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The 3-fold K-theoretic DT/PT vertex correspondence holds

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We prove the 3-fold DT/PT correspondence for K-theoretic vertices via wall-crossing techniques. We provide two different setups, following Mochizuki and following Joyce; both reduce the problem to q-combinatorial identities on word rearrangements. An important technical step is the construction of symmetric almost-perfect obstruction theories (APOTs) on auxiliary moduli stacks, e.g., master spaces, from the symmetric DT or PT obstruction theory. For this, we introduce symmetrized pullbacks of symmetric obstruction theories along smooth morphisms to Artin stacks.

MSC2020: 14N35.

1 Introduction

1.1 DT and PT theory

1.1.1 Let X be a smooth quasiprojective 3-fold over $\mathbb{C}$, with the action of a torus $\mathsf{T} := (\mathbb{C}^\times)^r$ (where r could be 0) such that the fixed locus X^{T} is proper. Donaldson–Thomas (DT) theory, as originally formulated by Thomas [2000], studies the moduli schemes

$$\mathrm{DT}_{\beta,\chi}(X) := \left\{ \begin{matrix} \text{ideal sheaves } \mathcal{I}_Z \subset \mathcal{O}_X \text{ of proper} \\ \text{1-dimensional subschemes } Z \end{matrix} \;\middle|\; [Z] = \beta,\ \chi(\mathcal{O}_Z) = \chi \right\}$$

for $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. Of particular importance in this subject are the generating series

$$\mathsf{Z}_\beta^{\mathrm{DT}}(X) := \sum_{\chi \in \mathbb{Z}} Q^\chi \int_{[\mathrm{DT}_{\beta,\chi}(X)]^{\mathrm{vir}}} 1 \in A_*^{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}((Q)) \tag{1}$$

of (equivariant) *DT invariants of curves* in X, where:

- $A_*^{\mathsf{T}}(-)$ is the T-equivariant Chow group, and $A_*^{\mathsf{T}}(-)_{\mathrm{loc}} := A_*^{\mathsf{T}}(-) \otimes_{A_*^{\mathsf{T}}(\mathrm{pt})} \operatorname{Frac} A_*^{\mathsf{T}}(\mathrm{pt})$,
- $[\mathrm{DT}_{\beta,\chi}(X)]^{\mathrm{vir}} \in A_*^{\mathsf{T}}(\mathrm{DT}_{\beta,\chi}(X))_{\mathrm{loc}}$ is the virtual fundamental class for the perfect obstruction theory $\operatorname{Ext}_X^*(\mathcal{I}_Z, \mathcal{I}_Z)_0$, defined via T-equivariant localization if necessary.

DT invariants of this flavor have deep connections to Gromov–Witten theory [Maulik et al. 2011], representation theory of quantum groups [Okounkov 2018], supersymmetric gauge and string theories [Nekrasov and Okounkov 2016], to give a (nonrepresentative) sample.

1.1.2 It is productive to refine cohomological enumerative invariants into K-theoretic ones. The K-theoretic analogue of $\mathsf{Z}_\beta^{\mathrm{DT}}(X)$ is the series

$$\mathsf{Z}_\beta^{\mathrm{DT},K}(X) := \sum_{\chi \in \mathbb{Z}} Q^\chi \chi(\mathrm{DT}_{\beta,\chi}(X), \widehat{\mathcal{O}}^{\mathrm{vir}}) \in K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}((Q)) \tag{2}$$

of *K-theoretic DT invariants of curves* in X, where, in analogy with (1):

- $K_{\mathsf{T}}(-) := K_0(\mathcal{C}oh_{\mathsf{T}}(-))$ is the Grothendieck K-group of T-equivariant coherent sheaves, and we set $K_{\mathsf{T}}(-)_{\mathrm{loc}} := K_{\mathsf{T}}(-) \otimes_{K_{\mathsf{T}}(\mathrm{pt})} \operatorname{Frac} K_{\mathsf{T}}(\mathrm{pt})$.
- $\mathcal{O}^{\mathrm{vir}} \in K_{\mathsf{T}}(\mathrm{DT}_{\beta,n}(X))_{\mathrm{loc}}$ is the virtual structure sheaf; see for example [Ciocan-Fontanine and Kapranov 2009].
- $\widehat{\mathcal{O}}^{\mathrm{vir}} := \mathcal{O}^{\mathrm{vir}} \otimes \mathcal{K}_{\mathrm{vir}}^{1/2}$ is the Nekrasov–Okounkov twist [2016] by a square root of the virtual canonical bundle, a special but important feature of K-theoretic DT-like theories.

The integrand in (2) can be much more general (see Section 1.3.5), though $\mathsf{Z}_\beta^{\mathrm{DT},K}(X)$ is already a very sophisticated quantity even in the simplest nontrivial examples.

While K-theoretic enumerative geometry is a rapidly growing subject, basic questions are often very difficult because vanishing arguments via dimension counts are unavailable. Only in special circumstances, including K-theoretic DT theory using $\widehat{\mathcal{O}}^{\mathrm{vir}}$ (but not $\mathcal{O}^{\mathrm{vir}}$), do *rigidity* arguments provide an appropriate substitute.

The remainder of this paper works in equivariant K-theory, but most statements and results hold equally well in equivariant cohomology/Chow.

1.1.3 Every DT invariant of curves in X implicitly contains contributions from freely roaming points, or 0-dimensional subschemes, in X. Algebraically, these can be formally removed by considering $\mathsf{Z}_\beta^{\mathrm{DT},K}(X)/\mathsf{Z}_0^{\mathrm{DT},K}(X)$. Geometrically, Pandharipande and Thomas [2009b] define the moduli schemes of *stable pairs*

$$\mathrm{PT}_{\beta,\chi}(X) := \{\mathcal{O}_X \xrightarrow{s} \mathcal{F} \mid \mathcal{F} \text{ pure of dim 1, proper support}, \dim \operatorname{supp} \operatorname{coker}(s) = 0,\ [\mathcal{F}] = \beta,\ \chi(\mathcal{F}) = \chi\}$$

for $\beta \in H_2(X, \mathbb{Z})$ and $\chi \in \mathbb{Z}$, which constrain point contributions to lie within the Cohen–Macaulay curve $\operatorname{supp} \mathcal{F}$. In contrast, and we take this perspective from here on, elements in DT moduli spaces are two-term complexes $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ for an arbitrary curve Z. Letting

$$I^\bullet := [\mathcal{O}_X \xrightarrow{s} \mathcal{F}] \in D^b\mathcal{C}oh_{\mathsf{T}}(X),$$

these PT moduli spaces carry a perfect obstruction theory $\operatorname{Ext}_X^*(I^\bullet, I^\bullet)_0$ analogous to that of DT moduli spaces where $I^\bullet = [\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z] \simeq \mathcal{I}_Z$. Exactly as in (1) and (2) but replacing the DT moduli space with PT, define the series

$$\mathsf{Z}_\beta^{\mathrm{PT}}(X) \in A_*^{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}((Q)) \quad \text{and} \quad \mathsf{Z}_\beta^{\mathrm{PT},K}(X) \in K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}((Q))$$

of (equivariant) *cohomological* and *K-theoretic PT invariants of curves* in X.

1.1.4 The question of comparing $\mathsf{Z}_\beta^{\mathrm{DT},K}(X)/\mathsf{Z}_0^{\mathrm{DT},K}(X)$ and $\mathsf{Z}_\beta^{\mathrm{PT},K}(X)$ is the *K-theoretic DT/PT correspondence*, to be addressed in this paper using wall-crossing techniques in equivariant K-theory. While our techniques are most naturally applied to smooth quasiprojective Calabi–Yau 3-folds, in this paper we focus on the case of smooth quasiprojective toric 3-folds (possibly Fano), and we prove a stronger DT/PT correspondence at the level of *K-theoretic vertices*.

1.2 DT and PT vertices

1.2.1 Suppose X is toric, with the action of its dense open torus T. Let $\Delta(X)$ be the toric 1-skeleton, whose vertices v are toric charts $\mathbb{C}^3$ with toric coordinates $\boldsymbol{t}_v$ and whose (half-)edges e are the nonempty double intersections $\mathbb{C}^\times \times \mathbb{C}^2$ with toric coordinates $\boldsymbol{t}_e$. Then T-equivariant localization and Čech cohomology imply

$$\begin{aligned} \mathsf{Z}_\beta^{\mathrm{DT},K}(X) &= \sum_{\boldsymbol{\lambda}} \prod_e \mathsf{E}_{\boldsymbol{\lambda}(e)}^K(\boldsymbol{t}_e) \prod_v \mathsf{V}_{\boldsymbol{\lambda}(e_1),\boldsymbol{\lambda}(e_2),\boldsymbol{\lambda}(e_3)}^{\mathrm{DT},K}(\boldsymbol{t}_v), \\ \mathsf{Z}_\beta^{\mathrm{PT},K}(X) &= \sum_{\boldsymbol{\lambda}} \prod_e \mathsf{E}_{\boldsymbol{\lambda}(e)}^K(\boldsymbol{t}_e) \prod_v \mathsf{V}_{\boldsymbol{\lambda}(e_1),\boldsymbol{\lambda}(e_2),\boldsymbol{\lambda}(e_3)}^{\mathrm{PT},K}(\boldsymbol{t}_v) \end{aligned} \tag{3}$$

factorize into *edge* contributions E^K and *vertex* contributions $\mathsf{V}^{\mathrm{DT},K}$ and $\mathsf{V}^{\mathrm{PT},K}$. Here e_1, e_2, e_3 denote the three incident edges or half-edges at each vertex v, and the sum is over all assignments $\boldsymbol{\lambda}$ of an integer partition to each edge and half-edge in $\Delta(X)$; see [Maulik et al. 2006a, Section 4] for details. One can arrange this factorization such that the edges E^K are simple combinatorial products which are the same for DT and PT, and indeed for any other flavor of curve-counting theory. The bulk of the complexity lies in the K-theoretic vertices $\mathsf{V}^{\mathrm{DT},K}_{\lambda,\mu,\nu}$ and $\mathsf{V}^{\mathrm{PT},K}_{\lambda,\mu,\nu}$, which we define below using a special geometry X.

1.2.2 Following [Maulik et al. 2011], let $\overline{X} := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the action of $\mathsf{T} := (\mathbb{C}^\times)^3$ by scaling with weights t_1, t_2, t_3. Let

$$X := \overline{X} \setminus (L_1 \cup L_2 \cup L_3),$$

where the L_i are the three T-invariant lines passing through (∞, ∞, ∞), and let

$$\iota_i \colon D_i \hookrightarrow \overline{X}$$

be the three T-invariant boundary divisors given by setting the i-th coordinate to ∞. Clearly each $D_i \cap X \cong \mathbb{C}^2$, and for $i \neq j$ the divisors $D_i \cap X$ and $D_j \cap X$ are disjoint in X. Set

$$D := D_1 \cup D_2 \cup D_3.$$

Our sheaves will live on $\overline{X}$ but have prespecified restrictions to D, so interesting things occur only in the toric chart $\overline{X} \setminus D \cong \mathbb{C}^3$.

1.2.3 Definition Fix integer partitions λ, μ, ν, corresponding to T-fixed points in $\mathrm{Hilb}(D_i \cap X)$ for $i = 1, 2, 3$ respectively. They specify a curve class

$$\beta_C := (|\lambda|, |\mu|, |\nu|) \in H_2(\overline{X}, \mathbb{Z}).$$

For $M \in \{\mathrm{DT}, \mathrm{PT}\}$, let

$$M_{(\lambda,\mu,\nu),\chi} := \bigl\{ I = [\mathcal{O}_{\overline{X}} \xrightarrow{s} \mathcal{E}] \mid (\iota_i^* I)_{i=1}^3 = (L\iota_i^* I)_{i=1}^3 \cong (\lambda, \mu, \nu) \bigr\} \subset M_{\beta_C,\chi}(\overline{X}), \tag{4}$$

where $\cong$ means isomorphic to the two-term complex $[\mathcal{O}_{D_i} \twoheadrightarrow \mathcal{O}_{Z_i}]$ denoted by λ or μ or ν. These $M_{(\lambda,\mu,\nu),\chi}$ carry symmetric[1] perfect obstruction theories given by $\mathrm{Ext}^*_{\overline{X}}(I, I(-D))$, using which we define the *equivariant K-theoretic DT* or *PT vertex*

$$\mathsf{V}^{M,K}_{\lambda,\mu,\nu} := \sum_{\chi \in \mathbb{Z}} Q^\chi \chi(M_{(\lambda,\mu,\nu),\chi}, \widehat{\mathcal{O}}^{\mathrm{vir}}) \in K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}((Q)); \tag{5}$$

cf. (2). This definition is equivalent to that of [Maulik et al. 2006a, Section 4] or [Pandharipande and Thomas 2009a, Section 4], which use a formal redistribution of vertex/edge contributions in a Čech cohomology computation. See Section 3.3 for details.

[1] For us, "symmetric" always means up to an overall equivariant weight; see Definition 2.1.2.

1.2.4 The notion of vertices and factorizations of the form (3) have a long history in mathematical physics, and the equivariant vertices (5) subsume many previous notions of vertex. Taking the *Calabi–Yau limit* $t_1t_2t_3 \to 1$ produces the *topological vertices* of [Aganagic et al. 2005; Iqbal et al. 2008; Li et al. 2009], or, with a bit more care [Nekrasov and Okounkov 2016] to incorporate an extra parameter into the CY limit, the *refined topological vertices* of [Iqbal et al. 2009].

The general principle is that the T-fixed points in $M_{(\lambda,\mu,\nu),n}$ are in bijection with plane partitions or other (labeled) configurations of boxes in $\mathbb{Z}^3$, with legs along the three positive axes. Various limits dramatically simplify their contributions in equivariant localization. Fixed loci for $M = \mathrm{PT}$ are especially complicated and can form positive-dimensional families if all three legs are nontrivial [Pandharipande and Thomas 2009a]. In this way, equivariant vertices may be viewed from the lens of enumerative combinatorics, with connections to statistical mechanics [Okounkov et al. 2006; Jenne et al. 2021].

Even if one only wants to study $\mathsf{V}^{M,K}$ or $\mathsf{Z}^{M,K}$ for $M = \mathrm{DT}$, the $M = \mathrm{PT}$ case usually has nicer properties and confers many technical advantages. For example, Z^{PT} is a rational function in Q [Maulik et al. 2006b, Conjecture 2; Bridgeland 2011, Theorem 1.1(b); Pandharipande and Pixton 2013] while Z^{DT} is not. The DT/PT correspondence implies that it is enough to study $M = \mathrm{PT}$.

1.3 The main theorem and possible extensions

1.3.1 Theorem (K-theoretic DT/PT vertex correspondence)

$$\mathsf{V}^{\mathrm{DT},K}_{\lambda,\mu,\nu} = \mathsf{V}^{\mathrm{PT},K}_{\lambda,\mu,\nu}\mathsf{V}^{\mathrm{DT},K}_{\varnothing,\varnothing,\varnothing}.$$

This theorem is the main goal of this paper. Using the factorization (3), it is an immediate corollary that, for smooth toric 3-folds X,

$$\mathsf{Z}^{\mathrm{DT},K}_{\beta}(X) = \mathsf{Z}^{\mathrm{PT},K}_{\beta}(X)\mathsf{Z}^{\mathrm{DT},K}_{0}(X),$$

which we refer to as the K-theoretic DT/PT correspondence *for partition functions* of X.

1.3.2 Theorem 1.3.1 first appeared as a conjecture for cohomological equivariant vertices [Pandharipande and Thomas 2009a, Conjecture 4]. Later, it was stated as a conjecture in its present K-theoretic form in [Nekrasov and Okounkov 2016, equation (16)], where the importance of using $\widehat{\mathcal{O}}^{\mathrm{vir}}$ instead of $\mathcal{O}^{\mathrm{vir}}$ was first recognized.

Some forms of Theorem 1.3.1 were previously known in the CY limit $t_1t_2t_3 \to 1$. For up to two nontrivial legs λ, μ, ν, there is a relatively elementary proof using transfer matrix techniques [Kononov et al. 2021], which also works for refined topological vertices. For three nontrivial legs, there is a much harder combinatorial proof via dimer models [Jenne et al. 2021], assuming a technical conjecture on the smoothness of PT fixed loci for the CY subtorus $\{t_1t_2t_3 = 1\} \subset \mathsf{T}$.

More geometrically, Bridgeland [2011] and Toda [2010; 2020] use Hall algebra and wall-crossing techniques to prove the (nonequivariant) cohomological DT/PT correspondence for partition functions of *arbitrary* smooth projective Calabi–Yau threefolds. Interestingly, it is known how to remove the CY restriction here if one uses nonvirtual, i.e., ordinary, fundamental classes [Stoppa and Thomas 2011]. In equivariant cohomology, an indirect proof of Theorem 1.3.1 is available by composing the GW/DT [Maulik et al. 2011] and GW/PT [Maulik et al. 2010, Theorem 21] vertex correspondences.

Morally, all these known cases take place in cohomology. Significantly more complexity appears in equivariant K-theory, and when working with vertices instead of partition functions.

1.3.3 From the lens of representation theory, there is an expectation that the critical cohomology (resp. K-theory) of $\bigsqcup_\chi M_{(\lambda,\mu,\nu),\chi}$, for $M \in \{\mathrm{DT}, \mathrm{PT}\}$, are modules for shifted $\widehat{\mathfrak{gl}}_1$ Yangians (resp. quantum loop algebras) [Feigin et al. 2017; Rapčák et al. 2020; Gaiotto and Rapčák 2022], and that fully equivariant DT and PT vertices arise as appropriate qq-characters [Nekrasov 2017] of these DT and PT modules. One can then imagine that the DT and PT modules are related in some way, e.g., taking duals followed by induction/restriction, and that the DT/PT vertex correspondence of Theorem 1.3.1 is what appears after taking qq-characters. In the one-legged case there is some preliminary work in this direction [Liu 2021a], but much of the general technical work remains to be done.

Such a representation-theoretic approach would provide a satisfying conceptual explanation for the simple form of Theorem 1.3.1 beyond our proof via geometric wall-crossing (Section 1.4).

1.3.4 There exist many variations on DT and PT vertices and the DT/PT correspondence. In what follows, we give some variations where the techniques that we use to prove Theorem 1.3.1 should be almost immediately applicable. Other variations, e.g., higher-rank [Lo 2012; Sheshmani 2016] or motivic DT/PT [Davison and Ricolfi 2021], may also be amenable to our techniques with some more work.

1.3.5 The integrands $\widehat{\mathcal{O}}^{\mathrm{vir}}$ in the vertices (5) may be tensored with *descendent classes* — tautological classes on $M_{(\lambda,\mu,\nu),\chi}$ of the form

$$f(\zeta) := R\pi_*(f(\mathcal{F}) \otimes \zeta),$$

where $[\mathcal{O} \to \mathcal{F}]$ is the universal pair on $\pi\colon M_{(\lambda,\mu,\nu),\chi} \times \overline{X} \to M_{(\lambda,\mu,\nu),\chi}$ and $\zeta \in K_{\mathsf{T}}(\overline{X})$ and f is some symmetric polynomial in the K-theoretic Chern roots of $\mathcal{F}$. Given some polynomial $\mathfrak{f}$ in descendent classes, let $\mathsf{V}^{M,K}_{\lambda,\mu,\nu}(\mathfrak{f})$ denote the resulting *descendent vertex*. A natural question is how descendents transform under the DT/PT correspondence; namely, whether

$$\mathsf{V}^{\mathrm{DT},K}_{\lambda,\mu,\nu}(\mathfrak{f}) = \sum_i \mathsf{V}^{\mathrm{PT},K}_{\lambda,\mu,\nu}(\mathfrak{f}_i^{(1)}) \mathsf{V}^{\mathrm{DT},K}_{\varnothing,\varnothing,\varnothing}(\mathfrak{f}_i^{(2)})$$

for some operation $\Delta(\mathfrak{f}) := \sum_i \mathfrak{f}_i^{(1)} \otimes \mathfrak{f}_i^{(2)}$, where $\mathfrak{f}_i^{(1)}, \mathfrak{f}_i^{(2)}$ are also polynomials in descendents. Theorem 1.3.1 says that $\Delta(1) = 1 \otimes 1$. For a special nontrivial $\mathfrak{f}$, a conjectural formula exists for cohomological equivariant capped vertices [Oblomkov 2019, Conjecture 5.3.1], roughly of the form $\Delta(\mathrm{ch}_k) = \sum_{k_1+k_2=k} \mathrm{ch}_{k_1} \otimes \mathrm{ch}_{k_2}$ up to some normalization.

We plan to tackle this problem in future work, using a variant of Joyce's recent [2021] universal wall-crossing machine adapted to moduli stacks with symmetric obstruction theories, in particular moduli stacks of (complexes of) coherent sheaves on CY 3-folds.

1.3.6 If X is the (singular) coarse moduli space of a projective CY 3-orbifold $\mathfrak{X}$ satisfying the hard Lefschetz condition, and $\pi\colon \widetilde{X} \to X$ is the preferred crepant resolution given by the derived McKay correspondence, then there is a notion of *Bryan–Steinberg* (*BS*) or π*-stable pairs* for π; see [Bryan and Steinberg 2016]. Roughly, in the same way that PT-stability disallows 0-dimensional components in DT-stable curves, BS-stability disallows those 1-dimensional components in PT-stable curves of $\widetilde{X}$ which are supported completely in the exceptional fibers of π. We expect, as with [Bryan and Steinberg 2016, Theorem 6] and [Beentjes et al. 2022, Section 1.1], that our wall-crossing techniques can be used to prove the *K-theoretic PT/BS correspondence*

$$\mathsf{Z}_\beta^{\mathrm{PT},K}(\widetilde{X}) = \mathsf{Z}_\beta^{\mathrm{BS},K}(\pi)\mathsf{Z}_{\mathrm{exc}}^{\mathrm{PT},K}(\pi),$$

where $\mathsf{Z}_\beta^{\mathrm{BS},K}(\pi)$ is the analogue of the partition function (2) for BS-stable pairs, and $\mathsf{Z}_{\mathrm{exc}}^{\mathrm{PT},K}(\pi)$ is the partition function for PT-stable pairs supported on exceptional fibers of π.

Furthermore, if X is toric, then the toric building blocks of π are crepant resolutions of $[\mathbb{C}^3/G]$ where $G \subset \mathrm{SO}(3)$ or $G \subset \mathrm{SU}(2)$ is a finite subgroup [Bryan and Gholampour 2009, Lemma 24], and for each such geometry, there should exist a BS vertex $\mathsf{V}_{\boldsymbol{\lambda}}^{\mathrm{BS},K}(G)$ such that the BS analogue of the factorization (3) holds; see [Liu 2021b] for the simplest case of the type-A surface singularity $X = (\mathbb{C}^2/\mathbb{Z}_n) \times \mathbb{C}$. We expect our wall-crossing techniques to also prove, for appropriate G, the *K-theoretic PT/BS vertex correspondence*

$$\mathsf{V}_{\boldsymbol{\lambda}}^{\mathrm{PT},K}(G) = \mathsf{V}_{\boldsymbol{\lambda}}^{\mathrm{BS},K}(G)\mathsf{V}_0^{\mathrm{PT},K}(G).$$

PT/BS correspondences, for vertices or otherwise, are an important step in understanding the DT *crepant resolution conjecture* (*CRC*) [Bryan et al. 2012, Section 4], which relates multiregular curve counts on $\mathfrak{X}$ with curve counts on $\widetilde{X}$.

1.3.7 A simpler version of the DT CRC,

$$\mathsf{Z}_0^{\mathrm{DT}}(\mathfrak{X}) = \frac{\mathsf{Z}_{\mathrm{exc}}^{\mathrm{DT}}(\widetilde{X})\,\mathsf{Z}_{\mathrm{exc}}^{\mathrm{DT},\vee}(\widetilde{X})}{\mathsf{Z}_0^{\mathrm{DT}}(\widetilde{X})},$$

relates point counts on $\mathfrak{X}$ with exceptional curve counts on $\widetilde{X}$, where $\mathsf{Z}_{\mathrm{exc}}^{\mathrm{DT},\vee}$ is a formal repackaging of the invariants in $\mathsf{Z}_{\mathrm{exc}}^{\mathrm{DT}}$. This (cohomological) "point-counting" CRC was proved by Calabrese [2016] via wall-crossing, by comparing DT invariants on $\widetilde{X}$ with counts of perverse coherent sheaves (so-called *noncommutative DT invariants*) for the resolution π. The relation between these noncommutative DT invariants of π and the DT invariants of $\widetilde{X}$ is also treated by Toda [2013] using wall-crossing. We expect that our techniques and the wall-crossing setup of [Toda 2013] will directly give generalizations of Calabrese's results to K-theory and for K-theoretic vertices.

1.3.8 If X is a Calabi–Yau 4-fold instead of a 3-fold, recent developments [Borisov and Joyce 2017; Oh and Thomas 2023] in constructing virtual cycles for moduli of sheaves on X mean that a conjectural 4*-fold K-theoretic DT/PT vertex correspondence* exists as well [Cao et al. 2022]:

$$\mathsf{V}_{\pi^1,\pi^2,\pi^3,\pi^4}^{\mathrm{DT},K} = \mathsf{V}_{\pi^1,\pi^2,\pi^3,\pi^4}^{\mathrm{PT},K}\mathsf{V}_{\varnothing,\varnothing,\varnothing,\varnothing}^{\mathrm{DT},K},$$

where the π^i are plane partitions. It subsumes the 3-fold DT/PT correspondence because these 4-fold vertices become equal to their 3-fold counterparts under a certain limit when the fourth leg is trivial. However, significantly less computational evidence is available: explicit calculation of general 4-fold PT vertices via equivariant localization, beyond the case of two nontrivial legs where all fixed points are isolated, only became possible very recently [Liu 2024].

We believe that our wall-crossing techniques can be extended to 4-fold vertices and partition functions, modulo some (difficult!) technical issues stemming from the usage of (-2)-shifted symplectic structures in the definition of 4-fold virtual cycles. Preliminary results on 4-fold versions of the DT crepant resolution conjecture also exist [Cao et al. 2023], and so it is not unreasonable to also expect 4-fold versions of the PT/BS correspondences of Section 1.3.6.

1.4 Strategy and tools

1.4.1 We give *two* proofs of Theorem 1.3.1 in the hopes of making this paper twice as exciting. Both proofs involve wall-crossing with respect to a family of weak stability conditions on an ambient moduli stack, following ideas of Toda [2010, Section 3], but they differ in how the wall-crossing step is implemented. The Mochizuki-style approach in Section 5 directly crosses the relevant wall, using a somewhat ad hoc setup stemming from [Mochizuki 2009; Nakajima and Yoshioka 2011; Kuhn and Tanaka 2021]. On the other hand, the Joyce-style approach in Section 6 is a simplified application of ideas from Joyce's recent *universal* wall-crossing machinery [Joyce 2021], which crosses the wall by first moving onto it and then off of it in two steps. We believe Section 5 is a good warm-up for the more sophisticated setting of Section 6. The content of Section 6 may also be viewed as a brief exposition of Joyce's universal wall-crossing machinery in its simplest form.

In both approaches, the end result is a fairly nontrivial identity which can be written in the language (Definition 4.2.5) of word rearrangements. In Section 5.3, we prove an explicit and fairly involved combinatorial result in order to simplify this identity into the desired DT/PT vertex correspondence. On the other hand, in Section 6.3, we provide a trick, stemming from ideas of Toda, which sidesteps all the combinatorics.

In the remainder of this subsection, we outline the wall-crossing ingredients which are common to both approaches.

1.4.2 To begin, in Section 3.1 we construct moduli stacks $\mathfrak{N}_{(\lambda,\mu,\nu),n}$, and Section 3.2 equips them with a family $(\tau_\xi)_\xi$ of weak stability conditions, with exactly one wall τ_0 where there are strictly semistable objects, such that the semistable loci

$$\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^-), \quad \mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^+),$$

for weak stability conditions $\tau^\pm$ on the two sides of this wall, are the DT and PT moduli schemes (4) respectively. On the wall τ_0, the strictly semistable objects are sums of the form

$$[\mathcal{O}_{\overline{X}} \to \mathcal{E}] \oplus [0 \to \mathcal{Z}_1] \oplus \cdots \oplus [0 \to \mathcal{Z}_k],$$

where all the $\mathcal{Z}_i$ have zero-dimensional support; for use in the wall-crossing argument, we therefore also define moduli stacks $\mathfrak{Q}_n$ of such coherent sheaves $\mathcal{Z}$. Finally, Section 3.3 equips all these stacks with symmetric obstruction theories, perfect on stable loci, such that the enumerative invariants $\mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^\pm) := \chi(\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^\pm), \hat{\mathcal{O}}^{\mathrm{vir}})$ form the desired DT/PT vertices (5).

1.4.3 Geometric wall-crossing arguments typically require that $\tau^\pm$-stable objects decompose into at most two pieces at τ_0, i.e., a so-called "simple" wall-crossing, so that *master space* techniques are applicable. This is not the case for $\mathfrak{N} = \mathfrak{N}_{(\lambda,\mu,\nu),n}$. The general strategy (Section 4.1) is to construct auxiliary *quiver-framed* stacks $\widetilde{\mathfrak{N}}$ such that:

(i) there are easily understood forgetful maps $\Pi_{\mathfrak{N}} \colon \widetilde{\mathfrak{N}} \to \mathfrak{N}$ (for us, flag variety fibrations);

(ii) there are weak stability conditions $\tilde{\tau}^\pm$ on $\widetilde{\mathfrak{N}}$, compatible with $\tau^\pm$ under $\Pi_{\mathfrak{N}}$, with no strictly $\tilde{\tau}^\pm$-semistable objects and proper T-fixed stable loci;

(iii) the symmetric obstruction theory on $\mathfrak{N}$ lifts in a compatible way along $\Pi_{\mathfrak{N}}$ to a symmetric obstruction theory on all semistable=stable loci in $\widetilde{\mathfrak{N}}$;

(iv) the wall-crossing problem between τ^- and τ^+ lifts to a wall-crossing problem between $\tilde{\tau}^-$ and $\tilde{\tau}^+$, which may (and will, for us) have more walls, such that stable objects decompose into at most two pieces at each wall.

The same must be done for the stacks $\mathfrak{Q}_n$. Properties (ii) and (iii) imply that enumerative invariants $\widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),n}(\tilde{\tau}^\pm) := \chi(\widetilde{\mathfrak{N}}^{\mathrm{sst}}(\tilde{\tau}^\pm), \hat{\mathcal{O}}^{\mathrm{vir}})$ are well-defined, and property (i) means that they relate to the original enumerative invariants $\mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^\pm)$ in explicit ways.

To get property (ii), we give a general argument in Section 4.3 to prove properness of stable loci in moduli stacks like $\mathfrak{N}^Q$, using the notion of elementary modification originated in [Langton 1975]. To get property (iii), which in the terminology of [Liu 2023] is asking for a *symmetrized pullback* of the obstruction theory on $\mathfrak{N}$, we will require a generalization of perfect obstruction theories due to Kiem and Savvas.

1.4.4 In Section 2.1, we review obstruction theories on Artin stacks, and what it means for them to be *perfect* and/or *symmetric* of T-weight κ. All objects and morphisms are T-equivariant unless stated otherwise. To emphasize, the natural symmetric obstruction theory on $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ is *not perfect*, and symmetry is what causes it to become perfect upon restriction to stable loci. Preserving the symmetry of auxiliary obstruction theories will therefore be a key technical step in this paper.

The strategies in [Liu 2023, Section 2] for constructing symmetrized pullbacks of obstruction theories are only applicable in very special settings, e.g., if $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ were affine or a shifted cotangent bundle. To get symmetrized pullbacks in general, we use Kiem and Savvas' notion [2022] of *almost-perfect obstruction theory* (*APOT*); we do not know how to prove part (i) of the following theorem using only perfect obstruction theories (POTs). Roughly, an APOT is an étale-local collection of POTs with the data of how to glue the local obstruction sheaves into a global obstruction sheaf. Every POT on a DM stack is an APOT, and every APOT induces a virtual structure sheaf.

The following theorem is the technical heart of this paper and should be of independent interest.

Theorem *Let $f\colon\mathfrak{X}\to\mathfrak{M}$ be a smooth morphism of Artin stacks over a base Artin stack $\mathfrak{B}$ which is smooth and of pure dimension. Suppose $\mathfrak{X}$ is a DM stack and $\mathfrak{M}$ has an obstruction theory $\phi_{\mathfrak{M}/\mathfrak{B}}\colon\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}\to\mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ which is symmetric of T-weight κ.*

(i) Symmetrized pullback (Theorem 2.3.4): *On $\mathfrak{X}$, there exists a symmetric APOT $\phi_{\mathfrak{X}/\mathfrak{B}}$ of weight κ which is compatible under f with $\phi_{\mathfrak{M}/\mathfrak{B}}$ in the sense of Definition 2.1.11.*

(ii) Virtual torus localization (Theorem 2.4.3): *Let $\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X}}$ be the virtual structure sheaf constructed from $\phi_{\mathfrak{X}/\mathfrak{B}}$. If $\mathfrak{X}$ is quasiseparated, locally of finite type, and has the resolution property,[2] then*

$$\hat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}}=\iota_*\frac{\hat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}^{\mathsf{T}}}}{\hat{\mathsf{e}}(\mathcal{N}^{\mathrm{vir}})}\in K_{\mathsf{T}}(\mathfrak{X})_{\mathrm{loc}},\tag{6}$$

where $\iota\colon\mathfrak{X}^{\mathsf{T}}\hookrightarrow\mathfrak{X}$ is the T-fixed locus, $\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X}^{\mathsf{T}}}$ is constructed from the T-fixed part of $\phi_{\mathfrak{X}/\mathfrak{B}}$,

$$\mathcal{N}^{\mathrm{vir}}:=\iota^*(f^*\mathbb{E}^\vee_{\mathfrak{M}/\mathfrak{B}}+\Omega_f^\vee-\kappa^{-1}\Omega_f)^{\mathrm{mov}}\in K_{\mathsf{T}}(\mathfrak{X}^{\mathsf{T}})$$

is the T-moving part of what the virtual normal bundle would be if it existed globally, and $\hat{\mathsf{e}}(\mathcal{N}^{\mathrm{vir}})$ is its symmetrized K-theoretic Euler class (Section 2.4).

Part (ii) of this theorem generalizes a result of Kiem and Savvas [2020, Theorem 5.13] by relaxing their assumption that $\mathcal{N}^{\mathrm{vir}}$ really must exist globally as a two-term complex of vector bundles, instead of just as a K-theory class.

1.4.5 Finally, with the constructions of Sections 1.4.3 and 1.4.4 finished, we can begin the actual wall-crossing procedure. To relate $\tilde{\mathsf{N}}_{(\lambda,\mu,\nu),n}(\tilde{\tau})$ and $\tilde{\mathsf{N}}_{(\lambda,\mu,\nu),n}(\tilde{\tau}')$ for $\tilde{\tau}'$ close by $\tilde{\tau}$, the strategy is to construct a *master space*, or possibly a sequence of master spaces, which interpolate between $\tilde{\tau}$-stable and $\tilde{\tau}'$-stable objects, and have appropriate analogues of properties (i)–(iii) from Section 1.4.3. "Interpolation" here means there is a $\mathbb{C}^\times$-action whose fixed loci consists of: a locus of the $\tilde{\tau}$-stable objects of interest, a locus of $\tilde{\tau}'$-stable objects of interest, and an "interaction" locus which contributes most of the complexity. Then apply the following to the master space.[3]

Proposition [Liu 2023, Proposition 5.2, Lemma 5.5] *In the situation of Theorem 1.4.4 with $(\mathbb{C}^\times\times\mathsf{T})$-equivariance, suppose in addition that:*

(i) *for the maximal torus $\mathsf{T}_w\subset\ker(w)$ associated to any $(\mathbb{C}^\times\times\mathsf{T})$-weight w which is nontrivial as a $\mathbb{C}^\times$-weight, the fixed locus $\mathfrak{X}^{\mathsf{T}_w}$ is proper;*

(ii) *the $\mathbb{C}^\times$-moving part of $f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}|_{\mathfrak{X}^{\mathbb{C}^\times}}$ has the form $\mathcal{F}-\kappa^{-1}\mathcal{F}^\vee$ for some $\mathcal{F}\in K_{\mathbb{C}^\times\times\mathsf{T}}(\mathfrak{X}^{\mathbb{C}^\times})$.*

[2] The condition that $\mathfrak{X}$ has the resolution property can actually be removed with more care; see the arguments of [Aranha et al. 2025, Theorem D].

[3] Section 1.4.5 also holds in equivariant cohomology, with (7) replaced by $0=\int_{[\mathfrak{X}^{\mathbb{C}^\times}]^{\mathrm{vir}}}(-1)^{\mathsf{ind}}\,\mathsf{ind}\in H_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$. Note that this is independent of κ.

Let $\mathcal{F}_{>0}$ and $\mathcal{F}_{<0}$ be the parts of $\mathcal{F}$ with positive and negative $\mathbb{C}^\times$-weight respectively, and set $\mathsf{ind} :=$ $\operatorname{rank}\mathcal{F}_{>0} - \operatorname{rank}\mathcal{F}_{<0}$. *Then*

$$0 = \chi\big(\mathfrak{X}^{\mathbb{C}^\times}, \widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}^{\mathbb{C}^\times}} \otimes (-1)^{\mathsf{ind}}(\kappa^{\frac{1}{2}\mathsf{ind}} - \kappa^{-\frac{1}{2}\mathsf{ind}})\big) \in K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}, \tag{7}$$

where $\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X}^{\mathbb{C}^\times}}$ is constructed from the $\mathbb{C}^\times$-fixed part of the APOT $\phi_{\mathfrak{X}/\mathfrak{B}}$.

We omit the proof because it is identical to that of [Liu 2023, Proposition 5.2, Lemma 5.5], namely: take the $(\mathbb{C}^\times\times\mathsf{T})$-equivariant localization formula on $\mathfrak{X}$, which exists by Theorem 1.4.4(ii), and apply an appropriate residue map in the $\mathbb{C}^\times$ coordinate. Symmetry of the (A)POT (and condition (ii) above) is crucial here, because then the contribution $1/\hat{\mathsf{e}}(\mathcal{N}^{\mathrm{vir}})$ to equivariant localization (6) is a product of rational functions of the form

$$\frac{(\kappa w)^{\frac{1}{2}} - (\kappa w)^{-\frac{1}{2}}}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}},$$

which have the well-defined limits $\pm\kappa^{\pm 1/2}$ as $w \to 0$ or $w \to \infty$. This elementary but very powerful observation allows us to explicitly identify the output of the residue map, namely the right-hand side of (7).

Note that κ must be a nontrivial T-weight in order for (7) to give a nontrivial relation between invariants. Additional work is needed to use our wall-crossing techniques in a setting where $\kappa = 1$; see, for instance, [Kiem and Li 2013], who treat the case of a simple wall-crossing in DT theory using $\mathbb{C}^\times$-intrinsic blowups and cosection localization.

1.4.6 After appropriate normalization, the wall-crossing formula (7) becomes a relation between enumerative invariants with coefficients which are the (symmetric) quantum integers

$$[n]_\kappa := (-1)^{n-1}\frac{\kappa^{\frac{1}{2}n} - \kappa^{-\frac{1}{2}n}}{\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}}}. \tag{8}$$

These include an unconventional sign $(-1)^{n-1}$ in order to save on signs elsewhere; see Proposition 4.2.2 for motivation. In particular, $\lim_{\kappa\to 1}[n]_\kappa = (-1)^{n-1}n$.

Many previous wall-crossing results in the CY3 setting use Joyce's machinery of Ringel–Hall algebras and motivic stack functions. We expect our equivariant techniques to provide quantizations/refinements of these results, upon working κ-equivariantly, in the sense that the original formulas appear in the $\kappa \to 1$ limit. In particular we expect this to be true for the contents of [Joyce and Song 2012].

2 Symmetrized pullback using APOTs

2.1 Obstruction theories and pullbacks

2.1.1 In this section, we construct a *symmetrized pullback* operation for symmetric obstruction theory along a smooth morphism from a Deligne–Mumford stack to an Artin stack. The main difficulty is that such a symmetrized pullback doesn't generally exist as a perfect obstruction theory – instead we allow the result to be a symmetric *almost perfect obstruction theory*, as introduced in [Kiem and Savvas 2022]. We also prove a localization formula for almost perfect obstruction theories obtained by symmetrized pullback, and establish that the construction is functorial under composition of smooth maps.

2.1.2 Definition Fix a base Artin stack $\mathfrak{B}$ which is smooth and of pure dimension. All stacks, morphisms, etc. in this section will be implicitly over $\mathfrak{B}$ and also T-equivariant unless stated otherwise. Let $\mathfrak{M}$ be an Artin stack and $D^{-}_{\mathcal{QCoh},\mathsf{T}}(\mathfrak{M})$ be its derived category of bounded-above T-equivariant complexes with quasicoherent cohomology [Olsson 2007]. Let $\mathbb{L}_{\mathfrak{M}/\mathfrak{B}} \in D^{-}_{\mathcal{QCoh},\mathsf{T}}(\mathfrak{M})$ denote the cotangent complex [Illusie 1971].

A T-equivariant *obstruction theory* of $\mathfrak{M}$ is an object $\mathbb{E} \in D^{-}_{\mathcal{QCoh},\mathsf{T}}(\mathfrak{M})$, together with a morphism $\phi\colon \mathbb{E} \to \mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ in $D^{-}_{\mathcal{QCoh},\mathsf{T}}(\mathfrak{M})$, such that $h^{\geq 0}(\phi)$ are isomorphisms and $h^{-1}(\phi)$ is surjective.

An obstruction theory is:

- *perfect* if $\mathbb{E}$ is a perfect complex of amplitude contained in $[-1, 1]$;
- *symmetric*, of T-weight κ, if $\mathbb{E}$ is a perfect complex and there is an isomorphism $\Theta\colon \mathbb{E} \xrightarrow{\sim} \kappa \otimes \mathbb{E}^\vee[1]$ such that $\kappa \otimes \Theta^\vee[1] = \Theta$.

This notion of perfect obstruction theory (POT), and symmetric POT, coincides with the one of Behrend and Fantechi [1997; 2008] if $\mathfrak{M}$ is a Deligne–Mumford stack, because of the requirement that $h^1(\phi)$ is an isomorphism.

2.1.3 Symmetric obstruction theories are a hallmark of DT-like theories in comparison to other enumerative theories of curves. Requiring an obstruction theory $\mathbb{E}$ on $\mathfrak{M}$ to be symmetric confers two important advantages.

First, although $h^{-2}(\mathbb{E}) \neq 0$ in general, if $\mathfrak{M}$ has a stability condition with no strictly semistable objects, then $\mathbb{E}$ is automatically perfect on the (open) stable locus $\mathfrak{M}^{\mathrm{st}} = \mathfrak{M}^{\mathrm{sst}}$. This is because stable objects have no automorphisms, so

$$h^1(\mathbb{E})|_{\mathfrak{M}^{\mathrm{st}}} \cong h^1(\mathbb{L}_{\mathfrak{M}})|_{\mathfrak{M}^{\mathrm{st}}} = h^1(\mathbb{L}_{\mathfrak{M}^{\mathrm{st}}}) = 0.$$

But then by symmetry

$$h^{-2}(\mathbb{E})|_{\mathfrak{M}^{\mathrm{st}}} \cong \kappa \otimes h^1(\mathbb{E})^\vee|_{\mathfrak{M}^{\mathrm{st}}} = 0$$

as well. Hence $\mathbb{E}$ induces a virtual cycle on $\mathfrak{M}^{\mathrm{st}}$, an important ingredient for enumerative geometry.

Second, the rational functions appearing in the T-equivariant localization formula for these virtual cycles are of a very special form and are amenable to *rigidity* arguments. In particular, our key computational tool, Proposition 1.4.5, would be drastically less effective using a nonsymmetric obstruction theory.

2.1.4 Definition Let $\mathfrak{X}$ be a Deligne–Mumford stack of finite presentation over $\mathfrak{B}$. Following [Kiem and Savvas 2020, Definition 5.1], a T-equivariant *almost-perfect obstruction theory* (APOT) on $\mathfrak{X}$ consists of:

(a) a T-equivariant étale cover $\{U_i \to \mathfrak{X}\}_{i \in I}$ where the U_i are schemes;

(b) for each i, a T-equivariant POT $\phi_i\colon \mathbb{E}_i \to \mathbb{L}_{U_i/\mathfrak{B}}$.

We often write $\phi = (U_i, \phi_i)_{i \in I}$ to denote this data. This data must satisfy the following conditions:

(i) There exist T-equivariant transition isomorphisms

$$\psi_{ij}: h^1(\mathbb{E}_i^\vee)|_{U_{ij}} \to h^1(\mathbb{E}_j^\vee)|_{U_{ij}}$$

which give descent data for a sheaf $\mathcal{O}b_\phi$ on $\mathfrak{X}$, called the obstruction sheaf.[4]

(ii) For each pair i, j, there exists a T-equivariant étale covering $\{V_k \to U_{ij}\}_{k \in K}$, such that the isomorphism ψ_{ij} of the obstruction sheaves lifts to a T-equivariant isomorphism of the perfect obstruction theories over each V_k.

