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Stability for the 3D Riemannian Penrose inequality

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We show that the Schwarzschild 3-manifold is stable for the 3-dimensional Riemannian Penrose inequality in the pointed measured Gromov–Hausdorff topology, modulo negligible domains and boundary area perturbations.

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1 Introduction

Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and ADM mass $m(g)$ whose outermost minimal boundary has total surface area A. Then the (Riemannian) Penrose inequality states that $m(g) \geq \sqrt{A/(16\pi)}$, with equality if and only if (M^3, g) is isometric to the Schwarzschild 3-manifold of mass $m(g)$. This theorem was proved by Huisken and Ilmanen [2001] in the case that the boundary is connected, and by Bray [2001] in the general case. See also [Bray and Lee 2009; Khuri et al. 2017; Agostiniani et al. 2022] for additional extensions and generalizations. As a corollary, the Penrose inequality implies the positive mass theorem that $m(g) \geq 0$, which was first proved by Schoen and Yau [1979]. See also other generalizations [Witten 1981; Schoen and Yau 1981].

A natural question to ask regarding the Penrose inequality is whether almost equality would imply that the manifold is close to the Schwarzschild manifold in some topology. This is called the stability problem for the Penrose inequality. Recently, there has been growing interest in studying similar stability problems. Notably, the author and Antoine Song [Dong and Song 2025] proved a stability result for the positive mass theorem. In this paper, we prove that, up to boundary area perturbations, the Schwarzschild manifold is stable for the Penrose inequality in the pointed measured Gromov–Hausdorff topology, modulo negligible domains; see Definition 2.3. More precisely, we have the following:

Theorem 1.1 *Let $A_0 \geq 0$ be a fixed constant and (M_i^3, g_i) be a sequence of complete one-ended asymptotically flat 3-manifolds, each of which has nonnegative scalar curvature, a compact outermost minimal boundary (possibly with multiple components) with total area $A_i \to A_0$. Suppose that the ADM mass $m(g_i) \to \sqrt{A_0/(16\pi)}$. Then for all i, there is a connected closed subset $N_i \subset M_i$ containing the end, such that its boundary $\partial N_i = \Sigma_i^{\mathrm{a}} \cup \Sigma_i^{\mathrm{s}}$, where the major part Σ_i^{a} is connected and satisfies*

$$\mathrm{Area}_{g_i}(\Sigma_i^{\mathrm{a}}) \to A_0,$$

and the minor part Σ_i^{s} satisfies

$$\mathrm{Area}_{g_i}(\Sigma_i^{\mathrm{s}}) \to 0.$$

Moreover, for any $p_i \in \Sigma_i^{\mathrm{a}}$, we have

$$(N_i, \hat{d}_{g_i,N_i}, p_i) \to (M^3_{\mathrm{Sch}}, g_{\mathrm{Sch}}, x_o)$$

in the pointed measured Gromov–Hausdorff topology, and

$$(\Sigma_i^{\mathrm{a}}, \hat{d}_{g_i,\Sigma_i^{\mathrm{a}}}) \to (\partial M^3_{\mathrm{Sch}}, \hat{d}_{g_{\mathrm{Sch}},\partial M^3_{\mathrm{Sch}}})$$

in the measured Gromov–Hausdorff topology. Here, $(M^3_{\mathrm{Sch}}, g_{\mathrm{Sch}}, x_o)$ denotes the standard Schwarzschild 3-manifold with boundary area $\mathrm{Area}(\partial M^3_{\mathrm{Sch}}) = A_0$ and mass $m(g_{\mathrm{Sch}}) = \sqrt{A_0/(16\pi)}$, $x_o \in \partial M^3_{\mathrm{Sch}}$ is a base point, and $\hat{d}_{g_i,\cdot}$ are the length metrics on the corresponding spaces induced by g_i (see Figure 1 for a simplified geometric picture).

We also have a similar stability result for the mass–capacity inequality. Recall that for an asymptotically flat 3-manifold (M^3, g) with outermost minimal boundary Σ and an end ∞_1, the capacity of Σ in (M^3, g) is defined by

$$\mathscr{C}(\Sigma, g) := \inf\Big\{\frac{1}{\pi}\int_M |\nabla\varphi|^2 \,\mathrm{dvol}_g : \varphi \in C^\infty(M), \varphi = \tfrac{1}{2} \text{ on } \Sigma, \lim_{x\to\infty_1} \varphi(x) = 1\Big\}.$$

Then, as a corollary of the positive mass theorem, it was shown by Bray [2001, Theorem 9] that $m(g) \geq \mathscr{C}(\Sigma, g)$, and equality holds if and only if (M^3, g) is the Schwarzschild 3-manifold. For more details, please refer to Section 3.

Theorem 1.2 *Let $m_0 > 0$ be a fixed constant and (M_i^3, g_i) be a sequence of complete one-ended asymptotically flat 3-manifolds, each of which has nonnegative scalar curvature and a compact connected outermost minimal boundary. Suppose that both $m(g_i) \to m_0$ and $\mathscr{C}(\partial M_i, g_i) \to m_0$. Then for all i, there is a connected closed subset $E_i \subset M_i$ containing the end, whose boundary is $\partial E_i = \Sigma_i^{\mathrm{b}} \cup \Sigma_i^{\mathrm{s}}$, where the major part Σ_i^{b} is connected and satisfies*

$$\sup_{x\in\Sigma_i^{\mathrm{b}}} d_{g_i}(x, \partial M_i) \to 0,$$

and the minor part Σ_i^{s} satisfies

$$\mathrm{Area}_{g_i}(\Sigma_i^{\mathrm{s}}) \to 0.$$

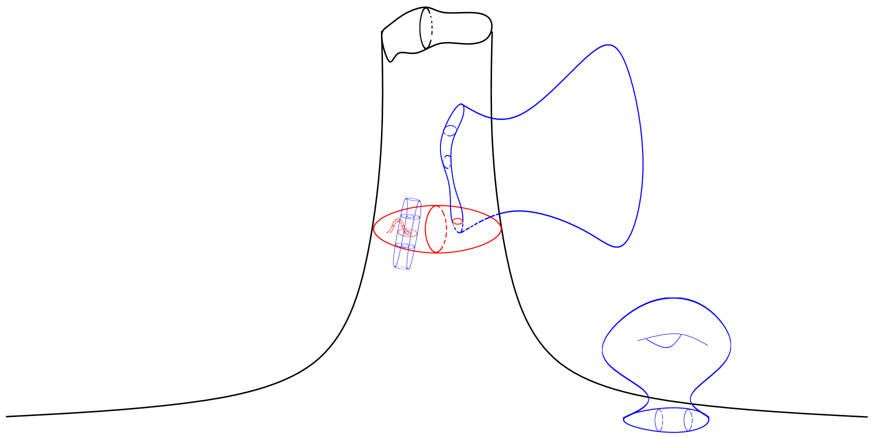

Figure 1: This simplified picture visually illustrates the conclusions of Theorem 1.1. The original manifold M is represented in black and blue. N refers to the region below the red part, excluding the blue part. Σ^{a} corresponds to the red part, resembling an almost standard sphere with some minor handles attached and a few small disks or annuli removed, while Σ^{s} is the boundary of the blue part, which has negligible area. In Theorem 1.2, the long neck between the boundary of the manifold and the red part in this picture will not occur.

Moreover, for any $p_i \in \Sigma_i^{\mathrm{b}}$, *we have*

$$(E_i, \hat{d}_{g_i,E_i}, p_i) \to (M^3_{\mathrm{Sch}}, g_{\mathrm{Sch}}, x_o)$$

in the pointed measured Gromov–Hausdorff topology, and

$$(\Sigma_i^{\mathrm{b}}, \hat{d}_{g_i,\Sigma_i^{\mathrm{b}}}) \to (\partial M^3_{\mathrm{Sch}}, \hat{d}_{g_{\mathrm{Sch}},\partial M^3_{\mathrm{Sch}}})$$

in the measured Gromov–Hausdorff topology. Here, $(M^3_{\mathrm{Sch}}, g_{\mathrm{Sch}}, x_o)$ *denotes the standard Schwarzschild 3-manifold with boundary area* $\mathrm{Area}(\partial M^3_{\mathrm{Sch}}) = 16\pi m_0^2$ *and mass* $m(g_{\mathrm{Sch}}) = m_0$, $x_o \in \partial M^3_{\mathrm{Sch}}$ *is a base point, and* $\hat{d}_{g_i,\cdot}$ *are the length metrics on the corresponding spaces induced by* g_i.

Our stability results can be viewed as generalizations of the main theorem in [Dong and Song 2025]. In particular, the case when $A_0 = 0$ in Theorem 1.1 was proved in [loc. cit.]. Our proof of the main theorems will build upon the tools developed in [loc. cit.], as well as some new techniques.

We give a remark on the boundary area perturbations in Theorem 1.1. Notice that in Theorem 1.2, we can ensure that Σ_i^{b}, the major part of the boundary of E_i, lies in a small neighborhood of the original boundary ∂M_i, but in Theorem 1.1, this is not necessarily true in general. Lee and Sormani [2012, Example 5.2] constructed manifolds with almost equality in the Penrose inequality that are close to Schwarzschild spaces with a cylinder of arbitrary length appended to the boundary. See also [Mantoulidis and Schoen 2015]. In other words, we need to cut out the long necks in these examples so that the remaining part has almost the same boundary area and is stable for the Penrose inequality. Such phenomena are consistent with our stability results.

It would also be an interesting question to prove similar results for other topologies. See, for example, the work of Lee and Sormani [2012; 2014], Lee, Naber and Neumayer [Lee et al. 2023], Kazaras, Khuri and Lee [Kazaras et al. 2024] and Dong [2024] for related results.

Outline of the proof We will mainly focus on the proof of Theorem 1.2. Up to a boundary area perturbation, Theorem 1.1 will follow as a corollary of Theorem 1.2 by using an argument from [Bray 2001].

Assume the conditions stated in Theorem 1.2, ie (M^3, g) is a complete asymptotically flat 3-manifold with nonnegative scalar curvature and a connected outermost minimal boundary Σ such that $m(g) > 0$ and $m(g) - \mathscr{C}(\Sigma, g) \ll 1$. By employing a doubling technique and allowing for a perturbation as in [Bray 2001], we can assume that $\overline{M} = M \cup_\Sigma M$ is a smooth asymptotically flat 3-manifold with nonnegative scalar curvature and two ends ∞_1, ∞_2. Then, the infimum in the definition of capacity is achieved by the Green's function f defined on $\overline{M}$, which satisfies

$$\Delta_g f = 0, \quad \lim_{x \to \infty_1} f(x) = 1, \quad \lim_{x \to \infty_2} f(x) = 0.$$

From symmetry of $\overline{M}$, f equals $\frac{1}{2}$ on Σ. We now consider the conformal metric $h = f^4 g$ on $\overline{M}$. Then, ∞_2 can be compactified such that $\overline{M}^* := \overline{M} \cup \{\infty_2\}$ is a smooth manifold, and $(\overline{M}^*, h)$ is an asymptotically flat 3-manifold with nonnegative scalar curvature. It can also be shown that the ADM mass $m(h) = m(g) - \mathscr{C}(\Sigma, g)$, as in [Bray 2001].

Based on our assumptions, $m(h) \ll 1$, the case discussed in [Dong and Song 2025]. So, modulo negligible domains, $(\overline{M}^*, h)$ is close to the Euclidean 3-space $\mathbb{R}^3$ in the pointed measured Gromov–Hausdorff topology. For the original metric $g = f^{-4} h$, we will show that f is uniformly close to the conformal factor in the Schwarzschild metric in the following two steps.

We first prove a new integral inequality involving the scalar curvature and the Hessian of the Green's function (Proposition 4.3). Let

$$u := \frac{2}{\mathscr{C}(\Sigma, g)} \cdot \frac{1 - f}{f}.$$

Then

$$1 - \frac{\mathscr{C}(\Sigma, g)}{m(g)} \geq \frac{\mathscr{C}(\Sigma, g)}{96\pi} \int_{(M,h)} \frac{\left|\nabla^2 u + |\nabla u|_h^{3/2} (h - 3\nu \otimes \nu)\right|_h^2}{|\nabla u|_h} + R_h |\nabla u|_h \, \mathrm{dvol}_h,$$

where $\nu = \nabla u / |\nabla u|_h$ and the integration is taken over regular set of u. This formula is very similar to the mass inequality proved by Bray, Kazaras, Khuri and Stern [Bray et al. 2022, Theorem 1.2]. One can follow the proof of [Bray et al. 2022, Theorem 1.2] and employ the technique of integration over level sets of u to control the integral in the inequality above. This method was initially explored by Stern [2022], and there have been many other applications and generalizations in recent studies, including by Bray, Hirsch, Kazaras, Khuri and Zhang [Bray et al. 2023], Agostiniani, Mazzieri and Oronzio [Agostiniani et al. 2024] and Agostiniani, Mantegazza, Mazzieri and Oronzio [Agostiniani et al. 2022]. In the proof, we will first use this method to obtain a preliminary integral inequality in Proposition 4.1. Then, using a corollary of that inequality, we can finally derive the desired integral inequality as stated above. As a corollary, we also provide another proof of the mass–area–capacity inequality (Theorem 4.2), which was first proved by Bray and Miao [2008].

Then, by using the integral inequality, together with the techniques used in [Dong and Song 2025], we are able to find a region $\mathcal{E} \subset \overline{M}^*$ with a small-area boundary. In this region $\mathcal{E}$, the behavior of $|\nabla f|$ closely resembles the conformal factor in the Schwarzschild metric, and particularly, $|\nabla f|$ is uniformly bounded. By the Arzelà–Ascoli theorem, along the convergence of $(\mathcal{E}, h)$, we can take the limit of such f and obtain a limit function f_∞ defined on $\mathbb{R}^3$. However, it is not immediately clear whether f_∞ is precisely the conformal factor in the Schwarzschild metric. To address this, we note that a favorable property of the pointed measured Gromov–Hausdorff convergence modulo negligible domains is that f also converges to f_∞ in the $W^{1,2}$-sense (Lemma 6.1). Therefore, the elliptic equations satisfied by f are preserved in this convergence, and f_∞ also satisfies an elliptic equation on $\mathbb{R}^3$. The fact that the area of the boundary $\partial\mathcal{E}$ converges to 0 has been essential here. The elliptic equation satisfied by f_∞, together with the control on the gradient, would imply the rigidity of f_∞ (Section 6).

Finally, using metric geometry tools and similar arguments from [Dong and Song 2025, Section 4], we can apply these properties of the functions f and metrics h to prove the main theorems.

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2 Preliminaries

2.1 Notation

We will use C, C' to denote a universal positive constant (which may be different from line to line); $\Psi(t), \Psi(t|a,b,\dots)$ denote small constants depending on $a, b, \dots$ and satisfying

$$\lim_{t\to 0} \Psi(t) = 0 \quad \text{and} \quad \lim_{t\to 0} \Psi(t|a,b,\dots) = 0$$

for each fixed $a, b, \dots$.

We denote the Euclidean metric by g_{Eucl} or δ, and the induced geometric quantities with subscript Eucl or δ.

For a general Riemannian manifold (M, g) and any $\pi \in M$, the geodesic ball with center p and radius r is denoted by $B_g(p,r)$, or $B(p,r)$ if the underlying metric is clear. Given a Riemannian metric, for a surface Σ and a domain Ω, $\mathrm{Area}(\Sigma)$ is the area of Σ, and $\mathrm{Vol}(\Omega)$ is the volume of Ω with respect to the metric.

Finally we introduce some notation about length metric that will be used later. Given a subset U in a Riemannian manifold (M, g), let $(U, \hat{d}_{g,U})$ be the induced length metric on U of the metric g, that is, for any $x_1, x_2 \in U$,

$$\hat{d}_{g,U}(x_1, x_2) := \inf\{L_g(\gamma) : \gamma \text{ is a rectifiable curve connecting } x_1, x_2 \text{ and } \gamma \subset U\},$$

where $L_g(\gamma) = \int_0^1 |\gamma'|_g$ is the length of γ with respect to metric g. By convention, two points in two different path-connected components of U are at infinite $\hat{d}_{g,U}$-distance.

