$, ie $B_{g(t)}(x_0, 8(n-1)r_0^{-1})$.

Consider the function $G = \Phi\varphi$. We fix $s \in (0, T \wedge \hat{T}]$. Considering the support of Φ , we see that G attains its maximum on $M \times [0, s]$ at some point (x_1, t_1) with $x_1 \in B_{g(t_1)}(x_0, 8(n-1)r_0^{-1})$. Because we showed that $\varphi(\cdot, 0) \equiv 0$, throughout $B_{g(0)}(x_0, 8(n-1)r_0^{-1})$, we see that $G(\cdot, 0) \equiv 0$ throughout M . Since our goal is to obtain an upper bound for G , we may assume $t_1 > 0$ and $G(x_1, t_1) > 0$; in particular both $\varphi(x_1, t_1) > 0$ (so our calculations above are valid) and $\Phi(x_1, t_1) > 0$. We may assume Φ and φ to be smooth when we apply the maximum principle; for example see [47, Section 7] for a detailed exposition.

In this case, at (x_1, t_1) we have $\Phi\nabla\varphi = -\varphi\nabla\Phi$ from $\nabla G = 0$, and hence

$$\begin{aligned} 0 &\leq \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)G = \varphi \cdot \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi + \Phi \cdot \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\varphi - 2\langle \nabla\varphi, \nabla\Phi \rangle \\ &\leq -\delta t^{-2}\varphi^2\Phi + \frac{2|\nabla\Phi|^2}{\Phi}\varphi \leq \varphi \left(-\delta t^{-2}G + \frac{2 \cdot 10^4}{A^2 r_0^{-2}}\right). \end{aligned}$$

Thus $\sup_{M \times [0, s]} G = G(x_1, t_1) \leq C_n r_0^2 \delta^{-1} s^2$ for some dimensional constant $C_n > 0$. Since s is arbitrary in $[0, T \wedge \hat{T}]$, we have

$$G(x, t) \leq C_n r_0^2 \delta^{-1} t^2$$

for all $(x, t) \in M \times [0, T \wedge \hat{T}]$. Since $r_0 \in (0, 1)$, evaluating at x_0 , where $\Phi(x_0, t) = e^{-10^2 t A^{-2} r_0^2}$, yields

$$(3-11) \quad \varphi(x_0, t) \leq e^{10^2 t A^{-2} r_0^2} \cdot C_n r_0^2 \delta^{-1} t^2 \leq L t^2,$$

for some constant $L < \infty$ depending only on n and C , for $t \in [0, T \wedge \hat{T}]$. By definition $S - \varphi(x_0, t)\xi(S) \in C$, and so we can compute at x_0

$$R + (4 - t \operatorname{scal}(R))I = S + 2I + n(n-1)tI = [S - \varphi(x_0, t)\xi(S)] + [2I + n(n-1)tI + \varphi(x_0, t)\xi(S)],$$

which must lie in C for $t \in [0, T \wedge \hat{T}]$, provided we constrain $\hat{T} > 0$ depending on n and C . Here we have used the convexity of the cone C and the fact that I lies in the interior of C , along with the estimate (3-11). □

We now use the cone almost improving nature (ie Lemma 3.7) to obtain curvature estimates.

Proposition 3.8 *Suppose M^n is a noncompact (connected) manifold for $n \geq 4$, and $g(t)$ for $t \in [0, T]$ is a smooth solution to the Ricci flow on M such that for some $x_0 \in M$, $s_0 > 0$ and $b \in (0, \Upsilon_n)$, we have*

- (i) $B_{g(t)}(x_0, 1) \Subset M$ for all $t \in [0, T]$,
- (ii) $R_{g(t)} + I \in \tilde{C}(b, s_0)$ on $B_{g(t)}(x_0, 1)$ for $t \in [0, T]$.

Then there exist $C_0(n, b, s_0), S_1(n, b, s_0) > 0$ such that for $t \in (0, S_1 \wedge T]$,

$$|R_{g(t)}(x_0)| \leq C_0 t^{-1}.$$

Thus we obtain C_0/t decay of the full curvature tensor analogous to the estimate in [35, Lemma 3.3].

Proof The proof of the curvature estimate uses the Perelman-inspired point-picking argument from [47, Lemma 2.1], and initially mirrors the proof of [35, Lemma 3.3].

Suppose the conclusion is false for some $n \geq 4$, $b \in (0, \Upsilon_n)$ and $s_0 > 0$. Then for any $a_k \rightarrow +\infty$, we can find a sequence of noncompact manifolds M_k^n , Ricci flows $g_k(t)$ for $t \in [0, T_k]$, and $x_k \in M_k$ satisfying the hypotheses but such that the curvature estimate fails with $C_0 = a_k$ in an arbitrarily short time. We may assume $a_k T_k \rightarrow 0$. By smoothness of each Ricci flow, we can choose $t_k \in (0, T_k]$ so that

- (i) $B_{g_k(t)}(x_k, 1) \Subset M_k$ for $t \in [0, t_k]$,
- (ii) $R_{g_k(t)} + I \in \tilde{C}(b, s_0)$ on $B_{g_k(t)}(x_k, 1)$ for $t \in [0, t_k]$,
- (iii) $|R_{g_k(t)}(x_k)| < a_k t^{-1}$ for $t \in (0, t_k)$,
- (iv) $|R_{g_k(t_k)}(x_k)| = a_k t_k^{-1}$.

By (iv) and the fact that $a_k t_k \rightarrow 0$, [47, Lemma 5.1] implies that for sufficiently large k , we can find $\beta(n) > 0$, times $\tilde{t}_k \in (0, t_k]$ and points $\tilde{x}_k \in B_{g_k(\tilde{t}_k)}(x_k, \frac{3}{4} - \frac{1}{2}\beta\sqrt{a_k\tilde{t}_k})$ such that

$$(3-12) \quad |R_{g_k(t)}(x)| \leq 4|R_{g_k(\tilde{t}_k)}(\tilde{x}_k)| = 4Q_k$$

whenever $d_{g_k(\tilde{t}_k)}(x, \tilde{x}_k) < \frac{1}{8}\beta a_k Q_k^{-1/2}$ and $\tilde{t}_k - \frac{1}{8}a_k Q_k^{-1} \leq t \leq \tilde{t}_k$ where $\tilde{t}_k Q_k \geq a_k \rightarrow +\infty$.

Included in the proof of [47, Lemma 5.1] is that for each such (x, t) we have $x \in B_{g_k(t)}(x_k, 1)$, ie the cylinder $B_{g_k(\tilde{t}_k)}(\tilde{x}_k, \frac{1}{8}\beta a_k Q_k^{-1/2}) \times [\tilde{t}_k - \frac{1}{8}a_k Q_k^{-1}, \tilde{t}_k]$ where (3-12) holds lies within the region where (ii) holds.

Consider the parabolic rescaling centred at $(\tilde{x}_k, \tilde{t}_k)$, namely $\tilde{g}_k(t) = Q_k g_k(\tilde{t}_k + Q_k^{-1}t)$ for $t \in [-\frac{1}{8}a_k, 0]$ such that

- (a) $|R_{\tilde{g}_k(0)}(\tilde{x}_k)| = 1$,
- (b) $|R_{\tilde{g}_k(t)}| \leq 4$ on $B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}\beta a_k) \times [-\frac{1}{8}a_k, 0]$, and
- (c) $R_{\tilde{g}_k(t)} + Q_k^{-1}I \in \tilde{C}(b, s_0)$ on $B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}\beta a_k) \times [-\frac{1}{8}a_k, 0]$.