In addition, if the following conditions are satisfied for some T-weight κ, then the APOT is *symmetric* with weight κ:

(iii) For each i, the POT $\phi_i: \mathbb{E}_i \to \mathbb{L}_{U_i/\mathfrak{B}}$ is symmetric with weight κ.

(iv) There exists a T-equivariant isomorphism $\kappa \otimes \mathcal{O}b_\phi \cong h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}})$.

We think of an APOT as an étale-local collection of POTs where only the local obstruction sheaves, not the POTs themselves, glue to form a global object. By [Alper et al. 2020, Theorem 4.3], after possibly passing to a multiple cover of T, a T-equivariant étale cover $\{U_i \to \mathfrak{X}\}_i$ always exists and moreover the U_i can be taken to be affine.

2.1.5 Theorem [Kiem and Savvas 2022, Definition 4.1] *A* T*-equivariant APOT on* $\mathfrak{X}$ *gives rise to a* T*-equivariant virtual structure sheaf* $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} \in K_{\mathsf{T}}(\mathfrak{X})$.

We recall the main ideas from Kiem and Savvas, for completeness and for the reader's convenience.

Proof sketch Recall from [Lee 2004, Definition 2.3] and [Ciocan-Fontanine and Kapranov 2009, Definition 3.2.1] that the usual construction of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}}$ using a POT $\mathbb{E}$ involves the cone stack $\mathfrak{E} := h^1/h^0(\mathbb{E}^\vee)$, an embedding $\mathfrak{C}_{\mathfrak{X}} \hookrightarrow \mathfrak{E}$ of the intrinsic normal cone $\mathfrak{C}_{\mathfrak{X}}$, and a K-theoretic Gysin map $0_{\mathfrak{E}}^!: K_{\mathsf{T}}(\mathfrak{E}) \to K_{\mathsf{T}}(\mathfrak{X})$.

For an APOT ϕ, Kiem and Savvas' idea is to work with the *sheaf stack* $\mathcal{O}b_\phi$, which is approximately the coarse moduli stack $h^1(\mathbb{E}^\vee)$ of $\mathfrak{E}$. To be precise, $\mathcal{O}b_\phi$ is (an abuse of notation for) the stack which, to a morphism $\rho: W \to \mathfrak{X}$ from a scheme W, assigns the abelian group $H^0(W, \rho^*\mathcal{O}b_\phi)$. This is not an algebraic stack, but its K-group $K_{\mathsf{T}}(\mathcal{O}b_\phi)$ and other objects can be defined in terms of what Kiem and Savvas call *affine local charts* of the form

$$(U \xrightarrow{\rho} \mathfrak{X}, \mathcal{E} \xrightarrow{r} \rho^*\mathcal{O}b_\phi). \tag{9}$$

Here everything is T-equivariant, ρ is an étale map from an affine scheme U, and r is a surjection from a vector bundle $\mathcal{E}$ on U. Working on affine local charts, homology sheaves of Koszul complexes on each U glue by standard descent arguments, and this suffices to construct the desired Gysin map

$$0_{\mathcal{O}b_\phi}^!: K_{\mathsf{T}}(\mathcal{O}b_\phi) \to K_{\mathsf{T}}(\mathfrak{X}).$$

[4]Here, we follow the presentation of Kiem and Savvas, although it is slightly inaccurate: the obstruction sheaf is really part of the *data* defining the almost perfect obstruction theory.

See [Kiem and Savvas 2022, Section 2]. Similarly, there is a sheaf stack $\mathfrak{n}_{\mathfrak{X}/\mathfrak{B}} := h^1(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}}^\vee)$ and a closed substack $\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}} \subset \mathfrak{n}_{\mathfrak{X}/\mathfrak{B}}$ (cf. [Chang and Li 2011]), which Kiem and Savvas call the *coarse intrinsic normal sheaf* and *coarse intrinsic normal cone*, respectively. Using the transition maps in Definition 2.1.4(ii), the closed embeddings $h^1(\phi_i^\vee)\colon \mathfrak{n}_{U_i/\mathfrak{B}} \hookrightarrow \mathcal{O}b_{\phi_i}$ glue to form a global closed embedding

$$j_\phi\colon \mathfrak{n}_{\mathfrak{X}/\mathfrak{B}} \hookrightarrow \mathcal{O}b_\phi \tag{10}$$

of sheaf stacks, and so $\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}} \subset \mathfrak{n}_{\mathfrak{X}/\mathfrak{B}}$ embeds as a closed substack $\mathfrak{c}_\phi := j_\phi(\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}})$ of $\mathcal{O}b_\phi$ [Kiem and Savvas 2022, Theorem 3.4]. Then, using the Gysin map, we define $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} := 0^!_{\mathcal{O}b_\phi}\mathcal{O}_{\mathfrak{c}_\phi}$ as usual. □

2.1.6 For the reader's convenience, we add some detail about a tricky step in the functoriality and deformation invariance proofs in [Kiem and Savvas 2022; 2020]. We continue with the notation in (the proof of) Theorem 2.1.5.

Theorem [Kiem and Savvas 2022, Theorem 4.2] *Consider a cartesian diagram*

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{u} & \mathfrak{X} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{v} & W \end{array}$$

where Z and W are smooth varieties and v is a regular closed immersion. Suppose we are given T*-equivariant APOTs*

$$\phi = (U_i, \phi_i\colon \mathbb{E}_i \to \mathbb{L}_{U_i})_{i\in I} \quad \text{and} \quad \phi' = (V_i := U_i \times_{\mathfrak{X}} \mathfrak{Y}, \phi_i'\colon \mathbb{E}_i' \to \mathbb{L}_{V_i})_{i\in I}$$

on $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, compatible with u in the sense that there are commutative diagrams of distinguished triangles

$$\begin{array}{ccccccc} \mathbb{E}_i|_{V_i} & \xrightarrow{g_i} & \mathbb{E}_i' & \longrightarrow & \mathcal{N}_{Z/W}|_{V_i}[1] & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \phi_i|_{V_i}} & & \downarrow{\scriptstyle \phi_i'} & & \downarrow & & \\ \mathbb{L}_{U_i}|_{V_i} & \longrightarrow & \mathbb{L}_{V_i} & \longrightarrow & \mathbb{L}_{V_i/U_i} & \xrightarrow{+1} & \end{array}$$

which are compatible with the lifted isomorphisms in Definition 2.1.4(ii). Then

$$[\mathcal{O}_{\mathfrak{Y}}^{\mathrm{vir}}] = v^![\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}}] \in K_{\mathsf{T}}(\mathfrak{Y}),$$

where $v^! := v_*\mathcal{O}_Z \otimes_{\mathcal{O}_{\mathfrak{X}}} -\colon K_{\mathsf{T}}(\mathfrak{X}) \to K_{\mathsf{T}}(\mathfrak{Y})$ is the Gysin map.

In this setting, we get exact sequences

$$\mathcal{N}_{Z/W}|_{V_i} \to \mathcal{O}b_{\phi_i'} \xrightarrow{h^1(g_i^\vee)} \mathcal{O}b_{\phi_i}|_{V_i} \to 0,$$

which glue to a global exact sequence of T-equivariant sheaves on $\mathfrak{Y}$.

2.1.7 Proof For a relatively DM morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$, let $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}$ and $\mathfrak{N}_{\mathfrak{X}/\mathfrak{Y}}$ denote the normal cone stack and its abelian hull; their coarse spaces are the sheaf stacks $\mathfrak{c}_{\mathfrak{X}/\mathfrak{Y}}$ and $\mathfrak{n}_{\mathfrak{X}/\mathfrak{Y}}$ respectively.

Let $\mathcal{M}_{\mathfrak{X}}^{\circ} \to \mathbb{P}^1$ denote the deformation of $\mathfrak{X}$ to the intrinsic normal cone $\mathfrak{C}_{\mathfrak{X}} = \mathfrak{C}_{\mathfrak{X}/\mathfrak{B}}$. Following a standard argument in [Kim et al. 2003, Theorem 1], Kiem and Savvas [2022, (4.11)] use a "double deformation space" $\mathfrak{W}$, which is the deformation of $\mathfrak{Y} \times \mathbb{P}^1$ to its normal cone stack in $\mathcal{M}_{\mathfrak{X}}^{\circ}$, to prove the equality

$$(11) \qquad [\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}}}] = [\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}}] \in K_{\mathsf{T}}(\mathfrak{N}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}).$$

To prove the desired functoriality statement, we require, furthermore, the equality [Kiem and Savvas 2022, (4.13)]

$$(12) \qquad [\mathcal{O}_{\mathfrak{c}_{\mathfrak{Y}}}] = [\mathcal{O}_{\mathfrak{c}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}}] \in K_{\mathsf{T}}(\mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}).$$

We will obtain this by pushing forward (11) in a suitable way along the coarse space morphism

$$\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}} \colon \mathcal{N}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}} \to \mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}.$$

This procedure clarifies a crucial step in the proof of [Kiem and Savvas 2022, Theorem 4.2].

2.1.8 Definition Let $\mathbb{F}$ be a T-equivariant perfect complex of amplitude $[0, 1]$, and let

$$\pi \colon h^1/h^0(\mathbb{F}) \to h^1(\mathbb{F})$$

be the coarse space morphism. We define an operation

$$\pi_{\diamond} \colon K_{\mathsf{T}}(h^1/h^0(\mathbb{F})) \to K_{\mathsf{T}}(h^1(\mathbb{F}))$$

as follows. Let $(U \xrightarrow{\rho} \mathfrak{X},\ \mathcal{E} \xrightarrow{r} \rho^* h^1(\mathbb{F}))$ be an affine local chart for $h^1(\mathbb{F})$, as in (9). Since U is affine, and $\mathcal{E}$ is locally free, it is projective, which allows us to extend the morphism $\mathcal{E} \twoheadrightarrow h^1(\mathbb{F})$ to a morphism $\xi \colon \mathcal{E} \to \mathbb{F}$. Note that $h^1(\operatorname{cone}(\xi))$ vanishes by construction. By [Stacks 2005–, Tag 0F8E], ξ extends to a 2-term resolution

$$\mathbb{F} = [\mathcal{F} \to \mathcal{E}]$$

by locally free sheaves. Let F and E be the corresponding vector bundles on $\mathfrak{X}$. Over U:

- a T-equivariant coherent sheaf on the cone stack $h^1/h^0(\mathbb{F})$ is an $(F \rtimes \mathsf{T})$-equivariant coherent sheaf on E;
- a T-equivariant coherent sheaf on the sheaf stack $h^1(\mathbb{F})$ is a T-equivariant coherent sheaf on E (by definition [Kiem and Savvas 2022, Section 2.2]).

Define $\pi_{\diamond}$ over U by forgetting the F-equivariance. As this operation is exact on every affine local chart, it maps exact sequences to exact sequences and induces the desired map of K-groups.

Note that we define this operation only for this specific type of morphism.

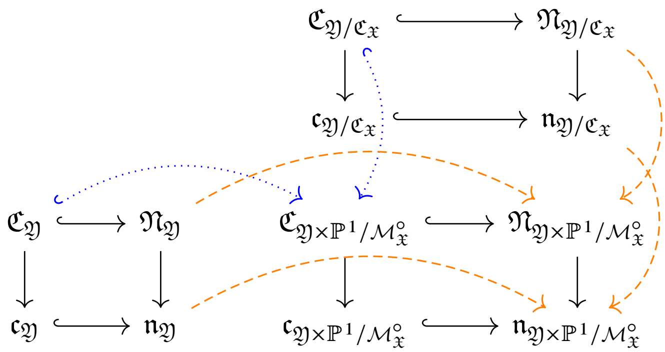
2.1.9 From (11), we therefore obtain

$$(13) \qquad (\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}})_{\diamond}[\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}}}] = (\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}})_{\diamond}[\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}}] \in K_{\mathsf{T}}(\mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}).$$

By construction, $(\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}})_{\diamond}$ maps structure sheaves of closed substacks to the structure sheaves of their images in $\mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}$. So we must study the images of $\mathfrak{C}_{\mathfrak{Y}}$ and $\mathfrak{C}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}$ in $\mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^{\circ}}$.

Consider the diagram

(14)



The dotted arrows are given by the inclusion of normal cones. We now explain how to get the two commutative squares involving the dashed arrows. For each square the setup is the same. Given a cartesian square

$$\begin{array}{ccc} Y & \xhookrightarrow{j} & \tilde{Y} \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ C & \xhookrightarrow{i} & M \end{array}$$

there are the induced distinguished triangles

$$a^*\mathbb{L}_{C/M} \to \mathbb{L}_{Y/M} \to \mathbb{L}_{Y/C} \xrightarrow{+1} \quad \text{and} \quad j^*\mathbb{L}_{\tilde{Y}/M} \to \mathbb{L}_{Y/M} \to \mathbb{L}_{Y/\tilde{Y}} \xrightarrow{+1}.$$

By functoriality of $h^1/h^0(-)$ [Behrend and Fantechi 1997, Section 2], these induce morphisms of abelian cone stacks

$$\mathfrak{N}_{Y/C} \to \mathfrak{N}_{Y/M} \to a^*\mathfrak{N}_{C/M} \quad \text{and} \quad \mathfrak{N}_{Y/\tilde{Y}} \to \mathfrak{N}_{Y/M} \to j^*\mathfrak{N}_{\tilde{Y}/M}.$$

We get the desired morphism $\mathfrak{N}_{Y/C} \to j^*\mathfrak{N}_{\tilde{Y}/M}$ by composition. By a similar composition, the distinguished triangles give a morphism

$$h^1(\mathbb{L}^\vee_{Y/C}) \to h^1(\mathbb{L}^\vee_{Y/M}) \to j^*h^1(\mathbb{L}^\vee_{\tilde{Y}/M}),$$

compatible with $\mathfrak{N}_{Y/C} \to j^*\mathfrak{N}_{\tilde{Y}/M}$ by construction. In sheaf stack and cone stack language, this yields the commutative diagram

$$\begin{array}{ccccc} \mathfrak{N}_{Y/C} & \longrightarrow & j^*\mathfrak{N}_{\tilde{Y}/M} = Y \times_{\tilde{Y}} \mathfrak{N}_{\tilde{Y}/M} & \hookrightarrow & \mathfrak{N}_{\tilde{Y}/M} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathfrak{n}_{Y/C} & \longrightarrow & j^*h^1(\mathbb{L}^\vee_{\tilde{Y}/M}) = Y \times_{\tilde{Y}} \mathfrak{n}_{\tilde{Y}/M} & \hookrightarrow & \mathfrak{n}_{\tilde{Y}/M} \end{array}$$

which yields the desired (dashed) maps in (14). These are naturally compatible with the (dotted) maps between cones in (14). Since $\mathfrak{c}$ is by definition the image of $\mathfrak{C}$ in $\mathfrak{n}$, we conclude

$$(\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}^\circ_{\mathfrak{X}}})_\diamond[\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}}}] = [\mathcal{O}_{\mathfrak{c}_{\mathfrak{Y}}}] \quad \text{and} \quad (\pi_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}^\circ_{\mathfrak{X}}})_\diamond[\mathcal{O}_{\mathfrak{C}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}}] = [\mathcal{O}_{\mathfrak{c}_{\mathfrak{Y}/\mathfrak{C}_{\mathfrak{X}}}}].$$

Combining this with (13), we obtain the desired (12).

2.1.10 From (12), we may follow Kiem and Savvas' argument verbatim; see also [Kim et al. 2003, Theorem 1]. Roughly, one constructs a sheaf stack $\mathfrak{K}$ and a closed embedding of sheaf stacks

$$\mathfrak{n}_{\mathfrak{Y}\times\mathbb{P}^1/\mathcal{M}_{\mathfrak{X}}^\circ} \hookrightarrow \mathfrak{K}$$

by gluing, in the same manner as in (10), so that the fiber of $\mathfrak{K}$ over $0 \in \mathbb{P}^1$ is $u^*\mathcal{O}b_\phi \oplus \mathcal{N}_{Z/W}|_{\mathfrak{Y}}$ and the fiber over $1 \in \mathbb{P}^1$ is $\mathcal{O}b_{\phi'}$. Applying the Gysin maps to (12), viewed as an identity in $K_{\mathsf{T}}(\mathfrak{K})$, gives the desired functoriality statement. □

Remark Equations (5.11) and (A.10) of [Kiem and Savvas 2020] can be proven with the same method detailed above.

2.1.11 Definition Let $\mathfrak{N}$ be another Artin stack over the base $\mathfrak{B}$, and let $f\colon \mathfrak{M} \to \mathfrak{N}$ be a smooth morphism of Artin stacks over $\mathfrak{B}$. Suppose $\phi_{\mathfrak{M}/\mathfrak{B}}\colon \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ and $\phi_{\mathfrak{N}/\mathfrak{B}}\colon \mathbb{E}_{\mathfrak{N}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{N}/\mathfrak{B}}$ are obstruction theories.

- The two obstruction theories $\phi_{\mathfrak{M}/\mathfrak{B}}$ and $\phi_{\mathfrak{N}/\mathfrak{B}}$ are *compatible under* f if there is a morphism of exact triangles

(15)
$$\begin{array}{ccccccc}
\mathbb{L}_f[-1] & \xrightarrow{\delta} & f^*\mathbb{E}_{\mathfrak{N}/\mathfrak{B}} & \longrightarrow & \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & \xrightarrow{+1} & \\
\| & & \downarrow{\scriptstyle f^*\phi_{\mathfrak{N}/\mathfrak{B}}} & & \downarrow{\scriptstyle \phi_{\mathfrak{M}/\mathfrak{B}}} & & \\
\mathbb{L}_f[-1] & \longrightarrow & f^*\mathbb{L}_{\mathfrak{N}/\mathfrak{B}} & \longrightarrow & \mathbb{L}_{\mathfrak{M}/\mathfrak{B}} & \xrightarrow{+1} &
\end{array}$$

- ([Liu 2023, Definition 2.3]; cf. [Park 2021, Theorem 0.1]) Suppose $\phi_{\mathfrak{M}/\mathfrak{B}}$ and $\phi_{\mathfrak{N}/\mathfrak{B}}$ are symmetric of weight κ. Then the two symmetric obstruction theories are *compatible under* f if there are morphisms of exact triangles

(16)
$$\begin{array}{ccccccc}
\mathbb{L}_f[-1] & \longrightarrow & \kappa\otimes\mathbb{F}^\vee[1] & \xrightarrow{\zeta} & \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & \xrightarrow{+1} & \\
\| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \zeta^\vee} & & \\
\mathbb{L}_f[-1] & \xrightarrow{\delta} & f^*\mathbb{E}_{\mathfrak{N}/\mathfrak{B}} & \xrightarrow{\eta^\vee} & \mathbb{F} & \xrightarrow{+1} & \\
\| & & \downarrow{\scriptstyle f^*\phi_{\mathfrak{N}/\mathfrak{B}}} & & \downarrow & & \\
\mathbb{L}_f[-1] & \longrightarrow & f^*\mathbb{L}_{\mathfrak{N}/\mathfrak{B}} & \longrightarrow & \mathbb{L}_{\mathfrak{M}/\mathfrak{B}} & \xrightarrow{+1} &
\end{array}$$

  such that the third column is $\phi_{\mathfrak{M}/\mathfrak{B}}$.

From a different perspective, if only the (symmetric) obstruction theory $\phi_{\mathfrak{N}/\mathfrak{B}}$ is given, then a (symmetric) object $\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ along with relevant morphisms fitting into (15) or (16) is called a *smooth pullback* or *symmetrized* (*smooth*) *pullback* along f of $\phi_{\mathfrak{N}/\mathfrak{B}}$, respectively. This is justified as the resulting map $\phi_{\mathfrak{M}/\mathfrak{B}}$ will automatically be a (symmetric) obstruction theory. Moreover, in the case of smooth pullback, if $\phi_{\mathfrak{N}/\mathfrak{B}}$ is perfect then so is $\phi_{\mathfrak{M}/\mathfrak{B}}$ by the five lemma. The symmetry of the upper-right square in (16) means that the output of symmetrized pullback is a symmetric obstruction theory of the same weight.

2.1.12 Symmetrized pullback was introduced in [Liu 2023] in order to equip certain moduli stacks in Vafa–Witten theory with symmetric obstruction theories. There, the stack $\mathfrak{N}$ was (the classical truncation of) a (-1)-shifted cotangent bundle $T^*[-1]\overline{\mathfrak{N}}$, so symmetrized pullback on $\mathfrak{N}$ was roughly equivalent to $T^*[-1]$ of a smooth pullback on $\overline{\mathfrak{N}}$, and so the only task in constructing a symmetrized pullback was to construct the lift δ in (15). This is not difficult in practice because $\phi_{\mathfrak{N}/\mathfrak{B}}$ arises from functorial constructions like the Atiyah class.

However, in general, symmetrized pullbacks are rather difficult to construct. In our situation, we resort to using APOTs; see Section 2.3. The étale-locally affine nature of APOTs forces some vanishings that we otherwise do not have.

2.2 Smooth pullback of APOTs

2.2.1 As a warm-up and for use in Section 2.4, we now construct smooth pullbacks of APOTs. The key idea is that, by working with APOTs, we no longer require the existence of a lift δ in the compatibility diagram (15).

Definition Let $f:\mathfrak{X}\to\mathfrak{Y}$ be a smooth morphism of DM stacks over $\mathfrak{B}$. Since f is smooth and of DM-type, $\mathbb{L}_f=\Omega_f$ is a locally free sheaf. An APOT $\phi_{\mathfrak{X}}$ on $\mathfrak{X}$ and an APOT $\phi_{\mathfrak{Y}}$ on $\mathfrak{Y}$ are *compatible under f* if:

(i) there exists a T-equivariant étale cover $\{U_i\to\mathfrak{Y}\}_{i\in I}$ and a T-equivariant étale cover $\{W_j\to\mathfrak{X}\}_{j\in J}$ refining $\{f^{-1}(U_i)\to\mathfrak{X}\}_{i\in I}$, such that $\phi_{\mathfrak{X}}|_{W_j}$ and $\phi_{\mathfrak{Y}}|_{U_i}$ are POTs which are compatible under $f|_{W_j}:W_j\to U_i$ for each j;

(ii) $f^*\mathcal{O}b_{\phi_{\mathfrak{Y}}}=\mathcal{O}b_{\phi_{\mathfrak{X}}}$.

As in Definition 2.1.11, if only $\phi_{\mathfrak{Y}}$ is given, we refer to any APOT $\phi_{\mathfrak{X}}$ satisfying these conditions as a *smooth pullback* of $\phi_{\mathfrak{Y}}$ along f.

In fact, after sufficient refinement of the étale cover, any smooth pullback APOT is étale-locally equivalent to the one constructed in the following Proposition 2.2.2, for an appropriate notion of equivalence of APOTs.

2.2.2 Proposition *An APOT on $\mathfrak{Y}$ admits a smooth pullback to $\mathfrak{X}$ such that $\mathcal{O}^{\mathrm{vir}}_{\mathfrak{X}}=f^*\mathcal{O}^{\mathrm{vir}}_{\mathfrak{Y}}$.*

Proof Let $(U_i,\phi_i:=\mathbb{E}_i\to\mathbb{L}_{U_i/\mathfrak{B}})_{i\in I}$ be the given APOT on $\mathfrak{Y}$. By [Alper et al. 2020, Theorem 4.3], after possibly passing to a multiple cover of T, there exists a T-equivariant affine étale cover $\{\widetilde{U}_j\to\mathfrak{X}\}_{j\in J}$ refining $\{f^{-1}(U_i)\to\mathfrak{X}\}_{i\in I}$, and moreover the $\widetilde{U}_j$ can be taken to be affine. As we may refine the cover $\{U_i\}_{i\in I}$ and restrict the ϕ_i, we may assume $J=I$ and every $\widetilde{U}_i$ gets mapped into U_i. Let $f_i:\widetilde{U}_i\to U_i$ be the natural maps.

By Lemma 2.3.3 the morphism $\Omega_{f_i}\to f_i^*\mathbb{L}_{U_i/\mathfrak{B}}[1]$ in the distinguished triangle

$$f_i^*\mathbb{L}_{U_i/\mathfrak{B}}\to\mathbb{L}_{\widetilde{U}_i/\mathfrak{B}}\to\Omega_{f_i}\xrightarrow{+1}$$

vanishes, which gives us a T-equivariant splitting

$$\mathbb{L}_{\widetilde{U}_i/\mathfrak{B}} \cong f_i^*\mathbb{L}_{U_i/\mathfrak{B}} \oplus \Omega_{f_i}. \tag{17}$$

We fix this splitting for each i and define the smooth pullback APOT on $\mathfrak{X}$ by taking the cover $\{\widetilde{U}_i\}_{i\in I}$ above and the morphisms

$$\widetilde{\phi}_i\colon \widetilde{\mathbb{E}}_i := f_i^*\mathbb{E}_i \oplus \Omega_{f_i} \xrightarrow{f_i^*\phi_i\oplus \mathrm{id}} f_i^*\mathbb{L}_{U_i/\mathfrak{B}} \oplus \Omega_{f_i} \xrightarrow{\sim} \mathbb{L}_{\widetilde{U}_i/\mathfrak{B}}.$$

By construction of $\widetilde{\mathbb{E}}_i$, the local obstruction sheaves $h^1(\widetilde{\mathbb{E}}_i^\vee)$ are isomorphic to $h^1(f^*\mathbb{E}_i^\vee)$ and glue to give the obstruction sheaf $\mathcal{O}b_{\widetilde{\phi}} = f^*\mathcal{O}b_\phi$.

2.2.3 It remains to verify that $\widetilde{\phi} := (\widetilde{U}_i, \widetilde{\phi}_i)_{i\in I}$ is indeed an APOT, by checking property (ii) in the definition. For a pair i, j, we may choose a T-equivariant cover by affine étale neighborhoods $\widetilde{V}_k \to \widetilde{U}_i \times_{\mathfrak{X}} \widetilde{U}_j$ so that $\widetilde{V}_k$ factors through a neighborhood V_l of $U_i \times_{\mathfrak{Y}} U_j$. We get a compatible isomorphism

$$\widetilde{\mathbb{E}}_i|_{\widetilde{V}_k} \simeq f^*\mathbb{E}_i|_{\widetilde{V}_k} \oplus \Omega_{\widetilde{V}_k/V_l} \to f^*\mathbb{E}_j|_{\widetilde{V}_k} \oplus \Omega_{\widetilde{V}_k/V_l} \simeq \widetilde{\mathbb{E}}_j|_{\widetilde{V}_k}$$

as the upper-triangular map determined as follows. Along the diagonal, we take (a lift of) $f^*\phi_{ij}$ and id_{Ω_f}. This guarantees compatibility with the obstruction sheaf. Off-diagonal, the map $\Omega_{\widetilde{V}_k/V_l} \to f^*\mathbb{E}_j|_{\widetilde{V}_k}$ is determined via

$$\Omega_{f_i} \hookrightarrow \mathbb{L}_{\widetilde{U}_i/\mathfrak{B}} \to h^0(\mathbb{L}_{\widetilde{U}_i/\mathfrak{B}}) \simeq h^0(\widetilde{\mathbb{E}}_j) \to h^0(f^*\mathbb{E}_j),$$

where the first inclusion comes from the splitting (17). Here we use that for a vector bundle W on an affine scheme U and an element C of $D^-_{\mathfrak{QCoh},\mathsf{T}}(U)$, we have $\mathrm{Hom}(W, C) \simeq \mathrm{Hom}(W, \tau_{\geq 0}C)$. This choice guarantees that the local isomorphism of obstruction theories is compatible with the maps to cotangent complexes.

2.2.4 Finally, we prove that the induced virtual structure sheaves satisfy $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} = f^*\mathcal{O}_{\mathfrak{Y}}^{\mathrm{vir}}$. By construction, $\mathcal{O}b_{\widetilde{\phi}} = f^*\mathcal{O}b_\phi$, and recall that $\mathcal{O}_{\mathfrak{Y}}^{\mathrm{vir}} := 0^!_{\mathcal{O}b_\phi}[\mathcal{O}_{\mathfrak{c}_\phi}]$. So, we want to show

$$f^*0^!_{\mathcal{O}b_\phi}\mathcal{O}_{\mathfrak{c}_\phi} = 0^!_{\mathcal{O}b_{\widetilde{\phi}}}\mathcal{O}_{\mathfrak{c}_{\widetilde{\phi}}}.$$

By the functoriality of $0^!$ [Kiem and Savvas 2020, Lemma 3.8], we only need to show that $f^*\mathcal{O}_{\mathfrak{c}_\phi} = \mathcal{O}_{\mathfrak{c}_{\widetilde{\phi}}}$. This reduces to proving the equality of closed substacks $\mathfrak{c}_{\widetilde{\phi}} = f^{-1}\mathfrak{c}_\phi$ in $\mathcal{O}b_{\widetilde{\phi}}$. Consider the distinguished triangle

$$f^*\mathbb{L}_{\mathfrak{Y}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{X}/\mathfrak{B}} \to \mathbb{L}_f \xrightarrow{+1}.$$

Using [Behrend and Fantechi 1997, Propositions 2.7 and 3.14] and smoothness of f, we obtain the vertical morphisms in the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{C}_{\mathfrak{X}/\mathfrak{B}} & \hookrightarrow & \mathfrak{N}_{\mathfrak{X}/\mathfrak{B}} & \longrightarrow & \mathfrak{n}_{\mathfrak{X}/\mathfrak{B}} & \xhookrightarrow{j_{\widetilde{\phi}}} & \mathcal{O}b_{\widetilde{\phi}} \\
\downarrow & & \downarrow & & \downarrow \wr & & \| \\
f^*\mathfrak{C}_{\mathfrak{Y}/\mathfrak{B}} & \hookrightarrow & f^*\mathfrak{N}_{\mathfrak{Y}/\mathfrak{B}} & \longrightarrow & f^*\mathfrak{n}_{\mathfrak{Y}/\mathfrak{B}} & \xhookrightarrow{f^*j_\phi} & f^*\mathcal{O}b_\phi
\end{array}$$

Here, the leftmost square is cartesian. By definition, $\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}}$ is the image of $\mathfrak{C}_{\mathfrak{Y}/\mathfrak{B}}$. For an affine test-scheme T, one has that $\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}}(T) \subset \mathfrak{n}_{\mathfrak{X}/\mathfrak{B}}(T)$ is the set of sections which admit a lift to $\mathfrak{N}_{\mathfrak{X}/\mathfrak{B}}$ that factors through $\mathfrak{C}_{\mathfrak{X}/\mathfrak{B}}$; see also [Chang and Li 2011, Lemma 2.3]. Using this, one checks immediately that $f^{-1}\mathfrak{c}_{\mathfrak{Y}/\mathfrak{B}} \subseteq f^*\mathfrak{N}_{\mathfrak{Y}/\mathfrak{B}}$ is the image of $f^*\mathfrak{C}_{\mathfrak{Y}/\mathfrak{B}}$ and that the latter agrees with $\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}}$ under the identification $\mathfrak{n}_{\mathfrak{X}/\mathfrak{B}} \simeq f^*\mathfrak{n}_{\mathfrak{Y}/\mathfrak{B}}$. We conclude that

$$\mathfrak{c}_{\tilde{\phi}} = j_{\tilde{\phi}}(\mathfrak{c}_{\mathfrak{X}/\mathfrak{B}}) = (f^* j_\phi)(f^{-1}\mathfrak{c}_{\mathfrak{Y}/\mathfrak{B}}) = f^{-1} j_\phi(\mathfrak{c}_{\mathfrak{Y}/\mathfrak{B}}) = f^{-1}\mathfrak{c}_\phi. \qquad \square$$

2.3 Symmetrized pullback of obstruction theories

2.3.1 Definition Let $f\colon \mathfrak{X} \to \mathfrak{M}$ be a smooth morphism of Artin stacks over $\mathfrak{B}$, where $\mathfrak{X}$ is a DM stack. Since f is smooth and of DM-type, $\mathbb{L}_f = \Omega_f$ is a locally free sheaf. A symmetric APOT $\phi_{\mathfrak{X}}$ on $\mathfrak{X}$ and a symmetric obstruction theory $\phi_{\mathfrak{M}}$ on $\mathfrak{M}$, both of weight κ, are *compatible under* f if there exists a T-equivariant étale cover $\{U_i \to \mathfrak{X}\}_{i\in I}$ such that $\phi_{\mathfrak{X}}|_{U_i}$ and $\phi_{\mathfrak{M}}$ are symmetric obstruction theories which are compatible under f for each i.

This definition should be compared to Definition 2.2.1 for the compatibility of two APOTs. We refer to any symmetric APOT $\phi_{\mathfrak{X}}$ satisfying these conditions as a *symmetrized pullback* of $\phi_{\mathfrak{M}}$ along f.

2.3.2 In [Liu 2023, Section 2.4], the following strategy for constructing a symmetrized pullback of obstruction theories (without using APOTs) was proposed. Let $\phi_{\mathfrak{M}/\mathfrak{B}}\colon \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ be a symmetric obstruction theory of weight κ. Recall from Definition 2.1.11 that, in order to construct a *smooth* pullback of $\phi_{\mathfrak{M}/\mathfrak{B}}$ along f, we would like to find a dashed map δ making the square commute in the following diagram of solid arrows:

(18)
$$\begin{array}{ccccccc}
 & & & \Omega_f^\vee[2]\otimes\kappa & & & \\
 & \overset{\delta^\vee[1]\circ\delta=0}{\nearrow} & & \uparrow{\scriptstyle \delta^\vee[1]} & & \overset{\xi}{\nwarrow} & \\
\Omega_f[-1] & \overset{\delta}{\dashrightarrow} & f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & \dashrightarrow & \operatorname{cone}(\delta) & \overset{+1}{\dashrightarrow} & \\
\| & & \downarrow{\scriptstyle f^*\phi_{\mathfrak{M}/\mathfrak{B}}} & & \downarrow & & \\
\Omega_f[-1] & \longrightarrow & f^*\mathbb{L}_{\mathfrak{M}/\mathfrak{B}} & \longrightarrow & \mathbb{L}_{\mathfrak{X}/\mathfrak{B}} & \overset{+1}{\longrightarrow} &
\end{array}$$

This induces $\operatorname{cone}(\delta)$ and all the other dashed maps, noncanonically. If in addition $\delta^\vee[1]\circ\delta = 0$, then the dotted map ξ exists, and the cocone of ξ would be a candidate for the *symmetrized* pullback of $\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ along f.

2.3.3 In general, without using APOTs, we neither have a lift δ, nor that $\delta^\vee[1]\circ\delta = 0$ if δ exists. However, observe that:

(i) The obstruction to the existence of the lift δ lies in

$$\operatorname{Hom}(\Omega_f[-1], f^*\operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}})) = \operatorname{Ext}^1(\Omega_f, f^*\operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}})).$$

(ii) If δ exists, then it is unique up to an element of

$$\mathrm{Hom}(\Omega_f[-1], f^*\operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}})[-1]) = \mathrm{Hom}(\Omega_f, f^*\operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}})).$$

(iii) For a given δ, the element $\delta^\vee[1]\circ\delta$ lies in

$$\mathrm{Hom}(\Omega_f[-1], \Omega_f^\vee[2]\otimes\kappa) = \mathrm{Ext}^3(\Omega_f, \Omega_f^\vee\otimes\kappa).$$

By the following lemma, if we only want to construct the symmetrized pullback on étale-local affine charts, then all these Ext groups will be zero.

Lemma *Let U be an affine scheme with T-action. Let V be a T-equivariant vector bundle on U and $E \in D^{\le a}_{\mathfrak{QCoh},\mathsf{T}}(U)$ for some $a\in\mathbb{Z}$. Then $\mathrm{Ext}^i(V,E)=0$ for $i>a$.*

Proof One can represent E by a complex of T-equivariant quasicoherent sheaves in degrees $\le a$. Then

$$\mathrm{Ext}^i(V,E) = h^i(R\Gamma(E\otimes V^\vee)),$$

and the latter vanishes for $i>a$ by [Stacks 2005–, Lemma 0G9R]. □

2.3.4 Theorem *In the setting of Definition 2.3.1, a symmetric obstruction theory on $\mathfrak{M}$ admits a symmetrized pullback to an APOT on $\mathfrak{X}$.*

Proof Let $\phi\colon \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ be the symmetric obstruction theory on $\mathfrak{M}$. Since ϕ is an obstruction theory, $\operatorname{cone}(\phi)$ is concentrated in degrees ≤ -2. Thus, Lemma 2.3.3 together with the observations 2.3.3(i) and 2.3.3(iii) respectively lets us conclude that the obstructions to finding δ and ξ vanish if $\mathfrak{X}$ is affine. In particular, on each affine U_i of an affine étale cover $\{U_i\to\mathfrak{X}\}_{i\in I}$, the obstructions vanish and we may choose δ_i and ξ_i filling in the diagram (18).

Let $\widehat{\mathbb{E}}_i := \operatorname{cocone}(\xi_i)$. By construction, this comes with a map $\widehat{\mathbb{E}}_i \to \operatorname{cone}(\delta_i)$ which is an isomorphism on h^1 and h^0 and surjective on h^{-1}. Composition with the chosen map $\operatorname{cone}(\delta_i)\to\mathbb{L}_{U_i/\mathfrak{B}}$ yields a map $\widehat{\phi}_i\colon\widehat{\mathbb{E}}_i\to\mathbb{L}_{U_i/\mathfrak{B}}$ which makes $\widehat{\mathbb{E}}_i$ into a T-equivariant POT on U_i over $\mathfrak{B}$. Next, we construct transition isomorphisms for these POTs.

2.3.5 By the definition of a symmetric APOT (of weight κ), we must construct transition data for the sheaves $h^1(\widehat{\mathbb{E}}_i^\vee)$ giving the global obstruction sheaf $h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}})\otimes\kappa^{-1}$, and construct for each i an isomorphism

$$\psi_i\colon h^1(\widehat{\mathbb{E}}_i^\vee)\to h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}})|_{U_i}\otimes\kappa^{-1}.$$

For a given i, we may choose a dashed arrow ζ_i giving a morphism of exact triangles (cf. the upper-right square in (16))

$$\begin{array}{ccccccc}
\operatorname{cocone}(\delta_i^\vee[1]) & \longrightarrow & f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}|_{U_i} & \xrightarrow{\delta_i^\vee[1]} & \Omega_f^\vee|_{U_i}[2]\otimes\kappa & \xrightarrow{+1} & \\
\Big\downarrow{\scriptstyle \zeta_i} & & \Big\downarrow & & \Big\| & & \\
\widehat{\mathbb{E}}_i & \longrightarrow & \operatorname{cone}(\delta_i) & \xrightarrow{\xi_i} & \Omega_f^\vee|_{U_i}[2]\otimes\kappa & \xrightarrow{+1} &
\end{array} \tag{19}$$

Here, we make a distinguished choice of the cocone of $\delta_i^\vee[1]$ by dualizing the dashed row of (18), namely $\operatorname{cocone}(\delta_i^\vee) := \operatorname{cone}(\delta_i)^\vee[2] \otimes \kappa$ with the induced maps.

Dualizing (19) and taking h^1, we get a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^1(\operatorname{cone}(\delta_i)^\vee) & \longrightarrow & h^1(\widehat{\mathbb{E}}_i^\vee) & \longrightarrow & \Omega_f|_{U_i} \otimes \kappa^{-1} & \longrightarrow & h^2(\operatorname{cone}(\delta_i)^\vee) \longrightarrow 0 \\ & & \downarrow & & \downarrow{\scriptstyle h^1(\zeta_i^\vee)} & & \| & & \downarrow \\ 0 & \longrightarrow & h^1(f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}^\vee|_{U_i}) & \longrightarrow & h^1(\operatorname{cocone}(\delta_i^\vee[1])^\vee) & \longrightarrow & \Omega_f|_{U_i} \otimes \kappa^{-1} & \longrightarrow & h^2(f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}^\vee|_{U_i}) \longrightarrow 0 \end{array}$$

Since $h^j(\operatorname{cone}(\delta_i)^\vee) \to h^j(f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}^\vee|_{U_i})$ are isomorphisms outside of $j = -1, 0$, it follows by the five lemma that $\zeta_i^\vee$ induces an isomorphism on h^1. Thus, we get isomorphisms

$$h^1(\widehat{\mathbb{E}}_i^\vee) \simeq h^1(\operatorname{cocone}(\delta_i^\vee[1])^\vee) \simeq h^0(\operatorname{cone}(\delta_i)) \otimes \kappa^{-1} \simeq h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}})|_{U_i} \otimes \kappa^{-1}.$$

We define $\psi_i\colon h^1(\widehat{\mathbb{E}}_i^\vee) \to h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}})|_{U_i} \otimes \kappa^{-1}$ to be this composition.

2.3.6 To conclude the construction of the symmetrized pullback APOT, we need to show that for any pair i, j, we can locally find étale neighborhoods $V_k \to U_i \times_{\mathfrak{X}} U_j$ and isomorphisms of obstruction theories $\eta_{ijk}\colon \widehat{\mathbb{E}}_i|_{V_k} \to \widehat{\mathbb{E}}_j|_{V_k}$ such that

$$h^1(\eta_{ijk}^\vee) = \psi_i^{-1}|_{V_k} \circ \psi_j|_{V_k}\colon h^1(\widehat{\mathbb{E}}_j|_{V_k}^\vee) \to h^1(\widehat{\mathbb{E}}_i|_{V_k}^\vee).$$

For any point in $U_i \times_{\mathfrak{X}} U_j$, choose some affine étale neighborhood V_k. By pulling everything back to V_k, we may, without loss of generality, assume that $U_i = U_j = V_k =: V$. Note that we have $\delta_i = \delta_j =: \delta$ since the difference $\delta_i - \delta_j$ lies in $\operatorname{Hom}_V(\Omega_f, f^* \operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}})) = 0$. This might a priori be problematic, since the choice of $\operatorname{cone}(\delta)$ and the map to $\mathbb{L}_{\mathfrak{X}/\mathfrak{B}}$ may be incompatible for different i and j. Lemma 2.3.7 shows that we may safely identify $\operatorname{cone}(\delta_i)$ and $\operatorname{cone}(\delta_j)$.

2.3.7 Lemma *If $\mathfrak{X}$ is affine, then the choice of* $\operatorname{cone}(\delta)$ *together with the morphism* $\operatorname{cone}(\delta) \to \mathbb{L}_{\mathfrak{X}/\mathfrak{B}}$ *in (18) is unique up to (not necessarily unique) isomorphism.*

Proof Since cones are unique up to isomorphism, we need to show the following: for any two choices $v_1, v_2\colon \operatorname{cone}(\delta) \to \mathbb{L}_{\mathfrak{X}/\mathfrak{B}}$ that give an morphism of exact triangles in (18), there is an automorphism $s\colon \operatorname{cone}(\delta) \to \operatorname{cone}(\delta)$ such that $v_2 = v_1 \circ s$ and such that

$$\begin{array}{ccccc} & & \operatorname{cone}(\delta) & & \\ & \overset{q}{\nearrow} & & \overset{r}{\searrow} & \\ f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & & \downarrow{\scriptstyle s} & & \Omega_f \\ & \underset{q}{\searrow} & & \underset{r}{\nearrow} & \\ & & \operatorname{cone}(\delta) & & \end{array}$$

is a commutative diagram, i.e., s is an isomorphism of cones.

Now consider the commutative diagram of solid arrows

$$(20)\qquad \begin{array}{ccccc} f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & \xrightarrow{q} & \operatorname{cone}(\delta) & \underset{r}{\longrightarrow} \overset{u_1}{\dashleftarrow} & \Omega_f \\ \big\downarrow f^*\phi & & v_1 \big\downarrow\big\downarrow v_2 & \overset{u_0}{\swarrow} & \big\| \\ f^*\mathbb{L}_{\mathfrak{M}/\mathfrak{B}} & \longrightarrow & \mathbb{L}_{\mathfrak{X}/\mathfrak{B}} & \xrightarrow{r'} & \Omega_f \end{array}$$

Let $u_v := v_2 - v_1$. Since both squares commute, $u_v \circ q = 0$ and $r' \circ u_v = 0$. The first vanishing implies there exists some u_0 such that $u_0 \circ r = u_v$. Since each v_i is itself an obstruction theory, we have $h^{\geq -1}(\operatorname{cone} v_1) = 0$, and therefore $\operatorname{Hom}(\Omega_f, \operatorname{cone} v_1) = 0$. Hence, postcomposition with v_1 induces an isomorphism

$$v_1\circ\colon \operatorname{Hom}(\Omega_f, \operatorname{cone}(\delta)) \xrightarrow{\sim} \operatorname{Hom}(\Omega_f, \mathbb{L}_{\mathfrak{X}/\mathfrak{B}}).$$

This implies that $u_0 = v_1 \circ u_1$ for a unique u_1. Let $u_2 := u_1 \circ r$. We claim that $s := \mathrm{id} + u_2$ has the desired properties. By construction, $v_1 \circ s = v_1 + u_v = v_2$. Moreover, $s \circ q = q + u_1 \circ r \circ q = q$. Finally, $r \circ s = r + r \circ u_2 = r$ because $r \circ u_2 = r' \circ v_1 \circ u_1 \circ r = r' \circ u_0 \circ r = r' \circ u_v = 0$ as observed earlier. □

2.3.8 After using Lemma 2.3.7 to identify the choice of $\operatorname{cone}(\delta)$ for i and j, we may assume that all dashed arrows in (18) are chosen identically for i and j. Now we must run the "dualized" version of the same argument again for $\widehat{\mathbb{E}} = \operatorname{cocone}(\xi)$. Namely, the lift ξ in (18) is unique up to an element of $\operatorname{Ext}^2(\Omega_f, \Omega_f^\vee \otimes \kappa) = 0$, just like in Section 2.3.3(ii), so we may also assume that $\xi_i = \xi_j =: \xi$. Again, this might a priori be problematic, since the choice of $\operatorname{cocone}(\xi)$ and the map ζ in (19) may be incompatible for different i and j. For instance, the isomorphisms ψ_i depend on the choice of ζ_i. The following Lemma 2.3.9 shows that we may safely identify $\operatorname{cocone}(\xi_i)$ and $\operatorname{cocone}(\xi_j)$.

