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Fukaya categories of hyperplane arrangements

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To a simple polarized hyperplane arrangement (not necessarily cyclic) $\mathbb{V}$, one can associate a stopped Liouville manifold (equivalently, a Liouville sector) $(M(\mathbb{V}), \xi)$, where $M(\mathbb{V})$ is the complement of finitely many hyperplanes in $\mathbb{C}^d$, obtained as the complexifications of the real hyperplanes in $\mathbb{V}$. The Liouville structure on $M(\mathbb{V})$ comes from a very affine embedding, and the stop ξ is determined by the polarization. In this article, we study the symplectic topology of $(M(\mathbb{V}), \xi)$. In particular, we prove that their partially wrapped Fukaya categories are generated by Lagrangian submanifolds associated to the bounded and feasible chambers of $\mathbb{V}$. A computation of the Fukaya A_∞-algebra of these Lagrangians then enables us to identify the partially wrapped Fukaya categories $\mathcal{W}(M(\mathbb{V}), \xi)$ with the $\mathbb{G}_m^d$-equivariant hypertoric convolution algebras $\widetilde{B}(\mathbb{V})$ associated to $\mathbb{V}$. This confirms a conjecture of Lauda, Licata and Manion (2024) and provides evidence for the general conjecture of Lekili and Segal (2023) on the equivariant Fukaya categories of symplectic manifolds with Hamiltonian torus actions.

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1 Introduction

1a Main results Very affine varieties are local models of divisor complements in proper Calabi–Yau varieties, so they provide important examples for the study of mirror symmetry and symplectic topology, and are studied extensively in the literature; see for example Abouzaid and Auroux [2], Abouzaid, Auroux, Efimov, Katzarkov and Orlov [3], Auroux [8], Gammage and Jeffs [21], Gammage and Shende [22] an Lekili and Polishchuk [34]. In this paper, we consider a particular class of very affine varieties, defined via the combinatorial data of a polarized hyperplane arrangement.

A polarized hyperplane arrangement in $\mathbb{R}^d$ consists of a triple $\mathbb{V} = (V, \eta, \xi)$, where the d-dimensional subspace $V \subset \mathbb{R}^n$ and the vector $\eta \in \mathbb{R}^n/V$ determine n hyperplanes $H_{\mathbb{R},i} \subset \mathbb{R}^d$, where $1 \le i \le n$, and $\xi \in (\mathbb{R}^n)^*/V^\perp$ is the polarization; see Definition 3.1. Assume that $\mathbb{V}$ is simple, meaning that the nonempty intersection of any k hyperplanes is codimension k, we can define a Weinstein sector $(M(\mathbb{V}), \xi)$ as follows. Let $M(\mathbb{V})$ be the complement

$$M(\mathbb{V}) := \mathbb{C}^d \setminus \bigcup_{i=1}^{n} H_i,$$

where $H_i \subset \mathbb{C}^d$ is the complexification of the hyperplane $H_{\mathbb{R},i} \subset \mathbb{R}^d$ in the arrangement $\mathbb{V}$. It follows from the definition that we have an affine algebraic embedding $M(\mathbb{V}) \hookrightarrow (\mathbb{C}^*)^n$, equipping $M(\mathbb{V})$ with the structure of a very affine variety. Symplectically, $M(\mathbb{V})$ is a Weinstein manifold whose Weinstein structure is inherited from the standard Stein structure on $(\mathbb{C}^*)^n$. Since the polarization ξ can be interpreted as a superpotential $\xi\colon M(\mathbb{V}) \to \mathbb{C}$, it defines a stop (ie a Weinstein hypersurface) in the ideal boundary $\partial_\infty M(\mathbb{V})$, which we will still denote by ξ by abuse of notation. Our goal in this paper is to identify the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ with a convolution algebra associated to the combinatorial data $\mathbb{V}$. In particular, we will find a generating set of objects in $\mathcal{W}(M(\mathbb{V}), \xi)$ and show that their endomorphism A_∞-algebra in $\mathcal{W}(M(\mathbb{V}), \xi)$ is formal.

To describe the generating Lagrangians in $\mathcal{W}(M(\mathbb{V}), \xi)$, note that the hyperplanes in the arrangement $\mathbb{V}$ divide $\mathbb{R}^d$ into a finite number of chambers. Since these hyperplanes come with their co-orientations, each of these chambers in $\mathbb{R}^d$ corresponds to a sign sequence $\alpha \in \{+, -\}^n$, which we call *feasible*. Among the set of feasible sign sequences $\mathscr{F}(\mathbb{V})$, those for which the linear functional $\xi\colon \Delta_\alpha \to \mathbb{R}$ is bounded above are called *bounded*, where $\Delta_\alpha \subset \mathbb{R}^d$ is the (closed) chamber corresponding to the sign sequence $\alpha \in \mathscr{F}(\mathbb{V})$. Denote by $\mathscr{P}(\mathbb{V}) \subset \mathscr{F}(\mathbb{V})$ the set of bounded-feasible sign sequences (or equivalently, chambers). See Figure 1 for a polarized hyperplane arrangement consisting of three lines in $\mathbb{R}^2$. To each $\alpha \in \mathscr{P}(\mathbb{V})$, we can associate a Lagrangian submanifold $L_\alpha \subset M(\mathbb{V})$, which is the connected component (open chamber) $\Delta_\alpha^\circ \subset \Delta_\alpha$ in the real locus

$$M(\mathbb{V}) \cap \mathbb{R}^d \subset \mathbb{C}^d.$$

This is a noncompact, exact Lagrangian submanifold which gives an object of the Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$.

Conventions In this paper, we will be working over the ring of integers $\mathbb{Z}$, which is where the $\mathbb{G}_m$-equivariant hypertoric convolution algebras are defined (see Lee, Li and Mak [32]), although these algebras are originally introduced over $\mathbb{C}$ in the work of Braden, Licata, Phan, Proudfoot and Webster [13]. On the other hand, the work of Ganatra, Pardon and Shende [25], which our work heavily relies on, is also done over $\mathbb{Z}$.

We will now state the main results of this paper.

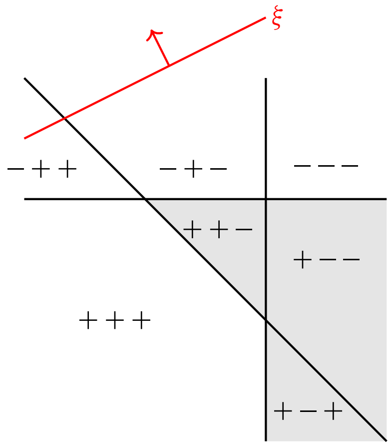

Figure 1: A polarized hyperplane arrangement in $\mathbb{R}^2$, with the bounded-feasible chambers shaded.

1.1 Theorem (generation, Theorem 4.13) *The partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ is generated by the Lagrangian submanifolds $\{L_\alpha\}_{\alpha \in \mathscr{P}(\mathbb{V})}$.*

The key ingredient of the proof is the sectoral descent established by Ganatra, Pardon and Shende [25] (cf Theorem 2.18). We will present the main idea of the proof using the explicit example given in Figure 1.

Theorem 1.1 enables us to identify the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ with the endomorphism A_∞-algebra of its objects $\{L_\alpha\}_{\alpha \in \mathscr{P}(\mathbb{V})}$. Consider the Fukaya A_∞-algebra

$$\widetilde{\mathcal{B}}_{\mathbb{V}} := \bigoplus_{\alpha, \beta \in \mathscr{P}(\mathbb{V})} \mathrm{CW}^*(L_\alpha, L_\beta),$$

where $\mathrm{CW}^*(-,-)$ denotes the partially wrapped Floer cochain complex with respect to the stop ξ. We will show that $\widetilde{\mathcal{B}}_{\mathbb{V}}$ is quasi-isomorphic to a convolution algebra $\widetilde{B}(\mathbb{V})$ introduced by Braden, Licata, Phan, Proudfoot and Webster [13], which is the universal flat graded deformation of the hypertoric convolution algebra $B(\mathbb{V})$; see Braden, Licata, Proudfoot and Webster [14]. More backgrounds and motivations about why this should be the case will be given in Section 1b.

1.2 Theorem (formality, Theorem 5.5) *There is a quasi-isomorphism between A_∞-algebras over $\mathbb{Z}$:*

$$\widetilde{\mathcal{B}}_{\mathbb{V}} \cong \widetilde{B}(\mathbb{V}).$$

In particular, the Fukaya A_∞-algebra $\widetilde{\mathcal{B}}_{\mathbb{V}}$ is formal.

This proves Lauda, Licata and Manion's [31, Conjecture 1.2] in the general case when the hyperplane arrangement $\mathbb{V}$ is not necessarily *cyclic*. Refer to [31] for the precise definition of a cyclic hyperplane arrangement. In particular, such a hyperplane arrangement cannot contain two parallel hyperplanes. When $\mathbb{V}$ is cyclic, the Weinstein manifold $M(\mathbb{V})$ has a convenient interpretation as a symmetric product of punctured spheres $\mathbb{C} \setminus \{z_1, \dots, z_n\}$, and Theorem 1.2 follows essentially from the computation of the (fully) wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}))$ by Lekili and Polishchuk [34].

In [14] and [31], some functorial properties of the category of perfect $\widetilde{B}(\mathbb{V})$-modules with respect to the deletion and restriction of hyperplane arrangements are discussed. In Section 6, we discuss these functors and describe some other natural geometric functors between Fukaya categories.

1b Motivation To a polarized hyperplane arrangement $\mathbb{V}$, one can associate another geometric object, namely a noncompact hyperkähler variety $\mathfrak{M}_{\mathbb{V}}$, called a hypertoric variety; see Bielawski and Dancer [9]. Under certain assumptions on $\mathbb{V}$, the variety $\mathfrak{M}_{\mathbb{V}}$ is smooth. After a suitable hyperkähler twist, one can equip $\mathfrak{M}_{\mathbb{V}}$ with the structure of a smooth affine variety. The induced Liouville structure on $\mathfrak{M}_{\mathbb{V}}$, together with a Weinstein hypersurface $\mathfrak{f}_{\mathbb{V}}$ in the ideal boundary $\partial_\infty \mathfrak{M}_{\mathbb{V}}$ determined by the polarization ξ, provides a stopped Liouville manifold $(\mathfrak{M}(\mathbb{V}), \mathfrak{f}_{\mathbb{V}})$. Moreover, there is the complex moment map

$$\bar{\mu}_{\mathbb{C}} : \mathfrak{M}_{\mathbb{V}} \to \mathbb{C}^d,$$

which defines an algebraic $(\mathbb{C}^*)^d$-fibration whose discriminant loci are precisely the union of the complex hyperplanes $\bigcup_{i=1}^n H_i$. In this situation, each feasible chamber Δ_α gives rise to a Lagrangian V_α that is diffeomorphic to the toric variety associated to the moment polytope Δ_α.

In [14], the authors introduced the hypertoric convolution algebra $B(\mathbb{V})$, which can be interpreted as the topological convolution algebra formed by the cohomologies of the complex Lagrangians in $\mathfrak{M}_{\mathbb{V}}$ associated to the bounded feasible chambers in $\mathbb{V}$. They also predicted that $B(\mathbb{V})$ can be realized as an appropriate version of the Fukaya category of $\mathfrak{M}_{\mathbb{V}}$. In the ongoing project [32], we investigate this question by examining the infinitesimal Fukaya category $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$ (ie the forward stopped subcategory of the partially wrapped Fukaya category in the sense of Ganatra, Pardon and Shende [24, Definition 6.4]) and show that $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$ is quasiequivalent to the category of perfect modules over $B(\mathbb{V})$.

It is a general philosophy due to Aganagic [5] and Teleman [44] that the symplectic topology of $\mathfrak{M}_{\mathbb{V}}$ is closely related to a half-dimensional space, which is exactly the Weinstein manifold $M(\mathbb{V})$ that we study in this paper. The precise connections between the Fukaya categories of $\mathfrak{M}_{\mathbb{V}}$ and $M(\mathbb{V})$ are explained in Lekili and Segal [35, Conjecture A]. They conjectured that there should be a quasiequivalence

$$\mathcal{W}_{\mathbb{G}_m^d}(\mathfrak{M}_{\mathbb{V}})_{-1} \cong \mathcal{W}(M(\mathbb{V})), \tag{1}$$

where $\mathcal{W}_{\mathbb{G}_m^d}(\mathfrak{M}_{\mathbb{V}})_{-1} \subset \mathcal{W}_{\mathbb{G}_m^d}(\mathfrak{M}_{\mathbb{V}})$ is the spectral component of the $\mathbb{G}_m^d$-equivariant wrapped Fukaya category at -1. We refer the reader to [35, Section 1.2] for more details; see also McBreen, Shende and Zhou [38] for a related recent work on multiplicative hypertoric varieties.

Here, what is relevant for us is a partially wrapped version of the conjectural equivalence (1). The corresponding conjecture relating partially wrapped Fukaya categories then takes the form

$$\mathcal{F}_{\mathbb{G}_m^d}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}}) \cong \mathcal{W}(M(\mathbb{V}), \xi), \tag{2}$$

where $\mathcal{F}_{\mathbb{G}_m^d}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$ consists of objects in $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$ which are invariant under the fiberwise $\mathbb{G}_m^d$-action with respect to the complex moment map $\bar{\mu}_{\mathbb{C}}$. Note that performing localizations (stop removals) on both sides of (2) would imply the equivalence (1) conjectured by Lekili and Segal.

1.3 Remark There are two versions of equivariant Fukaya categories that can be associated to the hypertoric variety $\mathfrak{M}_{\mathbb{V}}$, defined by using the geometric Hamiltonian T^d-action of the real torus $T^d \subset \mathbb{G}_m^d$ on the Weinstein manifold $\mathfrak{M}_{\mathbb{V}}$, and the algebraic $\mathbb{G}_m^d$-action on the A_∞-category $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$, respectively. For the former version, see Cazassus [15], Daemi and Fukaya [19], Hendricks, Lipshitz and Sarkar [27], Xiao [46], Kim, Lau and Zheng [29] and Lau, Leung and Li [30], and for the latter one, see Lekili and Pascaleff [33], Seidel [40] and Seidel and Solomon [41]. In our case, these two versions are expected to be the same, because under hyperkähler twist, the fiberwise Hamiltonian T^d-action associated to the complex moment map $\bar{\mu}_{\mathbb{C}} : \mathfrak{M}_{\mathbb{V}} \to \mathbb{C}^d$ corresponds to the fiberwise $\mathbb{G}_m^d$-action on the mirror, and the latter is expected to induce a $\mathbb{G}_m^d$-action on the Fukaya category $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$.

Consider the Fukaya A_∞-algebra

$$\mathcal{B}_{\mathbb{V}} := \bigoplus_{\alpha,\beta \in \mathscr{P}(\mathbb{V})} \mathrm{CF}^*(V_\alpha, V_\beta),$$

where CF^* denotes the infinitesimally wrapped Floer cochain complexes with respect to the stop $\mathfrak{f}_{\mathbb{V}} \subset \partial_\infty \mathfrak{M}_{\mathbb{V}}$ mentioned above. In [32], we plan to give a Floer-theoretic proof that there is a quasi-isomorphism between $\mathcal{B}_{\mathbb{V}}$ and the hypertoric convolution algebra $B(\mathbb{V})$, and that the Lagrangians $\{V_\alpha\}_{\alpha \in \mathscr{P}(\mathbb{V})}$ split-generate the Fukaya category $\mathcal{F}(\mathfrak{M}_{\mathbb{V}}, \mathfrak{f}_{\mathbb{V}})$. However, shortly after we put the current paper on the arXiv, we learned that Côté, Gammage and Hilburn [18] have proved this using microlocal perverse sheaves. Together with conjectural quasiequivalences (2), it follows that there should be a quasi-isomorphism between the partially wrapped Fukaya A_∞-algebra $\widetilde{\mathcal{B}}_{\mathbb{V}}$ of $(M(\mathbb{V}), \xi)$ and the $\mathbb{G}_m^d$-equivariant convolution algebra $\widetilde{B}(\mathbb{V})$. Theorem 1.2 confirms that this is indeed the case.

1c Idea of the proof

In this subsection, we give an outline of the proof of our generation result (Theorem 1.1). Let us start with a 1-dimensional hyperplane arrangement, whose complexified complement $M(\mathbb{V})$ is $\mathbb{C} \setminus \{r_1, r_2, \dots, r_n\}$, where $-\infty < r_1 < \cdots < r_n < \infty$, a complex plane with n points on the real line removed. Note that Lagrangian chambers are the real line segments connecting the consecutive punctures in $\mathbb{R}$ (colored in blue in Figure 2), together with a ray to positive or negative infinity depending on the choice of polarization ξ.

In this case, the generation is a priori clear: we can prove generation via its Weinstein handle structure, by arguing that the real line segments connecting punctures are cocores for a given Weinstein structure on $M(\mathbb{V})$, and applying the generation result of Ganatra, Pardon and Shende [25], which says that cocores of a Weinstein manifold generate the fully wrapped Fukaya category. To see that the line segments combined with the ray $(-\infty, r_1)$ generate the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$, we appeal to the computation of wrapped Floer cohomology shown by Abouzaid, Auroux, Efimov, Katzarkov and Orlov [3], where an exact triangle relating the blue arcs in Figure 2, left, to the Lagrangians $L_1^{\pm}$ in Figure 7 was established via an explicit A_∞-algebra computation.

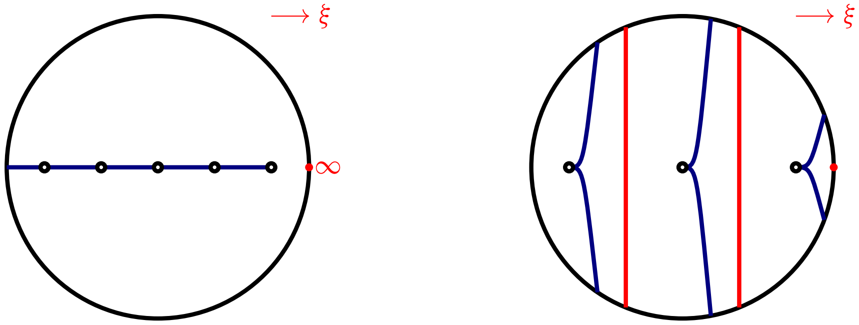

Figure 2: Left: 1-dimensional hyperplane arrangement. Right: cutting 1-dimensional hyperplane arrangements into sectors.

To motivate, we prove the generation via a "local-to-global" approach: consider vertical real lines cutting $M(\mathbb{V})$ into union of "sectors", as shown in Figure 2, right. Each piece can be identified with part of $\mathbb{C}^*$, together with a stop ∞ or two stops $\pm\infty$, so the wrapped Fukaya category of each sector is generated by the blue rays depicted in the figure, where the upper and lower ones are both Lagrangian cocores of $\mathbb{C}^*$, which together generate the linking disks to all the stops. Sectorial descent (cf Theorem 2.18) then allows us to obtain a collection of generators for $\mathcal{W}(M(\mathbb{V}), \xi)$, as the union of all the blue arcs. Observe that after gluing the sectors together to get the ambient punctured disk $(M(\mathbb{V}), \xi)$, the left most upper and lower blue Lagrangians are isomorphic via counterclockwise rotation, and the lower blue Lagrangians in the other sectors can be obtained from the upper ones by performing iterated surgeries. Therefore the upper collection of the blue arcs would suffice to generate $\mathcal{W}(M(\mathbb{V}), \xi)$.

The final step to the proof is to show that the real lines depicted in Figure 2, left, generate the upper blue arcs in Figure 2, right. The left most Lagrangians are canonically identified by rotation, and we observe that the upper Lagrangians in Figure 2, right, can be obtained from inductively concatenating the real lines in Figure 2, left, through the stop, or equivalently via "wrapped Lagrangian surgery" around the puncture; see Figure 3. We therefore conclude that the blue Lagrangians in Figure 2, left, generate $\mathcal{W}(M(\mathbb{V}), \xi)$.

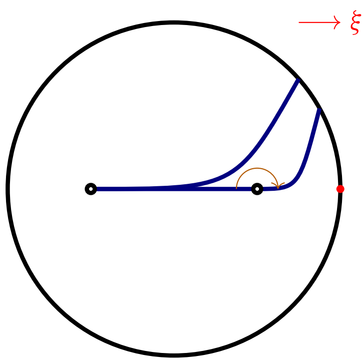

Figure 3: Wrapped surgery around a puncture.

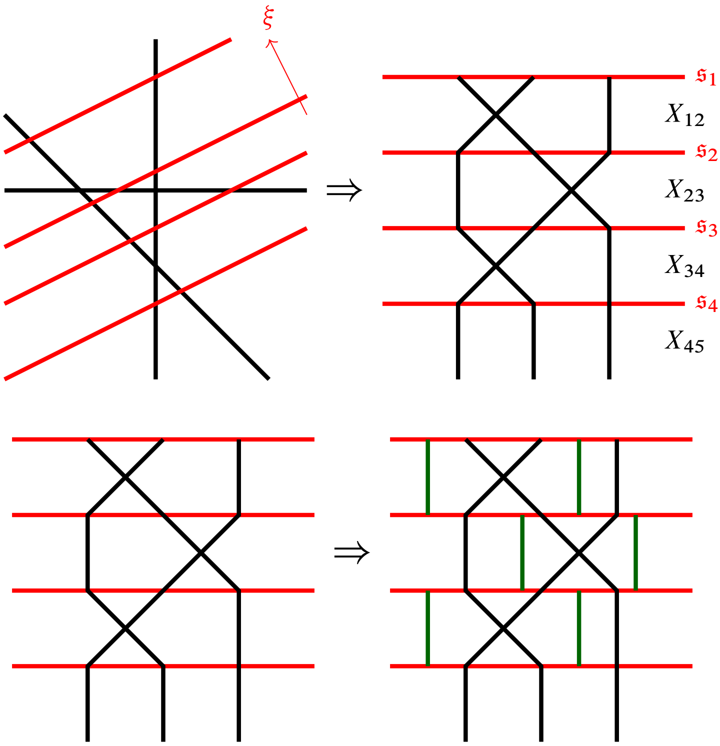

Figure 4: Top: decomposing 2-dimensional pair-of-pants into sectors. Bottom: second cut of the P_j into the standard pieces.

Things get more complicated as the dimension increases, but the key phenomenon can already be seen in complex dimension 2. To illustrate, we will explain our proof when $M(\mathbb{V})$ is the 2-dimensional pair-of-pants associated with the hyperplane arrangement shown in Figure 1. As a symmetric product of two 1-dimensional pair-of-pants, the generation of its wrapped Fukaya category can be computed from dimension 1 cases as explained by Auroux [6], but our alternative proof sketched here will also apply to general two-dimensional hyperplane arrangements. Using the polarization ξ, we can cut the Weinstein sector $(M(\mathbb{V}), \xi)$ into four subsectors as shown in the top diagram of Figure 4 (cf Propositions 3.14 and 3.22). For $i = 1, \dots, 4$, we denote by $\mathfrak{s}_i$ the i^{th} sectorial hypersurface and by $X_{i,i+1}$ the subsectors after cutting. Each of these subsectors looks pretty similar in the figure, containing of a single crossing point and several nonintersecting line segments. After curving the picture, we get four sectors on the right-hand side of the top diagram of Figure 4, and the generation problem in this case reduces to that of the wrapped Fukaya category for each subsector.

For each of these subsectors, we perform a further cut by hyperplanes which are transverse to the hyperplane defined by the polarization ξ, cutting each sector into "squares" as shown in the bottom diagram of Figure 4. Squares that surround an intersection point of the hyperplanes can be identified with $(\mathbb{C}^\times)^2$ with four stops given by the two linear functions, and the other squares are products of the

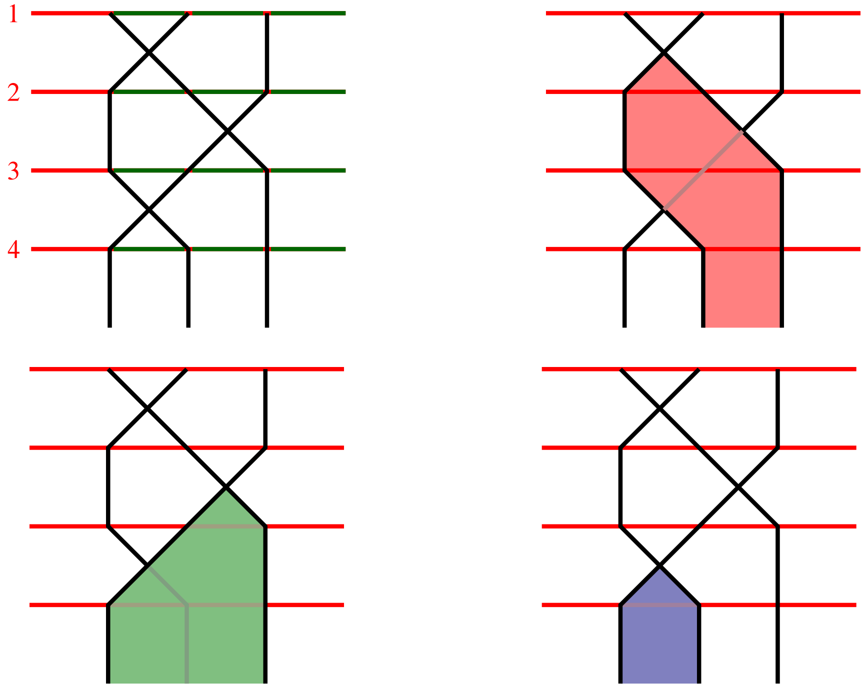

Figure 5: Generating Lagrangians for two-dimensional pairs of pants. Top: linking disks, left, and standard Lagrangian L_1^+, right. Bottom: standard Lagrangian L_2^+, left, and standard Lagrangian L_3^+, right.

red boundary with $T^*[0,1]$, whose Fukaya category is equivalent to the Fukaya category of one of the two red boundaries; see Ganatra, Pardon and Shende [25]. It is well-known (see Abouzaid [1]) that the wrapped Fukaya category of $(\mathbb{C}^*)^2$ is generated by one of the four real quadrants (a cotangent fiber), and adding stops adds linking disks associated to these stops as new generators. Moreover, the linking disks correspond to generating Lagrangians of the stops, which are $M(\mathbb{V})$ for some 1-dimensional hyperplane arrangement, and from the previous paragraph on dimension 1 hyperplane arrangements we know that they are generated by real lines connecting punctures.

By sectorial descent (Theorem 2.18), we get a collection of generating Lagrangians, which are depicted in Figure 5. The green line segments in Figure 5, top left, represent the linking disks associated to the corresponding generating Lagrangians of the stop, and the remaining three pictures describe the three generating Lagrangians obtained from each crossing point (after gluing together all the sectors). We label the three standard Lagrangians by L_1^+, L_2^+, L_3^+, respectively, where the $+$ indicates that the Lagrangian is the one shifted upwards (under the Hamiltonian flow) in direction of $\operatorname{Im}(\xi)$ in $\mathbb{C}^2$. We denote the linking disks by D_{ij}, where $1 \le i \le 4$ is the index of the stop and $1 \le j \le 3$ is the order of cocores, from left to right. It remains to show that the Lagrangians associated to the bounded-feasible chambers L_{++-}, L_{+--} and L_{+-+}, as depicted in Figure 6, generate all the standard Lagrangians L_1^+, L_2^+, L_3^+ and the

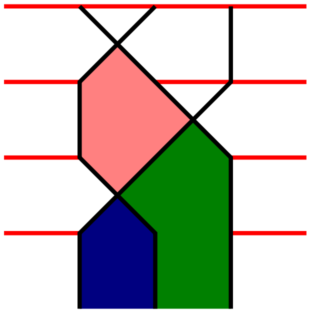

Figure 6: The bounded feasible chambers. The red one is L_{++-}, the green one is L_{+--} and the blue one is L_{+-+}.

linking disks D_{ij} in the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$. Applying the surgery exact triangle (cf Proposition 4.8) we have:

1.4 Lemma (cf Section 4c) *There are quasi-isomorphisms*

$$L_1^+ \simeq \mathrm{Cone}(L_{++-} \to L_{+--}), \quad L_2^+ \simeq \mathrm{Cone}(L_{+--} \to L_{+-+}), \quad L_3^+ \simeq L_{+-+}$$

between objects in the triangulated A_∞-category $\mathcal{W}(M(\mathbb{V}), \xi)^{\mathrm{perf}}$ (cf Section 2b), where Cone$(\bullet \to \bullet)$ *stands for the mapping cone.*

Thus it remains to show that the linking disks D_{ij} are generated by the "chamber Lagrangians" L_{++-}, L_{+--} and L_{+-+}. To prove this, let us first focus on the top sector X_{12}, shown at left of Figure 7.

Consider the Lagrangian L_1^+ in this sector. We have the wrapping exact triangle for the linking disk D_{11}

$$L_1^- \to D_{11} \to L_1^+ \to L_1^-[1],$$

where L_1^- is the triangular Lagrangian obtained by shifting L_1 down in the direction of $\mathrm{Im}(\xi)$; see Figure 7, right. This implies that D_{11} is generated by L_1^- and L_1^+. Consider the second sectorial hypersurface $\mathfrak{s}_2$, regarded as a stop in X_{12}, there is a similar wrapping exact triangle

$$\widetilde{L}_1^- \to D_{21} \to \widetilde{L}_1^+ \to \widetilde{L}_1^-[1].$$

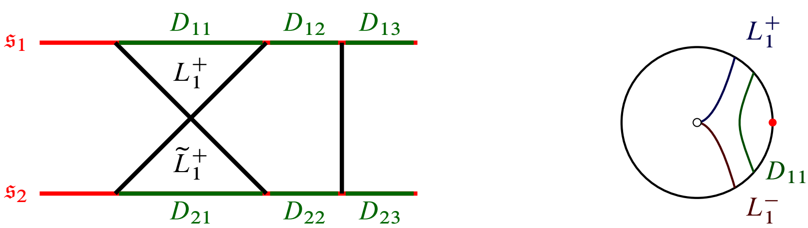

Figure 7: Left: description of X_{12}. $\widetilde{L}_1^+$ is a flipping of L_1^+. Right: The wrapping exact triangle.

Here, $\widetilde{L}_1^{\pm}$ is a flipping of $L_1^{\pm}$; cf **4e-2**. Since there is a quasi-isomorphism $\widetilde{L}_1^{\pm} \cong L_1^{\pm}$ in $\mathcal{W}(X_{12})$, we can abuse notation and write both as $L_1^{\pm}$. This shows that we could get L_1^- from D_{21}, therefore D_{11} is generated by L_1^+ and D_{21}, hence it can be removed from the collection of generating objects for $\mathcal{W}(X_{12})$. For the remaining linking disks in the first stop $\mathfrak{s}_1$, we can show via Lagrangian surgery that:

1.5 Lemma (cf Section 4e) *We have quasi-isomorphisms*

$$D_{22} \simeq \mathrm{Cone}(D_{11} \to D_{12}) \quad \textit{and} \quad D_{13} \simeq D_{23}$$

in the Fukaya category $\mathcal{W}(X_{12})^{\mathrm{perf}}$.

1.6 Corollary *The partially wrapped Fukaya category $\mathcal{W}(X_{12})$ is generated by the Lagrangian L_1^+ and the linking disks D_{21}, D_{22} and D_{23}.*

Note that the quasi-isomorphisms in Lemma 1.5 also hold in the wrapped Fukaya category of the ambient Liouville sector $(M(\mathbb{V}), \xi)$, by the existence of an inclusion functor associated to the embedding $X_{12} \hookrightarrow (M(\mathbb{V}), \xi)$. Under this inclusion, L_1^+ in $\mathcal{W}(X_{12})$ is sent to the standard Lagrangian at the top right of Figure 5.

Similar arguments can be carried out for the Liouville sectors X_{23} and X_{34}. We can identify linking disks at the top and the bottom via surgery exact sequences, therefore inductively replacing linking disks from the top with those at the bottom in the collection of generating objects. Finally, it remains to deal with the linking disks D_{41}, D_{42} and D_{43}. To do this, note that the subsector X_{45} at the bottom has a geometric decomposition:

1.7 Lemma *We have an isomorphism of Liouville sectors* $X_{45} \cong \mathfrak{s}_4 \times \mathbb{C}_{\mathrm{Re} \geq 0}$.