If we had a uniform positive lower bound on the injectivity radii $\text{inj}(\tilde{g}_k(0))(\tilde{x}_k)$, then Hamilton’s compactness theorem would let us extract a subsequence converging in the C^∞ Cheeger–Gromov sense to a complete ancient solution of Ricci flow $g_\infty(t)$ which would be noncompact, nonflat and have bounded curvature. Moreover, we would have $R_{g_\infty(t)} \in \tilde{C}(b, s_0)$ for all $t \leq 0$ so that [54, Lemma 15] would apply to give a contradiction to the noncompactness of the underlying manifolds M_k . To accommodate the lack of an injectivity radius lower bound, we can instead take a local limit $\tilde{g}_\infty(t)$ of $\tilde{g}_k(t)$ by lifting to a Euclidean ball via the exponential map of $\tilde{g}_k(0)$ as in the proof of [35, Lemma 3.3]. (We only need to consider the local limit $\tilde{g}_\infty(0)$, but we take the limit at each time for consistency with [35].) Since the limit is a priori only locally defined on a ball in Euclidean space, we need to extract more information along the sequence first.

We now improve the pinching behaviour at $t = 0$ by using the fact that the flow is *almost* a complete ancient solution with bounded curvature. We first show that it becomes almost PIC2 pinched.

Claim 3.9 For any $L, \varepsilon > 0$ and $s \geq s_0$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have

$$R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(b, s)$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$.

Proof of claim Let S be the set of $s' \in [s_0, +\infty)$ such that for all $L > 0$ and $\varepsilon \in (0, 1)$ we can find $N \in \mathbb{N}$ such that for all $k > N$ we have

$$(3-13) \quad R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(b, s')$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$. Clearly $s_0 \in S$ by (c) above. We want to show that S is open and closed so that $S = [s_0, +\infty)$.

We first show that S is closed. Let $s_i \in S$ be such that $s_i \rightarrow s_\infty \in \mathbb{R}$. By definition of the cones $\tilde{C}(b, \tau)$ we have the following type of continuity of the cones with respect to τ : for any $\varepsilon' > 0$, there exists $\delta > 0$ such that if $R \in \tilde{C}(b, s)$ for some $|s - s_\infty| < \delta$, then $R + \varepsilon' \text{scal}(R) \cdot I \in \tilde{C}(b, s_\infty)$.

For any given $L > 0$ and $\varepsilon \in (0, 1)$ for which we want (3-13) to be true with $s' = s_\infty$ for sufficiently large k , we choose $\varepsilon' = \varepsilon / (8n(n - 1))$ and obtain a corresponding $\delta > 0$ from the above continuity. We fix i_0 sufficiently large so that $|s_{i_0} - s_\infty| < \delta$. Since $s_{i_0} \in S$, we can apply the definition of $s_{i_0} \in S$ with ε replaced by $\frac{1}{4}\varepsilon$ to deduce that there exists $N \in \mathbb{N}$ such that for all $k > N$,

$$R_{\tilde{g}_k(t)} + \frac{1}{4}\varepsilon I \in \tilde{C}(b, s_{i_0})$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$. Hence, $\hat{R} = R_{\tilde{g}_k(t)} + \frac{1}{4}\varepsilon I$ satisfies

$$\hat{R} + \frac{1}{8n(n - 1)}\varepsilon \cdot \text{scal}(\hat{R})I \in \tilde{C}(b, s_\infty).$$

Since

$$\text{scal}(\hat{R}) = \text{scal}(R_{\tilde{g}_k(t)}) + \frac{1}{4}\varepsilon n(n - 1),$$

we can then unwind the definition of \hat{R} to give

$$R_{\tilde{g}_k(t)} + \left(\frac{\varepsilon}{4} + \frac{1}{8} \cdot \frac{\varepsilon^2}{4} + \frac{1}{8n(n - 1)}\varepsilon \cdot \text{scal}(R_{\tilde{g}_k(t)}) \right) I \in \tilde{C}(b, s_\infty).$$

A coarse consequence of the curvature bound $|R_{\tilde{g}_k(t)}| \leq 4$ from (b) is that $\text{scal}(R_{\tilde{g}_k(t)}) \leq 4n(n - 1)$, so keeping in mind that $\varepsilon \leq 1$, we find that

$$R_{\tilde{g}_k(t)} + \varepsilon I \in \tilde{C}(b, s_\infty)$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$ for sufficiently large k . Hence S is closed.

It remains to show that S is open. Let $s' \in S$. It suffices to show that for some $\sigma' > 0$, the following is true: For all $L > 0$ and $\varepsilon \in (0, 1)$, the modified curvature tensor $\check{R} = R_{\tilde{g}_k(t)} + \varepsilon I$ satisfies $\check{R} - \sigma' \cdot \text{scal}(\check{R})I \in \tilde{C}(b, s')$ on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0]$ for sufficiently large k . The result will then follow using the continuity of the cones.

Set $\varepsilon' = \frac{1}{32}\varepsilon$ and let $L' > L$ be a constant to be chosen later. Since $s' \in \mathcal{S}$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have

$$R_{\tilde{g}_k(t)} + \varepsilon' I \in \tilde{\mathcal{C}}(b, s')$$

on $B_{\tilde{g}_k(0)}(\tilde{x}_k, L') \times [-(L')^2, 0]$.

Define $\sigma' := \min\{\hat{T}(n, \tilde{\mathcal{C}}(b, s')), 1/(2n(n-1))\}$, where \hat{T} is the positive constant obtained from Lemma 3.7 with \mathbf{C} chosen to be $\tilde{\mathcal{C}}(b, s')$. Set $r_0 = \frac{1}{8}\sqrt{\varepsilon} \in (0, 1)$, let $(x, \tau) \in B_{\tilde{g}_k(0)}(\tilde{x}_k, L) \times [-L^2, 0] \Subset B_{\tilde{g}_k(0)}(\tilde{x}_k, L') \times [-(L')^2, 0]$ and define a rescaled Ricci flow

$$\hat{g}_k(t) = 4r_0^2 \tilde{g}_k\left(\tau + \frac{t - \sigma'}{4r_0^2}\right) \quad \text{for } t \in [0, \sigma'].$$

In order to ensure that $\hat{g}_k(t)$ is defined on the whole time interval $[0, \sigma']$, even if τ is chosen at its least value $-L^2$, we ask that L' is large enough that $L^2 + \sigma'/(4r_0^2) \leq (L')^2$. We would also like to choose L' large enough that for every $t \in [0, \sigma']$ there is enough space to fit balls $B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1})$ within $B_{\tilde{g}_k(0)}(\tilde{x}_k, L')$. By the overall curvature bound of part (b), we have $|R_{\tilde{g}_k(\cdot)}| \leq 4$, and this bound is enough to allow us to compare geodesic balls at different times. In particular, we have the inclusion

$$B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1}) \subset B_{\tilde{g}_k(0)}(x, L_0)$$

for all $t \in [0, \sigma']$, for some L_0 depending on n, ε, L and σ' . Thus it suffices to insist that $L' \geq L + L_0$. Furthermore since $a_k \rightarrow +\infty$, for sufficiently large k we have

$$B_{\tilde{g}_k(0)}(\tilde{x}_k, L') = B_{g_k(\tilde{t}_k)}(\tilde{x}_k, L' Q_k^{-1/2}) \subset B_{g_k(\tilde{t}_k)}(\tilde{x}_k, \frac{1}{8}\beta a_k Q_k^{-1/2}) \subset B_{g_k(\tilde{t}_k)}(x_k, 1),$$

so that the region under consideration is compactly contained in M_k .