2.3.9 Lemma *There exists an isomorphism $s\colon \widehat{\mathbb{E}}_i \to \widehat{\mathbb{E}}_j$ that is compatible with the composition to* $\operatorname{cone}(\delta)$ *and such that $s \circ \zeta_i = \zeta_j$.*

Proof This is essentially the dual of the argument in Lemma 2.3.7. Choose an isomorphism of $\widehat{\mathbb{E}} := \widehat{\mathbb{E}}_i$ with $\widehat{\mathbb{E}}_j$. By rotating (19), consider the commutative diagram of solid arrows

$$\begin{array}{ccccccc} \Omega_f^\vee|_V[1] \otimes \kappa & \xrightarrow{h'} & \operatorname{cocone}(\delta^\vee[1]) & \longrightarrow & f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}|_V & \xrightarrow{\delta^\vee[1]} & \\ \big\| & \overset{u_0}{\swarrow} & \zeta_i \big\downarrow\big\downarrow \zeta_j & & \big\downarrow & & \\ \Omega_f^\vee|_V[1] \otimes \kappa & \underset{u_1}{\overset{h}{\rightleftarrows}} & \widehat{\mathbb{E}} & \xrightarrow{g} & \operatorname{cone}(\delta) & \xrightarrow{+1} & \end{array}$$

Cf. (20). Let $u_\zeta := \zeta_j - \zeta_i$. Since $g \circ u_\zeta = 0$, there exists some u_0 such that $h \circ u_0 = u_\zeta$. On the other hand, $\operatorname{cocone}(\zeta_1) \simeq \Omega_f|_V[-1]$, so precomposition with ζ_1 induces an isomorphism

$$\circ\zeta_1\colon \operatorname{Hom}(\widehat{\mathbb{E}}_i, \Omega_f^\vee|_V[1]) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{cocone}(\delta^\vee[1]), \Omega_f^\vee|_V[1]).$$

This implies that $u_0 = u_1 \circ \zeta_1$ for a unique u_1. Let $u_2 := h \circ u_1$. The desired isomorphism is $s := \mathrm{id} + u_2$. By construction, $g \circ s = g + g \circ h \circ u_1 = g$. Moreover, $u_2 \circ \zeta_1 = h \circ u_1 = u_\xi$ so that $s \circ \zeta_1 = \zeta_1 + u_\xi = \zeta_2$. □

2.3.10 Finally, we show that each $\widehat{\mathbb{E}}_i$ is a *symmetric* POT. For this, we may assume that $\mathfrak{X} = U_i$ is affine and that we have an obstruction theory $\widehat{\mathbb{E}}$ obtained following the strategy of filling in (18) and taking the cocone of ξ. By what we have argued so far, this strategy works and the resulting $\widehat{\mathbb{E}}$ is unique up to isomorphism. In particular, after choosing δ, $\mathrm{cone}(\delta)$ and ξ, we may construct $\widehat{\mathbb{E}}$ in the following way: as a consequence of the octahedral axiom, we can expand only the upper-left corner of the following diagram to obtain the full commutative 3×3 diagram whose rows and columns are distinguished triangles

$$
\begin{array}{ccccccc}
\Omega_f[-1] & \xrightarrow{\xi^\vee} & \mathrm{cone}(\delta)^\vee[-1] & \xrightarrow{h} & \widehat{\mathbb{E}} & \xrightarrow{+1} & \\
\Vert & & \downarrow & & \downarrow{\scriptstyle g} & & \\
\Omega_f[-1] & \xrightarrow{\delta} & f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} & \longrightarrow & \mathrm{cone}(\delta) & \xrightarrow{+1} & \\
\downarrow & & \downarrow{\scriptstyle \delta^\vee} & & \downarrow{\scriptstyle \xi} & & \\
0 & \longrightarrow & \Omega_f^\vee[2] \otimes \kappa & = & \Omega_f^\vee[2] \otimes \kappa & \xrightarrow{+1} & \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & &
\end{array}
\tag{21}
$$

Here, we identified the middle-right and lower-middle entry by uniqueness of cones (up to isomorphism), which also identifies the lower-right entry as $\Omega_f^\vee[2] \otimes \kappa$. We also recover ξ as the *unique* lift of $\delta^\vee$ to a map from $\mathrm{cone}(\delta)$. We conclude that the upper-right corner of the diagram is indeed (isomorphic to) $\widehat{\mathbb{E}}$.

Applying the functor $- \mapsto (-)^\vee[1] \otimes \kappa$ and mirroring the diagram along the antidiagonal transforms the diagram into itself, except possibly for the upper-right corner.

By the same argument as in Lemma 2.3.7, we have:

Lemma *Consider only the upper two rows in* (21), *and suppose everything except for* $\widehat{\mathbb{E}}$ *is already fixed. The choice of* $\widehat{\mathbb{E}}$ *together with morphisms making the upper two rows into a morphism of exact triangles is unique up to isomorphism.*

In our situation, this shows that there is a unique isomorphism $\Theta : \widehat{\mathbb{E}} \to \widehat{\mathbb{E}}^\vee[1] \otimes \kappa$ which identifies $g^\vee$ with h and $h^\vee$ with g, and with respect to which the whole diagram (21) is invariant under $(-)^\vee[1] \otimes \kappa$. Indeed, the remaining maps involving $\widehat{\mathbb{E}}$ are induced from g and h by composition.

Furthermore, we see that $\Theta^\vee[1] \otimes \kappa$ satisfies the same defining property, and so does $\Theta^s := \Theta + \Theta^\vee[1] \otimes \kappa$. Moreover, the latter is again an isomorphism, since g and $h^\vee$ are isomorphisms in degrees ≥ 0, while $g^\vee$ and h are isomorphisms in degree ≤ -1. This concludes the proof of Theorem 2.3.4. □

2.3.11 Theorem 2.3.4 states that we can pull back a symmetric obstruction theory along a smooth morphism from a DM stack. If the base is already DM, then this also works for APOTs. Let $g \colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of DM stacks over the base $\mathfrak{B}$. We assume that everything is T-equivariant.

Proposition *A symmetric APOT on $\mathfrak{Y}$ admits a symmetric pullback to an APOT on $\mathfrak{X}$ such that*

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} = g^*\mathcal{O}_{\mathfrak{Y}}^{\mathrm{vir}} \otimes \mathsf{e}(\Omega_{\mathfrak{X}/\mathfrak{Y}} \otimes \kappa^{-1}).$$

Proof We construct the APOT $\widehat{\phi}$ on $\mathfrak{X}$ as follows: take $\mathcal{O}b_{\widehat{\phi}} := \Omega_{\mathfrak{X}/\mathfrak{B}} \otimes \kappa^{-1}$, and define local charts by applying Theorem 2.3.4 locally on each étale chart on $\mathfrak{Y}$. Then it remains to glue everything together, using the same strategy as for Proposition 2.2.2.

Explicitly, given a chart $U_i \to \mathfrak{Y}$ with symmetric POT $\phi_i\colon \mathbb{E}_i \to \mathbb{L}_{U_i/\mathfrak{B}}$, we can find charts $V_j \to \mathfrak{X}$ that factor through $U_i \times_{\mathfrak{Y}} \mathfrak{X}$ on which $\mathbb{L}_{V_j/\mathfrak{B}} \simeq \Omega_{V_j/U_i} \oplus g^*\mathbb{L}_{U_i/\mathfrak{B}}$. Then set

$$\widehat{\mathbb{E}}_j := \Omega_{V_j/U_i} \oplus g^*\mathbb{E}_i \oplus \Omega_{V_j/U_i}^\vee \otimes \kappa[1],$$

and take $\widehat{\phi}_j := \mathrm{id} \oplus g^*\phi \oplus 0$. Given two charts V_{j_1} and V_{j_2} over U_i, let W_k be an affine étale chart factoring through $V_{j_1} \times_{\mathfrak{X}} V_{j_2}$. Let $\eta\colon \Omega_{\mathfrak{X}/\mathfrak{Y}}|_{W_k} \to h^0(g^*\mathbb{E}_i)$ be the morphism defined as the composition

$$\Omega_{\mathfrak{X}/\mathfrak{Y}}|_{V_{j_1}} \to h^0(\widehat{\mathbb{E}}_{j_1}) \simeq h^0(\mathbb{L}_{\mathfrak{X}/\mathfrak{B}}) \simeq h^0(\widehat{\mathbb{E}}_{j_2}) \to h^0(g^*\mathbb{E}_i),$$

where the first and last morphism are the natural inclusion and projection coming from the definition of $\widehat{\mathbb{E}}_{j_1}$ and $\widehat{\mathbb{E}}_{j_2}$, respectively. Then the transition map between $\widehat{\mathbb{E}}_{j_1}|_{W_k}$ and $\widehat{\mathbb{E}}_{j_2}|_{W_k}$ is given by the matrix

$$\begin{bmatrix} \mathrm{id}_{\Omega_{W_k/U_i}} & 0 & 0 \\ \eta & \mathrm{id}_{g^*\mathbb{E}_i} & 0 \\ 0 & \eta^\vee[1] \otimes \kappa & \mathrm{id}_{\Omega_{W_k/U_i}^\vee[1]\otimes\kappa} \end{bmatrix}.$$

2.3.12 We prove the claimed identity for virtual structure sheaves. By the exact sequence of Kähler differentials, we have an exact sequence of coherent sheaves on $\mathfrak{X}$

$$0 \to g^*\mathcal{O}b_\phi \xrightarrow{\iota} \mathcal{O}b_{\widehat{\phi}} \to \Omega_{\mathfrak{X}/\mathfrak{Y}} \otimes \kappa^{-1} \to 0.$$

We claim that the induced pushforward $\iota_*\colon K_{\mathsf{T}}(g^*\mathcal{O}b_\phi) \to K_{\mathsf{T}}(\mathcal{O}b_{\widehat{\phi}})$ [Kiem and Savvas 2020, (3.16)] sends $g^*\mathcal{O}_{\mathfrak{c}_\phi}$ to $\mathcal{O}_{\mathfrak{c}_{\widehat{\phi}}}$. Assuming this is true, we have that

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} = 0^!_{\mathcal{O}b_{\widehat{\phi}}}\mathcal{O}_{\mathfrak{c}_\phi} = 0^!_{\mathcal{O}b_\phi}\iota^*\iota_*\mathcal{O}_{\mathfrak{c}_\phi},$$

by Lemmas 3.8 and 3.7 in [loc. cit.], respectively. Moreover, from [loc. cit., (3.10)] one can see that $\iota^*\iota_*$ equals multiplication with $\mathsf{e}(\Omega_{\mathfrak{X}/\mathfrak{Y}} \otimes \kappa^{-1})$ as desired.

Now to prove our claim, it is enough to show that the closed substack $g^{-1}\mathfrak{c}_\phi \subseteq g^*\mathcal{O}b_\phi$ is identified with $\mathfrak{c}_{\widehat{\phi}} \subseteq \mathcal{O}b_{\widehat{\phi}}$ when viewed as a closed substack of $\mathcal{O}b_{\widehat{\phi}}$ via ι (which, as a map of sheaf stacks, is a closed immersion). This can be checked locally. In particular, by the explicit local description of $\widehat{\phi}$ in Section 2.3.11, we may assume that the obstruction theory on $\mathfrak{X}$ is obtained by first taking the smooth pullback of ϕ and then modifying it by adding $\Omega_{\mathfrak{X}/\mathfrak{Y}} \otimes \kappa^{-1}$ to the obstruction sheaf. The result follows. □

2.3.13 Suppose that, in the setting of Definition 2.3.1, we have a diagram

$$\mathfrak{X} \xrightarrow{g} \mathfrak{Y} \xrightarrow{f} \mathfrak{M}$$

over $\mathfrak{B}$, where g is a smooth morphism of DM stacks. Then we have the APOTs $\phi_{\mathfrak{X}}$ on $\mathfrak{X}$ and $\phi_{\mathfrak{Y}}$ on $\mathfrak{Y}$ obtained from symmetrized pullback along $f \circ g$ and f, respectively. The following functoriality statement is very useful and follows from the same arguments as in the construction of symmetrized pullback.

Lemma *The symmetrized pullback of $\phi_{\mathfrak{Y}}$ along g, as in Proposition 2.3.11, agrees with the symmetrized pullback $\phi_{\mathfrak{X}}$ of the symmetric obstruction theory on $\mathfrak{M}$ along $f \circ g$.*

2.4 Virtual torus localization

2.4.1 Let $\mathfrak{X}$ be a DM stack with T-action, and let $\iota\colon \mathfrak{X}^{\mathsf{T}} \hookrightarrow \mathfrak{X}$ be the T-fixed locus. Suppose $\mathfrak{X}$ has an (A)POT, and let $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}}$ be the induced virtual structure sheaf. Recall that virtual T-localization requires the following two ingredients.

First, $\iota_*\colon K_{\mathsf{T}}(\mathfrak{X}^{\mathsf{T}}) \to K_{\mathsf{T}}(\mathfrak{X})$ must become an isomorphism of $K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$-modules after base change to $K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$. This holds if $\mathfrak{X}$ is DM of finite type with a T-equivariant étale atlas [Kiem and Savvas 2020, Proposition 5.12]. By [Alper et al. 2020, Theorem 4.3], such an atlas will always exist after possibly passing to a multiple cover of T.

Second, for locally free sheaves $\mathcal{E}$ on $\mathfrak{X}^{\mathsf{T}}$ of nontrivial T-weight, the *K-theoretic Euler class*

$$\mathsf{e}(\mathcal{E}) := \sum_i (-1)^i \wedge^i (\mathcal{E}^\vee) \in K_{\mathsf{T}}(\mathfrak{X}^{\mathsf{T}})$$

must be invertible in $K_{\mathsf{T}}(\mathfrak{X}^{\mathsf{T}})_{\mathrm{loc}}$, which is automatic for DM stacks. For two such locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$, we write $\mathsf{e}(\mathcal{E}_1 - \mathcal{E}_2) := \mathsf{e}(\mathcal{E}_1)/\mathsf{e}(\mathcal{E}_2)$. Generally we will only apply e to (some notion of) the virtual normal bundle of ι.

2.4.2 Kiem and Savvas prove a torus localization formula for APOTs on a DM stack $\mathfrak{X}$ under the assumption that the virtual normal bundle $\mathcal{N}_\iota^{\mathrm{vir}}$ exists globally on $\mathfrak{X}^{\mathsf{T}}$ [Kiem and Savvas 2020, Assumption 5.3]. We were not able to confirm this assumption in our setting, and it likely does not hold in general for the symmetrized pullbacks constructed in Section 2.3. However, if a virtual normal bundle existed in the situation of Definition 2.3.1, it would have K-theory class

$$\mathcal{N}^{\mathrm{vir}} := (f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}^\vee + \Omega_f^\vee - \Omega_f \otimes \kappa^{-1})|_{\mathfrak{X}^{\mathsf{T}}}^{\mathrm{mov}}, \tag{22}$$

where the superscript mov denotes the T-moving part. In this situation, we can show that the localization formula still holds using $\mathcal{N}^{\mathrm{vir}}$, at least if $\mathfrak{X}$ has the resolution property [Totaro 2004].

2.4.3 Theorem *In the situation of Definition 2.3.1, suppose that $\mathfrak{X}$ has a symmetric APOT $\phi_{\mathfrak{X}/\mathfrak{B}}$ given by symmetrized pullback along f of a symmetric obstruction theory on $\mathfrak{M}$. If $\mathfrak{X}$ has the resolution property, then*

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} = \iota_* \frac{\mathcal{O}_{\mathfrak{X}^{\mathsf{T}}}^{\mathrm{vir}}}{\mathsf{e}(\mathcal{N}^{\mathrm{vir}})} \in K_{\mathsf{T}}(\mathfrak{X})_{\mathrm{loc}}, \tag{23}$$

where $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}}$ is the virtual structure sheaf for $\phi_{\mathfrak{X}/\mathfrak{B}}$, and $\mathcal{O}_{\mathfrak{X}^{\mathsf{T}}}^{\mathrm{vir}}$ is the virtual structure sheaf for the T-fixed part of $\phi_{\mathfrak{X}/\mathfrak{B}}|_{\mathfrak{X}^{\mathsf{T}}}$.

In this setting, let $\mathcal{K}_{\mathrm{vir}} := \det(f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} + \Omega_f - \Omega_f^\vee \otimes \kappa)$ denote the (would-be) virtual canonical bundle on $\mathfrak{X}$. If the virtual canonical $\det \mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ for $\mathfrak{M}$ admits a square root, possibly up to passing to a multiple cover of T, then the Nekrasov–Okounkov *symmetrized virtual structure sheaf* is

$$\widehat{\mathcal{O}}_{\mathfrak{X}}^{\mathrm{vir}} := \mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} \otimes (\det \mathbb{E}_{\mathfrak{X}/\mathfrak{B}})^{1/2},$$

and the *symmetrized Euler class* is $\hat{\mathsf{e}}(-) := \mathsf{e}(-) \otimes \det(-)^{1/2}$. Then Theorem 2.4.3 immediately implies

$$\widehat{\mathcal{O}}_{\mathfrak{X}}^{\mathrm{vir}} = \iota_* \frac{\widehat{\mathcal{O}}_{\mathfrak{X}^{\mathsf{T}}}^{\mathrm{vir}}}{\hat{\mathsf{e}}(\mathcal{N}^{\mathrm{vir}})}.$$

2.4.4 Remark In [Kiem and Savvas 2020], the localization formula is only stated for the action of a rank-one torus. One deduces the general case by the following standard strategy: The argument in [loc. cit.] goes through when considering another torus T_1 acting compatibly with the $\mathbb{C}^*$-action. For general T, one can always choose a generic rank-one subtorus T_0 such that $\mathfrak{X}^T = \mathfrak{X}^{\mathsf{T}_0}$, and such that T splits as $\mathsf{T}_0 \times \mathsf{T}_1$. The localization formula for T then follows from the one for T_0 while remembering the T_1-equivariance.

2.4.5 Proof of Theorem 2.4.3 Let $\phi_{\mathfrak{M}/\mathfrak{B}} \colon \mathbb{E}_{\mathfrak{M}/\mathfrak{B}} \to \mathbb{L}_{\mathfrak{M}/\mathfrak{B}}$ be the original symmetric obstruction theory on $\mathfrak{M}$ over $\mathfrak{B}$. We divide the proof of this theorem into three steps. Throughout, we work T-equivariantly.

(i) Construct an affine bundle $a \colon \mathfrak{A} \to \mathfrak{X}$ so that, on $\mathfrak{A}$, there is a lift $\delta \colon a^*\Omega_f[-1] \to a^* f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ for which the composition $\delta^\vee[1] \circ \delta \colon a^*\Omega_f[-1] \to a^* f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}} \to a^*\Omega_f^\vee[2] \otimes \kappa$ vanishes, as in (18).

(ii) Show that the T-localization formula on $\mathfrak{A}$ holds for the APOT on $\mathfrak{A}$ obtained from smooth (non-symmetric) pullback along a of the APOT on $\mathfrak{X}$.

(iii) Use the Thom isomorphism $a^* \colon K_{\mathsf{T}}(\mathfrak{X}) \xrightarrow{\sim} K_{\mathsf{T}}(\mathfrak{A})$ [Chriss and Ginzburg 1997, Theorem 5.4.17] to obtain the desired localization formula on $\mathfrak{X}$ from the one on $\mathfrak{A}$.

2.4.6 Step 1

Lemma *Let $\mathfrak{X}$ be a stack with the resolution property. Let $\mathcal{E} \in D^{\le k_0}\mathcal{C}oh(\mathfrak{X})$, let $\mathcal{V}$ be a vector bundle on $\mathfrak{X}$ and let $e \in \operatorname{Ext}^k(\mathcal{V}, \mathcal{E})$ for $k > k_0$. Then there exists a surjection of vector bundles $p \colon \mathcal{V}_0 \twoheadrightarrow \mathcal{V}$ with the following property: if $a \colon \mathfrak{A} \to \mathfrak{X}$ denotes the fiber over* id *in the induced map $\mathcal{H}om(\mathcal{V}, \mathcal{V}_0) \to \mathcal{H}om(\mathcal{V}, \mathcal{V})$, then $a^*e = 0$.*

Therefore a is a $\mathcal{H}om(\mathcal{V}, \ker p)$-torsor over $\mathfrak{X}$. As $\ker p$ is a vector bundle, this is an affine bundle.

Proof Up to a shift, we may assume that $k_0 = 0$ and $k = 1$. Interpret e as a morphism $e \colon \mathcal{V} \to \mathcal{E}[1]$ and let $\mathcal{F} := \operatorname{cocone}(e)$. Then $e = 0$ if and only if the natural map $f \colon \mathcal{F} \to \mathcal{V}$ splits in the derived category. By assumption on $\mathcal{E}$ and the long exact sequence of cohomology groups, we have $\mathcal{F} \in D^{\le k_0}\mathcal{C}oh(\mathfrak{X})$, and $h^0(f) \colon h^0(\mathcal{F}) \to \mathcal{V}$ is surjective. By the resolution property, $\mathcal{F}$ can be represented by a complex of vector bundles $\cdots \to \mathcal{V}_1 \to \mathcal{V}_0$ in nonpositive degree, so that f is induced by a surjective map $p \colon \mathcal{V}_0 \twoheadrightarrow \mathcal{V}$. By construction, on the affine bundle $\mathfrak{A}$ there is a universal splitting of this surjection, which induces a splitting of $\mathcal{F} \to \mathcal{V}$. □

2.4.7 Apply Lemma 2.4.6 on $\mathfrak{X}$, to the class in $\mathrm{Ext}^1(\Omega_f, f^*\operatorname{cone}(\phi_{\mathfrak{M}/\mathfrak{B}}))$ that obstructs the existence of a lift δ. This yields an affine bundle $a'\colon \mathfrak{A}' \to \mathfrak{X}$ on which the desired lift $\delta\colon (a')^*\Omega_f[-1] \to (a')^* f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ exists.

Apply Lemma 2.4.6 again on $\mathfrak{A}'$ to the class $\delta^\vee \circ \delta \in \mathrm{Ext}^3((a')^*\Omega_f, (a')^*\Omega_f^\vee \otimes \kappa)$. This yields an affine bundle $\mathfrak{A} \to \mathfrak{A}'$ on which the pullback of this class vanishes. Let $a\colon \mathfrak{A} \to \mathfrak{A}' \to \mathfrak{X}$ denote the induced projection, and use δ again to denote the pullback of δ to $\mathfrak{A}$.

2.4.8 Step 2 Recall the construction of the APOT on $\mathfrak{X}$ from symmetrized pullback: on affine étale charts U_i, fill in the morphisms $\delta_i\colon \Omega_f|_{U_i}[-1] \to f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ and $\xi_i\colon \operatorname{cone}(\delta_i)|_{U_i} \to \Omega_f^\vee|_{U_i}[2] \otimes \kappa$ of (18), and take $\widehat{\mathbb{E}}_i := \operatorname{cocone}(\xi_i)$. On the other hand, on $\mathfrak{A}$, the lift $\delta\colon a^*\Omega_f|_{U_i}[-1] \to a^* f^*\mathbb{E}_{\mathfrak{M}/\mathfrak{B}}$ is globally defined, and the condition $\delta^\vee[1]\circ\delta = 0$ is satisfied globally and induces a global $\xi\colon \operatorname{cone}(\delta) \to a^*\Omega_f^\vee[2]\otimes\kappa$. Let $\widehat{\mathbb{E}}_{\mathfrak{A}} := \operatorname{cocone}(\xi)$.

By the same arguments as in Section 2.3.8, there are local isomorphisms of ξ with the pullbacks of the ξ_i which induce isomorphisms of cocones. In particular, the natural map $\widehat{\mathbb{E}}_{\mathfrak{A}} \to a^*\mathbb{L}_{\mathfrak{X}}$ is an isomorphism on h^0 and surjective on h^{-1}.

Now consider the diagram of solid arrows in $D^-_{\mathcal{QC}oh,\mathsf{T}}(\mathfrak{A})$

$$\begin{array}{ccccc} \Omega_a[-1] & \dashrightarrow & \widehat{\mathbb{E}}_{\mathfrak{A}} & & \\ \| & & \downarrow & & \\ \Omega_a[-1] & \longrightarrow & a^*\mathbb{L}_{\mathfrak{X}/\mathfrak{B}} & \longrightarrow & \mathbb{L}_{\mathfrak{A}/\mathfrak{B}} \end{array}$$

If we could find a dashed arrow that makes the diagram commute, then the cone would give a perfect obstruction theory on $\mathfrak{A}$. Since $a^*\mathbb{L}_{\mathfrak{X}/\mathfrak{B}}$ is in cohomological degree ≤ 0 and $\Omega_a[-1]$ is a vector bundle in degree 1, affine étale-locally the connecting map $\Omega_a[-1] \to a^*\mathbb{L}_{\mathfrak{X}}$ is zero by Lemma 2.3.3. Thus, just like in Section 2.2, we may choose POTs of the form $\Omega_a \oplus \widehat{\mathbb{E}}_{\mathfrak{A}} \to \mathbb{L}_{\mathfrak{A}/\mathfrak{B}}$ étale-locally on $\mathfrak{A}$, which fit together to form an APOT $(U_j, \widehat{\mathbb{F}}_j \to \mathbb{L}_{U_j/\mathfrak{B}})_j$ on $\mathfrak{A}$. More precisely, on each U_j, we choose a section of $\mathbb{L}_{\mathfrak{A}/\mathfrak{B}} \to \Omega_a$. It is straightforward to check that the APOT we obtain in this way agrees with the smooth pullback of the APOT on $\mathfrak{X}$ to $\mathfrak{A}$ using Proposition 2.2.2.

With this APOT on $\mathfrak{A}$, we claim that the T-fixed locus $\iota_{\mathfrak{A}}\colon \mathfrak{A}^{\mathsf{T}} \hookrightarrow \mathfrak{A}$ possesses a global virtual normal bundle in the sense of [Kiem and Savvas 2020, Assumption 5.4]. In fact, the virtual normal bundle is

$$\mathcal{N}^{\mathrm{vir}}_{\iota_{\mathfrak{A}}} := (\mathcal{T}_{\mathfrak{A}/\mathfrak{X}} \oplus \widehat{\mathbb{E}}_{\mathfrak{A}}^\vee)|^{\mathrm{mov}}_{\mathfrak{A}^{\mathsf{T}}}.$$

Namely, restricted to each U_i, this is isomorphic with the dual of the moving part of $\widehat{\mathbb{F}}_i$, and moreover, on a common refinement V of U_i and U_j, the isomorphism $\widehat{\mathbb{F}}_i|_V \to \widehat{\mathbb{F}}_j|_V$ defining the obstruction theory is the identity on the $\widehat{\mathbb{E}}_{\mathfrak{A}}^\vee$ summand, and therefore respects the obstruction sheaf.

It follows that the localization formula of [loc. cit., Theorem 5.15] is applicable to the APOT on $\mathfrak{A}$.

2.4.9 Step 3 The localization formula [loc. cit., Theorem 5.15] on $\mathfrak{A}$ gives

$$\mathcal{O}^{\mathrm{vir}}_{\mathfrak{A}} = (\iota_{\mathfrak{A}})_* \frac{\mathcal{O}^{\mathrm{vir}}_{\mathfrak{A}^{\mathsf{T}}}}{\mathsf{e}(\mathcal{N}^{\mathrm{vir}}_{\iota_{\mathfrak{A}}})} = (\iota_{\mathfrak{A}})_* \frac{\mathcal{O}^{\mathrm{vir}}_{\mathfrak{A}^{\mathsf{T}}}}{\mathsf{e}(\mathcal{T}_{\mathfrak{A}/\mathfrak{X}}|^{\mathrm{mov}}_{\mathfrak{A}^{\mathsf{T}}})\,\mathsf{e}(\widehat{\mathbb{E}}_{\mathfrak{A}}^\vee|^{\mathrm{mov}}_{\mathfrak{A}^{\mathsf{T}}})}. \tag{24}$$

Let $\mathfrak{F}_{\mathfrak{A}} := a^{-1}(\mathfrak{X}^{\mathsf{T}})$ and factor $\iota_{\mathfrak{A}} = \iota_{\mathfrak{F}} \circ \iota'_{\mathfrak{A}}$, where $\iota'_{\mathfrak{A}}\colon \mathfrak{A}^{\mathsf{T}} \hookrightarrow \mathfrak{F}_{\mathfrak{A}}$ and $\iota_{\mathfrak{F}}\colon \mathfrak{F}_{\mathfrak{A}} \hookrightarrow \mathfrak{A}$ are the natural closed immersions. Note that $\mathsf{e}(\widehat{\mathbb{E}}_{\mathfrak{A}}^{\vee}|_{\mathfrak{F}_{\mathfrak{A}}}^{\mathrm{mov}}) = a^*\mathsf{e}(\mathcal{N}^{\mathrm{vir}})$. Using this, and the projection formula, we rewrite (24) as

$$\mathcal{O}_{\mathfrak{A}}^{\mathrm{vir}} = (\iota_{\mathfrak{F}})_* \left(\left((\iota'_{\mathfrak{A}})_* \frac{\mathcal{O}_{\mathfrak{A}^{\mathsf{T}}}^{\mathrm{vir}}}{\mathsf{e}(\mathcal{T}_{\mathfrak{A}/\mathfrak{X}}|_{\mathfrak{A}^{\mathsf{T}}}^{\mathrm{mov}})} \right) \frac{1}{a^*\mathsf{e}(\mathcal{N}^{\mathrm{vir}})} \right).$$

On the other hand, one can check that $\iota'_{\mathfrak{A}}$ is a regular closed immersion with normal bundle $\mathcal{T}_{\mathfrak{A}/\mathfrak{X}}|_{\mathfrak{A}^{\mathsf{T}}}^{\mathrm{mov}}$. Applying the torus localization formula on $\mathfrak{F}_{\mathfrak{A}}$ gives

$$\mathcal{O}_{\mathfrak{F}_{\mathfrak{A}}}^{\mathrm{vir}} = (\iota'_{\mathfrak{A}})_* \frac{\mathcal{O}_{\mathfrak{A}^{\mathsf{T}}}^{\mathrm{vir}}}{\mathsf{e}(\mathcal{T}_{\mathfrak{A}/\mathfrak{X}}|_{\mathfrak{A}^{\mathsf{T}}}^{\mathrm{mov}})}.$$

Combining the last two equalities, and using that the APOTs on $\mathfrak{A}$ and $\mathfrak{F}_{\mathfrak{A}}$ are the smooth pullbacks of the APOTs on $\mathfrak{X}$ and $\mathfrak{X}^{\mathsf{T}}$ respectively, we conclude that

$$a^*\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}} = \mathcal{O}_{\mathfrak{A}}^{\mathrm{vir}} = (\iota_{\mathfrak{F}})_* \frac{\mathcal{O}_{\mathfrak{F}_{\mathfrak{A}}}^{\mathrm{vir}}}{a^*\mathsf{e}(\mathcal{N}^{\mathrm{vir}})} = a^*\iota_* \frac{\mathcal{O}_{\mathfrak{X}^{\mathsf{T}}}^{\mathrm{vir}}}{\mathsf{e}(\mathcal{N}^{\mathrm{vir}})}.$$

Since a^* is an isomorphism on K-theory, this proves (23). □

2.5 Comparison of virtual cycles

2.5.1 By [Thomas 2022], the virtual structure sheaf associated to a POT depends only on the *K-theory class* of the POT. In other words, if a space Z has two possibly different POTs, it is sufficient to check that their K-theory classes are equal in order to conclude that they induce the same $\widehat{\mathcal{O}}^{\mathrm{vir}}$. The same should be true for APOTs and their associated virtual structure sheaves (from Theorem 2.1.5).

In particular, in wall-crossing applications, we will have a master space $\mathbb{M}$ with symmetric APOT from symmetrized pullback, and we will need to identify the induced APOT on various torus-fixed loci $Z \subset \mathbb{M}$ (which arise in the virtual localization of Theorem 2.4.3) with symmetric APOTs constructed by symmetrized pullback along maps to the actual moduli spaces of interest.

2.5.2 For the following result, we work over a trivial base $\mathfrak{B} = \mathrm{pt}$.

Proposition *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Artin stacks with T-actions and with T-equivariant symmetric obstruction theories $\phi_{\mathfrak{M}}\colon \mathbb{E}_{\mathfrak{M}} \to \mathbb{L}_{\mathfrak{M}}$ and $\phi_{\mathfrak{N}}\colon \mathbb{E}_{\mathfrak{N}} \to \mathbb{L}_{\mathfrak{N}}$. Consider the T-equivariant setup*

$$\begin{array}{ccc} Z & \hookrightarrow & \mathbb{M} \circlearrowleft \mathbb{C}^* \\ \downarrow f & & \downarrow g \\ \mathfrak{N} & & \mathfrak{M} \end{array}$$

where f and g are smooth morphisms, $\mathbb{M}$ is an algebraic space with $\mathbb{C}^{\times}$-action compatible with the T-action, and $Z \subset \mathbb{M}$ is a $\mathbb{C}^{\times}$-fixed component. Then Z has two T-equivariant symmetric APOTs (*by* Theorem 2.3.4):

(i) *the one obtained by symmetrized pullback along f;*

(ii) *the $\mathbb{C}^\times$-fixed part of the restriction of the symmetric APOT on $\mathbb{M}$, obtained by symmetrized pullback along g.*

Assume $\mathbb{M}$ has the $(\mathsf{T}\times\mathbb{C}^\times)$-equivariant resolution property [Totaro 2004]*. If we have an identity of K-theory classes*

$$f^*\mathbb{E}_{\mathfrak{N}} + \Omega_f - \kappa\Omega_f^\vee = g^*\mathbb{E}_{\mathfrak{M}}|_Z^f + \Omega_g|_Z^f - \kappa\Omega_g^\vee|_Z^f \in K_{\mathsf{T}}(Z), \tag{25}$$

then the virtual structure sheaves of the two different APOTs on Z coincide.

A POT is in particular an APOT, by choosing any étale atlas and restricting to each étale chart, so this proposition can also be applied when Z has a POT whose K-theory class equals the right-hand side of (25).

Both sides of (25), particularly the right-hand side, should be compared with the expression (22) for the virtual normal bundle of an APOT.

2.5.3 Proof The idea of the proof is similar to the one for the localization formula. We pass to a $(\mathsf{T}\times\mathbb{C}^\times)$-equivariant affine bundle $b\colon B \to \mathbb{M}$, where all obstructions to the construction of a *global* symmetrized pullback vanish. In fact, we can use that $\mathbb{M}$ is an algebraic space to arrange for B to be affine, and for smooth pullback along b to produce a *POT* on B; note that in the more general case of Theorem 2.4.3, the smooth pullback of an APOT to B only yielded another *APOT*. There is an induced commutative diagram

$$\begin{array}{ccccc} A & \hookrightarrow & B_Z & \hookrightarrow & B \circlearrowleft \mathsf{T}\times\mathbb{C}^* \\ & \overset{a}{\searrow} & \downarrow & & \downarrow b \\ & & Z & \hookrightarrow & \mathbb{M} \circlearrowleft \mathsf{T}\times\mathbb{C}^* \end{array} \tag{26}$$

where the square is Cartesian and $A \subset B_Z$ is the $\mathbb{C}^\times$-fixed locus. So $a\colon A \to Z$ is a T-equivariant affine bundle and A is affine. On A, we will use [Thomas 2022] to show that the virtual structure sheaf of the pullback POTs to A only depend on the K-theory class, and pass back to Z via the Thom isomorphism $K_{\mathsf{T}}(Z) \xrightarrow{\sim} K_{\mathsf{T}}(A)$.

To produce the desired b, first, we need to find a $(\mathsf{T}\times\mathbb{C}^\times)$-equivariant isomorphism $\mathbb{M} \simeq [\widetilde{W}/\mathrm{GL}(n)]$ where $\widetilde{W}$ is a quasi-affine scheme. Because $\mathbb{M}$ has the $(\mathsf{T}\times\mathbb{C}^\times)$-equivariant resolution property, by [Gross 2017, Theorem A] we have $[\mathbb{M}/(\mathsf{T}\times\mathbb{C}^\times)] \simeq [W/\mathrm{GL}(n)]$ for some quasi-affine scheme W and some integer $n \geq 0$. Hence we can construct a Cartesian square

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \mathbb{M} \\ \downarrow & & \downarrow \\ W & \longrightarrow & [\mathbb{M}/(\mathsf{T}\times\mathbb{C}^\times)] \end{array}$$

where horizontal (resp. vertical) arrows are torsors for $\mathrm{GL}(n)$ (resp. $\mathsf{T}\times\mathbb{C}^\times$). The upper row gives the desired isomorphism $\mathbb{M} \simeq [\widetilde{W}/\mathrm{GL}(n)]$.

$\widetilde{W}$ comes with a $(\mathsf{T}\times\mathbb{C}^\times)$-action, which commutes with the $\mathrm{GL}(n)$-action by construction using the cartesian square above. So we follow the equivariant Jouanolou trick of [Totaro 2004, Section 3]

$\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariantly instead of $\mathrm{GL}(n)$-equivariantly. This gives a $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariant open embedding of $\widetilde{W}$ into a finite type affine scheme Y, so that its complement is the vanishing locus of the regular functions $f_1,\dots,f_r$, whose span is preserved by the $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-action.

Consider the $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariant exact sequence on $\widetilde{W}$

$$0\to\mathcal{O}_{\widetilde{W}}\to\mathcal{E}\to\mathcal{F}\to 0,$$

where $\mathcal{E}:=\mathcal{O}_{\widetilde{W}}^r$, the first morphism are the maps $f:=(f_1,\dots,f_r)$ and $\mathcal{F}$ is the cokernel. Since the f_i cut out the complement of $\widetilde{W}$ in Y, the first morphism is locally a split monomorphism, making it an injection and $\mathcal{F}$ locally free. Now take $\widetilde{B}$ to be the complement of $\mathbb{P}(\mathcal{F})$ in $\mathbb{P}(\mathcal{E})$. Following [Weibel 1989, Proposition 4.4], this is $\mathrm{Spec}_{\widetilde{W}}\big(\mathrm{Sym}(\mathcal{E})/(f-1)\big)$ and hence affine over $\widetilde{W}$ and a vector bundle torsor for $\mathcal{F}^\vee$, and $\widetilde{B}$ is affine. Since the exact sequence was $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariant, this works $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariantly. So, we have an affine $\mathrm{GL}(n)\times(\mathsf{T}\times\mathbb{C}^\times)$-equivariant vector bundle torsor $\widetilde{B}\to\widetilde{W}$. We quotient by $\mathrm{GL}(n)$ to get a $(\mathsf{T}\times\mathbb{C}^\times)$-equivariant vector bundle torsor

$$B:=[\widetilde{B}/\mathrm{GL}(n)]\to\mathbb{M}\simeq[\widetilde{W}/\mathrm{GL}(n)].$$

This morphism is representable to the algebraic space $\mathbb{M}$, making B an algebraic space. So, by [Totaro 2004, Remark (2)], the quotient B of the affine $\widetilde{B}$ by $\mathrm{GL}(n)$ is again affine.

Now, using $a\colon A\to Z$, we argue as in Section 2.4.8. Denote the resulting virtual structure sheaves of the two APOTs by $\mathcal{O}_Z^{\mathrm{vir}}$ and $\mathcal{O}_{Z\subset\mathbb{M}}^{\mathrm{vir}}$ respectively.

First, for the symmetrized pullback along f, the obstructions to constructing a global symmetrized pullback vanish on A. As in Section 2.4.8, this lets us construct a global $\widehat{\mathbb{E}}_A^Z\to a^*\mathbb{L}_Z$ which is an isomorphism on h^0 and surjective on h^{-1}. Moreover, because A is affine, any map $\Omega_a[-1]\to a^*\mathbb{L}_Z$ vanishes by degree reasons. So, the smooth pullback to A of the symmetrized pullback APOT on Z is the POT

$$\phi_A^Z\colon\widehat{\mathbb{E}}_A^Z\oplus\Omega_a\to\mathbb{L}_A. \tag{27}$$

Analogously, the smooth pullback along b to B of the symmetrized pullback APOT on $\mathbb{M}$ is a POT $\widehat{\mathbb{E}}_B^{\mathbb{M}}\oplus\Omega_b\to\mathbb{L}_B$. Restricting to A and taking the $\mathbb{C}^\times$-fixed part give the POT

$$\phi_A^B\colon\widehat{\mathbb{E}}_B^{\mathbb{M}}|_A^f\oplus\Omega_b|_A^f\to\mathbb{L}_A \tag{28}$$

on A. We need to show that ϕ_A^B agrees with the APOT $a^*\phi_Z^{\mathbb{M}}$ obtained by going the other way from $\mathbb{M}$ to A in the diagram (26), namely: restrict to Z and take the $\mathbb{C}^\times$-fixed part of the APOT on $\mathbb{M}$, and then the smooth pullback along a to A.

An affine étale cover $\{U_i\}_{i\in I}$ of B induces an affine étale cover $\{U_i^A\}_{i\in I}$ of A by taking fiber products. There are equivariant splittings $\mathbb{L}_{U_i}\cong b_i^*\mathbb{L}_{U_i}\oplus\Omega_{b_i}$ where $b_i:=b|_{U_i}$, and similarly for U_i^A and $a_i:=a|_{U_i^A}$. After possibly refining the cover, $a^*\phi_Z^{\mathbb{M}}$ and ϕ_A^B are given on U_i by

$$a_i^*(\widehat{\mathbb{E}}_{\mathfrak{M},i}|_Z^f)\oplus\Omega_{a_i}\quad\text{and}\quad(b_i^*\widehat{\mathbb{E}}_{\mathfrak{M},i})|_Z^f\oplus\Omega_{b_i}|_Z^f$$

respectively, where $(\widehat{\mathbb{E}}_{\mathfrak{M},i})_{i\in I}$ is the symmetrized pullback APOT on $\mathbb{M}$. But these two are equal because A is the $\mathbb{C}^\times$-fixed locus.

It remains to show that the two POTs ϕ_A^Z and ϕ_A^B, from (27) and (28) respectively, induce the same virtual structure sheaf. From the above global description, their K-theory classes are

$$a^*(f^*\mathbb{E}_{\mathfrak{N}} + \Omega_f - \kappa\Omega_f^\vee) + \Omega_a$$

and

$$b^*(g^*\mathbb{E}_{\mathfrak{M}} + \Omega_g - \kappa\Omega_g^\vee)|_A^f + \Omega_b|_A^f = a^*(g^*\mathbb{E}_{\mathfrak{M}}|_Z^f + \Omega_g|_Z^f - \kappa\Omega_g^\vee|_Z^f) + \Omega_a,$$

respectively. These two K-theory classes coincide by hypothesis. By [Thomas 2022], on a quasiprojective scheme, the virtual structure sheaf induced by a POT only depends on the K-theory class of the POT. By Proposition 2.2.2, we get

$$a^*\mathcal{O}_Z^{\mathrm{vir}} = a^*\mathcal{O}_{Z\subset\mathbb{M}}^{\mathrm{vir}}.$$

The proposition follows as a^* is an isomorphism on K-theory. □

3 The DT/PT setup

3.1 Moduli stacks

3.1.1 In this section we set up the moduli stacks, stability conditions, and finally the wall-crossing problem for the DT/PT vertex correspondence. For the wall-crossing, we follow [Toda 2010], where a triangulated category with weak stability conditions is defined, such that the DT and PT stability conditions have exactly one wall (where there are strictly semistables) separating them. We combine this with the specific quasiprojective geometry from [Maulik et al. 2011], to give us a wall-crossing problem for DT and PT *vertices* with specified triples of partitions (λ, μ, ν).