This implies that all the linking disks D_{41}, D_{42} and D_{43} become zero objects after pushing forward to the ambient sector $(M(\mathbb{V}), \xi)$. They can thus be removed from the collection of generating objects. We conclude from Lemma 1.4 and Corollary 1.6 that:

1.8 Proposition (cf Theorem 4.13) *The partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ is generated by the Lagrangians L_{++-}, L_{+--} and L_{+-+} corresponding to the bounded and feasible chambers in the polarized hyperplane arrangement $\mathbb{V}$.*

1d Outline The paper is organized as follows. In Section 2, we recall the basics of Liouville sectors and Fukaya categories based on the works [25; 23] of Ganatra, Pardon and Shende. Section 3 contains the main technical input of this paper. We first recall the notion of a polarized hyperplane arrangement in Section 3a, and then establish some basic facts about the symplectic geometry of the Weinstein manifold $M(\mathbb{V})$ in Section 3b. The sectorial cut of $(M(\mathbb{V}), \xi)$ is carried out rigorously in Sections 3d and 3e. This is followed by a discussion about the generation for each subsector in the sectorial decomposition in Section 3f. In Section 4, we prove Theorem 1.1 using the surgery exact triangle for Lagrangian submanifolds with clean intersections. Using a computational technique similar to that of [8], Theorem 1.2 is proved in Section 5. As an application, we study the functoriality of Fukaya categories with respect to the deletion and restriction of hyperplane arrangements in Section 6.

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2 Review of sector theory

In this section, we recall the definitions and some important structural results for Fukaya categories of Weinstein manifolds.

2a Liouville sectors Recall that a *symplectic manifold* is a pair (X, ω), where X is an even-dimensional real manifold and $\omega \in \Omega^2(X)$ is a closed nondegenerate 2-form on X. In this paper, we will study a particular class of noncompact symplectic manifolds.

2.1 Definition A symplectic manifold (X, ω) is a *Liouville manifold* if ω is exact and there exists a vector field Z on X such that

(i) Z is complete,

(ii) $\mathcal{L}_Z\omega = \omega$, or equivalently, Z is dual to a primitive λ of ω,

(iii) there exists a (codimension 0) compact symplectic submanifold with boundary $(X_0, \omega) \subset (X, \omega)$ such that Z is outward pointing along ∂X_0 and Z has no critical points outside X_0.

We call this vector field Z a *Liouville vector field* and the pair (ω, Z) a *Liouville structure* on X. The primitive λ of ω is called a *Liouville* 1*-form*.

In this paper, all Liouville manifolds are assumed to have vanishing first Chern class, ie $c_1(X) = 0$.

The compact symplectic submanifold (X_0, ω) in the item (iii) is called a *Liouville domain*. We can also construct a Liouville manifold X out of the given Liouville domain (X_0, ω) by taking its *completion*:

$$X = X_0 \cup_{\partial X_0} \partial X_0 \times [1, +\infty),$$

where the Liouville structure on $\partial X_0 \times [1, +\infty)_r$ is $(\omega = d(r\lambda|_{\partial X_0}), Z = r\partial_r)$.

2.2 Definition Let (X, ω, Z) be a Liouville manifold. We call X *Weinstein* if there exists an exhausting Morse function $\phi\colon X \to \mathbb{R}$ that is Lyapunov for Z. The triple (ω, Z, ϕ) is called a *Weinstein structure* on X.

2.3 Definition A *Stein manifold* is a complex manifold (X, J) which admits a proper holomorphic embedding $i\colon X \to \mathbb{C}^N$ for some N.

Given a Stein manifold (X, J) equipped with an embedding $i\colon X \hookrightarrow \mathbb{C}^N$, one can consider the squared distance function $\phi_{\mathbb{C}^N}(z) = \sum_{i=1}^N |z_i|^2$. The pullback function $\phi := \phi_{\mathbb{C}^N} \circ i$ defines a plurisubharmonic function on X, hence also a symplectic form $\omega_\phi = -dd^{\mathbb{C}}\phi$ and a Liouville vector field $\nabla\phi$ dual to $-d^{\mathbb{C}}\phi$. This assignment defines a map from the category of Stein manifolds with a fixed potential ϕ to the category of Weinstein manifolds with the same potential ϕ. It is shown in [17, Theorem 1.1] that this map is a weak homotopy equivalence.

A submanifold $S \subseteq X$ is *coisotropic* if $\dim S \geq \frac{1}{2}\dim X$ and for every $s \in S$, $\ker \omega|_{T_s S}$ has constant positive rank. It follows directly that every real hypersurface of X is coisotropic. If $S \subseteq X$ is coisotropic, write $\mathcal{D}_S := \ker \omega|_S \subseteq TS$ for the *characteristic foliation* of S.

2.4 Definition Let (X, ω, Z) be a Liouville manifold. A real oriented hypersurface $S \subseteq X$ is called *sectorial* if the following conditions are satisfied:

(i) Z is tangent to S near infinity.

(ii) There exists a Z-invariant neighborhood $\mathrm{Nbd}^Z(S)$ of $S \subset X$ and functions

$$R, I\colon \mathrm{Nbd}^Z(S) \to \mathbb{R}$$

such that $R^{-1}(0) = S$ and

$$(R, I)\colon \mathrm{Nbd}^Z(S) \to \mathbb{C}_{-\varepsilon<\mathrm{Re}<\varepsilon}$$

is a trivial symplectic fibration onto the open strip in $\mathbb{C}$ consisting of complex numbers with real part constrained in $(-\varepsilon, \varepsilon)$, where $\mathbb{C}_{-\varepsilon<\mathrm{Re}<\varepsilon}$ is equipped with the standard symplectic form restricted from $\mathbb{C}$;

(iii) Under the above decomposition $\mathrm{Nbd}^Z(S) \xrightarrow{\simeq} (R,I)^{-1}(0) \times \mathbb{C}_{-\varepsilon<\mathrm{Re}<\varepsilon}$, the Liouville 1-form λ_X on X is sent to $\lambda_X|_{(R,I)^{-1}(0)} + \lambda_{\mathbb{C}}^{\alpha} + df$ for some $0<\alpha<1$, where

$$\lambda_{\mathbb{C}}^{\alpha} = -\alpha y\,dx + (1-\alpha)x\,dy, \tag{3}$$

and f is some real-valued function with compact support.

A *Liouville sector* Y is the closure of one connected component of $X \setminus \bigsqcup_{i=1}^{k} S_i$, where the S_i are all disjoint sectorial hypersurfaces.

Alternatively, we can describe Liouville sectors in terms of pairs of Liouville manifolds [20]. A *Liouville pair* is a pair (X,F) consisting of a Liouville domain (X,ω,Z) and a Liouville hypersurface $(F,\omega|_F,Z_F)$ in ∂X. We say the pair is *Weinstein* if in addition X is a Weinstein domain and $F \subseteq \partial X$ is Weinstein with respect to the induced Weinstein structure of X. Ganatra, Pardon and Shende [23] showed that we have a canonical way to construct a Liouville sector out of a Liouville pair, so that F arises as one of the fibers $(R,I)^{-1}(0)$ for some chosen R- and I-functions. Such a hypersurface $F \subseteq \partial X$, or more generally a boundary component of a Liouville sector, is usually referred to as a *stop* of the sector.

2.5 Remark This definition is different from the one in [23], but by Proposition 2.25 in loc. cit., they are equivalent as definitions for Liouville sectors. In other words, it is not hard to show that our definition gives a Liouville sector, and conversely, for any given Liouville sector X as in [23], there is a convex completion of X into a Liouville manifold $\widetilde{X}$, so that $\partial X \hookrightarrow \widetilde{X}$ is a sectorial hypersurface in $\widetilde{X}$.

2.6 Definition [25, Definition 1.32] Let $S_1,\dots,S_k$ be a collection of transversely intersecting sectorial hypersurfaces in X. The collection $\{S_1,\dots,S_k\}$ is *sectorial* if there is a collection of pairs of real-valued functions $\{(R_i,I_i)\}_{1\le i\le k}$ together with positive real numbers $(\varepsilon_i)_{1\le i\le k}$ such that $R_i^{-1}(0)=S_i$ and:

(i) The intersections $\bigcap_{i\in J} S_i =: S_J$ for all $J \subseteq \{1,\dots,k\}$ are either empty or coisotropic.

(ii) When S_J is nonempty, there exists a Z-invariant neighborhood $\mathrm{Nbd}^Z(S_J)$ of S_J so that the functions R_i and I_i for $i\in J$ define a trivial symplectic fibration

$$((R_i,I_i)_{i\in J})\colon \mathrm{Nbd}^Z(S_J) \to \mathbb{C}^{|J|}_{-\varepsilon_i<\mathrm{Re}<\varepsilon_i}. \tag{4}$$

(iii) Under the decomposition (4), the Liouville 1-form λ_X of X projects to

$$\lambda_X|_{((R_i,I_i)_{i\in J})^{-1}(0)} + \sum_{i\in J} \lambda_{\mathbb{C}}^{\alpha_i} + df$$

for some finite collection of positive numbers $0<\alpha_i<1$ and a real-valued function f with compact support.

Similar to Definition 2.4, we define a *cornered Liouville sector* to be the closure of one connected component of the complement $X \setminus \bigcup_{i=1}^{k} S_i$, where $\{S_i\}_{1\le i\le k}$ is a transversely intersecting sectorial collection of real hypersurfaces in X.

Given a Liouville sector (X,ω,Z) and a submanifold $S \subseteq X$ which is cylindrical near infinity, we define the *boundary at infinity* of S to be the manifold

$$\partial_\infty S := \{[x] \mid x \in S \setminus X_0,\ x \sim y \iff \exists t \in \mathbb{R},\ \varphi_Z^t(x) = y\},$$

where $X_0 \hookrightarrow X$ is some subdomain such that Z has no zeroes outside X_0 (cf Definition 2.1) and φ_Z^t is the time-t flow of the Liouville vector field Z. In particular, $\partial_\infty X$ is a contact manifold with boundary.

2b Wrapped Fukaya categories In this subsection, we briefly recall the definition of wrapped Fukaya categories associated to a cornered Liouville sector, following [23] and [25]. Given a Liouville sector (X,ω,Z), recall that a *Lagrangian submanifold* of X is a submanifold $L \subseteq X$ of half dimension such that $\omega|_L = 0$.

2.7 Definition A Lagrangian submanifold $L \subseteq X$ is *exact* if the restriction of the primitive $\lambda|_L$ is exact on L. It is *cylindrical near infinity* if Z is tangent to L near infinity.

To any two transversely intersecting exact cylindrical Lagrangian submanifolds $L, K \subseteq X$ equipped with Spin structures, we can associate a $\mathbb{Z}$-graded free $\mathbb{Z}$-module called the Floer cochain complex

$$\mathrm{CF}^*(L,K) := \bigoplus_{p \in L \cap K} \mathfrak{o}_p,$$

where $\mathfrak{o}_p$ is the orientation line associated to $p \in L \cap K$. $\mathrm{CF}^*(L,K)$ admits a differential defined by an oriented count of holomorphic strips with boundary on $L \cap K$, whose cohomology gives the Lagrangian Floer cohomology group $\mathrm{HF}^*(L,K)$.

2.8 Definition An isotopy $\{L_t\}$ of exact cylindrical Lagrangian submanifolds $L_t \subseteq X$ is called *positive* if for some, equivalently any, contact form α on $\partial_\infty X$, we have $\alpha(\partial_t \partial_\infty L_t) > 0$.

A positive isotopy $(L_0 \rightsquigarrow L_1) := \{L_t\}_{0 \le t \le 1}$ of Lagrangian submanifolds L_t such that L_0, L_1 are transversal to K induces a chain map

$$\mathrm{CF}^*(L_0,K) \to \mathrm{CF}^*(L_1,K)$$

between Floer cochain complexes. We define the *wrapped Floer cohomology* associated to L, K by the homotopy colimit over positive isotopies

$$\mathrm{HW}^*(L,K) := \varinjlim_{L \rightsquigarrow L'} \mathrm{HF}^*(L',K).$$

Suppose we have a sequence of positive isotopies $L_0 \rightsquigarrow L_1 \rightsquigarrow \cdots$ and let $(L_t)_{t \in \mathbb{R}_{\ge 0}}$ be the concatenation of them. Ganatra, Pardon and Shende [23, Lemma 3.29] show that the sequence is cofinal if

$$\int_0^\infty \min_{\partial_\infty L_t} \alpha(\partial_t \partial_\infty L_t) = \infty. \tag{5}$$

In other words, when (5) is satisfied, we can compute the wrapped Floer cohomology as

$$\mathrm{HW}^*(L_0, K) = \varinjlim_n \mathrm{HF}^*(L_n, K).$$

For a smooth function $H\colon [0,1] \times X \to \mathbb{R}$, we denote $H(t,\cdot)$ by H_t. The *Hamiltonian vector field* $X_H = (X_{H_t})_{t\in[0,1]}$ is the unique t-dependent vector field on X so that $\omega(-, X_{H_t}) = dH_t(-)$. We denote the time 1 map of the flow by ϕ_H. As positive isotopies are in particular exact, for each $n \in \mathbb{N}$, we can write L_n as $\phi_{H_n}(L_0)$ for some Hamiltonian functions $H_n \in C^\infty([0,1]\times X)$. The generators of $\mathrm{CF}^*(L_n, K)$ can then be identified with time-1 Hamiltonian chords of X_{H_n} from L_0 to K. Moreover, it is well-known that for suitably defined ω-tamed almost complex structures, there is a bijective correspondence between finite-energy pseudoholomorphic strips with boundary on L_n, K and finite-energy solutions of the X_{H_n}-perturbed Cauchy–Riemann equation, so one can define a chain complex $\mathrm{CF}^*(L_0, K; H_n)$ using Hamiltonian chords as generators and there is a canonical isomorphism $\mathrm{CF}^*(L_n, K) \cong \mathrm{CF}^*(L_0, K; H_n)$. We will use these identifications interchangeably when we compute the wrapped Fukaya cohomology; see Section 5.

The *wrapped Fukaya category* $\mathcal{W}(X)$ of a Liouville sector X is an A_∞-category first introduced by Auroux [6], and later rigorously defined in [25, Definition 2.20, 2.32] and [42]. The objects of $\mathcal{W}(X)$ are exact cylindrical Lagrangian submanifolds $L \subseteq X$ equipped with Spin structures and the morphism space between L and K is the localization of $\mathrm{CF}^*(L, K)$ along continuation elements. In particular, the cohomology of the morphism space is $\mathrm{HW}^*(L, K)$.

The A_∞-category $\mathcal{W}(X)$ becomes easier to study from an algebraic perspective if we pass to its enlargement $\mathcal{W}(X)^{\mathrm{perf}}$, the A_∞-category of perfect modules over $\mathcal{W}(X)$. $\mathcal{W}(X)^{\mathrm{perf}}$ is a triangulated A_∞-category and there is a fully faithful embedding $\mathcal{W}(X) \hookrightarrow \mathcal{W}(X)^{\mathrm{perf}}$. We write $D^{\mathrm{perf}}\mathcal{W}(X) = H^0(\mathcal{W}(X)^{\mathrm{perf}})$ for the derived category of $\mathcal{W}(X)$. This is a genuine triangulated category.

2c Generation of wrapped Fukaya categories

In this section we recall the generation results for wrapped Fukaya categories of Weinstein sectors proved in [16; 25]. A Liouville sector (X, ω, Z) is said to be *Weinstein* if the convex completion of X is Weinstein and there exists a symplectic decomposition $\mathrm{Nbd}^Z_X(\partial X) \simeq F \times T^*(-\varepsilon, 0]$ such that F is a Weinstein hypersurface in ∂X. Similarly, a cornered Liouville sector (X, ω, Z) is Weinstein if each sectorial corner $\partial_I X$, which is the intersection of $|I| = k$ sectorial boundaries, decomposes into a product of $T^*(-\varepsilon, 0]^k$ and a Weinstein submanifold $F_I \subset \partial_I X$.

The wrapped Fukaya categories of Weinstein manifolds behave nicely since the Lagrangian cocores provide a natural collection of generating objects.

2.9 Proposition [17, Proposition 11.9] *If (X, ω, Z, ϕ) is a Weinstein manifold, then all the unstable manifolds of the pair (ϕ, Z) are coisotropic. In particular, if the index of a critical point of ϕ is $n = \frac{1}{2}\dim X$, then a corresponding unstable manifold is Lagrangian.*

2.10 Definition We call an unstable manifold of a Weinstein manifold $(X^{2n}, \omega, Z, \phi)$ with index n a *cocore* of X.

2.11 Theorem [16; 25] *The wrapped Fukaya category of a Weinstein manifold* (X, ω, Z, ϕ) *is generated by its cocores.*

In particular, wrapped Fukaya categories of Weinstein manifolds are finitely generated.

We then consider the more general case of a cornered Weinstein sector. In this case, each corner stratum is a product $\mathrm{Nbd}^Z_X(\partial_I X) \simeq F_I \times T^*[-\varepsilon, 0]^{|I|}$ for some $I \subseteq \{1, \dots, n\}$, hence for $L \subseteq F_I$ a cocore of F_I and $\gamma \subseteq T^*[-\varepsilon, 0]^{|I|}$ a product of cotangent fibers, we get an object $L \times \gamma$ in the wrapped Fukaya category of X, which we call a *linking disk* associated to the corner $\partial_I X$.

2.12 Theorem [16; 25] *The wrapped Fukaya category of a cornered Weinstein sector* (X, ω, Z, ϕ) *is generated by its cocores and linking disks associated to all its corner strata.*

We review some examples that will become the building blocks in the proof of our generation result (cf Proposition 3.24).

2.13 Example In [1], the author proves that for the cotangent bundle T^*Q over a compact manifold Q which is Spin, the wrapped Fukaya category $\mathcal{W}(T^*Q)$ is generated by a single cotangent fiber. This can be generalized to the case when the manifold Q is compact with corners, in which case the cotangent bundle of Q is a cornered Weinstein sector. In particular, consider the cotangent bundle of an interval, as drawn in Figure 8.

The red boundary components are sectorial boundaries, and the wrapped Fukaya category $\mathcal{W}(T^*[0,1])$ is generated by the cotangent fiber $L := T^*_0[0,1]$, which is colored in green in the picture. From this figure, it is not hard to see that $\mathrm{CW}^*(L, L) \cong \mathbb{Z}(0)$, therefore $\mathcal{W}(T^*[0,1]) \simeq \mathrm{Mod}_{\mathbb{Z}}$ is quasiequivalent to the category of modules over $\mathbb{Z}$. By the Künneth theorem, it follows that for any cornered Weinstein sector X, we have $\mathcal{W}(X \times T^*[0,1]) \simeq \mathcal{W}(X)$.

2.14 Example Let $X = (\mathbb{C}^*)^d$ be an algebraic torus, which we can regard symplectically as the cotangent bundle T^*T^d of the real torus T^d. By Example 2.13, we know that $\mathcal{W}(T^*T^d)$ is generated by one cotangent fiber, which corresponds to one of the 2^d quadrants of the real locus of $(\mathbb{C}^*)^d$. Cutting $(\mathbb{C}^*)^d$ by sectorial hypersurfaces simply adds linking disks associated to these sectorial hypersurfaces (and their intersections) to the generating set.

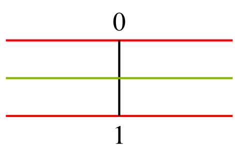

Figure 8: The cotangent bundle of [0, 1].

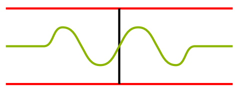

Figure 9: An example of generalized cocore.

It is a convenient viewpoint to regard linking disks as generalizations of cocores. To make it precise, we introduce the notion of a relative core.

2.15 Definition Let (X,ω,Z,ϕ) be a Weinstein manifold, then the union of the stable manifolds of all critical points of ϕ is called the *core* of X. For a Weinstein pair (X,F), the *relative core* of (X,F) is the union $\mathfrak{c}_X \cup (\mathfrak{c}_F \times \mathbb{R})$, where $\mathfrak{c}_F$ is the core of F and $\mathfrak{c}_F \times \mathbb{R} \subset X$ consists of points whose positive Liouville flow converge to $\mathfrak{c}_F$.

2.16 Definition [25] Let $\mathfrak{c}_{X,F}$ be the relative core of a Weinstein sector (X,F), a *generalized cocore* of (X,F) associated to a component of $\mathfrak{c}_{X,F}$ is a cylindrical exact Lagrangian submanifold intersecting $\mathfrak{c}_{X,F}$ transversely at a single point in this component.

Theorem 2.12 remains valid for generalized cocores: the wrapped Fukaya category of a cornered Weinstein sector (X,ω,Z,ϕ) is generated by a collection of generalized cocores, one for each component of the relative core.

In particular, instead of picking the standard generator of $T^*[0,1]$, we can pick any curve γ in $T^*[0,1]$ with a single transverse intersection with the zero section, and the product $\gamma \times L$ for any generating Lagrangian L in $\mathcal{W}(F)$ will give us a linking disk in $\mathcal{W}(X)$ to associated to F. Figure 9 shows one possible choice of a generalized cocore in $T^*[0,1]$.

2d Stopped inclusion and the Viterbo restriction Let (X,λ) and (X',λ') be two Liouville sectors. A (*stopped*) *inclusion* $i\colon X' \hookrightarrow X$ is a proper map that is a diffeomorphism onto its image, such that $i^*\lambda = \lambda' + df$ for some compactly supported function $f\colon X' \to \mathbb{R}$. Ganatra, Pardon and Shende [23] showed that an inclusion of Liouville sectors induces an A_∞-functor between the wrapped Fukaya categories

$$\mathcal{W}(X') \to \mathcal{W}(X),$$

which we call a *stopped inclusion functor*. On the other hand, given an embedding of Liouville domains $X_0 \hookrightarrow X_1$, one would expect that there is a *Viterbo restriction functor*

$$\mathcal{W}(X_1) \to \mathcal{W}(X_0)$$

between wrapped Fukaya categories, which is contravariant with respect to Liouville embedding. The geometric construction, under additional assumptions on cylindrical exact Lagrangians, has been carried out by Abouzaid and Seidel [4]. Algebraically, the existence of such a restriction functor for X_0 a

Weinstein domain is proved by Ganatra, Pardon and Shende [25]. The proof in [25, Section 11.1] suggests that such an A_∞-functor also exists for embeddings of Weinstein sectors $W_0 \hookrightarrow W_1$, provided that the Weinstein embedding is a diffeomorphism on connected components of sectorial boundaries. We record this fact in the following lemma.

2.17 Lemma *Let $X_0 \hookrightarrow X_1$ be a Weinstein embedding of Weinstein sectors that is a diffeomorphism on the boundary, then there exists a Viterbo restriction functor $\mathcal{W}(X_1) \to \mathcal{W}(X_0)$.*

2e The Orlov functor Recall from Definition 2.4 that given a Liouville sector (X, ω, Z), the boundary ∂X admits a Z-invariant neighborhood $\mathrm{Nbd}^Z(\partial X) \cong F \times T^*[-\varepsilon, 0]$ for some $\varepsilon > 0$, where $F \subset \partial X$ is a Liouville hypersurface. When F happens to be Weinstein, Theorem 2.12 tells us that the wrapped Fukaya category $\mathcal{W}(F)$ is generated by its cocores. Example 2.13 implies that the wrapped Fukaya category of $\mathrm{Nbd}^Z(\partial X)$, regarded as a Weinstein sector, is quasiequivalent to $\mathcal{W}(F)$. Together with the stopped inclusion $\mathrm{Nbd}^Z(F) \hookrightarrow X$, we obtain an A_∞-functor

$$\mathsf{Or}\colon \mathcal{W}(F) \to \mathcal{W}(X),$$

which is usually known as the *Orlov functor*. Note that the images of the Orlov functor is exactly the linking disks associated to the stop F.

2f Sectorial descent Suppose a Liouville sector X is covered by the other two Liouville sectors X_1 and X_2 such that the intersection $X_{12} := X_1 \cap X_2$ becomes a sectorial hypersurface of both X_1 and X_2; see Definition 2.6. Then the stopped inclusion functor gives us a diagram of wrapped Fukaya categories

$$\begin{array}{ccccc} & & \mathcal{W}(X_1) & & \\ & \nearrow & & \searrow & \\ \mathcal{W}(X_{12}) & & & & \mathcal{W}(X) \\ & \searrow & & \nearrow & \\ & & \mathcal{W}(X_2) & & \end{array} \tag{6}$$

In general, this diagram is an almost homotopy pushout, and if all the sectors are Weinstein, then it is a genuine homotopy pushout [25, Theorem 1.28]. More generally, suppose that X can be covered by a collection of cornered Liouville sectors, ie $X = \bigcup_{i=1}^k X_i$, the intersection poset of the cornered Liouville sectors $\{X_i\}_{i=1}^k$ gives a diagram of wrapped Fukaya categories, leading to the following is a generalization of (6).

2.18 Theorem (sectorial descent [25], Theorem 1.35) *For any Weinstein sectorial covering $X_1, \dots, X_k$ of a Liouville sector X, the induced functor*

$$\operatorname*{hocolim}_{\varnothing \neq I \subseteq \{1,\dots,k\}} \mathcal{W}\Big(\bigcap_{i \in I} X_i\Big) \xrightarrow{\simeq} \mathcal{W}(X)$$

is a pre-triangulated equivalence.

3 Symplectic geometry of hyperplane complements and generating Lagrangians

In this section, we first introduce our main object of study, a polarized hyperplane arrangement $\mathbb{V} = (V, \eta, \xi)$ (see Definition 3.5) and establish some basic geometric properties for the complement $M(\mathbb{V})$ of the complexified hyperplanes in $\mathbb{V}$. Then we apply Theorem 2.18 together with the generation results reviewed in Examples 2.13 and 2.14 to get a generating set of Lagrangians for the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$.

3a Hyperplane arrangements Let (V, η) be a pair of d-dimensional subspace $V \subset \mathbb{R}^n$ and an element $\eta \in \mathbb{R}^n / V$. This determines a hyperplane arrangement of n hyperplanes in $V \cong \mathbb{R}^d$. If we present V as the image of a linear map $\mathbb{R}^d \to \mathbb{R}^n$ induced by an $n \times d$ matrix $A = (a_{ij})$ and $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ represents η, then the hyperplanes in the arrangement are given by

$$H_{\mathbb{R},i} = \left\{ s = (s_1, \dots, s_d) \in \mathbb{R}^d \;\middle|\; \sum_{j=1}^{d} a_{ij} s_j + w_i = 0 \right\} \quad \text{for } i = 1, \dots, n. \tag{7}$$

We assume that (V, η) is chosen such that $H_{\mathbb{R},i} \neq \varnothing$ for all i so none of the hyperplane is redundant.

The positive half-spaces of $\mathbb{R}^d$ induce co-orientations on the hyperplanes so that we can assign a region (possibly empty) of the arrangement a sequence α of signs $\{+, -\}$ of length n. More precisely, given $\alpha = (\alpha(1), \dots, \alpha(n)) \in \{+, -\}^n$, the corresponding region (also called chamber) $\Delta_\alpha \subset V + \eta$ is the set of points s with

$$\alpha(i) \Bigl(w_i + \sum_{j=1}^{d} a_{ij} s_j \Bigr) \geq 0 \quad \text{for } 1 \leq i \leq n.$$

Such a sign sequence α is called *feasible*, if $\Delta_\alpha \neq \varnothing$. We denote the set of feasible sign sequences for $\mathbb{V}$ by $\mathscr{F}(\mathbb{V})$. Given two sign sequences α and β, define

$$d_{\alpha\beta} := \#\{1 \leq i \leq n \mid \alpha(i) \neq \beta(i)\} \tag{8}$$

to be the number of sign changes required to turn α into β. Due to the bijective correspondence, sign sequences and chambers will often be mentioned interchangeably.

3.1 Definition Let $\mathbb{V} = (V, \eta)$ be an arrangement of n hyperplanes as above. A *polarization* is an element $\xi \in V^* \cong (\mathbb{R}^n)^* / V^\perp$ such that each element of $(\mathbb{R}^n)^*$ representing ξ has at least d nonzero entries. We call the triple $\mathbb{V} = (V, \eta, \xi)$ a *polarized hyperplane arrangement* of n hyperplanes in V.

3.2 Remark The condition that ξ has at least d nonzero entries is added here to ensure that ξ is generic enough for our later purposes. This is usually not required in the literature.

Choose a lift $\tilde{\xi} \in (\mathbb{R}^n)^*$ of ξ. We say a sign sequence $\alpha \in \{+,-\}^n$ is *bounded* if the affine linear functional $\tilde{\xi}$ on $V+\eta$ is bounded above on Δ_α. Note that this is independent of the choice of the lift $\tilde{\xi}$. Denote the set of bounded sign sequences for $\mathbb{V}$ by $\mathscr{B}(\mathbb{V})$. Let $\mathscr{P}(\mathbb{V}) := \mathscr{B}(\mathbb{V}) \cap \mathscr{F}(\mathbb{V})$ be the set of *bounded and feasible* sign sequences.

3.3 Definition A hyperplane arrangement (V,η) is called *simple* if for every subset of k hyperplanes $H_{\mathbb{R},i_1},\dots,H_{\mathbb{R},i_k}$, the intersection $\bigcap_{l=1}^k H_{\mathbb{R},i_l}$ is either empty or has codimension k.

In fact, this is equivalent to saying that the intersection of hyperplanes is either transverse or empty (when they are parallel). The following combinatorial lemma will play a crucial role in the proof of our main result.

3.4 Lemma *Suppose $\mathbb{V}$ is a simple hyperplane arrangement equipped with a generic polarization ξ. Let $\mathcal{H} = \{H_{\mathbb{R},1},\dots,H_{\mathbb{R},n}\}$ be the corresponding collection of hyperplanes in $\mathbb{R}^d$. Then the number of bounded and feasible chambers is equal to the number of 0-dimensional strata in the union of all hyperplanes $\bigcup_{i=1}^n H_{\mathbb{R},i}$.*

Proof The proof is based on double induction on the pair $(d,n) \in \mathbb{N}^2$. Denote by $N_{\mathbb{V}}$ the number of bounded-feasible chambers of $\mathbb{V}$. Due to the genericity of ξ, it is straightforward that $N_{\mathbb{V}} = 0$ if and only if there is no 0-dimensional stratum in $\bigcup_{i=1}^n H_{\mathbb{R},i}$. Also, when $d=1$, it is clear that $N_{\mathbb{V}} = n$.

We first perform induction on d. Suppose that the claim holds when $d=r$. Consider a polarized arrangement $\mathbb{V}$ of $n \ge r$ hyperplanes in $\mathbb{R}^{r+1}$ and apply the induction on n. The initial step $n=r$ is obvious. Suppose that the claim holds for $n=k>r+1$. We add a new hyperplane $H_{\mathbb{R}}$ to $\mathcal{H}$ which is in general position, so that the restrictions of the hyperplanes in $\mathcal{H}$ to $H_{\mathbb{R}}$ is still a hyperplane arrangement in $H_{\mathbb{R}} \cong \mathbb{R}^r$. By induction hypothesis, we know that the number of bounded and feasible chambers on the induced hyperplane arrangement in $H_{\mathbb{R}}$ is the same as the number of 0-dimensional strata. Moreover, the simplicity assumption implies that each such chamber in $H_{\mathbb{R}}$ either divides one of the bounded and feasible chambers in $\mathbb{V}$ or adds a new bounded and feasible chamber to $\mathbb{V}$. This implies that after adding $H_{\mathbb{R}}$, the number of bounded and feasible chambers in the hyperplane arrangement is increased by the number of 0-dimensional strata in $H_{\mathbb{R}}$ created by the intersection $H_{\mathbb{R}} \cap \left(\bigcup_{i=1}^k H_{\mathbb{R},i}\right)$. □

3.5 Definition Let $\mathbb{V} = (V,\eta,\xi)$ be a polarized hyperplane arrangement. The complexified polarized hyperplane arrangement associated to $\mathbb{V}$ is a linear extension of $\mathbb{V}$ over $\mathbb{C}$, denoted by $\mathbb{V}_{\mathbb{C}} = (V_{\mathbb{C}},\eta_{\mathbb{C}},\xi_{\mathbb{C}})$. If it is clear from the context, we simply write it as $\mathbb{V}$.

For example, if we write one of the hyperplanes $H_{\mathbb{R},i} \subset \mathbb{R}^d$ as in (7), then its complexification $H_i \subset \mathbb{C}^d$ is given by

$$H_i = \left\{ z \in \mathbb{C}^d \;\middle|\; \sum_{j=1}^d a_{ij} z_j + w_i = 0 \right\}, \tag{9}$$

where $a_{ij}, w_j \in \mathbb{R}$.

3b Essential geometry In this subsection, we describe the Weinstein structure of $M(\mathbb{V})$ and show that $M(\mathbb{V})$ is of finite type (Corollary 3.10).

Let $\ell_i(z)$ be the defining equation of the (complex) hyperplane H_i in the arrangement $\mathbb{V}$ and $\xi(z) = \xi \cdot z$. We assume that the polarization ξ satisfies the following properties:

(i) ξ does not belong to the span of the normal vectors of any k hyperplanes in $\mathbb{V}$ if $k \le d-1$.