We conclude that, increasing N if necessary (possibly depending also on ε or L'), for $k > N$, $\hat{g}_k(t)$ for $t \in [0, \sigma']$ is a Ricci flow such that for all $t \in [0, \sigma']$,

- (1) $B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1}) \Subset M_k$,
- (2) $R_{\hat{g}_k(t)} + I \in \tilde{\mathcal{C}}(b, s')$ on $B_{\hat{g}_k(t)}(x, 8(n-1)r_0^{-1})$,
- (3) $\text{Ric}_{\hat{g}_k(t)} \leq (n-1)r_0^{-2}$ on $B_{\hat{g}_k(t)}(x, r_0)$.

Applying Lemma 3.7 to $\hat{g}_k(t)$ at (x, σ') gives

$$R_{\hat{g}_k(\sigma')}(x) + (4 - \sigma' \cdot \text{scal}(R_{\hat{g}_k(\sigma')}(x)))I \in \tilde{\mathcal{C}}(b, s').$$

Rescaling the metric back by the same factor $4r_0^2 = \frac{1}{16}\varepsilon$ then gives

$$R_{\tilde{g}_k(\tau)}(x) + \left(\frac{1}{4}\varepsilon - \sigma' \cdot \text{scal}(R_{\tilde{g}_k(\tau)}(x))\right)I \in \tilde{\mathcal{C}}(b, s').$$

Rewriting using $\check{R} = R_{\tilde{g}_k(\tau)}(x) + \varepsilon I$ gives

$$\check{R} + \left(\frac{1}{4}\varepsilon + \sigma'\varepsilon n(n-1) - \varepsilon\right)I - \sigma' \cdot \text{scal}(\check{R})I \in \tilde{\mathcal{C}}(b, s').$$

By definition, $\sigma' \leq 1/(2n(n - 1))$, so the part in brackets is negative and we deduce that for the given $\varepsilon, L > 0$,

$$\check{R} - \sigma' \cdot \text{scal}(\check{R})I \in \check{C}(b, s')$$

throughout $B_{\check{g}_k(0)}(\check{x}_k, L) \times [-L^2, 0]$ for sufficiently large k , as desired. This shows that \mathcal{S} is also open, and hence $S = [s_0, +\infty)$. □

Thanks to Claim 3.9, any local limit $\check{g}_\infty(0)$ (in the sense described before Claim 3.9) of the metrics $\check{g}_k(0)$ has curvature lying in $\check{C}(b, s) := \ell_b(\check{C}(s))$, for all s sufficiently large. By definition, being in $\check{C}(s)$ for all s sufficiently large is equivalent to being in C_{PIC2} . Thus $\check{g}_\infty(0) \in \ell_b(C_{\text{PIC2}}) \subset C_{\text{PIC2}}$ because $\ell_b := \ell_{a,b}$ for $a := b + \frac{1}{2}(n - 2)b^2 > b \geq 0$, eg using Lemma 3.2. Next, we will improve this control further by showing that $\check{g}_\infty(0)$ lies in $\hat{C}(s)$ for all s sufficiently large, which will ultimately show that any local blow-up limit $\check{g}_\infty(0)$ is in fact a space form.

Claim 3.10 *There exists $\hat{s}_0 > 0$ such that the following is true: for any $L, \varepsilon > 0$ and $s \geq \hat{s}_0$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have*

$$R_{\check{g}_k(t)} + \varepsilon I \in \hat{C}(s)$$

on $B_{\check{g}_k(0)}(\check{x}_k, L) \times [-L^2, 0]$.

Proof of claim Since $\check{C}(s) \rightarrow C_{\text{PIC2}}$ as $s \rightarrow \infty$, we have $\check{C}(b, s) \rightarrow \ell_b(C_{\text{PIC2}})$ by definition. Lemma 3.2 tells us that if $R \in \ell_b(C_{\text{PIC2}})$ then, for $\delta > 0$ depending on $b > 0$ and n , we have $R - \delta \text{scal}(R) \cdot I \in C_{\text{PIC2}}$. Thus for $\tilde{s} > 0$ large enough,

$$\check{C}(b, s) \subset \{S \in \mathcal{C}_B(\mathbb{R}^n) : S - \frac{1}{2}\delta \text{scal}(S) \cdot I \in C_{\text{PIC2}}\}$$

for every $s \geq \tilde{s}$, while for $\hat{s}_0 > 0$ small enough, by continuity of the cones $\hat{C}(s)$, and the fact that $\hat{C}(s) \rightarrow C_{\text{PIC2}}$ as $s \searrow 0$, we have

$$\{S \in \mathcal{C}_B(\mathbb{R}^n) : S - \frac{1}{2}\delta \text{scal}(S) \cdot I \in C_{\text{PIC2}}\} \subset \hat{C}(\hat{s}_0).$$

To summarise, for our given small $\hat{s}_0 > 0$, we have

$$(3-14) \quad \check{C}(b, s) \subset \hat{C}(\hat{s}_0)$$

for every $s \geq \tilde{s}$.

Mimicking the proof of Claim 3.9, we let S' be the set of $s' \in [\hat{s}_0, +\infty)$ such that for all $L > 0$ and $\varepsilon \in (0, 1)$, we can find $N \in \mathbb{N}$ such that for all $k > N$, we have

$$R_{\check{g}_k(t)} + \varepsilon I \in \hat{C}(s')$$

on $B_{\check{g}_k(0)}(\check{x}_k, L) \times [-L^2, 0]$. The set S' is nonempty because $\hat{s}_0 \in S'$ by Claim 3.9 and (3-14). Now we can carry out the same argument as in the proof of Claim 3.9 to show that $S' = [\hat{s}_0, +\infty)$. □

By Claim 3.10, any local limit $\tilde{g}_\infty(0)$ of the metrics $\tilde{g}_k(0)$ is a space-form with nonnegative curvature. As $|R_{\tilde{g}_k(0)}(\tilde{x}_k)| = 1$, this forces $\text{Ric}_{\tilde{g}_\infty(0)} = \alpha$ for some dimensional constant $\alpha > 0$. Now we can argue as in the proof of [35, Lemma 3.3] to draw a contradiction from the noncompactness of M_k for k sufficiently large. \square

4 Existence of Ricci flow under pinching

In this section, we will construct a smooth complete Ricci flow solution with scaling invariant estimates from metrics with pinched curvature. To do this, we will need a local existence theorem for the Ricci flow, which is one of the main goals of this section.