3.1.2 Definition Continue with the geometry and notation of Section 1.2.2. Let $\mathcal{C}oh_{\le 1}(\overline{X})$ be the abelian category of coherent sheaves on $\overline{X}$ of (support of) dimension ≤ 1. Define the abelian (resp. triangulated) category

$$\mathcal{A} := \langle \mathcal{O}_{\overline{X}}, \mathcal{C}oh_{\le 1}(\overline{X})[-1]\rangle_{\mathrm{ex}} \subset \mathcal{D} \quad (\text{resp. } \mathcal{D} := \langle \mathcal{O}_{\overline{X}}, \mathcal{C}oh_{\le 1}(\overline{X})[-1]\rangle_{\mathrm{tr}} \subset D^b\mathcal{C}oh(\overline{X}))$$

to be the smallest extension-closed (resp. triangulated) subcategory containing $\mathcal{O}_{\overline{X}}$ and $\mathcal{C}oh_{\le 1}(\overline{X})[-1]$. It is known that $\mathcal{A}$ is a heart of a bounded t-structure on $\mathcal{D}$; see [Toda 2010, Lemma 3.5]. Objects $I \in \mathcal{D}$ have *numerical class*

$$\mathrm{cl}\, I := (\mathrm{rk}(I), \mathrm{ch}_2(I), \mathrm{ch}_3(I)) \in \mathbb{Z} \oplus H_2(\overline{X}) \oplus \mathbb{Z},$$

where we identify $H^4(\overline{X}) \cong H_2(\overline{X})$ and $H^6(\overline{X}) \cong H_0(\overline{X}) = \mathbb{Z}$ by Poincaré duality. For $\mathcal{F} \in \mathcal{C}oh_{\le 1}(\overline{X})$, this identifies $\mathrm{ch}_2(\mathcal{F})$ with the (integral) class β_C of $\mathrm{supp}\,\mathcal{F}$, and Hirzebruch–Riemann–Roch yields

$$\mathrm{ch}_3(\mathcal{F}) = \chi(\mathcal{F}) - \frac{1}{2}\int_{\beta_C} c_1(\overline{X}), \tag{29}$$

which is obviously an integer.

Let $\mathfrak{M}_\alpha = \mathfrak{M}_{r,\beta_C,n}$ denote the moduli stack of objects in $\mathcal{A}$ of numerical class $\alpha = (r, -\beta_C, -n)$. For any β_C and n, the stacks $\mathfrak{M}_{1,\beta_C,n}$ and $\mathfrak{M}_{0,0,n}$ are known to be Artin and locally of finite type [Toda 2010, Lemma 3.15]. Any given instance of DT/PT wall-crossing involves a fixed curve class β_C, with $r \in \{0,1\}$ and varying $n \in \mathbb{Z}$. For short, let $\mathfrak{M}_{\mathrm{rank}=r} := \bigsqcup_{\beta_C,n} \mathfrak{M}_{r,\beta_C,n}$.

3.1.3 Lemma [Toda 2010, Lemma 3.11] *Let $[I] \in \mathfrak{M}_{\mathrm{rank}=1}$. Then there is an exact sequence in $\mathcal{A}$,*

$$0 \to \mathcal{I}_C \to I \to \mathcal{Q}[-1] \to 0,$$

where $\mathcal{I}_C$ is the ideal sheaf of a 1-dimensional subscheme $C \subset \overline{X}$ and $\dim \mathcal{Q} \le 1$. *There is an isomorphism*

$$I \cong [\mathcal{O}_{\overline{X}} \xrightarrow{s} \mathcal{E}], \tag{30}$$

where $\dim \mathcal{E} \le 1$ *and* $\dim \operatorname{coker} s = 0$, *if and only if* $\dim \mathcal{Q} = 0$.

Proof The proof of [Toda 2010, Lemma 3.11] applies verbatim without the semistability assumption on I. □

3.1.4 Definition In the setting of Lemma 3.1.3, let

$$\mathfrak{M}^\circ_{1,\beta_C,n} \subset \mathfrak{M}_{1,\beta_C,n}$$

denote the open substack where indeed $\dim \mathcal{Q} = 0$, and, as before, write $\mathfrak{M}^\circ_{\mathrm{rank}=1} := \bigsqcup_{\beta_C,n} \mathfrak{M}^\circ_{1,\beta_C,n}$. We view points in $\mathfrak{M}^\circ_{\mathrm{rank}=1}$ as two-term complexes (30). It is $\mathfrak{M}^\circ_{\mathrm{rank}=1}$, not $\mathfrak{M}_{\mathrm{rank}=1}$, which is most relevant to us.

3.1.5 Since $\mathcal{A}$ is a $\mathbb{C}$-linear category, every object in $\mathfrak{M}_\alpha$ has a group $\mathbb{C}^\times$ of automorphisms by scaling. Let $\mathfrak{M}^{\mathrm{pl}}_\alpha$ denote the $\mathbb{C}^\times$-rigidification of $\mathfrak{M}_\alpha$; see [Abramovich et al. 2008]. Roughly, this means to quotient away the group $\mathbb{C}^\times$ of scalar automorphisms from all automorphism groups. Then the canonical map

$$\Pi^{\mathrm{pl}}_\alpha \colon \mathfrak{M}_\alpha \to \mathfrak{M}^{\mathrm{pl}}_\alpha$$

is a principal $[\mathrm{pt}/\mathbb{C}^\times]$-bundle for all $\alpha \ne 0$. The notation pl stands for *projective linear*, exemplified by the moduli stack $[\mathrm{pt}/\mathrm{GL}(n)]^{\mathrm{pl}} = [\mathrm{pt}/\mathrm{PGL}(n)]$ of vector spaces up to projective automorphisms. Later, stable loci will be substacks of $\mathfrak{M}^{\mathrm{pl}}_\alpha$ instead of $\mathfrak{M}_\alpha$.

3.1.6 Lemma *The $\mathbb{C}^\times$-rigidification map*

$$\Pi^{\mathrm{pl}}_1 \colon \mathfrak{M}^\circ_{\mathrm{rank}=1} \cong \mathfrak{M}^{\circ,\mathrm{pl}}_{\mathrm{rank}=1} \times [\mathrm{pt}/\mathbb{C}^\times] \to \mathfrak{M}^{\circ,\mathrm{pl}}_{\mathrm{rank}=1}$$

is a ***trivial*** $[\mathrm{pt}/\mathbb{C}^\times]$*-bundle.*

Proof By definition, all objects in $\mathfrak{M}^\circ_{\mathrm{rank}=1}$ are pairs of the form

$$[\mathcal{L} \xrightarrow{s} \mathcal{E}], \quad \text{where } \mathcal{L} \cong \mathcal{O}_{\overline{X}},\ \mathcal{E} \in \mathcal{C}oh_{\le 1}(\overline{X}).$$

Since $\mathrm{Aut}(\mathcal{L}) \cong \mathbb{C}^\times$, the $\mathbb{C}^\times$-rigidification of such a pair is equivalent to fixing the extra data of an isomorphism $\phi\colon \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{\overline{X}}$. Hence Π_1^{pl} has a section given by forgetting this extra data, and this section trivializes the bundle Π_1^{pl}; see [Laumon and Moret-Bailly 2000, Lemma 3.21]. □

This argument applies equally well to Hilbert schemes. For instance, $\mathrm{Hilb}(D_i)$ is the $\mathbb{C}^\times$-rigidification of the associated Hilbert *stack* $\mathfrak{Hilb}(D_i) \cong \mathrm{Hilb}(D_i) \times [\mathrm{pt}/\mathbb{C}^\times]$ of objects of the form $[\mathcal{O}_{D_i} \twoheadrightarrow \mathcal{O}_Z]$.

Due to this lemma, later (e.g., Definition 4.1.1) we often denote elements of $\mathfrak{M}^\circ_{\mathrm{rank}=1}$ by $[\mathcal{O}_{\overline{X}} \otimes L \to \mathcal{E}]$, where L is a 1-dimensional vector space and corresponds to a $\mathbb{C}$-point in $[\mathrm{pt}/\mathbb{C}^\times]$.

3.1.7 Definition For a curve class $\beta_C = (\beta_1, \beta_2, \beta_3) \in H_2(\overline{X})$ and an integer $n \in \mathbb{Z}$, let

$$\mathfrak{N}_{\beta_C,n} \subset \mathfrak{M}^\circ_{1,\beta_C,n}$$

be the open locus of two-term complexes I such that, for each $i = 1, 2, 3$:

(i) $L^k \iota_i^* I = 0$ for $k > 0$, so that $L\iota_i^* = \iota_i^*$;

(ii) the *evaluation map*

$$\mathrm{ev}_i\colon \mathfrak{N}_{\beta_C,n} \to \mathfrak{Hilb}(D_i \cap X, \beta_i), \quad [I] \mapsto [\iota_i^* I],$$

lands in the Hilbert stack (see Lemma 3.1.6) of β_i points on $D_i \cap X$.

Similarly, viewing $\mathfrak{M}_{0,0,n}$ as a moduli stack of zero-dimensional sheaves, let

$$\mathfrak{Q}_n \subset \mathfrak{M}_{0,0,n}$$

be the open locus of sheaves whose restriction to D is zero, so that the analogous evaluation maps are zero.

Condition (i) is an open condition by upper semicontinuity of cohomology. Condition (ii) is also an open condition, since $X \subset \overline{X}$ is open and the Hilbert stack is a semistable (and stable) locus in the moduli stack of *all* coherent sheaves on $D_i \subset \overline{X}$ of Chern character $(1, 0, -\beta_i) \in H^0 \oplus H^2 \oplus H^4$. It should be compared with the *admissibility* condition for relative ideal sheaves [Maulik et al. 2006b; Li and Wu 2015].

3.1.8 To avoid dealing with each ev_i separately, it is convenient to introduce the open locus

$$\mathfrak{Hilb}^\circ(D) \subset \mathfrak{Hilb}(D)$$

consisting of rank-1 torsion-free sheaves on D which are locally free along the singular locus of D. Note the isomorphism

$$\mathrm{Hilb}^\circ(D) := (\mathfrak{Hilb}^\circ(D))^{\mathrm{pl}} \cong \mathrm{Hilb}(D_1) \times \mathrm{Hilb}(D_2) \times \mathrm{Hilb}(D_3).$$

We continue to use the same names to denote the pl version of all evaluation maps. The total evaluation map

$$\mathrm{ev} := \mathrm{ev}_1 \times \mathrm{ev}_2 \times \mathrm{ev}_3\colon \mathfrak{N}_{\beta_C,n} \to \mathfrak{Hilb}^\circ(D)$$

and its pl version are compatible with the rigidification maps.

3.1.9 Definition For points $p_i \in \mathrm{Hilb}(D_i \cap X)$ for $i = 1, 2, 3$, let $|p_i|$ be the lengths of the subschemes they define, set $\beta_C = (|p_1|, |p_2|, |p_3|)$, and let

$$\mathfrak{N}^{\mathrm{pl}}_{(p_1,p_2,p_3),n} = \mathrm{ev}^{-1}(p_1, p_2, p_3) \subset \mathfrak{N}^{\mathrm{pl}}_{\beta_C,n}$$

be the closed substack of objects with the prescribed intersection with the D_i. Set

$$\mathfrak{N}_{(p_1,p_2,p_3),n} := [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{N}^{\mathrm{pl}}_{(p_1,p_2,p_3),n} \subset \mathfrak{N}_{\beta_C,n},$$

where the $\mathbb{C}^\times$ is the group of scaling automorphisms, so that the pl superscript makes sense by Lemma 3.1.6. In other words, automorphisms of objects $[I] \in \mathfrak{N}^{\mathrm{pl}}_{(p_1,p_2,p_3),n}$ (resp. $\mathfrak{N}_{(p_1,p_2,p_3),n}$) must restrict to the *identity* (resp. any scaling automorphism) on $I|_D$.

In what follows, $(p_1, p_2, p_3) = (\lambda, \mu, \nu)$ is a triple of T-fixed points in $\mathrm{Hilb}(\mathbb{C}^2)$, i.e., a triple of integer partitions. The moduli substacks $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ and $\mathfrak{Q}_n$ will be the relevant moduli stacks involved in wall-crossing. Later in Section 3.3 we will equip them with obstruction theories and enumerative invariants.

3.2 Stability conditions

3.2.1 We endow $\mathcal{A}$ with the same weak stability condition as Toda [2010, Section 3.2]. Recall that a *weak stability condition* on an abelian category, in Joyce's sense [Joyce and Song 2012, Definition 3.5], is a function τ taking numerical classes into some poset, such that for any short exact sequence $0 \to F \to E \to G \to 0$, either

$$\tau(F) \le \tau(E) \le \tau(G) \quad \text{or} \quad \tau(F) \ge \tau(E) \ge \tau(G).$$

An object E is τ*-semistable* (resp. τ*-stable*) if for any short exact sequence $0 \to F \to E \to G \to 0$, we have $\tau(F) \le \tau(G)$ (resp. $\tau(F) < \tau(G)$). We caution that, for stable objects E, it is possible to have inequalities like $\tau(F) = \tau(E) < \tau(G)$.

3.2.2 Definition For a parameter $\xi := (z_1, z_0) \in \mathbb{C}^2$ where $\arg(z_i) \in (\pi/2, \pi)$, let

$$\tau_\xi \colon \mathbb{Z} \oplus H_2(\overline{X}) \oplus \mathbb{Z} \to \mathbb{R}, \qquad (r, -\beta_C, -n) \mapsto \begin{cases} \arg z_1 & \text{if } r \ne 0, \\ \pi/2 & \text{if } r = 0, \beta_C \ne 0, \\ \arg z_0 & \text{if } r = 0, \beta_C = 0. \end{cases}$$

Let $\mathfrak{M}^{\circ,\mathrm{sst}}_\alpha(\tau_\xi) \subset \mathfrak{M}^{\circ,\mathrm{pl}}_\alpha$ be the substack of τ_ξ-semistable objects, and similarly for $\mathfrak{N}^{\mathrm{sst}}_{\beta_C,n}(\tau_\xi) \subset \mathfrak{N}^{\mathrm{pl}}_{\beta_C,n}$ and $\mathfrak{Q}^{\mathrm{sst}}_n(\tau_\xi) \subset \mathfrak{Q}^{\mathrm{pl}}_n$.

The choice of the range $(\pi/2, \pi)$ is purely for the sake of agreement with Toda's more general construction of weak stability conditions on triangulated categories, and will not be important for this paper.

3.2.3 Lemma *Consider objects in* $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$.

(i) *If* $\arg(z_0) < \arg(z_1)$, *then all* τ_ξ*-semistable objects are* τ_ξ*-stable, and*

$$\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi) = \mathrm{DT}_{\beta_C,\chi}(\overline{X}).$$

In particular, $\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau_\xi) = \mathrm{DT}_{(\lambda,\mu,\nu),\chi}$ *from* (4).

(ii) *If* $\arg(z_0) > \arg(z_1)$, *then all* τ_ξ*-semistable objects are* τ_ξ*-stable, and*

$$\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi) = \mathrm{PT}_{\beta_C,\chi}(\overline{X}).$$

In particular, $\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau_\xi) = \mathrm{PT}_{(\lambda,\mu,\nu),\chi}$ *from* (4).

(iii) *If* $\arg(z_0) = \arg(z_1)$, *then all objects are semistable, i.e.,* $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi) = \mathfrak{M}^{\circ,\mathrm{pl}}_{1,\beta_C,n}$.

The integers $n = \mathrm{ch}_3$ *and* χ *are related by* (29).

Let τ^-, τ^+ and τ_0 denote τ_ξ for some choice of ξ in cases (i), (ii) and (iii) above, respectively. Then this lemma describes a wall in a space of weak stability conditions, and the moduli spaces on both sides of the wall.

Proof The third statement follows immediately from the definition of the stability conditions τ_ξ. The first two statements about $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$ were proven in [Toda 2010, Proposition 3.12]. Since the conditions defining $\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau_\xi)$ inside $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$ are the same conditions as the ones defining $M_{(\lambda,\mu,\nu),\chi}$ inside $M_{\beta_C,\chi}(\overline{X})$ in (4), the identifications of $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$ with the DT and PT moduli restricts to these loci. □

3.2.4 Lemma *For any* ξ, *the semistable loci* $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$, $\mathfrak{N}^{\mathrm{sst}}_{(p_1,p_2,p_3),n}(\tau_\xi)$ *and* $\mathfrak{Q}^{\mathrm{sst}}_n(\tau_\xi)$ *are finite type.*

Proof It is well-known that the moduli stack $\mathfrak{Q}$ of zero-dimensional sheaves is finite type, so we focus on $\mathfrak{N}$.

From [Toda 2010, Lemma 3.15], $\mathfrak{M}^{\circ}_{1,\beta_C,n}(\tau_\xi)$ is of finite type. Since

$$\mathfrak{N}^{\mathrm{sst}}_{(p_1,p_2,p_3),n}(\tau_\xi) \subset \mathfrak{N}^{\mathrm{sst}}_{\beta_C,n}(\tau_\xi) \subset \mathfrak{M}^{\circ,\mathrm{pl}}_{1,\beta_C,n}(\tau_\xi)$$

is a closed followed by open immersion, we are done. □

3.2.5 Corollary *The moduli stack* $\mathfrak{M}^{\circ}_{1,\beta_C,n}$ *is of finite type and there exists* $m_0 = m_0(\beta_C) \in \mathbb{Z}$ *such that* $\mathfrak{M}^{\circ}_{1,\beta_C,n} = \varnothing$ *for* $n \le m_0$. *The same is true for* $\mathfrak{N}_{\beta_C,n}$ *in place of* $\mathfrak{M}^{\circ}_{1,\beta_C,n}$.

Proof The second sentence follows from the first, since $\mathfrak{N}_{\beta_C,n}$ is open in $\mathfrak{M}^{\circ}_{1,\beta_C,n}$. That $\mathfrak{M}^{\circ}_{1,\beta_C,n}$ is finite type follows from Lemma 3.2.4 in view of Lemma 3.2.3(iii). It remains to show the existence of $m_0(\beta_C)$. It is well known that we can find m_0 such that $\mathrm{DT}_{\beta_C,n}(\overline{X}) = \varnothing$ for $n \le m_0$. But for any object $(\mathcal{O}_{\overline{X}} \xrightarrow{s} \mathcal{E})$ of $\mathfrak{M}^{\circ}_{1,\beta_C,n}$ we obtain an object of $\mathrm{DT}_{\beta_C,n'}(\overline{X})$ for $n' \le n$ by replacing $\mathcal{E}$ with the image of s. Hence, there are no objects for $n \le m_0$. □

3.2.5 A one-parameter family of our rank-1 semistable objects always has a limit. This will be important to show properness of auxiliary moduli stacks later.

Lemma *For any ξ, the stack $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$, as well as its preimage in $\mathfrak{M}^{\circ}_{1,\beta_C,n}$ under the rigidification map, satisfy the existence part of the valuative criterion for properness.*

Proof By Lemma 3.1.6, the existence part of the valuative criterion holds for $\mathfrak{M}^{\circ,\mathrm{sst}}_{1,\beta_C,n}(\tau_\xi)$ if and only if it holds for its preimage under the rigidification map. It is well-known that the DT and PT moduli spaces of a projective variety are projective. This leaves us with case (iii) in Lemma 3.2.3, and hence it is enough to show the existence part of the valuative criterion for $\mathfrak{M}^{\circ}_{1,\beta_C,n}$.

Let R be a DVR with generic point η and closed point ξ. Suppose we are given any family of pairs $\mathcal{O}_{\overline{X}_\eta} \xrightarrow{\phi_\eta} \mathcal{E}_\eta$ over the generic point of R, which lies in $\mathfrak{M}^{\circ}_{1,\beta_C,n}$. Then one can find an R-flat coherent sheaf $\mathcal{E}_R$ extending $\mathcal{E}_\eta$ such that the section ϕ_η extends to a section of $\mathcal{E}_R$, which gives a pair $\mathcal{O}_{\overline{X}_R} \xrightarrow{\phi_R} \mathcal{E}_R$ in $\mathfrak{M}_{1,\beta_C,n}$.

We will modify this pair so that the fiber over ξ lies in $\mathfrak{M}^{\circ}_{1,\beta_C,n}$. Indeed, if it isn't already the case, then $\mathcal{F} := \operatorname{coker}(\phi_R)$ has one-dimensional support along the special fiber. We can find a quotient $q\colon \mathcal{F} \to \mathcal{F}'$ such that $\mathcal{F}'|_\eta = 0$ and such that the kernel has zero-dimensional support along the fiber over ξ. Let $\mathcal{E}'_R$ be the kernel of the composition $\mathcal{E}_R \to \mathcal{F} \to \mathcal{F}'$. Then the pair $\mathcal{O}_R \to \mathcal{E}'_R$ lies in $\mathfrak{M}^{\circ}_{1,\beta_C,n}$. □

3.3 Enumerative invariants

3.3.1 Recall that $\mathsf{T} = (\mathbb{C}^\times)^3$ acts on $\overline{X}$ by scaling with weights denoted by $t_1, t_2, t_3 \in K_{\mathsf{T}}(\mathrm{pt})$. Let

$$\kappa := t_1 t_2 t_3.$$

The goal of this subsection is to establish the following definition and explain why it produces DT and PT vertices for the corresponding stability chambers.

Definition Fix integer partitions λ, μ, ν. For stability conditions τ on $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$ with no strictly semistables, let

$$\mathsf{N}_{(\lambda,\mu,\nu),n}(\tau) := \chi(\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau), \widehat{\mathcal{O}}^{\mathrm{vir}}) \in K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}},$$

where $\mathcal{O}^{\mathrm{vir}}$ is defined using the symmetric obstruction theory of Proposition 3.3.2, which is perfect on the stable locus by the discussion in Section 2.1.3, and the symmetrized $\widehat{\mathcal{O}}^{\mathrm{vir}}$ is well-defined by Lemma 3.3.7.

By Lemma 3.3.6, the series $\sum_n Q^n \mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^\pm)$ are exactly the DT and PT vertices (5) of interest, up to a shift by a power of Q, identical for DT and PT, coming from (29).

3.3.2 Proposition *The moduli stack $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$ has an obstruction theory given by $(\mathbb{F}[1])^\vee$, for*

$$\mathbb{F} := R\pi_* \mathcal{E}xt(\mathcal{I}, \mathcal{I}(-D)) \in D^b\mathcal{C}oh_{\mathsf{T}}(\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}), \tag{31}$$

where $\mathcal{E}xt := R\mathcal{H}om$ is the derived functor of $\mathcal{H}om$, and $\mathcal{I}$ is the universal family on the source of $\pi\colon \mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n} \times \overline{X} \to \mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$.

To be precise, a universal family $\widetilde{\mathcal{I}} \in D^b\mathcal{C}oh_{\mathsf{T}}(\mathfrak{M}_{\mathrm{rank}=1} \times \overline{X})$ certainly exists, and it descends to a universal family on $\mathfrak{M}^{\mathrm{pl}}_{\mathrm{rank}=1}$ after a twist by the weight-1 generator of the scaling $\mathbb{C}^\times$, by Lemma 3.1.6. Restriction to $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$ produces the desired universal family $\mathcal{I}$.

For any $I \in D^b\mathcal{C}oh(\overline{X})$, relative Serre duality applied to $R\pi_*\,\mathcal{E}xt$, along with the T-equivariant identification $\mathcal{K}_{\overline{X}} \cong \kappa \otimes \mathcal{O}_{\overline{X}}(-2D)$, shows that

$$\mathrm{Ext}^*_{\overline{X}}(I, I(-D))^\vee \cong \mathrm{Ext}^{3-*}_{\overline{X}}(I(-D), I \otimes \mathcal{K}_{\overline{X}}) = \kappa \otimes \mathrm{Ext}^{3-*}_{\overline{X}}(I, I(-D)),$$

where $\mathrm{Ext} = R\,\mathrm{Hom}$ is the derived functor of Hom. Hence $\mathbb{F}$ defines a *symmetric* obstruction theory.

3.3.3 Proof Recall that $\mathfrak{N}_{\beta_C,n} \subset \mathfrak{M}^\circ_{1,\beta_C,n} \subset \mathfrak{M}_{1,\beta_C,n}$ are open, and $\mathfrak{M}_{1,\beta_C,n}$ is an open substack of a moduli stack of perfect complexes on $\overline{X}$ by [Toda 2010, Section 6.3]. By [Kuhn 2024, Theorem 1.4], the Atiyah class of the universal complex $\mathcal{I}$ gives an obstruction theory

$$R\pi_*\,\mathcal{E}xt(\mathcal{I},\mathcal{I})^\vee[-1] \to \mathbb{L}_{\mathfrak{N}_{\beta,n}}.$$

In the same way, the obstruction theory on $\mathfrak{Hilb}^\circ(D)$ is given by the Atiyah class of the universal sheaf $\mathcal{I}_D$ on $\mathfrak{Hilb}^\circ(D) \times D$. Since the Atiyah class is compatible with pullbacks, the evaluation map induces a map Ξ and a commutative diagram

$$\begin{array}{ccccccc} \mathrm{ev}^*\, R\pi_*\,\mathcal{E}xt(\mathcal{I}_D,\mathcal{I}_D)^\vee[-1] & \xrightarrow{\ \Xi\ } & R\pi_*\,\mathcal{E}xt(\mathcal{I},\mathcal{I})^\vee[-1] & \dashrightarrow & ?? & \xrightarrow{[1]} & \\ \downarrow & & \downarrow & & \vdots & & \\ \mathrm{ev}^*\,\mathbb{L}_{\mathfrak{Hilb}^\circ(D)} & \longrightarrow & \mathbb{L}_{\mathfrak{N}_{\beta,n}} & \longrightarrow & \mathbb{L}_{\mathfrak{N}_{\beta,n}/\mathfrak{Hilb}^\circ(D)} & \xrightarrow{[1]} & \end{array} \tag{32}$$

By standard considerations (see, e.g., [Manolache 2012, Construction 3.13]), $\mathrm{cone}(\Xi)$ fills in ?? and the dotted arrows and gives a relative obstruction theory for $\mathfrak{N}_{\beta,n}$ over $\mathfrak{Hilb}^\circ(D)$, which pulls back to a relative obstruction theory for $\mathfrak{N}^{\mathrm{pl}}_{\beta_C,n}$ over $\mathrm{Hilb}^\circ(D)$ by Lemma 3.1.6. To determine the explicit formula (31) for $\mathbb{F} := \mathrm{cone}(\Xi)^\vee[-1]$, we need to determine the cocone of $\Xi^\vee[-1]$, which is the restriction morphism

$$R\pi_*\,\mathcal{E}xt(\mathcal{I},\mathcal{I}) \to R\pi_*\,\mathcal{E}xt(\mathcal{I}|_{\mathfrak{N}_{\beta_C,n}\times D}, \mathcal{I}|_{\mathfrak{N}_{\beta_C,n}\times D})$$

using Definition 3.1.7(i) and that $\mathcal{I}|_{\mathfrak{N}_{\beta_C,n}\times D} = (\mathrm{ev}\times\mathrm{id})^*\mathcal{I}_D$. By adjunction, this map is identified with the result of applying $R\pi_*\,\mathcal{E}xt(\mathcal{I},-)$ to the counit of adjunction $\mathcal{I} \to \iota_*\iota^*\mathcal{I}$ for the inclusion $\iota\colon D \hookrightarrow X$. Thus, the cocone is precisely $R\pi_*\,\mathcal{E}xt(\mathcal{I},\mathcal{I}(-D))$, as claimed. □

3.3.4 Remark Alternatively, it should be possible to construct the obstruction theory (31) by directly twisting the construction [Huybrechts and Thomas 2010] of the truncated Atiyah class by $\mathcal{O}_{\overline{X}}(-D)$ to disallow deformations along D. More generally, we expect that the technology of [Huybrechts and Lehn 1995; Huybrechts and Thomas 2010; Sala 2012] can be combined to show that enforcing a framing isomorphism $L\iota^*I = F$ for a given F changes the obstruction theory from $\mathrm{Ext}_{\overline{X}}(I,I)_0$ into the hyper-Ext group $\mathrm{Ext}_{\overline{X}}(I, I \to F)$. For us, in full agreement with Proposition 3.3.2,

$$\mathrm{Ext}_{\overline{X}}(I, I \to I|_D) \cong \mathrm{Ext}_{\overline{X}}(I, I(-D)).$$

The presence of the framing F removes the need for the tracelessness condition.

3.3.5 Remark Typical DT-style obstruction theories involve taking the *traceless* part of Ext groups like $\mathrm{Ext}(I, I)$. We can see explicitly why this is unnecessary for us. The cohomology long exact sequence of the (dual of the) top row in (32) begins as

$$0 \to \mathrm{Hom}_{\overline{X}}(I, I(-D)) \to \mathrm{Hom}_{\overline{X}}(I, I) \to \mathrm{Hom}_D(I|_D, I|_D) \to \cdots . \tag{33}$$

These three terms describe automorphisms of objects in $\mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{Hilb}^\circ(D)$, respectively. Clearly the first term $\mathrm{Hom}_{\overline{X}}(I, I(-D))$ is the group of automorphisms of I which are the identity on $I|_D$. When I is a stable object, the latter two terms here consist only of scalar automorphisms and are identified with each other, so $\mathrm{Hom}_{\overline{X}}(I, I(-D)) = 0$ as expected. Then, as discussed in Section 2.1.3, Serre duality gives $\mathrm{Ext}^3_{\overline{X}}(I, I(-D)) = 0$ as well, so Proposition 3.3.2 induces a *perfect* obstruction theory on stable loci of $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$, as expected.

3.3.6 Lemma *Let $I \in \mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^\pm)$ be a T-fixed point. Then the virtual tangent space*

$$T_I^{\mathrm{vir}} \mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^\pm)$$

agrees with the vertex contribution V_α of [Maulik et al. 2006a, Section 4.9] *or* [Pandharipande and Thomas 2009a, Section 4.6].

Proof The following computation is independent of stability condition and applies to the whole stack. For $\overline{X}$, the original vertex contribution is by definition the Laurent polynomial V in

$$T_I^{\mathrm{vir}} \mathfrak{N}^{\mathrm{pl}}_{\beta,n} = V + \sum_{i=1}^{3} \left(\frac{F_i}{1-t_i} + \frac{F_i}{1-t_i^{-1}} \right) = V + F_1 + F_2 + F_3, \tag{34}$$

where $F_i = T_{p_i} \mathrm{Hilb}(D_i) \in K_{\mathsf{T}}(\mathrm{pt})$; the terms in the brackets are edge contributions and correspond to deformations of $I|_D$. (Note that the equalities in (34) take place in $K_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$ but clearly both the left and right-hand sides lie in $K_{\mathsf{T}}(\mathrm{pt})$.) On the other hand, the top row in (32) gives

$$T_I^{\mathrm{vir}} \mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n} = T^{\mathrm{vir}} \mathfrak{N}^{\mathrm{pl}}_{\beta,n} - \sum_i T_{p_i} \mathrm{Hilb}(D_i)$$

in K-theory, thereby removing the edge contributions. Alternatively, the same Čech cohomology computation which gives (34) also gives

$$T_I^{\mathrm{vir}} \mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n} = V + \sum_{i=1}^{3} \left(\frac{F_i}{1-t_i} + \frac{t_i^{-1} F_i}{1-t_i^{-1}} \right) = V, \tag{35}$$

where now the edge terms in the brackets are zero. Note that the twist by D does not affect the $\mathbb{C}^3$ chart where terms in V originate, so V is unchanged between (34) and (35). □

3.3.7 Lemma [Nekrasov and Okounkov 2016, Section 6] *After a base change $\kappa \rightsquigarrow \kappa^{1/2}$, $\det \mathbb{F}$ admits a square root.*

Throughout, we implicitly take a double cover of T so that $\kappa^{1/2}$ exists. The Nekrasov–Okounkov symmetrization $\hat{\mathcal{O}}^{\mathrm{vir}} := \mathcal{O}^{\mathrm{vir}} \otimes (\det \mathbb{F})^{1/2}$ is therefore well-defined on any locus where $\mathcal{O}^{\mathrm{vir}}$ can be constructed.

Proof sketch On $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n} \times \mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n} \times \overline{X}$, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be universal families for the first and second factors, and consider

$$\mathbb{L} := \det R\pi_* \mathcal{E}xt(\mathcal{I}_1, \mathcal{I}_2(-D)),$$

where π is projection onto the first two factors. The desired $\det \mathbb{F}$ is the restriction of $\mathbb{L}$ to the diagonal Δ. By the argument in [Nekrasov and Okounkov 2016, Lemma 6.1], it suffices to show that $c_1(\mathbb{L}|_\Delta)$ is divisible by two. This follows from

$$(12)^*\mathbb{L} = \det R\pi_* \mathcal{E}xt(\mathcal{I}_2, \mathcal{I}_1(-D)) = \kappa^{\mathrm{rank}} \det R\pi_* \mathcal{E}xt(\mathcal{I}_1, \mathcal{I}_2(-D)) = \mathbb{L},$$

where (12) swaps the two copies of $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),n}$, and $\mathrm{rank} = 0$ is the rank of $R\pi_* \mathcal{E}xt(\mathcal{I}_2, \mathcal{I}_1(-D))$. □

3.3.8 For completeness, we mention here that the moduli stacks $\mathfrak{Q}_n$ are well known to carry a symmetric obstruction theory given by $(\mathbb{F}[1])^\vee$, where

$$\mathbb{F} := R\pi_* \mathcal{E}xt(\mathcal{E}, \mathcal{E}) \in D^b\mathcal{C}oh_{\mathsf{T}}(\mathfrak{Q}_n). \tag{36}$$

Here, since Lemma 3.1.6 is unavailable, the universal family $\mathcal{E}$ on $\mathfrak{Q}_n$ does not descend to $\mathfrak{Q}^{\mathrm{pl}}_n$, but $\mathcal{E}xt(\mathcal{E}, \mathcal{E})$ has trivial $\mathbb{C}^\times$-weight and therefore does descend. The symmetric obstruction theory on $\mathfrak{Q}^{\mathrm{pl}}_n$ which is compatible with the one on $\mathfrak{Q}_n$, in the sense of Definition 2.1.11, is therefore given by $(\mathbb{F}_\perp[1])^\vee$, where

$$\mathbb{F}_\perp := R\pi_* \mathcal{E}xt(\mathcal{E}, \mathcal{E})_\perp \in D^b\mathcal{C}oh_{\mathsf{T}}(\mathfrak{Q}^{\mathrm{pl}}_n) \tag{37}$$

is the "symmetrically traceless" version of $\mathbb{F}$ in the sense of [Tanaka and Thomas 2020, Theorem 6.1]. It is obtained from $\mathbb{F}$ by removing a copy of $\mathbb{C}$ in H^0 and, by Serre duality, a copy of $\mathbb{C} \otimes \kappa$ in H^3; see Section 3.3.9 for details.

Viewing objects in $\mathfrak{Q}_n$ or $\mathfrak{Q}^{\mathrm{pl}}_n$ as pairs $[0 \to \mathcal{E}]$, and recalling that they have zero intersection with $D \subset \overline{X}$, (36) is exactly the same formula as (31). For instance, Lemma 3.3.7 applies equally well for $\mathfrak{Q}^{\mathrm{pl}}_n$.

Note that, unlike $\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),m}(\tau^\pm)$, the moduli stacks $\mathfrak{Q}^{\mathrm{sst}}_m(\tau_\xi)$ have strictly semistable objects for any ξ and any $m > 1$, and so there is no direct analogue of the enumerative invariants $\mathsf{N}_{(\lambda,\mu,\nu),m}(\tau^\pm)$. One would have to follow the strategy of [Joyce 2021, Theorem 5.7] and [Liu 2023, Section 6] in order to define *semistable invariants* $\mathsf{Q}_m(\tau_\xi)$ intrinsic to the stack $\mathfrak{Q}^{\mathrm{sst}}_m(\tau_\xi)$.

3.3.9 To make the construction of $\mathbb{F}_\perp$ completely precise, let $R := \mathbb{C}[x, y, z]$, and consider the Koszul resolution of the diagonal $\mathbb{C}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$:

$$R \otimes R \to (R \otimes R)^{\oplus 3} \to (R \otimes R)^{\oplus 3} \to R \otimes R.$$

Letting R act on the right, this is split as a sequence of R-modules. Hence, for any R-module V, tensoring from the right provides a free resolution of V for the left R-action

$$R \otimes V \to R \otimes V^{\oplus 3} \to R \otimes V^{\oplus 3} \to R \otimes V.$$

This procedure works for families of quasicoherent sheaves on $\mathbb{C}^3$ over an arbitrary base. In particular, it provides a resolution of the universal sheaf $\mathscr{E}$ over $\mathfrak{Q}_n$, with $\mathcal{V} := \pi_* \mathscr{E}$,

$$\pi^* \mathcal{V} \to \pi^* \mathcal{V}^{\oplus 3} \to \pi^* \mathcal{V}^{\oplus 3} \to \pi^* \mathcal{V}.$$

Using this resolution, one computes $\mathbb{F} = R\pi_* \mathcal{E}xt(\mathscr{E}, \mathscr{E})$ explicitly as a complex in degrees $[0, 3]$:

$$\mathcal{V} \otimes_{\mathcal{O}_{\mathfrak{Q}_n}} \mathcal{V}^\vee \to \mathcal{V} \otimes_{\mathcal{O}_{\mathfrak{Q}_n}} \mathcal{V}^\vee \otimes_{\mathbb{C}} \mathbb{C}^{\oplus 3} \to \mathcal{V} \otimes_{\mathcal{O}_{\mathfrak{Q}_n}} \mathcal{V}^\vee \otimes_{\mathbb{C}} \mathbb{C}^{\oplus 3} \to \mathcal{V} \otimes_{\mathcal{O}_{\mathfrak{Q}_n}} \mathcal{V}^\vee. \tag{38}$$

One obtains $\mathbb{F}_\perp$ by removing the diagonal copy of $\mathcal{O}_{\mathfrak{Q}_n}$ from $\mathcal{V} \otimes_{\mathfrak{Q}_n} \mathcal{V}^\vee$ in degrees 0 and 3 and noting that (38) descends to the rigidification.

4 Ingredients for wall-crossing

4.1 Quiver-framed stacks

4.1.1 We first build auxiliary moduli stacks following the general strategy outlined in Section 1.4.3, for our moduli stacks $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ of pairs and $\mathfrak{Q}_n$ of zero-dimensional sheaves as described in Definition 3.1.9. This involves the following framing functors.

Definition Let $\mathcal{O}_{\overline{X}}(1)$ denote $\mathcal{O}_{\mathbb{P}^1}(1)^{\boxtimes 3}$, or any other ample line bundle on $\overline{X}$. Recall from Lemma 3.1.6 that $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ parametrizes pairs $[\mathcal{O}_{\overline{X}} \otimes L \to \mathcal{E}]$, where the 1-dimensional vector space L is a $\mathbb{C}$-point in $[\mathrm{pt}/\mathbb{C}^\times]$. For $k \in \mathbb{Z}_{>0}$, let ${}_k\mathfrak{N}_{(\lambda,\mu,\nu),n} \subset \mathfrak{N}_{(\lambda,\mu,\nu),n}$ be the open locus of such pairs where $\mathcal{E}$ is k-regular [Huybrechts and Thomas 2010, Section 1.7], i.e., such that $H^i(\mathcal{E}(k-i)) = 0$ for all $i > 0$. For $p \in \mathbb{Z}_{>0}$, define the functor

$$F_{k,p} \colon [\mathcal{O}_{\overline{X}} \otimes L \to \mathcal{E}] \mapsto \chi(\mathcal{E}(k)) \oplus L^{\oplus p}$$

which sends $\mathbb{C}$-valued objects of ${}_k\mathfrak{N}_{(\lambda,\mu,\nu),n} \subset \mathfrak{N}_{(\lambda,\mu,\nu),n}$ to vector spaces. The sheaves $\mathcal{E}$ parametrized by $\mathfrak{Q}_n$ are zero-dimensional and therefore automatically k-regular; we extend $F_{k,p}$ to objects in $\mathfrak{Q}_n$ as

$$F_{k,p} \colon [0 \to \mathcal{E}] \mapsto \chi(\mathcal{E}(k)) = \chi(\mathcal{E}),$$

which is independent of both k and p. Then $F_{k,p}$ becomes a functor which is exact on short exact sequences involving only objects in $\mathfrak{Q}$ and ${}_k\mathfrak{N}_{(\lambda,\mu,\nu)}$. Set $f_{k,p}(I) := \dim F_{k,p}(I)$ and note that it only depends on the numerical class $\mathrm{cl}\, I$.

The term $L^{\oplus p}$ ensures that the induced map $\mathrm{Hom}(-,-) \to \mathrm{Hom}(F_{k,p}(-), F_{k,p}(-))$ is an inclusion on ${}_k\mathfrak{N}_{(\lambda,\mu,\nu)}$, especially when $\lambda = \mu = \nu = \varnothing$. For most of the paper, we fix $p = 1$, but in Section 6.3.3 we take advantage of the freedom in choosing p in order to get a nice factorization of some generating series.

4.1.2 Definition Let (λ,μ,ν) be integer partitions, which specify $\beta_C = (|\lambda|,|\mu|,|\nu|)$, and let $\alpha = (1,-\beta_C,-n)$ be a numerical class. In the wall-crossing arguments of Sections 5 and 6, we always begin by fixing (λ,μ,ν) and n, which determines the class α; choices of parameters below will depend on α. Choose $k \gg 0$ so that ${}_k\mathfrak{N}_{\beta_C,m} = \mathfrak{N}_{\beta_C,m}$ for all $m \le n$; this is possible by Corollary 3.2.5. Choose $p > 0$ and set

$$N := f_{k,p}(\alpha).$$

For $m \le n$ and a dimension vector $\boldsymbol{d} = (d_1,\dots,d_N)$, let $\mathfrak{N}^{Q(N)}_{(\lambda,\mu,\nu),m,\boldsymbol{d}}$ be the moduli stack of triples $(I,\boldsymbol{V},\boldsymbol{\rho})$ where:

- I is an object in ${}_k\mathfrak{N}_{(\lambda,\mu,\nu),m} = \mathfrak{N}_{(\lambda,\mu,\nu),m}$;
- $\boldsymbol{V} = (V_i)_{i=1}^N$ are vector spaces with $\dim V_i = d_i$; set $V_{N+1} := F_{k,p}(I)$;
- $\boldsymbol{\rho} = (\rho_i)_{i=1}^N$ are linear maps $\rho_i\colon V_i \to V_{i+1}$.

Similarly define $\mathfrak{Q}^{Q(N)}_{m,\boldsymbol{d}}$ using objects in ${}_k\mathfrak{Q}_{m,\boldsymbol{d}}$. In other words, these moduli stacks parametrize objects of the underlying moduli stack $\mathfrak{N}_{(\lambda,\mu,\nu),m}$ or $\mathfrak{Q}_m$ along with quiver representations

$$\overset{V_1}{\blacksquare} \longrightarrow \overset{V_2}{\blacksquare} \longrightarrow \cdots \longrightarrow \overset{V_{N-1}}{\blacksquare} \longrightarrow \overset{V_N}{\blacksquare} \longrightarrow \overset{V_{N+1}=F_{k,p}(I)}{\bullet}.$$

Denote this quiver by $Q(N)$. Consequently, the forgetful maps

$$\Pi_{\mathfrak{N}}\colon \mathfrak{N}^{Q(N)}_{(\lambda,\mu,\nu),m,\boldsymbol{d}} \to \mathfrak{N}_{(\lambda,\mu,\nu),m} \quad\text{and}\quad \Pi_{\mathfrak{Q}}\colon \mathfrak{Q}^{Q(N)}_{m,\boldsymbol{d}} \to \mathfrak{Q}_m \tag{39}$$

are smooth, with fiber $\bigl[\bigoplus_{i=1}^N \operatorname{Hom}(V_i,V_{i+1}) / \prod_{i=1}^N \operatorname{GL}(V_i)\bigr]$. Write $(\beta,\boldsymbol{d})$ for the numerical class of $(I,\boldsymbol{V},\boldsymbol{\rho})$, where $\beta = \operatorname{cl}(I)$ and $\boldsymbol{d} := \dim \boldsymbol{V}$.

4.1.3 The same construction works to give relative quiver spaces

$$\Pi_{\mathfrak{M}} : \mathfrak{M}^{\circ,Q(N)}_{1,\beta_C,m,\boldsymbol{d}} \to \mathfrak{M}^{\circ}_{1,\beta_C,m} \quad\text{and}\quad \Pi_{\mathfrak{M}} : \mathfrak{M}^{\circ,Q(N)}_{0,0,m,\boldsymbol{d}} \to \mathfrak{M}^{\circ}_{0,0,m}.$$

The maps $\Pi_{\mathfrak{M}}$ base change to $\Pi_{\mathfrak{N}}$ over $\mathfrak{N}_{(\lambda,\mu,\nu),m} \subseteq \mathfrak{M}^{\circ}_{1,\beta_C,m}$ and to $\Pi_{\mathfrak{Q}}$ over $\mathfrak{Q}_m \subseteq \mathfrak{M}_{0,0,m}$, respectively.