For any $D > 0$ and $p \in U$, we use $\hat{B}_{g,U}(p, D)$ to denote the geodesic ball inside $(U, \hat{d}_{g,U})$, that is,

$$\hat{B}_{g,U}(p, D) := \{x \in U : \hat{d}_{g,U}(p, x) \le D\}.$$

2.2 Geometry of the Schwarzschild metric

For a positive number $m > 0$, the Schwarzschild 3-manifold $(M^3_{\mathrm{Sch}}, g_{\mathrm{Sch}})$ (with mass m) is given by the following warped product metric on $\mathbb{S}^2 \times [0, \infty)$:

$$g_{\mathrm{Sch}} = ds^2 + u_m(s)^2 g_{\mathbb{S}^2} \quad \text{for } s \in [0, \infty), \tag{1}$$

where $g_{\mathbb{S}^2}$ is the spherical metric with $\mathrm{Area}(\mathbb{S}^2, g_{\mathbb{S}^2}) = 4\pi$, and u_m is a positive increasing function satisfying

$$u_m(0) = 2m, \quad u_m'(0) = 0, \quad u_m'(s) = \left(1 - \frac{2m}{u_m(s)}\right)^{\frac{1}{2}}, \quad u_m''(s) = \frac{m}{u_m(s)^2}. \tag{2}$$

Then the scalar curvature of g_{Sch} is identically zero, and the boundary $\Sigma_{\mathrm{Sch}} := \partial M^3_{\mathrm{Sch}}$ is the only minimal surface inside M^3_{Sch}.

Under Cartesian coordinates, M^3_{Sch} is diffeomorphic to $\mathbb{R}^3 \setminus B(m/2)$, where $B(m/2)$ is the Euclidean ball with radius $m/2$ around the center, and we have

$$g_{\mathrm{Sch},ij}(x) = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} \quad \text{for all } |x| \ge \tfrac{1}{2}m. \tag{3}$$

This metric can also be extended to give a Schwarzschild metric g_{Sch} defined on $\mathbb{R}^3 \setminus \{0\}$.

Define $\rho_m(x) := \mathrm{dist}(x, \Sigma_{\mathrm{Sch}})$. Then

$$\rho_m(x) = \int_{m/2}^{|x|} \left(1 + \frac{m}{2t}\right)^2 dt = |x| - \frac{m^2}{4|x|} + m \log \frac{2|x|}{m},$$

which implies that

$$\rho_m(x) - m \log \frac{2\rho_m(x)}{m} \le |x| \le \rho_m(x). \tag{4}$$

Using these two representations (1) and (3) of g_{Sch} to compute the area of geodesic spheres, we have

$$u_m(\rho_m(x))^2 \cdot 4\pi = \left(1 + \frac{m}{2|x|}\right)^4 \cdot 4\pi |x|^2,$$

ie

$$u_m(\rho_m(x)) = \left(1+\frac{m}{2|x|}\right)^2\cdot|x|. \tag{5}$$

In particular, together with (4),

$$\lim_{r\to\infty}\frac{u_m(r)}{r} = 1. \tag{6}$$

Now we introduce another harmonic function f_m and rewrite above identities using f_m instead of $|x|$. Define

$$f_m(x) := \left(1+\frac{m}{2|x|}\right)^{-1}.$$

Standard computations imply that

$$\Delta_{g_{\mathrm{Sch}}} f_m = 0, \qquad f_m = \tfrac{1}{2} \quad \text{on } \Sigma_{\mathrm{Sch}}, \qquad \lim_{|x|\to\infty} f_m(x) = 1.$$

Then

$$|x| = \frac{m}{2}\cdot\frac{f_m(x)}{1-f_m(x)},$$

and

$$\rho_m(x) = \rho_m(f_m(x)) = \frac{m}{2}\left(\frac{1}{1-f_m(x)} - \frac{1}{f_m(x)}\right) + m\log\frac{f_m(x)}{1-f_m(x)}. \tag{7}$$

Moreover,

$$u_m(\rho_m(x)) = \frac{m}{2}\cdot\frac{1}{f_m(x)(1-f_m(x))}. \tag{8}$$

2.3 Asymptotically flat 3-manifolds

A smooth orientable connected complete Riemannian 3-manifold (M^3, g) is called asymptotically flat if there exists a compact subset $K\subset M$ such that $M\setminus K = \bigsqcup_{k=1}^N M^k_{\mathrm{end}}$ consists of finite pairwise disjoint ends, and for each $1\le k\le N$, there exist $C>0, \sigma>\frac{1}{2}$, and a C^∞-diffeomorphism $\Phi_k : M^k_{\mathrm{end}} \to \mathbb{R}^3\setminus B(1)$ such that under this identification,

$$|\partial^l(g_{ij}-\delta_{ij})(x)| \le C|x|^{-\sigma-|l|}$$

for all multi-indices $|l| = 0, 1, 2$ and any $x\in\mathbb{R}^3\setminus B(1)$. Furthermore, we always assume the scalar curvature R_g is integrable over (M^3, g). The ADM mass from general relativity of each end M^k_{end}, for $1\le k\le N$, is then well-defined (see [Arnowitt et al. 1961; Bartnik 1986]) and given by

$$m_k(g) := \lim_{r\to\infty}\frac{1}{16\pi}\int_{S_r}\sum_{i,j}(g_{ij,i}-g_{ii,j})\nu^j\,dA,$$

where ν is the unit outer normal to the coordinate sphere S_r of radius $|x| = r$ in the given end, and dA is its area element.

Definition 2.1 A surface $\Sigma \subset (M^3, g)$ is called a horizon if it is a minimal surface. It is called an outermost horizon if it is a horizon and it is not enclosed by another minimal surface in (M^3, g).

Let (M^3, g) be an asymptotically flat 3-manifold. By Lemma 4.1 in [Huisken and Ilmanen 2001], we know that inside M^3 there is a trapped compact region T whose topological boundary consists of smooth embedded minimal 2-spheres. An "exterior region" M^3_{ext} is defined as the metric completion of any connected component of $M \setminus T$ containing one end. Then M^3_{ext} is connected, asymptotically flat, has a compact minimal boundary $\partial M^3_{\text{ext}}$ ($\partial M^3_{\text{ext}}$ may be empty), and contains no other compact minimal surfaces, that is, M^3_{ext} is an asymptotically flat 3-manifold with outermost horizon boundary.

We will be able to perturb an asymptotically flat metric to a metric with nicer behavior at infinity in each end because of the following definition and proposition.

Definition 2.2 We say that (M^3, g) is harmonically flat at infinity if $(M^3 \setminus K, g)$ is isometric to a finite disjoint union of regions with zero scalar curvature which are conformal to $(\mathbb{R}^3 \setminus B, \delta)$ for some compact set K in M^3 and some ball B in $\mathbb{R}^3$ centered around the origin.

By definition, if (M^3, g) is harmonically flat, then on each end, $g_{ij}(x) = V(x)\delta_{ij}$ for some bounded positive δ-harmonic function $V(x)$, which satisfies that $\Delta_\delta V(x) = 0$ and (see [Bray 2001, equation (10)])

$$V(x) = a + \frac{b}{|x|} + O\left(\frac{1}{|x|^2}\right). \tag{9}$$

In this case, its ADM mass on this end is given by $2ab$.

Proposition 2.1 [Schoen and Yau 1981] *Let (M^3, g) be a complete, asymptotically flat 3-manifold with $R_g \geq 0$ and ADM mass $m_k(g)$ in the k^{th} end. For any $\epsilon > 0$, there exists a metric $\hat{g}$ such that $e^{-\epsilon} g \leq \hat{g} \leq e^{\epsilon} g$, $R_{\hat{g}} \geq 0$, $(M^3, \hat{g})$ is harmonically flat at infinity, and $|m_k(\hat{g}) - m_k(g)| \leq \epsilon$, where $m_k(\hat{g})$ is the ADM mass of $\hat{g}$ in the k^{th} end.*

2.4 pm-GH convergence modulo negligible domains

In this subsection, we recall some definitions for the pointed measured Gromov–Hausdorff topology.

Assume (X, d_X, x), (Y, d_Y, y) are two pointed metric spaces. The pointed Gromov–Hausdorff (or pGH-) distance is defined in the following way. A pointed map $f : (X, d_X, x) \to (Y, d_Y, y)$ is called an ε-pointed Gromov–Hausdorff approximation (or ε-pGH approximation) if it satisfies the following conditions:

(i) $f(x) = y$.

(ii) $B(y, 1/\varepsilon) \subset B_\varepsilon(f(B(x, 1/\varepsilon)))$.

(iii) $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$ for all $x_1, x_2 \in B(x, 1/\varepsilon)$.

The pGH-distance is defined by

$$d_{\mathrm{pGH}}\big((X,d_X,x),(Y,d_Y,y)\big) := \inf\{\varepsilon>0 : \exists\ \varepsilon\text{-pGH approximation } f\colon (X,d_X,x)\to(Y,d_Y,y)\}.$$

We say that a sequence of pointed metric spaces (X_i,d_i,p_i) converges to a pointed metric space (X,d,p) in the pointed Gromov–Hausdorff topology if

$$d_{\mathrm{pGH}}((X_i,d_i,p_i),(X,d,p))\to 0.$$

If (X_i,d_i) are length metric spaces, ie for any two points $x,y\in X_i$,

$$d_i(x,y)=\inf\{L_{d_i}(\gamma):\gamma \text{ is a rectifiable curve connecting } x,y\},$$

where $L_{d_i}(\gamma)$ is the length of γ induced by the metric d_i, then equivalently,

$$d_{\mathrm{pGH}}\big((X_i,d_i,p_i),(X,d,p)\big)\to 0$$

if and only if for all $D>0$,

$$d_{\mathrm{pGH}}\big((B(p_i,D),d_i),(B(p,D),d)\big)\to 0,$$

where $B(p_i,D)$ are the geodesic balls of metric d_i.

A pointed metric measure space is a structure (X,d_X,μ,x) where (X,d_X) is a complete separable metric space, μ a Radon measure on X and $x\in\operatorname{supp}(\mu)$.

We say that a sequence of pointed metric measure length spaces (X_i,d_i,μ_i,p_i) converges to a pointed metric measure length space (X,d,μ,p) in the pointed measured Gromov–Hausdorff (or pm-GH) topology, if for any $\varepsilon>0$, $D>0$, there exists $N(\varepsilon,D)\in\mathbb{Z}_+$ such that for all $i\geq N(\varepsilon,D)$, there exists a Borel ε-pGH approximation

$$f_i^{D,\varepsilon}\colon (B(p_i,D),d_i,p_i)\to(B(p,D+\varepsilon),d,p)$$

such that

$$(f_i^{D,\varepsilon})_\sharp(\mu_i|_{B(p_i,D)}) \text{ weakly converges to } \mu|_{B(p,D)} \text{ as } i\to\infty \text{ for a.e. } D>0.$$

In the case when X_i is an n-dimensional manifold, without extra explanations, we will always consider (X_i,d_i,p_i) as a pointed metric measure space equipped with the n-dimensional Hausdorff measure $\mathcal{H}^n_{d_i}$ induced by d_i.

Finally, we introduce a notation about the topology used in this paper. For simplicity, we only consider manifolds, but one can easily generalize it to general metric measured spaces.

Definition 2.3 For a sequence of pointed Riemannian n-manifolds (M_i^n,g_i,p_i) and (M^n,g,p), we say that (M_i,g_i,p_i) converges to (M,g,p) in the pointed measured Gromov–Hausdorff topology modulo negligible domains if there exist open subsets $Z_i\subset M_i$ such that $\mathcal{H}^{n-1}(\partial Z_i)\to 0$, $p_i\in M_i\setminus Z_i$ and

$$(M_i\setminus Z_i,\hat{d}_{g_i},p_i)\to(M,d_g,p)$$

in the pointed measured Gromov–Hausdorff topology for the induced length metric.

We have the following theorem, which states that C^0-convergence of metric tensors modulo negligible domains implies Gromov–Hausdorff convergence of length metrics modulo negligible domains after perturbation.

Theorem 2.2 *Assume that $n \geq 2$, and (E_i^n, g_i) is a sequence of n-dimensional complete Riemannian manifolds with compact boundaries. If (E_i, g_i) converges to $\mathbb{R}^n$ in the C^0-sense modulo negligible domains, that is, there exist embeddings $\boldsymbol{u}_i \colon E_i \to \mathbb{R}^n$ such that*

- *$\boldsymbol{u}_i(E_i)$ contains the end of $\mathbb{R}^n$,*
- *$\|(\boldsymbol{u}_i^{-1})^* g_i - g_{\mathrm{Eucl}}\|_{C^0} \to 0$, and*
- *$\mathcal{H}^{n-1}(\partial E_i) \to 0$,*

then there exist closed subsets $E_i'' \subset E_i$ with compact boundaries such that for any base points $p_i \in E_i''$, (E_i'', g_i, p_i) converges to $(\mathbb{R}^n, g_{\mathrm{Eucl}}, 0)$ in the pointed measured Gromov–Hausdorff topology modulo negligible domains in the sense of Definition 2.3.

Notice that this theorem was proved for dimension 3 in [Dong and Song 2025]. Using the same techniques as in [loc. cit.], along with inductive arguments, we can extend this result to a general dimensional version. For the reader's convenience, we provide a detailed proof in the appendix.

3 Capacity of horizon and Green's function

In this section, we introduce some properties of capacity and Green's function, which are known in the literature and will be used later in this paper. Most of this section follows from [Bray 2001].

Let's firstly introduce the capacity of a surface in the special case when it is the horizon of an asymptotically flat 3-manifold which is harmonically flat at infinity.

Definition 3.1 Given a complete, asymptotically flat 3-manifold (M^3, g) with a connected outermost horizon boundary Σ, nonnegative scalar curvature and one asymptotically flat end ∞_1, the capacity of Σ in (M^3, g) is defined by

$$\mathcal{C}(\Sigma, g) := \inf\left\{\frac{1}{\pi}\int_{M^3} |\nabla\varphi|^2 \,\mathrm{dvol}_g : \varphi \in C^\infty(M),\ \varphi = \tfrac{1}{2} \text{ on } \Sigma,\ \lim_{x\to\infty_1} \varphi(x) = 1\right\}.$$

From standard theory (see [Bartnik 1986]), the infimum in the definition of $\mathcal{C}(\Sigma, g)$ is achieved by the Green's function $\varphi \in C^\infty(M^3)$, which satisfies

$$\Delta_g \varphi = 0, \qquad \varphi = \tfrac{1}{2} \quad \text{on } \Sigma, \qquad \lim_{x\to\infty_1} \varphi(x) = 1. \tag{10}$$

By the maximum principle, $\varphi(x) \in \left[\frac{1}{2}, 1\right)$ for any $x \in M^3$. Define the level sets of φ to be

$$\Sigma_t^\varphi := \{x \in M^3 : \varphi(x) = t\}.$$

Then by Sard's theorem, Σ_t^{φ} is a smooth surface for almost all $t \in \left(\frac{1}{2}, 1\right)$. By the coarea formula,

$$\mathscr{C}(\Sigma, g) = \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \int_{\Sigma_t^{\varphi}} |\nabla \varphi|.$$

For any regular value $t \in \left(\frac{1}{2}, 1\right)$, integrating $\Delta\varphi = 0$ over $\left\{\frac{1}{2} \le \varphi \le t\right\}$, and using Stokes' theorem, we get

$$\int_{\Sigma} |\nabla \varphi| = \int_{\Sigma_t^{\varphi}} |\nabla \varphi|. \tag{11}$$

So

$$\mathscr{C}(\Sigma, g) = \frac{1}{2\pi} \int_{\Sigma} |\nabla \varphi|. \tag{12}$$

When (M^3, g) is harmonically flat at infinity, we have the following expansion of the Green's function (see [Bartnik 1986] and [Bray 2001, equation (80)]):

$$\varphi(x) = 1 - \frac{\mathscr{C}(\Sigma, g)}{2|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } x \to \infty_1. \tag{13}$$

Now we introduce another definition, which is closely related to the capacity of a horizon surface.