Theorem 4.1 *For any $s_0 > 0$, $n \geq 4$ and $b \in (0, \Upsilon_n)$, there exist $a_0(n, b, s_0) > 0$ and $T(n, b, s_0) > 0$ such that the following holds. Suppose (M^n, g_0) is a manifold such that*

- (a) $B_{g_0}(p, 4) \Subset M$,
- (b) $R_{g_0} \in \tilde{\mathcal{C}}(b, s_0)$ on $B_{g_0}(p, 4)$.

Then there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, 1) \times [0, T]$ such that $g(0) = g_0$ and

- (i) $R_{g(t)} + I \in \tilde{\mathcal{C}}(b, s_0)$,
- (ii) $|R_{g(t)}| \leq a_0 t^{-1}$.

We will need several ingredients for the proof of Theorem 4.1. The first of these is a result of Hochard that allows us to construct a local Ricci flow on regions with bounded curvature by modifying an incomplete Riemannian metric at its extremities in order to make it complete, without increasing the curvature too much, and without changing it in the interior.

Proposition 4.2 ([35, Proposition 4.2], based on [27]) *For $n \geq 2$ there exist constants $\alpha \in (0, 1]$ and $\Lambda > 1$ depending on n such that the following is true. Suppose (N^n, h_0) is a smooth manifold (not necessarily complete) that satisfies $|R_{h_0}| \leq \rho^{-2}$ throughout, for some $\rho > 0$. Then there exists a smooth Ricci flow $h(t)$ on N for $t \in [0, \alpha\rho^2]$, with the properties that*

- (i) $h(0) = h_0$ on $N_\rho = \{x \in N : B_{h_0}(x, \rho) \Subset N\}$,
- (ii) $|R_{h(t)}| \leq \Lambda\rho^{-2}$ throughout $N \times [0, \alpha\rho^2]$.

We also recall the shrinking balls lemma, which is one of the local ball inclusion results based on the distance distortion estimates of Hamilton and Perelman from [41, Lemma 8.3].

Lemma 4.3 [47, Corollary 3.3] *For $n \geq 2$ there exists a constant $\beta \geq 1$ depending only on n such that the following is true. Suppose $(N^n, g(t))$ is a Ricci flow for $t \in [0, S]$ and $x_0 \in N$ with $B_{g(0)}(x_0, r) \Subset N$ for some $r > 0$, and $\text{Ric}_{g(t)} \leq a/t$ on $B_{g_0}(x_0, r)$ for each $t \in (0, S]$. Then*

$$B_{g(t)}(x_0, r - \beta\sqrt{at}) \subset B_{g_0}(x_0, r).$$

We are now able to prove Theorem 4.1. We stay as close as possible to the proof of [35, Theorem 5.1].

Proof of Theorem 4.1 We start by specifying the positive constants that will be used in the construction:

- $\Lambda(n) > 1$ and $\alpha(n)$ from Proposition 4.2,
- $\beta(n)$ from Lemma 4.3,
- $C_0(n, b, s_0)$ from Proposition 3.8,
- $a_0(n, b, s_0) = \max\{1, \alpha\Lambda, \Lambda(\alpha + C_0)\}$,
- $S_0(n, a_0, b, s_0)$ from Lemma 3.6,
- $S_1(n, b, s_0)$ from Proposition 3.8,
- $\Lambda_0(n, b, s_0) := \max\{4S_1^{-1/2}a_0^{-1/2}, 4S_0^{-1/2}a_0^{-1/2}, 8\beta, 8C_0^{-1/2}a_0^{-1/2}\}$,
- $\mu(n, b, s_0) := \sqrt{1 + \alpha C_0^{-1}} - 1 > 0$.

Choose $\rho \in (0, 1)$ small enough that for all $x \in B_{g_0}(p, 4)$, we have $|R_{g_0}| \leq \rho^{-2}$. We have no uniform positive lower bound for ρ . Apply Proposition 4.2 with $N = B_{g_0}(p, 4)$ to find a smooth solution $g(t)$ to the Ricci flow defined on $B_{g_0}(p, 3) \times [0, \alpha\rho^2]$ with $|R_{g(t)}| \leq \Lambda\rho^{-2}$ and $g(0) = g_0$ on $B_{g_0}(p, 3)$. In particular, for all $(x, t) \in B_{g_0}(p, 3) \times (0, \alpha\rho^2]$, we have

$$|R_{g(t)}| \leq \Lambda\rho^{-2} \leq a_0t^{-1}.$$

Now define sequences of times t_k and radii r_k inductively as follows:

- (a) $t_1 = \alpha\rho^2$ and $r_1 = 3$, where ρ is as above.
- (b) $t_{k+1} = (1 + \mu)^2t_k$ for all $k \geq 1$.
- (c) $r_{k+1} = r_k - \Lambda_0\sqrt{a_0t_k}$ for all $k \geq 1$.

Consider the following statement:

$\mathcal{P}(k)$ There exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, r_k) \times [0, t_k]$ with $g(0) = g_0$ such that $|R_{g(t)}| \leq a_0t^{-1}$.

Clearly $\mathcal{P}(1)$ is true. We want to show that $\mathcal{P}(k)$ is true for every k while $r_k > 0$, and we do so by induction.

Suppose $\mathcal{P}(k)$ is true for some $k \in \mathbb{N}$, and consider a Ricci flow $g(t)$ that this provides. We want to show that $\mathcal{P}(k + 1)$ is true if $r_{k+1} > 0$ by extending $g(t)$ to a longer time interval.

Let $x \in B_{g_0}(p, r_{k+1} + \frac{3}{4}\Lambda_0\sqrt{a_0t_k})$, so that

$$B_{g_0}(x, \frac{1}{4}\Lambda_0\sqrt{a_0t_k}) \Subset B_{g_0}(p, r_k).$$

Consider the rescaled Ricci flow $\tilde{g}(t) = \lambda^{-2}g(\lambda^2t)$ for $t \in [0, \lambda^{-2}t_k]$ where $\lambda = \frac{1}{4}\Lambda_0\sqrt{a_0t_k}$ so that $B_{g(0)}(x, \frac{1}{4}\Lambda_0\sqrt{a_0t_k}) = B_{\tilde{g}(0)}(x, 1)$ and $\lambda^{-2}t_k = 16\Lambda_0^{-2}a_0^{-1}$. On the rescaled domain, the Ricci flow $\tilde{g}(t)$ is smooth and satisfies:

- (i) $R_{\tilde{g}(0)} \in \tilde{C}(b, s_0)$.
- (ii) $|R_{\tilde{g}(t)}| \leq a_0 t^{-1}$ on $B_{\tilde{g}(0)}(x, 1) \times (0, 16\Lambda_0^{-2}a_0]$.