4.1.4 Let $\mathcal{I}$ and $(\mathcal{V}_i)_{i=1}^{N+1}$ denote the universal families associated to I and $(V_i)_{i=1}^{N+1}$ in Definition 4.1.2 respectively, on both $\mathfrak{N}^{Q(N)}_{(\lambda,\mu,\nu),n,\boldsymbol{d}}$ and $\mathfrak{Q}^{Q(N)}_{n,\boldsymbol{d}}$. The relative cotangent complexes $\mathbb{L}_{\Pi_{\mathfrak{N}}}$ and $\mathbb{L}_{\Pi_{\mathfrak{Q}}}$ are both given by the restriction to the diagonal of $\mathbb{F}^\vee$, where

$$\mathbb{F} := \Bigl[\bigoplus_{i=1}^N \mathcal{V}_i^\vee \boxtimes \mathcal{V}_i \to \bigoplus_{i=1}^N \mathcal{V}_i^\vee \boxtimes \mathcal{V}_{i+1}\Bigr], \tag{40}$$

with second term in degree 0. The morphism is the derivative of the natural $\prod_{i=1}^N \operatorname{GL}(V_i)$ action. Note that the universal bundles $\mathcal{V}_i$ may not descend to $\mathfrak{N}^{Q(N),\mathrm{pl}}_{(\lambda,\mu,\nu),n,\boldsymbol{d}}$ or $\mathfrak{Q}^{Q(N),\mathrm{pl}}_{n,\boldsymbol{d}}$, but bilinear quantities like $\mathcal{V}_i^\vee \otimes \mathcal{V}_i$, which have weight zero with respect to the global $\mathbb{C}^\times$ stabilizer, do. For later use, denote the rank of $\mathbb{F}$ by

$$\chi_Q(\boldsymbol{e},\boldsymbol{f}) := \sum_{i=1}^N (e_i f_{i+1} - e_i f_i),$$

where $f_{N+1} := \operatorname{rank} \mathcal{V}_{N+1}$. Similarly, let $\chi(\gamma,\delta) := \operatorname{rank} \operatorname{Ext}_{\overline{X}}(I_\gamma, I_\delta(-D))$ be the analogous Euler pairing on the original moduli stacks $\mathfrak{N}$ and $\mathfrak{Q}$, where I_γ and I_δ denote any element I of numerical class γ and δ, respectively. Later in wall-crossing formulas,

$$c_Q(\boldsymbol{e},\boldsymbol{f}) := \chi_Q(\boldsymbol{e},\boldsymbol{f}) - \chi_Q(\boldsymbol{f},\boldsymbol{e}) = \sum_{i=1}^{N}(e_i f_{i+1} - f_i e_{i+1})$$

will appear as part of the quantity ind in Proposition 1.4.5 for the complicated $\mathbb{C}^\times$-fixed locus in the master space.

4.1.5 Remark The length of the quiver $Q(N)$ may actually be chosen to be greater than N: for $\mathfrak{X} = \mathfrak{N}_{(\lambda,\mu,\nu)}$ or $\mathfrak{Q}$, or in fact any other appropriate moduli stack, there is obviously an isomorphism of stacks (with obstruction theory)

$$\mathfrak{X}^{Q(N)}_{m,\boldsymbol{d}} \cong \mathfrak{X}^{Q(N+1)}_{m,(0,\boldsymbol{d})}.$$

Similarly, but less obviously, if $d_a = d_{a+1}$ for some a, then there is an isomorphism of stacks (with obstruction theory)

$$\mathfrak{X}^{Q(N)}_{m,\boldsymbol{d}} \supset \{\rho_a : V_a \xrightarrow{\sim} V_{a+1}\} \cong \mathfrak{X}^{Q(N-1)}_{m,(d_1,\dots,d_a,d_{a+2},\dots,d_N)}$$

for the open substack where ρ_a is an isomorphism. One can check that the isomorphism ρ_a splits off a two-term complex $[\mathcal{V}_a^\vee \boxtimes \mathcal{V}_a^\vee \to \mathcal{V}_a^\vee \boxtimes \mathcal{V}_{a+1}^\vee]$ in (40), and this complex is quasi-isomorphic to 0.

4.2 Full flags and words

4.2.1 Definition A class $(\beta,\boldsymbol{d})$ is a *full flag* if

- $\beta \neq 0$ and $\boldsymbol{d} \neq \boldsymbol{0}$;
- $d_1 \leq 1$, and $d_N = d_{N+1} := f_{k,p}(\beta)$, and $d_a \leq d_{a+1} \leq d_a + 1$ for all $1 \leq a < N$.

Given a quiver-framed stack $\mathfrak{X}^{Q(N)}_{\beta,\boldsymbol{d}}$, let

$$\mathfrak{X}^{\mathrm{fl}}_{\beta,\boldsymbol{d}} \subset \mathfrak{X}^{Q(N),\mathrm{pl}}_{\beta,\boldsymbol{d}}$$

be the open substack for which all maps in the quiver representation are injective. As in Remark 4.1.5, we have a canonical isomorphism

$$\mathfrak{X}^{\mathrm{fl}}_{\beta,\boldsymbol{d}} \simeq \mathfrak{X}^{\mathrm{fl}}_{\beta,(1,2,\dots,d_N)}.$$

In both wall-crossing approaches, the only relevant classes which appear are full flags with $\beta = (1,-\beta_C,-m)$ and $\beta = (0,0,-m)$. At walls, a full flag splits into two smaller full flags. We view objects of $\mathfrak{N}^{\mathrm{fl}}_{(\lambda,\mu,\nu),m,\boldsymbol{d}}$ or $\mathfrak{Q}^{\mathrm{fl}}_{m,\boldsymbol{d}}$ as objects I in $\mathfrak{N}_{(\lambda,\mu,\nu),m}$ or $\mathfrak{Q}_m$ together with a full flag of subspaces in $F_{k,p}(I)$.

4.2.2 Proposition *Let $\pi: \mathfrak{X} \to \mathfrak{Y}$ be a $\mathbb{P}^N$-bundle. Suppose both $\mathfrak{X}$ and $\mathfrak{Y}$ have symmetric APOTs, related by symmetrized pullback. Then the associated virtual cycles satisfy*

$$\pi_*(\widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}}) = [N+1]_\kappa \cdot \widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{Y}}.$$

In particular, let $(\beta, \boldsymbol{d})$ be a full flag, and take $\pi\colon \mathfrak{X}^{\mathrm{fl}}_{\beta,\boldsymbol{d}} \to \mathfrak{X}_\beta$ to be the forgetful map. Supposing that it lands in a stable locus $\mathfrak{X}^{\mathrm{st}}_\beta \subset \mathfrak{X}_\beta$, since full flag bundles are iterated projective bundles,

$$\pi_*(\widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}^{\mathrm{fl}}_{\beta,\boldsymbol{d}}}) = [d_N]_\kappa! \cdot \widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}_\beta}. \tag{41}$$

Proof Apply Proposition 2.3.11 to π, and twist by the square root $(\mathcal{K}^{\mathrm{vir}}_\pi)^{\frac{1}{2}} = \kappa^{-\frac{1}{2}\dim \pi} K_\pi$ of the relative virtual canonical to obtain that

$$\widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{X}} = \pi^* \widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{Y}} \otimes \mathsf{e}(\Omega_\pi \otimes \kappa^{-1}) \otimes \kappa^{-\frac{1}{2}\dim \pi} K_\pi.$$

By the projection formula, it remains to compute the pushforward

$$\pi_*\big(\mathsf{e}(\Omega_\pi \otimes \kappa^{-1}) \otimes \kappa^{-\frac{1}{2}\dim \pi} K_\pi\big).$$

This is a standard computation (see [Thomas 2020, Proposition 5.5] and [Liu 2023, Proposition 4.6]), which we briefly review. The pushforward is some combination of relative cohomology bundles $R^i\pi_*\Omega^j_\pi$, which are all canonically trivialized by powers of the hyperplane class. So it is enough to compute on fibers:

$$\chi\big(\mathbb{P}^N, \mathsf{e}(\Omega_{\mathbb{P}^N} \otimes \kappa^{-1}) \otimes K_{\mathbb{P}^N}\big)\kappa^{-\frac{1}{2}N} = (-1)^N \kappa^{-\frac{1}{2}N} \sum_{i,j=0}^{N} (-1)^{i+j} \dim H^{i,j}(\mathbb{P}^N) t^j = [N+1]_t,$$

using that the modules $H^{i,j}(\mathbb{P}^N) \subset H^{i+j}(\mathbb{P}^N;\mathbb{C})$ are T-equivariantly trivial by Hodge theory. This computation motivates the sign $(-1)^{N-1}$ in the definition (8) of $[N]_t$. □

4.2.3 Let $\mathbf{1}$ be the framing dimension vector $(1,1,\dots,1)$, and

$$\mathfrak{Q}^{\mathrm{fl,st}}_{m,\mathbf{1}} \subset \mathfrak{Q}^{\mathrm{fl}}_{m,\mathbf{1}}$$

be the open substack of triples $([0 \to \mathcal{E}], \boldsymbol{V}, \boldsymbol{\rho})$ where the composition $V_1 \to \cdots \to V_N \to F_{k,p}(\mathcal{E}) = H^0(\mathcal{E})$ induces a surjection $V_1 \otimes \mathcal{O}_{\overline{X}} \twoheadrightarrow \mathcal{E}$. We call such triples *stable*.

Proposition *There is an isomorphism*

$$\mathfrak{Q}^{\mathrm{fl,st}}_{m,\mathbf{1}} \cong \mathrm{Hilb}(\mathbb{C}^3, m),$$

which identifies the (*symmetrized*) *virtual structure sheaves.*

Here, we use Theorem 2.3.4 to equip the left-hand side with a symmetric APOT by symmetrized pullback along the forgetful map to $\mathfrak{Q}_m$, which is clearly smooth. We view $\mathrm{Hilb}(\mathbb{C}^3, m)$ as the moduli scheme of pairs $[\mathcal{O}_{\overline{X}} \twoheadrightarrow \mathcal{O}_Z]$ where $Z \subset \mathbb{C}^3 = \overline{X} \setminus D$ is a zero-dimensional subscheme of length m and the map is the natural surjection.

More generally, if $((0,0,-m), \boldsymbol{e})$ is a full flag, then it is clear that $\mathfrak{Q}^{\mathrm{fl,st}}_{m,\boldsymbol{e}}$ is naturally isomorphic to the full flag bundle over $\mathrm{Hilb}(\mathbb{C}^3, m)$ associated to the vector bundle $\mathrm{coker}(\mathcal{O}_{\mathrm{Hilb}(\mathbb{C}^3,m)} \to \pi_*\mathcal{E})$, where $\pi\colon \mathrm{Hilb}(\mathbb{C}^3,m) \times \mathbb{C}^3 \to \mathrm{Hilb}(\mathbb{C}^3,m)$ is the projection and the morphism is the one adjoint to the universal section $\mathcal{O}_{\mathrm{Hilb}(\mathbb{C}^3,m)\times\mathbb{C}^3} \to \mathcal{E}$. Consequently,

$$\chi(\mathfrak{Q}^{\mathrm{fl,st}}_{m,\boldsymbol{e}}, \widehat{\mathcal{O}}^{\mathrm{vir}}) = [m-1]_\kappa! \cdot \chi(\mathrm{Hilb}(\mathbb{C}^3,m), \widehat{\mathcal{O}}^{\mathrm{vir}}).$$

4.2.4 Proof It is clear from the moduli description of $\mathfrak{Q}^{\mathrm{fl,st}}_{m,\mathbf{1}}$ that the desired isomorphism exists, namely, if $N>1$ then use Remark 4.1.5 to shorten the chain of isomorphisms $V_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} V_N \to H^0(\mathcal{E})$ to the single map $V_1 \to H^0(\mathcal{E})$. To match virtual cycles, recall that virtual tangent space of $\mathrm{Hilb}(\mathbb{C}^3)$, i.e., the K-theory class of the dual of the symmetric obstruction theory, is

$$\begin{aligned}\mathrm{Ext}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_{\overline{X}}) - \mathrm{Ext}_{\overline{X}}([\mathcal{O}_{\overline{X}} \twoheadrightarrow \mathcal{O}_Z],[\mathcal{O}_{\overline{X}} \twoheadrightarrow \mathcal{O}_Z]) \\ = \mathrm{Ext}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_Z) + \mathrm{Ext}_{\overline{X}}(\mathcal{O}_Z,\mathcal{O}_{\overline{X}}) - \mathrm{Ext}_{\overline{X}}(\mathcal{O}_Z,\mathcal{O}_Z),\end{aligned}$$

where $\mathrm{Ext}_{\overline{X}}(-,-) := \sum_i (-1)^i \mathrm{Ext}^i_{\overline{X}}(-,-)$. On the other hand, the APOT on stable loci of $\mathfrak{Q}^{Q(N),\mathrm{pl}}_{m,\mathbf{1}}$ comes from symmetrized pullback, with virtual tangent space

$$-\mathrm{Ext}_{\overline{X}}(\mathcal{O}_Z,\mathcal{O}_Z)_\perp + \big(\mathrm{Hom}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_Z) - \mathbb{C}\big) - \big(\mathrm{Hom}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_Z) - \mathbb{C}\big)^\vee \kappa^{-1}$$

coming from (37) and (40). The $\mathbb{C} = \mathrm{Ext}^0_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_{\overline{X}})$ and $\mathbb{C}^\vee \otimes \kappa^{-1}$ terms remove the $\perp$ from the first term; see Section 3.3.8. Since Z is affine, $\mathrm{Hom}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_Z) = \mathrm{Ext}_{\overline{X}}(\mathcal{O}_{\overline{X}},\mathcal{O}_Z)$. So the two virtual tangent spaces match in K-theory. By Proposition 2.5.2 (with $\mathbb{M} = \mathfrak{Q}^{\mathrm{fl,st}}_{m,\mathbf{1}}$ and a trivial $\mathbb{C}^\times$-action), the virtual cycles match too. □

4.2.5 For later use, we collect here some notation, which make handling the combinatorial expressions in the upcoming wall-crossing formulas of Sections 5 and 6 a bit simpler. By iterated wall-crossing, we obtain decompositions of the full flag dimension vector $(1,2,\dots,N)$, of the form

$$(1,2,\dots,N) = \sum_{i=1}^k \boldsymbol{e}_i, \tag{42}$$

where $e_{i,a} \le e_{i,a+1} \le e_{i,a}+1$ for all i and $1 \le a < N$. This makes contact with the combinatorial subject of *word rearrangements*.

Definition For $\ell \ge 1$ and $\boldsymbol{m} \in \mathbb{Z}^\ell_{>0}$ such that $N = |\boldsymbol{m}| := \sum_{i=1}^\ell m_i$, let

$$R(\boldsymbol{m}) := \{\text{rearrangement of the word } 1^{m_1}2^{m_2}\cdots\ell^{m_\ell}\}.$$

Given a word $w \in R(\boldsymbol{m})$, let $O_i(w) \subset \{1,2,\dots,N\}$ be the set of indices where the letter i occurs in w, and let $o_i(w) := \min O_i(w)$ be the index of the first occurrence. There is a bijection between $R(\boldsymbol{m})$ and decompositions (42) such that $m_i = e_{i,N}$, where

$$e_{i,a} := \#\{b \in O_i(w) : b \le a\}.$$

Namely, given a decomposition (42), form the word whose a-th letter is the (unique) i such that $e_{i,a} = e_{i,a-1}+1$, with the convention that $e_{i,-1} = 0$. Under this bijection,

$$c_{i,j}(w) := c_Q(\boldsymbol{e}_i,\boldsymbol{e}_j) = \#\{(a,b) \in O_i(w)\times O_j(w) : a<b\} - \#\{(a,b) \in O_i(w)\times O_j(w) : a>b\}. \tag{43}$$

The equality is easy to check: it holds for $w = 1^{m_1}2^{m_2}\cdots\ell^{m_\ell}$, and then each inversion reduces both sides by 2. So $\sum_{j>i} c_Q(i,j)$ is related to the inversion number for i. It can be used to quantize the combinatorial identity $|R(\boldsymbol{m})| = \binom{N}{\boldsymbol{m}}$; see Remark 5.3.3.

4.2.6 Example Take the decomposition

$$(1,2,3,4,5,6) = \underbrace{(1,1,2,2,2,2)}_{e_1} + \underbrace{(0,0,0,1,1,1)}_{e_2} + \underbrace{(0,1,1,1,2,3)}_{e_3}.$$

The corresponding word rearrangement is $w = 131233 \in R(2,1,3)$. We have

$$c_{1,2}(w) = 2-0 = 2, \quad c_{1,3}(w) = 5-1 = 4, \quad c_{2,3}(w) = 2-1 = 1.$$

Indeed, this matches with $c_Q(\boldsymbol{e}_1, \boldsymbol{e}_2) = 2$ and $c_Q(\boldsymbol{e}_1, \boldsymbol{e}_3) = 1-1+2+2$ and $c_Q(\boldsymbol{e}_2, \boldsymbol{e}_3) = -1+1+1$.

4.3 Properness of moduli spaces

We describe a general strategy for proving the valuative criteria for properness and separatedness for moduli spaces of (semi)stable objects, originally due to Langton [1975] and nicely explained by Huybrechts and Lehn [1997]. This will be applied in Sections 5 and 6.

Throughout, we let R denote an arbitrary DVR over $\mathbb{C}$ with generic point η, closed point ξ, fraction field K and uniformizer π.

4.3.1 We consider the following general (although slightly imprecise) setup.

Let $\mathfrak{M}_\alpha$ be a moduli stack of objects in an abelian category $\mathcal{A}$ with a given numerical class α. We let τ be a weak stability condition on $\mathcal{A}$, and let $\mathfrak{M}_\alpha^{\mathrm{st}}(\tau) \subseteq \mathfrak{M}_\alpha^{\mathrm{sst}}(\tau) \subseteq \mathfrak{M}_\alpha^{\mathrm{pl}}$ be the moduli stacks of stable and semistable objects. We assume that $\mathcal{A}$ is noetherian and τ*-artinian* [Joyce 2021, 3.1] and that $\mathfrak{M}_\alpha^{\mathrm{sst}}$ is of finite type.

Remark For us, we will take the spaces $\mathfrak{M}_{\alpha,\boldsymbol{d}}^{\circ,Q}$ parametrizing pairs with quiver representations of Section 4.1.3, as well as some variants of them.

4.3.2 Quite generally, one has the following consequences of the properties of stable objects:

Lemma *The stack $\mathfrak{M}_\alpha^{\mathrm{st}}(\tau)$ is a separated algebraic space.*

Proof The automorphism group of a stable object is given by the scaling automorphisms, which are removed by rigidification. To see separatedness, let $\mathcal{E}_R$ and $\mathcal{E}'_R$ be two families of stable objects over the DVR R and $f\colon \mathcal{E}_\eta \to \mathcal{E}'_\eta$ an isomorphism over the generic point. Then there exists a unique integer k such that $\pi^k f$ extends to an morphism over Spec R whose restriction to the special fiber is nonzero. In particular, $\pi^k f$ induces a nonzero morphism $\mathcal{E}_\xi \to \mathcal{E}'_\xi$ of stable objects over the closed point of R. Since any morphism of stable objects in the same numerical class is an isomorphism, this shows that $\pi^k f$ extends to an isomorphism of the families $\mathcal{E}_R$.

Now suppose we are given two R-valued points x_R, y_R of $\mathfrak{M}_\alpha^{\mathrm{st}}(\tau)$, and an isomorphism $[f]\colon x_\eta \to y_\eta$. We want to check that $[f]$ extends uniquely to an isomorphism $x_R \to y_R$. In general the map $\mathfrak{M}_\alpha \to \mathfrak{M}_\alpha^{\mathrm{pl}}$ is not a split gerbe, so x_η, y_η may not be genuine families. This is remedied by passing to an étale extension of R, *after which* we may assume that x_R and y_R are represented by genuine families of stable objects $\mathcal{E}_R$ and $\mathcal{E}'_R$, and that $[f]$ is represented by a generic isomorphism $f\colon \mathcal{E}_\eta \to \mathcal{E}'_\eta$. By what we said above, there is a unique k for which $\pi^k f$ extends to an isomorphism $\mathcal{E}_R \to \mathcal{E}'_R$ (again uniquely). Passing to the projective linear stack exactly has the effect of identifying the generic isomorphisms f and $\pi^k f$. Thus $[\pi^k f]$ gives the desired extension of $[f]$ to an isomorphism over R. Note that we replaced R by some étale extension in order to extend $[f]$. By uniqueness of the extension, we obtain one over the original R by étale descent. □

4.3.3 We recall the notion of elementary modification originated in [Langton 1975]. We will use the following definition, which is somewhat imprecise in general, but makes sense when working with some abelian category of quasicoherent sheaves or complexes with extra structure.

Let $\mathscr{A}$ be an abelian category of sheaves or complexes, possibly with extra structure. Suppose we are given a family of objects A_R over a DVR R. Here, we always require a family to be R-flat, in particular multiplication by the uniformizer π should be injective. Let $j\colon\xi \to \operatorname{Spec} R$ denote the inclusion of the closed point.

Definition Let A_ξ denote the restriction of A_R to the closed fiber. Let $0 \to B \to A_\xi \to C \to 0$ be an exact sequence in $\mathcal{A}$, defined either by the subobject B or the quotient C of A_ξ. Then we call the family of objects A'_R, obtained as the cocone of the composition

$$A_R \to j_* A_\xi \to j_* C,$$

the *elementary modification of* A_R *along* B *or* C, respectively.

4.3.4 We describe a strategy to check the existence part of the valuative criterion for properness for $\mathfrak{M}_\alpha^{\mathrm{sst}}$, and we record the assumptions under which this strategy works. Let $\mathcal{E}_K$ be a semistable object of $\mathfrak{M}_\alpha$ over $\operatorname{Spec} K$. We make a first assumption, which roughly states that the moduli stack of objects in $\mathscr{A}$ satisfies the existence part of the valuative criterion.

(A0) There is an extension of $\mathcal{E}_K$ to a family $\mathcal{E}_R^{(0)}$ in $\mathfrak{M}_\alpha$, possibly after a base change on R and possibly with additionally desirable properties.

Now, suppose one has a family $\mathcal{E}_R^{(n)}$ extending $\mathcal{E}_K$ over R. If the central fiber $\mathcal{E}_\xi^{(n)}$ is semistable, we are finished. If not, by [Joyce 2021, Theorem 3.3] there exists a Harder–Narasimhan filtration of $\mathcal{E}_\xi^{(n)}$

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{L-1} \subset F_L = \mathcal{E}_\xi^{(n)}$$

such that the subquotients $\bar{F}_i := F_i/F_{i-1}$ are semistable for $1 \le i \le L$, with $\tau(\bar{F}_1) > \cdots > \tau(\bar{F}_L)$. Let $\tau_{\max}^n := \tau_{\max}(\mathcal{E}_\xi^{(n)}) := \tau(\bar{F}_1)$ and $\tau_{\min}^n := \tau_{\min}(\mathcal{E}_\xi^{(n)}) := \tau(\bar{F}_L)$. Then $\tau_{\max}^n \ge \tau(\alpha) = \tau(\mathcal{E}_\xi^{(n)}) \ge \tau_{\min}^n$, and at least one of the inequalities is strict.

If $\tau_{\max}^n > \tau(\alpha)$, let $G^{(n)} := \mathcal{E}_\xi^{(n)}/F_1$. Otherwise, let $G^{(n)} := \mathcal{E}_\xi^{(n)}/F_{L-1}$. Let $\mathcal{E}_R^{(n+1)}$ be the elementary modification along $G^{(n)}$. In other words, this is either the elementary modification along the maximal destabilizing subobject of $\mathcal{E}_\xi^{(n)}$, or along a maximally destabilizing quotient. Note that we have the quotient map $\mathcal{E}_\xi^{(n)} \to G^{(n)}$, and, since $\pi\mathcal{E}_R^{(n)} \subseteq \mathcal{E}_R^{(n+1)}$, also a natural map

$$G^{(n)} = \mathcal{E}_R^{(n)}/\mathcal{E}_R^{(n+1)} \simeq \pi\mathcal{E}_R^{(n)}/\pi\mathcal{E}_R^{(n+1)} \to \mathcal{E}_R^{(n+1)}/\pi\mathcal{E}_R^{(n+1)} \simeq \mathcal{E}_\xi^{(n+1)}.$$

This inductively defines a sequence of objects $\mathcal{E}_R^{(n)}$ with restrictions $\mathcal{E}_\xi^{(n)}$. We also let $\varphi^{(n)}\colon G^{(n)} \to G^{(n+1)}$ be the composition $G^{(n)} \to \mathcal{E}_\xi^{(n+1)} \to G^{(n+1)}$. One checks the following.

Lemma *Unless $\mathcal{E}_\xi^{(n)}$ is already semistable, we have $\tau_{\max}^n \geq \tau_{\max}^{n+1}$ and $\tau_{\min}^n \leq \tau_{\min}^{n+1}$.*

It follows that we are in exactly one of the following situations:

(i) $[\tau_{\min}^{n+1}, \tau_{\max}^{n+1}] \subsetneq [\tau_{\min}^n, \tau_{\max}^n]$.

(ii) $[\tau_{\min}^{n+1}, \tau_{\max}^{n+1}] = [\tau_{\min}^n, \tau_{\max}^n]$ and $\varphi^{(n)}\colon G^{(n)} \to G^{(n+1)}$ *is not* an isomorphism, i.e., it has a nontrivial kernel or cokernel.

(iii) $[\tau_{\min}^{n+1}, \tau_{\max}^{n+1}] = [\tau_{\min}^n, \tau_{\max}^n]$ and $\varphi^{(n)}\colon G^{(n)} \to G^{(n+1)}$ *is* an isomorphism.

Consequently, one needs to check the following assumptions for the existence part of the valuative criterion to hold:

(A1) There is no infinite sequence of consecutive steps in which situation (iii) occurs.

(A2) There is no infinite sequence of consecutive steps in which either only situation (ii) or (iii) occur, and in which (ii) occurs infinitely many times.

(A3) The ascending sequence of values $\tau_{\min}^n$ has to stabilize. So does the descending sequence $\tau_{\max}^n$.

4.3.5 Remark (a) In practice, assumption (A3) is often automatically satisfied, e.g., if the set of possible values that τ takes in the interval $[\tau_{\min}^{(0)}, \tau_{\max}^{(0)}] = [\tau_{\min}(\mathcal{E}_\xi), \tau_{\max}(\mathcal{E}_\xi)]$ when evaluated on classes which appear in effective decompositions of α is a priori finite.

(b) When objects in $\mathcal{A}$ consist of coherent sheaves and maps between them on a *projective* (or proper) variety, assumption (A0) can often be reduced to standard extension theorems for coherent sheaves.

(c) Similarly, assumption (A1) is often automatic when working with a moduli space parametrizing configurations of coherent sheaves as we do here. The reason (exactly as in [Huybrechts and Lehn 1995, Theorem 2.B.1]) is that an infinite sequence of steps (iii) gives rise to a section of a relative Quot-scheme in a formal neighborhood of ξ, which can be algebraized, contradicting the semistability of the generic fiber.

(d) If τ is a (nonweak) stability condition, then (A2) holds. This is not clear if one has only a weak stability condition, and requires an additional argument.

4.3.6 Remark For us, we only want to show properness of T-fixed loci of moduli spaces $\mathfrak{N}_\alpha^{\mathrm{sst}}$ when there are no properly semistable objects. We also have an open embedding $\mathfrak{N}_\alpha^{\mathrm{sst}} \subseteq \mathfrak{M}_\alpha^{\mathrm{sst}}$ (cf. Definition 3.1.7). In our situation, it is clear that when semistability equals stability for α (so that $\mathfrak{M}_\alpha$ is a separated algebraic space), then the inclusion on T-fixed loci $(\mathfrak{N}_\alpha^{\mathrm{sst}})^{\mathsf{T}} \subseteq (\mathfrak{M}_\alpha^{\mathrm{sst}})^{\mathsf{T}}$ is an inclusion of connected components. Thus, in order to establish properness of T-fixed loci, it suffices to show properness of $\mathfrak{M}_\alpha$ using the strategy of this subsection.

5 DT/PT via Mochizuki-style wall-crossing

5.1 Flag-framed invariants

5.1.1 In this section, we give a proof of the K-theoretic DT/PT vertex correspondence, using a somewhat ad hoc wall-crossing setup similar to the one used in [Nakajima and Yoshioka 2011] and [Kuhn and Tanaka 2021]. Since this sort of setup ultimately stems from work of Mochizuki [2009], we refer to it as *Mochizuki-style* wall-crossing. In contrast, in Section 6, we provide a different proof of the K-theoretic DT/PT vertex correspondence using a vastly more general and complicated wall-crossing setup due to Joyce. Both proofs crucially require the techniques developed in Section 2. We also give a formulation of the Mochizuki-style approach in terms of a weak stability condition on the quiver which, as far as we know, is new.

5.1.2 Throughout this section, fix the integer partitions (λ, μ, ν), and hence the class $\beta_C = (|\lambda|, |\mu|, |\nu|)$, and fix a class $\alpha = (1, -\beta_C, -n)$, where $n \in \mathbb{Z}$. Since (λ, μ, ν) will be fixed throughout this section, it is convenient to omit them from subscripts, e.g., denote by

$$\mathfrak{N}_m := \mathfrak{N}_{(\lambda,\mu,\nu),m}$$

the moduli stack parametrizing pairs $[\mathcal{O}_{\overline{X}} \otimes L \to \mathcal{E}]$ with prescribed restrictions to the boundary divisors. For short, let

$$\mathfrak{N}_m^{\mathrm{DT}} := \mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),m}(\tau^-) \quad \text{and} \quad \mathfrak{N}_m^{\mathrm{PT}} := \mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),m}(\tau^+)$$

denote the moduli stacks parametrizing PT and DT-stable pairs, respectively, where $\tau^{\pm}$ are as in Lemma 3.2.3.

Fix $k \gg 0$ large enough (depending on α) as described in Definition 4.1.2, and $p \geq 1$ arbitrary. The choice of p is unimportant in this section; it will only matter in Section 6. Let $N := f_{k,p}(\alpha)$. We will consider the quiver framed stacks $\mathfrak{N}^{Q(N)}_{m,\boldsymbol{d}}$ and $\mathfrak{Q}^{Q(N)}_{m,\boldsymbol{d}}$ of Definition 4.1.2 for various $m \in \mathbb{Z}$. The wall-crossing will begin with a framing dimension vector $\boldsymbol{d} = (d_i)_{i=1}^N$ such that $(\alpha, \boldsymbol{d})$ is a full flag, in the sense of Definition 4.2.1.

5.1.3 We will set up the wall-crossing step such that the full flag $(\alpha, \boldsymbol{d})$ splits into smaller full flags. Hence, for the remainder of this subsection, fix a class $\beta = (1, -\beta_C, -m)$ with $m \leq n$, and a dimension vector $\boldsymbol{e} \leq \boldsymbol{d}$ such that $(\beta, \boldsymbol{e})$ is a full flag. Let $M := f_{k,p}(\beta)$, and for $\ell \in \{0, 1, \dots, M\}$ let $0 \leq i_{\boldsymbol{e}}(\ell) \leq N$ be the minimal index for which $e_{i_{\boldsymbol{e}}(\ell)} = \ell$, with the convention that $V_0 = 0$ and $e_0 = 0$. Finally, let

$$\mathfrak{N}^{\mathrm{fl,DT}}_{m,\boldsymbol{e}}, \ \mathfrak{N}^{\mathrm{fl,PT}}_{m,\boldsymbol{e}} \subset \mathfrak{N}^{\mathrm{fl}}_{m,\boldsymbol{e}}$$

denote the preimages of $\mathfrak{N}_m^{\mathrm{DT}}$ and $\mathfrak{N}_m^{\mathrm{PT}}$ under the forgetful projection $\Pi_{\mathfrak{N}}\colon \mathfrak{N}^{\mathrm{fl}}_{m,\boldsymbol{e}} \to \mathfrak{N}^{\mathrm{pl}}_m$. Objects in these stacks are triples $(I, \boldsymbol{V}, \boldsymbol{\rho})$ where every map in $\boldsymbol{\rho} = (\rho_i)_i$ is injective, and $I = [\mathcal{O}_{\overline{X}} \to \mathcal{E}]$ is a DT- or PT-stable pair.

5.1.4 We introduce stability conditions on the stacks $\mathfrak{N}^{\mathrm{fl}}_{m,e}$.

Definition Let $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e} \subseteq \mathfrak{N}^{\mathrm{fl}}_{m,e}$ be the open substack of objects (I, V, ρ) satisfying:

(i) for a nonzero injection of pairs $[0 \to \mathcal{Z}] \hookrightarrow I$ with $\mathcal{Z}$ zero-dimensional, we have

$$V_{i_e(\ell)} \cap F_{k,p}(\mathcal{Z}) = \{0\} \subset F_{k,p}(I),$$

where we identify $V_{i_e(\ell)}$ with its image in $F_{k,p}(I)$;

(ii) for a nonzero surjection of pairs $I \twoheadrightarrow [0 \to \mathcal{Z}]$ with $\mathcal{Z}$ zero-dimensional, the composition

$$V_{i_e(\ell)} \to F_{k,p}(I) \twoheadrightarrow F_{k,p}(\mathcal{Z})$$

is nonzero.

We say the objects in $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$ are *ℓ-stable*. The name is justified by Proposition 5.1.5. By construction,

$$\mathfrak{N}^{\mathrm{fl},0\text{-st}}_{m,e} = \mathfrak{N}^{\mathrm{fl},\mathrm{DT}}_{m,e} \quad \text{and} \quad \mathfrak{N}^{\mathrm{fl},M\text{-st}}_{m,e} = \mathfrak{N}^{\mathrm{fl},\mathrm{PT}}_{m,e}.$$

5.1.5 Proposition *The stacks $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$ are separated algebraic spaces with proper T-fixed loci.*

Consequently, Theorem 2.3.4 endows each ℓ-stable locus with a symmetric APOT, by symmetrized pullback along the forgetful map to $\mathfrak{N}_m$.

Proof Let $(I = [\mathcal{O}_{\overline{X}} \xrightarrow{s} \mathcal{E}], V, \rho)$ be an object of $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$. We first show that it has no nontrivial automorphisms.

Consider the zero-dimensional sheaf $\mathcal{Y} := \operatorname{coker} s$, and let $\mathcal{Z} \subset \mathcal{E}$ be the largest zero-dimensional subsheaf. Since we are on the rigidified stack, any automorphism φ of the pair I induces the identity on $\mathcal{O}_{\overline{X}}$ (see the proof of Lemma 3.1.6), and therefore is of the form $\mathrm{id}_I + \delta_1[-1]$, where $\delta_1\colon \mathcal{E} \to \mathcal{E}$ factors through a map $\mathcal{Y} \to \mathcal{Z}$ via the natural compositions. In particular, $\mathrm{im}(\delta_1)$ is a zero-dimensional sheaf.

Suppose that $\mathrm{im}\,\delta_1 \neq 0$, so that $I \to [0 \to \mathrm{im}\,\delta_1]$ is a nonzero morphism. By Definition 5.1.4(ii), the image of $V_{i_e(\ell)}$ in $F_{k,p}(\mathrm{im}(\delta_1))$ is nonzero. But the automorphism of $F_{k,p}(I)$ induced by φ must respect the full flag (V, ρ), so in particular $V_{i_e(\ell)}$ is mapped to itself. This contradicts Definition 5.1.4(i). Hence $\delta_1 = 0$ and therefore $\varphi = \mathrm{id}_I$ is a trivial automorphism. Since every object has only trivial automorphisms, we see that $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$ is an algebraic space.

The statements about separatedness and properness can be checked via the strategy of Section 4.3.4 using the description via weak stability conditions in Lemma 5.1.8. This is the content of Proposition 5.1.9, in view of Remark 4.3.6. □

5.1.6 The spaces $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$ for varying ℓ are connected by spaces of "semistable" objects, as follows.

Definition For $\ell \in \{0, 1, \dots, M-1\}$, let $\mathfrak{N}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}_{m,e} \subseteq \mathfrak{N}^{\mathrm{fl}}_{m,e}$ denote the open substack parametrizing objects (I, V, ρ) satisfying:

(i) For a nonzero injection of pairs $[0 \to \mathcal{Z}] \hookrightarrow I$ with $\mathcal{Z}$ zero-dimensional, we have

$$V_{i_e(\ell)} \cap F_{k,p}(\mathcal{Z}) = \{0\} \subset F_{k,p}(I).$$

(ii) For a nonzero surjection of pairs $I \twoheadrightarrow [0 \to \mathcal{Z}]$ with $\mathcal{Z}$ zero-dimensional, the composition

$$V_{i_e(\ell+1)} \to F_{k,p}(I) \twoheadrightarrow F_{k,p}(\mathcal{Z})$$

is nonzero.

In other words, (i) imposes the corresponding condition from $\mathfrak{N}_{m,e}^{\mathrm{fl},\ell\text{-st}}$, and (ii) the condition from $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell+1)\text{-st}}$. This leads to a stability condition which is weaker than both ℓ- and $(\ell+1)$-stability. Consequently, the stack $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}$ contains both $\mathfrak{N}_{m,e}^{\mathrm{fl},\ell\text{-st}}$ and $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell+1)\text{-st}}$ as open substacks. We say the objects in $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}$ are $(\ell,\ell+1)$-*semistable*.

5.1.7 Through the stacks $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}$, we have reduced to a simple wall-crossing, meaning that semistable objects split into at most two pieces and the stabilizer dimension of semistable objects is at most one. Recall the locus $\mathfrak{Q}_{m,e}^{\mathrm{fl,st}}$ from Section 4.2.3.

Lemma *An object (I,V,ρ) in $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}$ has nontrivial automorphisms if and only if*

$$(I,V,\rho) = (I',V',\rho') \oplus (\mathcal{Z}[-1],V'',\rho''),$$

where, for some full flag $((1,-\beta_C,-m'),e')$ such that the first nonzero entry in $e-e'$ occurs at $i_e(\ell+1)$,

$$[(I',V',\rho')] \in \mathfrak{N}_{m',e'}^{\mathrm{fl},\ell\text{-st}} \quad \text{and} \quad (\mathcal{Z}[-1],V'',\rho'') \in \mathfrak{Q}_{m-m',e-e'}^{\mathrm{fl,st}}.$$

Each such "strictly polystable" object has stabilizer group $\mathbb{C}^\times$ given by scaling $\mathcal{Z}$.

Proof It is straightforward to check using the definition of ℓ-stability that any direct sum of the indicated form has automorphisms given by the scaling action on $\mathcal{Z}$.

The only thing left to prove is that given an object (I,V,ρ) with nontrivial automorphism group, it is of the claimed form. Indeed, let $\varphi\colon I \to I$ be a nontrivial automorphism compatible with the flag structure, and let $\delta := \varphi - \mathrm{id}_I$. We claim that this induces the desired decomposition of I with

$$\begin{aligned} I' &:= \ker\delta, & \mathcal{Z}[-1] &:= \operatorname{im}\delta, \\ V_i' &:= V_i \cap F_{k,p}(I'), & V_i'' &:= \operatorname{im}(V_i \to F_{k,p}(I) \to F_{k,p}(\mathcal{Z})), \end{aligned}$$

and the obvious induced quiver maps ρ' and ρ''. We showed in the proof of Proposition 5.1.5 that $\mathcal{Z}$ has zero-dimensional support. One then concludes that

- the composition $V_{i_e(\ell+1)} \to F_{k,p}(I) \twoheadrightarrow F_{k,p}(\mathcal{Z})$ is nonzero (by Definition 5.1.6(ii));
- $V_{i_e(\ell)}$ factors through $F_{k,p}(I')$ (by Definition 5.1.6(i));
- $V_{i_e(\ell+1)} \cap F_{k,p}(\mathcal{Z}) \neq 0$ (since $\delta(V_{i_e(\ell+1)}) \subseteq F_{k,p}(\mathcal{Z})$).

Thus, $W := V_{i_e(\ell+1)} \cap F_{k,p}(\mathcal{Z})$ is one-dimensional, and δ induces an isomorphism on W which is scaling by some $t \in \mathrm{Aut}(W) = \mathbb{C}^\times$. Since t is a nonzero element of $\mathrm{End}(W)$ and φ was arbitrary, the stabilizer group of (I,V,ρ) injects into $\mathrm{End}(W)$. Now δ/t is an idempotent in $\mathrm{Aut}(W)$, so δ/t must be an idempotent projection onto $\mathcal{Z}$ which gives the desired splitting. To conclude:

- $(\mathcal{Z}[-1], \boldsymbol{V}'', \boldsymbol{\rho}'')$ is stable in the sense of Section 4.2.3, namely the composition

$$V_{i_e(\ell+1)}/V_{i_e(\ell)} \otimes \mathcal{O}_{\overline{X}} \to V_{i_e(\ell+1)+1}/V_{i_e(\ell)} \otimes \mathcal{O}_{\overline{X}} \to \cdots \to \mathcal{Z}$$

is surjective (by applying Definition 5.1.6(ii) to quotients of $\mathcal{Z}$).

- $(I', \boldsymbol{V}', \boldsymbol{\rho}')$ is ℓ-stable (by comparing Definitions 5.1.4 and 5.1.6, using the splitting). □

5.1.8 We can characterize ℓ- and $(\ell, \ell+1)$-stability as weak stability conditions. Given a nonzero class $(\gamma, \boldsymbol{f})$ which is a full flag, let $i_{\min}(\boldsymbol{f})$ be the minimal index for which $f_i > 0$. Recall the definition of the weak stability condition τ_0 from Section 3.2.1 and Lemma 3.2.3. For a parameter $\mu \in \mathbb{R}$, define

$$\tau_\mu^{\mathrm{fl}}(\gamma, \boldsymbol{f}) := \begin{cases} (\tau_0(\gamma), \mu - i_{\min}(\boldsymbol{f})) & \text{if } \gamma = (0, 0, -\star), \\ (\tau_0(\gamma), 0) & \text{otherwise}, \end{cases}$$

where $\star$ denotes a positive integer. One checks that τ_μ^{fl} is a weak stability condition for any $\mu \in \mathbb{R}$, using that $i_{\min}(\boldsymbol{f}' + \boldsymbol{f}'') = \min(i_{\min}(\boldsymbol{f}'), i_{\min}(\boldsymbol{f}''))$. The following is straightforward to verify from the definitions.

Lemma *Let $(I, \boldsymbol{V}, \boldsymbol{\rho})$ be an object of $\mathfrak{N}_{m,\boldsymbol{e}}^{\mathrm{fl}}$.*

(i) *Let $0 \le \ell \le M$. Choose $i_{\boldsymbol{e}}(\ell) < \mu < i_{\boldsymbol{e}}(\ell+1)$. Then $(I, \boldsymbol{V}, \boldsymbol{\rho})$ is τ_μ^{fl}-stable if and only if it is ℓ-stable.*

(ii) *Let $0 \le \ell \le M-1$. Choose $\mu = i_{\boldsymbol{e}}(\ell+1)$. Then $(I, \boldsymbol{V}, \boldsymbol{\rho})$ is τ_μ^{fl}-stable if and only if it is $(\ell, \ell+1)$-stable.*

5.1.9 From Remark 4.3.6, properness of the T-fixed loci in the (semi)stable loci of $\mathfrak{N}^{\mathrm{fl}}$ follows immediately from properness of the same (semi)stable loci in the ambient stack $\mathfrak{M}^{\circ,\mathrm{fl}}$; see Section 3.1.1. Recall that $\mathfrak{M}^{\circ}$ is the analogue of $\mathfrak{N}$ except that one does not impose any conditions on the restriction of objects to the boundary of $\overline{X}$. Definitions 5.1.4 and 5.1.6 apply equally well to $\mathfrak{M}^{\circ,\mathrm{fl}}$.

Proposition *The stacks $\mathfrak{M}_{\beta,\boldsymbol{e}}^{\circ,\mathrm{fl},\ell\text{-sst}}$ and $\mathfrak{M}_{\beta,\boldsymbol{e}}^{\circ,\mathrm{fl},(\ell,\ell+1)\text{-sst}}$ satisfy the existence part of the valuative criterion of properness. In particular, the stacks $\mathfrak{M}_{\beta,\boldsymbol{e}}^{\circ,\mathrm{fl},\ell\text{-sst}}$ are proper.*

Proof Apply Lemma 5.1.8 to express ℓ- and $(\ell, \ell+1)$-(semi)stability using a weak stability condition τ_μ^{fl} for some μ, and follow the general strategy outlined in Section 4.3.4.

We first note that for any object in $\mathfrak{M}_{\beta,\boldsymbol{e}}^{\circ,\mathrm{fl}}$, the subquotients in a Harder–Narasimhan filtration with respect to τ_μ^{fl} have numerical classes which are again full flags, and lie in either the τ_μ^{fl}-semistable locus of $\mathfrak{M}_{\gamma,\boldsymbol{f}}^{\circ,\mathrm{fl}}$ or in $\mathfrak{Q}_{m',\boldsymbol{f}}^{\mathrm{fl,st}}$.

