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We exhibit moduli spaces of slope stable vector bundles on general polarized HK varieties (X, h) of type $K3^{[2]}$ which have an irreducible component of dimension $2a^2 + 2$, with a an arbitrary integer greater than 1. This is done by studying the case $X = S^{[2]}$ where S is an elliptic $K3$ surface. We show that in this case there is an irreducible component of the moduli space of stable vector bundles on $S^{[2]}$ which is birational to a moduli space of sheaves on S. We expect that if the moduli space of sheaves on S is a smooth HK variety (necessarily of type $K3^{[a^2+1]}$) then the following more precise version holds: the closure of the moduli space of slope stable vector bundles on (X, h) in the moduli space of Gieseker–Maruyama semistable sheaves with its GIT polarization is a general polarized HK variety of type $K3^{[a^2+1]}$.

1 Introduction

1.1 Background and motivation

Starting with Mukai's groundbreaking work of the 80s, moduli of (semistable) sheaves on a (polarized) $K3$ surface have played a prominent rôle in mathematics. These moduli spaces are varieties interesting in themselves (some of them are HK varieties of type $K3^{[n]}$, a few of them admit resolutions which are HK varieties of type OG10), and their geometry is intertwined with that of the $K3$ surface. One wonders whether moduli of sheaves on higher-dimensional HK varieties may also be the source of interesting geometry. In [O'Grady 2022] we introduced the notion of a modular (torsion-free) sheaf. The sheaf $\mathscr{F}$ on an HK variety X is modular if $\Delta(\mathscr{F}) := -2\,\mathrm{rk}(\mathscr{F})\,\mathrm{ch}_2(\mathscr{F}) + \mathrm{ch}_1(\mathscr{F})^2$ (the *discriminant of* $\mathscr{F}$) satisfies a topological condition (see Section 2.6 for details), for example it is modular if $\Delta(\mathscr{F})$ is a multiple of $c_2(X)$. We proved that variation of slope stability for modular sheaves behaves as variation of slope stability for sheaves on surfaces, and that slope (semi)stability of modular sheaves on a HK with a Lagrangian fibration can be tackled with methods similar to those employed when dealing with sheaves on elliptically fibered $K3$ surfaces. Building on these results, in [O'Grady 2022; 2024] we proved existence and uniqueness results analogous to existence and uniqueness results for stable spherical vector bundles on $K3$ surfaces. In this regard we mention that Mukai's beautiful one-line proof of uniqueness fails, and our unicity argument is substantially more involved — one may view this as a foreboding of difficulties to come.

The main result of the present paper is the following. Let (X, h) be a general polarized HK variety of type $K3^{[2]}$, with the exclusion of the case in which the divisibility of h is 1 and $q_X(h) \equiv 2 \pmod 8$. Then for all choices of a positive integer a (greater than 1) in an ideal of $\mathbb{Z}$ which depends on (X, h),

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there exists a choice of a triple $(r,m,s) \in \mathbb{N}_+ \oplus \mathbb{Z} \oplus \mathbb{Q}$ for which the moduli space of h slope stable vector bundles $\mathscr{F}$ on X with $\mathrm{rk}(\mathscr{F}) = r$, $c_1(\mathscr{F}) = mh$ and $\Delta(\mathscr{F}) = sc_2(X)$ contains an irreducible component of dimension $2a^2+2$.

In fact the proof suggests that the following holds: the moduli spaces that we consider (or, to be safe, their closure in the moduli of Gieseker–Maruyama semistable sheaves) are deformations of moduli spaces of sheaves on $K3$ surfaces. Moreover, we expect that in many cases the pair (mod. space, GIT pol.) is a general polarized HK variety of type $K3^{[a^2+1]}$ (or that this holds for a connected component of the moduli space). At first glance this appears to be a letdown, but the key word is *general*: we expect to realize a general polarized HK variety of type $K3^{[n]}$ (for certain values of n) as a moduli space of sheaves on a general polarized HK variety of type $K3^{[2]}$. Note that if $n > 1$ then a general polarized HK variety of type $K3^{[n]}$ cannot be a moduli space of sheaves on a general polarized $K3$ surface because the former has 20 moduli while the latter has 19 moduli. In [O'Grady $\geq$ 2026] we prove that our expectation is correct (it holds for a connected component of the moduli space; we do not know whether the moduli space is irreducible) when the moduli space has dimension 4 (the "missing case" $a = 1$).