Definition 3.2 Given a complete, asymptotically flat 3-manifold $(\overline{M}^3, \overline{g})$ with multiple asymptotically flat ends and one chosen end ∞_1, define

$$\mathscr{C}(\overline{g}) := \inf\left\{\frac{1}{2\pi} \int_{\overline{M}^3} |\nabla \phi|^2 \, \mathrm{dvol}_{\overline{g}} : \phi \in \mathrm{Lip}(\overline{M}),\ \lim_{x \to \infty_1} \phi(x) = 1,\ \lim_{x \to \{\infty_k\}_{k \ge 2}} \phi(x) = 0\right\}.$$

Similarly, the infimum in the definition of $\mathscr{C}(\overline{g})$ is achieved by the Green's function ϕ, which satisfies

$$\Delta_{\overline{g}} \phi = 0, \qquad \lim_{x \to \infty_1} \phi(x) = 1, \qquad \lim_{x \to \infty_k} \phi(x) = 0 \quad \text{for all } k \ge 2. \tag{14}$$

For a complete asymptotically flat 3-manifold (M^3, g) with a compact outermost horizon boundary Σ, nonnegative scalar curvature and one end ∞_1, we can take another copy of (M^3, g) and glue them together along the boundary Σ to get a new metric space $(\overline{M}, \overline{g})$. In general, $(\overline{M}, \overline{g})$ is only a Lipschitz manifold with two asymptotically flat ends $\{\infty_1, \infty_2\}$. From the proof of [Bray 2001, Theorem 9], for any $\delta > 0$ small enough we can smooth out $(\overline{M}, \overline{g})$ and construct a smooth complete 3-manifold $(\widetilde{M}_\delta, \widetilde{g}_\delta)$ with nonnegative scalar curvature and two asymptotically flat ends which, in the limit as $\delta \to 0$, approaches $(\overline{M}, \overline{g})$ uniformly. For the reader's convenience, we recall the details of [Bray 2001] in the following.

Let $(M_1^3, g), (M_2^3, g)$ be the two copies of (M^3, g). A first step is to construct a smooth manifold (see [Bray 2001, equation (92)])

$$(\widetilde{M}_\delta, \overline{g}_\delta) := (M_1^3, g) \sqcup (\Sigma \times (0, 2\delta), G) \sqcup (M_2^3, g),$$

where $\Sigma \times \{0\}$ and $\Sigma \times \{2\delta\}$ are identified with $\Sigma \subset (M^3, g)$, G is a warped product metric and symmetric about $t = \delta$, and $\Sigma \times \{\delta\} \subset (\Sigma \times (0, 2\delta), G)$ is totally geodesic. In general, the scalar curvature of G only satisfies $R_G \ge R_0$ for some constant $R_0 \le 0$ independent of δ, and may not be nonnegative.

Then a second step is to take a conformal deformation of $\overline{g}_\delta$ to get a new metric with nonnegative scalar curvature. Define a smooth function $\mathcal{R}_\delta$ which equals R_0 in $\Sigma\times[0,2\delta]$, equals 0 for x more than a distance δ from $\Sigma\times[0,2\delta]$, takes values in $[R_0,0]$ everywhere and is symmetric about $\Sigma\times\{\delta\}$. In particular, $R_{\overline{g}_\delta}(x)\geq\mathcal{R}_\delta(x)$ for any $x\in\widetilde{M}_\delta$. Define $u_\delta(x)$ such that (see [Bray 2001, equation (101)])

$$(-8\Delta_{\overline{g}_\delta}+\mathcal{R}_\delta(x))u_\delta(x)=0 \quad\text{and}\quad \lim_{x\to\{\infty_1,\infty_2\}}u_\delta(x)=1. \tag{15}$$

Then u_δ is a smooth function and satisfies (see [Bray 2001, equation (102)])

$$1\leq u_\delta(x)\leq 1+\epsilon(\delta),$$

where ϵ goes to 0 as $\delta\to 0$. Define

$$\widetilde{g}_\delta := u_\delta^4\cdot\overline{g}_\delta.$$

The scalar curvature of $\widetilde{g}_\delta$ satisfies

$$R_{\widetilde{g}_\delta}=u_\delta^{-5}(R_{\overline{g}_\delta}u_\delta-8\Delta_{\overline{g}_\delta}u_\delta)=u_\delta^{-4}(R_{\overline{g}_\delta}-\mathcal{R}_\delta)\geq 0.$$

By definition, $\lim_{\delta\to 0}m(\widetilde{g}_\delta)=m(\overline{g})$ and $\lim_{\delta\to 0}\mathcal{C}(\widetilde{g}_\delta)=\mathcal{C}(\overline{g})$.

To see the relation between $\mathcal{C}(\overline{g})$ and $\mathcal{C}(\Sigma,g)$, we define the reflection map

$$\Phi\colon M_1^3\cup_\Sigma M_2^3\to M_1^3\cup_\Sigma M_2^3$$

such that for any $x\in M_1^3$, $\Phi(x)\in M_2^3$ is the same point under the identification $M_1^3=M_2^3=M^3$, $\Phi^2=\mathrm{Id}$ and $\Phi|_\Sigma=\mathrm{Id}$. If ϕ satisfies (14), then $1-\phi\circ\Phi$ also satisfies (14) and by the uniqueness we have $\phi(x)=1-\phi\circ\Phi(x)$, which implies that $\phi|_\Sigma=\frac{1}{2}$. So $\phi|_{M_1^3}$ also satisfies (10), which implies that

$$\mathcal{C}(\overline{g})=\mathcal{C}(\Sigma,g). \tag{16}$$

Similarly, we can define the reflection map $\Phi_\delta\colon\widetilde{M}_\delta\to\widetilde{M}_\delta$ and from the equation (15) and the fact that $\overline{g}_\delta=\overline{g}_\delta\circ\Phi_\delta, \mathcal{R}_\delta=\mathcal{R}_\delta\circ\Phi_\delta$, we know u_δ is also symmetric about $\Sigma\times\{\delta\}$ and particularly, $\langle\nabla u_\delta,\vec{n}\rangle_{\overline{g}_\delta}=0$ on $\Sigma\times\{\delta\}$, where $\vec{n}$ is the normal vector of $\Sigma\times\{\delta\}\subset(\widetilde{M}_\delta,\overline{g}_\delta)$. Thus, $\widetilde{g}_\delta$ is symmetric about $\Sigma\times\{\delta\}$ and the mean curvature of $\Sigma\times\{\delta\}\subset(\widetilde{M}_\delta,\widetilde{g}_\delta)$ is

$$H_{(\Sigma\times\{\delta\},\widetilde{g}_\delta)}=u_\delta^{-2}H_{(\Sigma\times\{\delta\},\overline{g}_\delta)}-2\langle\nabla u_\delta^{-2},\vec{n}\rangle_{\overline{g}_\delta}=0.$$

Let $(M_\delta,\widetilde{g}_\delta)$ be one half of $(\widetilde{M}_\delta,\widetilde{g}_\delta)$ with minimal boundary $\Sigma_\delta:=\Sigma\times\{\delta\}$ and one asymptotically flat end. Then $(M_\delta,\widetilde{g}_\delta,\Sigma_\delta)$ converges to (M^3,g,Σ) uniformly as $\delta\to 0$.

Without loss of generality, by applying Proposition 2.1 we can assume that $(\widetilde{M}_\delta,\widetilde{g}_\delta)$ is also harmonically flat at infinity.

In summary, we have the following proposition. For more details see [Bray 1997].

Proposition 3.1 *Given a complete one-ended asymptotically flat* 3*-manifold* (M^3, g) *with a connected outermost horizon boundary* Σ *and nonnegative scalar curvature, there is a sequence of smooth complete* 3*-manifolds* $(\widetilde{M}^3_\delta, \tilde{g}_\delta)$ *which have nonnegative scalar curvature and two harmonically flat ends, and are symmetric about a minimal surface* $\Sigma_\delta \subset (\widetilde{M}^3_\delta, \tilde{g}_\delta)$*, such that* $(\widetilde{M}_\delta, \tilde{g}_\delta) \to (\overline{M}, \overline{g})$ *and* $(M_\delta, \tilde{g}_\delta) \to (M, g)$ *uniformly as* $\delta \to 0$*, where* $(\overline{M}, \overline{g})$ *is the doubling of* (M, g) *along the boundary* Σ*, and* $(M_\delta, \tilde{g}_\delta)$ *is one half of* $(\widetilde{M}_\delta, \tilde{g}_\delta)$ *with minimal boundary* Σ_δ*.*

To conclude this section, we give a remark about the relations between the mass, capacity, and boundary area of the outermost horizon by briefly recalling Bray's proof [2001] of the Penrose inequality. Given a complete smooth 3-manifold (M^3, g_0) with a harmonically flat end, nonnegative scalar curvature, an outermost minimizing horizon Σ_0 of total area A_0 and total mass m_0, then for all $t \geq 0$, we can construct a continuous family of conformal metrics g_t on M^3 which are asymptotically flat with nonnegative scalar curvature and total mass $m(t)$. Let $\Sigma(t)$ be the outermost minimal enclosure of Σ_0 in (M^3, g_t), and $M(t)$ the asymptotically flat manifold with boundary $\Sigma(t)$. Then $\Sigma(t)$ is a smooth outermost horizon in $(M(t), g_t)$ with area $A(t)$ being a constant function about t. It was shown that $m(t)$ is decreasing, and as $t \to \infty$, $(M(t), g_t)$ approaches a Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, g_{\mathrm{Sch}})$ with total mass $\lim_{t\to\infty} m(t) = \sqrt{A_0/(16\pi)}$. In particular, $m_0 \geq \sqrt{A_0/(16\pi)}$, which proves the Penrose inequality.

4 Integral estimate of the Hessian of the Green's function

In this and the following section, we assume that $(\widetilde{M}^3, g)$ is a complete, asymptotically flat 3-manifold with two harmonically flat ends $\{\infty_1, \infty_2\}$ and nonnegative scalar curvature obtained as in Proposition 3.1. In particular, the topology of $\widetilde{M}^3$ is $\mathbb{R}^3 \setminus \{0\}$, and there is a minimal surface $\Sigma \subset (\widetilde{M}^3, g)$ such that g is symmetric about Σ. Let (M^3, g) be the half of $(\widetilde{M}^3, g)$ which contains the end ∞_1 and has minimal boundary Σ. So Σ is diffeomorphic to a 2-sphere, M^3 is diffeomorphic to $\mathbb{R}^3 \setminus B(1)$, and $(\widetilde{M}^3, g) = (M^3, g) \cup_\Sigma (M^{3,\prime}, g)$, where M' denotes a copy of M containing the other end ∞_2.

Let $f(x)$ be the solution to (14) on $(\widetilde{M}^3, g)$, that is,

$$\Delta_g f = 0, \quad \lim_{x\to\infty_1} f(x) = 1, \quad \lim_{x\to\infty_2} f(x) = 0. \tag{17}$$

Then f is a smooth function satisfying $0 < f < 1$ and at infinity has the expansion (see [Bartnik 1986; Bray 2001])

$$f(x) = \begin{cases} 1 - \dfrac{c_1}{|x|} + O\Big(\dfrac{1}{|x|^2}\Big) & \text{as } x \to \infty_1, \\ \dfrac{c_2}{|x|} + O\Big(\dfrac{1}{|x|^2}\Big) & \text{as } x \to \infty_2, \end{cases} \tag{18}$$

where c_k are positive constants for $k = 1, 2$.

Moreover, for some $\tau \in (0,1)$,

$$\begin{aligned} \partial_j f(x) &= \frac{c_1}{|x|^2}\cdot\frac{x^j}{|x|} + O\Big(\frac{1}{|x|^{2+\tau}}\Big), \\ \partial_j\partial_k f(x) &= \frac{c_1\delta_{jk}}{|x|^3} - \frac{3c_1}{|x|^3}\cdot\frac{x^j x^k}{|x|^2} + O\Big(\frac{1}{|x|^{3+\tau}}\Big). \end{aligned} \tag{19}$$

By the symmetry of $\tilde{g}$ about Σ, we know that on (M^3, g), f satisfies (10), that is,

$$\Delta_g f = 0, \qquad f = \tfrac{1}{2} \quad \text{on } \Sigma, \qquad \lim_{x\to\infty_1} f(x) = 1. \tag{20}$$

So by (12) and (13),

$$c_1 = \tfrac{1}{2}\mathscr{C}(\Sigma, g) = \frac{1}{4\pi}\int_\Sigma |\nabla f|. \tag{21}$$

Similarly, on (M', g), $1-f$ also satisfies (10), so

$$c_2 = \tfrac{1}{2}\mathscr{C}(\Sigma, g) = c_1. \tag{22}$$

We now introduce an auxiliary function

$$u := t_0 \cdot \frac{1-f}{f}$$

for some $t_0 > 0$, and the conformal metric $h = f^4 g$. We always assume $m(g) > 0$ in the following. Then $R_h = f^{-4}R_g \geq 0$. Notice that $u \in (0, t_0]$ on M, $u = t_0$ on Σ and $u(x) \to 0$ as $x \to \infty_1$. Moreover,

$$\Delta_h u = 0.$$

Since u is a proper smooth map on M, by Sard's theorem, the regular values of u is an open dense subset of $(0, t_0]$. For any regular value $t \in (0, t_0)$, we define

$$M_t := \{t \leq u \leq t_0\} \quad \text{and} \quad \Sigma_t := \{u = t\}.$$

Since Σ is connected by assumptions, using the maximum principle, we know that a regular level set Σ_t is also connected and separates Σ from ∞_1. In particular, Σ_t is a connected 2-sphere.

Proposition 4.1 *We have the following integration inequality for u on (M^3, h):*

$$1 - \frac{\mathscr{C}(\Sigma, g)^2}{m(g)^2} \geq \frac{1}{8\pi t_0}\int_{(M,h)}\bigg(\frac{\big|\nabla^2 u + u^{-1}|\nabla u|_h^2(h - 3\nu\otimes\nu)\big|_h^2}{|\nabla u|_h} + R_h|\nabla u|_h\bigg)\,\mathrm{dvol}_h,$$

where $\nu = \nabla u/|\nabla u|_h$, and the integral is taken over the regular set of u.