Suppose that $(I_K, \boldsymbol{V}_K, \boldsymbol{\rho}_K)$ is a one-parameter family of ℓ- or $(\ell, \ell+1)$-(semi)stable objects over the generic point Spec K of a DVR R. By Lemma 3.2.5, we can extend I_K to a family of objects I_R in $\mathfrak{M}_\beta^{\circ,\mathrm{sst}}(\tau_0) = \mathfrak{M}_\beta^{\circ,\mathrm{pl}}$ over R. The quiver data is equivalent to a full flag in $F_{k,p}(I_K)$, which we can extend to a full flag in $F_{k,p}(I_R)$. This guarantees that the slopes of a Harder–Narasimhan filtration of $(I_\xi, \boldsymbol{V}_\xi, \boldsymbol{\rho}_\xi)$ are of the form $(\tau_0(\beta), a)$, for $a \in \{\mu - i\}_{0 \le i \le N} \cup \{0\}$. In particular, we only need to check assumption (A2) in Section 4.3.4; (A1) and (A3) are automatic.

Thus, suppose we have constructed a sequence $(I_R^{(n)}, V_R^{(n)}, \rho_R^{(n)})$ as in Section 4.3.4. Say, to go from n to $n+1$, we are doing an elementary modification along a (semi)stable subobject

$$(\mathcal{Z}[-1], W, \sigma) \subset (I_\xi^{(n)}, V_\xi^{(n)}, \rho_\xi^{(n)}) \quad \text{such that} \quad \tau_\mu^{\mathrm{fl}}((\mathcal{Z}[-1], W, \sigma)) > (\tau_0(\beta), 0),$$

with $\mathcal{Z}$ zero-dimensional, and say that we are in situation (ii), i.e., the maximally destabilizing subobject $(\mathcal{Z}'[-1], W', \sigma')$ of $(I_\xi^{(n+1)}, V_\xi^{(n+1)}, \rho_\xi^{(n+1)})$ has the same slope as $(\mathcal{Z}[-1], W, \sigma)$. By the definition of τ_μ, if i_0 is the minimal index such that $W_{i_0} \neq 0$, it must also be the minimal index i such that $W'_i \neq 0$. In particular, W_{i_0} and W'_{i_0} both have rank one. The induced map $(\mathcal{Z}'[-1], W', \sigma') \to (\mathcal{Z}[-1], W, \sigma)$ induces an isomorphism $W'_{i_0} \to W_{i_0}$. Hence:

- The map $\mathcal{Z}' \to \mathcal{Z}$ must be surjective, since the induced maps $W_{i_0} \otimes \mathcal{O}_{\overline{X}} \to \mathcal{Z}$ and $W'_{i_0} \otimes \mathcal{O}_{\overline{X}} \to \mathcal{Z}'$ are both surjective by (semi)stability.
- If $\mathcal{Z}' \to \mathcal{Z}$ is an isomorphism, then all induced maps $W'_i \to W_i$ are injective.

Consequently, exactly one of two possibilities occurs:

- The map $\mathcal{Z}' \to \mathcal{Z}$ has nontrivial kernel, and $\operatorname{length}(\mathcal{Z}') > \operatorname{length}(\mathcal{Z})$.
- The map $\mathcal{Z}' \to \mathcal{Z}$ is an isomorphism, and there exists $i > i_0$ such that $\dim W'_i < \dim W_i$.

Hence the quantity $\operatorname{length}(\mathcal{Z}) - \sum_i \dim W_i$ increases at each such step. But it is a priori bounded above by N, hence such a situation can only occur finitely many times, proving (A2) in this case. The case of a quotient object with slope lower than $(\tau_0(\beta), 0)$ is analogous. □

5.2 Wall-crossing formula

5.2.1 Recall that we fixed a full flag (β, e) with $\beta = (1, -\beta_C, -m)$, and let $M := f_{k,p}(\beta)$. Fix $0 \leq \ell \leq M-1$. We construct a master space $\mathbb{M}^{\ell,\ell+1}$ relating $\mathfrak{N}_{m,e}^{\mathrm{fl},\ell\text{-st}}$ and $\mathfrak{N}_{m,e}^{\mathrm{fl},(\ell+1)\text{-st}}$.

Definition For a full flag (γ, f) with $\gamma = (1, -\beta_C, -m')$, and an integer $f_{-1} \geq 0$, consider the (unrigidified) moduli stack parametrizing tuples $(I, (\widehat{V}_{-1}, V), (\widehat{\rho}_{-1}, \widehat{\rho}_0, \rho))$, where:

- (I, V, ρ) is an object of the unrigidified stack $\mathfrak{N}_{m',f}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}} \times [\mathrm{pt}/\mathbb{C}^\times]$; see Lemma 3.1.6.
- $\widehat{V}_{-1}$ is an f_{-1}-dimensional vector space.
- $\widehat{\rho}_{-1}$ and $\widehat{\rho}_0$ are linear maps

$$V_{i_e(\ell+1)}/V_{i_e(\ell)} \xrightarrow{\widehat{\rho}_{-1}} \widehat{V}_{-1} \xleftarrow{\widehat{\rho}_0} L,$$

  where $I = [\mathcal{O}_{\overline{X}} \otimes L \to \mathcal{E}]$.

The group $\mathbb{C}^\times$ of scaling automorphisms acts diagonally on I, V and $\widehat{V}_{-1}$. Write (β, f, f_{-1}) for the numerical class of $(I, (\widehat{V}_{-1}, V), (\widehat{\rho}_{-1}, \widehat{\rho}_0, \rho))$, where $\beta := \operatorname{cl} I$ and $f := \dim V$.

Let $\widehat{\mathfrak{N}}^{\mathrm{fl},\ell,\ell+1}_{m',\boldsymbol{f},f_{-1}}$ be the $\mathbb{C}^\times$-rigidified moduli stack, and let $\mathbb{M}^{\ell,\ell+1} \subset \widehat{\mathfrak{N}}^{\mathrm{fl},\ell,\ell+1}_{m,\boldsymbol{e},1}$ denote the open substack of objects satisfying the following conditions:

(i) If $\hat{\rho}_{-1} = 0$, then $(I, \boldsymbol{V}, \boldsymbol{\rho})$ belongs to $\mathfrak{N}^{\mathrm{fl},(\ell+1)\text{-st}}_{m,\boldsymbol{e}}$.

(ii) If $\hat{\rho}_0 = 0$, then $(I, \boldsymbol{V}, \boldsymbol{\rho})$ belongs to $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,\boldsymbol{e}}$.

(iii) At least one of $\hat{\rho}_{-1}$ and $\hat{\rho}_0$ is nonzero.

In the literature, this construction of $\mathbb{M}^{\ell,\ell+1}$ is also called an *enhanced master space* [Mochizuki 2009, Sections 1.6.3 and 4.3].

5.2.2 Here is an alternative description of $\mathbb{M}^{\ell,\ell+1}$. Let $\mathcal{L} := \mathcal{V}_{i_e(\ell+1)}/\mathcal{V}_{i_e(\ell)}$ on $\mathfrak{N}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}_{m,\boldsymbol{e}}$, and let $\mathcal{L}^\times$ denote the associated principal $\mathbb{C}^\times$-bundle. The line bundle $\mathcal{L}$ has the following useful properties.

(i) The stabilizer groups at the stacky points of $\mathfrak{N}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}_{m,\boldsymbol{e}}$ act nontrivially, in fact freely on $\mathcal{L}^\times$. In particular, the total space of $\mathcal{L}^\times$ is a Deligne–Mumford stack (in fact an algebraic space).

(ii) Let x be a point in $\mathcal{L}^\times$ lying over an object $(I, \boldsymbol{V}, \boldsymbol{\rho})$, normalized so that $I = [\mathcal{O}_{\overline{X}} \to \mathcal{E}]$. Then $\lim_{t\to 0} tx$ exists if and only if there exists a zero-dimensional quotient $\mathcal{E} \twoheadrightarrow \mathcal{Z}$ such that $V_{i_e(\ell)}$ lies in the kernel of the map $V_{i_e(\ell+1)} \to F_{k,p}(\mathcal{Z})$, i.e., if and only if the pair $(I, \boldsymbol{V}, \boldsymbol{\rho})$ does *not* lie in $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,\boldsymbol{e}}$.

(iii) Similarly, $\lim_{t\to\infty} tx$ exists if and only if $(I, \boldsymbol{V}, \boldsymbol{\rho})$ does not lie in $\mathfrak{N}^{\mathrm{fl},(\ell+1)\text{-st}}_{m,\boldsymbol{e}}$.

Then $\mathbb{M}^{\ell,\ell+1}$ is constructed from $\mathcal{L}^\times$ by gluing in the total space of $\mathcal{L}|_{\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,\boldsymbol{e}}}$ and the total space of $\mathcal{L}^\vee|_{\mathfrak{N}^{\mathrm{fl},(\ell+1)\text{-st}}_{m,\boldsymbol{e}}}$ to $\mathcal{L}^\times$, thereby adding in the missing limits of tx for $t \to 0$ and for $t \to \infty$, respectively.

Via this description, we see that $\mathbb{M}^{\ell,\ell+1}$ is smooth over $\mathfrak{N}^{\mathrm{fl},(\ell,\ell+1)\text{-sst}}_{m,\boldsymbol{e}}$ and hence over $\mathfrak{N}_m$.

5.2.3 Here is yet another description of $\mathbb{M}^{\ell,\ell+1}$ in terms of a weak stability condition, building on Section 5.1.8. For objects in the moduli stack $\widehat{\mathfrak{N}}^{\mathrm{fl},\ell,\ell+1}$ from Definition 5.2.1, define

$$\hat{\tau}^{\mathrm{fl}}_{i_e(\ell+1)}(\gamma, \boldsymbol{f}, f_{-1}) := \begin{cases} ((-\infty,-\infty),0) & \text{if } \gamma = 0,\ \boldsymbol{f} = \boldsymbol{0} \text{ and } f_{-1} > 0, \\ \left(\tau^{\mathrm{fl}}_{i_e(\ell+1)}(\gamma, \boldsymbol{f}), \tfrac{1}{2} - f_{-1}\right) & \text{if } \gamma = (0,0,-\star) \text{ and } i_{\min}(\boldsymbol{f}) = i_{\boldsymbol{e}}(\ell+1), \\ (\tau^{\mathrm{fl}}_{i_e(\ell+1)}(\gamma, \boldsymbol{f}), 0) & \text{otherwise,} \end{cases}$$

where $(\gamma, \boldsymbol{f})$ is a full flag and $\star$ denotes a positive integer. One checks that $\hat{\tau}^{\mathrm{fl}}_{i_e(\ell+1)}$ is a weak stability condition for any $0 \le \ell \le M-1$, using that it refines the weak stability condition $\tau^{\mathrm{fl}}_{i_e(\ell+1)}$ and some casework.

Lemma *The master space $\mathbb{M}^{\ell,\ell+1}$ is canonically isomorphic to the substack of $\widehat{\mathfrak{N}}^{\mathrm{fl},\ell,\ell+1}_{m,\boldsymbol{e},1}$ of $\hat{\tau}^{\mathrm{fl}}_{i_e(\ell+1)}$-stable objects.*

Proof sketch Straightforward to verify from the definitions, using the observations that for any object in class $(\beta, \boldsymbol{e}, 1)$, we have:

- If $\hat{\rho}_{-1} = 0$ and $\hat{\rho}_0 = 0$, then there is a quotient-object of numerical class $(0, \boldsymbol{0}, 1)$.

- If $\widehat{\rho}_0 \neq 0$, then all subobjects of class $(\gamma, \boldsymbol{f}, 0)$ must have $\gamma = (0, 0, -\star)$.
- If $\widehat{\rho}_{-1} \neq 0$, then all subobjects of class $(\gamma, \boldsymbol{f}, 0)$ must have $f_{i_e(\ell+1)} = f_{i_e(\ell)}$. □

5.2.4 In addition to the T-action that it inherits from $\mathfrak{N}_m$, the master space $\mathbb{M}^{\ell,\ell+1}$ has a natural $\mathbb{C}^\times$-action given by scaling $\widehat{\rho}_{-1}$ with weight z^{-1}.

Proposition *The master space $\mathbb{M}^{\ell,\ell+1}$ is a separated algebraic space with an induced $\mathbb{C}^\times \times \mathsf{T}$ action. If w is a $\mathbb{C}^\times \times \mathsf{T}$ weight with nontrivial $\mathbb{C}^\times$ component, the fixed locus for the maximal torus $\mathsf{T}_w \subset \ker(w)$ is proper.*

Proof It follows directly from Lemma 4.3.2 that the master space is a separated algebraic space.

To show properness of T_w-fixed loci, we can argue, like we did in Section 5.1.9, that they are components of the T_w-fixed loci of a proper space $\overline{\mathbb{M}}^{\ell,\ell+1}$ obtained by working with $\mathfrak{M}^\circ$ instead of $\mathfrak{N}$ in the constructions of Definition 5.2.1 and Section 5.2.3. In particular, we can realize $\overline{\mathbb{M}}^{\ell,\ell+1}$ as an open substack of $\widehat{\tau}^{\mathrm{fl}}_{i_e(\ell+1)}$-stable objects in an ambient stack $\widehat{\mathfrak{M}}^{\circ,\mathrm{fl},\ell,\ell+1}_{\beta,\boldsymbol{e},1}$.

It remains to check the valuative criterion of properness for $\overline{\mathbb{M}}^{\ell,\ell+1}$. Note that any family of $\widehat{\tau}^{\mathrm{fl}}_{i_e(\ell+1)}$-stable objects has an underlying family of $(\ell,\ell+1)$-semistable objects. Thus, by Proposition 5.1.9, we can complete a family over the generic point of a DVR to some family of objects in $\widehat{\mathfrak{M}}^{\circ,\mathrm{fl},\ell,\ell+1}_{\beta,\boldsymbol{e},1}$, which yields an $(\ell,\ell+1)$-semistable family under the forgetful map to $\mathfrak{M}^{\circ,\mathrm{fl}}_{\beta,\boldsymbol{e}}$. Applying the strategy of Section 4.3.4 using the description of Section 5.2.3, it remains to show that assumption (A2) holds. This follows in the same way as in Proposition 5.1.9. □

5.2.5 From Section 5.2.2, $\mathbb{M}^{\ell,\ell+1}$ is smooth over $\mathfrak{N}_m$, and Proposition 5.2.4 shows it is a separated algebraic space with proper T-fixed loci. Then Theorem 2.3.4 endows $\mathbb{M}^{\ell,\ell+1}$ with a symmetric APOT by symmetrized pullback along the forgetful map to $\mathfrak{N}_m$. By Theorem 2.4.3 and Lemma 3.3.7, we obtain a symmetrized virtual structure sheaf $\widehat{\mathcal{O}}^{\mathrm{vir}}$ amenable to $(\mathbb{C}^\times \times \mathsf{T})$-equivariant localization.

5.2.6 Proposition *The $\mathbb{C}^\times$-fixed locus of $\mathbb{M}^{\ell,\ell+1}$ is the disjoint union of the following pieces.*

(i) *Let $Z_{\widehat{\rho}_{-1}=0} := \{\widehat{\rho}_{-1} = 0\} \subset \mathbb{M}^{\ell,\ell+1}$. By definition, $\widehat{\rho}_0 \neq 0$. There is a natural isomorphism of stacks*

$$Z_{\widehat{\rho}_{-1}=0} \xrightarrow{\sim} \mathfrak{N}^{\mathrm{fl},(\ell+1)\text{-st}}_{m,\boldsymbol{e}}, \quad [(\boldsymbol{I}, (\widehat{V}_{-1}, \boldsymbol{V}), (0, \widehat{\rho}_0, \boldsymbol{\rho}))] \mapsto [(\boldsymbol{I}, \boldsymbol{V}, \boldsymbol{\rho})],$$

which identifies the $\widehat{\mathcal{O}}^{\mathrm{vir}}$. The virtual normal bundle is $z \otimes \mathcal{L}^\vee$.

(ii) *Let $Z_{\widehat{\rho}_0=0} := \{\widehat{\rho}_0 = 0\} \subset \mathbb{M}^{\ell,\ell+1}$. By definition, $\widehat{\rho}_{-1} \neq 0$. There is a natural isomorphism of stacks*

$$Z_{\widehat{\rho}_0=0} \xrightarrow{\sim} \mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,\boldsymbol{e}}, \quad [(\boldsymbol{I}, (\widehat{V}_{-1}, \boldsymbol{V}), (\widehat{\rho}_{-1}, 0, \boldsymbol{\rho}))] \mapsto [(\boldsymbol{I}, \boldsymbol{V}, \boldsymbol{\rho})],$$

which identifies the $\widehat{\mathcal{O}}^{\mathrm{vir}}$. The virtual normal bundle is $z^{-1} \otimes \mathcal{L}$.

(iii) *For each splitting $(\beta, \boldsymbol{e}) = (\gamma, \boldsymbol{f}) + (\delta, \boldsymbol{g})$ appearing in Lemma 5.1.7, let*

$$Z_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})} := \{[(\boldsymbol{I}' \oplus \boldsymbol{I}'', (\widehat{V}_{-1}, \boldsymbol{V}' \oplus \boldsymbol{V}''), (\widehat{\rho}'_{-1}, \widehat{\rho}'_0, \boldsymbol{\rho}') \oplus (\widehat{\rho}''_{-1}, \widehat{\rho}''_0, \boldsymbol{\rho}''))] : \widehat{\rho}_{-1} \neq 0, \widehat{\rho}_0 \neq 0\} \subset \mathbb{M}^{\ell,\ell+1},$$

where $\mathrm{cl}((I',V',\rho'))=(\gamma,f)$ *and* $\mathrm{cl}((I'',V'',\rho''))=(\delta,g)$*. If* $\delta=(0,0,-m')$*, then there is a natural isomorphism of stacks*

$$Z_{(\gamma,f),(\delta,g)}\xrightarrow{\sim}\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m-m',f}\times\mathfrak{Q}^{\mathrm{st}}_{m',g},\quad [(I,(\hat{V}_{-1},V),(\hat{\rho}_{-1},\hat{\rho}_0,\rho))]\mapsto([(I',V',\rho')],[(I'',V'',\rho'')]),$$

which identifies the $\hat{\mathcal{O}}^{\mathrm{vir}}$*. Under this isomorphism, the virtual normal bundle is*

$$\mathcal{N}^{\mathrm{vir}}_{(\gamma,f),(\delta,g)}=\\ (z^{-1}\mathbb{F}(f,g)+z\mathbb{F}(g,f))-\kappa\cdot(\cdots)^{\vee}-\big(z^{-1}R\pi_*\,\mathcal{E}xt(\mathcal{I}_\gamma,\mathcal{I}_\delta(-D))+zR\pi_*\,\mathcal{E}xt(\mathcal{I}_\delta,\mathcal{I}_\gamma(-D))\big),$$

where $\mathbb{F}(f,g)$ *is the restriction of (40) to numerical class* (f,g)*, and* $\mathcal{I}_\gamma$ *and* $\mathcal{I}_\delta$ *are pullbacks of universal families for numerical classes* γ *and* δ*, respectively. The* $(\cdots)^\vee$ *indicates the dual of the preceding bracketed term.*

Proof The proof is analogous to that in [Kuhn and Tanaka 2021]. We will sketch the main parts. The descriptions of the first and second types of fixed loci are immediate from the construction. The identification of normal bundles is also straightforward: the induced APOT on these fixed loci agrees with the natural ones on $\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e}$ and $\mathfrak{N}^{\mathrm{fl},(\ell+1)\text{-st}}_{m,e}$ respectively induced by symmetrized pullback from $\mathfrak{N}_m$.

The description of the third type of fixed locus is essentially Lemma 5.1.7 and an argument that this gives the right subscheme structure on the fixed locus; see [Kuhn and Tanaka 2021, Proposition 4.59 and Lemma 4.28]. By Proposition 2.5.2, to show the identification of virtual structure sheaves, it is enough to check that the K-theory class of the fixed obstruction theory is the one on the target. This, and showing the virtual normal bundle is of the claimed form are slightly tedious, but straightforward, checks. We do this in Proposition 6.2.7 in detail. □

5.2.7 By Propositions 5.1.9 and 5.2.4, the enumerative invariants

$$\tilde{\mathsf{N}}_{m,e}(\ell):=\chi(\mathfrak{N}^{\mathrm{fl},\ell\text{-st}}_{m,e},\hat{\mathcal{O}}^{\mathrm{vir}})\quad\text{and}\quad\tilde{\mathsf{Q}}_{m,e}:=\chi(\mathfrak{Q}^{\mathrm{fl,st}}_{m,e},\hat{\mathcal{O}}^{\mathrm{vir}}),$$

and also $\chi(\mathbb{M}^{\ell,\ell+1},\hat{\mathcal{O}}^{\mathrm{vir}})$, are well-defined by T-equivariant localization. Using Proposition 5.2.6 and applying appropriate residues to the $\mathbb{C}^\times$-localization formula on $\mathbb{M}^{\ell,\ell+1}$, as written in Proposition 1.4.5, we immediately obtain the "single-step" wall-crossing formula

$$0=(-1)(\kappa^{1/2}-\kappa^{-1/2})\tilde{\mathsf{N}}_{m,e}(\ell+1)+(-1)(\kappa^{-1/2}-\kappa^{1/2})\tilde{\mathsf{N}}_{m,e}(\ell)\\ +\sum_{\substack{(\beta,e)=(\gamma,f)+(\delta,g)\\ \gamma=:(1,-\beta_C,-m+m')\\ \delta=:(0,0,-m')\\ i_{\min}(g)=i_e(\ell+1)}}(-1)^{\mathsf{ind}}(\kappa^{\frac12\mathsf{ind}}-\kappa^{-\frac12\mathsf{ind}})\tilde{\mathsf{N}}_{m-m',f}(\ell)\tilde{\mathsf{Q}}_{m',g},$$

where, according to Proposition 1.4.5,

$$\mathsf{ind}=-\chi((1,-\beta_C,-m+m'),(0,0,-m'))+\chi_Q(f,g)-\chi_Q(g,f)=m'+c_Q(f,g).$$

Rearranging and rewriting in terms of the quantum integers (8), we get

$$\widetilde{\mathsf{N}}_{m,\boldsymbol{e}}(\ell+1)-\widetilde{\mathsf{N}}_{m,\boldsymbol{e}}(\ell) = -\sum_{\substack{(\beta,\boldsymbol{e})=(\gamma,\boldsymbol{f})+(\delta,\boldsymbol{g})\\ \gamma=:(1,-\beta_C,-m+m')\\ \delta=:(0,0,-m')\\ i_{\min}(\boldsymbol{g})=i_e(\ell+1)}} [m'+c_Q(\boldsymbol{f},\boldsymbol{g})]_\kappa\cdot\widetilde{\mathsf{N}}_{m-m',\boldsymbol{f}}(\ell)\widetilde{\mathsf{Q}}_{m',\boldsymbol{g}}. \tag{44}$$

5.2.8 Recall the full flag $(\alpha,\boldsymbol{d})$, where $\alpha=(1,-\beta_C,-n)$ was the original class of interest and $\boldsymbol{d}=(1,2,\dots,N)$. We now iteratively apply (44) to express $\widetilde{\mathsf{N}}_{n,\boldsymbol{d}}(N)$ in terms of invariants $\widetilde{\mathsf{N}}_{m,\boldsymbol{e}}(0)$ for varying $m\le n$ and full flags $((1,-\beta_C,-m),\boldsymbol{e})$. The resulting wall-crossing formula is

$$\widetilde{\mathsf{N}}_{n,\boldsymbol{d}}(N) = \sum_{\substack{j>0,\,(\alpha,\boldsymbol{d})=\sum_{i=1}^j(\gamma_i,\boldsymbol{e}_i)\\ \{\gamma_i=:(0,0,-m_i)\}_{i=1}^{j-1},\,\gamma_j=:(1,-\beta_C,-m_j)\\ \forall i:\,(\gamma_i,\boldsymbol{e}_i)\text{ full flag}\\ \boldsymbol{e}_1<\cdots<\boldsymbol{e}_j}} (-1)^j\widetilde{\mathsf{N}}_{m_j,\boldsymbol{e}_j}(0)\prod_{i=1}^{j-1}\Bigl[m_i-c_Q\Bigl(\boldsymbol{e}_i,\sum_{\ell=i+1}^{j}\boldsymbol{e}_\ell\Bigr)\Bigr]_\kappa\widetilde{\mathsf{Q}}_{m_i,\boldsymbol{e}_i}. \tag{45}$$

Recall that $\mathsf{N}^*_{(\lambda,\mu,\nu),m}:=\chi(\mathfrak{N}^*_m,\widehat{\mathcal{O}}^{\mathrm{vir}})$, for $*\in\{\mathrm{DT},\mathrm{PT}\}$, were the original DT and PT invariants of interest. Apply (41) and Proposition 4.2.3 to get

$$\widetilde{\mathsf{N}}_{n,\boldsymbol{d}}(N)=[N]_\kappa!\cdot\mathsf{N}^{\mathrm{PT}}_{(\lambda,\mu,\nu),n},\quad \widetilde{\mathsf{N}}_{m,\boldsymbol{e}}(0)=[N-n+m]_\kappa!\cdot\mathsf{N}^{\mathrm{DT}}_{(\lambda,\mu,\nu),m},\quad \widetilde{\mathsf{Q}}_{m,\boldsymbol{e}}=[m-1]_\kappa!\cdot\mathsf{N}^{\mathrm{DT}}_{(\varnothing,\varnothing,\varnothing),m}.$$

In the word notation of Definition 4.2.5, the wall-crossing formula (45) then becomes

$$\mathsf{N}^{\mathrm{PT}}_{(\lambda,\mu,\nu),n}= \sum_{j\ge0,\,\boldsymbol{m}\in\mathbb{Z}^j_{>0}}(-1)^j\frac{[N-|\boldsymbol{m}|]_\kappa!\prod_{i=1}^j[m_i-1]_\kappa!}{[N]_\kappa!}c'_>(m_1,\dots,m_j,N-|\boldsymbol{m}|)\cdot\mathsf{N}^{\mathrm{DT}}_{(\lambda,\mu,\nu),n-|\boldsymbol{m}|}\prod_{i=1}^j\mathsf{N}^{\mathrm{DT}}_{(\varnothing,\varnothing,\varnothing),m_i},$$

where

$$c'_>(m_1,\dots,m_{j+1}):=\sum_{\substack{w\in R(m_1,\dots,m_{j+1})\\ o_1(w)>\cdots>o_j(w)}}\prod_{i=1}^j\Bigl[m_i-\sum_{\ell=i+1}^{j+1}c_{i,j}(w)\Bigr]_\kappa.$$

Applying Proposition 5.3.1 below, which we will prove below using purely combinatorial methods, this becomes

$$\mathsf{N}^{\mathrm{PT}}_{(\lambda,\mu,\nu),n}=\sum_{j\ge0,\,\boldsymbol{m}\in\mathbb{Z}^j_{>0}}(-1)^j\mathsf{N}^{\mathrm{DT}}_{(\lambda,\mu,\nu),n-|\boldsymbol{m}|}\prod_{i=1}^j\mathsf{N}^{\mathrm{DT}}_{(\varnothing,\varnothing,\varnothing),m_i},$$

which translates into the identity of generating series

$$\sum_n Q^n\mathsf{N}^{\mathrm{PT}}_{(\lambda,\mu,\nu),n}=\Bigl(\sum_n Q^n\mathsf{N}^{\mathrm{DT}}_{(\lambda,\mu,\nu),n}\Bigr)\Bigl(\sum_n Q^n\mathsf{N}^{\mathrm{DT}}_{(\varnothing,\varnothing,\varnothing),n}\Bigr)^{-1}.$$

This concludes the Mochizuki-style wall-crossing proof of the DT/PT vertex correspondence (Theorem 1.3.1). □

5.3 Combinatorial reduction

5.3.1 Proposition *For integers* $\ell \geq 0$ *and* $N > 0$, *and* $\boldsymbol{m} = (m_1, \dots, m_\ell) \in \mathbb{Z}_{>0}^\ell$,

$$\sum_{\sigma \in S_\ell} c'_>(m_{\sigma(1)}, \dots, m_{\sigma(\ell)}, N - |\boldsymbol{m}|) = \frac{[N]_\kappa!}{[N - |\boldsymbol{m}|]_\kappa! \prod_i [m_i - 1]_\kappa!} \cdot \ell!. \tag{46}$$

5.3.2 We will prove this combinatorial result by induction on ℓ. Note that the symmetrization is really necessary in order to obtain a nice formula; the individual terms $c'_>(m_{\sigma(1)}, \dots, m_{\sigma(\ell)}, N - |\boldsymbol{m}|)$ are significantly more complicated.

Proof The base case is $\ell = 1$ with $m := m_1$. The left-hand side of (46) becomes

$$\sum_{w \in R(m, N-m)} [m - c_{1,2}(w)]_\kappa.$$

Our quantum integers satisfy

$$[a - b]_\kappa = (-1)^b \kappa^{\frac{1}{2}b}[a]_\kappa - (-1)^a \kappa^{\frac{1}{2}a}[b]_\kappa,$$

so this sum becomes

$$\sum_{w \in R(m, N-m)} (-1)^{c_{1,2}(w)} \kappa^{\frac{1}{2}c_{1,2}(w)}[m]_\kappa + (-1)^m \kappa^{\frac{1}{2}m}[c_{1,2}(w)]_\kappa.$$

The sum over the second term is zero: there is an involution on $R(m, N - m)$ which takes a word and reverses it, and this acts as $c_{1,2} \mapsto -c_{1,2}$. In the first term, the quantity $c_{1,2}(w) \bmod 2$ is constant on $R(m, N - m)$ because, by (43), every inversion changes $c_{1,2}(w)$ by 2. So $(-1)^{c_{1,2}(w)}$ factors out of the sum; it is enough to evaluate it on the word $w = 1^m 2^{N-m}$, for which $c_{1,2}(w) = m(N - m)$. We conclude the base case by applying the following result with $n := N - m$.

Proposition *For integers* $m, n > 0$,

$$(-1)^{mn} \sum_{w \in R(m,n)} \kappa^{\frac{1}{2}c_{1,2}(w)} = \frac{[m+n]_\kappa!}{[m]_\kappa! \, [n]_\kappa!}.$$

Proof Let $f(m, n)$ denote the left-hand side. By induction on $m + n$, it suffices to verify the recursion

$$f(m, n) = (-1)^n \kappa^{\frac{1}{2}n} f(m - 1, n) + (-1)^m \kappa^{-\frac{1}{2}m} f(m, n - 1).$$

It is an easy algebraic exercise to verify that the right-hand side satisfies the same recursion. Clearly $f(m, 0) = 1 = f(0, n)$ for any $m, n > 0$, so the necessary base cases hold.

View $w \in R(m, n)$ as a string of m ones and n twos. Let w' be w without the first letter w_1, so that $w' \in R(m - 1, n)$ if $w_1 = 1$ or $w' \in R(m, n - 1)$ if $w_1 = 2$. Using (43), the contribution of the first letter to $c_{1,2}$ is

$$c_{1,2}(w) - c_{1,2}(w') = \begin{cases} n & \text{if } w_1 = 1, \\ -m & \text{if } w_1 = 2. \end{cases}$$

The desired recursion follows immediately. □

5.3.3 Remark Proposition 5.3.2 is a special case of a more general model for κ-multinomial coefficients:

$$(-1)^{\sum_{i=1}^{k} m_i \sum_{j=1}^{i-1} m_j} \sum_{w \in R(m_1,\dots,m_k)} \kappa^{\frac{1}{2} c_{i,j}(w)} = \frac{\left[\sum_{i=1}^{k} m_i\right]_\kappa!}{\prod_{i=1}^{k} [m_i]_\kappa!}$$

for integers $k > 0$ and $m_1, \dots, m_k > 0$. This can be proved using the explicit bijection

$$R(m_1,\dots,m_k) \xrightarrow{\sim} R(m_1+\dots+m_{k-1}, m_k) \times R(m_1,\dots,m_{k-1}), \quad w \mapsto (w', w''),$$

where w' arises from w by replacing all the letters $< k$ (resp. $= k$) by 1 (resp. 2), and w'' is the subword of w given by deleting all the letters k. We leave the details to the reader.

Note, however, that this identity is not so useful for proving Proposition 5.3.2 when $k > 2$, because the ordering condition $o_1(w) > \dots > o_\ell(w)$ becomes a nontrivial constraint and the symmetrization becomes important.

5.3.4 Lemma *Let $\{m_i\}_{i=1}^{\ell}$ and $\{c_{i,j}\}_{i,j=1}^{\ell+1}$ be arbitrary sets of integers satisfying $c_{i,j} = -c_{j,i}$. Fix $\sigma \in S_\ell$, viewed as an element $\sigma \in S_{\ell+1}$ such that $\sigma(\ell+1) = \ell+1$, and let $i_0 := \sigma^{-1}1$. Then*

$$\prod_{i=1}^{\ell} \Bigg[m_i + \sum_{\substack{j \in [1,\ell+1] \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa$$

$$= \sum_{\Omega \subset \sigma[1,i_0-1]} \prod_{i \in [2,\ell] \setminus \Omega} \Bigg[m_i + \sum_{\substack{j \in [2,\ell+1] \setminus \Omega \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa \prod_{i \in \Omega} \Bigg[\sum_{\substack{j \in \Omega \cup \{1\} \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa \Bigg[\sum_{i \in \Omega \cup \{1\}} m_i + \sum_{\ell \in [2,\ell+1] \setminus \Omega} c_{j,i} \Bigg]_\kappa.$$

Proof The identity is the case $i_1 = 1$ of the formula

$$\prod_{i=1}^{\ell} \Bigg[m_i + \sum_{\substack{j \in [1,\ell+1] \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa$$

$$= \prod_{1 \le \sigma^{-1}i < i_1} \Bigg[m_i + \sum_{\substack{j \in [1,\ell+1] \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa$$

$$\sum_{\Omega \subset \sigma[i_1,i_0-1]} \prod_{\substack{i \in [2,\ell] \setminus \Omega \\ i_1 \le \sigma^{-1}i}} \Bigg[m_i + \sum_{\substack{j \in [2,\ell+1] \setminus \Omega \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa \prod_{i \in \Omega} \Bigg[\sum_{\substack{j \in \Omega \cup \{1\} \\ \sigma^{-1}i < \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa \Bigg[\sum_{i \in \Omega \cup \{1\}} m_i + \sum_{\substack{j \in [2,\ell+1] \setminus \Omega \\ i_1 \le \sigma^{-1}j}} c_{j,i} \Bigg]_\kappa,$$

which holds for all $1 \le i_1 \le i_0$, and which one shows by descending induction on i_1. To go from i_1 to $i_1 - 1$, one takes the term

$$[m_{\sigma i_1} + (c_{\sigma(i_1+1),\sigma i_1} + \dots + c_{\sigma i_0,\sigma i_1}) + c_{\sigma(i_0+1),\sigma i_1} + \dots + c_{\ell+1,\sigma i_1}]_\kappa,$$

denoting the bracketed terms by C, and applies the quantum integer "Jacobi identity"

$$[A+C]_\kappa [B]_\kappa = [A]_\kappa [B-C]_\kappa + [A+B]_\kappa [C]_\kappa,$$

which holds for arbitrary $A, B, C \in \mathbb{Z}$. Here $[B]_\kappa$ is whichever quantum integer currently contains m_1. □

5.3.5 We begin the inductive proof of Proposition 5.3.1. The case $\ell = 0$ is trivially true, and $\ell = 1$ was already done in Section 5.3.2. Suppose that $\ell > 1$. Applying Lemma 5.3.4 with $w \in R(m_1, \dots, m_\ell, m_{\ell+1})$ and $c_{i,j} = c_{i,j}(w)$, and doing some reordering, the left-hand side of Proposition 5.3.1 becomes

$$\sum_{\Omega\subset[2,\ell]} \sum_{\substack{w\in R(m_1,\dots,m_{\ell+1})\\ \forall i\in\Omega:\, o_i(w)>o_1(w)}} \Biggl[\sum_{i\in\Omega\cup\{1\}} m_i + \sum_{j\in[2,\ell+1]\setminus\Omega} c_{j,i}\Biggr]_\kappa \prod_{i\in[2,\ell]\setminus\Omega}\Biggl[m_i + \sum_{\substack{j\in[2,\ell+1]\setminus\Omega\\ (j=\ell+1 \text{ or } o_i(w)>o_j(w))}} c_{j,i}\Biggr]_\kappa \prod_{i\in\Omega}\Biggl[\sum_{\substack{j\in\Omega\cup\{1\}\\ o_i(w)>o_j(w)}} c_{j,i}\Biggr]_\kappa.$$

Given $\Omega \subset [2,\ell]$, let $\boldsymbol{m}_\Omega := \{m_i : i \in \Omega\cup\{1\}\}$ and $\boldsymbol{m}_{\overline{\Omega}} := \{m_i : i \in [2,\ell]\setminus\Omega\}$. Using the linearity of $c_{j,i} = c_Q(\boldsymbol{e}_j, \boldsymbol{e}_i)$ in both $\boldsymbol{e}_j$ and $\boldsymbol{e}_i$, we can rewrite this sum as

$$\sum_{\Omega\subseteq[2,j]} \sum_{w'\in R(|\boldsymbol{m}_\Omega|,N-|\boldsymbol{m}_\Omega|)} \Biggl[\sum_{i\in\Omega\cup\{1\}} m_i + c_{2,1}(w')\Biggr]_\kappa \tag{47}$$

$$\sum_{\substack{w_\Omega\in R(\boldsymbol{m}_\Omega)\\ \forall i\in\Omega:\, o_i(w_\Omega)>o_1(w_\Omega)}} \prod_{i\in\Omega}\Biggl[\sum_{\substack{j\in\Omega\cup\{1\}\\ o_i(w_\Omega)>o_j(w_\Omega)}} c_{j,i}(w_\Omega)\Biggr]_\kappa \tag{48}$$

$$\sum_{w_{\overline{\Omega}}\in R(\boldsymbol{m}_{\overline{\Omega}},N-|\boldsymbol{m}|)} \prod_{i\in[2,\ell]\setminus\Omega}\Biggl[m_i + c_{\ell+1,i}(w_{\overline{\Omega}}) + \sum_{\substack{j\in[2,\ell]\setminus\Omega\\ o_i(w_{\overline{\Omega}})>o_j(w_{\overline{\Omega}})}} c_{j,i}(w_{\overline{\Omega}})\Biggr]_\kappa. \tag{49}$$

Here we are treating elements of $R(\boldsymbol{m}_\Omega, m_1)$ as words on the alphabet $\Omega\cup\{1\}$, i.e., rearrangements of the word with m_j instances of the letter j for $j \in \Omega\cup\{1\}$, and similarly for $R(\boldsymbol{m}_{\overline{\Omega}}, N-|\boldsymbol{m}|)$. Evidently $w \mapsto (w', w_\Omega, w_{\overline{\Omega}})$ is a bijection.

5.3.6 We analyze the three lines (47), (48) and (49). First, (49) is an instance of the left-hand side of (46) for a lower value of ℓ. By inductive assumption, it is equal to

$$(\ell-|\Omega|-1)!\cdot\frac{[N-|\boldsymbol{m}_\Omega|]_\kappa!}{[N-|\boldsymbol{m}|]_\kappa!\prod_{i\in[2,\ell]\setminus\Omega}[m_i-1]_\kappa!}.$$

Second, for the sum (48), since $o_1(w_\Omega)$ is minimal among the $o_i(w_\Omega)$, the word w_Ω begins with the letter 1. Removing this 1 changes the terms $c_{1,i}(w_\Omega)$ to $m_i + c_{1,i}(w'_\Omega)$ and also removes the ordering condition $o_i(w_\Omega) > o_1(w_\Omega)$ for all $i \in \Omega$. This exhibits (48) as another instance of the left-hand side of (46), so by induction it equals

$$|\Omega|!\cdot\frac{[|\boldsymbol{m}_\Omega|-1]_\kappa!}{\prod_{i\in\Omega\cup\{1\}}[m_i-1]_\kappa!}.$$

Finally, note that from what we have shown, (48) and (49) are independent of the choice of the element $w \in R(|\boldsymbol{m}_\Omega|, N-|\boldsymbol{m}_\Omega|)$ in (47). Therefore, we may evaluate the sum in (47) directly, which is the special case $\ell = 1$ of Section 5.3, and therefore equals

$$\frac{[N]_\kappa!}{[N - |\boldsymbol{m}_\Omega|]_\kappa!\,[|\boldsymbol{m}_\Omega| - 1]_\kappa!}.$$

In total, and taking the sum over Ω, the left-hand side of (46) equals

$$\sum_{\omega=0}^{\ell-1} \binom{\ell-1}{\omega} \omega!(\ell-1-\omega)! \frac{[N]_\kappa!}{[N - |\boldsymbol{m}|]_\kappa! \prod_i [m_i - 1]_\kappa!} = j(j-1)! \frac{[N]_\kappa!}{[N - |\boldsymbol{m}|]_\kappa! \prod_i [m_i - 1]_\kappa!}.$$

This concludes the proof of Proposition 5.3.1. □

6 DT/PT via Joyce-style wall-crossing

6.1 Quiver-framed invariants

6.1.1 In this section, we give a proof of the K-theoretic DT/PT vertex using Joyce's so-called *dominant wall-crossing* setup [Joyce 2021, Theorem 5.8], which relates invariants within a stability chamber with invariants on a wall. We avoid most of the generalities of Joyce's machine,[5] including all appearances of vertex and/or Lie algebras, and we hope the content of this section serves as a concrete illustration of Joyce's machinery in its simplest form.

The content of this section differs from the Mochizuki-style wall-crossing of Section 5 in two important ways.

First, the family $(\tau_\xi)_\xi$ of weak stability conditions is genuinely lifted to a family of weak stability conditions on auxiliary stacks, so that it becomes possible to study enumerative invariants *on* the wall at τ_0. We will consider two separate but similar setups: wall-crossing from τ^- to τ_0, and wall-crossing from τ_0 to τ^+. Both of these wall-crossings are of individual interest; see Section 6.4. This is in contrast to Section 5 where the "intermediate" stability conditions have no relation to τ_0, and the wall-crossing from τ^- to τ^+ is done all in one go.

Second, in Section 6.3, we explain a trick to avoid the combinatorics that appeared in Section 5.3. The idea is, after proving a wall-crossing formula for a fixed $\alpha = (1, -\beta_C, -n)$, to sum over all n to get a wall-crossing formula for generating series. The unknown combinatorial quantity in this formula for (λ, μ, ν) can be made equal (up to arbitrary order, as generating series) to a similar quantity in the same formula for $(\varnothing, \varnothing, \varnothing)$, using the extra freedom to choose the framing parameter p in the abstract wall-crossing setup of Section 4.1. But on the other hand every piece of the formula for $(\varnothing, \varnothing, \varnothing)$ is explicit and well-understood, so we have avoided any actual combinatorial work.

To emphasize, we will fix an arbitrary $p \geq 1$ in Definition 6.1.3, and all results in Sections 6.1 and 6.2 hold for any such choice.

[5] In fact, as written in [Joyce 2021], Joyce did not have our technology of symmetrized pullback using APOTs and therefore does not deal with wall-crossing in the CY3 setting.

6.1.2 As in Section 4.1, fix the integer partitions (λ,μ,ν) and therefore the class $\beta_C=(|\lambda|,|\mu|,|\nu|)$. Furthermore, fix a class $\alpha=(1,-\beta_C,-n)$ with $\mathfrak{N}_{(\lambda,\mu,\nu),n}\neq\varnothing$. We will prove a wall-crossing formula for objects in class α. In particular, in what follows, several choices of parameters depend on α.

Recall the family of stability conditions on $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ from Section 3.2; let τ^-,τ^+ be weak stability conditions in the DT and PT stability chambers respectively, and τ_0 be on the (only) wall separating the two chambers.

6.1.3 Definition (cf. [Joyce 2021, Definition 5.5]) Fix $k\gg 0$ large enough (depending on $\alpha=(1,-\beta_C,-n)$) as described in Definition 4.1.2, and $p\geq 1$ arbitrary. Let $N:=f_{k,p}(\alpha)$. For $m\leq n$ and a dimension vector $\boldsymbol{e}=(e_i)_{i=1}^N$, recall that the auxiliary moduli stacks $\mathfrak{N}^{Q(N)}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}$ and $\mathfrak{Q}^{Q(N)}_{m,\boldsymbol{e}}$ from Definition 4.1.2 are defined using a quiver of length N. We will put a family of weak stability conditions on these stacks. Let n_0 be the maximal integer such that $\mathfrak{N}_{(\lambda,\mu,\nu),m'}=\varnothing$ for all $m'\leq n_0$, and assume that $m>n_0$ when considering classes $(1,-\beta_C,-m)$. Fix the following parameters.