Applying Lemma 3.6 to $\tilde{g}(t)$, we deduce that

$$R_{\tilde{g}(t)}(x) + I \in \tilde{C}(b, s_0)$$

for $t \leq \min\{S_0, 16\Lambda_0^{-2}a_0^{-1}\} = 16\Lambda_0^{-2}a_0^{-1}$ thanks to the choice of Λ_0 . Hence for all

$$(x, t) \in B_{g_0}(p, r_{k+1} + \frac{3}{4}\Lambda_0\sqrt{a_0 t_k}) \times [0, t_k],$$

we have

$$R_{\tilde{g}(t)} + (\frac{1}{4}\Lambda_0\sqrt{a_0 t_k})^{-2}I \in \tilde{C}(b, s_0).$$

We now use this to obtain an improved curvature estimate on a slightly smaller ball. For $x \in \Omega := B_{g_0}(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k})$, we have

$$B_{g_0}(x, \frac{1}{2}\Lambda_0\sqrt{a_0 t_k}) \Subset B_{g_0}(p, r_{k+1} + \frac{3}{4}\Lambda_0\sqrt{a_0 t_k}).$$

By Lemma 4.3 and our choice of Λ_0 , we deduce that

$$B_{\tilde{g}(t)}(x, \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}) \Subset B_{g_0}(p, r_{k+1} + \frac{3}{4}\Lambda_0\sqrt{a_0 t_k}).$$

Therefore, the rescaled Ricci flow $\tilde{g}(t)$ for $t \in [0, 16\Lambda_0^{-2}a_0^{-1}]$ satisfies

- (I) $B_{\tilde{g}(t)}(x, 1) \Subset M$,
- (II) $R_{\tilde{g}(t)} + I \in \tilde{C}(b, s_0)$ on $B_{\tilde{g}(t)}(x, 1)$,

for each $t \in [0, 16\Lambda_0^{-2}a_0^{-1}]$, and hence Proposition 3.8 applies to show that

$$(4-1) \quad |R_{\tilde{g}(t)}(x)| \leq C_0 t^{-1}$$

for $0 < t \leq \min\{S_1, 16\Lambda_0^{-2}a_0^{-1}\} = 16\Lambda_0^{-2}a_0^{-1}$. Since this estimate is scaling invariant, we have improved the curvature decay of $g(t)$ from $a_0 t^{-1}$ to $C_0 t^{-1}$ on $\Omega \times (0, t_k]$, where we recall that $\Omega := B_{g_0}(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k})$.

Now we construct an extension of $g(t)$. For $h_0 = g(t_k)$, (4-1) implies $\sup_{\Omega} |R_{h_0}| \leq \rho_0^{-2}$ where $\rho_0 = \sqrt{C_0^{-1}t_k}$. Note that by definition of Λ_0 , we have $\rho_0 \leq \frac{1}{8}\sqrt{a_0 t_k}\Lambda_0$. Moreover, for $x \in B_{g_0}(p, r_{k+1})$, Lemma 4.3 (using only the original $a_0 t^{-1}$ curvature decay rather than the refined $C_0 t^{-1}$ decay) and the choice of Λ_0 imply

$$B_{g(t_k)}(x, \rho_0) \subset B_{g_0}(x, \rho_0 + \beta\sqrt{a_0 t_k}) \subset B_{g_0}(x, \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}) \Subset \Omega.$$

This shows that $B_{g_0}(p, r_{k+1}) \subset \Omega_{\rho_0}$ where Ω_{ρ_0} , as in Proposition 4.2, is computed using h_0 . By applying Proposition 4.2, we find a Ricci flow $g(t)$ on $B_{g_0}(p, r_{k+1}) \times [t_k, t_{k+1}]$, extending $g(t)$ on this smaller ball, with $t_{k+1} = t_k + \alpha\rho_0^2 = (1 + \mu)^2 t_k$ and

$$|R_{g(t)}| \leq \Lambda\rho_0^{-2} = \Lambda C_0 t_k^{-1} \leq a_0 t_{k+1}^{-1} \leq a_0 t^{-1},$$

thanks to the choice of a_0 . This shows that $\mathcal{P}(k + 1)$ is true if $r_{k+1} > 0$.

Since $\lim_{k \rightarrow +\infty} r_k = -\infty$ and $r_1 = 3$, there exists $i \in \mathbb{N}$ such that $r_i \geq 2$ and $r_{i+1} < 2$. Since $\mathcal{P}(i)$ holds, we now wish to estimate the corresponding t_i from below:

$$2 > r_{i+1} = 3 - \Lambda_0 \sqrt{a_0} \cdot \sum_{k=1}^i \sqrt{t_k} \geq 3 - \Lambda_0 \sqrt{a_0 t_i} \cdot \sum_{k=0}^{\infty} (1 + \mu)^{-k} = 3 - \sqrt{t_i} \cdot \frac{\Lambda_0 \sqrt{a_0} (1 + \mu)}{\mu}.$$

This implies

$$t_i > \frac{\mu^2}{a_0 \Lambda_0^2 (1 + \mu)^2} =: T(n, b, s_0),$$

showing that there exists a smooth Ricci flow solution $g(t)$ on $B_{g_0}(p, 2) \times [0, T]$ such that $g(0) = g_0$ and $|R_{g(t)}| \leq a_0 t^{-1}$. The conclusion on pinching follows from applying Lemma 3.6 on $B_{g_0}(x, 1)$ where $x \in B_{g_0}(p, 1)$, provided that we shrink T further if necessary. \square

We can now establish the existence of a Ricci flow on $M \times [0, +\infty)$ as claimed in Theorem 1.3, using Theorem 4.1.

Proof of Theorem 1.3 By Lemma 3.4, the pinching hypothesis (1-3) implies that $R_{g_0} \in \tilde{\mathcal{C}}(b, s_0)$ for some $s_0 > 0$ and $b \in (0, \Upsilon_n)$ depending only on n and ε_0 .

Fix $p \in M$. Pick $R_i \rightarrow +\infty$ and write $h_{i,0} = R_i^{-2} g_0$. Then $R_{h_{i,0}} \in \tilde{\mathcal{C}}(b, s_0)$ for all i . We apply Theorem 4.1 to $h_{i,0}$ to obtain a Ricci flow solution $h_i(t)$ on $B_{h_{i,0}}(p, 1) \times [0, T]$ with

- (i) $|R_{h_i(t)}| \leq a_0 t^{-1}$,
- (ii) $R_{h_i(t)} + I \in \tilde{\mathcal{C}}(b, s_0)$,

for some $a_0(n, s_0), T(n, s_0) > 0$. Define the rescaled Ricci flow solution $g_i(t) = R_i^2 h_i(R_i^{-2} t)$ on $B_{g_0}(p, R_i) \times [0, TR_i^2]$ with

$$\begin{cases} g_i(0) = g_0, \\ |R_{g_i(t)}| \leq a_0 t^{-1}, \\ R_{g_i(t)} + R_i^{-2} I \in \tilde{\mathcal{C}}(b, s_0), \end{cases}$$

on each $B_{g_0}(p, R_i) \times (0, TR_i^2]$.