Proof We first smooth $|\nabla u|_h$ by defining for any $\epsilon > 0$,

$$\phi_\epsilon := \sqrt{|\nabla u|_h^2 + \epsilon}.$$

If Σ_t is a regular level set of u, then the Gauss–Codazzi equation implies that

$$R_h - 2\,\mathrm{Ric}_h(\nu,\nu) = R_{\Sigma_t} + |\mathrm{II}|^2 - H^2,$$

where $\nu = \nabla u/|\nabla u|_h$, and where $\mathrm{II} = \nabla^2_{\Sigma_t} u/|\nabla u|_h$ and $H = \mathrm{tr}_{\Sigma_t}\mathrm{II}$ are the second fundamental form and mean curvature of Σ_t in (M^3, h), respectively. So

$$(R_h - R_{\Sigma_t})|\nabla u|_h^2 = 2\,\mathrm{Ric}_h(\nabla u,\nabla u) + |\nabla^2 u|_h^2 - 2\big|\nabla|\nabla u|_h\big|_h^2 - (\Delta_h u)^2 + 2\Delta_h u\cdot\nabla_h^2 u(\nu,\nu) \tag{23}$$
$$= 2\,\mathrm{Ric}_h(\nabla u,\nabla u) + |\nabla^2 u|_h^2 - 2\big|\nabla|\nabla u|_h\big|_h^2,$$

where we used the equation $\Delta_h u = 0$. Together with the Bochner formula

$$\Delta_h|\nabla u|_h^2 = 2|\nabla^2 u|_h^2 + 2\langle\nabla\Delta_h u,\nabla u\rangle_h + 2\,\mathrm{Ric}_h(\nabla u,\nabla u),$$

we have

$$\Delta_h|\nabla u|^2 = |\nabla^2 u|_h^2 + 2\big|\nabla|\nabla u|_h\big|_h^2 + (R_h - R_{\Sigma_t})|\nabla u|_h^2.$$

At any point on a regular level set Σ_t,

$$\begin{aligned}\Delta_h\phi_\epsilon &= \tfrac12\phi_\epsilon^{-1}\Delta|\nabla u|_h^2 - \tfrac14\phi_\epsilon^{-3}\big|\nabla|\nabla u|_h^2\big|_h^2\\ &= \tfrac12\phi_\epsilon^{-1}\big(|\nabla^2 u|_h^2 + (R_h - R_{\Sigma_t})|\nabla u|_h^2\big) + \frac{\epsilon}{\phi_\epsilon^3}\big|\nabla|\nabla u|_h\big|_h^2\\ &\ge \tfrac12\phi_\epsilon^{-1}\big(|\nabla^2 u|_h^2 + (R_h - R_{\Sigma_t})|\nabla u|_h^2\big).\end{aligned}$$

Taking integration on M_t and using integration by parts and the coarea formula, we have

$$\int_\Sigma\langle\nabla\phi_\epsilon,\nu\rangle_h - \int_{\Sigma_t}\langle\nabla\phi_\epsilon,\nu\rangle_h \ge \frac12\int_{M_t}\phi_\epsilon^{-1}(|\nabla^2 u|_h^2 + R_h|\nabla u|_h^2) - \frac12\int_t^{t_0}\int_{\Sigma_s} R_{\Sigma_s}\frac{|\nabla u|_h}{\phi_\epsilon}\,ds. \tag{24}$$

Define the tensor

$$\mathcal{T}_1 := \nabla^2 u + u^{-1}|\nabla u|_h^2(h - 3\nu\otimes\nu).$$

Then

$$|\mathcal{T}_1|_h^2 = |\nabla^2 u|_h^2 + 6u^{-2}|\nabla u|_h^4 - 6u^{-1}|\nabla u|_h^2\nabla^2 u(\nu,\nu).$$

Notice that

$$\begin{aligned}u^{-1}|\nabla u|_h\nabla^2 u(\nu,\nu) &= u^{-1}|\nabla u|_h\langle\nabla|\nabla u|_h,\nu\rangle_h\\ &= \mathrm{div}(|\nabla u|_h u^{-1}\nabla u) - |\nabla u|\,\mathrm{div}(u^{-1}\nabla u)\\ &= \mathrm{div}(|\nabla u|_h u^{-1}\nabla u) + u^{-2}|\nabla u|_h^3.\end{aligned}$$

So

$$|\mathcal{T}_1|_h^2 = |\nabla^2 u|_h^2 - 6|\nabla u|_h\,\mathrm{div}(|\nabla u|_h u^{-1}\nabla u). \tag{25}$$

Substituting (25) into (24), and using the Gauss–Bonnet theorem, we have

$$\begin{aligned}(26)\quad &\int_\Sigma \langle \nabla\phi_\epsilon,\nu\rangle_h - \int_{\Sigma_t}\langle\nabla\phi_\epsilon,\nu\rangle_h \\ &\geq \frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}(|\mathcal{T}_1|_h^2+R_h|\nabla u|_h^2)+3\int_{M_t}\frac{|\nabla u|_h}{\phi_\epsilon}\operatorname{div}(|\nabla u|_h u^{-1}\nabla u)-\frac{1}{2}\int_t^{t_0}\int_{\Sigma_s}R_{\Sigma_s}\frac{|\nabla u|_h}{\phi_\epsilon}\,ds \\ &\geq \frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}(|\mathcal{T}_1|_h^2+R_h|\nabla u|_h^2)+3\int_\Sigma u^{-1}|\nabla u|_h^2-3\int_{\Sigma_t}u^{-1}|\nabla u|_h^2 \\ &\qquad -3\epsilon\cdot\int_{M_t}\frac{\operatorname{div}(|\nabla u|u^{-1}\nabla u)}{\phi_\epsilon^2+\phi_\epsilon|\nabla u|_h}-4\pi(t_0-t)+\frac{\epsilon}{2}\cdot\int_t^{t_0}\int_{\Sigma_s}\frac{R_{\Sigma_s}}{\phi_\epsilon^2+\phi_\epsilon|\nabla u|_h}\,ds.\end{aligned}$$

Using the asymptotical estimates (19), we know

$$\epsilon\cdot\int_{M_t}\frac{\operatorname{div}(|\nabla u|u^{-1}\nabla u)}{\phi_\epsilon^2+\phi_\epsilon|\nabla u|_h}\leq\epsilon\cdot\int_{M_t}\frac{u^{-1}|\nabla u|_h\nabla^2u(\nu,\nu)}{\phi_\epsilon^2+\phi_\epsilon|\nabla u|_h}\leq C(h)\epsilon\cdot t^{-3}, \tag{27}$$

$$\epsilon\cdot\int_t^{t_0}\int_{\Sigma_s}\frac{R_{\Sigma_s}}{\phi_\epsilon^2+\phi_\epsilon|\nabla u|_h}\,ds\leq C(h)\epsilon\cdot t^{-3}. \tag{28}$$

For any regular value $t\in(0,1)$ of u, as in the proof of [Bray et al. 2022, Theorem 1.2], we can divide the integrals into two disjoint parts so that one is the integral over preimage of an open set containing the critical values of u. First letting $\epsilon\to 0$, and then choosing a sequence of regular values $t_i\to 0$, together with the asymptotical estimates, we have

$$\int_\Sigma\langle\nabla|\nabla u|_h,\nu\rangle_h+4\pi t_0-3t_0^{-1}\int_\Sigma|\nabla u|_h^2\geq\frac{1}{2}\int_M\frac{1}{|\nabla u|_h}\cdot(|\mathcal{T}_1|_h^2+R_h|\nabla u|_h^2)\,\mathrm{dvol}_h\,.$$

Since the mean curvature of Σ in (M,g) is zero, $\nabla_g^2 f(\nu,\nu)=0$, which implies that

$$\int_\Sigma\langle\nabla|\nabla u|_h,\nu\rangle_h=\frac{2}{t_0}\int_\Sigma|\nabla u|_h^2.$$

So we have

$$4\pi t_0-t_0^{-1}\int_\Sigma|\nabla u|_h^2\geq\frac{1}{2}\int_M\frac{1}{|\nabla u|_h}\cdot(|\mathcal{T}_1|_h^2+R_h|\nabla u|_h^2)\,\mathrm{dvol}_h\,. \tag{29}$$

In particular, since $R_h\geq 0$,

$$\int_\Sigma|\nabla u|_h^2\leq 4\pi t_0^2. \tag{30}$$

On the other hand,

$$\left(\int_\Sigma|\nabla u|_h\right)^2\leq\int_\Sigma|\nabla u|_h^2\cdot\mathrm{Area}_h(\Sigma)=\frac{1}{16}\int_\Sigma|\nabla u|_h^2\cdot\mathrm{Area}_g(\Sigma)$$

and

$$\int_\Sigma|\nabla u|_h\,\mathrm{dA}_h=t_0\int_\Sigma|\nabla f|_g\,\mathrm{dA}_g=2\pi t_0\mathscr{C}(\Sigma,g),$$

so

$$\int_\Sigma |\nabla u|_h^2 \geq \frac{64\pi^2 t_0^2 \mathscr{C}(\Sigma,g)^2}{\mathrm{Area}_g(\Sigma)}.$$

Together with (29), we have

$$4\pi t_0\left(1-\frac{16\pi\mathscr{C}(\Sigma,g)^2}{\mathrm{Area}_g(\Sigma)}\right) \geq \frac{1}{2}\int_M \frac{1}{|\nabla u|_h}\cdot(|\mathscr{T}_1|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h\,. \tag{31}$$

By the Penrose inequality $\mathrm{Area}_g(\Sigma)\leq 16\pi m(g)^2$, we have

$$4\pi t_0\left(1-\frac{\mathscr{C}(\Sigma,g)^2}{m(g)^2}\right) \geq \frac{1}{2}\int_M \frac{1}{|\nabla u|_h}\cdot(|\mathscr{T}_1|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h\,.$$

This concludes the proof. □

Notice that (31) gives another proof of the following mass–area–capacity inequality; cf [Bray and Miao 2008, Theorem 1].

Theorem 4.2 *Let (M^3,g) be a complete, one-ended asymptotically flat 3-manifold with nonnegative scalar curvature and a compact connected outermost minimal boundary Σ. Then*

$$\mathscr{C}(\Sigma,g)^2 \leq \frac{1}{16\pi}\mathrm{Area}_g(\Sigma)\leq m(g)^2. \tag{32}$$

By the symmetry of $(\widetilde{M},g)$, applying Proposition 4.1 to $\tilde{u}:=t_0^2/u$ and $\tilde{h}:=(1-f)^4g=(u/t_0)^4h$, we get

$$1-\frac{\mathscr{C}(\Sigma,g)^2}{m(g)^2}\geq\frac{1}{8\pi t_0}\int_{(M',h)}\left(\frac{\left|\nabla^2\tilde{u}-\tilde{u}^{-1}|\nabla\tilde{u}|_h^2(h-\nu\otimes\nu)\right|_h^2}{|\nabla\tilde{u}|_h}+R_h|\nabla\tilde{u}|_h\right)\mathrm{dvol}_h\,. \tag{33}$$

Proposition 4.1 is not sufficient for our control of f around infinity. For this purpose, using a similar argument together with (30), we prove the following integration inequality.

Proposition 4.3 *For*

$$u=\frac{2}{\mathscr{C}(\Sigma,g)}\cdot\frac{1-f}{f},$$

we have

$$1-\frac{\mathscr{C}(\Sigma,g)}{m(g)}\geq\frac{\mathscr{C}(\Sigma,g)}{96\pi}\int_{(M,h)}\frac{1}{|\nabla u|_h}\left(\left|\nabla^2 u+|\nabla u|_h^{\frac{3}{2}}(h-3\nu\otimes\nu)\right|_h^2+R_h|\nabla u|_h^2\right)\mathrm{dvol}_h\,. \tag{34}$$

Proof The proof is almost the same as the proof of Proposition 4.1. The difference is that instead of considering the tensor $\mathscr{T}_1$, we define

$$\mathscr{T}_2 := \nabla^2 u + |\nabla u|_h^{3/2}(h-3\nu\otimes\nu).$$

Then

$$|\mathscr{T}_2|_h^2 = |\nabla^2 u|_h^2 + 6|\nabla u|_h^3 - 6|\nabla u|_h^{3/2}\nabla^2 u(\nu,\nu). \tag{35}$$

Notice that

$$|\nabla u|_h^{\frac{1}{2}}\nabla^2 u(\nu,\nu) = |\nabla u|_h^{\frac{1}{2}}\langle \nabla|\nabla u|_h,\nu\rangle_h = \operatorname{div}(|\nabla u|_h^{\frac{1}{2}}\nabla u) - |\nabla u|_h\langle \nabla|\nabla u|_h^{-\frac{1}{2}},\nabla u\rangle_h$$
$$= \operatorname{div}(|\nabla u|_h^{\frac{1}{2}}\nabla u) + \tfrac{1}{2}|\nabla u|_h^{\frac{1}{2}}\langle \nabla|\nabla u|_h,\nu\rangle_h,$$

which implies that

$$|\nabla u|_h^{\frac{1}{2}}\nabla^2 u(\nu,\nu) = 2\operatorname{div}(|\nabla u|_h^{\frac{1}{2}}\nabla u). \tag{36}$$

Substituting (35) and (36) into (24), and using the Gauss–Bonnet theorem and asymptotical estimates, gives

$$\int_\Sigma \langle \nabla\phi_\epsilon,\nu\rangle_h - \int_{\Sigma_t}\langle \nabla\phi_\epsilon,\nu\rangle_h \tag{37}$$
$$\geq \frac{1}{2}\int_{M_t}\phi_\epsilon^{-1}(|\mathcal{T}_2|_h^2 + R_h|\nabla u|_h^2) - 4\pi(t_0-t) - 3\int_{M_t}|\nabla u|_h^2 + 6\int_\Sigma |\nabla u|_h^{\frac{3}{2}} - 6\int_{\Sigma_t}|\nabla u|_h^{\frac{3}{2}} - C(g,t)\cdot\epsilon.$$

First taking $\epsilon\to 0$, and then taking $t\to 0$, we have

$$\int_\Sigma \langle \nabla|\nabla u|_h,\nu\rangle_h + 4\pi t_0 + 3\int_M |\nabla u|_h^2 - 6\int_\Sigma |\nabla u|_h^{\frac{3}{2}} \geq \frac{1}{2}\int_M \frac{1}{|\nabla u|_h}(|\mathcal{T}_2|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h.$$

Notice that

$$\int_{\Sigma_t}|\nabla u|_h = \int_\Sigma |\nabla u|_h = 2\pi t_0\mathscr{C}(\Sigma,g), \quad \text{so} \quad \int_M |\nabla u|_h^2 = \int_0^{t_0}\int_{\Sigma_t}|\nabla u|_h = 2\pi t_0^2\mathscr{C}(\Sigma,g).$$

Using (30), we have

$$12\pi t_0 + 6\pi t_0^2\mathscr{C}(\Sigma,g) - 6\int_\Sigma |\nabla u|_h^{\frac{3}{2}} \geq \frac{1}{2}\int_M \frac{1}{|\nabla u|_h}(|\mathcal{T}_2|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h.$$

By the Hölder inequality,

$$\int_\Sigma |\nabla u|_h \leq \left(\int_\Sigma |\nabla u|_h^{\frac{3}{2}}\right)^{\frac{2}{3}}\cdot \mathrm{Area}_h(\Sigma)^{\frac{1}{3}},$$

we know

$$\int_\Sigma |\nabla u|_h^{\frac{3}{2}} \geq \frac{4(2\pi t_0\mathscr{C}(\Sigma,g))^{\frac{3}{2}}}{\mathrm{Area}_g(\Sigma)^{\frac{1}{2}}}.$$

Thus,

$$1 + \frac{t_0\mathscr{C}(\Sigma,g)}{2} - 2\left(\frac{8\pi t_0\mathscr{C}(\Sigma,g)^3}{\mathrm{Area}_g(\Sigma)}\right)^{\frac{1}{2}} \geq \frac{1}{24\pi t_0}\int_M \frac{1}{|\nabla u|_h}(|\mathcal{T}_2|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h.$$

If we take $t_0 = 2/\mathscr{C}(\Sigma,g)$, then

$$1 - \left(\frac{16\pi\mathscr{C}(\Sigma,g)^2}{\mathrm{Area}_g(\Sigma)}\right)^{\frac{1}{2}} \geq \frac{\mathscr{C}(\Sigma,g)}{96\pi}\int_M \frac{1}{|\nabla u|_h}(|\mathcal{T}_2|_h^2 + R_h|\nabla u|_h^2)\,\mathrm{dvol}_h.$$

Together with the Penrose inequality, this concludes the proof. □

Since at any regular point,

$$\big|\nabla(|\nabla u|_h^{\frac{1}{2}}-u)\big|_h \le \frac{\big|\nabla|\nabla u|_h - 2|\nabla u|_h^{\frac{3}{2}}\nu\big|_h}{2|\nabla u|_h^{\frac{1}{2}}} \le \frac{\big|\nabla^2 u + |\nabla u|_h^{\frac{3}{2}}(h-3\nu\otimes\nu)\big|_h}{2|\nabla u|_h^{\frac{1}{2}}},$$

we have the estimate

$$\int_{(M,h)} \big|\nabla(|\nabla u|_h^{\frac{1}{2}}-u)\big|_h^2 \,\mathrm{dvol}_h \le C\cdot\left(\frac{1}{\mathscr{C}(\Sigma,g)}-\frac{1}{m(g)}\right). \tag{38}$$

Similarly, for $\widetilde{u}=4/(\mathscr{C}(\Sigma,g)^2u)$ on (M',h), we have

$$\big|\nabla|\nabla\widetilde{u}|_h^{\frac{1}{2}}\big|_h = \frac{|\nabla|\nabla\widetilde{u}|_h|_h}{2|\nabla\widetilde{u}|_h^{\frac{1}{2}}} \le \frac{\big|\nabla^2\widetilde{u}-\widetilde{u}^{-1}|\nabla\widetilde{u}|_h^2(h-\nu\otimes\nu)\big|_h}{2|\nabla\widetilde{u}|_h^{\frac{1}{2}}},$$

which together with (33) implies that

$$\int_{(M',h)} \left|\nabla\left(|\nabla\widetilde{u}|_h^{\frac{1}{2}}-\frac{2}{\mathscr{C}(\Sigma,g)}\right)\right|_h^2 \mathrm{dvol}_h \le \frac{C}{\mathscr{C}(\Sigma,g)}\cdot\left(1-\frac{\mathscr{C}(\Sigma,g)^2}{m(g)^2}\right). \tag{39}$$

Notice that $|\nabla u|_h^{\frac{1}{2}}-u$ and $|\nabla\widetilde{u}|_h^{\frac{1}{2}}-2/\mathscr{C}(\Sigma,g)$ agree on the middle sphere Σ. Define

$$P(x) := \begin{cases} \big||\nabla u|_h^{1/2}(x)-u(x)\big| & \text{when } x\in M, \\ \left||\nabla\widetilde{u}|_h^{1/2}(x)-\dfrac{2}{\mathscr{C}(\Sigma,g)}\right| & \text{when } x\in M'. \end{cases} \tag{40}$$

Then P is a continuous function, smooth around the end ∞_1 in M, and we have proved the following L^2-estimate for its gradient.