- Pick a generic $\boldsymbol{\mu}=(\mu_i)_{i=1}^N\in\mathbb{R}^N$ so that $1>\mu_1\gg\mu_2\gg\cdots\gg\mu_N>0$. Genericity means that the μ_i must be $\mathbb{Z}$-linearly independent, so that if $\boldsymbol{\mu}\cdot\boldsymbol{f}=0$ for an integer vector $\boldsymbol{f}$ then $\boldsymbol{f}=0$. The symbols $\gg$ mean that the ratios $1/\mu_1$ and μ_i/μ_{i+1} for $1\leq i<N$ must satisfy finitely many lower bounds.
- Pick an additive function $\lambda^\pm$ on numerical classes so that $\lambda^\pm(\alpha)=0$ and $\lambda^-(0,0,-1)\ll 0$ (for DT wall-crossing) and $\lambda^+(0,0,-1)\gg 0$ (for PT wall-crossing).[6] The symbols $\ll$ and $\gg$ stand for the finitely many upper/lower bounds such that the proofs of Lemmas 6.1.8 and 6.1.11 hold.
- Pick an additive function r on numerical classes such that $r(1,-\beta_C,-n_0)=0$ and $r(0,0,-1)=1$, so that $r(1,-\beta_C,-m)=m-n_0>0$.

For $1\leq a,b\leq N$, let $\mathbf{1}_{[a,b]}:=(0,\dots,0,1,\dots,1,0,\dots,0)$, where the 1's appear exactly in the interval $[a,b]$. For brevity, write $\mathbf{1}:=\mathbf{1}_{[1,N]}$. For effective classes $(\beta,\boldsymbol{e})$ with $\beta\leq\alpha$, define the family of weak stability conditions

$$\tau_0^\pm(s,x)\colon(\beta,\boldsymbol{e})\mapsto\begin{cases}\left(\tau_0(\beta),\dfrac{s\lambda^\pm(\beta)+(\boldsymbol{\mu}+x\mathbf{1})\cdot\boldsymbol{e}}{r(\beta)}\right) & \text{if } \beta=(1,-\beta_C,-\star)\text{ or }(0,0,-\star),\\ (\tau_0(\beta),\infty) & \text{if } \beta=(1,-\beta'_C,-\star),\ \beta'_C\neq\beta_C,\\ (\tau_0(\beta),-\infty) & \text{if } \beta=(0,-\beta'_C,-\star),\ \beta'_C\neq 0,\\ \left(\infty,\dfrac{(\boldsymbol{\mu}+x\mathbf{1})\cdot\boldsymbol{e}}{\mathbf{1}\cdot\boldsymbol{e}}\right) & \text{if } \beta=0,\ (\boldsymbol{\mu}+x\mathbf{1})\cdot\boldsymbol{e}>0,\\ \left(-\infty,\dfrac{(\boldsymbol{\mu}+x\mathbf{1})\cdot\boldsymbol{e}}{\mathbf{1}\cdot\boldsymbol{e}}\right) & \text{if } \beta=0,\ (\boldsymbol{\mu}+x\mathbf{1})\cdot\boldsymbol{e}\leq 0\end{cases}\tag{50}$$

[6] Joyce [2021, Assumption 5.2(d)] requires $\lambda^\pm(\beta)>0$ (resp. $\lambda^\pm(\beta)<0$) if and only if $\tau^\pm(\beta)>\tau^\pm(\alpha)$ (resp. $\tau^\pm(\beta)<\tau^\pm(\alpha)$), but this is not satisfied by our choice of $\lambda^\pm$. Joyce uses this assumption to prove Lemma 6.1.8 in a more uniform way; see [loc. cit., Proposition 10.5], which is the only place where he uses this assumption. Our proof is rather hands-on but does not rely on this assumption.

for $s \in [0,1]$ and $x \in [-1,0]$. Here $\star$ stands for an arbitrary integer. This depends continuously on s but possibly *discontinuously* on x because of the transition between $\pm\infty$ in the last two cases.[7]

Take the lexicographic ordering on $(\mathbb{R} \cup \{\pm\infty\})^2$. We leave it to the reader to verify that $\tau_0^{\pm}(s,x)$ is a weak stability condition, using the additivity of $\lambda^{\pm}$ and r. (The second and third cases in (50) are there to ensure that this claim is true.) Let

$$\mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(\tau_0^{\pm}(s,x)) \subset \mathfrak{N}^{Q(N),\mathrm{pl}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}} \quad \text{and} \quad \mathfrak{Q}^{Q(N),\mathrm{sst}}_{m,\boldsymbol{e}}(\tau_0^{\pm}(s,x)) \subset \mathfrak{Q}^{Q(N),\mathrm{pl}}_{m,\boldsymbol{e}} \tag{51}$$

be the semistable loci, following the notation in Section 3.1.5 for $\mathbb{C}^\times$-rigidified stacks. One checks easily that the forgetful maps $\Pi_{\mathfrak{N}}$ and $\Pi_{\mathfrak{Q}}$ of (39) take $\tau_0^{\pm}(s,x)$-semistable loci to τ_0-semistable loci.

6.1.4 Lemma *A $\tau_0^{\pm}(s,x)$-semistable object of class $(\beta,\boldsymbol{e})$ with $\beta \neq 0$ must have all framing maps ρ_i injective.*

Proof Let $(I,\boldsymbol{V},\boldsymbol{\rho})$ be the object. If $\ker\rho_i \neq 0$ for some i, then it induces a nontrivial subobject and one has a splitting

$$(I,\boldsymbol{V},\boldsymbol{\rho}) = (I,\boldsymbol{V}',\boldsymbol{\rho}') \oplus (0,\boldsymbol{V}'',\boldsymbol{\rho}''),$$

where $V_j'' := \ker(\rho_i \circ \cdots \circ \rho_{j+1} \circ \rho_j)$ for $j \leq i$. Comparing $\tau_0^{\pm}(s,x)$ for the two summands, it is clear that one of the two summands must be destabilizing, a contradiction. □

6.1.5 Proposition *Suppose there are no strictly semistable objects in $\mathfrak{N}^{Q(N),\mathrm{sst}}_{\beta_C,m,\boldsymbol{e}}(\tau_0^{\pm}(s,x))$. Then,*

$$\mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(\tau_0^{\pm}(s,x)) \subset \mathfrak{N}^{Q(N),\mathrm{sst}}_{\beta_C,m,\boldsymbol{e}}(\tau_0^{\pm}(s,x))$$

is a separated algebraic space with proper T*-fixed loci.*

Proof By Lemma 4.3.2, we only need to address properness of T-fixed loci. As in Section 5.1.9, it is enough to show that

$$\mathfrak{M}^{\circ,Q(N),\mathrm{sst}}_{1,\beta_C,m,\boldsymbol{e}}(\tau_0^{\pm}(s,x)) \supset \mathfrak{N}^{Q(N),\mathrm{sst}}_{\beta_C,m,\boldsymbol{e}}(\tau_0^{\pm}(s,x))$$

is proper; this open inclusion induces an inclusion of connected components of T-fixed loci.

We follow the strategy of Section 4.3.4: let R be a DVR with fraction field K and closed point ξ, and assume we are given a family $A_K := (I_K,\boldsymbol{V}_K,\boldsymbol{\rho}_K)$ of $\tau_0^{\pm}(s,x)$-semistable objects in $\mathfrak{M}^{\circ,Q(N),\mathrm{pl}}_{1,\beta_C,m,\boldsymbol{e}}$. The underlying object I_K is τ_0-semistable, so can be extended to a family of objects $I_R^{(0)}$ in $\mathfrak{M}^{\circ}_{1,\beta_C,m}$. Since the quiver $Q(N)$ has no oriented cycles, the quiver data can also be extended compatibly. So we obtain an extension $A_R^{(0)} = (I_R^{(0)},\boldsymbol{V}_R^{(0)},\boldsymbol{\rho}_R^{(0)})$.

This addresses (A0) in Section 4.3.4. By our choice of $I_\xi^{(0)}$, all destabilizing sub- or quotient-objects of $A_\xi^{(0)}$ have numerical class of the form $(\gamma,\boldsymbol{f})$ with either $\gamma = (1,-\beta_C,-\star)$ or $\gamma = (0,0,-\star)$. Throughout, $\star$ stands for an arbitrary integer.

[7] The x-dependence of $\tau_0^{\pm}(s,x)$ is a remnant from Joyce's machine, particularly his construction of *semistable invariants*. We only really use it in the proof of Lemma 6.1.11. In principle, with a bit more work, we could have remained at $x=0$ throughout this paper.

Define a sequence of families $A_R^{(n)}$ by following the following procedure, which is a slight variation on Section 4.3.4.

(i) As long as $\tau_{\max}(A_\xi^{(n)}) = (\infty, \star)$, i.e., $A_\xi^{(n)}$ has a subobject of class $(0, \boldsymbol{f})$ and $(\boldsymbol{\mu} + x\mathbf{1}) \cdot \boldsymbol{f} > 0$, perform an elementary modification along a maximally destabilizing subobject $B^{(n)}$ of $A_\xi^{(n)}$ to obtain $A_R^{(n+1)}$. This means a subobject whose class $(0, \boldsymbol{f}')$ maximizes $(\boldsymbol{\mu} + x\mathbf{1}) \cdot \boldsymbol{f}'/\mathbf{1} \cdot \boldsymbol{f}'$ and, among such subobjects, maximizes $\mathbf{1} \cdot \boldsymbol{f}'$.

(ii) Once $\tau_{\max}(A_\xi^{(n)}) < (\infty, \star)$, if $\tau_{\min}(A_\xi^{(n)}) = (-\infty, \star)$, then perform the dual procedure to (i), i.e., an elementary modification along a maximally destabilizing quotient object $C^{(n)}$ of $A_\xi^{(n)}$ to obtain $A_R^{(n+1)}$. This means a quotient object whose class $(0, \boldsymbol{f}')$ minimizes $(\mu + x\mathbf{1}) \cdot \boldsymbol{f}'/\mathbf{1} \cdot \boldsymbol{f}'$ and, among such quotient objects, maximizes $\mathbf{1} \cdot \boldsymbol{f}'$.

(iii) Once $\tau_{\min}(A_\xi^{(n)}), \tau_{\max}(A_\xi^{(n)}) \in \{\tau_0(\beta)\} \times \mathbb{R}$, but $A_\xi^{(n)}$ is not yet stable, perform an elementary modification along a maximally destabilizing subobject $B^{(n)}$ of $A_\xi^{(n)}$ to obtain $A_R^{(n+1)}$. This means a subobject whose class $(\gamma, \boldsymbol{f}')$ maximizes the tuple

$$\left(\frac{\lambda^{\pm}(B^{(n)}) + (\boldsymbol{\mu} + x\mathbf{1}) \cdot \boldsymbol{f}'}{r(B^{(n)})}, r(B^{(n)}), \mathbf{1} \cdot \boldsymbol{f}'\right)$$

with respect to the lexicographic ordering.

Then, as in Remark 4.3.5, assumption (A1) holds for this sequence. Assumption (A3) holds, since the possible values of the quantities $\lambda^{\pm}(\beta)$, $r(\beta)$, $\mathbf{1} \cdot \boldsymbol{e}$ and $\boldsymbol{\mu} \cdot \boldsymbol{f}'$ that appear for effective classes $(\gamma, \boldsymbol{f}') \leq (\beta, \boldsymbol{e})$ are a priori bounded in terms of N and $\boldsymbol{e}$.

It remains to check (A2). In case (i), one checks that if $\tau_0^{\pm}(s, x)$ is equal for both $B^{(n+1)}$ and $B^{(n)}$, then the induced map $B^{(n+1)} \to B^{(n)}$ must be injective, thus is either an isomorphism or it holds that $\mathbf{1} \cdot \boldsymbol{f}'(B^{(n+1)}) < \mathbf{1} \cdot \boldsymbol{f}'(B^{(n)})$, which can only happen finitely many times. An analogous argument works in case (ii) (using the quotients $C^{(n)}$), and in case (iii), where one shows that the tuple $(r(B^{(n)}), \mathbf{1} \cdot \boldsymbol{f}'(B^{(n)}))$ decreases. □

6.1.6 We now define enumerative invariants for the semistable loci (51) for any s, x such that there are no strictly $\tau_0^{\pm}(s, x)$-semistables. Under this assumption, Theorem 2.3.4 yields a symmetric APOT on the semistable (= stable) loci by symmetrized pullback along $\Pi_{\mathfrak{N}}$ or $\Pi_{\mathfrak{Q}}$. By Theorem 2.4.3 and Lemma 3.3.7, we obtain a symmetrized virtual structure sheaf $\widehat{\mathcal{O}}^{\text{vir}}$ amenable to T-equivariant localization. This allows us to define the enumerative invariants

$$\widetilde{\mathsf{Z}}_{\beta,\boldsymbol{e}}^{\pm}(s, x) := \begin{cases} \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}^{\pm}(s, x) := \chi\big(\mathfrak{N}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}^{Q(N),\text{sst}}(\tau_0^{\pm}(s, x)), \widehat{\mathcal{O}}^{\text{vir}}\big) & \text{if } \beta = (1, -\beta_C, -m), \\ \widetilde{\mathsf{Q}}_{m,\boldsymbol{e}}^{\pm}(s, x) := \chi(\mathfrak{Q}_{m,\boldsymbol{e}}^{Q(N),\text{sst}}(\tau_0^{\pm}(s, x)), \widehat{\mathcal{O}}^{\text{vir}}) & \text{if } \beta = (0, 0, -m) \end{cases}$$

by T-equivariant localization, because Proposition 6.1.5 guarantees that the T-fixed loci are proper. Although $\widetilde{\mathsf{Z}}_{\beta,\boldsymbol{e}}^{\pm}(s, x)$ depends on (λ, μ, ν), this notation will only be used in contexts where (λ, μ, ν)

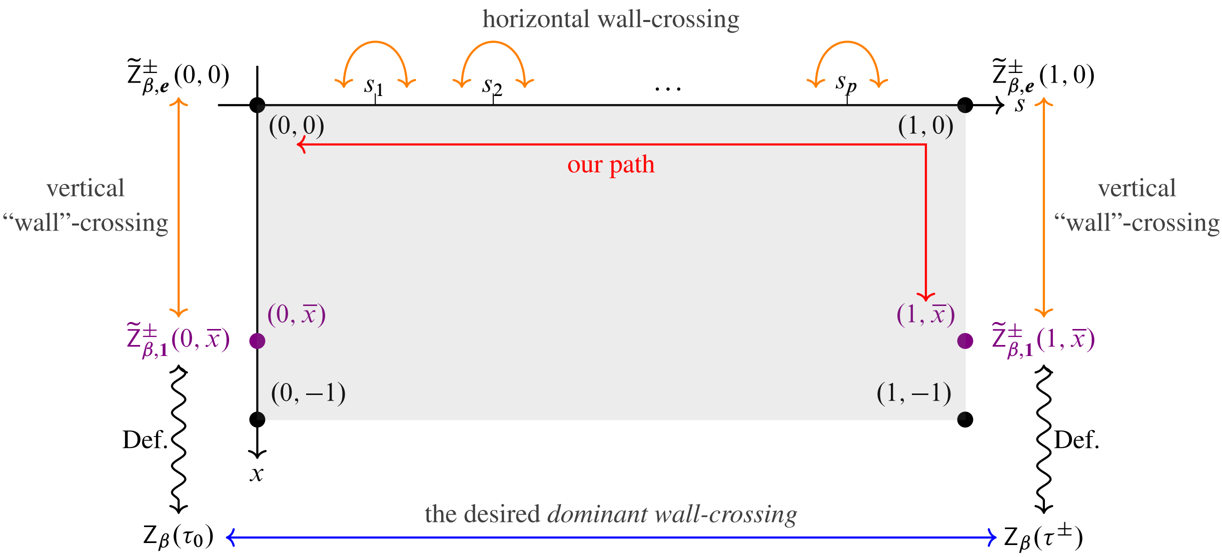


Figure 1: The steps involved in Joyce's dominant wall-crossing.

are fixed, i.e., only in Sections 6.1 and 6.2, and so there is no ambiguity. Later β will arise from splittings where it may be either of the two forms $\beta = (1, -\beta_C, -m)$ or $\beta = (0, 0, -m)$, and so it is useful to have some unifying notation.

To prove the desired DT/PT vertex correspondence, we will obtain wall-crossing formulas between $\tilde{Z}^{\pm}_{\beta,e}(s,x)$ with varying parameters (s,x). The core wall-crossing result will be for $\tilde{N}^{\pm}_{(\lambda,\mu,\nu),m,e}(s,0)$ for $s \in [0,1]$ and appropriate e (Section 6.2).

6.1.7 Joyce's strategy for wall-crossing between τ_0 and $\tau^{\pm}$, using the auxiliary invariants $\tilde{Z}^{\pm}_{\beta,e}(s,x)$, consists of the steps illustrated in Figure 1.

In Joyce's original setup, there is a special $\bar{x} \in (-1,0)$ where the auxiliary invariants $\tilde{Z}^{\pm}_{\beta,1}(s,\bar{x})$ for $s = 0, 1$ are used to formally define *semistable invariants* $Z_\beta(\tau_0)$ and $Z_\beta(\tau^{\pm})$ for the original moduli stack. These semistable invariants are well-defined regardless of whether β has strictly τ_0- or $\tau^{\pm}$-semistable objects, and Joyce's wall-crossing formulas are written in terms of them. For their definition in our situation, see [Liu 2023, Section 6]. However, semistable invariants are irrelevant for this paper because Lemma 6.1.8 directly relates $\tilde{N}^{\pm}_{(\lambda,\mu,\nu),m,1}(1,\bar{x})$ to the DT/PT invariants $N^{\pm}_{(\lambda,\mu,\nu),m}$ in a simple way.

For us, the first step (Lemma 6.1.11) is to move from $(s,x) = (1,\bar{x})$ to $(s,x) = (1,0)$. This gives formulas relating $\tilde{Z}^{\pm}_{\beta,e}(1,0)$ and $\tilde{Z}^{\pm}_{\beta,1}(1,\bar{x})$ for a framing dimension e such that (β, e) is a full flag. The argument here is not geometric and only involves some fiddling around with the weak stability condition $\tau^{\pm}_0(s,x)$.

The second step (Section 6.2) is to vary the parameter s with $x = 0$. This is a truly geometric wall-crossing argument. The advantage of wall-crossing at $x = 0$, instead of $x = \bar{x}$ or even $x = -1$, is that objects framed by full flags destabilize into at most two pieces at each of the walls s_i (Lemma 6.2.3), which makes master space techniques applicable.

6.1.8 For integers $1 \le a \le N$, let

$$x_0(a) := -\frac{1}{N-a+1}\sum_{i=a}^{N}\mu_i \in (-1,0).$$

This is the point where $\tau_0^\pm(s,x)$ has a discontinuity in x for classes $(0,\mathbf{1}_{[a,N]})$. In particular, for $x \le x_0(a)$, $(0,\mathbf{1}_{[a,N]})$ becomes a $\tau_0^\pm(s,x)$-destabilizing quotient class for any numerical class $(\beta,\boldsymbol{e})$ with $\beta \neq 0$; cf. [Joyce 2021, (10.30)].

Lemma *Let $1 \le a \le N$ and $\bar{x} \in (x_0(a),0]$. Then*

$$[(I,\boldsymbol{V},\boldsymbol{\rho})] \in \mathfrak{N}^{Q(N)}_{(\lambda,\mu,\nu),m,\mathbf{1}_{[a,N]}}$$

is $\tau_0^\pm(1,\bar{x})$-semistable if and only if I is $\tau^\pm$-semistable and

$$V_a \xrightarrow{\sim} V_{a+1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} V_N \hookrightarrow V_{N+1} = F_{k,p}(I) \tag{52}$$

is a chain of isomorphisms followed by an inclusion.

This relates our auxiliary invariants to the DT and PT invariants (Section 3.3) in the corresponding stability chambers: $\Pi_{\mathfrak{N}}$ restricted to the class $(\beta,\mathbf{1}_{[a,N]})$, where $\beta = (1,-\beta_C,-m)$, is therefore a $\mathbb{P}^{f_{k,p}(\beta)-1}$-bundle and so

$$\tilde{\mathsf{N}}^\pm_{(\lambda,\mu,\nu),m,\mathbf{1}_{[a,N]}}(1,\bar{x}) = [f_{k,p}(\beta)]_\kappa \cdot \mathsf{N}_{(\lambda,\mu,\nu),m}(\tau^\pm). \tag{53}$$

6.1.9 Proof The basic ideas are from [Joyce 2021, Propositions 10.3(a), 10.5, 10.13(a), 10.14(v)], which the reader should consult for more details. Let $(I,\boldsymbol{V},\boldsymbol{\rho})$ be an object of numerical class $(\beta,\mathbf{1}_{[a,N]})$.

We begin with the framing data. Given $(I,\boldsymbol{V},\boldsymbol{\rho})$ with framing of the form (52), there are subobjects of numerical class $(\beta,\mathbf{1}_{[b,N]})$ for $a < b \le N$, given by the object I along with a subchain of framing isomorphisms. None of these subobjects are $\tau_0^\pm(1,\bar{x})$-destabilizing by our choice of $\bar{x} > x_0(a)$.

On the other hand, if the quiver framing is not of the form (52), then $\rho_b = 0$ for some $a \le b \le N$ and the object $(I,\boldsymbol{V},\boldsymbol{\rho})$ splits as a direct sum of objects of class $(0,\mathbf{1}_{[a,b]})$ and $(\beta,\mathbf{1}_{[b+1,N]})$. One of these subobjects is $\tau_0^\pm(1,\bar{x})$-destabilizing.

6.1.10 Now we consider the object I. Note first that a subobject $I' \subset I$ induces a subobject $(I',\mathbf{0},\mathbf{0}) \subset (I,\boldsymbol{V},\boldsymbol{\rho})$. If I' has class $(1,-\beta'_C,-m')$ for $\beta'_C \neq \beta_C$, then I' and $(I',\mathbf{0},\mathbf{0})$ are $\tau^\pm$- and $\tau_0^\pm(s,x)$-destabilizing, respectively. Similarly, if I' has class $(0,-\beta'_C,-m')$ for $\beta'_C \neq 0$, then neither I' nor any subobject $(I',\boldsymbol{V}',\boldsymbol{\rho}')$ can cause $\tau^\pm$- or $\tau_0^\pm(s,x)$-instability or strict semistability. Hence it is enough to assume that I' has class $(1,-\beta_C,-m')$ or $(0,0,-m')$.

In what follows, we work in the DT chamber with stability condition τ^-. The proof for τ^+ in the PT chamber is analogous. Recall from Definition 3.2.2 that τ_0 is the wall where $\arg z_1 = \arg z_0$. Let θ denote this angle, so that $\theta = \tau_0(1,-\beta_C,-m') = \tau_0(0,0,-m')$.

For an effective splitting $(\beta, \mathbf{1}) = ((1, -\beta_C, -m'), \boldsymbol{e}') + ((0, 0, m'-m), \boldsymbol{e}'')$,

$$\tau_0^-(1,\bar{x})((1,-\beta_C,-m'),\boldsymbol{e}') = \left(\theta,\ \frac{m'-n}{m'-n_0}\lambda^-(0,0,-1) + \frac{(\boldsymbol{\mu}+\bar{x}\mathbf{1})\cdot\boldsymbol{e}'}{m'-n_0}\right),$$

$$\tau_0^-(1,\bar{x})((0,0,m'-m),\boldsymbol{e}'') = \left(\theta,\ \lambda^-(0,0,-1) + \frac{(\boldsymbol{\mu}+\bar{x}\mathbf{1})\cdot\boldsymbol{e}''}{m-m'}\right).$$

Compare the second entries. The absolute values of the second terms $((\boldsymbol{\mu}+\bar{x}\mathbf{1})\cdot\boldsymbol{e}')/(m'-n_0)$ and $((\boldsymbol{\mu}+\bar{x}\mathbf{1})\cdot\boldsymbol{e}'')/(m-m')$ are bounded above by N. By our choice of a sufficiently negative $\lambda^-(0,0,-1)$ in Definition 6.1.3, the first terms dominate. Since $n_0 < m' \le n$, it follows that

$$\tau_0^-(1,\bar{x})((1,-\beta_C,-m'),\boldsymbol{e}') > \tau_0^-(1,\bar{x})((0,0,m'-m),\boldsymbol{e}''). \tag{54}$$

Suppose I is τ^--unstable. Then its destabilizing subobject must have class $(1, -\beta_C, -m')$. So $(I, \boldsymbol{V}, \boldsymbol{\rho})$ has a subobject of class $((1, -\beta_C, -m'), \mathbf{0})$, which is $\tau_0^-(1, \bar{x})$-destabilizing by (54).

Conversely, suppose $(I, \boldsymbol{V}, \boldsymbol{\rho})$ is $\tau_0^-(1, \bar{x})$-unstable, with destabilizing subobject of class $(\gamma, \boldsymbol{d}')$. If $\gamma = \beta$, then the choice of $\bar{x}$ in Section 6.1.9 means the framing data cannot be of the form (52). Similarly if $\gamma = 0$, the framing maps cannot all be injective and therefore (52) cannot hold again. Finally, if $\gamma \ne 0, \beta$, then (54) means $\gamma = (1, -\beta_C, -m')$ is a τ^--destabilizing subobject of I. □

6.1.11 Lemma *Let $(\beta, \boldsymbol{e})$ be an effective numerical class with $\beta \ne 0$, $0 \le e_1 \le 1$, $e_N > 0$ and $e_i \le e_{i+1} \le e_i + 1$ for all $1 \le i < N$. If a is the smallest index where $e_a = e_N$, then for $s \in \{0, 1\}$ and $x \in (x_0(a), 0]$:*

(i) *there are no strictly $\tau_0^\pm(s, x)$-semistable objects, and $\widetilde{\mathsf{Z}}^\pm_{\beta,\boldsymbol{e}}(s, x)$ is independent of x;*

(ii) *if $e_N > 1$, then*

$$\widetilde{\mathsf{Z}}^\pm_{\beta,\boldsymbol{e}}(s,x) = [f_{k,p}(\beta) - e_a + 1]_\kappa \cdot \widetilde{\mathsf{Z}}^\pm_{\beta,\boldsymbol{e}-\mathbf{1}_{[a,N]}}(s,x). \tag{55}$$

The inequalities $1 > \mu_1 > \mu_2 > \cdots > \mu_N > 0$ imply that $-1 < x_0(1) < x_0(2) < \cdots < x_0(N) < 0$, so (55) may be applied recursively.

Proof We sketch the main ideas and refer the reader to [Joyce 2021, Proposition 10.14] for details.

We first show (i). By Lemma 6.1.4, it suffices to consider objects $(I, \boldsymbol{V}, \boldsymbol{\rho})$ of class $(\beta, \boldsymbol{e})$ with all ρ_i injective. Assume that $(I, \boldsymbol{V}, \boldsymbol{\rho})$ has weakly destabilizing sub- and quotient-objects of classes $(\gamma, \boldsymbol{f})$ and $(\delta, \boldsymbol{g})$, respectively. Equating the values of $\tau_0^\pm(s, x)$ on these classes, we get

$$\frac{s\lambda^\pm(\gamma)}{r(\gamma)} - \frac{s\lambda^\pm(\delta)}{r(\delta)} = (\boldsymbol{\mu} + x\mathbf{1}) \cdot \left(\frac{\boldsymbol{g}}{r(\delta)} - \frac{\boldsymbol{f}}{r(\gamma)}\right). \tag{56}$$

We argue that this cannot hold for $s \in \{0, 1\}$ and $x \in (x_0(a), 0]$. For $s = 1$ and $|x| \le 1$, this follows from our choice of sufficiently positive/negative $\lambda^\pm$ in Definition 6.1.3 unless the terms on the left-hand side of (56) cancel. In this latter case, and for $s = 0$, (56) has at most one solution. Suppose $\tilde{x} \in \mathbb{R}$ is a solution, and let $1 \le b \le N$ be the minimal index with $e_b > 0$. If $a > b$, then by our choice of $\boldsymbol{\mu}$, the μ_b-term in (56) dominates, and $\tilde{x}$ is close to a nonzero integer multiple of μ_b. In particular, it cannot be in $(x_0(a), 0]$.

Finally, if $a = b$, then $\boldsymbol{e} = \mathbf{1}_{[b,N]}$, and by Lemma 6.1.4, $\boldsymbol{f} = \mathbf{1}_{[c,N]}$ for some $b \le c \le N+1$. Then $\tilde{x}$ can be checked to either be $\le x_0(a)$ or positive. So, in particular, (56) does not hold for $x \in (x_0(a), 0]$. We conclude that there are no strictly semistable objects.

Next, we claim that $\tau_0^{\pm}(s,x)$ is continuous in $x \in (x_0(a), 0]$ for sub- and quotient-objects of $(I, \boldsymbol{V}, \boldsymbol{\rho})$. Indeed, $\tau_0^{\pm}(s,x)$ is only discontinuous for classes of the form $(0, \boldsymbol{g})$, for which the point of discontinuity is $x(\boldsymbol{g}) := -\boldsymbol{\mu}\cdot\boldsymbol{g}/\mathbf{1}\cdot\boldsymbol{g}$. By injectivity of the framing maps, such classes can only come from quotient objects and moreover, if $g_a = 0$ then $g_b = 0$ for all $b > a$. It follows that $x(\boldsymbol{g})$ is maximized, among quotient classes $(0,\boldsymbol{g})$, by $\boldsymbol{g} = \mathbf{1}_{[a,N]}$, and this maximum is exactly $x(\boldsymbol{g}) = x_0(a)$. From continuity and the absence of strictly semistables, we conclude that $\widetilde{\mathsf{Z}}_{\beta,\boldsymbol{e}}(s,x)$ is independent of $x \in (x_0(a), 0]$.

For (ii), it is enough to prove (55) for $x > x_0(a)$ very close to $x_0(a)$, and then to apply (i). Let $(I, \boldsymbol{V}, \boldsymbol{\rho})$ be a $\tau_0^{\pm}(s,x)$-semistable object of class $(\beta, \boldsymbol{e})$. Note that our assumptions imply $a > 1$. We canonically obtain a subobject $(I, \boldsymbol{V}', \boldsymbol{\rho}')$ defined as follows: for $i < a$, take $V_i' := V_i$ and for $i \ge a$, take V_i' to be the image of V_{a-1} in V_i. The corresponding quotient is of class $(0, \mathbf{1}_{[a,N]})$ and is given by a chain of isomorphisms of 1-dimensional vector spaces $L_a \cong L_{a+1} \cong \cdots \cong L_N$. It is easy to check numerically that $(I, \boldsymbol{V}', \boldsymbol{\rho}')$ is still $\tau_0^{\pm}(s,x)$-semistable for x very close to $x_0(a)$. Hence $\Pi\colon (I, \boldsymbol{V}, \boldsymbol{\rho}) \mapsto (I, \boldsymbol{V}', \boldsymbol{\rho}')$ is a morphism of moduli stacks. The construction can be reversed after choosing a generator for L_a among nonzero vectors in $F_{k,p}(I)/\operatorname{im}(V_{a-1})$ up to scaling. This means Π is a $\mathbb{P}^{f_{k,p}(\beta)-e_a+1}$-bundle. Then (55) follows from Proposition 4.2.2 and Lemma 2.3.13. □

6.2 Wall-crossing formulas

6.2.1 The strategy to obtain the desired DT/PT wall-crossing formula for the class $\alpha = (1, -\beta_C, -n)$ is to perform the wall-crossing in s twice: wall-cross using $\tau_0^-(s,0)$ from the DT chamber onto the wall, by varying s from 1 to 0, and then wall-cross using $\tau_0^+(s,0)$ from the wall into the PT chamber, by varying s from 0 to 1. Schematically, we follow the path in Figure 1 twice:

$$\widetilde{\mathsf{N}}^-_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(1,0) \rightsquigarrow \widetilde{\mathsf{N}}^-_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(0,0) = \widetilde{\mathsf{N}}^+_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(0,0) \rightsquigarrow \widetilde{\mathsf{N}}^+_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(1,0).$$

We begin with the class $(\alpha, \boldsymbol{d})$ where $\boldsymbol{d} = (1, 2, \ldots, N)$. Recall that this is a full flag, in the sense of Definition 4.2.1. All other classes, such as the classes $(\beta, \boldsymbol{e})$ with $\beta = (1, -\beta_C, -m)$ or $\beta = (0, 0, -m)$ from Section 6.1, will occur as pieces of splittings of $(\alpha, \boldsymbol{d})$. In particular, $m \le n$ and $\boldsymbol{e} \le \boldsymbol{d}$.

6.2.2 It follows immediately from Lemma 6.1.11 that if $(\beta, \boldsymbol{e})$ is a full flag, then iterated applications of (55) with $x = 0$ gives the full vertical "wall"-crossing

$$\widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(1,0) = [f_{k,p}(\beta)]_\kappa! \cdot \mathsf{N}_{(\lambda,\mu,\nu),m}(\tau^{\pm}), \tag{57}$$

$$\widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(0,0) = [f_{k,p}(\beta) - 1]_\kappa! \cdot \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m}, \tag{58}$$

where in (57) we further applied (53) and in (58) we set $\widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m} := \widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\mathbf{1}}(0,0)$. We can suppress $\pm$ from the notation here since $\widetilde{\mathsf{N}}^+_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(0,x) = \widetilde{\mathsf{N}}^-_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(0,x)$ by the definition of $\tau_0^{\pm}(0,x)$.

Analogously, $\widetilde{\mathsf{Q}}^{+}_{m,\boldsymbol{e}}(0,0) = \widetilde{\mathsf{Q}}^{-}_{m,\boldsymbol{e}}(0,0)$ by definition, and Lemma 6.2.3(ii) shows that there is no wall-crossing in s for the invariants $\widetilde{\mathsf{Q}}^{\pm}_{m,\boldsymbol{e}}(s,0)$. So we suppress both $\pm$ and s from the notation. If $((0,0,-m),\boldsymbol{e})$ is a full flag, iterated applications of (55) give

$$\widetilde{\mathsf{Q}}^{\pm}_{m,\boldsymbol{e}}(s,0) =: \widetilde{\mathsf{Q}}_{m,\boldsymbol{e}} = [m-1]_\kappa! \cdot \widetilde{\mathsf{Q}}_m, \tag{59}$$

where $\widetilde{\mathsf{Q}}_m := \widetilde{\mathsf{Q}}^{\pm}_{m,\mathbf{1}}(0,0)$ and we used that $f_{k,p}((0,0,-m)) = m$. Note that there is no analogue of (53) here; see also Section 3.3.8.

6.2.3 Lemma *Suppose $(\beta,\boldsymbol{e})$ is a full flag and there is an integer $M \geq 1$ and a splitting $(\beta,\boldsymbol{e}) = \sum_{i=1}^{M}(\beta_i,\boldsymbol{e}_i)$ such that for some $s \in [0,1]$,*

$$\tau_0^{\pm}(s,0)(\beta_1,\boldsymbol{e}_1) = \cdots = \tau_0^{\pm}(s,0)(\beta_M,\boldsymbol{e}_M), \tag{60}$$

and there exist $\tau_0^{\pm}(s,0)$-semistable objects of class $(\beta_i,\boldsymbol{e}_i)$ for all i. Then:

(i) *$s \in (0,1)$, $M \leq 2$ with equality at only finitely many s, and each $(\beta_i,\boldsymbol{e}_i)$ is a full flag with no strictly $\tau_0^{\pm}(s,0)$-semistable objects of class $(\beta_i,\boldsymbol{e}_i)$;*

(ii) *if $\beta = (0,0,-m)$, then $M = 1$;*

(iii) *if $\beta = (1,-\beta_C,-m)$ and $M = 2$, then $(\beta,\boldsymbol{e}) = ((1,-\beta_C,-m'),\boldsymbol{e}_1) + ((0,0,m'-m),\boldsymbol{e}_2)$, where*

   (a) (DT) *$\boldsymbol{e}_1 < \boldsymbol{e}_2$ if using $\tau_0^{-}(s,0)$;*
   (b) (PT) *$\boldsymbol{e}_1 > \boldsymbol{e}_2$ if using $\tau_0^{+}(s,0)$.*

   Here $<$ and $>$ refer to the lexicographical order.

Classes of the form $\beta_i = (0,-\beta'_C,-m')$ for $\beta'_C \neq 0$ will never arise in (60) because their $\tau_0 = \pi/2$ is already smaller than the τ_0 of classes of the form $(1,-\beta'_C,-m')$ or $(0,0,-m')$.

Later, in the situation of (iii) above, we will take $\{\boldsymbol{f},\boldsymbol{g}\} = \{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ such that $\boldsymbol{f} > \boldsymbol{g}$. Then a DT splitting is $((0,0,m'-m),\boldsymbol{f}) + ((1,-\beta_C,-m'),\boldsymbol{g})$ while a PT splitting is $((1,-\beta_C,-m'),\boldsymbol{f}) + ((0,0,m'-m),\boldsymbol{g})$.

6.2.4 Proof Part (i) is [Joyce 2021, Proposition 10.16], which we briefly review. By the definition of $\tau_0^{\pm}(s,0)$, evidently $\beta_i \neq 0$ for all i. If $\boldsymbol{e}_i = \mathbf{0}$ for some i, then $e_{j,N} > e_{j,N+1} := f_{k,p}(\beta_j)$ for some j, contradicting Lemma 6.1.4. Consequently, Lemma 6.1.4 implies that each $(\beta_i,\boldsymbol{e}_i)$ must be a full flag. No two $\boldsymbol{e}_i$ can be proportional, otherwise their sum cannot satisfy the condition $e_a \leq e_{a+1} \leq e_a + 1$ for all $1 \leq a < N$. Hence the condition (60) yields a collection of $M-1$ linear equations of the form

$$A_i s + B_i = 0, \quad \text{where } B_i \neq 0, \tag{61}$$

for the variable s. Genericity of $\boldsymbol{\mu}$ implies that there is no solution if $M > 2$, and that $s \in (0,1)$. Finally, if there were a strictly $\tau_0^{\pm}(s,0)$-semistable object of class $(\beta_i,\boldsymbol{e}_i)$, then there would be a splitting $(\beta_i,\boldsymbol{e}_i) = (\beta'_i,\boldsymbol{e}'_i) + (\beta''_i,\boldsymbol{e}''_i)$ such that, writing $\{i,j\} = \{1,2\}$, the splitting $(\beta,\boldsymbol{e}) = (\beta'_i,\boldsymbol{e}'_i) + (\beta''_i,\boldsymbol{e}''_i) + (\beta_j,\boldsymbol{e}_j)$ also satisfies the conditions of the lemma, contradicting $M \leq 2$.

Parts (ii) and (iii) are clear after writing out (60) explicitly. Namely, $((0,0,-m),\boldsymbol{e})$ can only split into two classes $((0,0,-m'),\boldsymbol{e}_1)$ and $((0,0,-m+m'),\boldsymbol{e}_2)$, in which case (61) reads

$$\frac{sm'\lambda^{\pm}(0,0,-1)+\boldsymbol{\mu}\cdot\boldsymbol{e}_1}{m'}=\frac{s(m-m')\lambda^{\pm}(0,0,-1)+\boldsymbol{\mu}\cdot\boldsymbol{e}_2}{m-m'}.$$

This is independent of s, i.e., $A_i=0$, and therefore has no solution. Similarly, for (60) to hold, $((1,-\beta_C,-m),\boldsymbol{e})$ can only split into two classes $((1,-\beta_C,-m'),\boldsymbol{e}_1)$ and $((0,0,-m+m'),\boldsymbol{e}_2)$, in which case (61) reads

$$\frac{s(m'-n)\lambda^{\pm}(0,0,-1)+\boldsymbol{\mu}\cdot\boldsymbol{e}_1}{m'-n_0}=\frac{s(m-m')\lambda^{\pm}(0,0,-1)+\boldsymbol{\mu}\cdot\boldsymbol{e}_2}{m-m'}.$$

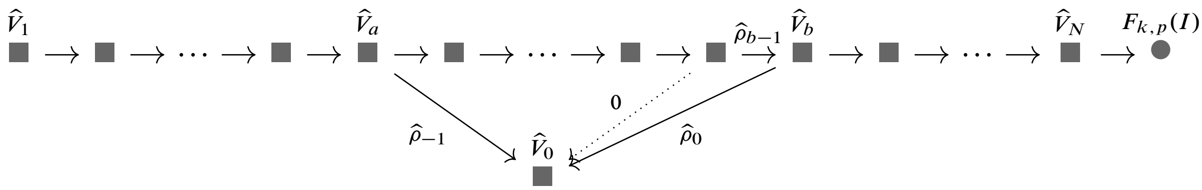
This simplifies to the equation

$$-s(n-n_0)\lambda^{\pm}(0,0,-1)=\boldsymbol{\mu}\cdot\left(\frac{m'-n_0}{m-m'}\boldsymbol{e}_2-\boldsymbol{e}_1\right), \tag{62}$$

which defines the walls s_i in the wall-crossing along s. The left-hand side is positive for τ^- and negative for τ^+. Since $\mu_1\gg\mu_2\gg\cdots$, and $n_0<m'<m$, this implies $\boldsymbol{e}_1<\boldsymbol{e}_2$ and $\boldsymbol{e}_1>\boldsymbol{e}_2$ in the two cases, respectively. □

6.2.5 Definition [Joyce 2021, Definition 10.19] Fix a full flag $(\beta,\boldsymbol{e})$ where $\beta=(1,-\beta_C,-m)$. Let $s\in(0,1)$ be a wall for $\widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(s,0)$ from Lemma 6.2.3(iii), i.e., a solution s to (62). For this s, genericity of $\boldsymbol{\mu}$ implies that the integer vectors $\boldsymbol{e}_1,\boldsymbol{e}_2$ solving (62) are unique.

Let $\{\boldsymbol{f},\boldsymbol{g}\}=\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$, ordered so that $\boldsymbol{f}>\boldsymbol{g}$. Let a (resp. b) be the smallest index where $f_a>0$ (resp. $g_b>0$) and consider the quiver $\widehat{Q}(k,a,b)$ given by



In complete analogy with Definition 4.1.2, let

$$\mathfrak{N}^{\widehat{Q}(k,a,b)}_{(\lambda,\mu,\nu),m,\hat{\boldsymbol{e}}}:=\{[(I,\widehat{\boldsymbol{V}},\widehat{\boldsymbol{\rho}})]:I\in\mathfrak{N}_{(\lambda,\mu,\nu),m},\ \dim\widehat{\boldsymbol{V}}=\hat{\boldsymbol{e}}\}$$

be the moduli stack of triples. We consider only $\hat{\boldsymbol{e}}:=(1,\boldsymbol{e})$, i.e., $\dim\widehat{V}_0=1$. Take the weak stability condition $\hat{\tau}_0^{\pm}(s,x)$ defined using the formula (50) but with the parameter $\hat{\boldsymbol{\mu}}:=(-\epsilon,\boldsymbol{\mu})$ for a very small $\epsilon>0$. Only a finite set of classes can occur in splittings by Lemma 6.2.3. So, we obtain a finite set of smallness conditions on ϵ as in [Joyce 2021, 10.6].

The dotted arrow is a relation, not a quiver map; it means to take the closed substack

$$\mathbb{M}\subset\mathfrak{N}^{\widehat{Q}(k,a,b),\mathrm{sst}}_{(\lambda,\mu,\nu),m,\hat{\boldsymbol{e}}}(\hat{\tau}_0^{\pm}(s,0)),$$

where $\widehat{\rho}_0\circ\widehat{\rho}_{b-1}=0$. Since $e_b=e_{b-1}+1$, this is the substack where $\widehat{\rho}_0$ factors through a morphism $\widehat{V}_b/\widehat{V}_{b-1}\to\widehat{V}_0$.

6.2.6 The moduli stack $\mathbb{M}$ is the master space for Joyce-style wall-crossing. Let $\mathbb{C}^\times$ act by scaling the map $\hat{\rho}_{-1}$ with weight denoted by z. We work $(\mathbb{C}^\times \times \mathsf{T})$-equivariantly on $\mathbb{M}$.

In any splitting $(\beta, (1, \boldsymbol{e})) = (\gamma, (1, \boldsymbol{f})) + (\delta, (0, \boldsymbol{g}))$, the $\hat{\tau}_0^\pm(s, 0)$ of the two terms cannot be equal because ϵ is small and contributes to only one of them. Hence $\mathbb{M}$ contains no strictly semistable objects. By the same reasoning as in Proposition 6.1.5, one concludes that the master space is a separated algebraic space with proper fixed loci for the maximal torus $\mathsf{T}_w \subset \ker(w)$ associated to any $(\mathbb{C}^\times \times \mathsf{T})$-weight w with nontrivial $\mathbb{C}^\times$ component (cf. Proposition 5.2.4). In particular, it has proper T-fixed locus. Then Theorem 2.3.4 endows $\mathbb{M}$ with a symmetric APOT by symmetrized pullback along the forgetful map to $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),m}$. By Theorem 2.4.3 and Lemma 3.3.7, we obtain a symmetrized virtual structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}$ amenable to $(\mathbb{C}^\times \times \mathsf{T})$-equivariant localization.