By [15, Corollary 3.2] (see also [44]) and a modification of Shi’s higher-order estimate given in [18, Theorem 14.16], we deduce that for any $k \in \mathbb{N}$, $S > 0$ and $\Omega \Subset M$, there exists $C(k, \Omega, g_0, a_0, S) > 0$ such that for sufficiently large i , we have

$$\sup_{\Omega \times [0, S]} |\nabla^k R_{g_i(t)}| \leq C(k, \Omega, g_0, a_0, S).$$

By applying the Ascoli–Arzelà theorem in coordinate charts, a subsequence converges to a smooth solution $g(t) = \lim_{i \rightarrow +\infty} g_i(t)$ of the Ricci flow on $M \times [0, +\infty)$, so that $g(0) = g_0$, $|R_{g(t)}| \leq a_0 t^{-1}$ and

$$R_{g(t)} \in \tilde{\mathcal{C}}(b, s_0)$$

throughout $M \times [0, +\infty)$. This implies the pinching conclusion (a) of the theorem by Lemma 3.2, because $\tilde{\mathcal{C}}(b, s_0) \subset \ell_b(C_{\text{PIC1}})$. Moreover, since (M, g_0) is complete, $g(t)$ is a complete solution by Lemma 4.3. \square

We are ready to prove the main theorem, Theorem 1.2, which we restate in the following equivalent form:

Theorem 4.4 *Suppose, for $n \geq 4$, that (M^n, g_0) is a complete noncompact manifold such that*

- (i) $R_{g_0} - \varepsilon_0 \operatorname{scal}(R_{g_0}) \cdot I \in C_{\text{PIC1}}$ for some $\varepsilon_0 \in (0, 1)$,
- (ii) $R_{g_0} \in C_{\text{PIC2}}$.

Then (M, g_0) is flat.

Before we begin the proof, we observe the following basic local fact that we will need more than once:

Lemma 4.5 *A manifold of dimension at least three that splits isometrically into a nontrivial product cannot be PIC1.*

To clarify, being PIC1 means that the curvature tensor lies in the interior of C_{PIC1} , or alternatively that the complex sectional curvatures corresponding to PIC1 sections are strictly positive.

Proof Consider a manifold $M_1 \times M_2$, where $\dim(M_1) \geq 1$ and $\dim(M_2) \geq 2$. Pick a unit vector v in some tangent space $T_x M_1$ and two orthonormal vectors e_1 and e_2 in a tangent space $T_y M_2$. Then the complex vectors v and $(1/\sqrt{2})(e_1 + ie_2)$ span a PIC1 section with complex sectional curvature

$$R\left(v, \frac{1}{\sqrt{2}}(e_1 + ie_2), v, \frac{1}{\sqrt{2}}(e_1 - ie_2)\right) = \frac{1}{2}(R(v, e_1, v, e_1) + R(v, e_2, v, e_2)) = 0,$$

which is not strictly positive. □

Proof of Theorem 4.4 By working on the universal cover, we may assume M to be simply connected. Suppose on the contrary, we have $\operatorname{scal}(R_{g_0}(p)) > 0$ for some $p \in M$. By Theorem 1.3, we can find a long-time solution $g(t)$ to the Ricci flow on $M \times [0, +\infty)$ such that

- (a) $R_{g(t)} - \varepsilon'_0 \operatorname{scal}(R_{g(t)}) \cdot I \in C_{\text{PIC1}}$ for some $\varepsilon'_0 \in (0, 1/(n(n-1)))$,
- (b) $|R_{g(t)}| \leq a_0 t^{-1}$ for some $a_0 > 0$.

Moreover, it follows from [34, Theorem 3.1] that $g(t)$ also satisfies $R_{g(t)} \in C_{\text{PIC2}}$ for all $t > 0$. Furthermore, the strong maximum principle implies that $\operatorname{scal}_{g(t)} > 0$ for all $t > 0$.

Claim 4.6 *We have $K(g(t)) > 0$ for all $t > 0$. That is, all real sectional curvatures are strictly positive for positive times.*

Proof of Claim Fix a time $t_0 > 0$ at which to consider the sectional curvatures. Because $R_{g(t_0)} \in C_{\text{PIC2}}$, Cabezas-Rivas and Wilking [10, Theorem 5.1] tell us that $(M, g(t_0))$ splits isometrically as $\Sigma^k \times F^{n-k}$, where Σ is the (closed) soul, and F is diffeomorphic to \mathbb{R}^{n-k} and carries a complete metric h with $R_h \in C_{\text{PIC2}}$. Because Σ is closed but M is noncompact, we must have $k < n$. By (a) and the positivity of the scalar curvature, $(M, g(t_0))$ is (strictly) PIC1, which is incompatible with being a nontrivial product

by Lemma 4.5, so $k = 0$ and hence M^n is diffeomorphic to \mathbb{R}^n . Similarly, considering the de Rham decomposition of M , (a) implies that $(M, g(t_0))$ is irreducible. Meanwhile, if $(M, g(t_0))$ is symmetric, then the scalar curvature is a positive constant, which by (a) implies a uniform positive lower bound for the Ricci curvature; Bonnet–Myers then forces M to be compact, which is assumed not to be the case. Hence $(M, g(t_0))$ is of positive scalar curvature, diffeomorphic to \mathbb{R}^n , nonsymmetric and irreducible. In particular, Berger’s holonomy classification theorem implies that $\text{Hol}(M, g(t_0))$ is either $\text{SO}(n)$ or, possibly if n is even, $U(\frac{1}{2}n)$. This is because all other options would be Ricci flat or Einstein (and hence compact). If $\text{Hol}(M, g(t_0)) = \text{SO}(n)$, it follows from the strong maximum principle argument in [7, Proposition 9] that $K(g(t_0)) > 0$. Indeed, if any sectional curvature were zero, then every parallel translation of that section would also have zero sectional curvature, and because the holonomy group is $\text{SO}(n)$ the manifold would have to be flat, violating the positivity of the scalar curvature.

If $\text{Hol}(M, g(t_0)) = U(\frac{1}{2}n)$, we can still deduce $K(g(t_0)) > 0$, as pointed out in the proof of [10, Corollary 7.6]. We include the argument for convenience. First, we note that $g(t_0)$ is Kähler by the holonomy. Suppose $K(\sigma) = 0$ for some $x \in M$ and real plane $\sigma \subset T_x M$. If σ is a complex holomorphic plane, ie $\sigma = \text{span}\{u, Jv\}$ for some $v \in T_x M$, then the strong maximum principle [7, Proposition 9] implies that the holomorphic sectional curvature of $g(t_0)$ vanishes on M since $\text{Hol}(M, g(t_0)) = U(\frac{1}{2}n)$, and hence $g(t_0)$ is flat [28, Chapter IX, Proposition 7.1], which contradicts the positive scalar curvature. Suppose now $\sigma = \text{span}\{u, v\}$ for some $u, v \in T_x M$ where $\{u, v\}$ are orthonormal but $Ju \notin \sigma$. We fix an orthonormal frame $\{e_i, Je_i\}_{i=1}^{n/2}$ such that defining $u_i := (1/\sqrt{2})(e_i - \sqrt{-1}Je_i)$ makes $\{u_i\}_{i=1}^{n/2}$ a unitary frame with $u = e_1$ and $v = \cos \theta \cdot Je_1 + \sin \theta \cdot e_2$ for some $\theta \in (0, 2\pi)$. Then $K(\sigma) = R(u, v, u, v) = 0$ is equivalent to

$$(4-2) \quad \cos^2 \theta \cdot R(e_1, Je_1, e_1, Je_1) + \sin 2\theta \cdot R(e_1, e_2, e_1, Je_1) + \sin^2 \theta \cdot R(e_1, e_2, e_1, e_2) = 0.$$