(ii) No two 0-dimensional strata in the intersections of the complex hyperplanes $\{H_i\}_{i=1}^n$ lie in the same level set $\operatorname{Re}(\xi)^{-1}(c)$ for some $c \in \mathbb{R}$.

The set of polarizations satisfying these two properties are generic in the space of all linear forms on V^*. Consider the affine canonical embedding $\iota\colon \mathbb{C}^d \to \mathbb{C}^n$ given by $\iota(z) = (\ell_1(z), \dots, \ell_n(z))$. This induces an embedding of the complement $M(\mathbb{V}) := \mathbb{C}^d \setminus \bigcup_{i=1}^n H_i$ into $(\mathbb{C}^*)^n$, which we will denote by $\iota\colon M(\mathbb{V}) \to (\mathbb{C}^*)^n$.

Next, we equip $M(\mathbb{V})$ with a symplectic form ω as follows: First, let ω_ϕ be the symplectic form on $(\mathbb{C}^*)^n$ defined by the pullback of the standard symplectic form on $\mathbb{C}^{2n}$ via the embedding $i : (\mathbb{C}^*)^n \hookrightarrow \mathbb{C}^{2n}$ given by $i(z_1, \dots, z_n) = (z_1, 1/z_1, \dots, z_n, 1/z_n)$. We have

$$\begin{aligned}\omega_\phi &= 2\sqrt{-1}\sum_{i=1}^n\left(dz_i \wedge d\overline{z}_i + d\left(\frac{1}{z_i}\right) \wedge d\left(\frac{1}{\overline{z}_i}\right)\right) = 2\sqrt{-1}\sum_{i=1}^n\left(1+\frac{1}{|z_i|^4}\right) dz_i \wedge d\overline{z}_i \\ &= 4\sum_{i=1}^n\left(1+\frac{1}{(x_i^2+y_i^2)^2}\right) dx_i \wedge dy_i.\end{aligned}$$

Equivalently, we equip $(\mathbb{C}^*)^n$ with the Kähler potential $\phi = \sum_{i=1}^n (|z_i|^2 + 1/|z_i|^2)$ so that

$$-d^{\mathbb{C}}\phi = 2\sum_{i=1}^n\left(1-\frac{1}{(x_i^2+y_i^2)^2}\right)x_i\, dy_i - 2\sum_{i=1}^n\left(1-\frac{1}{(x_i^2+y_i^2)^2}\right)y_i\, dx_i$$

and $\omega_\phi = -dd^{\mathbb{C}}\phi$. Take ω to be the pullback of the form ω_ϕ under the embedding $\iota\colon M(\mathbb{V}) \hookrightarrow (\mathbb{C}^*)^n$, which gives

$$\omega := \iota^*\omega_\phi = 4\sum_{i=1}^n\left(1+\frac{1}{|\ell_i|^4}\right) d\operatorname{Re}(\ell_i) \wedge d\operatorname{Im}(\ell_i).$$

Notice that ω depends on $M(\mathbb{V})$ as well as the choice of the embedding $\iota\colon \mathbb{C}^d \to \mathbb{C}^n$. However, different choices yield deformation-equivalent Weinstein structures.

3.6 Lemma *For any $\vec{c} = (c_1, \dots, c_n) \in \mathbb{R}^n_{>0}$, the rescaled embedding*

$$\iota_{\vec{c}} := \vec{c}\iota = (c_1\ell_1, \dots, c_n\ell_n)\colon M(\mathbb{V}) \to (\mathbb{C}^*)^n$$

gives us a new Weinstein structure $(M(\mathbb{V}), \iota_{\vec{c}}^*\phi)$ that is Weinstein deformation equivalent to $(M(\mathbb{V}), \iota^*\phi)$.

Proof Consider an isotopy of Stein embeddings $\iota_t\colon M(\mathbb{V}) \to \mathbb{C}^n$ via

$$\iota_t(z) = t\vec{c}\iota(z) + (1-t)\iota(z)$$

for $t \in [0,1]$. Then we have

$$\iota_t^*\phi(z) = \phi(t\vec{c}\iota(z) + (1-t)\iota(z)).$$

It is straightforward to compute the differential, and easy to see that the critical points of $d\phi_t(z)$, for all $0 \le t \le 1$, lie in a compact subset (by rescaling the previous critical points). Therefore $(M(\mathbb{V}), \iota^*\phi)$ and $(M(\mathbb{V}), \iota_{\vec{c}}^*\phi)$ are Weinstein deformation equivalent. □

From now on, if the context is clear, we will abuse notation by denoting the restriction of ϕ to $M(\mathbb{V})$ simply as ϕ.

Let $J \subset \{1,\dots,n\}$ be an index set such that $\bigcap_{i\in J} H_i \neq \varnothing$. Since all the ℓ_i are linear, near the intersection $\bigcap_{i\in J} H_i$, the symplectic form ω can be written as the sum of the dominant terms and bounded terms:

$$\omega = 4\sum_{i\in J}\left(1+\frac{1}{|\ell_i|^4}\right)d\mathrm{Re}(\ell_i)\wedge d\mathrm{Im}(\ell_i) + 4\sum_{i\notin J}\left(1+\frac{1}{|\ell_i|^4}\right)d\mathrm{Re}(\ell_i)\wedge d\mathrm{Im}(\ell_i).$$

Geometrically, the symplectic form ω asymptotically behaves like the standard symplectic form near $\bigcap_{i\in J} H_i \neq \varnothing$. The following lemma gives precise formulae for this fact.

3.7 Lemma *Let $J \subset \{1,\dots,n\}$ with $|J| = d$. Enumerate the elements in J by $1,\dots,d$. Consider the linear change of coordinates $\Phi\colon \mathbb{C}^d \to \mathbb{C}^d$ which pulls ℓ_i back to z_i for $i \in J$. Write*

$$\ell_j = \sum_{i\in J} a_{ij} z_i,$$

where the coefficients $a_{ij} \in \mathbb{R}$ are as in (9).

Then $M(\Phi^{-1}(\mathbb{V})) = \Phi^{-1}(M(\mathbb{V}))$ and we have the following formulae:

(10)
$$\Phi^*\omega = 4\sum_{i\in J}\left(1+\frac{1}{|z_i|^4}+\sum_{j\notin J} a_{ij}^2\left(1+\frac{1}{|\ell_j|^4}\right)\right)dx_i\wedge dy_i + 4\sum_{\substack{i\neq j\\ i,j\in J}}\sum_{k\notin J} a_{ik}a_{jk}\left(1+\frac{1}{|\ell_k|^4}\right)dx_i\wedge dy_j.$$

$$\Phi^*d\phi = \sum_{i=1}^{d} 2\left(\left(1-\frac{1}{|z_i|^4}\right)x_i + \sum_{j\notin J}\left(1-\frac{1}{|\ell_j|^4}\right)\mathrm{Re}(\ell_j)a_{ij}\right)dx_i + \sum_{i=1}^{d} 2\left(\left(1-\frac{1}{|z_i|^4}\right)y_i + \sum_{j\notin J}\left(1-\frac{1}{|\ell_j|^4}\right)\mathrm{Im}(\ell_j)a_{ij}\right)dy_i,$$

$$\Phi^* d^{\mathbb{C}}\phi = -\sum_{i=1}^{d} 2\left(\left(1-\frac{1}{|z_i|^4}\right)x_i + \sum_{j\notin J}\left(1-\frac{1}{|\ell_j|^4}\right)\operatorname{Re}(\ell_j)\, a_{ij}\right)dy_i + \sum_{i=1}^{d} 2\left(\left(1-\frac{1}{|z_i|^4}\right)y_i + \sum_{j\notin J}\left(1-\frac{1}{|\ell_j|^4}\right)\operatorname{Im}(\ell_j)\, a_{ij}\right)dx_i. \tag{11}$$

The proof of Lemma 3.7 follows from a straightforward computation, so we omit the details. For simplicity, we will drop Φ^* from the notation if it is clear from the context. We will find it convenient to use Lemma 3.7 when doing computations.

To write the formulae in the lemma above in more compact forms, let A be the $(n-d)\times d$-matrix whose $(i,j)^{\text{th}}$ entry is $(a_{ji}\sqrt{1+(1/|\ell_i|^4)})$ and let $D = ((1+1/|z_i|^4)\delta_{ij})$ be the diagonal matrix. Under the standard basis $\{\partial/\partial\vec{x}, \partial/\partial\vec{y}\}$, we can write the symplectic form ω in the matrix form

$$\omega = \begin{pmatrix} 0 & 4(D+A^TA) \\ -4(D+A^TA) & 0 \end{pmatrix}, \tag{12}$$

where $D + A^TA$ is invertible.

Similarly, consider the matrices

$$\tilde{A} = \left(a_{ji}\sqrt{1-\frac{1}{|\ell_i|^4}}\right), \qquad \tilde{D} = \left(\left(1-\frac{1}{|z_i|^4}\right)\delta_{ij}\right),$$

$$[d^{\mathbb{C}}\phi]_x = 2\vec{x}^T(\tilde{D}+\tilde{A}^T\tilde{A}), \qquad [d^{\mathbb{C}}\phi]_y = 2\vec{y}^T(\tilde{D}+\tilde{A}^T\tilde{A}).$$

Then we have $d^{\mathbb{C}}\phi = -[d^{\mathbb{C}}\phi]_x d\vec{y} + [d^{\mathbb{C}}\phi]_y d\vec{x}$. We can also compute the Hamiltonian vector fields $X_{\operatorname{Re}(\xi)}$ and $X_{\operatorname{Im}(\xi)}$ associated to the real and imaginary parts of ξ as follows.

3.8 Lemma *Write $\xi = \sum_{i=1}^d b_i z_i$, where $b_i \in \mathbb{R}$. Then $X_{\operatorname{Re}(\xi)}$ is a purely imaginary vector field given by the matrix formula*

$$X_{\operatorname{Re}(\xi)} = \frac{1}{4}\left(\frac{\partial}{\partial\vec{y}}\right)^T (D+A^TA)^{-1}\vec{b}, \tag{13}$$

where $\vec{b}$ refers to the vector $(b_1,\dots,b_d)^T$. Similarly, $X_{\operatorname{Im}(\xi)}$ is given by

$$X_{\operatorname{Im}(\xi)} = -\frac{1}{4}\left(\frac{\partial}{\partial\vec{x}}\right)^T (D+A^TA)^{-1}\vec{b}. \tag{14}$$

As a consequence, the square of the length of the vector field $X_{\operatorname{Re}(\xi)}$ is

$$\|X_{\operatorname{Re}(\xi)}\|^2 = \omega(X_{\operatorname{Re}(\xi)}, X_{\operatorname{Im}(\xi)}) = \tfrac{1}{4}(\vec{b})^T(D+A^TA)^{-1}\vec{b}.$$

Proof Since $d\,\mathrm{Re}(\xi)=\sum_{i=1}^d b_i\,dx_i$ and $d\,\mathrm{Im}(\xi)=\sum_{i=1}^d b_i\,dy_i$, we get

$$\begin{aligned} b_i &= d\,\mathrm{Re}(\xi)\Big(\frac{\partial}{\partial x_i}\Big)=\omega\Big(\frac{\partial}{\partial x_i},X_{\mathrm{Re}(\xi)}\Big)\\ &=4\sum_{j=1}^d\Big(1+\frac{1}{|z_j|^4}+\sum_{k\notin I}\Big(1+\frac{1}{|\ell_k|^4}\Big)a_{ik}a_{jk}\Big)dy_j(X_{\mathrm{Re}(\xi)}),\\ 0&=d\,\mathrm{Re}(\xi)\Big(\frac{\partial}{\partial y_i}\Big)=\omega\Big(\frac{\partial}{\partial y_i},X_{\mathrm{Re}(\xi)}\Big)\\ &=-4\sum_{j=1}^d\Big(1+\frac{1}{|z_j|^4}+\sum_{k\notin I}\Big(1+\frac{1}{|\ell_k|^4}\Big)a_{ik}a_{jk}\Big)dx_j(X_{\mathrm{Re}(\xi)}),\end{aligned}$$

Using the matrices A and D, the equations above can be rewritten as

$$\vec{b}=4(D+A^TA)\,d\vec{y}(X_{\mathrm{Re}(\xi)}),\quad \vec{0}=4(D+A^TA)\,d\vec{x}(X_{\mathrm{Re}(\xi)}).$$

The case of $X_{\mathrm{Im}(\xi)}$ can be dealt with similarly. Using (12) and the nondegeneracy of the matrix $D+A^TA$, we conclude the result. □

3.9 Lemma *Suppose that*

$$\bigcap_{j\in K,\ |K|=d+1}\{|\ell_j|\le 1\}=\varnothing. \tag{15}$$

Then there are functions $C_I(z), C_R(z)>0$ *and two constants* $0<c_1<c_2$ *such that for* $|\mathrm{Im}\,\xi|$ *sufficiently large, we have* $c_1<C_I(z)<c_2$ *and*

$$-d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)})=C_I(z)\,\mathrm{Im}(\xi).$$

Similarly, for $|\mathrm{Re}(\xi)|$ *sufficiently large, we have* $c_1<C_R(z)<c_2$ *and*

$$-d^{\mathbb{C}}\phi(X_{\mathrm{Re}(\xi)})=C_R(z)\,\mathrm{Re}(\xi).$$

Proof For any $J\subset\{1,\dots,n\}$ such that $|J|=d$ and $\epsilon>0$, we define

$$M(\mathbb{V})_{J,\epsilon}=\{z\in M(\mathbb{V})\mid |\ell_j(z)|>\epsilon \text{ for all } j\in J\}. \tag{16}$$

By assumption (15), the open sets $\{M(\mathbb{V})_{J,1}\mid |J|=d\}$ form a finite cover of $M(\mathbb{V})$. Consider a single open set $M(\mathbb{V})_{J,1}$ and let $\{\ell_j\mid j\in J\}$ be the coordinate functions $\{z_i\mid i=1,\dots,d\}$.

By (11), (13) and (14), we can write $d^{\mathbb{C}}\phi(X_{\mathrm{Re}(\xi)})$ and $d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)})$ as

$$\begin{aligned} d^{\mathbb{C}}\phi(X_{\mathrm{Re}(\xi)})&=-\tfrac12\vec{x}^T(\widetilde{D}+\widetilde{A}^T\widetilde{A})(D+A^TA)^{-1}\vec{b},\\ d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)})&=-\tfrac12\vec{y}^T(\widetilde{D}+\widetilde{A}^T\widetilde{A})(D+A^TA)^{-1}\vec{b}.\end{aligned}$$

When $|\mathrm{Im}(\xi)| \to \infty$, the inequality $|\mathrm{Im}(\xi)| = \left|\sum_{i=1}^d b_i y_i\right| \le \left(\sum b_i^2\right)^{1/2}\left(\sum y_i^2\right)^{1/2}$ implies that there must be at least one coordinate y_i with $y_j \to \infty$. Using the decomposition

$$\widetilde{D} + \widetilde{A}^T \widetilde{A} = (D + A^T A) - (D - \widetilde{D} + A^T A - \widetilde{A}^T \widetilde{A}),$$

we can write $-d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)})$ as

$$-d^{\mathbb{C}}\phi(X_{\mathrm{Im}\,\xi}) = \vec{y}^T\vec{b} - \underbrace{\vec{y}^T(D - \widetilde{D} + A^T A - \widetilde{A}^T \widetilde{A})(D + A^T A)^{-1}\vec{b}}_{\text{remainder term } R} = \mathrm{Im}(\xi) - R.$$

Note that the term

$$\vec{y}^T(D - \widetilde{D} + A^T A - \widetilde{A}^T \widetilde{A}) = \left(\frac{2y_i}{|z_i|^4} + \sum_{j \notin J} \frac{2}{|\ell_j|^4}\,\mathrm{Im}(\ell_j)a_{ij}\right)$$

in the remainder matrix R is bounded when all the $|z_i|$ are bounded away from 0 since $|\ell_j| \ge 1$ for all $j \notin J$. Since the vector $(D + A^T A)^{-1}\vec{b}$ is also bounded, it follows that the remainder term is bounded as $|z_i|$ are bounded away from 0. Let $K \subset \{1, \dots, d\}$ be a subset such that $|z_i| \to 0$ for $i \in K$ and $|z_j|$ is bounded away from 0 for $j \notin K$, we can write $D + A^T A$ as a 2×2 block matrix

$$D + A^T A = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where the first block E records all the entries coming from K. By [36, Theorem 2.1], the inverse $(D + A^T A)^{-1}$ is given by

$$\begin{pmatrix} E^{-1} + E^{-1}F(H - GE^{-1}F)^{-1}GE^{-1} & -E^{-1}F(H - GE^{-1}F)^{-1} \\ -(H - GE^{-1}F)^{-1}GE^{-1} & (H - GE^{-1}F)^{-1} \end{pmatrix}. \tag{17}$$

When the coordinates $(z_i)_{i \in K}$ are sufficiently small, E^{-1} is close to a zero matrix and the determinants $|F|$, $|G|$ and $|H|$ are bounded so $(D + A^T A)^{-1}$ is close to a 2×2 block matrix where only the bottom right block is nonzero (except when $K = \{1, \dots, d\}$, in which case the whole matrix $(D + A^T A)^{-1}$ is close to the zero matrix). On the other hand, when $(z_i)_{i \in K}$ are sufficiently small, the blow-up entries in $(D - \widetilde{D} + A^T A - \widetilde{A}^T \widetilde{A})$ are the diagonal entries $1/|z_i|^4$ for $i \in K$, which go to infinity at the same rate that E^{-1} is going to 0. As a result, the terms in $\vec{y}^T(D - \widetilde{D} + A^T A - \widetilde{A}^T \widetilde{A})(D + A^T A)^{-1}$ are still bounded even when some z_i are approaching 0. Therefore R is bounded for all possible $|z_i|$, which implies that as $|\mathrm{Im}(\xi)| \to \infty$, $-d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)}) \approx \mathrm{Im}(\xi)$. This proves the claim for $-d^{\mathbb{C}}\phi(X_{\mathrm{Im}(\xi)})$. The case of $-d^{\mathbb{C}}\phi(X_{\mathrm{Re}(\xi)})$ can be dealt with in a similar way. □

3.10 Corollary *The critical points of the function $\phi\colon M(\mathbb{V}) \to \mathbb{R}$ lie in a compact subset of $M(\mathbb{V})$. As a consequence, $(M(\mathbb{V}), \omega, \nabla\phi, \phi)$ is a Weinstein manifold of finite type.*

Proof By Lemma 3.6, at the cost of replacing ℓ_i by $c_i\ell_i$, we can assume that (15) is satisfied without changing the Weinstein deformation type. Then by Lemma 3.9, $d\phi$ is nonzero when $|\mathrm{Re}(\xi)|$ or $|\mathrm{Im}(\xi)|$ is large. This is true for any polarization ξ. Since the zeros of $d\phi$ do not depend on ξ, we conclude that they lie in a compact set. □

3c Some useful lemmas Before discussing the symplectic geometry of $M(\mathbb{V})$, we collect here some lemmas that will be used later. We start with a simple fact in linear algebra.

3.11 Lemma *The (pointwise) inverse matrix ω^{-1} of ω, as a section of $\mathrm{Hom}(T^*M(\mathbb{V}), TM(\mathbb{V}))$ over $M(\mathbb{V})$, extends smoothly over $\mathbb{C}^d$. Moreover, for a point $p \in \bigcap_{i\in J} H_i$, the image of $\omega^{-1}|_p$ lies in $\bigcap_{i\in J} T_pH_i$.*

Proof Since the complex structure $J_{\mathbb{V}}$ on $M(\mathbb{V})$ is ω-compatible, we have

$$-g \circ J_{\mathbb{V}} = \omega \colon TM(\mathbb{V}) \to T^*M(\mathbb{V}),$$

where g is the Riemannian metric. It follows that $\omega^{-1} = J_{\mathbb{V}} \circ g^{-1} \colon T^*M(\mathbb{V}) \to TM(\mathbb{V})$. Since $J_{\mathbb{V}}$ extends smoothly over $\mathbb{C}^d$, it suffices to check that g^{-1} extends smoothly over $\mathbb{C}^d$.

Let $p \in \bigcap_{i\in J} H_i$ and $p \notin H_i$ if $i \notin J$. In a neighborhood of p, we write $g = (g_{ij})$ as a 2×2 block matrix

$$R = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

such that $E = (g_{ij})_{i,j\in J}$. Note that R is only well-defined on the complement $M(\mathbb{V})$ of the hyperplanes. By [36, Theorem 2.1], R^{-1} is given by (17).

On the other hand, we know that as we approach the point p from the complement $M(\mathbb{V})$, E^{-1} converges to the zero matrix and the determinants $|F|, |G|$ and $|H|$ are bounded. Thus the inverse matrix R^{-1} converges to

$$\begin{pmatrix} 0 & 0 \\ 0 & H^{-1} \end{pmatrix},$$

which clearly extends smoothly over p.

Moreover, the image of R^{-1} at p is spanned by $\{\partial_{x_i}, \partial_{y_i}\}_{i\notin J}$, which gives precisely the subspace $\bigcap_{i\in J} T_pH_i$. Since $\bigcap_{i\in J} T_pH_i$ is $J_{\mathbb{V}}$-invariant, the image of ω^{-1} is contained in $\bigcap_{i\in J} T_pH_i$. □

3.12 Corollary *For any smooth function $H\colon [0,1]\times\mathbb{C}^d \to \mathbb{R}$, the Hamiltonian vector field of $H|_{M(\mathbb{V})}$ on $M(\mathbb{V})$ extends to a smooth vector field over $\mathbb{C}^d$ that is tangent to the strata in the union $\bigcup_{i=1}^n H_i$.*

Proof The 1-form dH_t is smooth over $\mathbb{C}^d$, so the result follows from Lemma 3.11. □

The following lemma deals with the completeness of vector fields away from lower-dimensional strata.

3.13 Lemma *Let $X \subseteq \overline{X}$ be an open submanifold of a smooth manifold $\overline{X}$ (possibly with boundary) such that $\overline{X} \setminus X$ is a stratified space of lower dimension and V a vector field on X. If V can be extended to a complete C^1 (eg smooth) vector field $\overline{V}$ on $\overline{X}$ such that $\overline{V}|_{\overline{X}\setminus X}$ is tangent to the strata of $\overline{X}\setminus X$, then the flow of V on X will not escape to $\overline{X}\setminus X$.*

Proof Since $\overline{V}$ is complete in $\overline{X}$, for every $t > 0$ the flow $\psi^t \colon \overline{X} \to \overline{X}$ is a well-defined diffeomorphism sending the stratified space $\overline{X}\setminus X$ to itself. Therefore ψ^t restricts to a diffeomorphism on X, which is also the flow of V on X. This shows that the flow of V on X will not escape to $\overline{X}\setminus X$. □

3d Homotopy sectoriality In Section 3b, we equipped $M(\mathbb{V})$ with a natural Weinstein structure. In order to apply the sectorial descent (Theorem 2.18) to study the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$, we need to find a sectorial collection of real hypersurfaces in $M(\mathbb{V})$. In this subsection, we will prove a slightly more general result, which is stated as follows.

Given any collection of transversely intersecting $k(\leq d)$ complex hyperplanes $\{S_i = \xi_i^{-1}(c_i)\}_{i=1}^k$ in $\mathbb{C}^d$ for some $c_i \in \mathbb{R}$, which are also transverse to the hyperplanes at infinity (ie the complex hyperplanes $H_1, \dots, H_n \subset \mathbb{C}^d$), there exists a Liouville deformation of $M(\mathbb{V})$ such that the real hypersurfaces $\mathrm{Re}(\xi_i)^{-1}(c_i)$ form a sectorial collection after deformation.

The proof is based on a lengthy calculation, so it is helpful to start with a sketch of the argument. Pick a subcollection $\{S_{i_1}, \dots, S_{i_s}\}$ of the hyperplanes $\{S_i\}_{1 \leq i \leq k}$ such that the intersection $\bigcap_{j=1}^s S_{i_j} \neq \varnothing$, we explain how to deform the Liouville structure near $\bigcap_{j=1}^s S_{i_j}$ in a way that is compatible with all other subcollections with nonempty intersections. We first construct a collection of commuting vector fields $\{X_{\mathrm{Re}(\xi_j)}, V_j\}_{j=1}^s$ near $\bigcap_{j=1}^s S_{i_j}$ such that $\omega(V_j, X_{\mathrm{Re}(\xi_{j'})}) = \delta_{j,j'}$ and $\omega(X_{\mathrm{Re}(\xi_j)}, X_{\mathrm{Re}(\xi_{j'})}) = \omega(V_j, V_{j'}) = 0$ (cf Lemmas 3.15 and 3.21). By integrating along (the normalization of) these vector fields, it gives the corresponding s pairs of R and I functions (in the sense of Definition 2.6), denoted by $(R_j, I_j)_{1 \leq j \leq s}$, which in turn produce a symplectic trivial fibration from a neighborhood of $\bigcap_{j=1}^s S_{i_j}$ to $\mathbb{C}_\varepsilon^s := ((-2\varepsilon, 2\varepsilon) \times \mathbb{R})^s$ (cf Corollary 3.17). We will also show that I_j is close to $\mathrm{Im}(\xi_j)$ when ε is small (cf Corollary 3.18 and Lemma 3.19). We can then apply Lemma 3.9 to get control of $d^{\mathbb{C}}\phi$ in terms of I_j outside of a compact subset. We write down an explicit deformation of the Liouville structure using the product coordinates obtained from the trivial fibration to $\mathbb{C}_\varepsilon^s$ (cf (20) and (21)), after an explicit calculation together with the estimates of $d^{\mathbb{C}}\phi$, we show that the deformation is a Liouville deformation such that the skeleton lies in a compact set for all time (cf Propositions 3.14 and 3.22). The construction of the commuting vector fields and the explicit deformation of the Liouville structure are canonical enough in the sense that they are compatible when we consider different subcollections of the complex hyperplanes $\{S_i\}_{1 \leq i \leq k}$.

We start with the case of a single hyperplane.

3.14 Proposition *Let $c \in \mathbb{R}$ be a real number such that $\mathrm{Re}(\xi)^{-1}(c)$ intersects all hyperplanes $H_1, \dots, H_n$ transversely. Then there is a Liouville deformation $(M(\mathbb{V}), \omega, Z_t)$ with $Z_0 = \nabla\phi$ and Z_1 is tangent to the sectorial hypersurface $\{\mathrm{Re}(\xi) = c\}$ outside of a compact subset. Moreover, the skeleton*

$$\overline{\bigcup_{0 \leq t \leq 1} \mathrm{Skel}(M(\mathbb{V}), Z_t)}$$

is compact.

As explained earlier, the first step is to construct the commuting vector fields X_R and V.

3.15 Lemma *Let $c \in \mathbb{R}$ be such that $\mathrm{Re}(\xi)^{-1}(c)$ intersects all the hyperplanes $H_1, \dots, H_n$ transversely. Then there is a small neighborhood $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$ of $\{\mathrm{Re}(\xi) = c\}$ and a vector field V in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$ such that $\omega(V, X_{\mathrm{Re}(\xi)}) = 1$ and the flow of V is well-defined in the punctured neighborhood $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \setminus \left(\bigcup_{i=1}^n H_i\right)$ for some positive and negative time.*

Proof Take $V = -JX_{\mathrm{Re}(\xi)}/\|X_{\mathrm{Re}(\xi)}\|^2$. Then

$$\omega(V, X_{\mathrm{Re}(\xi)}) = g\left(\frac{X_{\mathrm{Re}(\xi)}}{\|X_{\mathrm{Re}(\xi)}\|^2}, X_{\mathrm{Re}(\xi)}\right) = 1.$$

Thus it suffices to show that

(i) $\|X_{\mathrm{Re}(\xi)}\|$ is nonvanishing, and

(ii) the flow of V is well-defined for some small time $t \neq 0$,

in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \setminus \bigcup_{i=1}^n H_i$.

To see this, we consider the open cover $\{M(\mathbb{V})_{J,\epsilon}\}$ of $M(\mathbb{V})$ defined by (16) and identify the linear forms in ℓ_j for $j \in J$ with the coordinate functions $z_1, \dots, z_d$. Then the 1-form $d^{\mathbb{C}}\phi$ on $M(\mathbb{V})_{J,\epsilon}$ takes the form (11). Looking at the entries, we see that there is a constant $c > 0$ such that the matrix $(D + A^T A) - c\,\mathrm{Id}$ is positive definite. Recall that we are working over the subset $M(\mathbb{V})_{J,\epsilon}$, where $|\ell_j| > \varepsilon$ for $j \notin J$, therefore the matrix $D + A^T A$ blows up only when some z_i approaches 0. We apply the same trick used in the proof of Lemma 3.9.

Let $K \subset \{1, \dots, d\}$ and consider a region $\mathcal{D}_K \subset M(\mathbb{V})$ where $(z_i)_{i \in K}$ can be arbitrarily close to 0 but $(z_i)_{i \notin K}$ are bounded away from 0. Write $D + A^T A$ as a 2×2 block matrix, where the first block records all the coordinates with indices in K. As in the proof of Lemma 3.11, we can apply [36, Theorem 2.1] to compute the inverse of $D + A^T A$. When the coordinates $(z_i)_{i \in K}$ are sufficiently small, $(D + A^T A)^{-1}$ converges to a 2×2 block matrix where only the bottom right block, denoted by P, is nonzero (except when $K = \{1, \dots, d\}$, in which case $(D + A^T A)^{-1}$ converges to the 0 matrix). In fact, we also know that there are constants $C' > c' > 0$ (depending on $\mathcal{D}_K$) such that $P - c'\,\mathrm{Id}$ and $C'\,\mathrm{Id} - P$ are positive definite. The genericity of our choice of the polarization $\xi = \sum_{i=1}^d b_i z_i$ (see Definition 3.1) implies that all the b_i are nonzero. It follows that we have

$$0 < (\min_i |b_i|)\sqrt{c'} \leq \|X_{\mathrm{Re}(\xi)}\| \leq (\max_i |b_i|)\sqrt{C'} \tag{18}$$

in $\mathcal{D}_K$ as long as $K \neq \{1, \dots, d\}$. The transversality assumption on $\{\mathrm{Re}(\xi) = c\}$ implies that $\{\mathrm{Re}(\xi) = c\} \subset \mathbb{C}^d$ is away from 0-dimensional strata (ie $\bigcap_{i \in J'} H_i$ for some $|J'| = d$) formed by the hyperplanes $\{H_i\}_{i=1}^n$. As a consequence, we can cover a small neighborhood of $\{\mathrm{Re}(\xi) = c\}$ using the regions $\mathcal{D}_K$ with $K \neq \{1, \dots, d\}$. Altogether, they give us a uniform upper bound and a uniform lower bound on the norm $\|X_{\mathrm{Re}(\xi)}\|$. This proves the existence of an open neighborhood of $\{\mathrm{Re}(\xi) = c\}$ satisfying (i).

By Corollary 3.12, we know that $JX_{\mathrm{Re}(\xi)}$ extends to a vector field on $\mathbb{C}^d$. By abuse of notation, we denote its extension by $JX_{\mathrm{Re}(\xi)}$. Moreover, by (18) and the 2×2 block matrix argument above, we see that $\|X_{\mathrm{Re}(\xi)}\|$ also extends to $\mathbb{C}^d$ and it is 0 only at the 0-dimensional strata. As a result,

$V = -JX_{\mathrm{Re}(\xi)}/\|X_{\mathrm{Re}(\xi)}\|^2$ extends to a well-defined vector field in $\mathbb{C}^d$ except possibly at the 0-dimensional strata, and it is tangent to the strata in $\bigcup_{i=1}^n H_i$. By Lemma 3.13, to show the completeness of V on $M(\mathbb{V})$, it suffices to exclude the possibility that its flowlines escape from the open subset $M(\mathbb{V}) \subset \mathbb{C}^d$ in a finite time. Since in a neighborhood of $\{\mathrm{Re}(\xi) = c\}$, the norm $\|V\| = 1/\|X_{\mathrm{Re}(\xi)}\|$ is uniformly bounded above by a constant, the flow of V on $M(\mathbb{V})$ is complete by Grönwall's inequality [45, Theorem 2.8]. □

3.16 Remark The proof of Lemma 3.15 shows that there are positive constants c_R and C_R such that $c_R < \|X_{\mathrm{Re}(\xi)}\| < C_R$ in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$; see (18). Moreover, if we choose $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$ such that it is a union of $X_{\mathrm{Re}(\xi)}$-integral curves, then the flow of $X_{\mathrm{Re}(\xi)}$ in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$ exists for all positive and negative time.