6.2.7 Proposition [Joyce 2021, Propositions 10.20 and 10.21] *The $\mathbb{C}^\times$-fixed locus of $\mathbb{M}$ is the disjoint union of the following pieces.*

(i) *Let $Z_{\hat{\rho}_{-1}=0} := \{\hat{\rho}_{-1} = 0\} \subset \mathbb{M}$. By stability, $\hat{\rho}_0 \neq 0$. For $s_- < s$ very close to s, there is a natural isomorphism of stacks*

$$Z_{\hat{\rho}_{-1}=0} \xrightarrow{\sim} \mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(\tau_0^\pm(s_-, 0)), \quad [(\boldsymbol{I}, (\hat{V}_0, \boldsymbol{V}), (0, \hat{\rho}_0, \boldsymbol{\rho}))] \mapsto [(\boldsymbol{I}, \boldsymbol{V}, \boldsymbol{\rho})],$$

which identifies the $\hat{\mathcal{O}}^{\mathrm{vir}}$. The virtual normal bundle $\mathcal{N}^{\mathrm{vir}}_{\hat{\rho}_{-1}=0}$ is a line bundle with $\mathbb{C}^\times$-weight z.

(ii) *Let $Z_{\hat{\rho}_0=0} := \{\hat{\rho}_0 = 0\} \subset \mathbb{M}$. By stability, $\hat{\rho}_{-1} \neq 0$. For $s_+ > s$ very close to s, there is a natural isomorphism of stacks*

$$Z_{\hat{\rho}_0=0} \xrightarrow{\sim} \mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(\tau_0^\pm(s_+, 0)), \quad [(\boldsymbol{I}, (\hat{V}_0, \boldsymbol{V}), (\hat{\rho}_{-1}, 0, \boldsymbol{\rho}))] \mapsto [(\boldsymbol{I}, \boldsymbol{V}, \boldsymbol{\rho})],$$

which identifies the $\hat{\mathcal{O}}^{\mathrm{vir}}$. The virtual normal bundle $\mathcal{N}^{\mathrm{vir}}_{\hat{\rho}_0=0}$ is a line bundle with $\mathbb{C}^\times$-weight z^{-1}.

(iii) *For each splitting $(\beta, \boldsymbol{e}) = (\gamma, \boldsymbol{f}) + (\delta, \boldsymbol{g})$ satisfying Lemma 6.2.3, let*

$$Z^\pm_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})} := \big\{\big[\big(\boldsymbol{I}' \oplus \boldsymbol{I}'', (\hat{V}_0, \boldsymbol{V}' \oplus \boldsymbol{V}''), (\hat{\rho}_{-1}, \hat{\rho}_0, \boldsymbol{\rho}' \oplus \boldsymbol{\rho}'')\big)\big] : \hat{\rho}_{-1} \neq 0,\ \hat{\rho}_0|_{V_b'} = 0,\ \hat{\rho}_0|_{V_b''} \neq 0\big\} \subset \mathbb{M},$$

where

$$\mathrm{cl}((\boldsymbol{I}', \boldsymbol{V}', \boldsymbol{\rho}')) = (\gamma, \boldsymbol{f}) \quad \text{and} \quad \mathrm{cl}((\boldsymbol{I}'', \boldsymbol{V}'', \boldsymbol{\rho}'')) = (\delta, \boldsymbol{g}).$$

There are natural isomorphisms of stacks

$$\begin{aligned} Z^-_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})} &\xrightarrow{\sim} \mathfrak{Q}^{Q(N),\mathrm{sst}}_{m',\boldsymbol{f}}(\tau_0^-(s,0)) \times \mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m-m',\boldsymbol{g}}(\tau_0^-(s,0)), \\ Z^+_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})} &\xrightarrow{\sim} \mathfrak{N}^{Q(N),\mathrm{sst}}_{(\lambda,\mu,\nu),m-m',\boldsymbol{f}}(\tau_0^+(s,0)) \times \mathfrak{Q}^{Q(N),\mathrm{sst}}_{m',\boldsymbol{g}}(\tau_0^+(s,0)), \end{aligned} \tag{63}$$

given by

$$\big[\big(\boldsymbol{I}, (\hat{V}_0, \boldsymbol{V}), (\hat{\rho}_{-1}, \hat{\rho}_0, \boldsymbol{\rho})\big)\big] \mapsto ([(\boldsymbol{I}', \boldsymbol{V}', \boldsymbol{\rho}')], [(\boldsymbol{I}'', \boldsymbol{V}'', \boldsymbol{\rho}'')]),$$

which identifies the $\widehat{\mathcal{O}}^{\mathrm{vir}}$. The target stacks have no strictly $\tau_0^{\pm}(s,0)$-semistable objects by Lemma 6.2.3(i). Under this isomorphism, the virtual normal bundle is

$$\mathcal{N}^{\mathrm{vir}}_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})} = \\ (z^{-1}\mathbb{F}(\boldsymbol{f},\boldsymbol{g}) + z\mathbb{F}(\boldsymbol{g},\boldsymbol{f})) - \kappa\cdot(\cdots)^\vee - \big(z^{-1}R\pi_*\,\mathcal{E}xt(\mathcal{I}_\gamma,\mathcal{I}_\delta(-D)) + zR\pi_*\,\mathcal{E}xt(\mathcal{I}_\delta,\mathcal{I}_\gamma(-D))\big),$$

where $\mathbb{F}(\boldsymbol{f},\boldsymbol{g})$ is the restriction of (40) to numerical class $(\boldsymbol{f},\boldsymbol{g})$, and $\mathcal{I}_\gamma$ and $\mathcal{I}_\delta$ are pullbacks of universal families for numerical classes γ and δ, respectively. The $(\cdots)^\vee$ indicates the dual of the preceding bracketed term.

Proof The proof exactly follows Joyce's, except when showing that the various identifications of fixed loci also identify their $\widehat{\mathcal{O}}^{\mathrm{vir}}$. While Joyce carefully shows that the virtual cycles agree, in our case it is easier to apply Proposition 2.5.2 and match K-theory classes of APOTs. We show how this works in case (iii), for $Z := Z^-_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})}$; the other cases are similar, but simpler.

On the left-hand side of (63), there is an APOT which is the $\mathbb{C}^\times$-fixed part of the APOT on $\mathbb{M}$, which was obtained by symmetrized pullback along the forgetful morphism $g\colon \mathbb{M}\to\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),m}$. So, in K-theory,

$$(64)\quad (\mathbb{E}_{\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),m}} + \Omega_g - \kappa\Omega_g^\vee)|_Z = \big(R\pi_*\,\mathcal{E}xt(\mathcal{I}_\beta,\mathcal{I}_\beta(-D)) + \mathbb{F}(\boldsymbol{e},\boldsymbol{e}) - \kappa\mathbb{F}(\boldsymbol{e},\boldsymbol{e})^\vee\big)|_Z \\ = -R\pi_*\,\mathcal{E}xt((z\mathcal{I}_\gamma)\oplus\mathcal{I}_\delta,(z\mathcal{I}_\gamma)\oplus\mathcal{I}_\delta) \\ +(\mathbb{F}(\boldsymbol{f},\boldsymbol{f})+\mathbb{F}(\boldsymbol{g},\boldsymbol{g})+z^{-1}\mathbb{F}(\boldsymbol{f},\boldsymbol{g})+z\mathbb{F}(\boldsymbol{g},\boldsymbol{f}))-\kappa\otimes(\cdots)^\vee,$$

where the second equality passes through the isomorphism (63); for instance, $\mathcal{I}_\beta$ splits as $z\mathcal{I}_\gamma\oplus\mathcal{I}_\delta$. We omitted some obvious pullbacks.

On the right-hand side of (63), there is an APOT which is the $\boxplus$ of two APOTs obtained by symmetrized pullback along forgetful morphisms to $\mathfrak{Q}^{\mathrm{pl}}_{m'}$ and $\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),m-m'}$ respectively. This is equivalent to symmetrized pullback along the map f which is the $\boxplus$ of the two forgetful morphisms. So

$$\mathbb{E}_{\mathfrak{Q}^{\mathrm{pl}}_{m'}\times\mathfrak{N}^{\mathrm{pl}}_{(\lambda,\mu,\nu),m-m'}} + \Omega_f - \kappa\Omega_f^\vee \\ = -R\pi_*\,\mathcal{E}xt(\mathcal{I}_\gamma,\mathcal{I}_\gamma) + R\pi_*\,\mathcal{E}xt(\mathcal{I}_\delta,\mathcal{I}_\delta) + (\mathbb{F}(\boldsymbol{f},\boldsymbol{f})+\mathbb{F}(\boldsymbol{g},\boldsymbol{g})) - \kappa\otimes(\cdots)^\vee.$$

This is evidently the $\mathbb{C}^\times$-fixed part of (64). Hence the natural $\widehat{\mathcal{O}}^{\mathrm{vir}}$ on both sides of (63) are identified by Proposition 2.5.2.

It is also clear that the non-$\mathbb{C}^\times$-fixed part of (64) is the claimed virtual normal bundle $\mathcal{N}^{\mathrm{vir}}_{(\gamma,\boldsymbol{f}),(\delta,\boldsymbol{g})}$. □

6.2.8 Using Proposition 6.2.7 and applying appropriate residues to the $\mathbb{C}^\times$-localization formula on $\mathbb{M}$, as written in Proposition 1.4.5, we immediately obtain the "single-step" wall-crossing formula

$$(65)\quad 0 = \widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(s_-,0) - \widetilde{\mathsf{N}}^{\pm}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(s_+,0) \\ + \sum_{\substack{(\beta,\boldsymbol{e})=(\gamma,\boldsymbol{f})+(\delta,\boldsymbol{g})\\ \tau_0^{\pm}(s,0)(\gamma,\boldsymbol{f})=\tau_0^{\pm}(s,0)(\delta,\boldsymbol{g})\\ \boldsymbol{f}>\boldsymbol{g}}} [\chi(\gamma,\delta) - c_Q(\boldsymbol{f},\boldsymbol{g})]_\kappa\cdot\widetilde{\mathsf{Z}}^{\pm}_{\gamma,\boldsymbol{f}}(s,0)\widetilde{\mathsf{Z}}^{\pm}_{\delta,\boldsymbol{g}}(s,0),$$

using notation from Section 4.1.4. Each term corresponds to a $\mathbb{C}^\times$-fixed locus. Note that the splittings have $\gamma = (0,0,-m')$ for τ^- and $\delta = (0,0,-m')$ for τ^+, so we use the more uniform notation $\widetilde{\mathsf{Z}}^\pm$ in the sum.

Collecting the wall-crossings at each wall s which occurs in Lemma 6.2.3(iii) yields a combinatorial wall-crossing formula relating $\widetilde{\mathsf{Z}}^\pm_{\beta,\boldsymbol{e}}(0,0)$ and $\widetilde{\mathsf{Z}}^\pm_{\beta,\boldsymbol{e}}(1,0)$. These formulas depend on the direction in which we are moving along the s axis.

6.2.9 For the DT wall-crossing, recursively apply the "$-$" case of the single-step wall-crossing formula (65) to move from $s=1$ to $s=0$, starting from the class $(\alpha,\boldsymbol{d})$ where $\boldsymbol{d} = (1,2,\dots,N)$. We claim that the resulting formula is

$$(66)\quad \widetilde{\mathsf{N}}^-_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(1,0) = \sum_{\substack{\ell>0,(\alpha,\boldsymbol{d})=\sum_{i=1}^{\ell}(\gamma_i,\boldsymbol{e}_i)\\ \{\gamma_i=:(0,0,-m_i)\}_{i=1}^{\ell-1},\\ \gamma_\ell=:(1,-\beta_C,-m_\ell)\\ \forall i:(\gamma_i,\boldsymbol{e}_i)\text{ full flag}\\ \boldsymbol{e}_1>\cdots>\boldsymbol{e}_\ell}} \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m_\ell,\boldsymbol{e}_\ell}(0,0)\prod_{i=1}^{\ell-1}\Big[\chi\Big(\gamma_i,\sum_{j=i+1}^{\ell}\gamma_j\Big)-c_Q\Big(\boldsymbol{e}_i,\sum_{j=i+1}^{\ell}\boldsymbol{e}_j\Big)\Big]_\kappa \widetilde{\mathsf{Q}}_{m_i,\boldsymbol{e}_i}.$$

We explain the conditions in the sum. By Lemma 6.2.3, all full flags split into smaller full flags at each wall. Since we started with a full flag $(\alpha,\boldsymbol{d})$, note that all sums $\sum_{i\in I}(\gamma_i,\boldsymbol{e}_i)$ for any $I\subset\{1,\dots,\ell\}$ must be full flags. Conversely, every splitting into two smaller full flags occurs at some $s\in(0,1)$ because $|\lambda^\pm(0,0,-1)|$ in (62) is very big. Finally, the ordering condition $\boldsymbol{e}_1>\cdots>\boldsymbol{e}_\ell$ arises by considering two consecutive splittings. Namely, take walls $s'<s$ where numerical classes split as

$$\begin{aligned}((1,-\beta_C,-m),\boldsymbol{f}_{i-1}) &= ((0,0,-m+m'),\boldsymbol{e}_i)+((1,-\beta_C,-m'),\boldsymbol{f}_i) && \text{at } s\\ ((1,-\beta_C,-m'),\boldsymbol{f}_i) &= ((0,0,-m'+m''),\boldsymbol{e}_{i+1})+((1,-\beta_C,-m''),\boldsymbol{f}_{i+1}) && \text{at } s'.\end{aligned}$$

Since $\boldsymbol{e}_i>\boldsymbol{f}_i=\boldsymbol{e}_{i+1}+\boldsymbol{f}_{i+1}$, it is clear that $\boldsymbol{e}_i>\boldsymbol{e}_{i+1}$. Iterating this over all walls yields the ordering condition. In the word notation of Definition 4.2.5,

$$(67)\quad \widetilde{\mathsf{N}}^-_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(1,0) = \sum_{\substack{k>0,\,\boldsymbol{m}\in\mathbb{Z}^k_{>0}\\ w\in R(\boldsymbol{m},N-|\boldsymbol{m}|)\\ o_1(w)<\cdots<o_{k+1}(w)}} \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),n-|\boldsymbol{m}|,\boldsymbol{e}_{k+1}}(0,0)\prod_{i=1}^{k}\Big[m_i-\sum_{j=i+1}^{k+1}c_{i,j}(w)\Big]_\kappa \widetilde{\mathsf{Q}}_{m_i,\boldsymbol{e}_i},$$

where $(\boldsymbol{e}_i)_{i=1}^{k+1}$ is the decomposition of $\boldsymbol{d}$ associated to the word w, and we used that

$$\chi((0,0,-m),(1,-\beta_C,-m'))=m.$$

The condition that $\boldsymbol{e}_1>\cdots>\boldsymbol{e}_\ell$ becomes the condition that $o_1(w)<\cdots<o_{k+1}(w)$.

6.2.10 For the PT wall-crossing, recursively apply the "+" case of the single-step wall-crossing formula (65) to move along s in the other direction, from $s=0$ to $s=1$. We start from a full flag $(\beta,\boldsymbol{e})$ with

$\beta = (1, -\beta_C, -m)$; later this will be $(\gamma_\ell, \boldsymbol{e}_\ell)$ from (66). We claim that the resulting formula is

$$\text{(68)}\quad \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(0,0) = \sum_{\substack{\ell>0,\ (\beta,\boldsymbol{e})=\sum_{i=1}^{\ell}(\gamma_i,\boldsymbol{f}_i)\\ \{\gamma_i=:(0,0,-m_i)\}_{i=1}^{\ell-1},\\ \gamma_\ell=:(1,-\beta_C,-m_\ell)\\ \forall i:(\gamma_i,\boldsymbol{f}_i)\text{ full flag}\\ \boldsymbol{f}_\ell>\boldsymbol{f}_1>\cdots>\boldsymbol{f}_{\ell-1}}} \widetilde{\mathsf{N}}^{+}_{(\lambda,\mu,\nu),m_\ell,\boldsymbol{f}_\ell}(1,0)\prod_{i=1}^{\ell-1}\Bigl[\chi\Bigl(\gamma_i,\sum_{j=i+1}^{\ell}\gamma_j\Bigr)-c_Q\Bigl(\boldsymbol{f}_i,\sum_{j=i+1}^{\ell}\boldsymbol{f}_j\Bigr)\Bigr]_\kappa \widetilde{\mathsf{Q}}_{m_i,\boldsymbol{f}_i},$$

where we absorbed a sign $(-1)^{\ell-1}$ into the quantum integers using the skew-symmetry of χ and c_Q.

To explain the ordering condition in the above sum, we again consider two consecutive walls and splittings

$$\begin{aligned} ((1,-\beta_C,-m),\boldsymbol{e}_{i-1}) &= ((1,-\beta_C,-m'),\boldsymbol{e}_i)+((0,0,-m+m'),\boldsymbol{f}_i) && \text{at } s,\\ ((1,-\beta_C,-m'),\boldsymbol{e}_i) &= ((1,-\beta_C,-m''),\boldsymbol{e}_{i+1})+((0,0,-m'+m''),\boldsymbol{f}_{i+1}) && \text{at } s', \end{aligned}$$

but now $s' > s$ since we are moving in the other direction along s, and $\boldsymbol{e}_{i+1}+\boldsymbol{f}_{i+1}=\boldsymbol{e}_i>\boldsymbol{f}_i$ and we must compare $\boldsymbol{f}_i$ with $\boldsymbol{f}_{i+1}$. Using (62), we get

$$\frac{-1}{(n-n_0)\lambda^+}\boldsymbol{\mu}\cdot\Bigl(\frac{m''-n_0}{m'-m''}\boldsymbol{f}_{i+1}-\boldsymbol{e}_{i+1}\Bigr)=s'>s=\frac{-1}{(n-n_0)\lambda^+}\boldsymbol{\mu}\cdot\Bigl(\frac{m'-n_0}{m-m'}\boldsymbol{f}_i-\boldsymbol{e}_{i+1}-\boldsymbol{f}_{i+1}\Bigr),$$

where λ^+ is short for $\lambda^+(0,0,-1)$. As the prefactor $-1/((n-n_0)\lambda^+)$ is negative, this is equivalent to

$$\frac{m'-n_0}{m'-m''}\boldsymbol{\mu}\cdot\boldsymbol{f}_{i+1}<\frac{m'-n_0}{m-m'}\boldsymbol{\mu}\cdot\boldsymbol{f}_i,$$

which yields $\boldsymbol{f}_i > \boldsymbol{f}_{i+1}$ since $\boldsymbol{\mu}$ is generic. Iterating this over all walls yields the ordering condition $\boldsymbol{f}_1>\cdots>\boldsymbol{f}_{\ell-1}$. Finally, in a sequence of splittings

$$\boldsymbol{e}=\boldsymbol{e}_1+\boldsymbol{f}_1=(\boldsymbol{e}_2+\boldsymbol{f}_2)+\boldsymbol{f}_1=\cdots=((\boldsymbol{e}_{\ell-1}+\boldsymbol{f}_{\ell-1})+\cdots)+\boldsymbol{f}_1,$$

where $\boldsymbol{e}_i > \boldsymbol{f}_i$ by Lemma 6.2.3(iii), every $\boldsymbol{e}_i$ has the same first nonzero entry as $\boldsymbol{e}$ since $(\beta,\boldsymbol{e})$ is a full flag. Hence $\boldsymbol{f}_\ell := \boldsymbol{e}_{\ell-1} > \boldsymbol{f}_1$. In the word notation of Definition 4.2.5,

$$\text{(69)}\quad \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m,\boldsymbol{e}}(0,0) = \sum_{\substack{k>0,\ \boldsymbol{m}\in\mathbb{Z}^k_{>0}\\ w\in R(\boldsymbol{m},M-|\boldsymbol{m}|)\\ o_{k+1}(w)<o_1(w)<\cdots<o_k(w)}} \widetilde{\mathsf{N}}^{+}_{(\lambda,\mu,\nu),m-|\boldsymbol{m}|,\boldsymbol{f}_{k+1}}(1,0)\prod_{i=1}^{k}\Bigl[m_i-\sum_{j=i+1}^{k+1}c_{i,j}(w)\Bigr]_\kappa \widetilde{\mathsf{Q}}_{m_i,\boldsymbol{f}_i},$$

where $M := f_{k,p}((1,-\beta_C,-m))$ and $(\boldsymbol{f}_i)_{i=1}^{k+1}$ is the decomposition of $\boldsymbol{e}$ associated to the word w.

6.2.11 Proposition *In the word notation of Definition 4.2.5,*

(70)
$$\widetilde{\mathsf{N}}^{-}_{(\lambda,\mu,\nu),n,\boldsymbol{d}}(1,0) = \sum_{\substack{k>0,\,\boldsymbol{m}\in\mathbb{Z}^k_{>0}\\ w\in R(\boldsymbol{m},N-|\boldsymbol{m}|)\\ o_1(w)<\cdots<o_k(w)}} \widetilde{\mathsf{N}}^{+}_{(\lambda,\mu,\nu),n-|\boldsymbol{m}|,\boldsymbol{e}_{k+1}}(1,0)\prod_{i=1}^{k}\Big[m_i - \sum_{j=i+1}^{k+1} c_{i,j}(w)\Big]_\kappa \widetilde{\mathsf{Q}}_{m_i,\boldsymbol{e}_i}.$$

Proof Plug (68) into (66). The composite of a DT splitting in (66) with a PT splitting in (68) has the form

$$\boldsymbol{d} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{\ell_--1} + \boldsymbol{e}_{\ell_-} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{\ell_--1} + (\boldsymbol{f}_{\ell_+} + \boldsymbol{f}_{\ell_+-1} + \cdots + \boldsymbol{f}_1)$$

on framing dimensions, where $\boldsymbol{e}_1 > \cdots > \boldsymbol{e}_{\ell_--1} > \boldsymbol{e}_{\ell_-}$ (DT ordering) and $\boldsymbol{f}_{\ell_+} > \boldsymbol{f}_1 > \cdots > \boldsymbol{f}_{\ell_+-1}$ (PT ordering). Consequently,

$$\boldsymbol{e}_{\ell_--1} > \boldsymbol{e}_{\ell_-} > \boldsymbol{f}_{\ell_+},$$

because $\boldsymbol{f}_{\ell_+}$ is a summand of $\boldsymbol{e}_{\ell_-}$. Hence the composite ordering condition is

$$\boldsymbol{e}_1 > \cdots > \boldsymbol{e}_{\ell_--1} > \boldsymbol{f}_{\ell_+} > \boldsymbol{f}_1 > \cdots > \boldsymbol{f}_{\ell_+-1}.$$

Since we sum over all $\ell_- > 0$ and $\ell_+ > 0$, the result is that there is no constraint on $\boldsymbol{f}_{\ell_+}$. Setting $\boldsymbol{e}_{\ell_--1+i} := \boldsymbol{f}_i$ and $\ell := \ell_- + \ell_+ - 1$, the composite splitting therefore has the ordering $\boldsymbol{e}_1 > \cdots > \boldsymbol{e}_{\ell-1}$. In word notation this becomes $o_1(w) < \cdots < o_{\ell-1}(w)$.

Conversely, the position of $o_\ell(w)$ specifies the splitting $\ell = (\ell_- - 1) + \ell_+$ into the DT and PT parts of the splitting. □

6.3 "Combinatorial reduction"

6.3.1 We can use the formulas (57) and (59) to remove the dependence on framing dimensions $\boldsymbol{e}$ from the auxiliary invariants $\widetilde{\mathsf{N}}$ and $\widetilde{\mathsf{Q}}$ in (70). The result is

(71)
$$\mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^-) = \sum_{m\in\mathbb{Z}} \mathsf{N}_{(\lambda,\mu,\nu),n-m}(\tau^+)\cdot \mathsf{W}_{m,N},$$

where

$$\mathsf{W}_{m,N} := \sum_{\substack{k>0,\,\boldsymbol{m}\in\mathbb{Z}^k_{>0}\\ |\boldsymbol{m}|=m}} c'_<(\boldsymbol{m}, N-m)\frac{[N-m]_\kappa!}{[N]_\kappa!}\prod_{i=1}^{k}[m_i-1]_\kappa!\cdot\widetilde{\mathsf{Q}}_{m_i},$$

$$c'_<(m_1,\dots,m_\ell) := \sum_{\substack{w\in R(m_1,\dots,m_\ell)\\ o_1(w)<\cdots<o_{\ell-1}(w)}} \prod_{i=1}^{\ell-1}\Big[m_i - \sum_{j=i+1}^{\ell} c_{i,j}(w)\Big]_\kappa.$$

The combinatorial quantity $c'_<$ should be compared with the similar quantity $c'_>$ from Section 5.3.

6.3.2 In principle, one can now evaluate $\mathsf{W}_{m,N}$ explicitly in order to obtain the desired DT/PT vertex correspondence; see Proposition 6.3.4. The combinatorics involved is similar to that of Section 5.3, so, instead, we provide a different method of obtaining the DT/PT vertex correspondence from (71).

The idea is as follows. All the wall-crossing so far was done in a setup for fixed $\alpha = (1, -\beta_C, -n)$, especially the single-step wall-crossing (65) for full flags $(\beta, \boldsymbol{e})$. We now wish to allow n to vary because the DT/PT vertex correspondence is an equality of generating series. However, the N in (71) depends nontrivially on n (and on the curve class β_C). We will remove this dependence as follows.

6.3.3 Fix an integer M. The DT/PT vertex correspondence is an equality of Laurent series in Q, and we will prove it modulo Q^M. Since $\mathfrak{N}_{(\lambda,\mu,\nu),m} = \varnothing$ for $m \le n_0$ and $\mathfrak{Q}_m = \varnothing$ for $m < 0$, this means that only the enumerative invariants associated to the finite collection of moduli stacks

$$\{\mathfrak{N}_{(\lambda,\mu,\nu),m} : n_0 < m < M\} \quad \text{and} \quad \{\mathfrak{Q}_m : 0 \le m < M - n_0\}$$

are involved. Hence:

- We may choose a single $k \in \mathbb{Z}$ which satisfies the regularity condition of Definition 4.1.1 for each $\mathfrak{N}_{(\lambda,\mu,\nu),m}$ for $n_0 < m < M$.
- Choosing $\tilde{N} \ge f_{k,1}(1, -\beta_C, -M)$, we set $p_m := \tilde{N} - (M - m)$. This is bounded below by $f_{k,1}(1, \beta_C, -m)$, hence positive for $n_0 < m < M$. This choice guarantees that $\tilde{N} = f_{k,p_m}(\alpha)$ is independent of $n_0 < m < M$.

For each $n_0 < m < M$, using the framing functor F_{k,p_m}, we therefore obtain the wall-crossing formula (71). Collecting them into a generating series, we obtain the factorization

$$\sum_{n \ge n_0} Q^n \mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^-) \equiv \Bigl(\sum_{n \ge n_0} Q^n \mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^+)\Bigr)\Bigl(\sum_{n \ge 0} Q^n \mathsf{W}_{n,\tilde{N}}\Bigr) \mod Q^M. \tag{72}$$

This is almost Theorem 1.3.1, since the first two bracketed terms are $\mathsf{V}^{\mathrm{DT},K}_{\lambda,\mu,\nu}$ and $\mathsf{V}^{\mathrm{PT},K}_{\lambda,\mu,\nu}$, respectively, and it remains to evaluate the third bracketed term. The formula (72) is stated for a given (λ, μ, ν), and the constant $\tilde{N}$ depends a priori on the curve class $\beta_C = (|\lambda|, |\mu|, |\nu|)$. We apply the same sort of trick again to eliminate this dependence. For this, we may choose k, and $\tilde{N}$ large enough so that they work for both the original β_C and chosen M, as well as for $\beta_C = 0$ and M replaced by $M + n_0$. This yields the same formula (72) when $(\lambda, \mu, \nu) = (\varnothing, \varnothing, \varnothing)$, i.e.,

$$\sum_{n \in \mathbb{Z}} Q^n \mathsf{N}_{(\varnothing,\varnothing,\varnothing),n}(\tau^-) \equiv 1 \cdot \sum_{n \in \mathbb{Z}} Q^n \mathsf{W}_{n,\tilde{N}} \mod Q^{M+n_0} \tag{73}$$

by the triviality of the PT vertex for $(\varnothing, \varnothing, \varnothing)$. To emphasize, it is the same $\tilde{N}$ in (72) which appears in (73). So, inserting (73) into (72) gives

$$\Bigl(\sum_{n \in \mathbb{Z}} Q^n \mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^-)\Bigr) \equiv \Bigl(\sum_{n \in \mathbb{Z}} Q^n \mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^+)\Bigr)\Bigl(\sum_{n \in \mathbb{Z}} Q^n \mathsf{N}_{(\varnothing,\varnothing,\varnothing),n}(\tau^-)\Bigr) \mod Q^M.$$

The dependence on $\widetilde{N}$ has been completely eliminated. Since M was arbitrary, this equality holds up to any finite order. This concludes the Joyce-style wall-crossing proof of the DT/PT vertex correspondence (Theorem 1.3.1). □

6.3.4 For completeness, we record the explicit formula for $\mathsf{W}_{m,N}$, even though we did not require it for the DT/PT proof. It should be compared with (46).

Proposition *For integers $k \geq 0$ and $N > 0$, and $\boldsymbol{m} = (m_1, \dots, m_k) \in \mathbb{Z}_{>0}^k$,*

$$\sum_{\sigma \in S_k} c'_<(m_{\sigma(1)}, \dots, m_{\sigma(k)}, N - |\boldsymbol{m}|) = \frac{[N]_\kappa!}{[N - |\boldsymbol{m}|]_\kappa! \prod_i [m_i - 1]_\kappa!} \cdot \begin{cases} 1 & \text{if } k \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Exercise for the reader, using the same ideas as in the proof of (46). □

Plugging this back into $\mathsf{W}_{m,N}$ shows that $\mathsf{W}_{m,N} = \widetilde{\mathsf{Q}}_m$, which, in particular, is independent of N. This independence is an alternative way to obtain the factorization (72), as an exact equality of series instead of merely up to some finite order.

6.3.5 For use in Section 6.4, we can also apply the formulas (57), (58) and (59) to the individual DT and PT wall-crossing formulas (67) and (69). For DT, the result is

$$\mathsf{N}_{(\lambda,\mu,\nu),n}(\tau^-) = \sum_{m \in \mathbb{Z}} \widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),n-m} \cdot \mathsf{W}^-_{m,N}, \tag{74}$$

where

$$\mathsf{W}^-_{m,N} := \sum_{\substack{k>0,\, \boldsymbol{m} \in \mathbb{Z}^k_{>0} \\ |\boldsymbol{m}|=m}} c_<(\boldsymbol{m}, N-m) \frac{[N-m-1]_\kappa!}{[N]_\kappa!} \prod_{i=1}^k [m_i - 1]_\kappa! \cdot \widetilde{\mathsf{Q}}_{m_i},$$

$$c_<(m_1, \dots, m_\ell) := \sum_{\substack{w \in R(m_1,\dots,m_\ell) \\ o_1(w) < \cdots < o_\ell(w)}} \prod_{i=1}^{\ell-1} \Big[m_i - \sum_{j=i+1}^{\ell} c_{i,j}(w) \Big]_\kappa.$$

For PT, the result is

$$\widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),n} = \sum_{m \in \mathbb{Z}} \mathsf{N}_{(\lambda,\mu,\nu),n-m}(\tau^+) \cdot \mathsf{W}^+_{m,N}, \tag{75}$$

where

$$\mathsf{W}^+_{m,N} := \sum_{\substack{k>0,\, \boldsymbol{m} \in \mathbb{Z}^k_{>0} \\ |\boldsymbol{m}|=m}} b_<(\boldsymbol{m}, N-m) \frac{[N-m]_\kappa!}{[N-1]_\kappa!} \prod_{i=1}^k [m_i - 1]_\kappa! \cdot \widetilde{\mathsf{Q}}_{m_i},$$

$$b_<(m_1, \dots, m_\ell) := \sum_{\substack{w \in R(m_1,\dots,m_\ell) \\ o_\ell(w) < o_1(w) < \cdots < o_{\ell-1}(w)}} \prod_{i=1}^{\ell-1} \Big[m_i - \sum_{j=i+1}^{\ell} c_{i,j}(w) \Big]_\kappa.$$

6.3.6 Proposition *For integers $k > 0$ and $N > 0$, and $\boldsymbol{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k_{>0}$,*

$$\sum_{\sigma \in S_k} b_<(m_{\sigma(1)}, \dots, m_{\sigma(k)}, N - |\boldsymbol{m}|) = \frac{[N-1]_\kappa!}{[N-|\boldsymbol{m}|-1]_\kappa! \prod_i [m_i - 1]_\kappa!} \cdot \begin{cases} (-\kappa^{\frac{1}{2}})^{m_1} + (-\kappa^{\frac{1}{2}})^{-m_1} & \text{if } k = 1, \\ (-\kappa^{\frac{1}{2}})^{m_1 - m_2} + (-\kappa^{\frac{1}{2}})^{m_2 - m_1} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Exercise for the reader, using Proposition 6.3.4. □

6.4 Invariants at the wall

6.4.1 One advantage of the Joyce-style wall-crossing setup, compared to the Mochizuki-style setup of Section 5, is that we get for free some wall-crossing formulas (74) and (75) for the invariants $\widetilde{\mathsf{Q}}_m$ and $\widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m}$ *on* the wall τ_0. These wall-crossing formulas are interesting in their own right and have not appeared in the literature, to the best of our knowledge. We can explicitly identify $\widetilde{\mathsf{Q}}_m$ and $\widetilde{\mathsf{N}}_{(\lambda,\mu,\nu),m}$ as enumerative invariants of $\mathrm{Hilb}(\mathbb{C}^3)$ (Proposition 6.4.2) and of $\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}^{\oplus 2})$ (Proposition 6.4.4) respectively.

6.4.2 Proposition *Let $1 \le a \le N$. In the notation of Lemma 6.1.11, for $\bar{x}(a) \ge x_0(a)$ there is an isomorphism*

$$\mathfrak{Q}^{Q(N),\mathrm{sst}}_{\boldsymbol{m},\mathbf{1}_{[a,N]}}(\tau_0^{\pm}(0, \bar{x}(a))) \cong \mathfrak{Q}^{\mathrm{st}}_{\boldsymbol{m},\mathbf{1}_{[a,N]}}.$$

Proof This is basically an analogue of Lemma 6.1.8 for $\mathfrak{Q}^{Q(N)}$ instead of $\mathfrak{N}^{Q(N)}$. Let $(\mathcal{E}[-1], \boldsymbol{V}, \boldsymbol{\rho})$ be an object on the left-hand side. By Lemma 6.1.4, the framing is a chain of isomorphisms followed by an inclusion, like (52). Since $\mathcal{E}$ is a zero-dimensional sheaf, $\mathcal{E}(k) \cong \mathcal{E}$ and $V_{N+1} = H^0(\mathcal{E})$. Hence the data $(\mathcal{E}[-1], \boldsymbol{V}, \boldsymbol{\rho})$ is equivalent to the data $(\mathcal{E}, s)$ where $s\colon \mathcal{O}_{\mathbb{C}^3} \to \mathcal{E}$ is a section. If s were not surjective, it factors through a nontrivial subsheaf $\mathcal{E}' \subset \mathcal{E}$ of length $m' < m$. Then

$$\tau_0^{\pm}(0, \bar{x}(a))((\mathcal{E}'[-1], \boldsymbol{V}, \boldsymbol{\rho})) = \frac{(\boldsymbol{\mu} + \bar{x}(a)) \cdot \mathbf{1}_{[a,N]}}{m'} > 0 = \tau_0^{\pm}(0, \bar{x}(a))((\mathcal{E}/\mathcal{E}'[-1], \mathbf{0}, \mathbf{0})),$$

and therefore the subobject $(\mathcal{E}'[-1], \boldsymbol{V}, \boldsymbol{\rho})$ is destabilizing. Furthermore, this is the only way in which $(\mathcal{E}, s)$ can have a destabilizing subobject. This gives the desired isomorphism. □

6.4.3 Combining the isomorphisms of Propositions 6.4.2 and 4.2.3 and taking enumerative invariants, we get the equality $\widetilde{\mathsf{Q}}_m = \mathsf{N}_{(\varnothing,\varnothing,\varnothing),m}(\tau^-)$. Applying this to the left-hand side of the $(\varnothing, \varnothing, \varnothing)$ case of DT/PT wall-crossing formula (71) yields

$$\widetilde{\mathsf{Q}}_n = \mathsf{W}_{n,N}.$$

Matching coefficients of $\widetilde{\mathsf{Q}}_m$ on both sides would reproduce the combinatorial statement of Proposition 6.3.4, but note that this argument alone is *not sufficient* to provide an alternative proof of Proposition 6.3.4, since the $\widetilde{\mathsf{Q}}_m$ are not algebraically independent.

6.4.4 Proposition *Let $1 \le a \le N$. In the notation of Lemma 6.1.11, for $\bar{x}(a) \ge x_0(a)$ there is an isomorphism*

$$\mathfrak{N}^{Q(N),\mathrm{sst}}_{(\varnothing,\varnothing,\varnothing),\boldsymbol{m},\mathbf{1}_{[a,N]}}(\tau_0^{\pm}(0, \bar{x}(a))) \cong \mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}^{\oplus 2}, m)$$

which identifies the (symmetrized) virtual structure sheaves.

We view $\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}, m)$ as the moduli scheme of pairs $[\mathcal{O}_{\overline{X}}^{\oplus 2} \twoheadrightarrow \mathcal{E}]$ where $\mathcal{E}$ is a zero-dimensional of length m and support on $\mathbb{C}^3 = \overline{X} \setminus D$, and the map is a surjection. It carries a natural symmetric perfect obstruction theory coming from its presentation as a critical locus in an associated noncommutative Quot scheme. The generating series

$$\sum_{m\in\mathbb{Z}} Q^m \widetilde{\mathsf{N}}_{(\varnothing,\varnothing,\varnothing),m,\mathbf{1}_{[a,N]}}(0,\bar{x}(a)) = \sum_{m\in\mathbb{Z}} Q^m \chi(\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}^{\oplus 2}, m), \widehat{\mathcal{O}}^{\mathrm{vir}}) =: \mathsf{V}^{\mathrm{DT}(2),K}_{\varnothing,\varnothing,\varnothing}$$

is therefore the K-theoretic rank-2 degree-0 DT vertex computed in [Fasola et al. 2021, Theorem A].

6.4.5 Proof This will be a longer version of the proof of Proposition 6.4.2. By the same argument as in that proof, the relevant objects on the left-hand side are pairs

$$\big([L \otimes \mathcal{O}_{\overline{X}} \xrightarrow{s} \mathcal{E}], V \xrightarrow{\rho} L \oplus H^0(\mathcal{E})\big), \tag{76}$$

where L and V are 1-dimensional vector spaces and $\mathcal{E}$ is a zero-dimensional sheaf on $\mathbb{C}^3$. Let ρ_1 and ρ_2 be the L and $H^0(\mathcal{E})$ components of s, respectively. The component ρ_1 cannot be zero, because otherwise there would be the destabilizing subobject $([0 \to \mathcal{E}], V \xrightarrow{\rho_1} H^0(E))$. Hence ρ_1 is an isomorphism and we only need to study s and ρ_2.

Note that $s\colon L \otimes \mathcal{O} \to \mathcal{E}$ is equivalent to a morphism $s\colon L \to H^0(\mathcal{E})$. Abusing notation, we write s for both. Similarly, write $\rho_2\colon V \otimes \mathcal{O} \to \mathcal{E}$. Now assume for the sake of contradiction that

$$s \times \rho_2 \colon (L \oplus V) \otimes \mathcal{O} \to E$$

is not surjective. Take $\mathcal{E}'$ to be its image. Then the pair $([L \otimes \mathcal{O} \to \mathcal{E}'], V \to L \oplus H^0(\mathcal{E}'))$ is a subobject because both s and ρ_2 factor through $\mathcal{E}'$. By the same calculation as in the proof of Proposition 6.4.2, this subobject is destabilizing. Furthermore, this is the only way in which (76) can have a destabilizing subobject, given that ρ_1 is an isomorphism. This gives the desired isomorphism.

6.4.6 To match virtual cycles, recall that the virtual tangent space of $\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}^{\oplus 2})$ used in [Fasola et al. 2021] is

$$\begin{aligned}\mathrm{Ext}_{\overline{X}}(W \otimes \mathcal{O}, W \otimes \mathcal{O}) - \mathrm{Ext}_{\overline{X}}([W \otimes \mathcal{O} \to \mathcal{E}], [W \otimes \mathcal{O} \to \mathcal{E}])& \\ = \mathrm{Ext}_{\overline{X}}(W \otimes \mathcal{O}, \mathcal{E}) + \mathrm{Ext}_{\overline{X}}(\mathcal{E}, W \otimes \mathcal{O}) - \mathrm{Ext}_{\overline{X}}(\mathcal{E}, \mathcal{E}),&\end{aligned}$$

where $W := V \oplus L \cong \mathbb{C}^2$ and $\mathrm{Ext}_{\overline{X}}(-,-) := \sum_i (-1)^i \mathrm{Ext}^i_{\overline{X}}(-,-)$. On the other hand, abbreviating $I := [L \otimes \mathcal{O} \to \mathcal{E}]$, the APOT on stable loci of $\mathfrak{N}^{Q(N),\mathrm{pl}}_{m,\mathbf{1}}$ comes from symmetrized pullback, with virtual tangent space

$$-\mathrm{Ext}_{\overline{X}}(I, I(-D)) + (\mathrm{Hom}(V \otimes \mathcal{O}, L \otimes \mathcal{O} \oplus \mathcal{E}) - \mathbb{C}) - (\mathrm{Hom}(V \otimes \mathcal{O}, L \otimes \mathcal{O} \oplus \mathcal{E}) - \mathbb{C})^\vee \kappa^{-1} \tag{77}$$

coming from (40) (see also Remark 4.1.5). From (33),

$$\mathrm{Ext}_{\overline{X}}(I, I) = \mathrm{Ext}_{\overline{X}}(I, I(-D)) + \mathrm{Ext}_{\overline{X}}(I|_D, I|_D) = \mathrm{Ext}_{\overline{X}}(I, I(-D)) + \mathrm{Ext}_{\overline{X}}(L \otimes \mathcal{O}_D, L \otimes \mathcal{O}_D).$$

Evidently this last term is equal to $\mathrm{Ext}_{\overline{X}}(L\otimes\mathcal{O}, L\otimes\mathcal{O})$. The $\mathbb{C}=\mathrm{Hom}(V\otimes\mathcal{O}, V\otimes\mathcal{O})$ may be identified with $\mathrm{Hom}(V\otimes\mathcal{O}, L\otimes\mathcal{O})$ using the isomorphism ρ_1. Putting it all together, (77) becomes

$$(\mathrm{Ext}_{\overline{X}}(L\otimes\mathcal{O},\mathcal{E})+\mathrm{Ext}_{\overline{X}}(\mathcal{E},L\otimes\mathcal{O})-\mathrm{Ext}_{\overline{X}}(\mathcal{E},\mathcal{E}))+\mathrm{Ext}_{\overline{X}}(V\otimes\mathcal{O},\mathcal{E})+\mathrm{Ext}_{\overline{X}}(\mathcal{E},V\otimes\mathcal{O}).$$

So the K-theory classes of the two virtual tangent spaces match, proving (25), so that we can use Proposition 2.5.2 to identify $\widehat{\mathcal{O}}^{\mathrm{vir}}$. □

6.4.7 Applying the identifications of Propositions 6.4.4 and 4.2.3 to the wall-crossing formula (75) when $(\lambda,\mu,\nu)=(\varnothing,\varnothing,\varnothing)$, the combinatorial result of Proposition 6.3.6 yields the identity

$$\begin{aligned}\mathsf{V}^{\mathrm{DT}(2),K}_{\varnothing,\varnothing,\varnothing} &= \Big(\sum_{n\in\mathbb{Z}} Q^n(-\kappa^{\frac{1}{2}})^n\widetilde{\mathsf{Q}}_n\Big)\Big(\sum_{n\in\mathbb{Z}} Q^n(-\kappa^{\frac{1}{2}})^{-n}\widetilde{\mathsf{Q}}_n\Big)\\ &= \mathsf{V}^{\mathrm{DT},K}_{\varnothing,\varnothing,\varnothing}|_{Q\mapsto -Q\kappa^{1/2}}\cdot\mathsf{V}^{\mathrm{DT},K}_{\varnothing,\varnothing,\varnothing}|_{Q\mapsto -Q\kappa^{-1/2}}.\end{aligned} \tag{78}$$

This is exactly the rank-2 case of [Fasola et al. 2021, Theorem A], though of course their proof via a rigidity argument is very different, much simpler, and more conceptual than the path we took. Conversely, matching coefficients of $\widetilde{\mathsf{Q}}_{m_1}\widetilde{\mathsf{Q}}_{m_2}$ on both sides would reproduce the combinatorial statement of Proposition 6.3.6, but again note that this argument alone is *not sufficient* to provide an alternative proof of Proposition 6.3.6, since the $\widetilde{\mathsf{Q}}_m$ are not algebraically independent.

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