By considering the linear transformation of $T_x M$ that fixes each u_i for $i \neq 2$, but sends u_2 to $-u_2$, which is an element in $U(\frac{1}{2}n)$, we deduce by the strong maximum principle [7, Proposition 9] that (4-2) also holds with the sign of the middle term reversed. Adding both gives

$$\cos^2 \theta \cdot R(e_1, Je_1, e_1, Je_1) + \sin^2 \theta \cdot R(e_1, e_2, e_1, e_2) = 0,$$

and hence $R(e_1, e_2, e_1, e_2) = 0$ since $K \geq 0$. Similarly, since $u_2 \mapsto \sqrt{-1}u_2$ is an element of $U(\frac{1}{2}n)$, we deduce that $R(e_1, Je_2, e_1, Je_2) = 0$. Therefore by the Bianchi identity and Kähler symmetries we have

$$\begin{aligned} R(u_1, \bar{u}_1, u_2, \bar{u}_2) &= -R(e_1, Je_1, e_2, Je_2) = R(e_1, e_2, Je_2, Je_1) + R(e_1, Je_2, Je_1, e_2) \\ &= -R(e_1, e_2, e_1, e_2) - R(e_1, Je_2, e_1, Je_2) = 0. \end{aligned}$$

Using once again the invariance under $U(\frac{1}{2}n)$ coming from the strong maximum principle, we deduce that the orthogonal bisectional curvature of $g(t_0)$ vanishes. This then contradicts the positivity of the scalar curvature because the scalar curvature can be written as an average of orthogonal bisectional curvatures. For example, according to a formula of Berger [40, (2.1)], one can write the scalar curvature in terms of the orthogonal Ricci curvature Ric^\perp , which vanishes when the orthogonal bisectional curvature vanishes. \square

Since $K(g(t)) > 0$ and $|R_{g(t)}| \leq a_0 t^{-1}$ for all $t > 0$, a result of Gromoll and Meyer [19, Theorem B.65] implies that there exists $c_0(n, a_0) > 0$ such that $\text{inj}(g(t)) \geq c_0 \sqrt{t}$ for all $t > 0$. We claim that the asymptotic volume ratio is positive, ie for any $x \in M$,

$$\text{AVR}(g_0) := \lim_{r \rightarrow \infty} \frac{\text{Vol}_{g_0}(B_{g_0}(x, r))}{\omega_n r^n} > 0,$$

where ω_n is the volume of the unit ball in \mathbb{R}^n . For $t_0 > 0$ sufficiently large, Lemma 4.3 implies that

$$B_{g(t_0)}(x, c_0 \sqrt{t_0}) \subset B_{g_0}(x, c_1 \sqrt{t_0})$$

for some $c_1(n, a_0) > 0$. Additionally, $g(t_0) \leq g_0$ because $\text{Ric}_{g(t)} \geq 0$, so

$$\text{Vol}_{g_0}(B_{g_0}(x, c_1 \sqrt{t_0})) \geq \text{Vol}_{g(t_0)}(B_{g(t_0)}(x, c_0 \sqrt{t_0})) \geq \text{Vol}_{g(t_0)}(B_{g(t_0)}(x, c_0 \sqrt{t_0})) \geq c_2(n, a_0) t_0^{n/2},$$

where the last inequality follows by Günther's theorem (see [23, Theorem 3.101(ii)]) because $\text{inj}(g(t_0)) \geq c_0 \sqrt{t_0}$ and $|R_{g(t_0)}| \leq a_0 t_0^{-1}$.

Since t_0 is arbitrarily large, $\text{AVR}(g_0) > 0$. It is well known that the asymptotic volume ratio $\text{AVR}(g)$ is preserved under Ricci flow with $\text{Ric}_{g(t)} \geq 0$ and $|R_{g(t)}| \leq a_0 t^{-1}$; for instance, see the proof of [53, Theorem 7]. Therefore $\text{AVR}(g(t)) = \text{AVR}(g_0) > 0$ for all $t > 0$.

On the other hand, since $R_{g(t)} \in C_{\text{PIC}2}$, $g(t)$ satisfies Hamilton's differential Harnack inequality for all $t \geq 0$ by [4]. Together with the fact that $t \cdot \text{scal}_{g(t)}$ is uniformly bounded for all $t > 0$, we can use an argument of Schulze and Simon [42, Theorem 1.2] to deduce that $(M, g_i(t), p)$ where $g_i(t) = i^{-2} g(i^2 t)$ converges subsequentially in the Cheeger–Gromov sense to $(M_\infty, g_\infty(t), p_\infty)$ which is an expanding gradient soliton with the same asymptotic volume ratio as g_0 ; see also [4, Proposition 12]. By (a), the Ricci curvature of $g_\infty(t)$ is pinched, and hence $g_\infty(t)$ is flat for all $t > 0$ using [38, Corollary 3.1]. This implies $\text{AVR}(g_\infty(t)) = 1$ for all $t > 0$, which forces $\text{AVR}(g_0) = 1$; hence $R_{g_0} \equiv 0$ by the rigidity of volume comparison. This contradicts the nonflatness we have assumed at p . \square

It remains to prove the corollary combining our main PIC1 pinching Theorem 1.2 with earlier work.

Proof of Corollary 1.5 Because (M, g_0) is assumed to be not everywhere flat, by the PIC1 pinching Theorem 1.2 it must be compact. Without loss of generality we may reduce ε_0 so that $\varepsilon_0 \in (0, 1/(n(n-1)))$. By Lemma 3.3, applied with $C = C_{\text{PIC}1}$, the pinching hypothesis implies that there exists $b > 0$ such that $R_{g_0} \in \ell_b(C_{\text{PIC}1})$. But $\ell_b(C_{\text{PIC}1})$ is invariant under the Hamilton ODE by [2, Proposition 3.2], so if we start the Ricci flow (which is always possible on a closed manifold) then for later times $t > 0$ we still have $R_{g(t)} \in \ell_b(C_{\text{PIC}1})$; thus by Lemma 3.2 we have

$$R_{g(t)} - \delta \text{scal}(R_{g(t)}) \cdot I \in C_{\text{PIC}1}$$

for some $\delta > 0$. Because g_0 is assumed not to be flat, for $t > 0$ we have $\text{scal}(R_{g(t)}) > 0$ by the maximum principle, and thus $g(t)$ is strictly PIC1. The result then follows from Brendle's PIC1 version of the sphere theorem [3, Theorem 3]. \square

Appendix A geometric interpretation of $\check{C}(s)$

In Section 2, for each $s > 0$, $\check{C}(s)$ was defined to be the cone of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ satisfying

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} + \frac{1}{s}(1 - \lambda^2)(1 - \mu^2) \cdot \text{scal}(R) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and $\lambda, \mu \in [0, 1]$. In this appendix we interpret this definition in terms of complex sectional curvatures and use the insight in order to give a quantitative relationship between C_{PIC1} and $\check{C}(s)$.

As in Section 1, given an algebraic curvature tensor $R \in \mathcal{C}_B(\mathbb{R}^n)$, we can extend by complex linearity and consider complex sectional curvatures of two-complex-dimensional subspaces $\Sigma \subset \mathbb{C}^n$.