Proposition 4.4 *Let $(\widetilde{M},g)$ be a two-ended asymptotically flat 3-manifolds obtained as in Proposition 3.1, and assume $m(g)\ge m_0>0$. Let f be a solution of (14), and $h:=f^4g$ a conformal metric on $\widetilde{M}$. For the function P defined in (40), there exists a uniform constant $C>0$ such that*

$$\int_{(\widetilde{M},h)} |\nabla P|_h^2 \,\mathrm{dvol}_h \le \frac{C}{\mathscr{C}(\Sigma,g)}\left(1-\frac{\mathscr{C}(\Sigma,g)^2}{m(g)^2}\right). \tag{41}$$

Moreover,

$$\lim_{x\to\infty_1} P(x) = \lim_{x\to\infty_2} P(x) = 0. \tag{42}$$

5 Harmonic coordinate for conformal metric

Let $(\widetilde{M},g)$ be a two-ended asymptotically flat 3-manifolds obtained as in Proposition 3.1, f the harmonic function defined by (17) on $(\widetilde{M}^3,g)$, and $h:=f^4g$ the conformal metric. Then on the end ∞_2, since g is harmonically flat, we have $h_{ij}(x)=f^4(x)V^4(x)\delta_{ij}$, where $V(x)$ is a positive bounded δ-harmonic function. So on the end ∞_2, h is conformal to a punctured ball with the conformal factor $(fV)^4(x)$, where $(fV)(x)$ is a bounded δ-harmonic function in the punctured ball. Hence, by the removable singularity theorem, fV can be extended to the whole ball, which together with the expansion (18) and (9) implies that h can be extended smoothly over the one-point compactification $\widetilde{M}^* := \widetilde{M} \cup \{\infty_2\}$. Also the function P defined in (40) can be extended continuously to $\widetilde{M}^*$.

By standard computations, $(\widetilde{M}^*, h)$ also has nonnegative scalar curvature and has a single harmonically flat end ∞_1 with ADM mass $m(h) = m(g) - \mathscr{C}(\Sigma, g) \geq 0$; see [Bray 2001, equation (84)].

In the following, we assume that $|m(g) - m_0| \ll 1$ and $|\mathscr{C}(\Sigma, g) - m_0| \ll 1$ for a fixed $m_0 > 0$. By the mass-area-capacity inequality, $|\mathrm{Area}_g(\Sigma) - 16\pi m_0^2| \ll 1$.

Let $\{x^j\}_{j=1}^3$ denote the asymptotically flat coordinate system of the end ∞_1. We firstly solve the harmonic coordinate functions u^j, for each $j \in \{1, 2, 3\}$, so that

$$\Delta_h u^j = 0 \qquad \text{and} \qquad |u^j(x) - x^j| = o(|x|^{1-\sigma}) \quad \text{as } x \to \infty_1, \tag{43}$$

where $\sigma > \frac{1}{2}$ is the order of the asymptotic flatness. Denote by $\boldsymbol{u}$ the resulting harmonic map

$$\boldsymbol{u} := (u^1, u^2, u^3) \colon (\widetilde{M}^*, h) \to \mathbb{R}^3.$$

For any fixed small $0 < \epsilon \ll 1$, by [Dong and Song 2025], we know that there exists a connected region $\mathscr{E}_1 \subset (\widetilde{M}^*, h)$ containing ∞_1, with smooth boundary, such that

$$\mathrm{Area}_h(\partial \mathscr{E}_1) \leq m(h)^{2-\epsilon},$$

and $\boldsymbol{u} \colon \mathscr{E}_1 \to \mathscr{Y}_1 := \boldsymbol{u}(\mathscr{E}_1) \subset \mathbb{R}^3$ is a diffeomorphism with the Jacobian satisfying

$$|\mathrm{Jac}\, \boldsymbol{u} - \mathrm{Id}| \leq \Psi(m(h)),$$

and under the identification by $\boldsymbol{u}$, the metric tensor satisfies

$$\sum_{j,k=1}^{3} (h_{jk} - \delta_{jk})^2 \leq m(h)^{2\epsilon}.$$

Moreover, $d_{\mathrm{pGH}}\big((\mathscr{E}_1, \hat{d}_{\mathscr{E}_1}), (\mathbb{R}^3, d_{\mathrm{Eucl}})\big) \leq m(h)^{\epsilon}$.

Applying [Dong and Song 2025, Lemmas 3.2 and 3.3] to the function P defined in (40), and using a uniform approximation by smooth functions (see [Evans and Gariepy 2015, Theorem 4.3]) and Proposition 4.4, we can find a connected closed region $\mathscr{E}_2 \subset \mathscr{E}_1 \cap \{P \leq m(h)^{\epsilon}\}$, which contains a neighborhood of the infinity, and $\partial \mathscr{E}_2$ is a compact smooth surface satisfying

$$\mathrm{Area}_h(\partial \mathscr{E}_2) \leq C(m_0) \cdot m(h)^{1-\epsilon}. \tag{44}$$

We claim that $\mathscr{E}_2 \cap \Sigma \neq \varnothing$. Otherwise, since a neighborhood of the infinity is included in $\mathscr{E}_2$, a connected component of $\partial \mathscr{E}_2$ will enclose Σ in M. By the outermost minimal property of Σ, we know $\mathrm{Area}_g(\Sigma) \leq \mathrm{Area}_g(\partial \mathscr{E}_2) \leq 16\mathrm{Area}_h(\partial \mathscr{E}_2) \leq C(m_0) \cdot m(h)^{1-\epsilon}$, which contradicts with the fact that $|\mathrm{Area}_g(\Sigma) - m_0| \ll 1$ and $m(h) \ll 1$.

Notice that for any x such that $P(x) \le m(h)^\epsilon$, from (40), if $f(x) \le \frac{1}{2}$, we have

$$\Big|(1-f)^{-1}|\nabla f|_h^{\frac{1}{2}} - \left(\frac{2}{\mathscr{C}(\Sigma,g)}\right)^{\frac{1}{2}}\Big| \le C(m_0)\cdot m(h)^\epsilon;$$

if $f(x) \ge \frac{1}{2}$, we have

$$f^{-1}(1-f)\Big|(1-f)^{-1}|\nabla f|_h^{\frac{1}{2}} - \left(\frac{2}{\mathscr{C}(\Sigma,g)}\right)^{\frac{1}{2}}\Big| \le C(m_0)\cdot m(h)^\epsilon.$$

In summary, we have the following proposition.

Proposition 5.1 *There exists a connected region $\mathscr{E} \subset (\widetilde{M}^*, h)$ containing ∞_1, with smooth boundary, such that*

$$\mathrm{Area}_h(\partial\mathscr{E}) \le \Psi(m(h)),$$

and $\boldsymbol{u}: \mathscr{E} \to \mathscr{Y} := \boldsymbol{u}(\mathscr{E}) \subset \mathbb{R}^3$ is a diffeomorphism with the Jacobian satisfying

$$|\mathrm{Jac}\,\boldsymbol{u} - \mathrm{Id}| \le \Psi(m(h)),$$

and under the identification by $\boldsymbol{u}$, the metric tensor satisfies

$$\sum_{j,k=1}^{3}(h_{jk}-\delta_{jk})^2 \le \Psi(m(h)).$$

For any base point $p \in \mathscr{E} \cap \Sigma$, any $x_0 \in \mathbb{R}^3$ and any fixed $D > 0$,

$$d_{\mathrm{pGH}}\big((\widehat{B}_{h,\mathscr{E}}(p,D), \hat{d}_{h,\mathscr{E}}, p), (B_{\mathrm{Eucl}}(x_o,D), d_{\mathrm{Eucl}}, x_o)\big) \le \Psi(m(h)|D),$$

and $\Phi_{\boldsymbol{u}(p)} \circ \boldsymbol{u}$ gives a $\Psi(m(h)|D)$-pGH approximation, where $\Phi_{\boldsymbol{u}(p)}$ is the translation diffeomorphism of $\mathbb{R}^3$ mapping $\boldsymbol{u}(p)$ to x_o.

Moreover, inside $\mathscr{E}$, we have

$$\left|\,|\nabla f|_h - \frac{2(1-f)^2}{\mathscr{C}(\Sigma,g)}\right| \le \Psi(m(h)). \tag{45}$$

6 $W^{1,2}$-convergence of elliptic equations

Assume that $(\widetilde{M}_i^3, g_i)$ is a sequence of complete two-ended asymptotically flat 3-manifolds obtained as in Proposition 3.1 with $m(g_i) \to m_0$ and $\mathscr{C}(\Sigma_i, g_i) \to m_0$ for some positive constant $m_0 > 0$, where $\Sigma_i \subset (\widetilde{M}_i^3, g_i)$ is the minimal surface such that $(\widetilde{M}_i^3, g_i)$ is symmetric about Σ_i. Let

$$\varepsilon_i := m(g_i) - \mathscr{C}(\Sigma_i, g_i) \to 0.$$

Notice that from (32), $|\mathrm{Area}_{g_i}(\Sigma_i) - 16\pi m_0^2| \to 0$, and particularly $\mathrm{Area}_{g_i}(\Sigma_i) \ge A_0 > 0$ for some uniform $A_0 > 0$.

Let f_i be the harmonic functions defined by (17) on $(\widetilde{M}_i^3, g_i)$, (M_i, g_i) be the half of $(\widetilde{M}_i^3, g_i)$ such that f_i satisfies (20), and $h_i := f_i^4 g_i$ the conformal metrics. Using the same notation as in previous section, let $\widetilde{M}_i^*$ be the one-point compactification $\widetilde{M}_i \cup \{\infty_2\}$, then $(\widetilde{M}_i^*, h_i)$ is a sequence of one-ended asymptotically flat 3-manifolds with nonnegative scalar curvature and ADM mass $m(h_i) = \varepsilon_i \to 0$.

Let $\mathscr{E}_i$ be regions given by Proposition 5.1. Taking any base point $p_i \in \mathscr{E}_i \cap \Sigma_i = \mathscr{E}_i \cap \{f_i = \frac{1}{2}\}$, then

$$(\mathscr{E}_i, \hat{d}_{h_i,\mathscr{E}_i}, p_i) \to (\mathbb{R}^3, d_{\mathrm{Eucl}}, x_o)$$

in the pointed measured Gromov–Hausdorff topology, where $x_o \in \mathbb{R}^3$, the harmonic maps $\boldsymbol{u}_i$ with $\boldsymbol{u}_i(p_i) = x_o$ are $\Psi(\varepsilon_i)$-pGH approximation, and for any $D > 0$, $(\boldsymbol{u}_i)_\sharp(\mathrm{dvol}_{h_i}|_{\hat{B}_{h_i,\mathscr{E}_i}(p_i,D)})$ weakly converges to $\mathrm{dvol}_{\mathrm{Eucl}}|_{B(x_o,D)}$ as $i \to \infty$.

By (45) and the Arzelà–Ascoli theorem, up to a subsequence, $f_i \to f_\infty$ locally uniformly for some nonnegative bounded Lipschitz function f_∞ on $\mathbb{R}^3$.

Lemma 6.1 *Up to a subsequence, f_i converges to f_∞ in the weakly $W^{1,2}$-sense. That is, for any uniformly converging sequence of compactly supported Lipschitz functions $\psi_i \to \psi$ and $\nabla\psi_i \to \nabla\psi$ in L^2, we have*

$$\lim_{i\to\infty} \int_{\mathscr{E}_i} f_i \psi_i \,\mathrm{dvol}_{h_i} = \int_{\mathbb{R}^3} f_\infty \psi \,\mathrm{dvol}_{\mathrm{Eucl}},$$

$$\lim_{i\to\infty} \int_{\mathscr{E}_i} \langle \nabla f_i, \nabla \psi_i \rangle_{h_i} \,\mathrm{dvol}_{h_i} = \int_{\mathbb{R}^3} \langle \nabla f_\infty, \nabla\psi \rangle_{\mathrm{Eucl}} \,\mathrm{dvol}_{\mathrm{Eucl}}\,.$$

Proof Under the diffeomorphism $\boldsymbol{u}_i$, we can identify $\mathscr{E}_i$ as a subset in $\mathbb{R}^3$. Suppose that ψ_i and ψ have support in $U \subset B(x_o, D)$ for some $D > 0$. Then since $f_i \to f_\infty$ uniformly, by Proposition 5.1 we know

$$\begin{aligned}
\lim_{i\to\infty} \left| \int_{U\cap\mathscr{E}_i} f_i\psi_i \,\mathrm{dvol}_{h_i} - \int_U f_\infty\psi \,\mathrm{dvol}_{\mathrm{Eucl}} \right| &\\
\le \lim_{i\to\infty} \int_{U\cap\mathscr{E}_i} |f_i\psi_i - f_\infty\psi| \,\mathrm{dvol}_{h_i} &+ \lim_{i\to\infty} \left| \int_{U\cap\mathscr{E}_i} f_\infty\psi \,\mathrm{dvol}_{h_i} - \int_U f_\infty\psi \,\mathrm{dvol}_{\mathrm{Eucl}} \right| \\
\le C \lim_{i\to\infty} \int_{U\cap\mathscr{E}_i} |f_i\psi_i - f_\infty\psi| \,\mathrm{dvol}_{\mathrm{Eucl}} &= 0.
\end{aligned}$$

Similarly, since $|\nabla f_i|_{h_i}$ and $|\nabla \psi_i|_{h_i}$ are uniformly bounded and h_i converges uniformly to g_{Eucl},

$$\begin{aligned}
\lim_{i\to\infty} \left| \int_{U\cap\mathscr{E}_i} \langle \nabla f_i, \nabla\psi_i \rangle_{h_i} \,\mathrm{dvol}_{h_i} - \int_U \langle \nabla f_\infty, \nabla\psi \rangle_{\mathrm{Eucl}} \,\mathrm{dvol}_{\mathrm{Eucl}} \right| &\\
\le C \lim_{i\to\infty} \int_{U\cap\mathscr{E}_i} |\langle \nabla f_i, \nabla\psi_i \rangle_{h_i} - \langle \nabla f_\infty, \nabla\psi \rangle_{\mathrm{Eucl}}| \,\mathrm{dvol}_{\mathrm{Eucl}} &\\
= C \lim_{i\to\infty} \int_{U\cap\mathscr{E}_i} |h_i^{jk} \partial_j f_i \partial_k \psi_i - \delta^{jk} \partial_j f_\infty \partial_k \psi| \,\mathrm{dvol}_{\mathrm{Eucl}} &\\
\le C \lim_{i\to\infty} \sum_{j=1}^3 \int_{U\cap\mathscr{E}_i} \big(|\langle \partial_j f_i - \partial_j f_\infty \rangle| \cdot |\partial_j \psi| + |\partial_j \psi_i - \partial_j \psi| \big) \,\mathrm{dvol}_{\mathrm{Eucl}} &= 0,
\end{aligned}$$

where in the first inequality we used $\mathrm{Vol}(U \setminus \mathscr{E}_i) \to 0$, and in the last step, we used the assumption that $\partial_j \psi_i \to \partial_j \psi$ in L^2 and the fact that uniformly Lipschitz sequence f_i has a locally $W^{1,2}$-weak convergent subsequence. □

Proposition 6.2 *The limit f_∞ satisfies the following equation weakly on $\mathbb{R}^3$:*

$$\Delta_{\mathrm{Eucl}} f_\infty^2 = \frac{24}{m_0^2} \cdot (1 - f_\infty)^4. \tag{46}$$

Proof It's enough to show that for any smooth function ψ on $\mathbb{R}^3$ with compact support in $U \subset B(x_o, D)$,

$$-\int_U \langle \nabla f_\infty^2, \nabla \psi \rangle_{\mathrm{Eucl}} \, \mathrm{dvol}_{\mathrm{Eucl}} = \frac{24}{m_0^2} \int_U (1 - f_\infty)^4 \psi \, \mathrm{dvol}_{\mathrm{Eucl}}.$$

Define $\psi_i := \psi \circ \boldsymbol{u}_i$ on $\mathscr{E}_i$. Then ψ_i are uniformly Lipschitz and $\psi_i \to \psi$ uniformly.