3.17 Corollary *Under the assumption of Lemma 3.15, there is a constant $\varepsilon > 0$ and a pair of functions*

$$(R, I)\colon \mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \to \mathbb{C}_\varepsilon := (-2\varepsilon, 2\varepsilon) \times \mathbb{R} \tag{19}$$

such that $(R, I)^{-1}(0) = \{\xi = c\}$, $\nabla I = X_{\mathrm{Re}(\xi)}$ and $\nabla R = V$.

In particular, the vector fields X_R and X_I commute and hence (19) defines a trivial symplectic fibration.

Proof Since $\omega(V, X_{\mathrm{Re}(\xi)}) = 1$ is a constant, the vector fields V and $X_{\mathrm{Re}(\xi)}$ commute and define an integrable 2-plane bundle transverse to $\{\xi = c\}$. By Lemma 3.15, the flow of V is defined for some nonzero (positive or negative) time. On the other hand, the flow of $X_{\mathrm{Re}(\xi)}$ is defined for all time; see Remark 3.16. Therefore, by integrating the vector fields V and $X_{\mathrm{Re}(\xi)}$, we get the functions R and I respectively. Since

$$\omega(X_R, X_I) = \omega(JX_R, JX_I) = \omega(-\nabla R, -\nabla I) = \omega(V, X_{\mathrm{Re}(\xi)}) = 1,$$

(19) defines a trivial symplectic fibration. □

3.18 Corollary *The function I in Corollary 3.17 agrees with* $\mathrm{Im}(\xi)$.

Proof The gradient vector field of $\mathrm{Im}(\xi)$ is given by $\nabla\,\mathrm{Im}(\xi) = -JX_{\mathrm{Im}(\xi)} = X_{\mathrm{Re}(\xi)}$, which is the same as the gradient vector field of I.

Recall from Lemma 3.15 that $V = -JX_{\mathrm{Re}(\xi)}/\|X_{\mathrm{Re}(\xi)}\|^2 = -X_{\mathrm{Im}(\xi)}/\|X_{\mathrm{Re}(\xi)}\|^2$, which is tangent to $\{\mathrm{Im}(\xi) = 0\}$. As a result, $I|_{\{\mathrm{Im}(\xi)=0\}} = 0$. The uniqueness of integral curve with the same initial condition implies that $I = \mathrm{Im}(\xi)$. □

3.19 Lemma *Let R and I be functions as in Lemma 3.15 and Corollary 3.17. Suppose also that*

$$\bigcap_{j\in K,\ |K|=d+1} \{|\ell_j| \le 1\} = \varnothing.$$

Then inside the open neighborhood $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \cong F \times \mathbb{C}_\varepsilon$, we have a function $C(z) > 0$ and two positive constants c_1, c_2 such that $c_1 < C(z) < c_2$ and, for $|I|$ sufficiently large,

$$-d^{\mathbb{C}}\phi(X_I) = C(z) I.$$

Proof This is a consequence of Lemma 3.9 and Corollary 3.18. □

We will now start to prove Proposition 3.14, the main result of this subsection, using Lemma 3.19.

Proof of Proposition 3.14 The proof is divided into six steps.

Step 0: preparation Pick $c' \gg 0$ large enough so that $(M(\mathbb{V}), c'\phi)$ is covered by the open subsets $M(\mathbb{V})_{J,1}$ (ie the assumption (15) holds). By Lemma 3.6, working with $(M(\mathbb{V}), c'\phi)$ does not change the Weinstein deformation type, so it suffices to prove the proposition for $(M(\mathbb{V}), c'\phi)$ instead.

Using the cover $\{M(\mathbb{V})_{J,1}\}$, and under the convention of Lemma 3.15, we write $\xi = \sum_{i=1}^d b_i \ell_i = \sum_{i=1}^d b_i z_i$. Recall R and I from Lemma 3.15 and Corollary 3.17. Let $F = (R, I)^{-1}(0)$, we have a symplectic product decomposition

$$\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \cong F \times (-\varepsilon, \varepsilon) \times \mathbb{R},$$

which is given by symplectic parallel transports along the second and the third factors. Denote the Liouville 1-form on $M(\mathbb{V})$ by $\lambda_{\mathbb{V}}$. Let $\lambda_F = \lambda_{\mathbb{V}}|_{F\times\{0\}}$ and recall the definition of the 1-form $\lambda_{\mathbb{C}}^{\alpha}$ from (3). For any $0 < \alpha < 1$, the difference between our Liouville 1-form $\lambda_{\mathbb{V}}$ and $\lambda_F + \lambda_{\mathbb{C}}^{\alpha}$ is exact in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$, so we can write

$$\lambda_{\mathbb{V}} = \lambda_F + \lambda_{\mathbb{C}}^{\alpha} + df,$$

where $f : \mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\}) \to \mathbb{R}$ is a smooth function in $\mathrm{Nbd}(\{\mathrm{Re}(\xi) = c\})$, not necessarily compactly supported, which depends on α.

Step 1: an explicit deformation Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function depicted on the right-hand side of Figure 10, which is zero near the origin, particularly $\rho'(0) = \rho''(0) = 0$, and $\rho(x) = x - c''$ for some $c'' > 0$ near $x = \varepsilon$, $\rho(x) = x + c''$ near $x = -\varepsilon$ and $\rho(x) = x$ when $|x| \geq 2\varepsilon$. We assume that $0 \leq \rho' \leq 1$ when $|x| \leq \varepsilon$ and $1 \leq \rho' \leq 1 + \delta_{c''}$ when $|x| \geq \varepsilon$, where $\delta_{c''} > 0$ is a small number depending on c''. By possibly replacing c'' with a smaller one, we can make $\delta_{c''}$ as close to 0 as we want. Set

$$\tilde{f}(x, p) := f(x, \rho(R(p)), \rho'(R(p))I(p)) \quad \text{for } |R(p)| \leq \varepsilon, \tag{20}$$

$$\tilde{f}(x, p) := f(x, \rho(R(p)), I(p)) \quad \text{for } \varepsilon \leq |R(p)| \leq 2\varepsilon. \tag{21}$$

This is well-defined because $\rho' = 1$ near $|R| = \varepsilon$. Moreover, $\tilde{f} = f$ when $|R| \geq 2\varepsilon$.

Let $\lambda_1 := \lambda_F + \lambda_{\mathbb{C}}^{\alpha} + d\tilde{f}$. Since $\tilde{f} = 0$ near $R = 0$, the symplectic dual of λ_1 is tangent to the hypersurface $F \times \{0\} \times \mathbb{R}$ near infinity, such that $F \times \{0\} \times \mathbb{R}$ is sectorial with defining function I. For $0 \leq t \leq 1$, consider the family of 1-forms

$$\lambda_t = t\lambda_1 + (1-t)\lambda_{\mathbb{V}} = \lambda_F + \lambda_{\mathbb{C}}^{\alpha} + d(t\tilde{f} + (1-t)f),$$

which induces a Liouville homotopy from $(X, \omega, \lambda_{\mathbb{V}})$ to (X, ω, λ_1). Our goal is to show that for an appropriate choice of $\alpha \in (0, 1)$, the skeleton of this family of Liouville 1-forms is compact.

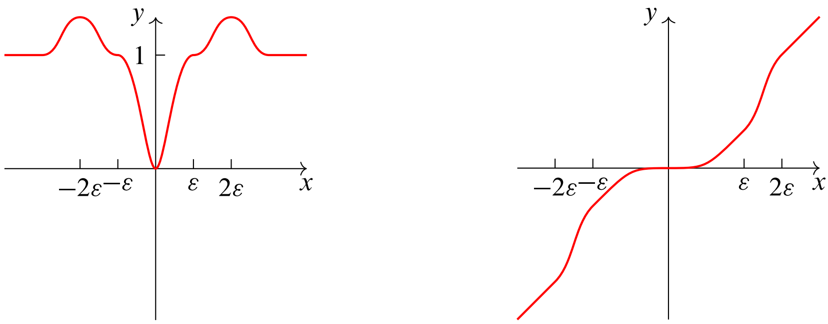

Figure 10: The bump function ρ, right, and its derivative, left.

Recall the notation $\mathbb{C}_\varepsilon = (-2\varepsilon, 2\varepsilon) \times \mathbb{R}$. To show that the skeleton is compact, our strategy is first to compute

$$d\tilde{f}|_{\{z\}\times\mathbb{C}_\varepsilon} \quad \text{and} \quad df|_{\{z\}\times\mathbb{C}_\varepsilon}$$

which shows that for every z, the zeros of $\lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon}$ lie in a compact set. Then we will argue that the zeros of $\lambda_t|_{F\times\{p\}}$ lie in a compact subset for every p to conclude that the zeros of λ_t lie in a compact subset.

Step 2: analyzing the critical points using Lemma 3.19 We implement the strategy described above, and consider first the region where $|R(p)| \le \varepsilon$.

On $\{z\} \times \mathbb{C}_\varepsilon$, by definition ∂_R is the unique vector field in $\mathrm{Span}\{V, X_{\mathrm{Re}(\xi)}\} = \mathrm{Span}\{X_R, X_I\}$ such that $dR(\partial_R) = 1$ and $dI(\partial_R) = 0$. In other words, $\omega(X_R, \partial_R) = -1$ and $\omega(X_I, \partial_R) = 0$. Therefore, $\partial_R = -X_I$ and similarly, $\partial_I = X_R$. Since $\partial_R \tilde{f} = \rho' \partial_R f + \rho'' I \partial_R f$ and $\partial_I \tilde{f} = \rho' \partial_I f$, we have

$$d\tilde{f}|_{\{z\}\times\mathbb{C}_\varepsilon} = \rho'(-df(X_I)\,dR + df(X_R)\,dI) - \rho'' I\, df(X_I)\,dR. \tag{22}$$

Recall that

$$\lambda_{\mathbb{C}}^\alpha = -\alpha I\,dR + (1-\alpha) R\,dI, \tag{23}$$

so df is given by

$$df|_{\{z\}\times\mathbb{C}_\varepsilon} = -d^{\mathbb{C}}\phi\Big|_{\{z\}\times\mathbb{C}_\varepsilon} - \lambda_{\mathbb{C}}^\alpha = \overbrace{(-d^{\mathbb{C}}\phi(X_R) - (1-\alpha)R)}^{df(X_R)} dI - \overbrace{(-d^{\mathbb{C}}\phi(X_I) - \alpha I)}^{df(X_I)} dR. \tag{24}$$

Consider the zeros of the restriction of the 1-form λ_t to $\{z\} \times \mathbb{C}_\varepsilon$ for $0 \le t \le 1$. By (23), (24) and (22),

$$\begin{aligned} \lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon} &= \lambda_{\mathbb{C}}^\alpha + t\,d\tilde{f}|_{\{z\}\times\mathbb{C}_\varepsilon} + (1-t)\,df|_{\{z\}\times\mathbb{C}_\varepsilon} \\ &= ((1-\alpha)R + (1 + t(\rho'-1))\,df(X_R))\,dI \\ &\qquad - (\alpha I + t\rho'' I\,df(X_R) + (1 + t(\rho'-1))\,df(X_I))\,dR. \end{aligned} \tag{25}$$

It suffices to look at the zeroes of the coefficients of dR and dI. For dI, the coefficient being zero implies that

$$df(X_R) = -\frac{(1-\alpha)R}{1 - t(1-\rho')}.$$

When $|I|$ is sufficiently large, it follows from Lemma 3.19 that the coefficient of dR in (25) being 0 means

$$\begin{aligned}0&=(\alpha I+t\rho'' I df(X_R)+(1+t(\rho'-1))df(X_I))\\&=\Big(\alpha I-t\rho'' I\frac{(1-\alpha)R}{1-t(1-\rho')}+(1+t(\rho'-1))(C(z)I-\alpha I)\Big)\\&=\Big(t(1-\rho')\alpha-t\rho''\frac{(1-\alpha)R}{1-t(1-\rho')}+(1-t(1-\rho'))C(z)\Big)I.\end{aligned}$$

The only possible solution is

$$\alpha=\frac{t\rho''R-C(1-t(1-\rho'))^2}{t\rho''R+t(1-\rho')(1-t(1-\rho'))}=1-\frac{C(1-t(1-\rho'))^2+t(1-\rho')(1-t(1-\rho'))}{t\rho''R+t(1-\rho')(1-t(1-\rho'))}. \tag{26}$$

Step 3: analyzing the critical points by a direct calculation Let $A=t(1-\rho')$. Then the right-hand side of (26) reads

$$1-\frac{C(1-A)^2+A(1-A)}{t\rho''R+A(1-A)}. \tag{27}$$

Since we are considering the region where $|R|\le\varepsilon$ and $|\rho''|$ is bounded (cf Figure 10, left, we have $0\le\rho'\le 1$, so

$$\frac{C(1-A)^2+A(1-A)}{t\rho''R+A(1-A)}\ge 0.$$

When $A\to 0$, we have $\rho'\to 1$ or $t\to 0$, hence

$$\frac{C(1-A)^2+A(1-A)}{t\rho''R+A(1-A)}\to\infty,$$

while if $A\to 1$, we must have $\rho'\to 0$ (thus also $\rho''\to 0$) and $t\to 1$, and a variation of the formula for α gives

$$\frac{1-\alpha}{1-A}=\frac{C(1-A)+A}{t\rho''R+A(1-A)},$$

which approaches ∞ as $A\to 1$. Looking back into the coefficient of dI, we have

$$(1-A)df(X_R)=-(1-\alpha)R,$$

which is equivalent to the equation

$$\frac{A(1-\alpha)}{1-A}R=d^{\mathbb{C}}\phi(X_R).$$

Since $|d^{\mathbb{C}}\phi(X_R)|\le C<\infty$ is bounded, $R/(1-A)$ must also be bounded. Traveling back to the coefficient for dR, we have

$$1=\frac{C(1-A)+A}{t\Big(\dfrac{1-\alpha}{1-A}R\Big)\rho''+A(1-\alpha)}.$$

If $A \to 1$, the numerator converges to 1, while

$$t\frac{1-\alpha}{1-A}R\rho'' \to 0 \quad \text{and} \quad A(1-\alpha) < 1,$$

so we derive that $1 < 1$ as $A \to 1$, a contradiction. Thus A cannot converge to 1, so there is some $0 < M < 1$ with $0 \le A \le M < 1$, such that (27) is away from 1. Therefore if α is sufficiently close to 1, the equation (26) has no solution.

We conclude that there is an $\alpha \in (0,1)$ such that the Liouville 1-form $\lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon}$ has no critical points in the region $|R| \le \varepsilon$ when $|I|$ is large. Note that Lemma 3.19 and the estimates above are uniform in z so there is an $\alpha \in (0,1)$ such that when $|I|$ is large, $\lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon}$ has no critical points in the region $|R| \le \varepsilon$ for all $z \in F$.

Step 4: the easier region Next we consider the region where $|R(p)| \ge \varepsilon$. When $|R(p)| \ge \varepsilon$, we have

$$d\tilde{f}|_{\{z\}\times\mathbb{C}_\varepsilon} = -\rho'(R)\,df(X_I)\,dR + df(X_R)\,dI.$$

In this case,

$$\begin{aligned}\lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon} &= \lambda_{\mathbb{C}}^{\alpha} + t\,d\tilde{f}|_{\{z\}\times\mathbb{C}_\varepsilon} + (1-t)\,df|_{\{z\}\times\mathbb{C}_\varepsilon} \\ &= \big((1-\alpha)R + df(X_R)\big)\,dI - \big(\alpha I + (1+t(\rho'-1))\,df(X_I)\big)\,dR.\end{aligned}$$

Applying Lemma 3.19 again, the coefficient of dR being zero means, for $|I|$ sufficiently large,

$$\begin{aligned}0 &= -d^{\mathbb{C}}\phi(X_I) - t(\rho'-1)\,d^{\mathbb{C}}\phi(X_I) - t(\rho'-1)\,\alpha I \\ &= \big((1+t(\rho'-1))C - t(\rho'-1)\big)\alpha I \\ &= \big(C + t(\rho'-1)(C-1)\big)\alpha I.\end{aligned}$$

Since C is uniformly bounded from above, we can choose ρ' to be very close to 1 such that the equation above has no solution for all $t \in [0,1]$. Therefore, $\lambda_t|_{\{z\}\times\mathbb{C}_\varepsilon}$ has no critical points in the region $|R(p)| \ge \varepsilon$ as well.

Step 5: concluding the proof We conclude that there is an $\alpha \in (0,1)$ such that the critical points of the Liouville 1-form λ_t lie in a compact subset of $\{z\}\times\mathbb{C}_\varepsilon$ for any $z \in F$, since the previous discussion does not depend on the choice of z. Compactness of the critical points of λ_t along the fiber follows from the fact that we are deforming the Liouville structure so that it remains complete in the complement $M(\mathbb{V})$, which we know from Corollary 3.10 to have compact set of critical points. □

We now generalize the argument above to the case where there are transversely intersecting hyperplanes $\xi_i^{-1}(c_i^{\pm}) \subset \mathbb{C}^d$ with $c_i^- < c_i^+$ for $i = 1, \ldots, d$.

3.20 Lemma *For any subcollection of hypersurfaces in* $\{\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm}) \mid i = 1, \ldots, d\}$, *their intersection is either coisotropic or empty.*

Proof This is a direct consequence of the fact that the vector fields $X_{\mathrm{Re}(\xi_i)}$ are purely imaginary and the characteristic foliation of $\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm})$ is generated by $X_{\mathrm{Re}(\xi_i)}$. □

Lemma 3.20 verifies the first item of Definition 2.6. To proceed, we need to show the orthogonality (4) in the definition of sectorial collections (cf Definition 2.6). In fact, we can prove the following generalization of Lemma 3.15.

3.21 Lemma *There exists a d-tuple of vector fields $\{V_1^{\pm}, \dots, V_d^{\pm}\}$, defined in the product neighborhood of $\bigcap_{i=1}^d \{\mathrm{Re}(\xi_i) = c_i^{\pm}\}$, satisfying the following properties*:

(i) $\omega(V_i^{\pm}, V_j^{\pm}) = 0$ *for all* i, j.

(ii) $\omega(V_i^{\pm}, X_{\mathrm{Re}(\xi_j)}) = \delta_{ij}$.

(iii) *The flow of $V_i^{\pm}$ exists for some positive time.*

Proof With the same conventions as in Lemma 3.7, we write $\xi_i = \sum_{j=1}^d b_{ij} z_j$. Then the Hamiltonian vector field of $\mathrm{Re}(\xi_i)$ is given by

$$X_{\mathrm{Re}(\xi_i)} = \tfrac{1}{4}\vec{b}_i^T (D + A^T A)^{-1} \frac{\partial}{\partial \vec{y}},$$

and similarly

$$X_{\mathrm{Im}(\xi_i)} = -\tfrac{1}{4}\vec{b}_i^T (D + A^T A)^{-1} \frac{\partial}{\partial \vec{x}},$$

so that the symplectic pairing between them is given by

$$\omega_{ij} := \omega(X_{\mathrm{Re}(\xi_i)}, X_{\mathrm{Im}(\xi_j)}) = 4X_{\mathrm{Re}(\xi_i)}^T (D + A^T A) X_{\mathrm{Im}(\xi_j)} = \tfrac{1}{4}\vec{b}_i^T (D + A^T A)^{-1} \vec{b}_j,$$

which is bounded for all $z \in M(\mathbb{V})$. Given a nonempty subset $J \subseteq \{1, \dots, d\}$ with $|J| = r < d$, without loss of generality we may assume that $J = \{1, \dots, r\}$. The restriction of the symplectic form ω to the subbundle of $T^*M(\mathbb{V})$ generated by $\{X_{\mathrm{Re}(\xi_i)}, X_{\mathrm{Im}(\xi_i)}\}_{1 \le i, j \le r}$ is given by the matrix

$$\Omega = \begin{pmatrix} 0 & (\omega_{ij}) \\ (-\omega_{ij}) & 0 \end{pmatrix}.$$

Let $\{V_i\}_{i=1}^r$ be a family of vector fields defined as linear combinations of $\{X_{\mathrm{Im}(\xi_j)}\}$ which satisfy $\omega(V_i, X_{\mathrm{Re}(\xi_i)}) = 1$ and $\omega(V_i, X_{\mathrm{Re}(\xi_j)}) = 0$ for $i \neq j$. Such V_i can be obtained by taking the inverse matrix of Ω. Also, since the coefficients of Ω are polynomials in ω_{ij}, which are bounded for $z \in M(\mathbb{V})$, the coefficients of the vector field V_i are Laurent polynomials in ω_{ij}. These coefficients are bounded unless the determinant of Ω goes to 0. However, Lemma 3.11 shows that this cannot happen near $\bigcap_{j \in J} H_j$. This allows us to apply the completeness criterion (Lemma 3.13) to conclude that all such V_i admit flows in the complement of $\bigcup_{j=1}^d H_j$ such that they do not escape to H_j. Now we have obtained V_i's for each open neighborhood of the intersection $\bigcap_{j \in J} \mathrm{Re}(\xi_j)^{-1}(c_j^{\pm})$, removing a neighborhood of some deeper strata, and we can patch them together using a partition of unity to get vector fields V_i defined in a neighborhood of $\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm})$, which still have well-defined flows in the complement of $\bigcup_{j=1}^d H_j$ such that they do not escape to H_j. There is a uniform upper bound for the norms $\|V_i\|$ so they are complete by Grönwall's inequality. It follows from the definition that $\omega(V_i^{\pm}, X_{\mathrm{Re}(\xi_j)}) = \delta_{ij}$, which completes the proof. □

As a consequence, by the same reasoning in Corollary 3.17, we can integrate the vector fields $V_i^{\pm}$ and $X_{\mathrm{Re}(\xi_i)}$ to obtain functions $\{R_i, I_i\}$, and a symplectic product decomposition of a neighborhood of the intersection $S^\sigma := \bigcap_{i=1}^d \mathrm{Re}(\xi_i)^{-1}(c_i^{\sigma_i})$,

$$((R_i, I_i)_{i=1,\dots,d})\colon \mathrm{Nbd}(S^\sigma) \to \mathbb{C}_\varepsilon^d$$

for each $\sigma = (\sigma_1, \dots, \sigma_d) \in \{\pm\}^d$. This verifies that Definition 2.6(ii) holds for the collection of hypersurfaces $\{\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm})\}_{i=1,\dots,d}$. Note that both I_i and $\mathrm{Im}(\xi_i)$ are obtained by integrating X_{R_i}. This time, $I_i|_{\mathrm{Im}(\xi_i)=0}$ is not necessarily 0 because $X_{\mathrm{Im}(\xi_j)}$ is not necessarily tangent to $\{\mathrm{Im}(\xi_i) = 0\}$. However, I_i and $\mathrm{Im}(\xi_i)$ are arbitrarily C^∞ close to each other when ε is small because $I_i|_{S^\sigma} = 0 = \mathrm{Im}(\xi_i)|_{S^\sigma}$ and they are obtained by integrating the same vector field. The direct generalization of Lemma 3.19 is therefore also true by replacing ξ with ξ_i and I with I_i.

We can adapt the proof of Proposition 3.14 to the collection of hypersurfaces $\{\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm})\}_{i=1}^d$ because we can use an explicit product type deformation in Step 1 (with respect to the product decomposition of $\mathbb{C}_\varepsilon^d$), apply the generalization of Lemma 3.19 in Step 2, and the rest follows by the same computations. Thus we have:

3.22 Proposition *There exists a Liouville deformation $(M(\mathbb{V}), \omega, Z_t)$ with compact skeleton such that $Z_0 = \nabla\phi$ and the real hypersurfaces $\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm}) \subset M(\mathbb{V})$ are all sectorial with respect to the deformed Liouville structure $(M(\mathbb{V}), \omega, Z_1)$.*

We can actually decompose this Liouville deformation into a finite number of steps, each of which makes one of the hypersurfaces $\mathrm{Re}(\xi_i)^{-1}(c_i^{\pm})$ sectorial. By [17, Proposition 11.8], we know that there exists an exact symplectomorphism

$$f\colon (M(\mathbb{V}), \omega, \lambda_\phi) \to (M(\mathbb{V}), \omega, \lambda_1)$$

such that $f^*\lambda_1 = \lambda_\phi + dg$, where $g\colon X \to \mathbb{R}$ is a compactly supported function. This implies that the wrapped Fukaya category is unchanged up to quasiequivalence under this deformation [28, Lemma 2].

3e Sectorial decomposition Given a polarized hyperplane arrangement $\mathbb{V} = (V, \eta, \xi)$, in this subsection we introduce sectorial cuts of the Weinstein manifold $M(\mathbb{V})$, so that it is divided into standard pieces with known generating Lagrangians. Note that the existence of sectorial cuts relies on the simplicity assumption of $\mathbb{V}$. The construction is divided into two steps.

Step 1 Let $\{\xi_i\}_{1 \le i \le d}$ be a basis of V^* with $\xi_1 = \xi$. Note that each ξ_i can be identified with a linear hyperplane in V, and we require that none of the linear hyperplanes associated to the ξ_i are parallel to any of the hyperplanes $H_{\mathbb{R},j}$ in the arrangement, where $1 \le j \le n$. Let $N_{\mathbb{V}}$ be the number of 0-dimensional strata in the hyperplane arrangement $\mathbb{V}$. Pick real numbers $(c_{1j})_{1 \le j \le N_{\mathbb{V}}}$ and $(c_{ik})_{2 \le i \le d, 1 \le k \le 2N_{\mathbb{V}}}$ with the following properties:

(i) For each $1 \le j \le N_{\mathbb{V}}$, there exists a unique q_{1j} so that $\mathrm{Re}(\xi_1)^{-1}(q_{1j})$ intersects with the hyperplanes in $\mathbb{V}$ at a unique 0-dimensional stratum. In this case, choose real numbers (c_{1j}) so that $c_{1,j-1} < q_{1j} < c_{1j}$ holds (when $j = 1$, we set $c_{1,0} = -\infty$).

(ii) For each $1 \le j \le N_{\mathbb{V}}$ and $i > 1$, there exists a q_{ij} so that $\mathrm{Re}(\xi_i)^{-1}(q_{ij})$ intersects with some hyperplanes in $\mathbb{V}$ at a 0-dimensional stratum. Pick real numbers (c_{ik}) with $c_{i,2j-1} < q_{ij} < c_{i,2j}$ so that

$$\big(\mathrm{Re}(\xi_i)^{-1}(c_{i,2j-1}) \cup \mathrm{Re}(\xi_i)^{-1}(c_{i,2j})\big) \cap \mathrm{Re}(\xi_1)^{-1}([c_{1j}, c_{1,j+1}])$$

intersects transversely with at most $d-2$ hyperplanes in $\mathbb{V}$.

By Proposition 3.22, after a Liouville deformation with compact skeleta, we can arrange that all the real hypersurfaces $\{\mathrm{Re}(\xi_i) = c_{ik}\}_{i,k}$, including $i = 1$, form a sectorial collection. Thus we can use them to cut the Weinstein manifold $M(\mathbb{V})$ into subsectors.

Step 2 We first cut $M(\mathbb{V})$ using the hypersurfaces defined by ξ_1, which leads to the decomposition

$$M(\mathbb{V}) = \bigcup_{j=1}^{N_{\mathbb{V}}} \mathrm{Re}(\xi_1)^{-1}([c_{1,j-1}, c_{1j}]) =: \bigcup_{j=1}^{N_{\mathbb{V}}} P_j,$$

so that we get a total number of $N_{\mathbb{V}}$ sectors with similar behaviors as depicted in Figure 11. See also the related discussions in the introduction (Section 1c).

Next, we perform further cuts on each subsector P_j using the real hypersurfaces $\mathrm{Re}(\xi_i)^{-1}(c_{i,2j-1})$ and $\mathrm{Re}(\xi_i)^{-1}(c_{i,2j})$ for $i > 1$, which gives the decomposition

$$P_j = \bigcup_{\alpha \in \{-,0,+\}^{d-1}} \Big(\bigcap_{i=2}^{d} \mathrm{Re}(\xi_i)(\alpha_i)\Big) \cap \mathrm{Re}(\xi_1)^{-1}([c_{1,j-1}, c_{1j}]) =: \bigcup_{\alpha \in \{-,0,+\}^{d-1}} P_{j\alpha},$$

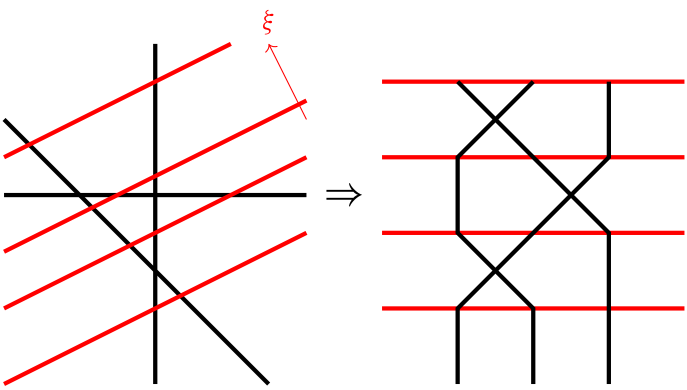

Figure 11: Decomposing 2-dimensional pair-of-pants into sectors.

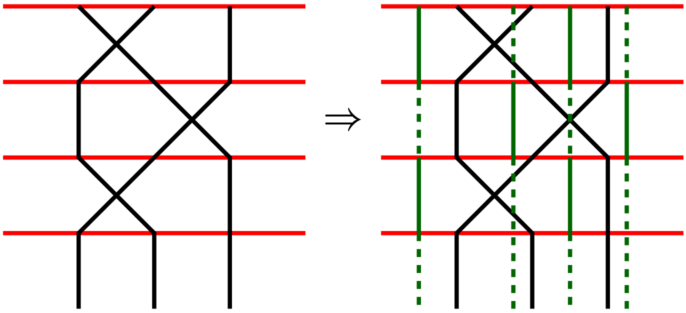

Figure 12: Second cut of the P_j into standard pieces.

where $\alpha = (\alpha_2, \dots, \alpha_d)$ and

$$\begin{aligned}\operatorname{Re}(\xi_i)(-) &= \operatorname{Re}(\xi_i)^{-1}((-\infty, c_{i,2j-1}]),\\ \operatorname{Re}(\xi_i)(0) &= \operatorname{Re}(\xi_i)^{-1}([c_{i,2j-1}, c_{i,2j}]),\\ \operatorname{Re}(\xi_i)(+) &= \operatorname{Re}(\xi_i)^{-1}([c_{i,2j}, +\infty)).\end{aligned}$$

An intuitive picture in dimension two is shown in Figure 12, where the green lines represent the second family of sectorial hypersurfaces that we introduced above, and the dashed parts indicate that we do not use these hypersurfaces to cut other irrelevant pieces.

Now we can apply sectorial descent (Theorem 2.18) with respect to the covering $\{P_{j\alpha}\}$, which gives the following.

3.23 Proposition *The partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ is generated by that of its covering sectors $\mathcal{W}(P_{j\alpha})$, where $1 \le j \le N_{\mathbb{V}}$ and $\alpha \in \{-, 0, +\}^{d-1}$.*

This reduces the generation problem of $\mathcal{W}(M(\mathbb{V}), \xi)$ to that of the Fukaya categories $\mathcal{W}(P_{j\alpha})$.

3f Generation for each piece In this subsection, we study the generation for the wrapped Fukaya categories of the Weinstein subsectors $P_{j\alpha} \subset M(\mathbb{V})$ defined in the last subsection. Notice that there are two types of such subsectors: those containing a crossing (ie a 0-dimensional intersection point formed by d complex hyperplanes in $\mathbb{V}$), denoted simply by $P_\times$, and those do not contain a crossing, which we denote by $P_|$.

3.24 Proposition (i) *The Liouville sector $P_\times$ is symplectomorphic to*

$$\Big((\mathbb{C}^*)^d, \bigcup_{i=1}^{d} (f_i)^{-1}(c_i^{\pm})\Big),$$

where $c_i^{\pm} \in \{c_{ik} \mid 1 \le k \le 2N_{\mathbb{V}}\}$ and f_i is the pullback of ξ_i under the symplectic embedding $P_\times \hookrightarrow (\mathbb{C}^*)^d$.

(ii) *The other Liouville sectors $P_|$ are symplectomorphic to the stabilizations*

$$(P_| \cap (\xi_1)^{-1}(c_1^+)) \times T^*[c_1^-, c_1^+],$$

where $c_1^\pm \in \{c_{1j} : 1 \le j \le N_{\mathbb{V}}\}$.

3.25 Remark It follows from (i) of the proposition above that the wrapped Fukaya category $\mathcal{W}(P_\times)$ is generated by a cotangent fiber of $(\mathbb{C}^*)^d \cong T^*T^d$, together with the linking disks associated to the stops $(\xi_i)^{-1}(c_i^\pm)$. The cotangent fiber $L \subset P_\times$ will be called a *standard Lagrangian* associated to $P_\times$. Order the crossing points from the bottom to the top according to the values of $\mathrm{Re}(\xi)$, the sector $P_\times$ corresponds to a unique crossing point, say the j^{th} crossing, and we will write the corresponding standard Lagrangian as L_j.