Every section Σ contains an isotropic vector $v \in \Sigma$, ie so that $(v, v) = 0$, where (\cdot, \cdot) is the complex linear extension of the standard inner product on \mathbb{R}^n . To see this directly, pick any basis \tilde{v}, \tilde{w} of Σ . Either \tilde{w} is isotropic, in which case we set $v = \tilde{w}$, or we can solve the quadratic polynomial $(\tilde{v} + z\tilde{w}, \tilde{v} + z\tilde{w}) = 0$, and then set $v = \tilde{v} + z\tilde{w}$. By scaling, we may assume that $|v| = 1$.

Because v is isotropic and of unit length, we can pick orthonormal $e_1, e_2 \in \mathbb{R}^n$ so that $v = (1/\sqrt{2})(e_1 + ie_2)$. Note that $\bar{v} = (1/\sqrt{2})(e_1 - ie_2)$ is orthogonal to v , eg $\langle v, \bar{v} \rangle = (v, v) = 0$.

As stated earlier, we refer to Σ as a PIC1 section if \bar{v} is orthogonal to Σ . More generally \bar{v} will lie at an angle $\theta \in [0, \frac{1}{2}\pi]$ to Σ , and defining $\alpha = \cos \theta \in [0, 1]$ we can define a unit vector $w \in \Sigma$ orthogonal to v by writing

$$w = \alpha \bar{v} + \sqrt{1 - \alpha^2} u$$

for some unit $u \in \mathbb{C}^n$ that is orthogonal to both v and \bar{v} . Because $(v, w) = \alpha$, we see that $\alpha = 0$ precisely when Σ is a PIC1 section. Indeed, α can be viewed as a measure of how far Σ is from being a PIC1 section that depends only on Σ (and not on our choices of vectors above) with $\alpha = 1$ being precisely the case that Σ is a real section (that is, the complexification of a two-real-dimensional plane in \mathbb{R}^n , or equivalently a section that contains a vector v such that $\{v, \bar{v}\}$ gives an orthonormal basis for Σ). We thus view α as a function on the set of sections.

A calculation then reveals that an alternative characterisation of $\check{C}(s)$ is the cone of all curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ whose complex sectional curvature satisfies

$$K^{\mathbb{C}}(\Sigma) + \frac{\alpha(\Sigma)^2}{s} \text{scal}(R) \geq 0.$$

We see very clearly the inclusions $C_{\text{PIC2}} \subset \check{C}(s) \subset C_{\text{PIC1}}$ mentioned in Section 2.

It might be initially a little surprising that a curvature tensor in C_{PIC1} , which is assumed only to have nonnegative complex sectional curvature for very special sections (the PIC1 sections), does in fact enjoy a lower bound for *all* complex sectional curvatures.

Lemma A.1 Suppose $R \in C_{\text{PIC1}} \subset C_B(\mathbb{R}^n)$. Then for all complex sections $\Sigma \subset \mathbb{C}^n$ we have

$$K^{\mathbb{C}}(\Sigma) \geq -C(n)\alpha(\Sigma) \text{scal}(R).$$

Before we prove Lemma A.1, we record the following more basic control that is equivalent to every $R \in C_{\text{PIC1}} \setminus \{0\}$ having positive scalar curvature.

Lemma A.2 If $R \in C_{\text{PIC1}}$, then

$$|R| \leq C(n) \text{scal}(R).$$

Proof Take any orthonormal basis $\{e_i\}$. Then

$$\text{scal}(R) = \sum_{i \neq j} R(e_i, e_j, e_i, e_j) = \frac{1}{2(n-2)} \sum_{i, j, k \text{ distinct}} [R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k)].$$

Each term in square brackets can be written

$$R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k) = R(e_i, e_j + ie_k, e_i, e_j - ie_k) = 2K^{\mathbb{C}}(\Sigma) \geq 0,$$

where Σ is the PIC1 section spanned by e_i and $(1/\sqrt{2})(e_j + ie_k)$. Because all these terms are nonnegative and sum to $2(n-2) \text{scal}(R)$, we have

$$0 \leq R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k) \leq 2(n-2) \text{scal}(R).$$

To control the sectional curvature of a plane spanned by e_i and e_j for arbitrary $i \neq j$, we pick any $k \neq i, j$ and compute

$$2R(e_i, e_j, e_i, e_j) = [R(e_i, e_j, e_i, e_j) + R(e_i, e_k, e_i, e_k)] + [R(e_j, e_i, e_j, e_i) + R(e_j, e_k, e_j, e_k)] - [R(e_k, e_j, e_k, e_j) + R(e_k, e_i, e_k, e_i)].$$

Therefore

$$-(n-2) \text{scal}(R) \leq R(e_i, e_j, e_i, e_j) \leq 2(n-2) \text{scal}(R). \quad \square$$

Proof of Lemma A.1 We are interested in the complex sectional curvature corresponding to Σ , which is spanned by the orthonormal basis $\{v, w\}$ of the type considered earlier. We compute

$$\begin{aligned} K^{\mathbb{C}}(\Sigma) &= R(v, w, \bar{v}, \bar{w}) = R(v, \alpha\bar{v} + \sqrt{1-\alpha^2}u, \bar{v}, \alpha v + \sqrt{1-\alpha^2}\bar{u}) \\ &= \alpha^2 R(v, \bar{v}, \bar{v}, v) + (1-\alpha^2)R(v, u, \bar{v}, \bar{u}) + \alpha\sqrt{1-\alpha^2}R(v, \bar{v}, \bar{v}, \bar{u}) + \alpha\sqrt{1-\alpha^2}R(v, u, \bar{v}, v) \\ &\geq (1-\alpha^2)R(v, u, \bar{v}, \bar{u}) - C(n)\alpha \text{scal}(R), \end{aligned}$$

where we have used Lemma A.2.

Now note that the section spanned by v and u is a PIC1 section because \bar{v} is orthogonal to both v (because v is isotropic) and u (by construction). Therefore $R(v, u, \bar{v}, \bar{u}) \geq 0$ because $R \in C_{\text{PIC1}}$.

We conclude that

$$R(v, w, \bar{v}, \bar{w}) \geq -C(n)\alpha \text{scal}(R),$$

as required. □

Corollary A.3 For every $n \geq 4$ and $\varepsilon > 0$ there exists $s_0 > 0$ such that for every $R \in C_{\text{PIC1}}$ we have

$$R + \varepsilon \operatorname{scal}(R) \cdot I \in \check{C}(s_0).$$

Proof Take an arbitrary section $\Sigma \subset \mathbb{C}^n$. Then by Lemma A.1 we have

$$K^{\mathbb{C}}(\Sigma) \geq -C(n)\alpha(\Sigma) \operatorname{scal}(R) \geq -\varepsilon \operatorname{scal}(R) - c_1(n, \varepsilon)\alpha(\Sigma)^2 \operatorname{scal}(R),$$

for some $c_1(n, \varepsilon) > 0$, by Young's inequality. If we write $\hat{R} := R + \varepsilon \operatorname{scal}(R) \cdot I$, then $\operatorname{scal}(\hat{R}) \geq \operatorname{scal}(R)$, so we find that

$$K_{\hat{R}}^{\mathbb{C}}(\Sigma) + c_1(n, \varepsilon)\alpha(\Sigma)^2 \operatorname{scal}(\hat{R}) \geq 0,$$

and we can choose $s_0 = 1/c_1(n, \varepsilon)$ to deduce that $\hat{R} \in \check{C}(s_0)$. \square

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