Notice that

$$\Delta_{h_i} f_i^2 = 6 |\nabla f_i|_{h_i}^2.$$

Using integration by parts, for $U_i := \boldsymbol{u}_i^{-1} U \subset \mathscr{E}_i$, we have

$$-\int_{U_i} \langle \nabla f_i^2, \nabla \psi_i \rangle_{h_i} \, \mathrm{dvol}_{h_i} = \int_{U_i} 6 \psi_i |\nabla f_i|_{h_i}^2 \, \mathrm{dvol}_{h_i} - \int_{\partial U_i} \psi_i \langle \nabla f_i, \vec{n} \rangle \, \mathrm{dA}_{h_i}.$$

Since $|\psi_i|, |\nabla f_i|_{h_i} \le C$ on U_i,

$$\left| \int_{\partial U_i} \psi_i \langle \nabla f_i, \vec{n} \rangle \, \mathrm{dA}_{h_i} \right| \le C \cdot \mathrm{Area}_{h_i}(\partial \mathscr{E}_i) \to 0.$$

Using (45), we have

$$\lim_{i \to \infty} \int_{U_i} 6 \psi_i |\nabla f_i|_{h_i}^2 \, \mathrm{dvol}_{h_i} = \lim_{i \to \infty} \int_{U_i} \frac{24}{\mathscr{C}(\Sigma_i, g_i)^2} (1 - f_i)^4 \psi_i = \frac{24}{m_0^2} \int_U (1 - f_\infty)^4 \psi \, \mathrm{dvol}_{\mathrm{Eucl}}.$$

From Lemma 6.1, we know

$$\lim_{i \to \infty} \int_{U_i} \langle \nabla f_i^2, \nabla \psi_i \rangle_{h_i} \, \mathrm{dvol}_{h_i} = \int_U \langle \nabla f_\infty^2, \nabla \psi \rangle_{\mathrm{Eucl}} \, \mathrm{dvol}_{\mathrm{Eucl}},$$

which concludes the proof. □

From standard elliptic theory, we know $f_\infty^2 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$. Also from (45), we have

$$|\nabla f_\infty^2|_{\mathrm{Eucl}} \le \frac{4 f_\infty (1 - f_\infty)^2}{m_0},$$

that is, $|\nabla (1 - f_\infty^2)^{-1}|_{\mathrm{Eucl}} \le C(m_0)$ at any point $x \in \mathbb{R}^3$ such that $f_\infty(x) \ne 1$. Since $f_\infty(x_o) = \frac{1}{2}$, for any $D > 0$ and any $x \in B_{\mathrm{Eucl}}(x_o, D)$, we have

$$(1 - f_\infty^2)^{-1}(x) \le \tfrac{4}{3} + C(m_0) D.$$

Together with the uniform convergence of f_i, it holds that

$$\widehat{B}_{h_i,\mathscr{E}_i}(p_i, D) \subset \left\{ x \in \widetilde{M}_i^* : 1 - f_i(x) \geq \frac{1}{C(m_0)(1+D)} \right\}. \tag{47}$$

Consider another function $\xi_i(x) := f_i^2/(1-f_i)^2$ defined on $(\widetilde{M}_i^*, h_i)$, which satisfies $\xi_i(p_i) = 1$ and

$$\Delta_{h_i}\xi_i = 6(1-f_i)^{-4}|\nabla f_i|_{h_i}^2. \tag{48}$$

For any fixed $D > 0$ and any $x \in \widehat{B}_{h_i,\mathscr{E}_i}(p_i, D)$, by (45) and (47), we have

$$|\nabla \xi_i|_{h_i}(x) = \frac{2f_i|\nabla f_i|_{h_i}(x)}{(1-f_i)^3} \leq C(m_0, D). \tag{49}$$

Similarly, by the Arzelà–Ascoli theorem, up to a subsequence, $\xi_i \to \xi_\infty$ locally uniformly and weakly $W^{1,2}$ for some Lipschitz function ξ_∞ on $\mathbb{R}^3$, and $\xi_\infty = f_\infty^2/(1-f_\infty)^2$. We also have

$$\Delta_{\mathrm{Eucl}}\xi_\infty = \frac{24}{m_0^2}, \tag{50}$$

so ξ_∞ is a smooth function. Using those properties, we prove the following rigidity:

Lemma 6.3
$$\xi_\infty(x) = \frac{4}{m_0^2}|x|^2$$
and

$$f_\infty(x) = \left(1 + \frac{m_0}{2|x|}\right)^{-1}. \tag{51}$$

Proof Using the relation $\xi_\infty = f_\infty^2/(1-f_\infty)^2$, (46) and (50), at any point x such that $f_\infty(x) \neq 0$, we have

$$\begin{aligned}\Delta_{\mathrm{Eucl}}\xi_\infty &= (1-f_\infty)^{-3}\Delta_{\mathrm{Eucl}} f_\infty^2 + 6f_\infty(1-f_\infty)^{-4}|\nabla f_\infty|^2 \\ &= \frac{24}{m_0^2}\cdot(1-f_\infty) + 6f_\infty(1-f_\infty)^{-4}|\nabla f_\infty|^2 = \frac{24}{m_0^2},\end{aligned}$$

which implies that $|\nabla f_\infty|^2 = (4/m_0^2)\cdot(1-f_\infty)^4$. So

$$|\nabla \xi_\infty|^2 = 4(1-f_\infty)^{-6} f_\infty^2|\nabla f_\infty|^2 = \frac{16}{m_0^2} f_\infty^2(1-f_\infty)^{-2} = \frac{16}{m_0^2}\xi_\infty,$$

which also holds at any $x \in \mathbb{R}^3$ since ξ_∞ is smooth.

From the Bochner formula, we have

$$\left|\nabla^2\xi_\infty - \frac{8}{m_0^2}\delta\right|^2 = 0.$$

Consider the smooth function

$$\eta(x) := \xi_\infty(x) - \frac{4}{m_0^2}|x|^2.$$

Then $\nabla^2\eta = 0$, ie η is a linear function, and $\eta(x_o) = 0$. Assuming that $\eta(x) = (x - x_o)\cdot b$ for some $b \in \mathbb{R}^3$. Using $|\nabla\xi_\infty|^2 = (16/m_0^2)\xi_\infty$, we have $|b|^2 + (16/m_0^2)x_o \cdot b = 0$. So

$$\xi_\infty(x) = \frac{4}{m_0^2}\left|x + \frac{m_0^2}{8}b\right|^2.$$

Up to a translation of the coordinates, we can take $x_o = (m_0/2, 0, 0)$ and $\xi(x) = (4/m_0^2)|x|^2$. □

7 Proof of main theorems

We first prove the stability of the mass–capacity inequality.

Proof of Theorem 1.2 Assume that (M_i^3, g_i) is a sequence of complete one-ended asymptotically flat 3-manifolds with nonnegative scalar curvature and compact connected outermost horizon boundaries Σ_i. Suppose that for some constant $m_0 > 0$, both $m(g_i) \to m_0$ and $\mathscr{C}(\Sigma_i, g_i) \to m_0$.

By Proposition 3.1, without loss of generality, we can assume that there exists a doubling $(\widetilde{M}_i, g_i)$ of each (M_i, g_i) such that $(\widetilde{M}_i, g_i)$ is symmetric about minimal surface Σ_i, and has nonnegative scalar curvature and two harmonically flat ends. Let f_i be harmonic functions defined by (14), $h_i = f_i^4 g_i$ and $\widetilde{M}_i^*$ the compactification. Set $\varepsilon_i := m(h_i) \to 0$. Using the same notation in last section, we have proved that there exist smooth regions $\mathscr{E}_i \subset \widetilde{M}_i^*$ such that $\mathrm{Area}_{h_i}(\partial\mathscr{E}_i) \to 0$, and for base point $p_i \in \mathscr{E}_i \cap \Sigma_i$,

$$(\mathscr{E}_i, \hat{d}_{h_i,\mathscr{E}_i}, p_i) \xrightarrow{\text{pm-GH}} (\mathbb{R}^3, d_{\mathrm{Eucl}}, x_o),$$

where $x_o = (m_0/2, 0, 0)$ and

$$f_i \to f_\infty(x) = \left(1 + \frac{m_0}{2|x|}\right)^{-1}$$

locally uniformly.

In the following, we will only consider the manifold M_i, which is equivalent to $\left\{\frac{1}{2} \le f_i < 1\right\}$. Since $h_i = f_i^4 g_i$, we know g_i and h_i are two uniformly equivalent metrics on M_i. When discussing uniform upper or lower bounds on volume, area, or distance, etc, there is no difference between using either metric, so we will omit the subscript for simplicity. Additionally, we always identify $\mathscr{E}_i$ as a subset of $\mathbb{R}^3$ using the diffeomorphism $\boldsymbol{u}_i$.

Notice that on $\left\{f_\infty \ge \frac{1}{2}\right\}$, the standard Schwarzschild metric is given by

$$g_{\mathrm{Sch}} = f_\infty^{-4} g_{\mathrm{Eucl}}.$$

Since $g_i = f_i^{-4}h_i$, for any $D > 0$ and any $x \in \hat{B}_{h_i,\mathscr{E}_i}(p_i, D) \cap \left\{f_i \ge \frac{1}{2}\right\}$, we have

$$\|g_i - g_{\mathrm{Sch}}\|(x) \le \Psi(\varepsilon_i|D).$$

Fix $0 < \epsilon \ll 1$. By the coarea formula,

$$\int_{\frac{1}{2}+2\epsilon}^{\frac{1}{2}+4\epsilon} \mathrm{Length}(\partial\mathscr{E}_i \cap \{f_\infty = t\})\,dt \le \int_{\partial\mathscr{E}_i\cap\{\frac{1}{2}\le f_\infty\le\frac{3}{4}\}} |\nabla f_\infty| \le C\cdot \mathrm{Area}(\partial\mathscr{E}_i).$$

So there exists a regular value $t_0 \in \left(\frac{1}{2}+2\epsilon, \frac{1}{2}+4\epsilon\right)$ such that $\partial\mathscr{E}_i \cap \{f_\infty = t_0\}$ consists of smooth curves, and the total length satisfies

$$\mathrm{Length}(\partial\mathscr{E}_i \cap \{f_\infty = t_0\}) \leq \frac{C \cdot \mathrm{Area}(\partial\mathscr{E}_i)}{\epsilon} = \Psi(\varepsilon_i|\epsilon).$$

We can assume $t_0 = \frac{1}{2}+3\epsilon$ for simplicity. By the uniform convergence $f_i \to f_\infty$, for all large enough i, we have

$$\mathscr{E}_i \cap \left\{\tfrac{1}{2}+3\epsilon \leq f_\infty \leq 1-\tfrac{1}{\epsilon}\right\} \subset M_i.$$

Define $\mathscr{E}_i(\epsilon)$ as the noncompact component of

$$\mathscr{E}_i \cap \left\{f_\infty \geq \tfrac{1}{2}+3\epsilon\right\} \cap M_i.$$

We make some modifications on $\mathscr{E}_i(\epsilon)$ for later usage, as in the proof of [Dong and Song 2025, Lemma 4.3]. Let $\{D_k\}_{k=0}^N \subset \mathscr{E}_i$ be the components of $\left\{f_\infty = \frac{1}{2}+3\epsilon\right\} \cap \mathscr{E}_i$, and assume D_0 has the largest area. Then $\sum_{k\geq 1} \mathrm{diam} D_k \leq \Psi(\varepsilon_i|\epsilon)$. Choose $x_k \in D_k$ for each $k \geq 1$. Notice that the Euclidean Hausdorff distance between D_0 and D_k is bounded by $\Psi(\varepsilon_i|\epsilon)$. Then there exists $y_k \in D_0$ such that $\hat{d}_{h_i,\mathscr{E}_i}(x_k, y_k) \leq \Psi(\varepsilon_i|\epsilon)$. For each $k \geq 1$, let γ_k be a geodesic between x_k and y_k for the metric $\hat{d}_{h_i,\mathscr{E}_i}$. By thickening each γ_k, we can get thin solid tubes T_k inside $\Psi(\varepsilon_i|\epsilon)$-neighborhood of γ_k with arbitrarily small boundary area. Let $\mathscr{E}_i(\epsilon)' := \mathscr{E}_i(\epsilon) \cup (\cup_k T_k)$. We then get a smooth connected subset by smoothing corners of $\mathscr{E}_i(\epsilon)'$.

For simplicity, we still denote the modified $\mathscr{E}_i(\epsilon)'$ by $\mathscr{E}_i(\epsilon)$. For all large enough i, we have $\mathscr{E}_i(\epsilon) \subset M_i \cap \mathscr{E}_i$. Write

$$\Sigma_i(\epsilon) = \overline{\partial\mathscr{E}_i(\epsilon) \setminus \partial\mathscr{E}_i}.$$

Then $\Sigma_i(\epsilon)$ is a connected closed surface with boundary, whose total length is bounded by $\Psi(\varepsilon_i|\epsilon)$, and $\partial\mathscr{E}_i(\epsilon) = \Sigma_i(\epsilon) \cup \left(\partial\mathscr{E}_i \cap \left\{f_\infty \geq \frac{1}{2}+3\epsilon\right\}\right)$. Notice that $\Sigma_i(\epsilon)$ lies inside the $\Psi(\varepsilon_i|\epsilon)$-neighborhood of $\left\{f_\infty = \frac{1}{2}+3\epsilon\right\} \cap \mathscr{E}_i$, which is contained in the $\left(\Psi(\varepsilon_i|\epsilon)+C\cdot\epsilon\right)$-neighborhood of $\left\{f_i = \frac{1}{2}\right\} \cap \mathscr{E}_i$ with respect to both distances $\hat{d}_{\mathrm{Eucl}}$ and $\hat{d}_{g_i}$. See also Figure 2.

Denote by $M_{\mathrm{Sch}}(\epsilon) := \left\{f_\infty \geq \frac{1}{2}+3\epsilon\right\}$, and $x_o(\epsilon) \in \left\{f_\infty = \frac{1}{2}+3\epsilon\right\}$ a base point closest to x_o.

Proposition 7.1 *It holds that*

$$(\mathscr{E}_i(\epsilon), \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}, q_i) \to (M_{\mathrm{Sch}}(\epsilon), d_{g_{\mathrm{Sch}}}, x_o(\epsilon)) \tag{52}$$

in the pointed measured Gromov–Hausdorff topology for some base points $q_i \in \Sigma_i(\epsilon)$, and

$$(\Sigma_i(\epsilon), \hat{d}_{g_i,\Sigma_i(\epsilon)}) \to (\partial M_{\mathrm{Sch}}(\epsilon), \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}) \tag{53}$$

in the measured Gromov–Hausdorff topology for the induced length metrics.