Note that here we use the term "standard Lagrangian" so that it is consistent with the term used in geometric representation theory, where the standard objects in the hypertoric category $\mathcal{O}$ (see [14]) correspond exactly (by our main theorem) to these Lagrangians.

On the other hand, statement (ii) of the proposition above implies that the wrapped Fukaya category $\mathcal{W}(P_\alpha) \cong \mathcal{W}\big((\xi_1)^{-1}(c_1^+) \cap P_\alpha\big)$ is generated by the linking disks associated to the stop $P_\alpha \cap (\xi_1)^{-1}(c_1^+)$.

Proof of Proposition 3.24(i) Let $J \subset \{1, \dots, n\}$ be the set of hyperplanes going through the dimension 0 crossing point contained in $P_\times$. We apply Lemma 3.7 to get a corresponding symplectic structure on $P_\times$, regarded as a (codimension 0) symplectic submanifold of $M(\mathbb{V}[J])$, where $\mathbb{V}[J]$ is the hyperplane arrangement whose first d members are coordinate hyperplanes. Recall also the symplectomorphism $\Phi\colon M(\mathbb{V}[J]) \to M(\mathbb{V})$ from Lemma 3.7. Consider the symplectic isotopy

$$\omega_t := 4\begin{pmatrix} 0 & D + (1-t)A^T A \\ -D - (1-t)A^T A & 0 \end{pmatrix},$$

whose derivative (with respect to t) is

$$\dot{\omega}_t = 4\begin{pmatrix} 0 & -A^T A \\ A^T A & 0 \end{pmatrix} = -4d\bigg(\sum_{j \notin J}\Big(1 - \frac{1}{|\ell_j|^4}\Big)\big(\mathrm{Re}(\ell_j)d\,\mathrm{Im}(\ell_j) - \mathrm{Im}(\ell_j)d\,\mathrm{Re}(\ell_j)\big)\bigg) = d\theta_t.$$

Let V be the symplectic dual to the 1-form θ_t; we need to prove that V is complete.

Consider a partial compactification $M(\mathbb{V}_J)$ of $M(\mathbb{V})$, where $\mathbb{V}_J$ is hyperplane arrangement obtained by removing those with indices in J. The stratification of the complement $M(\mathbb{V}_J) \setminus M(\mathbb{V})$ is determined by equations $\{\ell_i = 0 \mid \forall i \in J'\}_{J' \subset J}$. With respect to the symplectic structure ω_t, the vector field V is given by the formula

$$V = -[d\phi_J](D + (1-t)A^T A)^{-1}\begin{pmatrix} \dfrac{\partial}{\partial \vec{x}} \\ \dfrac{\partial}{\partial \vec{y}} \end{pmatrix}, \tag{28}$$

where

$$\phi_J = \sum_{j \notin J} \left(|\ell_j|^2 + \frac{1}{|\ell_j|^2} \right)$$

is the squared sum and $[d\phi_J]$ is the matrix corresponding to its differential, which is given by

$$d\phi_J = \sum_{j \notin J} \sum_{i=1}^{d} \left(1 - \frac{1}{|\ell_j|^4} \right) \mathrm{Re}(\ell_j) a_{ij} \, dx_i .$$

In the sector $P_\times$, we have $|\ell_j|$ bounded away from zero, and $\mathrm{Re}(\ell_j)$ is bounded for all $j \notin J$. If $|z_i| \to 0$ for some $1 \le i \le d$, the denominator of the coefficients of ∂_{x_i} and ∂_{y_i} in the expression (28) go to ∞, so the vector field V extends to a vector field over the open strata $\{z_i = 0\}$ and is tangent to it. To show the completeness of V, it suffices to look at the case when some $\mathrm{Im}(\ell_i) \to \infty$. Under this limit, the coefficient of ∂_{x_i} is always bounded. Since all the $|\ell_j|$ are bounded away from 0, for any sequence of points in $\Phi(P_\times)$ converging to any strata at infinity,

$$[d\phi_J](D + (1-t)A^T A)^{-1}$$

converges. It follows that near infinity, we have

$$\|[d\phi_J](D + (1-t)A^T A)^{-1}\| \le M$$

for some constant $M > 0$, which then implies that the coefficient of ∂_{y_i} is controlled by $CM|z|$, where $C > 0$ is another constant. It is a consequence of Grönwall's inequality [45, Theorem 2.8] that the vector field V is complete. Thus the time-1 flow of V induces a symplectic embedding $\phi_V^1 \colon \Phi(P_\times) \hookrightarrow (\mathbb{C}^*)^d$, which can be extended to an exact symplectomorphism. It implies that

$$\mathcal{W}(P_\times) \simeq \mathcal{W}(\Phi(P_\times)) \simeq \mathcal{W}\left((\mathbb{C}^*)^d, \bigcup_{i=1}^{d} (f_i)^{-1}(c_i^{\pm}) \right)$$

is generated by a cotangent fiber of T^*T^d and linking disks associated to the stops $(\xi_i)^{-1}(c_i^{\pm})$. □

Proof of Proposition 3.24(ii) Recall that in the proof of Lemma 3.21, we only use the fact that the hypersurfaces $\{\mathrm{Re}(\xi_1)^{-1}(c_1^{\pm})\}$ intersect with the removed hyperplanes transversely, and we cut the Liouville sector so that the sets $\mathrm{Re}(\xi_1)^{-1}(r)$, for $c_1^- \le r \le c_1^+$, intersect the removed hyperplanes in the closure of P_α transversely, so we conclude that for every $c_1^- \le r \le c_1^+$, there exists a product decomposition of the Z-invariant neighborhood

$$\mathrm{Nbd}^Z(\{\mathrm{Re}(\xi_1) = r\}) \cap P_| \xrightarrow{\simeq} \{\xi_1 = r\} \times T^*[-\varepsilon_r, \varepsilon_r] \cap P_| \tag{29}$$

for some small $\varepsilon_r > 0$. The product decomposition (29) for each r gives us a locally defined vector field $X_{I_i^r}$, for some function $I_1^r \colon \mathrm{Nbd}^Z(\{\mathrm{Re}(\xi_1) = r\}) \cap P_| \to \mathbb{R}$ from Lemma 3.21. The open intervals $(r - \varepsilon_c, r + \varepsilon_c)$ form a covering of $[c^-, c^+]$, so we can find a finite subcollection of the $(c - \varepsilon_c, c + \varepsilon_c)$'s covering the same interval. Let $\{\psi_r\}$ be a partition of unity defined on $P_\alpha = \bigcup_r \mathrm{Nbd}^Z(\{\mathrm{Re}(\xi_1) = r\}) \cap P_|$

that is subordinate to the finite cover, then the vector field $V = \sum_r \psi_r X_{I_1^r}$ satisfies $d\,\mathrm{Re}(\xi_1)(V) = 1$ everywhere and is locally complete by Lemma 3.13. Therefore the vector fields $(X_{\mathrm{Re}(\xi_1)}, V)$ induce the required product decomposition in the second part of the Proposition 3.24. □

Proposition 3.24 together with sectorial descent (Theorem 2.18) imply that the wrapped Fukaya category of $(M(\mathbb{V}), \xi = \xi_1)$ is generated by the union of all standard Lagrangians $\{L_j\}_{j=1}^{N_{\mathbb{V}}}$ and the image under the cup functors $\cup\colon \mathcal{W}(\xi_i^{-1}(c_{ik}) \cap P_{j\alpha}) \to \mathcal{W}(P_{j\alpha}) \to \mathcal{W}(M(\mathbb{V}), \xi_1)$ for all possible i and j (ie the linking disks corresponding to the cornered stops $P_{j\alpha}$). Here we want to rule out the generators coming from the stops $\xi_i^{-1}(c_{ik})$ for $i \geq 2$; see Theorem 3.26. In the next section, contributions to the generating collection of $\mathcal{W}(M(\mathbb{V}), \xi)$ from the stops $\xi_1^{-1}(c_{1j})$ will also be ruled out based on several formulae provided by Lagrangian surgery, so that the wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ will be shown to be generated by the standard Lagrangian submanifolds. In particular, since the standard Lagrangians are quasi-isomorphic to iterated mapping cones of bounded-feasible chamber Lagrangians (**4e-4**), this concludes the proof Theorem 1.1.

We end this subsection with the following theorem.

3.26 Theorem *The partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ is generated by the standard Lagrangians $\{L_j\}_{1 \leq j \leq N_{\mathbb{V}}}$ constructed in Proposition 3.24 and the linking disks associated to the stops $\xi_1^{-1}(c_{1j})$.*

Proof For simplicity, we call a sector $P_{j\alpha}$ of type $\times$ if it contains a crossing point (ie $P_\times$), and of type $|$ if it is a stabilization (ie $P_|$). The wrapped Fukaya categories of sectors of type $|$ are generated by the image of the cup functor $\cup\colon \mathcal{W}(P_{j\alpha} \cap \xi_1^{-1}(c_{1j})) \to \mathcal{W}(P_{j\alpha})$. On the other hand, for each j, the wrapped Fukaya categories of the type $\times$ sector $\mathcal{W}(P_{j0})$ is generated by the distinguished Lagrangian L_j associated to the crossing point together with the images of the cup functors from $\mathcal{W}(P_{j\alpha} \cap \xi_i^{-1}(c_{i,2j-1}))$, $\mathcal{W}(P_{j\alpha} \cap \xi_i^{-1}(c_{i,2j}))$ and $\mathcal{W}(P_{j\alpha} \cap \xi_1^{-1}(c_{1j}))$.

To see that the contributions from the functors $\cup\colon \mathcal{W}(P_{j0} \cap \xi_i^{-1}(c_{i,2j-1})) \to \mathcal{W}(P_{j0}) \to \mathcal{W}(P_j)$ and $\cup\colon \mathcal{W}(P_{j0} \cap \xi_i^{-1}(c_{i,2j})) \to \mathcal{W}(P_{j0}) \to \mathcal{W}(P_j)$ for $i \geq 2$ are redundant, we consider the gluing diagram

$$\begin{array}{ccc} \mathcal{W}(P_{j0} \cap \xi_i^{-1}(c_{i,2j-1})) & \longrightarrow & \mathcal{W}(P_{j,(0,\dots,0,-,0,\dots,0)}) \\ \downarrow & & \downarrow \\ \mathcal{W}(P_{j0}) & \longrightarrow & \mathcal{W}(P_{j0} \cup P_{j,(0,0,\dots,0,-,0,\dots,0)}), \end{array} \tag{30}$$

where the top arrow and the left arrows are cup functors. Since the sector $P_{j,(0,\dots,0,-,0,\dots,0)}$ is of type $|$, the cup functor $\mathcal{W}(P_{j,(\dots)} \cap \xi_1^{-1}(c_{1j})) \to \mathcal{W}(P_{j,(\dots)})$ is a quasiequivalence, and hence images of the top arrow of (30) for $i \geq 2$, are generated by images of Lagrangians in $P_{j,(\dots)} \cap \xi_1^{-1}(c_{1j})$ in $\mathcal{W}(P_{j,(\dots)})$.

By taking homotopy colimit, the wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ is generated by the union of all images of the cup functor $\cup\colon \mathcal{W}(P_{j\alpha} \cap \xi_1^{-1}(c_{1j})) \to \mathcal{W}(P_{j\alpha}) \to \mathcal{W}(M(\mathbb{V}), \xi)$ together with $\{L_j\}_{1 \leq j \leq N_{\mathbb{V}}}$. The remaining task is to compare these cup functors with linking disks associated to $\xi_1^{-1}(c_{1j})$.

Fix j, note that the union $\bigcup_{\alpha\in\{-,0,+\}} P_{j\alpha}\cap\xi_1^{-1}(c_{1j})=\xi_1^{-1}(c_{1j})$ provides a sectorial cover of $\xi_1^{-1}(c_{1j})$, so by sectorial descent (Theorem 2.18) the wrapped Fukaya category $\mathcal{W}(\xi_1^{-1}(c_{1j}))$ is the homotopy colimit of the Fukaya categories $\mathcal{W}(P_{j\alpha}\cap\xi_1^{-1}(c_{1j}))$ via the inclusion functor $\mathcal{W}(P_{j\alpha}\cap\xi_1^{-1}(c_{1j}))\to\mathcal{W}(\xi_1^{-1}(c_{1j}))$. Since the maps of Liouville sectors form a commutative diagram

$$\begin{array}{ccc} P_{j\alpha}\cap\xi_1^{-1}(c_{1j}) & \hookrightarrow & \xi_1^{-1}(c_{1j}) \\ \downarrow & & \downarrow \\ P_{j\alpha} & \hookrightarrow & P_j \end{array}$$

the corresponding functors of wrapped Fukaya categories also fit into a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{W}(P_{j\alpha}\cap\xi_1^{-1}(c_{1j})) & \longrightarrow & \mathcal{W}(\xi_1^{-1}(c_{1j})) \\ \downarrow & & \downarrow \\ \mathcal{W}(P_{j\alpha}) & \longrightarrow & \mathcal{W}(P_j) \end{array}$$

Taking the homotopy colimit for α, we get a diagram

$$\begin{array}{ccc} \operatorname{hocolim}_\alpha \mathcal{W}(P_{j\alpha}\cap\xi_1^{-1}(c_{1j})) & \longrightarrow & \mathcal{W}(\xi_1^{-1}(c_{1j})) \\ \downarrow & & \downarrow \\ \operatorname{hocolim}_\alpha \mathcal{W}(P_{j\alpha}) & \longrightarrow & \mathcal{W}(P_j) \end{array}$$

where the top and bottom functors are quasiequivalences of A_∞-categories, and the diagram is still commutative up to homotopy. The bottom left composition contributes linking disks associated to the sectorial corners to the generating collection, and we can invert the top arrow to see that these generating collections actually lie in the full A_∞-subcategory of $\mathcal{W}(P_j)$ spanned by generating collections of $\mathcal{W}(\xi_1^{-1}(c_{1j}))$. □

4 Exact triangle, surgery and isomorphic objects

In Theorem 3.26, we showed that the partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}),\xi)$ is generated by the standard Lagrangians together with the all the linking disks associated to the stop $\xi_1=\xi$. In this section, we will use surgery exact sequences to rule out the contributions of all the linking disks and show that the standard Lagrangians are generated by Lagrangians associated to bounded feasible chambers of $\mathbb{V}$. We start with a discussion about McLean-type neighborhoods in the special case corresponding to a (complex) hyperplane arrangement, which is a slight modification of [39].

4a A McLean-type neighborhood. Consider a projective completion of $M(\mathbb{V})\subset\mathbb{C}^d$, denoted by $\overline{M}(\mathbb{V})\subset\mathbb{CP}^d$. Let D_∞ denote the complement $\mathbb{CP}^d\setminus\mathbb{C}^d\cong\mathbb{CP}^{d-1}$. By abuse of notation, we will still use $\{H_i\}_{i=1}^n$ to denote the collection of the completions of the complexified hyperplanes in the arrangement $\mathbb{V}$. The intersection loci $\bigcup_{i=1}^n H_i\cap D_\infty$ can be seen as a generic hyperplane arrangement $\mathbb{V}_\infty$ in $\mathbb{CP}^{d-1}$. In general, this contains a singular locus of $\overline{M}(\mathbb{V})$, denoted by $\mathbb{V}_{\mathrm{sing}}$. For example, singular locus will arise when there are parallel hyperplanes in $\mathbb{V}$.

One can desingularize $\overline{M}(\mathbb{V})$ by iterated blow-ups along $\mathbb{V}_{\mathrm{sing}}$, starting from the deepest strata, which we describe as follows. First, blow up $\overline{M}(\mathbb{V})$ along all 0-dimensional strata of the intersections of the hyperplanes H_i in D_∞. The proper transforms of the hyperplanes can only intersect along dimension 1 or higher. We then blow up along 1-dimensional strata of the intersections of proper transforms of the hyperplanes H_i and D_∞, and so on. Since the blow-up loci do not intersect with each other, this iterated blow-up is independent of the choice of order among the strata of the same dimension in $\mathbb{V}_\infty$. Performing this procedure inductively until the $(d-2)^{\mathrm{th}}$ step, we obtain a nonsingular projective variety $X_{\mathbb{V}}$ with a simple normal crossing divisor $D_{\mathbb{V}}$ such that $M(\mathbb{V}) = X_{\mathbb{V}} \setminus D_{\mathbb{V}}$. The divisor $D_{\mathbb{V}}$ is linearly equivalent to

$$H_1 + \cdots + H_n - \sum a_i E_i$$

for some positive integers $a_i \in \mathbb{Z}_{>0}$, where the E_i are exceptional divisors. This is because at each step, there are at least two proper transforms of the hyperplanes H_i intersecting with the blow-up locus, hence the coefficient of the corresponding exceptional divisor of the proper transform of H_i is always negative. It follows that $D_{\mathbb{V}}$ supports an ample divisor, after a rescaling of the coefficients if necessary. The corresponding ample line bundle $\mathcal{L} \to X_{\mathbb{V}}$ has a section s with $s^{-1}(0) = D_{\mathbb{V}}$. We equip $X_{\mathbb{V}}$ with a symplectic form $\omega_X := -dd^{\mathbb{C}} \log \|s\|$, where $\|\cdot\|$ is a metric on $\mathcal{L}$ whose associated curvature form F has the property that iF is a positive $(1,1)$-form.

Let $\omega_{\log}$ be the restriction of ω_X to $M(\mathbb{V})$, and write $M(\mathbb{V})_{\log}$ for $M(\mathbb{V})$ equipped with the 2-form $\omega_{\log}$. We remove a small regular neighborhood N_D of $D_{\mathbb{V}}$ from $X_{\mathbb{V}}$ so that its complement is a Liouville domain with respect to the 1-form $\lambda_{\log} := -d^{\mathbb{C}} \log \|s\|$. Denote its completion by $\widehat{M}(\mathbb{V})_{\log}$, which is a Liouville manifold.

Recall that the hyperplanes H_i are complexification of real hyperplanes $H_{\mathbb{R},i}$; see Definition 3.5. Therefore, the blow-ups considered above can be defined over $\mathbb{R}$ and $X_{\mathbb{V}}$ has an induced real structure. In other words, there is a real algebraic variety $X_{\mathbb{V}}(\mathbb{R})$ such that $X_{\mathbb{V}} = X_{\mathbb{V}}(\mathbb{R}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$, so $X_{\mathbb{V}}$ comes with an antiholomorphic involution (lifting the complex conjugation on $\mathbb{CP}^d$), which we denote by τ. By construction, we have $\tau(D_{\mathbb{V}}) = D_{\mathbb{V}}$. By possibly replacing $\|\cdot\|$ with a τ-equivariant metric on $\mathcal{L}$ (after lifting τ to $\mathcal{L}$), we may assume that $\tau^*\lambda_{\log} = -\lambda_{\log}$ and $\tau^*\omega_X = -\omega_X$. By choosing the neighborhood N_D to be τ-invariant and using the cone coordinates, we can extend τ to an antisymplectic involution over $\widehat{M}(\mathbb{V})_{\log}$, which we continue to denote by τ by abuse of notation.

4.1 Lemma *For any two sufficiently small choices of the neighborhood N_D, the resulting Liouville manifolds $\widehat{M}(\mathbb{V})_{\log}$ are τ-equivariantly symplectomorphic.*

Proof Suppose that $N_{D,1}$ and $N_{D,2}$ are two different choices. There is an even smaller τ-invariant neighborhood $N'_D \subset N_{D,1} \cap N_{D,2}$, so by transitivity of an equivalence relation, it suffices to assume that $N_{D,2} = N'_D \subset N_{D,1}$. In this case, there is a canonical isomorphism between the respective completions, given by the identity outside of $N_{D,1}$, and matches with the Liouville flow starting from the boundary of $N_{D,1}$. It is clearly τ-equivariant. □

4.2 Lemma [39, Lemma 5.18] *$\widehat{M}(\mathbb{V})_{\log}$ is τ-equivariantly convex deformation equivalent to $M(\mathbb{V})$, where the latter is equipped with the Liouville structure specified in Section 3b.*

Proof The fact that $\widehat{M}(\mathbb{V})_{\log}$ is convex deformation equivalent to $M(\mathbb{V})$ is proved in [39, Lemma 5.18]. The proof directly generalizes to the τ-equivariant setting. □

We recall the McLean-type neighborhood theorem.

4.3 Proposition [39, Lemma 5.14] *Let $\widehat{M}(\mathbb{V})_{\log}$ be defined as above. For each $i \in \{1,\dots,n\}$, there is an open neighborhood $U_i \subset \mathbb{C}^d$ of H_i, together with a projection $\pi_i : U_i \to H_i$, such that the following statements are true:*

(i) *For all i, j, we have $\pi_i \circ \pi_j = \pi_j \circ \pi_i$ in $U_i \cap U_j$.*

(ii) *For any $I \subset \{1,\dots,n\}$, the composition of $(\pi_i)_{i\in I}$, which we denote by $\pi_I : U_I := \bigcap_{i\in I} U_i \to H_I := \bigcap_{i\in I} H_i$, has fibers symplectomorphic to*

$$\Bigl(\prod_{i\in I} \mathbb{D}, r_i\,dr_i \wedge d\theta_i\Bigr), \tag{31}$$

where $\mathbb{D}$ is the unit disk centered at the origin, and the intersection of a fiber with $U_I \cap H_j$ is mapped to $\{0\}_j \times \prod_{i\in I\setminus\{j\}} \mathbb{D}$ under the symplectomorphism.

(iii) *For any $j \in I$, the restriction of the map π_j to any fiber $\prod_{i\in I}\mathbb{D}$ of π_I is the natural projection*

$$\prod_{i\in I} \mathbb{D}_\epsilon \to \{0\}_j \times \prod_{i\in I\setminus\{j\}} \mathbb{D},$$

where $\mathbb{D}_\epsilon$ is the disk of radius ϵ centered at the origin.

(iv) *The symplectic parallel transport map of $\pi_I : U_I \to H_I$ has holonomy in $\prod_{i\in I} U(1)$.*

(v) *Let $\tau : \mathbb{C}^d \to \mathbb{C}^d$ and $\tau_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ be complex conjugations, and let $\tau_I : H_I \to H_I$ be the restriction of τ. For any $I \subset \{1,\dots,n\}$, there is an open τ_I-invariant neighborhood B_I of $L_I^\circ := \mathrm{Fix}(\tau_I) \setminus \bigl(\bigcup_{I\subset J} U_J\bigr)$ in H_I, together with a $\bigl(\tau_I \times \prod_{i\in I} \tau_{\mathbb{D}}, \tau\bigr)$-equivariant fiber-preserving symplectomorphism*

$$\Bigl(B_I \times \prod_{i\in I} \mathbb{D}, \omega_{H_I}|_{B_I} + \sum_{i\in I} r_i\,dr_i \wedge d\theta_i\Bigr) \simeq (U_I|_{B_I}, \omega_{M(\mathbb{V})}|_{U_I^\circ}), \tag{32}$$

where $U_I|_{B_I} := \pi_I^{-1}(B_I)$. In particular, it implies that for any sign sequence $\alpha \in \mathscr{F}(\mathbb{V})$ with $U_I \cap L_\alpha \neq \varnothing$, after possibly rotating the disk factors by $e^{\sqrt{-1}\pi}$, the $\bigl(\tau_I \times \prod_{i\in I}\tau_{\mathbb{D}}, \tau\bigr)$-equivariant symplectomorphism sends

$$L_I^\circ \times \prod_{i\in I} \{(r_i,\theta_i) \in \mathbb{D}^* \mid \alpha(i) e^{\sqrt{-1}\theta_i} = 1\},$$

which is an open subset of $\mathrm{Fix}(\tau_I \times \prod_{i\in I}\tau_{\mathbb{D}})$, to $U_I|_{B_I} \cap L_\alpha$.

Proof The proposition can be proved by generalizing [39, Lemma 5.14] to incorporate the antisymplectic involutions. For the sake of completeness, we present here the details of the proof.

Define a total order for subsets of $\{1,\dots,n\}$ as follows: if $|I_1|>|I_2|$, then $I_1<I_2$, and if $|I_1|=|I_2|$ and $\min(I_1\setminus I_2)<\min(I_2\setminus I_1)$, then $I_1<I_2$. We are going to construct U_i, π_i and B_I inductively, starting from its description near H_I for small I and progressing to large I.

We will construct U_i so that $U_I=\varnothing$ if $H_I=\varnothing$, so the base case is when $|I|=d$ and H_I is a point. Define $B_I=H_I$. Using a local $\prod_{i\in I}\tau_{\mathbb{D}}$-invariant metric and the corresponding exponential map, we can find a small neighborhood U_I of the point H_I and a $\bigl(\prod_{i\in I}\tau_{\mathbb{D}},\tau\bigr)$-equivariant diffeomorphism $\phi_I:\prod_{i\in I}\mathbb{D}\to U_I$ such that $\phi_I(\{0\}_j\times\prod_{i\in I\setminus j}\mathbb{D})=U_I\cap H_j$ for all j. Moreover, there are constants $a_i>0$ such that $\phi_I^*\omega_{\log}=\sum_{i\in I}a_ir_i\,dr_i\wedge d\theta_i$. By rescaling ϕ_I factor by factor, we may assume that $a_i=1$ for all i. Since τ is an antisymplectic involution, by possibly shrinking $\prod_{i\in I}\mathbb{D}$, we can isotope ϕ_I to another $\bigl(\prod_{i\in I}\tau_{\mathbb{D}},\tau\bigr)$-equivariant symplectomorphism $\phi_I:\prod_{i\in I}\mathbb{D}\to U_I$ such that $\phi_I\bigl(\{0\}_j\times\prod_{i\in I\setminus j}\mathbb{D}\bigr)=U_I\cap H_j$ and $\phi_I^*\omega_{\log}$ is of the form (31). We define $\pi_i:U_I\to H_I$ by $\phi_I\circ p_i\circ\phi_I^{-1}$, where $p_i:\prod_{j\in I}\mathbb{D}\to\prod_{j\in I\setminus\{i\}}\mathbb{D}$ is the natural projection. Since ϕ_I is $\bigl(\prod_{i\in I}\tau_{\mathbb{D}},\tau\bigr)$-equivariant, it sends the fixed-point set of $\prod_{i\in I}\tau_{\mathbb{D}}$ to that of τ. By possibly rotating the disk factors by $e^{\sqrt{-1}\pi}$, it implies that $\phi_I^{-1}(U_I\cap L_\alpha)$ equals $\prod_{i\in I}\{(r_i,\theta_i)\in\mathbb{D}^*\mid\alpha(i)e^{\sqrt{-1}\theta_i}=1\}$. This completes the base case of the induction.

Let $I\subset\{1,\dots,n\}$. Assume the induction hypothesis holds for all $J<I$. More precisely, it means that we have constructed $\{\pi_i\}_{i=1}^n$ and $\{U_i\}_{i=1}^n$ near H_J such that all the listed properties are satisfied. We need to extend the definitions over a neighborhood of H_I so that all the listed properties continue to hold. In fact, we only need to extend π_i and U_i for $i\in I$ in a neighborhood of H_I.

First note that, as a complete intersection, the structure group of the normal bundle ν_{H_I} of $H_I\subset X_{\mathbb{V}}$ is $\prod_{i\in I}U(1)$. Using a partition of unity, the exponential map and the induction hypothesis, we can construct a diffeomorphism ϕ_I from a polydisk subbundle of N_{H_I} to a neighborhood of H_I such that near $\phi_I^{-1}(H_I\cap H_J)$ for $J<I$, $\phi_I^*\omega_{\log}$ and $\phi_I^*\pi_i$ already satisfy all the listed properties and $\phi_I^*\pi_I$ coincides with the projection $\nu_{H_I}\to H_I$. By an abuse of notation, we denote the polydisk subbundle by ν_{H_I}. Moreover, since every connected component of L_I° is contractible, we can choose a smooth trivialization of $\nu_{H_I}\to H_I$ over a τ_I-invariant neighborhood B_I of L_I°, which is of the form $B_I\times\prod_{i\in I}\mathbb{D}$ (where the structure group $\prod_{i\in I}U(1)$ respects the product decomposition), and assume that ϕ_I is $\bigl(\tau_I\times\prod_{i\in I}\tau_{\mathbb{D}},\tau\bigr)$-equivariant over $B_I\times\prod_{i\in I}\mathbb{D}$.

On the other hand, we can construct a symplectic form ω' on ν_{H_I} such that

- $\omega'=\phi_I^*\omega_{\log}$ near $\phi_I^{-1}(H_I\cap H_J)$, for J less than I,
- over $H_I^\circ:=H_I\setminus\bigl(\bigcup_{J<I}\phi_I^{-1}(H_I\cap H_J)\bigr)$, the bundle $\nu_{H_I}|_{H_I^\circ}\to H_I^\circ$ has fibers diffeomorphic to (31), with the structure group respecting the product structure,
- the symplectic parallel transport map of $\nu_{H_I}|_{H_I^\circ}\to H_I^\circ$ has holonomy in $\prod_{i\in I}U(1)$,
- over the trivialization $B_I\times\prod_{i\in I}\mathbb{D}$, ω' is of the form (32).

The construction is called a "bundle compatible" symplectic form in [39, Lemma 5.14]. Note that, in our setup, the action of $\tau_I \times \prod_{i\in I} \tau_{\mathbb{D}}$ on $B_I \times \prod_{i\in I} \mathbb{D}$ is antisymplectic with respect to ω'.

Since $\omega'|_{H_I^\circ} = \omega_{H_I^\circ}$ and $\omega' = \phi_I^*\omega_{\log}$ near $\phi_I^{-1}(H_I \cap H_J)$ for $J < I$, we can apply Moser's trick to the linear interpolation of ω' and $\phi_I^*\omega_{\log}$. The upshot is that we are able to modify ϕ_I so that $\phi_I^*\omega_{\log} = \omega'$. Inside N_{H_I}, there is a natural projection map corresponding to the i^{th} factor of the structure group, denoted by p_i, which extends $\phi_I^*\pi_i$ from an open neighborhood of $\phi_I^{-1}(H_I \cap H_J)$ to the entire bundle. We can therefore extend U_i, for $i \in I$, to a neighborhood of H_I such that $U_I = \phi_I(N_{H_I})$. The maps π_i for $i \in I$ can also be extended to a neighborhood of H_I by $\phi_I \circ p_i \circ \phi_I^{-1}$. It is immediate to check that they satisfy all the listed properties. □

4b Clean surgery formula and Lagrangian cobordisms In this subsection, we briefly review the work on clean surgery formulae for closed Lagrangian submanifolds [37], and extend it to cylindrical exact Lagrangian submanifolds. Combining this with the recent work of Bosshard [12], we show that clean surgery of cylindrical exact Lagrangian submanifolds will induce exact triangles in the wrapped Fukaya category. We begin with a review of Lagrangian cobordisms and Lagrangian surgeries.

A *Lagrangian cobordism* $V \subseteq \mathbb{C} \times X$ is a cylindrical exact Lagrangian submanifold with respect to $\lambda + \lambda_{\mathbb{C}}$, where λ is the Liouville 1-form on (X, ω, Z) dual to Z, and $\lambda_{\mathbb{C}}$ is the standard Liouville 1-form on $\mathbb{C}$, with the following additional requirements: there exist (sectorial) Liouville subdomains $\mathbb{C}^{\text{int}} \subseteq \mathbb{C}$ and $X^{\text{int}} \subseteq X$ such that

(a) V is cylindrical outside $\mathbb{C}^{\text{int}} \times X^{\text{int}}$,

(b) $V \cap (\mathbb{C} \setminus \mathbb{C}^{\text{int}}) \times X$ consists only of finitely many submanifolds that are cylindrizations $\gamma_j \tilde{\times} L_j$ of product Lagrangian submanifolds of cylindrical rays γ_j in $\mathbb{C}$ and exact Lagrangian submanifolds $L_j \subseteq X$. These cylindrized products are called *ends* of the cobordism,

(c) V is disjoint from $\mathbb{C} \times \partial X$, where ∂X is the sectorial boundary of X.

In the above, the *cylindrization* $\gamma_j \tilde{\times} L_j$ is a deformation of the product $\gamma_j \times L_j$, so that it becomes cylindrical in $\mathbb{C} \times X$. We refer the reader to [25, Section 7.2] for its construction.

4.4 Theorem ([12]; see also [10; 11; 43]) *Let V be a Lagrangian cobordism with ends $L_0, L_1, \ldots, L_n$ (oriented counterclockwise), there is an iterated cone decomposition*

$$L_0 \cong [L_n \to L_{n-1} \to \cdots \to L_2 \to L_1]$$

in the derived wrapped Fukaya category $D^{\text{perf}}\mathcal{W}(X)$.