Proof We firstly show (53). By the construction of $\mathscr{E}_i(\epsilon)$, we know $\Sigma_i(\epsilon)$ is inside $\Psi(\varepsilon_i|\epsilon)$-neighborhood of $\partial\mathscr{E}_i(\epsilon) \cap \partial M_{\mathrm{Sch}}(\epsilon)$, so

$$(\Sigma_i(\epsilon), \hat{d}_{g_{\mathrm{Sch}},\Sigma_i(\epsilon)}) \to (\partial M_{\mathrm{Sch}}(\epsilon), \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)})$$

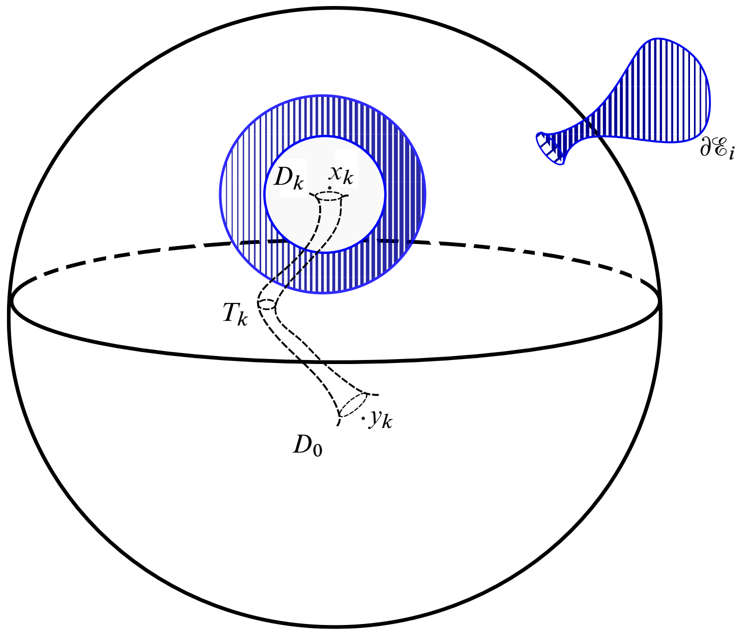

Figure 2: The constructed region $\mathscr{E}_i(\epsilon)$ near $\Sigma_i(\epsilon)$. The sphere is $\{f_\infty = \frac{1}{2} + 3\epsilon\}$, and the blue shaded region is included in the complement of $\mathscr{E}_i$. The boundary $\partial\mathscr{E}_i(\epsilon)$ consists of two parts: one is $\partial\mathscr{E}_i \cap \{f_\infty \geq \frac{1}{2} + 3\epsilon\}$, the boundary of the blue region; the other is $\Sigma_i(\epsilon)$, which is connected and consists of the connected sum of D_0 and D_k using the boundary of the tubes T_k.

in the Gromov–Hausdorff topology. It remains to show that for any $x, y \in \Sigma_i(\epsilon)$,

$$\lim_{i\to\infty} \hat{d}_{g_i,\Sigma_i(\epsilon)}(x, y) = \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}(x, y).$$

Let $\gamma \subset \partial M_{\mathrm{Sch}}(\epsilon)$ be a geodesic between x and y for distance $\hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}$. By our construction of $\mathscr{E}_i(\epsilon)$, we can always perturb γ to get $\tilde{\gamma} \subset \partial\mathscr{E}_i(\epsilon) \setminus \partial\mathscr{E}_i$ such that $|\mathrm{Length}_{\mathrm{Sch}}(\gamma) - \mathrm{Length}_{\mathrm{Sch}}(\tilde{\gamma})| \to 0$. Then

$$\hat{d}_{g_i,\Sigma_i(\epsilon)}(x, y) \leq \int_0^1 |\tilde{\gamma}'|_{g_i} \leq (1 + \Psi(\varepsilon_i)) \int_0^1 |\tilde{\gamma}'|_{g_{\mathrm{Sch}}}.$$

Taking $i \to \infty$, we have

$$\lim_{i\to\infty} \hat{d}_{g_i,\Sigma_i(\epsilon)}(x, y) \leq \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}(x, y).$$

Similarly, it's easy to check that

$$\hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}(x, y) \leq \lim_{i\to\infty} \hat{d}_{g_i,\Sigma_i(\epsilon)}(x, y).$$

This completes the Gromov–Hausdorff convergence in (53).

Notice that for any $x, y \in \partial M_{\mathrm{Sch}}(\epsilon)$, there is a uniform constant $C > 0$ such that

$$d_{\mathrm{Eucl}}(x, y) \leq \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}(\epsilon)}(x, y) \leq C d_{\mathrm{Eucl}}(x, y),$$

which implies that, for any $x, y \in \Sigma_i(\epsilon)$, for all large enough i,

$$\tfrac{1}{2}\hat{d}_{h_i,\mathscr{E}_i}(x,y) \le \hat{d}_{g_i,\Sigma_i(\epsilon)}(x,y) \le C\hat{d}_{h_i,\mathscr{E}_i}(x,y) + \Psi(\varepsilon_i|\epsilon). \tag{54}$$

Now we prove (52). It is enough to show that for any fixed $D > 0$ and any $x, y \in \mathscr{E}_i(\epsilon) \cap B(q_i, D)$,

$$|\hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) - d_{g_{\mathrm{Sch}}}(x,y)| \to 0 \quad \text{as } i \to \infty.$$

It's easy to check that $d_{g_{\mathrm{Sch}}}(x,y) \le \lim_{i\to\infty} \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y)$. In the following, we will show the other inequality.

For any fixed $\delta_0 > 0$. We firstly assume that

$$d_{g_{\mathrm{Sch}}}(x,y) \ge \delta_0, \quad d_{g_{\mathrm{Sch}}}(x, \partial M_{\mathrm{Sch}}(\epsilon)) \ge \delta_0 \quad \text{and} \quad d_{g_{\mathrm{Sch}}}(y, \partial M_{\mathrm{Sch}}(\epsilon)) \ge \delta_0.$$

If γ is a g_{Sch}-geodesic between x and y, then $d_{g_{\mathrm{Sch}}}(\gamma, \partial M_{\mathrm{Sch}}(\epsilon)) \ge \delta_0$ by the geometry of Schwarzschild metric. For any $0 < \delta \ll \delta_0$, we can use piecewise line segments $\{\gamma_j\}_{j=1}^{N(\delta,D)}$ to approximate γ so that $\Sigma_{j=1}^{N} \mathrm{Length}_{g_{\mathrm{Sch}}}(\gamma_j) \le \mathrm{Length}_{g_{\mathrm{Sch}}}(\gamma) + \Psi(\delta)$, and $\mathrm{Length}_{\mathrm{Eucl}}(\gamma_j) \ge \delta$. For each γ_j with $\gamma_j(0) = x_j$ and $\gamma_j(1) = y_j$, from the proof of [Dong and Song 2025, Lemma 4.5], we can find perturbed points x_j' and y_j' such that the straight line segment $\tilde{\gamma}_j$ between x_j', y_j' lies in $\mathscr{E}_i(\epsilon)$ and $d_{\mathrm{Eucl}}(x_j, x_j') + d_{\mathrm{Eucl}}(y_j, y_j') \le \Psi(\varepsilon_i)$. Then

$$\begin{aligned} \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) &\le \sum_j \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x_j', y_j') + C\sum_j (d_{\mathrm{Eucl}}(x_j, x_j') + d_{\mathrm{Eucl}}(y_j, y_j')) \\ &\le \sum_j \mathrm{Length}_{g_i}(\tilde{\gamma}_j) + \Psi(\varepsilon_i|\delta, D) \\ &\le (1 + \Psi(\varepsilon_i|D)) \sum_j \mathrm{Length}_{g_{\mathrm{Sch}}}(\gamma_j) + \Psi(\varepsilon_i|\delta, D). \end{aligned}$$

Taking $i \to \infty$, we have

$$\lim_{i\to\infty} \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) \le d_{g_{\mathrm{Sch}}}(x,y) + \Psi(\delta).$$

Taking $\delta \to 0$ gives

$$\lim_{i\to\infty} \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) \le d_{g_{\mathrm{Sch}}}(x,y). \tag{55}$$

If $d_{g_{\mathrm{Sch}}}(x,y) \le \delta_0$ and $d_{g_{\mathrm{Sch}}}(x, \partial M_{\mathrm{Sch}}(\epsilon)) \le \delta_0$, then $d_{\mathrm{Eucl}}(x, \partial M_{\mathrm{Sch}}(\epsilon)) \le \delta_0$ and $d_{\mathrm{Eucl}}(x,y) \le \delta_0$. We can take an almost $\hat{d}_{h_i,\mathscr{E}_i}$-geodesic $\gamma \subset \mathscr{E}_i$ between x, y, so $\mathrm{Length}_{h_i}(\gamma) \le 2\delta_0$. If $\gamma \subset \mathscr{E}_i(\epsilon)$, we have $\hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) \le C\delta_0$; otherwise, let x' be the first intersection point of γ and $\Sigma_i(\epsilon)$ and y' be the last intersection point. By (54), we have

$$\hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x', y') \le \hat{d}_{g_i,\Sigma_i(\epsilon)}(x', y') \le C\hat{d}_{h_i,\mathscr{E}_i}(x', y') + \Psi(\varepsilon_i|\epsilon) \le C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

So

$$\hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,y) \le \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x,x') + \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(x', y') + \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}(y', y) \le C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

This shows that the pointed Gromov–Hausdorff distance

$$d_{\mathrm{pGH}}\big((\mathscr{E}_i(\epsilon), \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}, q_i), (\mathscr{E}_i(\epsilon) \cap \{x : d_{g_{\mathrm{Sch}}}(x, \partial M_{\mathrm{Sch}}(\epsilon)) \geq \delta_0\}, \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}, q_i)\big) \leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

Together with (55), we have

$$d_{\mathrm{pGH}}\big((\mathscr{E}_i(\epsilon), \hat{d}_{g_i,\mathscr{E}_i(\epsilon)}, q_i), (M_{\mathrm{Sch}}(\epsilon), d_{g_{\mathrm{Sch}}}, x_o(\epsilon))\big) \leq C\delta_0 + \Psi(\varepsilon_i|\epsilon).$$

Choosing a sequence $\delta_0 \to 0$ and by a diagonal argument, for a subsequence $i \to \infty$, we get the conclusion on the pointed Gromov–Hausdorff convergence in (52).

Since the Hausdorff measure induced by $\hat{d}_{g_i,\mathscr{E}_i(\epsilon)}$ and $\hat{d}_{g_i,\Sigma_i(\epsilon)}$ are the same as the volume element dvol_{g_i} and the area element dA_{g_i} respectively, together with the isoperimetric inequality, it's standard to check that these measures also converge weakly; see [Dong and Song 2025, page 22]. In particular, $\mathrm{Area}_{g_i}(\Sigma_i(\epsilon)) \to \mathrm{Area}_{g_{\mathrm{Sch}}}(\partial M_{\mathrm{Sch}}(\epsilon))$. □

Choosing a sequence $\epsilon_i \to 0$, and using a diagonal argument, we can take a subsequence such that

$$(\mathscr{E}_i(\epsilon_i), \hat{d}_{g_i,\mathscr{E}_i(\epsilon_i)}, q_i) \to (M_{\mathrm{Sch}}, d_{g_{\mathrm{Sch}}}, x_o)$$

in the pointed measured Gromov–Hausdorff topology, and

$$(\Sigma_i(\epsilon_i), \hat{d}_{g_i,\Sigma_i(\epsilon_i)}) \to (\partial M_{\mathrm{Sch}}, \hat{d}_{g_{\mathrm{Sch}},\partial M_{\mathrm{Sch}}})$$

in the measured Gromov–Hausdorff topology.

We take $E_i = \mathscr{E}_i(\epsilon_i)$, $\Sigma_i^1 = \Sigma_i(\epsilon_i)$ and $\Sigma_i^2 = \partial\mathscr{E}_i \cap \{f_\infty \geq \frac{1}{2} + 3\epsilon_i\}$. This completes the proof. □

Finally, we prove the stability of the Penrose inequality as a corollary of the stability of the mass–capacity inequality.

Proof of Theorem 1.1 The case when $A_0 = 0$ was proved in [Dong and Song 2025]. Let $A_0 > 0$ be a fixed constant and (M_i^3, g_i) be a sequence of one-ended asymptotically flat 3-manifolds with nonnegative scalar curvature, whose boundaries are compact outermost minimal surfaces with area $A_i \to A_0$. Suppose that the ADM mass $m(g_i)$ converges to $\sqrt{A_0/(16\pi)}$.

As proved in [Bray 2001] (see also the remark at the end of Section 3), for each i, we can find a sequence of conformal metrics $g_i(t)$ and smooth subset $M_i(t) \subset M_i$ such that $(M_i(t), g_i(t))$ is an one-ended asymptotically flat 3-manifold with nonnegative scalar curvature and an outermost minimal boundary $\Sigma_i(t)$. From [Bray 2001, Theorem 4], for some large enough $T_i > 0$, for all $t > T_i$, $M_i(t)$ is diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$. In particular, $\Sigma_i(t)$ is connected. Moreover,

$$\mathrm{Area}_{g_i(t)}(\Sigma_i(t)) = A_i, \qquad m(g_i) \geq m(g_i(t)) \to \sqrt{\frac{A_i}{16\pi}} \quad \text{as } t \to \infty.$$

From [Bray 2001, Section 7], we know that for a.e. t,

$$m'(g_i(t)) = 2\big(\mathscr{C}(\Sigma_i(t), g_i(t)) - m(g_i(t))\big) \le 0.$$

So we can find $t_i \in (T_i, T_i + 1)$ such that

$$2\big(m(g_i(t_i)) - \mathscr{C}(\Sigma_i(t_i), g_i(t_i))\big) \le m(g_i(T_i)) - m(g_i(T_i + 1)) \le m(g_i) - \sqrt{\frac{A_i}{16\pi}}.$$

Then we can apply Theorem 1.2 to $(M_i(t_i), g_i(t_i))$. The conclusion follows if we take N_i to be the good region E_i associated with $(M_i(t_i), g_i(t_i))$. □

Appendix GH convergence modulo negligible domains in general dimensions

In this appendix, we provide a detailed proof of Theorem 2.2 for reader's convenience. In fact, we will prove a stronger quantitative version of the theorem. The proof involves a slight modification of the arguments presented in [Dong and Song 2025], supplemented by an induction argument.

To make the proof clearer, we will prove the theorem for subsets of $\mathbb{R}^n$ in the following. Note that the same proof also applies to the general case (see also the remark at the end of this section).

Theorem A.1 *Assume that $n \ge 2$, $Y_i \subset \mathbb{R}^n$ is a sequence of domains with smooth compact boundaries Σ_i, Y_i contains the end of $\mathbb{R}^n$, and $\mathscr{H}^{n-1}(\Sigma_i) \to 0$. Then there exists a sequence of smooth closed subsets $Y_i'' \subset Y_i$ such that $\mathscr{H}^{n-1}(\partial Y_i'') \to 0$ and for any base point $q_i \in Y_i''$,*

$$(Y_i'', q_i, \hat{d}_{\mathrm{Eucl}, Y_i''}) \xrightarrow{\mathrm{pmGH}} (\mathbb{R}^n, 0, \hat{d}_{\mathrm{Eucl}}).$$

Moreover, we have a quantitative version: there exists $0 < \varepsilon(n) \ll 1$ such that if $Y \subset \mathbb{R}^n$ is a domain with smooth compact boundary Σ, Y contains the end of $\mathbb{R}^n$, and $\mathscr{H}^{n-1}(\Sigma) \le \varepsilon(n)$, then we can find a perturbation $Y'' \subset Y$ such that

$$\mathscr{H}^{n-1}(\partial Y'') \le (\mathscr{H}^{n-1}(\Sigma))^{1-10^{-4}n^{-1}}, \tag{56}$$

and for any base point $q \in Y''$,

$$d_{\mathrm{pGH}}\big((Y'', \hat{d}_{Y''}, q), (\mathbb{R}^n, \hat{d}_{\mathrm{Eucl}}, 0)\big) \le (\mathscr{H}^{n-1}(\Sigma))^{2^{-n}}. \tag{57}$$

Proof Notice that the case when $n = 2$ is obvious by definition, and the case when $n = 3$ has been proved in [Dong and Song 2025]. So we assume that $n \ge 4$. In the following, we will prove the quantitative version by induction and assume that the conclusion holds for all dimensions less than or equal to $n - 1$.