We recall how to construct Lagrangian cobordisms from surgeries. We follow the terminology in [37]. A function $\nu_\lambda \colon [0, \varepsilon] \to [0, \lambda]$ is called *λ-admissible* if

(1) $\nu_\lambda(0) = \lambda > 0$ and $\nu_\lambda'(r) < 0$ for $r \in (0, \varepsilon)$,

(2) $\nu_\lambda^{-1}(r)$ and $\nu_\lambda(r)$ have vanishing derivatives of all orders at $r = \lambda$ and $r = \varepsilon$, respectively.

The graph of a λ-admissible function is depicted in Figure 13.

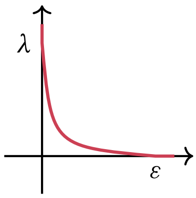

Figure 13: A λ-admissible function.

The next lemma is a generalization of Weinstein neighborhood for cylindrical Lagrangian submanifolds.

4.5 Lemma *Let L be a properly embedded exact Lagrangian submanifold of (X,ω,Z) that is cylindrical near infinity. There exists an open neighborhood W of L that is invariant under the Z-action outside of a compact subset and is symplectomorphic to an open neighborhood of L in T^*L.*

Proof Consider the Lagrangian submanifold with boundary $\bar{L} = L \cap \bar{X}$ in the Liouville domain $\bar{X}$. Then $\partial\bar{L}$ is a closed Legendrian submanifold in the contact manifold $(\partial\bar{X}, \alpha := \lambda|_{\partial\bar{X}})$. By the Legendrian neighborhood theorem [26, Corollary 2.5.9], there is a neighborhood U of $\partial\bar{L}$ such that U is contactomorphic to a neighborhood V of $\partial\bar{L}$ in $J^1(\partial\bar{L}) = \mathbb{R}\times T^*(\partial\bar{L})$ with the canonical contact form α_{can}. Denote the contactomorphism by $f: U \to V$ so we have $f^*\alpha_{\mathrm{can}} = g\alpha$, where $g: U \to \mathbb{R}_{>0}$ is a positive function. The map $F: \mathbb{R}_{>0}\times U \to \mathbb{R}_{>0}\times V$ given by $(r,u)\mapsto(g(u)r, f(u))$ satisfies $F^*d(r\alpha_{\mathrm{can}}) = d((r/g)g\alpha) = d(r\alpha)$ so it is a symplectomorphism between the respective symplectizations. Note that there is a symplectomorphism from the symplectization of $J^1(\partial\bar{L})$ to $T^*(\mathbb{R}_{>0}\times\partial\bar{L})$ which is the identity on $\mathbb{R}_{>0}\times\partial\bar{L}$; see the Lemma 4.6 below. Therefore, the neighborhood $\mathbb{R}_{>0}\times U$ of $\mathbb{R}_{>0}\times\partial\bar{L}$ is symplectomorphic to a neighborhood of the zero section in $T^*(\mathbb{R}_{>0}\times\partial\bar{L})$. Set $W_{>1} := \mathbb{R}_{>1}\times U$, which is a Z-invariant neighborhood of $L\setminus\bar{L}$ and is symplectomorphic to a neighborhood of the zero section in $T^*(\mathbb{R}_{>1}\times\partial\bar{L})$. We can extend $W_{>1}$ to a neighborhood W of L and apply the Moser trick to get a Weinstein neighborhood of L. Since we have already built a symplectomorphism from $W_{>1}$ to a neighborhood of the zero section in $T^*(\mathbb{R}_{>1}\times\partial\bar{L})$, we can ensure that the Moser flow in the Moser trick is compactly supported so the result follows. □

4.6 Lemma *Let K be a closed manifold. Then there is a symplectomorphism from the symplectization $(\mathbb{R}_{>0}\times J^1(K), d(r\alpha_{\mathrm{can}}))$ to the cotangent bundle $T^*(\mathbb{R}_{>0}\times K)$ sending $\mathbb{R}_{>0}\times K$ to the zero section.*

Proof We can write $J^1(K)$ as $\mathbb{R}\times T^*K$ and α_{can} as $-dz+\lambda_{\mathrm{can}}$ where λ_{can} is the canonical Liouville form on T^*K. So

$$d(r\alpha_{\mathrm{can}})) = dz\wedge dr + dr\wedge d\lambda_{\mathrm{can}} + r\,d\lambda_{\mathrm{can}}.$$

Use the obvious diffeomorphism to identify $\mathbb{R}_{>0}\times\mathbb{R}\times T^*K$ with $T^*\mathbb{R}_{>0}\times T^*K = T^*(\mathbb{R}_{>0}\times K)$ so the cotangent bundle symplectic form is given by $dz\wedge dr + d\lambda_{\mathrm{can}}$. For $t\in[0,1]$, let

$$\omega_t := dz\wedge dr + (1-t)(dr\wedge d\lambda_{\mathrm{can}} + r\,d\lambda) + t\,d\lambda_{\mathrm{can}} \quad\text{and}\quad X_t := \frac{1-r}{(1-t)r+t}Z,$$

where Z is the Liouville vector field satisfying $\iota_Z d\lambda_{\text{can}} = \lambda_{\text{can}}$. It is clear that

$$d\iota_{X_t}\omega_t = d((1-r)\lambda_{\text{can}}) = \frac{d}{dt}\omega_t$$

and X_t is integrable. The result follows from the Moser argument. □

Using Lemma 4.5, we can identify an open neighborhood of L in X as a codisk bundle over L of radius $\varepsilon > 0$, with respect to the Riemannian metric on T^*L induced by an appropriately chosen Riemannian metric on L (one can use the symplectomorphism in Lemma 4.6 to show that there is a strictly positive ε even though L is noncompact). Let $D \subseteq L$ be a submanifold that is also cylindrical near infinity, and write $N_L^* D$ to be the conormal bundle of D in T^*L.

4.7 Definition [37, Definition 2.20] We define the *flow handle* $H_\lambda(L, D)$ associated to the pair (L, D) to be the space

$$H_\lambda(L, D) := \{\varphi^{\|\zeta\|}_{\nu_\lambda(\|\zeta\|)}(\zeta) \mid \zeta \in N^*_{L,\varepsilon} D \setminus D\},$$

where $\varphi^{\|\zeta\|}_{\nu_\lambda(\|\zeta\|)}$ is the time-1 Hamiltonian flow of a function $\tilde{\nu}_\lambda(\|\zeta\|)$, with $\tilde{\nu}'_\lambda(r) = \nu_\lambda(r)$.

Let L_1, L_2 be two exact cylindrical Lagrangian submanifolds cleanly intersecting along a submanifold D. In [37], they showed that there is a symplectic automorphism of T^*L_1 so that L_2 becomes $N^*_{L_1} D$ in a given neighborhood of L_1. The *flow surgery* of L_1 and L_2 along D is then the submanifold given by attaching the flow handle $H_\lambda(L, D)$ defined in an open neighborhood U of D to $(L_1 \cup L_2) \setminus U$. It follows from [37, Corollary 2.22 and Lemma 6.2] (see also [12, Lemma 3.10]) that we have:

4.8 Proposition *Let $L = L_1 \#_D L_2$ be the flow surgery of L_1 and L_2 along D. Then there is a Lagrangian cobordism* (*the trace cobordism*) *from L_1 and L_2 to L.*

In particular, taking the flow surgery of two cylindrical Lagrangians $L_1, L_2 \subset X$ along their clean intersection loci D induces an exact triangle

$$L_1 \to L_1 \#_D L_2 \to L_2 \xrightarrow{+1} L_1[1]$$

in $D^{\text{perf}}\mathcal{W}(X)$.

4c Gluing adjacent chambers We keep the notation in the previous sections and consider the Weinstein manifold $(M(\mathbb{V}), \omega)$ with the symplectic form ω defined in Section 3b. The real locus $M(\mathbb{V}) \cap \mathbb{R}^d$ consists of (the interiors of) feasible chambers $\{\Delta_\alpha\}_{\alpha \in \mathscr{F}(\mathbb{V})}$ (cf Section 3a). It is straightforward that the interior of each chamber Δ°_α, as a connected component of the real locus, is a Lagrangian submanifold of $(M(\mathbb{V}), \omega)$. We call such a Lagrangian submanifold a *chamber Lagrangian*, denoted by L_α. Also, such Lagrangians are preserved under the (complete) convex deformation from $(M(\mathbb{V}), \omega)$ to $(M(\mathbb{V}), \omega_{\log})$ in Lemma 4.2, since they are τ-equivariant. We shall abuse notation and denote the corresponding Lagrangians in $(M(\mathbb{V}), \omega_{\log})$ by the same notation L_α.

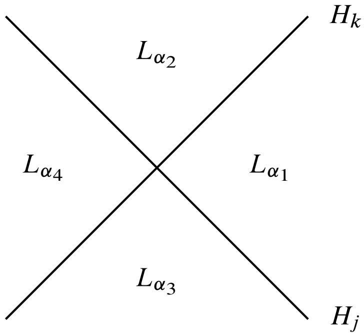

Figure 14: Lagrangians $L_{\alpha_1}, L_{\alpha_2}, L_{\alpha_3}, L_{\alpha_4}$.

For two sign sequences $\alpha_1, \alpha_2 \in \mathscr{F}(\mathbb{V})$, we say the chamber Lagrangian L_{α_1} is adjacent to L_{α_2} if α_1 differs from α_2 only at one element. By Lemma 4.2 and Proposition 4.3, we can glue two adjacent chamber Lagrangians via clean surgery. Suppose that L_{α_1} and L_{α_2} are two adjacent chamber Lagrangians with $\alpha_1(k) \neq \alpha_2(k)$ for some $1 \leq k \leq n$. Take a neighborhood U_k of H_k as in Proposition 4.3 and a fiberwise Hamiltonian function $\rho(r)$ supported in U_k with Hamiltonian vector field

$$X_{\rho(r)} = \frac{\rho'(r)}{r}\frac{\partial}{\partial\theta},$$

where (r,θ) are polar coordinates on the disk fibers of $U_k \to H_k$. Let $\widetilde{L}_{\alpha_1} := \phi^1_{X_{\rho(r)}}(L_{\alpha_1})$ be the time-1 Hamiltonian flow of L_{α_1}. If there is another hyperplane H_j intersecting with H_k and bounding both L_{α_1} and L_{α_2} as depicted in Figure 14, we do not perform a modification along the H_j-coordinate in the McLean-type neighborhood $U_{k,j}$ of the corner $H_{k,j}$. By Proposition 4.3(v), up to rotations in the disk fibers of the McLean neighborhood U_k, the Lagrangians L_{α_1} and L_{α_2} are given by $\mathbb{R}_+ \cap \mathbb{D}$ and $\mathbb{R}_- \cap \mathbb{D}$, respectively, and are identified as the fixed locus of the antisymplectic involution τ_k in the hyperplane H_k. Thus by choosing the fiberwise function $\rho\colon U_k \to \mathbb{R}$ so that $X_{\rho(r)}$ rotates L_{α_1} to L_{α_2} in the subdisk fibers of some shrink of U_k, we can achieve that $\widetilde{L}_{\alpha_1}$ is cylindrical, $\widetilde{L}_{\alpha_1} \cap L_{\alpha_2}$ is connected and the intersection is clean. The clean surgery of $\widetilde{L}_{\alpha_1}$ with L_{α_2} along their intersection, as introduced in the previous section, gives us a new Lagrangian $L_{\alpha_1} \# L_{\alpha_2}$, which is disjoint from a small tubular neighborhood of the hyperplane H_k, and the locus near the other hyperplanes is unchanged. We will consider the image of the Lagrangian $L_{\alpha_1} \# L_{\alpha_2}$ under the (completed) convex deformation from $(M(\mathbb{V}), \omega_{\log})$ back to $(M(\mathbb{V}), \omega)$, and (by the same rule of abusing notation as above) the corresponding Lagrangian in $(M(\mathbb{V}), \omega)$ is still denoted by $L_{\alpha_1} \# L_{\alpha_2}$.

Moreover, by Proposition 4.3, the Lagrangian surgery we considered above is consistent in the following sense: consider four nearby feasible sign sequences $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where α_1 (resp. α_4) differs from α_2 (resp. α_3) at k, and differs from α_3 (resp. α_2) at j. See Figure 14. Over the intersection H_{jk}, we have a polydisk bundle U_{jk} with fiber $(\mathbb{D}^2, r_j\,dr_j \wedge d\theta_j + r_k\,dr_k \wedge d\theta_k)$. With such a choice of fiberwise coordinates, the clean surgery $(L_{\alpha_1} \# L_{\alpha_2}) \# (L_{\alpha_3} \# L_{\alpha_4})$ is Hamiltonian isotopic to $(L_{\alpha_1} \# L_{\alpha_3}) \# (L_{\alpha_2} \# L_{\alpha_4})$.

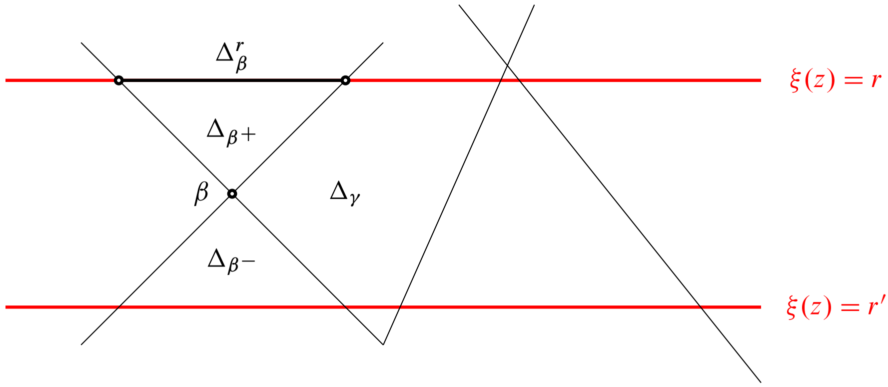

Figure 15: Description of the real locus $P(\mathbb{R})$ of the Liouville sector P.

4.9 Proposition *There is an exact triangle*

$$L_{\alpha_1} \to L_{\alpha_1} \# L_{\alpha_2} \to L_{\alpha_2} \xrightarrow{[1]}$$

in $D^{\mathrm{perf}}\mathcal{W}(M(\mathbb{V}),\xi)$, where L_{α_1} and L_{α_2} are adjacent chamber Lagrangians.

4.10 Remark By possibly replacing $\rho(r)$ with $-\rho(r)$ we can achieve that $L_{\alpha_1} \# L_{\alpha_2} = \widetilde{L}_{\alpha_1} \# L_{\alpha_2}$ lie in the region where $\mathrm{Im}(\xi) \geq 0$.

4d Lagrangians in a Liouville sector

In Section 3e, we constructed the Liouville sectors $\{P_j\}_{1\leq j\leq N_{\mathbb{V}}}$, each of which contains a unique 0-dimensional strata in the hyperplane arrangement $\mathbb{V}$. From now on, we simply write P for such a sector P_j, and denote the 0-dimensional intersection point by β. We write $\xi^{-1}(r) \cup \xi^{-1}(r')$ for the stops coming from the polarization ξ, where $-\infty < r' < r$.

We regard the real locus $P(\mathbb{R}) := P \cap \mathbb{R}^d$ as the union of the (open) chambers $\Delta_\bullet^\circ$ bounded by the hyperplanes and the stops $\xi^{-1}(r) \cup \xi^{-1}(r')$. For later purposes, we denote by $\Delta_{\beta+}$ (resp. $\Delta_{\beta-}$) the chamber with β as one of its vertices and the real projection of $\xi^{-1}(r)$ (resp. $\xi^{-1}(r')$) as its facet. We also denote the facet $\Delta_{\beta+} \cap \xi^{-1}(r)$ of $\Delta_{\beta+}$ (resp. the facet $\Delta_{\beta-} \cap \xi^{-1}(r')$ of $\Delta_{\beta-}$) by Δ_β^r (resp. $\Delta_\beta^{r'}$). See Figure 15.

We describe the properly embedded Lagrangians in P (disjoint from the sectorial boundary) lying over the interior of each chamber $\Delta_\bullet$. First, consider the case $\Delta_\bullet = \Delta_{\beta+}$. We further cut P into a union of cornered Liouville sectors described in Section 3e and use the same notation. By Proposition 3.24, the cornered Liouville sector $P = P_\times$ is symplectomorphic to $\big((T^*T^d), \bigcup_{i=1}^d (f_i)^{-1}(c_i^\pm)\big)$, where f_i is the pullback of the functions $\mathrm{Re}(\xi_i)$ and $\xi_1 = \xi$. Since this symplectomorphism preserves the real locus, the chamber $\Delta_{\beta+}^\circ$ gets mapped to a cotangent fiber (or equivalently, standard Lagrangian) in T^*T^d under this symplectomorphism, denoted by $T^*_{\beta+}T^d$. However, the cotangent fiber $T^*_{\beta+}T^d$ intersects the sectorial hypersurface $(f_1)^{-1}(c_1^+)$ corresponding to $\xi^{-1}(r)$, hence one needs to modify it near the stop $F_{c_1^+} \subset (f_1)^{-1}(c_1^+)$ to get an object in the wrapped Fukaya category $\mathcal{W}(P)$. Note that $T^*_{\beta+}T^d$ is

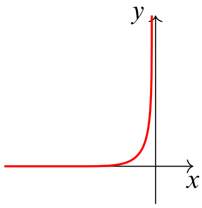

Figure 16: The function ρ.

disjoint from other added stops by our assumptions on $P_\times$. Choose a product decomposition of a small neighborhood of the sectorial hypersurface

$$(R_1, I_1)\colon \mathrm{Nbd}\big((f_1)^{-1}(c_i^{\pm})\big) \cong F_{c_1^{\pm}} \times \mathbb{C}_{-\epsilon<\mathrm{Re}(z)\le 0},$$

and take the Hamiltonian vector field $X_{\rho(R_1)}$, where $\rho(x)$ is a smooth function that is 0 near $x=-\varepsilon$ and equals x^α for some $-1<\alpha<0$ near $x=0$; see Figure 16. We can slightly perturb $T^*_{\beta^+}T^d$ by pushing it off along $X_{\rho(R_1)}$. Note that the function R_1 is the pullback of $\mathrm{Re}(\xi_1)$ under the symplectomorphism, and we know that

$$X_{\mathrm{Re}(\xi_1)} = \tfrac{1}{4}\vec{b}^T(D+A^TA)^{-1}\frac{\partial}{\partial\vec{y}},$$

which is bounded as $|z_j|, |\ell_k| \to 0$. Hence the vector field $X_{\rho(R_1)}$ is also complete. Define $L^{\pm}_{\Delta_{\beta^+}} \subset P_\times$ to be the corresponding Lagrangian submanifold under the perturbation of $X_{\rho(R_1)}$, where the sign $\pm$ indicates that the cotangent fiber $T^*_{\beta^+}T^d$ is pushed off to a positive (resp. negative) direction of $X_{\rho(R_1)}$. When it is clear from the context, we will simply denote these Lagrangian submanifolds as $L^{\pm}_{\beta^+}$. Note that the Lagrangian $L^{+}_{\beta^+}$ (resp. $L^{-}_{\beta^+}$) lies in the area where $\mathrm{Im}(\xi)\ge 0$ (resp. $\mathrm{Im}(\xi)\le 0$).

More generally, for any other chamber $\Delta_\gamma \subset P(\mathbb{R})$ which is different from $\Delta_{\beta^\pm}$, it intersects nontrivially with other sectorial hypersurfaces $(f_i)^{-1}(c_i^{\pm})$. In this case, we choose a compatible product decomposition of the sectorial hypersurfaces

$$(R_i, I_i)\colon \mathrm{Nbd}\big((f_i)^{-1}(c_i^{\pm})\big) \cong F_{c_i^{\pm}} \times \mathbb{C}_{-\epsilon<\mathrm{Re}(z)<\epsilon}$$

and take the Hamiltonian vector field of $R=\sum R_i$. Then a small perturbation of the cotangent fiber $T^*_\gamma T^d$ along $X_{\rho(R)}$ defines a Lagrangian $L^{\pm}_{\Delta_\gamma} \subset P_\times$ as before. Similar constructions work for the other Liouville subsectors $P_|$ by Proposition 3.24(ii), therefore we can glue these Lagrangians via surgery to obtain a globally defined Lagrangian $L^{\pm}_{\Delta_\gamma} \subset (M(\mathbb{V}),\xi)$. We will simplify the notation and denote it by $L^{\pm}_\gamma$. Again, the convention here is that L^{+}_γ lies in the area where $\mathrm{Im}(\xi)\ge 0$.

The following lemma will be used frequently in the next subsection.

4.11 Lemma *Let L be an exact cylindrical Lagrangian submanifold in $(M(\mathbb{V}),\xi)$ whose closure $\overline{L}$ in $\mathbb{C}^d$ does not contain any 0-dimensional intersections of the hyperplanes in $\mathbb{V}$, and it lies entirely in the half-space $\{\mathrm{Im}(\xi)\ge 0\}$, then L is the zero object in the derived Fukaya category $D^{\mathrm{perf}}\mathcal{W}(M(\mathbb{V}),\xi)$.*

Proof Note that the vector field $X_{\mathrm{Re}(\xi)}$ is purely imaginary and only converges to 0 when z goes to the 0-dimensional strata in the hyperplane arrangement. Flowing along $X_{\mathrm{Re}(\xi)}$, L is sent to a subspace $\{\mathrm{Im}(z) \geq c\}$ for some constant $c > 0$. It follows that we can use Hamiltonian isotopy to move L away from any compact subset of $(M(\mathbb{V}), \xi)$, which implies that L is the zero object by [25]. □

4e Chamber moves In this subsection, we prove several results about chamber moves that will be used in the proof of Theorem 1.1.

4e-1 Wrapping exact triangles Let $L_\beta^\pm = L_{\beta+}^\pm$ be the Lagrangians constructed in Section 4d. We want to describe an exact triangle of Lagrangians

$$L_\beta^+ \to L_\beta^- \to D_\beta \xrightarrow{[1]} \tag{33}$$

in the wrapped Fukaya category $\mathcal{W}^{\mathrm{perf}}(P)$, where D_β is the linking disk corresponding to the facet Δ_β^r; see Figure 15. A similar exact triangle was proved in [25], which we will refer to as the "wrapping exact triangle". In our situation, we do not know a priori whether L_β^+ is obtained from L_β^- via wrapping through the stop, so we cannot directly apply the result in [25]. Instead, we will prove (33) using clean surgery described in Section 4b. Consider two Lagrangians L_β^+ and L_β^- intersecting along the interior of a chamber in the real loci, except near the sectorial boundary, where they are shifted upwards and downwards in the direction of $\mathrm{Im}(\xi)$, respectively, and denote the clean surgery of them by $L_\beta^+ \# L_\beta^-$. Near the sectorial boundary that is identified with $\xi^{-1}(r) \times T^*[-\epsilon, 0]$, the Lagrangian $L_\beta^+ \# L_\beta^-$ cleanly intersects with $L_{\Delta_\beta} \subset P(\mathbb{R})$ along $L_{\Delta_\beta^r} \times \{-\epsilon/2\}$; see Figure 15. This follows from our choice of the product structure near the sectorial boundary, where $R = \mathrm{Re}(\xi)$. By definition, one can see that $L_\beta^+ \# L_\beta^-$ can be identified with the linking disk associated to the stop $\xi^{-1}(r)$.

4e-2 Flipping Lagrangians The Lagrangian submanifold $L_{\beta+}^+$ is Hamiltonian isotopic to $L_{\beta-}^-$, since by construction, the corresponding Lagrangians in T^*T^d are Hamiltonian isotopic to each other. Similarly, $L_{\beta+}^-$ is Hamiltonian isotopic to $L_{\beta-}^+$.

4e-3 Identifying linking disks Let $\Delta_\gamma \subset P(\mathbb{R})$ be a chamber adjacent to both $\Delta_{\beta+}$ and $\Delta_{\beta-}$. To this chamber, we have associated two Lagrangians $L_\gamma^\pm$ (cf Section 4d), and two linking disks D_γ and $D_{\gamma'}$ (cf 4e-1). An illustration is given in Figure 17. We claim that $D_{\gamma'}$ is generated by D_γ and D_β in $\mathcal{W}(P)$.

We have proved in 4e-1 that the Lagrangian $L_\gamma^+ \# L_\gamma^-$ is Hamiltonian isotopic to $D_\gamma \oplus D_\gamma'$. Moreover, we have $L_{\beta+}^+ \# L_{\beta+}^- \sim D_\beta$. By our discussions in Section 4c, we can glue the Lagrangians $L_{\beta+}^+$ and L_γ^+ via surgery. It follows from Lemma 4.11 that the Lagrangian $L_{\beta+}^+ \# L_\gamma^+$, which is disjoint from the crossing β, can be pushed off to infinity, hence becoming the zero object in the derived Fukaya category $D^{\mathrm{perf}}\mathcal{W}(P)$. A parallel argument holds for $L_{\beta+}^-$ and L_γ^-, which implies that D_γ' is generated by D_γ and D_β.

On the other hand, if a chamber Δ_γ is disjoint from the crossing β, the linking disks D_γ and D_γ' can be identified by the decomposition argument; see Proposition 3.24.

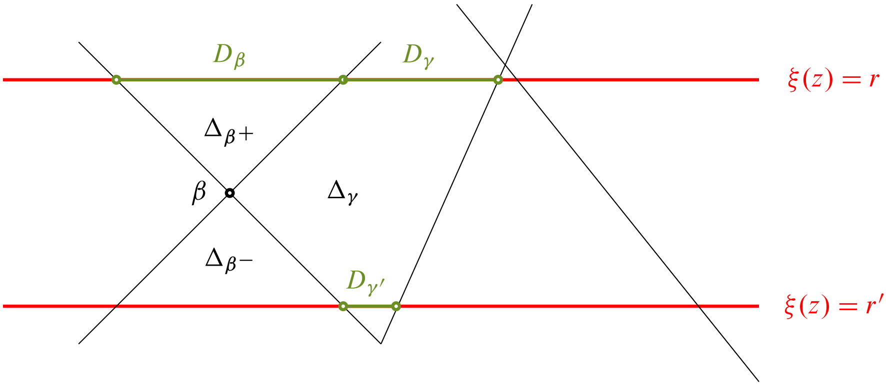

Figure 17: Linking disks in $\mathcal{W}(P)$.

4e-4 Decomposing chamber Lagrangians Let $I_\beta \subset \{1, \dots, n\}$ be the index set of the hyperplanes in $\mathbb{V}$ passing through the crossing β. Consider the corresponding polarized hyperplane arrangement $\mathbb{V}_\beta$, which consists of d hyperplanes $\{H_i\}_{i \in I_\beta}$ in $\mathbb{R}^d$ by the simplicity of $\mathbb{V}$. In other words, $\mathbb{V}_\beta$ is the iterated deletion of $\mathbb{V}$ by the hyperplanes $\{H_j\}_{j \notin I_\beta}$; see Section 6a. There exists a unique sign sequence $\tilde{\beta}\colon I_\beta \to \{+, -\}$ which singles out the bounded and feasible chamber of $\mathbb{V}_\beta$ with respect to the polarization ξ. In the Liouville sector P, this chamber is given by $\Delta_{\beta-}$. Let $L_{\beta-} \subset M(\mathbb{V})$ be the pushforward of $L^-_{\beta-} \subset P$ under the sectorial gluing. Consider the restriction map on the set of sign sequences $\mathrm{Res}_{I_\beta}\colon \{\pm\}^{|I|} \to \{\pm\}^{|I_\beta|}$, there is a collection of chamber Lagrangians $\{L_\alpha\}$ with $\mathrm{Res}_{I_\beta}(\alpha) = \tilde{\beta}$. We can glue these Lagrangians together via clean surgery to obtain an object in the Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$, which has the form of an iterated mapping cone built out of the L_α. We denote the object by $\bigodot_{\alpha:\mathrm{Res}_{I_\beta}(\alpha)=\tilde{\beta}} L_\alpha$. The polarization ξ induces a partial order on the set of sign sequences $\mathrm{Res}^{-1}_{I_\beta}(\tilde{\beta})$: two sign sequences $\alpha_1, \alpha_2 \in \mathrm{Res}^{-1}_{I_\beta}(\tilde{\beta})$ satisfy $\alpha_1 < \alpha_2$ if $d_{\alpha_1\beta} < d_{\alpha_2\beta}$, ie α_2 differs with β in more entries than α_1. We iteratively apply the surgery procedure described in Section 4c such that all clean surgeries are carried from the smaller chamber to the larger one. Note that the result of the surgery belongs to the subset $\{\mathrm{Im}(\xi) \geq 0\}$; see Remark 4.10.

4.12 Proposition *For each 0-dimensional crossing β, the Lagrangian*

$$L := L_{\beta-} \# \bigodot_{\alpha:\mathrm{Res}_{I_\beta}(\alpha)=\tilde{\beta}} L_\alpha$$

obtained by surgery is a trivial object in $D^{\mathrm{perf}}\mathcal{W}(M(\mathbb{V}), \xi)$.

Proof We first perform a surgery between $L_{\beta-}$ and L_β. Since they intersect cleanly, we can perform a clean surgery as we did in Section 4c. Then we glue the result $L_{\beta-} \# L_\beta$ with other chamber Lagrangians with labels in $\mathrm{Res}^{-1}_{I_\beta}(\tilde{\beta})$. As all the glued objects lie in the region $\{\mathrm{Im}(\xi) \geq 0\}$, the resulting Lagrangian L also lies in $\{\mathrm{Im}(\xi) \geq 0\}$. As a surgery, L is also disjoint from the 0-dimensional intersections of the complex hyperplanes and the stop, so it becomes the zero object by Lemma 4.11. ◻

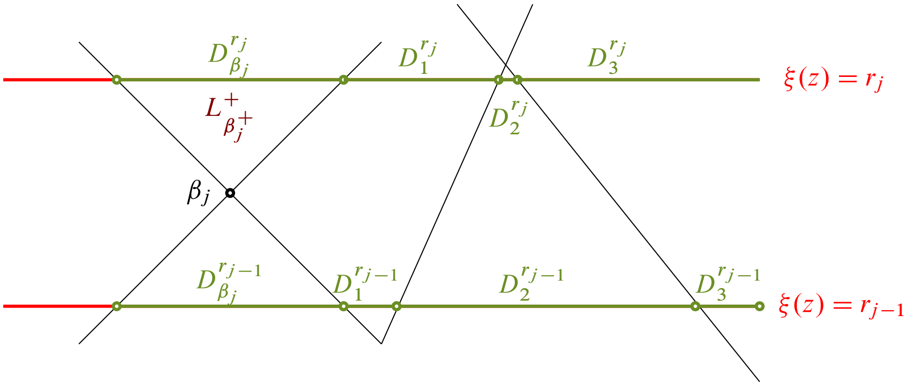

Figure 18: Description of objects in P_j.

4f Proof of Theorem 1.1 We are ready to prove one of our main theorems (Theorem 1.1), restated below for convenience.

4.13 Theorem *The partially wrapped Fukaya category $\mathcal{W}(M(\mathbb{V}),\xi)$ is generated by the collection of chamber Lagrangians $\{L_\alpha\}_{\alpha\in\mathscr{P}(\mathbb{V})}$.*

Let $P_j \subset (M(\mathbb{V}),\xi)$ be the Liouville subsector containing the 0-dimensional intersection β_j defined in Section 3e, where $1 \le j \le N_{\mathbb{V}}$. Due to the genericity of the polarization ξ, the stop $\xi^{-1}(r_j)$ consists of preimages of the open chambers in $\mathbb{R}^{d-1}$ given by the restriction of the chambers of $\mathbb{V}$. To the open chambers in $\xi^{-1}(r_j)$, we associate the linking disks, which are denoted by $D_\bullet^{r_j}$, where the subscript indicates to which open chamber in the stop the linking disk is associated. For example, we write $D_{\beta_j}^{r_j}$ for the linking disk associated to the chamber corresponding to the 0-dimensional stratum β_j. See Figure 18 for a description.

Recall that we have proved in Section 3f that the wrapped Fukaya category $\mathcal{W}(P_j)$ is generated by all the linking disks $D_\bullet^{r_j}$ and the standard Lagrangian $L_{\beta_j^+}^{+}$. When $j = 1$, we can further simplify the set of generating Lagrangians.