We will find $\varepsilon(n)$ by induction. Set $\eta := \mathscr{H}^{n-1}(\Sigma) \le \varepsilon(n) \ll 1$. Let $\mathscr{W}$ be the domain bounded by Σ. By the isoperimetric inequality,

$$\mathscr{H}^n(\mathscr{W}) \le C(n)\mathscr{H}^{n-1}(\Sigma)^{n/(n-1)} \le C(n)\eta^{n/(n-1)} \le \eta.$$

Take $\delta_0 = \eta^{10^{-2}n^{-1}}$ and $\delta_1 = \eta^{10^{-4}n^{-1}}$. For any $\boldsymbol{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, consider the closed cube $\boldsymbol{C}_{\boldsymbol{k}}(\delta_1)$ defined by

$$\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) := [k_1\delta_1, (k_1+1)\delta_1] \times \cdots \times [k_n\delta_1, (k_n+1)\delta_1] \subset \mathbb{R}^n.$$

For $t \in \mathbb{R}$, define the plane

$$A_{\boldsymbol{k},\delta_1}(t) := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = (k_n + t)\delta_1\}.$$

By definition, $\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \subset \bigcup_{t \in [0,1]} A_{\boldsymbol{k},\delta_1}(t)$.

By the coarea formula, there exists $t_{\boldsymbol{k}} \in \left(\frac{1}{2}, \frac{1}{2} + \delta_0\right)$ such that $\Sigma^{n-2}(t_{\boldsymbol{k}}) := A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \cap \Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)$ consists of $(n-2)$-submanifolds and

$$\mathcal{H}^{n-2}(\Sigma^{n-2}(t_{\boldsymbol{k}})) \le \frac{\mathcal{H}^{n-1}(\Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1))}{\delta_0\delta_1} \le 4\eta^{1-(10n)^{-1}}. \tag{58}$$

If $A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \cap \Sigma \cap \partial\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \neq \varnothing$, then there exist components $\mathcal{P}$ of $A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \cap \partial\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \setminus \Sigma$ and a bounded open subset $Z(t_{\boldsymbol{k}}) \subset A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)$ such that $\mathcal{H}^{n-2}(\mathcal{P}) \le 2(n-1) \cdot \mathcal{H}^{n-2}(\Sigma^{n-2}(t_{\boldsymbol{k}}))$, $\partial Z(t_{\boldsymbol{k}}) \subset \mathcal{P} \cup \Sigma^{n-2}(t_{\boldsymbol{k}})$, and $\Sigma^{n-2}(t_{\boldsymbol{k}}) \subset \overline{Z(t_{\boldsymbol{k}})}$. In particular,

$$\mathcal{H}^{n-2}(\partial Z(t_{\boldsymbol{k}})) \le 2n \cdot \mathcal{H}^{n-2}(\Sigma^{n-2}(t_{\boldsymbol{k}})) \le 8n \cdot \eta^{1-(10n)^{-1}}. \tag{59}$$

Take $\varepsilon(n)$ small enough such that $8n \cdot \varepsilon(n)^{1-(10n)^{-1}} \le \varepsilon(n-1)$. Applying the induction assumption to $A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \setminus Z(t_{\boldsymbol{k}})$ and using (59), we can find a perturbation $\widetilde{\Sigma}^{n-2}(t_{\boldsymbol{k}}) \subset A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}})$ of $\partial Z(t_{\boldsymbol{k}})$ satisfying the following properties:

- Denote by $\widetilde{Z}(t_{\boldsymbol{k}}) \subset A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}})$ the domain bounded by $\widetilde{\Sigma}^{n-2}(t_{\boldsymbol{k}})$. Then $Z(t_{\boldsymbol{k}}) \subset \widetilde{Z}(t_{\boldsymbol{k}})$, and $A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \setminus \widetilde{Z}(t_{\boldsymbol{k}})$ contains the end of $\mathbb{R}^{n-1}$.
- The area satisfies

$$\mathcal{H}^{n-2}(\widetilde{\Sigma}^{n-2}(t_{\boldsymbol{k}})) \le \mathcal{H}^{n-2}(\partial Z(t_{\boldsymbol{k}}))^{1-10^{-4}(n-1)^{-1}} \le (2n \cdot \mathcal{H}^{n-2}(\Sigma^{n-2}(t_{\boldsymbol{k}})))^{1-10^{-4}(n-1)^{-1}}. \tag{60}$$

- For any two points $x, y \in \widetilde{Y}(t_{\boldsymbol{k}}) := A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \setminus \widetilde{Z}(t_{\boldsymbol{k}})$, we have

$$|\hat{d}_{\widetilde{Y}(t_{\boldsymbol{k}})}(x,y) - \hat{d}_{\mathbb{R}^{n-1}}(x,y)| \le (8n \cdot \eta^{1-(10n)^{-1}})^{2^{-(n-1)}} \le 10^{-4}\eta^{2^{-n}}. \tag{61}$$

Define $D''_{\boldsymbol{k}} := \widetilde{Z}(t_{\boldsymbol{k}}) \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)$, and $D'_{\boldsymbol{k}} := A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}}) \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \setminus \overline{D''_{\boldsymbol{k}}}$. By the relative isoperimetric inequality and (58), we know that

$$\begin{aligned} \mathcal{H}^{n-1}(D''_{\boldsymbol{k}}) &\le C(n) \cdot \mathcal{H}^{n-2}(\widetilde{\Sigma}^{n-2}(t_{\boldsymbol{k}}))^{(n-1)/(n-2)} \\ &\le C(n) \cdot \mathcal{H}^{n-2}(\Sigma^{n-2}(t_{\boldsymbol{k}}))^{1+(1-10^{-4})/(n-2)} \\ &\le C(n) \cdot \eta^{1/(4(n-2))}\delta_0^{-1}\delta_1^{-1} \cdot \mathcal{H}^{n-1}(\Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)) \\ &\le \mathcal{H}^{n-1}(\Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)). \end{aligned} \tag{62}$$

We can take a finite covering $\{B(x_\alpha, r_\alpha) : r_\alpha < \eta\}$ of $\overline{D'_{\boldsymbol{k}}}$ such that for all $x \in B(x_\alpha, r_\alpha) \cap D'_{\boldsymbol{k}}$, there is a line segment inside $D'_{\boldsymbol{k}}$ connecting x and x_α. For any pair x_α, x_β, let $\gamma_{\alpha,\beta} \subset \widetilde{Y}(t_{\boldsymbol{k}})$ be an almost $\hat{d}_{\widetilde{Y}(t_{\boldsymbol{k}})}$-geodesic between x_α and x_β. Define $\widetilde{D}'_{\boldsymbol{k}}$ by

$$\widetilde{D}'_{\boldsymbol{k}} := D'_{\boldsymbol{k}} \cup \{\gamma_{\alpha,\beta}\}.$$

In this way, using (61), for any $x, y \in \widetilde{D}'_{\boldsymbol{k}}$, we have a curve $\gamma \subset \widetilde{D}'_{\boldsymbol{k}}$ connecting x and y such that

$$|\mathrm{Length}_{\mathrm{Eucl}}(\gamma) - d_{\mathrm{Eucl}}(x, y)| \le \eta + 10^{-4}\eta^{2^{-n}} \le 10^{-3}\eta^{2^{-n}}. \tag{63}$$

For simplicity, we still denote $\widetilde{D}'_{\boldsymbol{k}}$ by $D'_{\boldsymbol{k}}$. Let $\pi_{\boldsymbol{k}} : \mathbb{R}^n \to A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}})$ be the orthogonal projection.

Define

$$\boldsymbol{C}_{\boldsymbol{k}}(\delta_1)' := D'_{\boldsymbol{k}} \cup \big(\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \cap \pi_{\boldsymbol{k}}^{-1}\big(D'_{\boldsymbol{k}} \setminus \pi_{\boldsymbol{k}}(\Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1))\big)\big).$$

We then proceed by following the same arguments as presented in [Dong and Song 2025, Section 4]. The main difference here is that we need to explicitly compute the quantitative upper bounds for the purpose of induction. Therefore, we will omit most of the proofs, providing them only when modifications are necessary.

Lemma A.2 [Dong and Song 2025, Lemma 4.3] $\mathcal{H}^n(\boldsymbol{C}_{\boldsymbol{k}}(\delta_1) \setminus \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)') \le 2\delta_1 \mathcal{H}^{n-1}(\Sigma \cap \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)) \le 2\eta^{1+10^{-4}n^{-1}}$.

Lemma A.3 [Dong and Song 2025, Lemma 4.4] *$\boldsymbol{C}_{\boldsymbol{k}}(\delta_1)'$ is path-connected and $\boldsymbol{C}_{\boldsymbol{k}}(\delta_1)' \subset Y$.*

Define

$$Y' := \bigcup_{\boldsymbol{k} \in \mathbb{Z}^n} \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)' \subset Y.$$

Notice that when $|\boldsymbol{k}|$ is big enough, one can certainly ensure that $\boldsymbol{C}_{\boldsymbol{k}}(\delta_1)' = \boldsymbol{C}_{\boldsymbol{k}}(\delta_1)$, so that $Y \setminus Y'$ is a bounded set.

For any subset $V \subset Y$, let V_t be the t-neighborhood of V inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$ in terms of the length metric $\hat{d}_{\mathrm{Eucl},Y}$, ie

$$V_t := \{y \in Y : \text{there exists a } z \in V \text{ such that } \hat{d}_{\mathrm{Eucl},Y}(y, z) \le t\}.$$

So $(Y')_t$ is the t-neighborhood of Y' inside $(Y, \hat{d}_{\mathrm{Eucl},Y})$.

Lemma A.4 [Dong and Song 2025, Lemma 4.5] *There exists a closed subset Y'' with smooth boundary such that $Y' \subset Y'' \subset (Y')_{6\delta_0}$,*

$$\mathcal{H}^{n-1}(\partial Y'') \le \delta_0^{-1} \mathcal{H}^{n-1}(\partial Y) \le \eta^{1-10^{-2}n^{-1}},$$

and Y'' is contained in the $6\delta_0$-neighborhood of Y' inside Y'', with respect to its length metric $\hat{d}_{\mathrm{Eucl},Y''}$.

Let Y'' be as in Lemma A.4. Recall that $\hat{d}_{\mathrm{Eucl},Y''}$ is defined as the length metric on Y'' induced by g_{Eucl}. Since $Y' \subset Y'' \subset Y$, we have $d_{\mathrm{Eucl}} \le \hat{d}_{\mathrm{Eucl},Y} \le \hat{d}_{\mathrm{Eucl},Y''}$.

Lemma A.5 [Dong and Song 2025, Lemma 4.6] $\operatorname{diam}_{\hat{d}_{\mathrm{Eucl},Y''}}(\boldsymbol{C_k}(\delta_1)') \le (n+2)\delta_1 + 10^{-3}\eta^{2^{-n}}$.

Proof For any two points $x_1, x_2 \in \boldsymbol{C_k}(\delta_1)'$, let L_{x_1}, L_{x_2} be the line segments inside $\boldsymbol{C_k}(\delta_1)$ through x_1, x_2 and orthogonal to $A_{\boldsymbol{k},\delta_1}(t_{\boldsymbol{k}})$ respectively. Let $x_1' = L_{x_1} \cap D_{\boldsymbol{k}}'$, $x_2' = L_{x_2} \cap D_{\boldsymbol{k}}'$. Then by (63) we can find a curve γ between x_1', x_2' inside $D_{\boldsymbol{k}}'$ such that

$$\mathrm{Length}_{\mathrm{Eucl}}(\gamma) \le d_{\mathrm{Eucl}}(x_1', x_2') + 10^{-3}\eta^{2^{-n}}.$$

Consider the curve $\tilde{\gamma}$ consisting of three parts: the line segment $[x_1x_1']$ between x_1, x_1', γ, and the line segment $[x_2'x_2]$ between x_2', x_2. We have $\tilde{\gamma} \subset \boldsymbol{C_k}(\delta_1)' \subset Y'$, so

$$\hat{d}_{\mathrm{Eucl},Y''}(x_1, x_2) \le L_{\mathrm{Eucl}}(\tilde{\gamma}) \le (n+2)\delta_1 + 10^{-3}\eta^{2^{-n}}. \qquad \square$$

Lemma A.6 [Dong and Song 2025, Lemma 4.7] *For any base point $q \in Y'$ and any $D > 0$,*

$$d_{\mathrm{pGH}}\big((Y' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)\big) \le 10^{-2}\eta^{2^{-n}}.$$

Proof The same proof of [Dong and Song 2025, Lemma 4.7] shows that

$$\begin{aligned} d_{\mathrm{pGH}}\big((Y' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)\big) &\le 4 \cdot ((n+2)\delta_1 + 10^{-3}\eta^{2^{-n}}) \\ &\le 10^{-2}\eta^{2^{-n}}. \end{aligned} \qquad \square$$

So we have:

Proposition A.7 [Dong and Song 2025, Proposition 4.8] *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{\mathrm{pGH}}\big((Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q)\big) \le \tfrac{1}{4}\eta^{2^{-n}}.$$

For any $p \in Y''$ and $D > 0$, denote by $\hat{B}_{Y''}(p, D)$ the geodesic ball in $(Y'', \hat{d}_{\mathrm{Eucl},Y''})$, that is,

$$\hat{B}_{Y''}(p, D) := \{x \in Y'' : \hat{d}_{\mathrm{Eucl},Y''}(p, x) \le D\}.$$

Lemma A.8 [Dong and Song 2025, Lemma 4.9] *For any base point $q \in Y''$ and any $D > 0$,*

$$d_{\mathrm{pGH}}\big((Y'' \cap B_{\mathrm{Eucl}}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q), (\hat{B}_{\mathrm{Eucl},Y''}(q, D), \hat{d}_{\mathrm{Eucl},Y''}, q)\big) \le \tfrac{1}{4}\eta^{2^{-n}}.$$

To compare those metric spaces to the Euclidean 3-space $(\mathbb{R}^3, g_{\mathrm{Eucl}})$, we need the following lemma, which is a corollary of the fact that $\mathcal{H}^{n-1}(\partial Y'') \le \delta_1^{-1}\eta$.

Lemma A.9 [Dong and Song 2025, Lemma 4.12] *For any $q \in Y''$ and $D > 0$,*

$$d_{\mathrm{pGH}}\big((Y'' \cap B_{\mathrm{Eucl}}(q, D), d_{\mathrm{Eucl}}, q), (B_{\mathrm{Eucl}}(0, D), d_{\mathrm{Eucl}}, 0)\big) \le \tfrac{1}{4}\eta^{2^{-n}}.$$

Proof Under a translation diffeomorphism, we can assume $q=0$. It suffices to show that $B_{\mathrm{Eucl}}(q,D)$ lies in a $\frac{1}{4}\eta^{2^{-n}}$-neighborhood of Y''. If that were not the case, there would be a $\mu > \frac{1}{4}\eta^{2^{-n}}$ and an $x \in B_{\mathrm{Eucl}}(q,D)$ with $B_{\mathrm{Eucl}}(x,\mu)\cap Y'' = \varnothing$. But from the isoperimetric inequality, we have

$$\mathcal{H}^n(\mathbb{R}^3 \setminus Y'') \le C(n)\mathcal{H}^{n-1}(\partial Y'')^{n/(n-1)} \le C(n)\eta^{(1-10^{-4}n^{-1})\cdot(n/(n-1))},$$

which would imply that

$$\omega_n \cdot \frac{\eta^{n/2^n}}{4^n} \le \omega_n \mu^n = \mathcal{H}^n(B_{\mathrm{Eucl}}(x,\mu)) \le C(n)\eta^{n/(2(n-1))},$$

a contraction when $\eta \le \varepsilon(n) \ll 1$. □

This concludes the proof of the theorem. □

Remark A.1 In the proof above, since the underlying metric is the Euclidean metric, we don't need to use [Dong and Song 2025, Lemma 4.10], where the estimate depends on a diameter upper bound. In the general case when the underlying metric is C^0 close to the Euclidean metric, we need to apply [Dong and Song 2025, Lemma 4.10] to each small cube whose diameter is much smaller than 1, so the conclusion still holds except that we need to replace the explicit error term (57) by

$$d_{\mathrm{GH}}(\widehat{B}_{g,Y''}(q,D), B_{\mathrm{Eucl}}(0,D)) \le \Psi(\eta|n,D) \quad \text{for all } D>0.$$

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