4.14 Lemma *The wrapped Fukaya category $\mathcal{W}(P_1)$ is generated by a single chamber Lagrangian $L_{\beta_1}^{+}$.*

Proof We add an auxiliary stop $\xi^{-1}(r')$ for some $-\infty < r' < r_1$, and denote by P_0' the resulting Liouville sector. This gives rise to additional linking disks $D_{\beta_1}^{r'}$ and $D_\bullet^{r'}$, which become trivial objects under the stop removal functor $\mathcal{W}(P_0') \to \mathcal{W}(P_0)$. It is therefore enough to show that the old linking disks $D_{\beta_1}^{r_1}$ and $D_\bullet^{r_1}$ are generated by the Lagrangian $L_{\beta_1^+}^{+}$ and the new linking disks $D_{\beta_1}^{r'}$ and $D_\bullet^{r'}$. First, it follows from our discussions in 4e-1 that $D_{\beta_1^+}^{r'} \cong \mathrm{Cone}(L_{\beta_1^+}^{+} \to L_{\beta_1^+}^{-})$. Applying the flipping argument in 4e-2, this is isomorphic to $\mathrm{Cone}(L_{\beta_1^-}^{-} \to L_{\beta_1^-}^{+}) \cong D_{\beta_1}^{r'}$. It follows that $D_{\beta_1}^{r_1} \simeq D_{\beta_1}^{r'}$ in $\mathcal{W}(P_1)^{\mathrm{perf}}$. Next, take a linking disk $D_1^{r_1}$ adjacent to $D_{\beta_1}^{r_1}$. By 4e-3, $D_{\beta_1}^{r_1} \oplus D_1^{r_1} \simeq D_1^{r'}$. For other linking disks $D_\bullet^{r_1}$, we can directly identify them with the corresponding ones in $\xi^{-1}(r')$ by Proposition 3.24(ii). $\square$

Proof of Theorem 4.13 We prove the theorem by induction. For $i \geq 1$, we write $P_{\leq i}$ for the union of all subsectors P_j with $1 \leq j \leq i$. We claim that the wrapped Fukaya category $\mathcal{W}(P_{\leq i})$ is generated by the chamber Lagrangians $L_\alpha \cap P_{\leq i}$, where $\alpha \in \mathscr{P}(\mathbb{V})$.

When $i = 1$, the claim follows from Lemma 4.14. In other words, by the flipping argument in Sections 4e-2 and 4e-4 (cf Proposition 4.12), we have quasi-isomorphisms $L^+_{\beta_1^+} \cong L^-_{\beta_1^-} \cong L_{\beta_1}$. Suppose that the statement holds for $i = k$. Since the wrapped Fukaya category $\mathcal{W}(P_{\leq k+1})$ can be realized as the sectorial gluing of $\mathcal{W}(P_{\leq k})$ and $\mathcal{W}(P_{k+1})$, it is enough to show that $\mathcal{W}(P_{k+1})$ is generated by the chamber Lagrangians $\{L_\alpha \cap P_{\leq i}\}_{\alpha \in \mathscr{P}(\mathbb{V})}$. First, take the generator $L^+_{\beta_k^+}$, arguing in the same way as in the proof of Lemma 4.14, we have

$$L^+_{\beta^+_{k+1}} \simeq L^-_{\beta^-_{k+1}} \simeq \bigodot_{\alpha:\mathrm{Res}_{I_{\beta_{k+1}}}(\alpha)=\tilde{\beta}_{k+1}} L_\alpha[1],$$

which verifies the generation of $L^+_{\beta^+_{k+1}}$ by the Lagrangians in $\{L_\alpha \cap P_{\leq i}\}_{\alpha \in \mathscr{P}(\mathbb{V})}$. Appling the same argument to $L^-_{\beta^+_{k+1}}$ and $D^{r_{k+1}}_{\beta_{k+1}}$ gives the same conclusion. Moreover, in Lemma 4.14 we have proved that each linking disk $D^{r_{k+1}}_\bullet$ can be identified with the corresponding linking disk $D^{r_k}_\bullet$ up to quasi-isomorphism in $\mathcal{W}(P_{\leq k+1})$. (For an adjacent one, we take the mapping cone with $D^{r_{k+1}}_\beta$, as in 4e-3.) Then the induction hypothesis implies that the linking disks $D^{r_k}_\bullet$ are generated by chamber Lagrangians. □

5 Computing Floer cohomologies

In this section, we compute Fukaya A_∞-algebra of the Lagrangians $\{L_\alpha\}_{\alpha \in \mathscr{P}(\mathbb{V})}$ in $\mathcal{W}(M(\mathbb{V}), \xi)$ and prove Theorem 1.2.

5a The convolution algebra $\tilde{B}(\mathbb{V})$ We first recall the definition of the convolution algebra $\tilde{B}(\mathbb{V})$ associated to a polarized hyperplane arrangement $\mathbb{V} = (V, \eta, \xi)$ introduced in [13]. For $S = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, let u_S be the monomial $u_{i_1} \cdots u_{i_m} \in \mathbb{Z}[u_1, \dots, u_n]$, and let $H_{\mathbb{R},S} \subset V + \eta$ be the intersection $\bigcap_{i \in S} H_{\mathbb{R},i}$ of hyperplanes indexed by the elements of S. For $\alpha, \beta \in \mathscr{P}(\mathbb{V})$, introduce the ring

$$\tilde{R}_{\alpha\beta} := \frac{\mathbb{Z}[u_1, \dots, u_n]}{(u_S, \Delta_\alpha \cap \Delta_\beta \cap H_{\mathbb{R},S} = \varnothing)}.$$

Let $f_{\alpha\beta} \in \tilde{R}_{\alpha\beta}$ be the element corresponding to $1 \in \mathbb{Z}[u_1, \dots, u_n]$. For $\alpha, \beta, \gamma \in \mathscr{P}(\mathbb{V})$, we introduce the notation

$$S(\alpha\beta\gamma) := \{i \in \{1, \dots, n\} \mid \alpha(i) = \gamma(i) \neq \beta(i)\}.$$

Define

$$\tilde{B}(\mathbb{V}) := \bigoplus_{\alpha,\beta \in \mathscr{P}(\mathbb{V})} \tilde{R}_{\alpha\beta},$$

with multiplication

$$f_{\alpha\beta} \cdot f_{\beta\gamma} = u_{S(\alpha\beta\gamma)} f_{\alpha\gamma}, \tag{34}$$

which is extended bilinearly over $\mathbb{Z}[u_1,\dots,u_n]$. The algebra $\widetilde{B}(\mathbb{V})$ admits a $\mathbb{Z}$-grading given by

$$|f_{\alpha\beta}| = d_{\alpha\beta} \quad \text{and} \quad |u_i| = 2,$$

where the number $d_{\alpha\beta} \in \mathbb{N}$ is defined by (8). This grading can be refined to a multigrading by $\mathbb{Z}\langle e_1,\dots,e_n\rangle$, in which case

$$|f_{\alpha\beta}| = e_{i_1} + \dots + e_{i_k} \quad \text{and} \quad |u_i| = 2e_i$$

if β is obtained from α by changing the signs in positions $i_1,\dots,i_k$. Thus $\widetilde{B}(\mathbb{V})$ can be viewed as an algebra over $\mathbb{Z}[u_1,\dots,u_n]$. Note that the single grading on $\widetilde{B}(\mathbb{V})$ is recovered by setting all e_i to be 1.

5.1 Remark In [14, Section 4], the geometric description of this convolution algebra comes from the relative core $\mathfrak{X}$ in the associated hypertoric variety $\mathfrak{M}_{\mathbb{V}}$ associated to the polarized hyperplane arrangement $\mathbb{V}$. Note that $\mathfrak{X}$ is a union of toric varieties $\{X_\alpha\}_{\alpha\in\mathscr{P}(\mathbb{V})}$ associated to each (closed) chamber Δ_α. Take a normalization of $\mathfrak{X}$, denoted by $\pi\colon \widetilde{\mathfrak{X}} \to \mathfrak{X}$. Then the equivariant cohomology $H_T^*(\widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}})$ becomes a $\mathbb{Z}$-graded algebra equipped with the usual convolution product with respect to the orientation on $\widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}}$, twisted by the combinatorial signs for $\mathbb{V}$. It turns out that there is a $\mathbb{Z}$-algebra isomorphism

$$\widetilde{B}(\mathbb{V}) \cong H_T^*(\widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}}).$$

5b Proof of Theorem 1.2 We first compute the partially wrapped Floer cohomologies of the generating chamber Lagrangians $\{L_\alpha\}_{\alpha\in\mathscr{P}(\mathbb{V})}$ in the Liouville sector $(M(\mathbb{V}),\xi)$. To do so, it would be more convenient to view $M(\mathbb{V})$ as the complement of $n+1$ hyperplanes in $\mathbb{CP}^d$. By abuse of notation, we denote these hyperplanes by $H_0, H_1, \dots, H_n \subset \mathbb{CP}^d$, where H_0 is the hyperplane at infinity. After taking the iterated blow-ups as in Section 4a, we obtain a nonsingular projective variety $X_{\mathbb{V}}$ with a simple normal crossing divisor $D_{\mathbb{V}}$ such that $M(\mathbb{V}) = X_{\mathbb{V}} \setminus D_{\mathbb{V}}$. We may write $D_{\mathbb{V}} = \bigcup_{i=0}^{n+m} D_i$, where $D_i = \widetilde{H}_i$ is a strict transform of H_i for $0 \le i \le n$, and the divisors $D_{n+1},\dots,D_{n+m}$ are exceptional divisors. For $I \subset \{0,\dots,n+m\}$, denote by D_I the intersection of the divisors D_i for $i \in I$, and introduce the notation

$$D_I^\circ := D_I \setminus \bigcup_{j\notin I} D_j.$$

As in Section 4a, we can equip $M(\mathbb{V})$ with the structure of a Liouville manifold so that for each $i \in \{0,\dots,n+m\}$ there is a tubular neighborhood U_i of D_i, and the $|I|$-fold intersection of these tubular neighborhoods, $U_I := \bigcap_{i\in I} U_i$, admits the structure of a symplectic disk bundle with structure group $U(1)^{|I|}$ over D_I; see Proposition 4.3. Let $\pi_I : U_I \to D_I$ be the projection map, and let $S_I \to D_I$ be the associated $T^{|I|}$-bundle. Denote by U_I° the restriction of U_I, and by S_I° the restriction of S_I, to the open stratum $D_I^\circ \subset D_I$. As a convention, we set

$$D_\varnothing = X_{\mathbb{V}},\ D_\varnothing^\circ = S_\varnothing = S_\varnothing^\circ = M(\mathbb{V}).$$

In order to compute the wrapped Floer cohomology with respect to the stop defined by the polarization ξ, we need to understand the hypersurface $\mathrm{Re}(\xi)^{-1}(C)$ for $C \gg 1$ in the McLean neighborhood of the divisor $D_{\mathbb{V}}$. Let $\bar{\xi}\colon \mathbb{CP}^d \to \mathbb{C}$ be the homogenization of the polarization $\xi\colon M(\mathbb{V}) \to \mathbb{C}$, and let

$\ell_\infty\colon \mathbb{CP}^d \to \mathbb{C}$ be the linear function defining the hyperplane $H_0 \subset \mathbb{CP}^d$. We have a rational map $\phi\colon \mathbb{CP}^d \dashrightarrow \mathbb{CP}^1$, given by $\phi(z) := [\bar{\xi}(z) : \ell_\infty(z)]$ with the base locus $B = \{\bar{\xi} = 0\} \cap H_0$. Composing ϕ with the blow-up map $\pi\colon X_{\mathbb{V}} \to \mathbb{CP}^d$, we get a rational map $\phi_\pi := \pi \circ \phi\colon X_{\mathbb{V}} \dashrightarrow \mathbb{CP}^1$ with the base locus $\pi^* B$. Notice that $\pi^* B \subset \bigcup_{i \in \{0,n+1,\dots,n+m\}} D_i$, and

$$\phi_\pi^{-1}([1:0]) = \bigcup_{i \in \{0,n+1,\dots,n+m\}} D_i \setminus \pi^* B,$$
$$\overline{\phi_\pi^{-1}([1:0])} = \bigcup_{i \in \{0,n+1,\dots,n+m\}} D_i.$$

In particular, a punctured McLean-type neighborhood $\mathrm{Nbd}^\circ(D_{\mathbb{V}}) := \mathrm{Nbd}(D_{\mathbb{V}}) \setminus D_{\mathbb{V}}$ lies in $X_{\mathbb{V}} \setminus \pi^* B$. It follows that we have a well-defined morphism $\phi_\pi\colon \mathrm{Nbd}^\circ(D_{\mathbb{V}}) \to \mathbb{CP}^1$. By possibly replacing ξ with $\xi + c$ for some $c \in \mathbb{R}$, we may assume that $\xi^{-1}(0) \cap L_\alpha = \varnothing$, so that $\bar{L}_\alpha \cap \pi^* B = \varnothing$ for all $\alpha \in \mathscr{P}(\mathbb{V})$.

For C sufficiently large and $I \subset \{0, n+1, \dots, n+m\}$, the restriction of the bundle projection

$$\pi_I|_{\xi^{-1}(C) \cap U_I} \colon \xi^{-1}(C) \cap U_I \to D_I$$

is submersive onto its image. In fact, the fibers of $\pi_I|_{\xi^{-1}(C)\cap U_I}$ are diffeomorphic to $T^{|I|-1}$. The closure $\overline{\xi^{-1}(C)}$ in $X_{\mathbb{V}}$ is given by $\overline{\xi^{-1}(C)} = \xi^{-1}(C) \cup \pi^* B$. As a result, $\overline{\xi^{-1}(C)}$ can be written as a union of the image of sections $s_i\colon \pi_i(\xi^{-1}(C) \cap U_i) \to U_i|_{\pi_i(\xi^{-1}(C)\cap U_i)}$ for $i = 0, n+1, \dots, n+m$ such that $s_i(z) = 0$ if and only if $z \in \pi^* B \cap D_i$. To compute the wrapped Floer cohomologies of the chamber Lagrangians with respect to the sectorial hypersurface $\mathrm{Re}(\xi)^{-1}(C)$, it suffices to compute the wrapped Floer cohomologies with respect to the stop $\mathfrak{s}_{\mathbb{V}} := \xi^{-1}(C) = \bigcup_{i=0,n+1,\dots,n+m} \mathrm{im}(s_i) \cap M(\mathbb{V})$.

For $\epsilon > 0$, let $\rho_\epsilon\colon (0,\infty) \to [0,\infty)$ be a smooth function such that

$$\rho_\epsilon(r) = 0 \quad \text{for } r > \epsilon \qquad \text{and} \qquad \rho_\epsilon(r) = \frac{1}{r}. \tag{35}$$

We choose a Hamiltonian $h\colon M(\mathbb{V}) \to \mathbb{R}$ so that it takes the form $h_I = \sum_{i \in I} \rho_\epsilon(h_i)$ in the tubular neighborhood U_I° for each I, where each h_i generates the S^1-action which rotates the fibers of $S_i^\circ \to D_i^\circ$. It follows that the Hamiltonian orbits of X_h near D_I° are given by orbits of the circle actions, therefore they appear in families and the collection of them can then be identified with the torus fibers of $S_I^\circ \to D_I^\circ$. To make sure that the Hamiltonian chords between the chamber Lagrangians are nondegenerate, we further perturb the Hamiltonian h with a C^2-small Morse function on the chamber $\Delta_\alpha \subset \mathbb{R}^d$, which reaches its maximum at the corners (ie 0-dimensional strata) and whose restriction to each stratum $\Delta_\alpha \cap D_I^\circ$ has a single critical point which is a minimum for any $I \subset \{1, \dots, n\}$. Since Δ_α is contractible, such a function is easy to construct. We denote such a (perturbed) Hamiltonian by $\tilde{h}\colon M(\mathbb{V}) \to \mathbb{R}$.

To determine the gradings of the generators of the wrapped Floer cochain complexes, we need to choose a trivialization of the canonical bundle $K_{M(\mathbb{V})}$. The trivialization that we shall use here is the one induced by the restriction of the trivialization of the canonical bundle $K_{\mathbb{C}^d}$. Note that this is different from the grading convention of [34], in the case of higher-dimensional of pair-of-pants, where one uses the trivialization coming from the restriction of that of the canonical bundle of $(\mathbb{C}^*)^d$.

5.2 Proposition *Given $\alpha \in \mathscr{P}(\mathbb{V})$, we have an isomorphism*

$$\mathrm{HW}^*(L_\alpha, L_\alpha) \cong \widetilde{R}_{\alpha\alpha} \tag{36}$$

as $\mathbb{Z}$-graded rings.

Proof The proof is divided into three steps.

Step 1: graded module structure We first show that (36) is an isomorphism of $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-graded modules. For each $I \subset \{1, \dots, n\}$, denote the local coordinates on the fibers of the symplectic disk bundle $U_I^\circ \to D_I^\circ$ by $x_{I,i}$, where $i \in I$. Assume that $D_I \cap \Delta_\alpha \neq \varnothing$. By our choice of the wrapping Hamiltonian $\widetilde{h}$, for each $|I|$-tuple of positive integers $(k_i)_{i \in I}$, there is a single nondegenerate time-1 chord of $X_{\widetilde{h}}$ from L_α to itself which wraps k_i times around the hyperplane $D_i \subset X_\mathbb{V}$, which is locally defined by the equation $x_{I,i} = 0$. We label the corresponding generator of $\mathrm{CW}^*(L_\alpha, L_\alpha)$ by $\prod_{i \in I} u_i^{k_i}$, reflecting the fact that it is a product of the generators $u_i^{k_i}$, which wrap k_i times along a single hyperplane D_i, a fact which will be established shortly. By our grading conventions, each generator u_i of $\mathrm{CW}^*(L_\alpha, L_\alpha)$ has degree 2. On the other hand, if $D_I \cap \Delta_\alpha = \varnothing$, then for $i \in I$ there is no time-1 chord of $X_{\widetilde{h}}$ which wraps around each of the hyperplane D_i by a positive number of times. Note also that by our choice of the stop $\mathfrak{s}_\mathbb{V} \subset \partial_\infty M(\mathbb{V})$, no wrapping can occur along the compactifying divisors $D_0, D_{n+1}, \dots, D_{n+m} \subset X_\mathbb{V}$.

Observe that $H_0(L_\alpha; \mathbb{Z})$ injects into $H_0(M(\mathbb{V}); \mathbb{Z})$ so we have the isomorphisms

$$H_1(M(\mathbb{V}), L_\alpha) \cong H_1(M(\mathbb{V}); \mathbb{Z}) \cong \mathbb{Z}^n = \mathbb{Z}\langle e_1, \dots, e_n\rangle, \tag{37}$$

whose generators correspond to meridian loops around the hyperplanes $H_1, \dots, H_n \subset \mathbb{C}^d$. Under this isomorphism, the generator $\prod_{i \in I} u_i^{k_i} \in \mathrm{CW}^*(L_\alpha, L_\alpha)$ represents the (relative) homology class $(k_1, \dots, k_n) \in \mathbb{Z}^n$, where $k_i = 0$ if $i \notin I$. This enables to give a topological $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-grading on the generators of $\mathrm{CW}^*(L_\alpha, L_\alpha)$. As a $\mathbb{Z}$-graded module, $\mathrm{CW}^*(L_\alpha, L_\alpha)$ is supported in even degrees, so the Floer differential vanishes and (36) holds as an isomorphism of $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-graded modules.

Step 2: locality of the Floer product Next we consider the triangle product on the Floer cochain complexes $\mathrm{CW}^*(L_\alpha, L_\alpha)$. Whenever there is a perturbed holomorphic curve contributing to the triangle product

$$\mu^2 \colon \mathrm{CW}^*(L_\alpha, L_\alpha) \otimes \mathrm{CW}^*(L_\alpha, L_\alpha) \to \mathrm{CW}^*(L_\alpha, L_\alpha),$$

the relative homology class of the output Hamiltonian chord must be equal to the sum of those of the input chords. Any generator which appears in the expression of the product of the generators $\prod_{i \in I} u_i^{k_i}$ and $\prod_{i \in I} u_i^{l_i}$, where $(l_i)_{i \in I}$ is another $|I|$-tuple of positive integers, must have grading $\sum_{i \in I}(2k_i + 2l_i)$ and represents the (relative) homology class $(k_1 + l_1, \dots, k_n + l_n) \in \mathbb{Z}^n$ under the isomorphism (37). By our grading convention, we conclude that such a generator must be an integer multiple of $\prod_{i \in I} u_i^{k_i + l_i}$.

To determine the integer coefficients of the Floer product, it is more convenient to modify the wrapping (with the same stop $\mathfrak{s}_{\mathbb{V}}$) according to the generators. Indeed, by [23, Lemma 3.29], the Floer cochain complex $\mathrm{CW}^*(L_\alpha, L_\alpha)$ can be computed as the direct limit $\varinjlim_t \mathrm{CF}^*((L_\alpha)_t, L_\alpha)$ for a cofinal Lagrangian isotopy $(L_\alpha)_{t\geq 0}$ starting from $(L_\alpha)_0 = L_\alpha$. As explained in Step 1, no two generators in $\mathrm{CW}^*(L_\alpha, L_\alpha)$ have the same $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-grading, so if we choose a cofinal Lagrangian isotopy such that no two generators in $\varinjlim_t \mathrm{CF}^*((L_\alpha)_t, L_\alpha)$ have the same $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-grading, then we can guarantee that under the quasi-isomorphism $\mathrm{CW}^*(L_\alpha, L_\alpha) \to \varinjlim_t \mathrm{CF}^*((L_\alpha)_t, L_\alpha)$, a generator of the former cochain complex is sent to the unique generator of the latter cochain complex with the same $\mathbb{Z}\langle e_1, \dots, e_n\rangle$-grading.

We can determine the product $\mu^2\left(\prod_{i\in I} u_i^{k_i}, \prod_{i\in I} u_i^{l_i}\right)$ by working with a local model near $\bigcup_{i\in J} D_i$, where $|J| = d$ and $I \subset J \subset \{1, \dots, n\}$. By relabeling, we may assume that $J = \{1, \dots, d\}$. Consider the map Φ from Lemma 3.7, under the action of which the symplectic form and its primitive become (10) and (11) respectively. Since the difference between (11) and the Liouville 1-form

$$\lambda_0 := -\sum_{i=1}^{d} 2\left(\left(1 - \frac{1}{|z_i|^4}\right)x_i\right)dy_i + \sum_{i=1}^{d} 2\left(\left(1 - \frac{1}{|z_i|^4}\right)y_i\right)dx_i$$

is bounded near $\bigcup_{i\in J} D_i$, by taking a linear interpolation between (11) and λ_0, based on Lemmas 3.11 and 3.13, we can use Moser's trick to show that $M(\mathbb{V})$ can be equipped with a symplectic form that is symplectomorphic to ω, and equals $d\lambda_0$ in a neighborhood N_J of $\bigcup_{i\in J} D_i$.

Note that locally the symplectic structure $d\lambda_0$ is of product type and for each $i \in J$. We can choose a Hamiltonian $\hbar_i$ which is defined to be $\rho_\epsilon(|z_i|)$ (cf (35)) near D_i and has its support in N_j. The Hamiltonian flow of $\hbar = \sum_{i=1}^d \hbar_i$ wraps L_α around $\bigcup_{i=1}^d D_i$. We use a product-type perturbation of the form as before so that for each $(n_1, \dots, n_d) \in \mathbb{N}$, there is exactly one time-1 Hamiltonian chord with multigrading $(n_1, \dots, n_d, 0, \dots, 0)$, where the zeros indicate that there are no wrappings around D_i for $i \notin J$. These Hamiltonian chords correspond to the generators $\prod_{i\in J} u_i^{n_i}$. Since $I \subset J$, we can compute $\mu^2\left(\prod_{i\in I} u_i^{k_i}, \prod_{i\in I} u_i^{l_i}\right)$ under this wrapping. Since $d\ell_i(X_{\hbar_j}) = 0$ when $i, j \in J$ and $i \neq j$, for any Floer triangle $u\colon S \to M(\mathbb{V})$ contributing to $\mu^2\left(\prod_{i\in I} u_i^{k_i}, \prod_{i\in I} u_i^{l_i}\right)$, we can show that the image of u is contained in N_J using the open mapping theorem. See for example [10, Remark 4.4.3]. More precisely, choose strip-like ends

$$\varepsilon_{0,+}, \varepsilon_{1,+}\colon [0,\infty) \to S \quad \text{and} \quad \varepsilon_-\colon (-\infty, 0] \to S$$

and consider the map $\ell_i \circ u\colon S \to \mathbb{C}$. Since

- $\ell_i \circ u$ is holomorphic at $z \in S$ as long as $|\ell_i \circ u(z)| > \epsilon$ (recall that $\hbar_i = \rho_\epsilon(|z_i|)$, so the ℓ_i-projection of the support of the perturbation $\hbar_i$ lies in $\{|z_i| \leq \epsilon\}$),
- the limits $\lim_{s\to\infty} \varepsilon_{j,+}^*(\ell_i \circ u)(s,t)$ for $j = 0, 1$ and $\lim_{s\to-\infty} \varepsilon_-^*(\ell_i \circ u)(s,t)$ exist, where s, t are coordinates on the strip-like ends, and they lie inside the neighborhood $\{|z_i| \leq \epsilon\}$;
- the projections under ℓ_i of the Lagrangian boundaries are contained in $\{|z_i| \leq \epsilon\}$,

we conclude that $\mathrm{im}(\ell_i \circ u) \cap \{|z_i| > \epsilon\} = \varnothing$. In other words, the image of $\ell_i \circ u$ is contained in $\{|z_i| \leq \epsilon\}$.

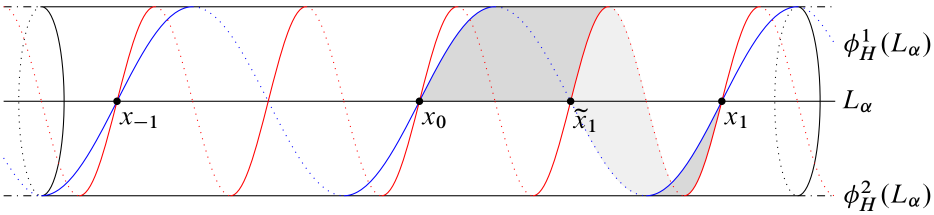

Figure 19: Wrapping in the cylinder, the shaded region is a triangle contributing to the triangle product between x_0 and x_1. Under the identification $\mathrm{CW}^*(L_\alpha, \phi_H^1(L_\alpha)) \cong \mathrm{CW}^*(L_\alpha, \phi_H^2(L_\alpha))$, the generator x_1 maps to $\widetilde{x}_1$; see [7, Figure 12].

Step 3: coefficients of the product structure Near the tubular neighborhood N_J, the wrapping Hamiltonian is modeled on a standard product Hamiltonian in a neighborhood of the origin in $(\mathbb{C}^*)^d$, and the Lagrangian L_α maps locally to one of the orthants in the real locus; see Example 2.14. As argued above, any perturbed holomorphic curve u contributing to the product μ^2 must remain entirely within the local chart N_J. Moreover, the projection of u to each coordinate factor z_i is a perturbed holomorphic curve in a neighborhood of the origin in $\mathbb{C}^*$ with boundary on the appropriate arcs. Conversely, every tuple of index 0 perturbed holomorphic curves in the coordinate factors lifts to an index 0 perturbed holomorphic curve in the local chart N_J. Note that the generator $\prod_{i\in I} u_i^{k_i}$ projects to the generator x_{-k_j} of the wrapped Floer complex of $\mathbb{R}\times\{1\}\subset\mathbb{R}\times S^1$ on the z_j factor; see Figure 19 and the reference therein. On the cylinder, the generators x_{-k_j} and x_{-l_j} are the inputs of a unique triangle contributing to the Floer product, whose output can be identified with $x_{-k_j-l_j}$. We have proved that

$$\mu^2\Biggl(\prod_{i\in I} u_i^{k_i}, \prod_{i\in I} u_i^{l_i}\Biggr) = \prod_{i\in I} u_i^{k_i+l_i},$$

hence the ring structure on $\mathrm{HW}^*(L_\alpha, L_\alpha)$ is as expected. □

5.3 Proposition *Given $\alpha,\beta\in\mathscr{P}(\mathbb{V})$, we have an isomorphism*

$$\mathrm{HW}^*(L_\alpha, L_\beta) \cong \widetilde{R}_{\alpha\beta}\cdot f_{\alpha\beta} \tag{38}$$

as $\mathbb{Z}$-graded $(\mathrm{HW}^*(L_\alpha, L_\alpha), \mathrm{HW}^*(L_\beta, L_\beta))$ and $(\widetilde{R}_{\alpha\alpha}, \widetilde{R}_{\beta\beta})$-bimodules.

Proof We argue as in the proof of Proposition 5.2. As before, if $\Delta_\alpha\cap\Delta_\beta\cap H_I=\varnothing$, which means the closures of L_α and L_β in $X_\mathbb{V}$ do not intersect with each other near U_I°, then there is no generator of $\mathrm{CW}^*(L_\alpha, L_\beta)$ in U_I°.

Now assume that $\Delta_\alpha\cap\Delta_\beta\cap H_I\neq\varnothing$. In this case, the local coordinates $z_i=\ell_i$ define a local projection $U_I\to\mathbb{C}^{|I|}$, under which L_α and L_β map to orthants in the real locus. These orthants correspond to real points whose coordinates have the same signs for those $i\in I$ with $\alpha(i)=\beta(i)$, and have different signs otherwise. Let $K\subset I$ be the subset defined by the elements $i\in I$ with $\alpha(i)=\beta(i)$. Given any tuple $(k_i)_{i\in I}$, where $k_i\in\mathbb{N}$ for $i\in K$, and $k_i\in\mathbb{Z}_{\geq 0}+\frac{1}{2}$ for $i\in I\setminus K$, near D_I° there is a family of time-1 trajectories of X_h from L_α to L_β that wraps k_i times around the hyperplane D_i for each $i\in I$. After perturbing the Hamiltonian slightly from h to $\widetilde{h}$, there is a single nondegenerate such trajectory, and we label the corresponding generator of $\mathrm{CW}^*(L_\alpha, L_\beta)$ by $\prod_{i\in I} u_i^{k_i}$. Note that a key difference between the situation of Proposition 5.2 and the situation here is that some of the k_i are half-integers, and we introduce the notation $f_{\alpha\beta}$ by requiring

$$\prod_{i\in I} u_i^{k_i} = \prod_{i\in I} u_i^{\lfloor k_i \rfloor} \cdot f_{\alpha\beta}.$$

Note that by our grading conventions, the generator u_i has degree 2, therefore $u_i^{1/2}$ has degree 1. As a consequence, $f_{\alpha\beta}$ has degree $d_{\alpha\beta}$. Note that when $I = \{0\}$, L_α and L_β represent the same orthant in the local projection to $\mathbb{C}$, so by our choice of the stop $\mathfrak{s}_{\mathbb{V}}$, wrapping at infinity does not add any new generators to $\mathrm{CW}^*(L_\alpha, L_\beta)$. We have proved that as a graded vector space, $\mathrm{CW}^*(L_\alpha, L_\beta)$ is isomorphic to the right-hand side of (38).

We need to prove the vanishing of the Floer differential. Since L_α and L_β are contractible, by choosing basepoints $*_\alpha \in L_\alpha$, $*_\beta \in L_\beta$ and $* \in M(\mathbb{V})$, and a reference path from $*$ to $*_\alpha$, and from $*$ to $*_\beta$, we can complete any arc connecting L_α to L_β into a closed loop in $M(\mathbb{V})$, which is unique up to homotopy. We can use this to assign classes in $H_1(M(\mathbb{V});\mathbb{Z}) \cong \mathbb{Z}^n$ to the generators of $\mathrm{CW}^*(L_\alpha, L_\beta)$. In fact, in our case it would be more convenient to consider $H_1\big(M(\mathbb{V}); \frac{1}{2}\mathbb{Z}\big) \cong \big(\frac{1}{2}\mathbb{Z}\big)^n$ so that the class associated to the generator $\prod_{i\in I} u_i^{k_i} \in \mathrm{CW}^*(L_\alpha, L_\beta)$ is $(k_1, \dots, k_n) \in \big(\frac{1}{2}\mathbb{Z}\big)^n$, where $k_i = 0$ if $i \notin I$. Any two generators of $\mathrm{CW}^*(L_\alpha, L_\beta)$ related by the Floer differential must represent the same class in $\big(\frac{1}{2}\mathbb{Z}\big)^n$, and their gradings differ by 1, which is impossible unless the differential vanishes identically.

The claim about the module structures is a consequence of the following proposition. □

5.4 Proposition *Indexing the generators of* $\mathrm{HW}^*(L_\alpha, L_\beta)$ *by monomials in* $u_1, \dots, u_n$ *and* $f_{\alpha\beta}$ *as in Proposition 5.3, the Floer product*

$$\mathrm{HW}^*(L_\beta, L_\gamma) \otimes \mathrm{HW}^*(L_\alpha, L_\beta) \to \mathrm{HW}^*(L_\alpha, L_\gamma)$$

coincides with the product in the algebra $\widetilde{B}(\mathbb{V})$.

Proof The argument is similar to the special case of $\alpha = \beta = \gamma$ treated in Proposition 5.2, except for the relation (34) involving half-integer powers of u_i. We again work near $\bigcup_{i\in J} D_i$, where $|J| = d$ and $I \subset J \subset \{1, \dots, n\}$. By a Moser argument as in the proof of Proposition 5.2, the neighborhood N_J of $\bigcup_{i\in J} D_i$ is modeled on the standard product $(\mathbb{C}^*)^d$ and the wrapping Hamiltonian $\sum_{i\in J} \hbar_i$ has been chosen accordingly. Now suppose that $\alpha(j) = \gamma(j) \neq \beta(j)$. On the j^{th} coordinate factor z_j, the Lagrangians L_α, L_β and L_γ project to the arcs $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{R}_+$ in $\mathbb{C}^*$, respectively. The generators $f_{\alpha\beta} \in \mathrm{CW}^*(L_\alpha, L_\beta)$, $f_{\beta\gamma} \in \mathrm{CW}^*(L_\beta, L_\gamma)$ and $f_{\alpha\gamma} \in \mathrm{CW}^*(L_\alpha, L_\gamma)$ project to $x_{-\frac{1}{2}} \in \mathrm{CW}^*(\mathbb{R}_+, \mathbb{R}_-)$, $x_{\frac{1}{2}} \in \mathrm{CW}^*(\mathbb{R}_-, \mathbb{R}_+)$ and $x_{-1} \in \mathrm{CW}^*(\mathbb{R}_+, \mathbb{R}_+)$, respectively, where the wrapped Floer cochain complexes CW^* are taken inside $\mathbb{C}^*$. Since there is a unique holomorphic triangle with inputs $x_{-\frac{1}{2}}$, $x_{\frac{1}{2}}$ and output x_{-1}, we get an additional u_j factor before $f_{\alpha\gamma}$ in the expression of $f_{\alpha\beta} \cdot f_{\beta\gamma}$. In terms of half-integer powers of u_i, this simply corresponds to the identity $u_i^{1/2} \cdot u_i^{1/2} = u_i$. Taking into account all such $j \in J$ proves (34). □

We have proved that the partially wrapped Floer cohomology algebra is isomorphic to $\widetilde{B}(\mathbb{V})$, ie

$$\widetilde{B}(\mathbb{V}) \cong \bigoplus_{\alpha,\beta \in \mathscr{P}(\mathbb{V})} \mathrm{HW}^*(L_\alpha, L_\beta).$$

In order to obtain a statement on the level of A_∞-algebras, we need the following formality result.

5.5 Theorem *The partially wrapped Fukaya A_∞-algebra*

$$\widetilde{\mathcal{B}}_{\mathbb{V}} := \bigoplus_{\alpha,\beta \in \mathscr{P}(\mathbb{V})} \mathrm{CW}^*(L_\alpha, L_\beta)$$

is formal. As a consequence, it is quasi-isomorphic to the $\mathbb{Z}$-graded associative algebra $\widetilde{B}(\mathbb{V})$.

Proof As before, label the generators of $\widetilde{\mathcal{B}}_{\mathbb{V}}$ by monomials in the variables $u_1, \dots, u_n$ with half-integral powers. Suppose that there is an A_∞-operation

$$\mu^j \Bigl(\prod_{i \in I_1} u_i^{k_{1,i}}, \dots, \prod_{i \in I_j} u_i^{k_{j,i}} \Bigr) \in \widetilde{\mathcal{B}}_{\mathbb{V}}$$

with nontrivial output in $\widetilde{\mathcal{B}}_{\mathbb{V}}$. Homological considerations as in the proof of Proposition 5.3 tells us that the right-hand side of the above expression must have the same homology class in $H_1\bigl(M(\mathbb{V}); \tfrac{1}{2}\mathbb{Z}\bigr) \cong \bigl(\tfrac{1}{2}\mathbb{Z}\bigr)^n$ as the sum of the inputs on the left-hand side. This, however, forces the grading of the right-hand side of μ^j to be the sum of the gradings of the inputs. Since μ^j has degree $2-j$, we then conclude that μ^j vanishes unless $j = 2$. □

5.6 Corollary *There is a quasiequivalence between A_∞-categories*

$$\mathcal{W}^{\mathrm{perf}}(M(\mathbb{V}), \xi) \cong \mathrm{Perf}(\widetilde{B}(\mathbb{V})),$$

where $\mathrm{Perf}(\widetilde{B}(\mathbb{V}))$ is the dg category of perfect modules over the graded algebra $\widetilde{B}(\mathbb{V})$.

6 Functoriality

In this section, we study the geometric interpretation of the combinatorial operations, such as deletion and restriction, on the polarized hyperplane arrangement $\mathbb{V}$.

6a Deletion and restriction We recall the notions from [31].

6.1 Definition Let $\mathbb{V} = (\mathbb{R}^d, \eta, \xi)$ be a polarized hyperplane arrangement and $H_{\mathbb{R},i}$ be one of the hyperplanes in $\mathbb{V}$.

(i) The (i^{th}) *deletion* of $\mathbb{V} = (\mathbb{R}^d, \eta, \xi)$ is the polarized hyperplane arrangement

$$\mathbb{V}_i := (\pi_i(\mathbb{R}^d), \pi_i(\eta), \xi \circ \pi_i^{-1}),$$

where $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the i^{th} projection. Equivalently, this is obtained by deleting $H_{\mathbb{R},i}$ from the set of hyperplanes in $\mathbb{V}$.

(ii) The (i^{th}) *restriction* of $\mathbb{V} = (\mathbb{R}^d, \eta, \xi)$ is the polarized hyperplane arrangement

$$\mathbb{V}^i := (\iota_i^{-1}\mathbb{R}^d, \iota_i^{-1}\eta, \iota_i^*\xi),$$

where $\iota_i : \mathbb{R}^{d-1} \to \mathbb{R}^d$ is the inclusion into the i^{th} coordinate hyperplane. Equivalently, this is obtained by taking the intersection with $H_{\mathbb{R},i}$.

Fix i and choose a sign $s \in \{+,-\}$. Given a sign sequence $\alpha \in \mathscr{P}(\mathbb{V}_i)$, we define α_s^i to be the sign sequence with s inserted at i^{th} position. Consider the map

$$\mathrm{del}_i^s : \mathscr{P}(\mathbb{V}_i) \to \mathscr{P}(\mathbb{V}), \quad \alpha \mapsto \begin{cases} \alpha_s^i & \text{if } \alpha_s^i \in \mathscr{P}(\mathbb{V}), \\ 0 & \text{if } \alpha_s^i \notin \mathscr{P}(\mathbb{V}). \end{cases}$$

Similarly, for $\alpha \in \mathscr{P}(\mathbb{V})$, we define $\alpha^{(i)}$ to be the sign sequence with the i^{th} sign removed. Consider the map

$$\mathrm{rest}_i^s : \mathscr{P}(\mathbb{V}) \to \mathscr{P}(\mathbb{V}^i), \quad \alpha \mapsto \begin{cases} \alpha^{(i)} & \text{if } \alpha(i) = s, \\ 0 & \text{if } \alpha(i) \neq s. \end{cases}$$

These maps on the sign sequences induce well-defined graded algebra morphisms between the corresponding convolution algebras.

6.2 Proposition [14; 31] *Let $\mathbb{V}$ be a polarized hyperplane arrangement as above. For each i and sign $s \in \{+,-\}$,*

(i) *the map $\mathrm{del}_i^s : \mathscr{P}(\mathbb{V}_i) \to \mathscr{P}(\mathbb{V})$ induces a graded algebra morphism*

$$\mathrm{del}_i^s : \widetilde{B}(\mathbb{V}_i) \to \widetilde{B}(\mathbb{V}),$$

(ii) *the map $\mathrm{rest}_i^s : \mathscr{P}(\mathbb{V}) \to \mathscr{P}(\mathbb{V}^i)$ induces a graded algebra morphism*

$$\mathrm{rest}_i^s : \widetilde{B}(\mathbb{V}) \to \widetilde{B}(\mathbb{V}^i).$$

Note that these graded algebra morphisms induce natural functors on the categories of perfect modules. For instance, the restriction morphism rest_i^s induces a functor $\mathrm{rest}_i^s : \mathrm{Perf}(\widetilde{B}(\mathbb{V})) \to \mathrm{Perf}(\widetilde{B}(\mathbb{V}^i))$. By Corollary 5.6, there is also an A_∞-functor

$$\mathrm{rest}_i^s : \mathcal{W}(M(\mathbb{V}), \xi) \to \mathcal{W}(M(\mathbb{V}^i), \iota_i^*\xi)$$

given as follows. For any chamber Lagrangian L_α, if $\alpha(i) = -s$, then $\mathrm{rest}_i^s(L_\alpha) = 0$; if $\alpha(i) = s$, then $\mathrm{rest}_i^s(L_\alpha) = (\overline{L}_\alpha \cap H_i) \setminus \bigcup_{j \neq i} H_j$ is the chamber Lagrangian in H_i which is the boundary strata of $\overline{L}_\alpha$. Given two chamber Lagrangians L_α and L_β, we can define a chain map $\mathrm{CW}^*(L_\alpha, L_\beta) \to \mathrm{CW}^*(\mathrm{rest}_i^s(L_\alpha), \mathrm{rest}_i^s(L_\beta))$ as the homomorphism $\mathrm{rest}_i^s : \widetilde{R}_{\alpha\beta} \to \widetilde{R}_{\alpha^{(i)}\beta^{(i)}}$. The formality of the Fukaya A_∞-algebra $\widetilde{\mathcal{B}}_{\mathbb{V}}$, together with the generation of the Fukaya category $\mathcal{W}(M(\mathbb{V}), \xi)$ by chamber Lagrangians ensures that this gives a well-defined A_∞-functor. It follows directly from the construction that:

6.3 Corollary *The functor $\mathrm{rest}_i^s : \mathcal{W}(M(\mathbb{V}), \xi) \to \mathcal{W}(M(\mathbb{V}^i), \iota_i^*\xi)$ is strict.*

If it is clear from the context, we simply write $\iota_i^*\xi$ as ξ. The deletion functor $\mathrm{del}_i^s : \mathcal{W}(M(\mathbb{V}_i), \xi) \to \mathcal{W}(M(\mathbb{V}), \xi)$ can be defined in a similar way. We leave the precise construction for the reader.

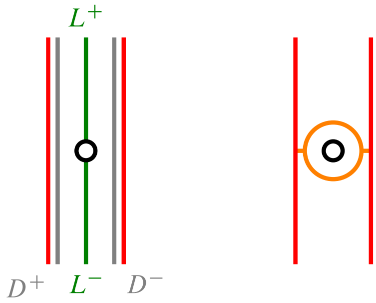

Figure 20: Cocore and linking disks, left, and core, right, of the sector $T^*[-\varepsilon,\varepsilon]^\times$.

6.4 Remark It is interesting to ask whether the restriction functor rest_i^s defined in [31] has an interpretation in terms of symplectic geometry. However, it turned out that all functors we obtained from the restriction of hyperplane arrangements in the following sections are neither rest_i^s nor the adjoints to it. It remains to see whether rest_i^s has geometric interpretation or not.

6b Geometric functors from restriction and deletion In this subsection, we discuss several functors arising from restriction and deletion of polarized hyperplane arrangements. Let $\mathcal{H}_{\mathbb{C}} = \{H_1,\dots,H_n\}$ be the collection of the complexifications of the hyperplanes in $\mathbb{V}$. Fix a hyperplane $H_i \in \mathcal{H}_{\mathbb{C}}$, and let ℓ_i be the defining function of H_i. Then for $\varepsilon > 0$, after a possible Liouville deformation, we have a stopped inclusion

$$\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon]) \cap (M(\mathbb{V}),\xi) \hookrightarrow (M(\mathbb{V}),\xi),$$

where $\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon]) \cap (M(\mathbb{V}),\xi)$ splits into a product

$$\begin{aligned}\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon]) \cap (M(\mathbb{V}),\xi) &\simeq \Big(H_i \setminus \bigcup_{H\in\mathcal{H}_{\mathbb{C}}} H_i\cap H, \xi|_{H_i}\Big) \times (T^*[-\varepsilon,\varepsilon]^\times)\\ &\simeq (M(\mathbb{V}^i),\xi)\times(T^*[-\varepsilon,\varepsilon]^\times),\end{aligned}$$

so the Künneth functor gives us a quasi-isomorphism

$$\mathcal{W}\big(\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon]) \cap (M(\mathbb{V}),\xi)\big) \xrightarrow{\simeq} \mathcal{W}(M(\mathbb{V}^i),\xi)\otimes\mathcal{W}(T^*[-\varepsilon,\varepsilon]^\times).$$

Here we regard $T^*[-\varepsilon,\varepsilon]$ as a Liouville sector with two disjoint sectorial hypersurfaces corresponding to the cotangent fibers at $\pm\varepsilon$, and $T^*[-\varepsilon,\varepsilon]^\times := T^*[-\varepsilon,\varepsilon]\setminus\{(0,0)\}$. Therefore, there are two linking disks in $T^*[-\varepsilon,\varepsilon]^\times$, one for the left end and the other for the right end. We write D^s to be the left or right linking disks, depending on the chosen normal vectors of the hyperplane H_i. Since the normal vector defines the "positive" and "negative" part of the two half-spaces separated by $\mathrm{Re}(\ell_i)^{-1}(0)$, we let s be the corresponding sign.

$T^*[-\varepsilon,\varepsilon]^\times$ has four natural Lagrangians: two generating Lagrangians L^s and two linking disks D^s as depicted in Figure 20, so we get four functors by tensoring any chamber Lagrangians in $\mathcal{W}(M(\mathbb{V}^i),\xi)$ with any one of them (on the level of morphisms, it is given by tensoring with the unit), denoted by

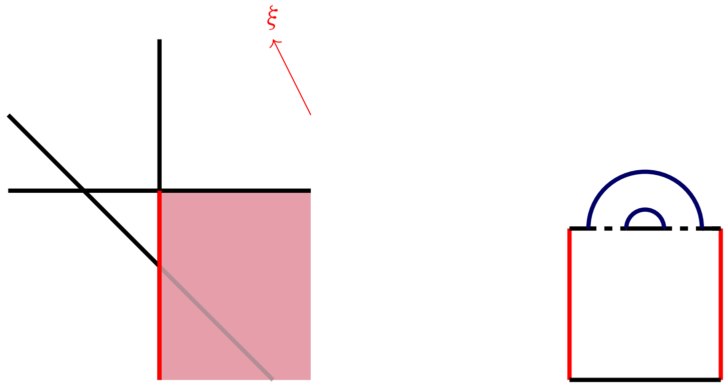

Figure 21: The restriction functor, left, and the shopping bag construction, right.

$\Psi_i^{s,*}\colon \mathcal{W}(M(\mathbb{V}^i),\xi)\to\mathcal{W}\big(\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon])\cap(M(\mathbb{V}),\xi)\big)$. As $\Psi_i^{s,L}$ and $\Psi_i^{s,D}$ would determine $\Psi_i^{-s,D}$ and $\Psi_i^{-s,L}$ via mapping cones, it suffices to compute two of them; without loss of generality, consider $\Psi_i^{+,L}$ and $\Psi_i^{s,D}$. Composing them with the stopped inclusion $\mathcal{W}\big(\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon])\cap(M(\mathbb{V}),\xi)\big)\to\mathcal{W}(M(\mathbb{V}),\xi)$ we obtain the restriction functors

$$\mathrm{Res}_i^{s,*}\colon \mathcal{W}(M(\mathbb{V}^i),\xi)\to\mathcal{W}(M(\mathbb{V}),\xi). \tag{39}$$

Given a standard Lagrangian L_j in $\mathcal{W}(M(\mathbb{V}^i),\xi)$, there exists a unique standard Lagrangian $\bar{L}_j$ in $\mathcal{W}(M(\mathbb{V}),\xi)$, whose closure in $\mathbb{C}^d$ contains L_j as one of its faces and $\mathrm{Im}(\bar{L}_j)\geq 0$ corresponding to the choice Ψ_i^{+}. Figure 21, left, shows one such pair in two-dimensional pairs-of-pants.

Proposition 4.12 then implies that $\mathrm{Res}_i^{+,L}(L_j)=\bar{L}_j$ (on the level of morphisms, this functor is faithful but not full). Thus this functor sends standard Lagrangians to standard Lagrangians, although it does not necessarily preserve chamber Lagrangians.

The deletion functor Let $H_i\in\mathcal{H}_{\mathbb{C}}$ be a hyperplane with the defining equation ℓ_i, and let $\mathbb{V}_i$ be the hyperplane arrangement obtained by deleting H_i from $\mathbb{V}$. Similar as before, we consider a product decomposition

$$\mathrm{Re}(\ell_i)^{-1}([-\varepsilon,\varepsilon])\cong\ell_i^{-1}(0)\times T^*[-\varepsilon,\varepsilon].$$

We describe the operation of deleting the hyperplane H_i from $\mathbb{V}$ as the following "shopping bag construction".

6.5 Lemma *There exists a Weinstein cobordism Φ_s from sector $T^*[-1,1]$ to $T^*[-1,1]^\times$, depending on the parity $s\in\{+,-\}$.*

Proof Attach a 1-handle to $[-1,1]\times[-1,1]$ on the top edge ($s=-$), the resulting space is isomorphic to $T^*[-1,1]^\times\cong(\mathbb{C}^*,\pm\infty)$. See Figure 21, right, where the red line segments are sectorial boundaries, the black line segments are ideal boundaries for $T^*[-1,1]$, while the blue arcs denote the 1-handles. Similarly, one can attach a 1-handle to $[-1,1]\times[-1,1]$ at the bottom edge ($s=+$). □

By Lemma 6.5, for $s \in \{+,-\}$, we define the deletion functor

$$\mathrm{Del}_i^s \colon \mathcal{W}(M(\mathbb{V}),\xi) \to \mathcal{W}(M(\mathbb{V}_i),\xi),$$

as the gluing of the Viterbo restriction functor

$$\Phi_s \colon \mathcal{W}\big(\ell^{-1}(0)\times(\mathbb{C}^*,\pm\infty)\big) \to \mathcal{W}\big(\ell^{-1}(0)\times T^*[-\varepsilon,\varepsilon]\big) \cong \mathcal{W}(\ell^{-1}(0))$$

with the identity functors of the remaining parts. More precisely, we have the sectorial decompositions

$$\begin{aligned}(M(\mathbb{V}_i),\xi) &= (\ell_i^{-1}(-\infty,-\varepsilon],\xi)\cup(\ell_i^{-1}(0)\times T^*[-\varepsilon,\varepsilon],\xi)\cup(\ell_i^{-1}[\varepsilon,+\infty),\xi),\\ (M(\mathbb{V}),\xi) &= (\ell_i^{-1}(-\infty,-\varepsilon],\xi)\cup(\ell_i^{-1}(0)\times(\mathbb{C}^*,\pm\infty),\xi)\cup(\ell_i^{-1}[\varepsilon,+\infty),\xi).\end{aligned}$$

We then consider the diagram

$$\begin{array}{ccccccc} & & \mathcal{W}\big(\ell_i^{-1}(-\infty,-\varepsilon],\xi\big) & \xrightarrow{\mathrm{Id}} & \mathcal{W}\big(\ell_i^{-1}(-\varepsilon,-\varepsilon],\xi\big) & & \\ & \swarrow & & & & \searrow & \\ \mathcal{W}(M(\mathbb{V}),\xi) & \longleftarrow & \mathcal{W}(\ell_i^{-1}[-\varepsilon,\varepsilon],\xi) & \xrightarrow{\Phi_s} & \mathcal{W}(\ell_i^{-1}(0),\xi) & \longrightarrow & \mathcal{W}(M(\mathbb{V}_i),\xi) \\ & \nwarrow & & & & \nearrow & \\ & & \mathcal{W}\big(\ell_i^{-1}(\varepsilon,+\infty],\xi\big) & \xrightarrow{\mathrm{Id}} & \mathcal{W}\big(\ell_i^{-1}[\varepsilon,+\infty),\xi\big) & & \end{array}$$

Since the left- and right-hand diagrams are homotopy colimit diagrams and the functors commute with the colimit diagrams, this diagram induces a well-defined A_∞-functor $\mathrm{Del}_i^s \colon \mathcal{W}(M(\mathbb{V}),\xi) \to \mathcal{W}(M(\mathbb{V}_i),\xi)$. Let us compute this cobordism functor explicitly. Recall from the construction of Viterbo restriction functor in [25] that if we consider the product $(T^*[-1,1]^\times)\times\mathbb{C}_{\mathrm{Re}\geq 0}$ of the larger sector, then the inclusion $T^*[-1,1]^\times \hookrightarrow T^*[-1,1]$ induces a backward stopped inclusion

$$\big((T^*[-1,1]^\times)\times\mathbb{C}_{\mathrm{Re}\geq 0}, T^*[-1,1]^\times\times\mathbb{C}_{\mathrm{Re}=+\infty}\big) \hookrightarrow (T^*[-1,1]^\times\times\mathbb{C}_{\mathrm{Re}\geq 0}, T^*[-1,1]\times\mathbb{C}_{\mathrm{Re}=+\infty}).$$

Note that the left-hand side is the stabilization of $T^*[-1,1]^\times$, so the wrapped Fukaya category is quasiequivalent to that of $T^*[-1,1]^\times$, which is generated by the two purely imaginary rays as shown in Figure 20.

It is easier to keep track of the Viterbo restriction functor using relative cores: depending on the parity, the relative core of $T^*[-1,1]$ includes into the upper or lower part of the relative core of $T^*[-1,1]^\times$, and the Viterbo restriction functor would send the imaginary ray that does not intersect with the relative core of $T^*[-1,1]$ to 0, while the other imaginary ray would get sent to the canonical generator of $T^*[-1,1]$; see Example 2.13. Together with the Künneth functor, we are able to compute the image of deletion functor on objects. For each standard Lagrangian L in $\ell_i^{-1}(0)$, we write L^+ for the product of L with positive imaginary ray, L^- for product with negative imaginary ray, and L^0 for product with the cotangent fiber at 0 of $T^*[-1,1]$.

6.6 Lemma *Φ_s sends L^s to L^0 and L^{-s} to 0.*

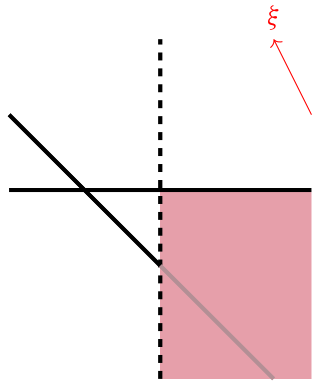

Figure 22: The deletion functor.

To determine Del_i^s, we need to keep track of the Lagrangians L^0 and L^s after gluing. Recall from the previous discussions that the inclusion $(\ell_i^{-1}(0) \times T^*[-1,1]^\times, \xi) \hookrightarrow (M(\mathbb{V}), \xi)$ sends the Lagrangian L^s to the standard Lagrangian $\widetilde{L}$ whose closure contains L as one of its faces, and the Lagrangian L^0 is away from all crossing points of $M(\mathbb{V}_i)$. Therefore we know that L^0 gets sent to 0 after gluing back the remaining pieces, and hence the deletion functor sends all the standard Lagrangians $\widetilde{L}$ to 0. See Figure 22 for an example in the case of two-dimensional pairs-of-pants.

6.7 Remark The restriction and deletion functors discussed here do not arise directly from the algebraic restriction and deletion operations outlined in Section 6a. This is because they send standard Lagrangians to standard Lagrangians, whereas their algebraic counterparts do not do so in general. To describe the corresponding algebraic framework, one needs to generalize the restriction and deletion operators by imposing signs on each chamber separately, rather than uniformly for all chambers. This induces a well-defined morphism of graded algebra on the relevant convolution algebras $\widetilde{B}$. However, the natural pullback functor on the module categories still does not send standard objects to standard objects. Instead, one needs to construct a functor at the 0-categorical level by ensuring that the standard object is sent to the desired standard object. Then our generation and formality results will provide functors we want. We refrain from elaborating on this approach in this paper because it may appear somewhat unnatural from an algebraic viewpoint.

References

[1] **M Abouzaid**, *A cotangent fibre generates the Fukaya category*, Adv. Math. 228 (2011) 894–939 MR

[2] **M Abouzaid**, **D Auroux**, *Homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$*, Geom. Topol. 28 (2024) 2825–2914 MR

[3] **M Abouzaid**, **D Auroux**, **A I Efimov**, **L Katzarkov**, **D Orlov**, *Homological mirror symmetry for punctured spheres*, J. Amer. Math. Soc. 26 (2013) 1051–1083 MR

[4] **M Abouzaid**, **P Seidel**, *An open string analogue of Viterbo functoriality*, Geom. Topol. 14 (2010) 627–718 MR

[5] **M Aganagic**, *Knot categorification from mirror symmetry, II: Lagrangians*, Adv. Theor. Math. Phys. 29 (2025) 557–749 MR

[6] **D Auroux**, *Fukaya categories and bordered Heegaard–Floer homology*, from "Proceedings of the International Congress of Mathematicians, II" (R Bhatia, A Pal, G Rangarajan, V Srinivas, M Vanninathan, editors), Hindustan, New Delhi (2010) 917–941 MR

[7] **D Auroux**, *A beginner's introduction to Fukaya categories*, from "Contact and symplectic topology" (F Bourgeois, V Colin, A Stipsicz, editors), Bolyai Soc. Math. Stud. 26, János Bolyai Math. Soc., Budapest (2014) 85–136 MR

[8] **D Auroux**, *Speculations on homological mirror symmetry for hypersurfaces in* $(\mathbb{C}^*)^n$, from "Surveys in differential geometry 2017: celebrating the 50th anniversary of the Journal of Differential Geometry" (H-D Cao, J Li, R M Schoen, S-T Yau, editors), Surv. Differ. Geom. 22, International, Somerville, MA (2018) 1–47 MR

[9] **R Bielawski**, **A S Dancer**, *The geometry and topology of toric hyperkähler manifolds*, Comm. Anal. Geom. 8 (2000) 727–760 MR

[10] **P Biran**, **O Cornea**, *Lagrangian cobordism, I*, J. Amer. Math. Soc. 26 (2013) 295–340 MR

[11] **P Biran**, **O Cornea**, *Lagrangian cobordism and Fukaya categories*, Geom. Funct. Anal. 24 (2014) 1731–1830 MR

[12] **V Bosshard**, *Lagrangian cobordisms in Liouville manifolds*, J. Topol. Anal. 16 (2024) 777–831 MR

[13] **T Braden**, **A Licata**, **C Phan**, **N Proudfoot**, **B Webster**, *Localization algebras and deformations of Koszul algebras*, Selecta Math. 17 (2011) 533–572 MR

[14] **T Braden**, **A Licata**, **N Proudfoot**, **B Webster**, *Gale duality and Koszul duality*, Adv. Math. 225 (2010) 2002–2049 MR

[15] **G Cazassus**, *Equivariant Lagrangian Floer homology via cotangent bundles of* EG_N, J. Topol. 17 (2024) art. id. ,e12328 MR

[16] **B Chantraine**, **G Dimitroglou Rizell**, **P Ghiggini**, **R Golovko**, *Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors*, Ann. Sci. Éc. Norm. Supér. 57 (2024) 1–85 MR

[17] **K Cieliebak**, **Y Eliashberg**, *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications 59, Amer. Math. Soc., Providence, RI (2012) MR

[18] **L Côté**, **B Gammage**, **J Hilburn**, *Hypertoric Fukaya categories and categories O*, preprint (2024) arXiv 2406.01379

[19] **A Daemi**, **K Fukaya**, *Atiyah–Floer conjecture: a formulation, a strategy of proof and generalizations*, from "Modern geometry: a celebration of the work of Simon Donaldson" (V Muñoz, I Smith, R P Thomas, editors), Proc. Sympos. Pure Math. 99, Amer. Math. Soc., Providence, RI (2018) 23–57 MR

[20] **Y Eliashberg**, *Weinstein manifolds revisited*, from "Modern geometry: a celebration of the work of Simon Donaldson" (V Muñoz, I Smith, R P Thomas, editors), Proc. Sympos. Pure Math. 99, Amer. Math. Soc., Providence, RI (2018) 59–82 MR

[21] **B Gammage**, **M Jeffs**, *Homological mirror symmetry for functors between Fukaya categories of very affine hypersurfaces*, J. Topol. 18 (2025) art. id. e70012 MR

[22] **B Gammage**, **V Shende**, *Mirror symmetry for very affine hypersurfaces*, Acta Math. 229 (2022) 287–346 MR

[23] **S Ganatra**, **J Pardon**, **V Shende**, *Covariantly functorial wrapped Floer theory on Liouville sectors*, Publ. Math. Inst. Hautes Études Sci. 131 (2020) 73–200 MR

[24] **S Ganatra**, **J Pardon**, **V Shende**, *Microlocal Morse theory of wrapped Fukaya categories*, Ann. of Math. 199 (2024) 943–1042 MR

[25] **S Ganatra**, **J Pardon**, **V Shende**, *Sectorial descent for wrapped Fukaya categories*, J. Amer. Math. Soc. 37 (2024) 499–635 MR

[26] **H Geiges**, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge Univ. Press (2008) MR

[27] **K Hendricks**, **R Lipshitz**, **S Sarkar**, *A simplicial construction of G-equivariant Floer homology*, Proc. Lond. Math. Soc. 121 (2020) 1798–1866 MR

[28] **M Jeffs**, *Mirror symmetry and Fukaya categories of singular hypersurfaces*, Adv. Math. 397 (2022) art. id. 108116 MR

[29] **Y Kim**, **S-C Lau**, **X Zheng**, *T-equivariant disc potential and SYZ mirror construction*, Adv. Math. 430 (2023) art. id. 109209 MR

[30] **S-C Lau**, **N-C C Leung**, **Y-L L Li**, *Equivariant Lagrangian correspondence and a conjecture of Teleman*, preprint (2023) arXiv 2312.13926

[31] **A D Lauda**, **A M Licata**, **A Manion**, *From hypertoric geometry to bordered Floer homology via the $m = 1$ amplituhedron*, Selecta Math. 30 (2024) art. id. 43 MR

[32] **S Lee**, **Y Li**, **C Y Mak**, *Hypertoric convolution algebras as Fukaya categories*, *I*, in preparation

[33] **Y Lekili**, **J Pascaleff**, *Floer cohomology of $\mathfrak{g}$-equivariant Lagrangian branes*, Compos. Math. 152 (2016) 1071–1110 MR

[34] **Y Lekili**, **A Polishchuk**, *Homological mirror symmetry for higher-dimensional pairs of pants*, Compos. Math. 156 (2020) 1310–1347 MR

[35] **Y Lekili**, **E Segal**, *Equivariant Fukaya categories at singular values*, preprint (2023) arXiv 2304.10969

[36] **T-T Lu**, **S-H Shiou**, *Inverses of 2×2 block matrices*, Comput. Math. Appl. 43 (2002) 119–129 MR

[37] **C Y Mak**, **W Wu**, *Dehn twist exact sequences through Lagrangian cobordism*, Compos. Math. 154 (2018) 2485–2533 MR

[38] **M McBreen**, **V Shende**, **P Zhou**, *The Hamiltonian reduction of hypertoric mirror symmetry*, preprint (2024) arXiv 2405.07955

[39] **M McLean**, *The growth rate of symplectic homology and affine varieties*, Geom. Funct. Anal. 22 (2012) 369–442 MR

[40] **P Seidel**, *Picard–Lefschetz theory and dilating $\mathbb{C}^*$-actions*, J. Topol. 8 (2015) 1167–1201 MR

[41] **P Seidel**, **J P Solomon**, *Symplectic cohomology and q-intersection numbers*, Geom. Funct. Anal. 22 (2012) 443–477 MR

[42] **Z Sylvan**, *On partially wrapped Fukaya categories*, J. Topol. 12 (2019) 372–441 MR

[43] **H L Tanaka**, *Surgery induces exact sequences in Lagrangian cobordisms*, preprint (2018) arXiv 1805.07424

[44] **C Teleman**, *Gauge theory and mirror symmetry*, from "Proceedings of the International Congress of Mathematicians, II" (S Y Jang, Y R Kim, D-W Lee, I Yie, editors), Kyung Moon Sa, Seoul (2014) 1309–1332 MR

[45] **G Teschl**, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics 140, Amer. Math. Soc., Providence, RI (2012) MR

[46] **Y Xiao**, *Equivariant Lagrangian Floer theory on compact toric manifolds*, preprint (2023) arXiv 2310.20202

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