1.2 Main result

Let (X,h) be a polarized HK variety (polarizations are always primitive). A *mock Mukai vector* is given by

$$w = (r,l,s) \in \mathbb{N}_+ \oplus \mathrm{NS}(X) \oplus H^{2,2}_{\mathbb{Z}}(X). \tag{1.2.1}$$

We let $M_w(X,h)$ be the moduli space of h slope stable vector bundles $\mathscr{F}$ such that

$$w(\mathscr{F}) := (\mathrm{rk}(\mathscr{F}), c_1(\mathscr{F}), \Delta(\mathscr{F})) = w. \tag{1.2.2}$$

By Maruyama's classical results, $M_w(X,h)$ is a scheme of finite type over $\mathbb{C}$.

Remark 1.1 Let (X,h) be a polarized $K3$ surface and let $v = (r,l,s) \in \mathbb{N} \oplus \mathrm{NS}(X) \oplus H^{2,2}_{\mathbb{Z}}(X)$ be a Mukai vector. We denote by $\mathscr{M}_v(X,h)$ the moduli space of h Gieseker–Maruyama semistable sheaves on X with Mukai vector v. If w is the mock Mukai vector $(r,l,2r^2+v^2)$, where $v^2 = \langle v,v\rangle$ is the Mukai square of v, then $M_w(X,h)$ is the open subset of $\mathscr{M}_v(X,h)$ parametrizing slope stable locally free sheaves. In order to avoid confusion we use the notation $M_w(X,h)$ only if $\dim X > 2$.

Now suppose that X is of type $K3^{[2]}$. Then $q_X(h)$, the value of the Beauville–Bogomolov–Fujiki quadratic form on h, is a positive even integer. The divisibility of h, i.e., the positive generator of $q_X(h, H^2(X;\mathbb{Z}))$, that we denote by $\mathrm{div}(h)$, is either 1 or 2; if the latter holds, then $q_X(h) \equiv -2 \pmod 8$. In both cases (i.e., divisibility 1 and 2) the corresponding moduli space of polarized varieties is irreducible of dimension 20. For $a, r_1 \in \mathbb{N}_+$ we let

$$w := ar_1\left(2r_1, \frac{2}{\mathrm{div}(h)}h, \frac{ar_1^3 c_2(X)}{3}\right). \tag{1.2.3}$$

Theorem 1.2 *Let r_1 be a positive integer. Let (X,h) be a polarized HK variety of type $K3^{[2]}$ such that*

$$\operatorname{div}(h) = \begin{cases} 1 & \text{if } r_1 \equiv 0 \pmod 2, \\ 2 & \text{if } r_1 \equiv 1 \pmod 2, \end{cases} \tag{1.2.4}$$

$$q_X(h) \equiv \begin{cases} -2 \pmod{2r_1} & \text{if } r_1 \equiv 0 \pmod 4, \\ -2r_1 - 8 \pmod{8r_1} & \text{if } r_1 \equiv 1 \pmod 4, \\ r_1 - 2 \pmod{2r_1} & \text{if } r_1 \equiv 2 \pmod 4, \\ 2r_1 - 8 \pmod{8r_1} & \text{if } r_1 \equiv 3 \pmod 4. \end{cases} \tag{1.2.5}$$

Let a be a positive integer greater than 1 such that $2a$ is a multiple of r_1, and let w be the mock Mukai vector given by (1.2.3). Then the moduli space $M_w(X,h)$ is nonempty, and for (X,h) general it has an irreducible component of dimension $2a^2+2$.

Following is an outline of the main steps that go into the proof of Theorem 1.2. Given sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ on a $K3$ surface S, there is a natural involution of the sheaf $\mathscr{E}_1 \boxtimes \mathscr{E}_2 \oplus \mathscr{E}_2 \boxtimes \mathscr{E}_1$ on S^2 which lifts the involution of S^2 exchanging the factors. We denote by $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ the sheaf on $S^{[2]}$ which corresponds via the BKR correspondence to the $\mathscr{S}_2$-sheaf $\mathscr{E}_1 \boxtimes \mathscr{E}_2 \oplus \mathscr{E}_2 \boxtimes \mathscr{E}_1$. In Section 2 we study basic properties of the sheaf $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$. (We assume that $\mathscr{E}_1$ is locally free and $\mathscr{E}_2$ is torsion free; this guarantees that $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is torsion free. If also $\mathscr{E}_2$ is locally free then $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is locally free.) We show that $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is modular if $(\operatorname{rk}(\mathscr{E}_1), c_1(\mathscr{E}_1))$ is proportional to $(\operatorname{rk}(\mathscr{E}_2), c_1(\mathscr{E}_2))$ and the Mukai vectors $v(\mathscr{E}_1)$, $v(\mathscr{E}_2)$ are orthogonal. These conditions imply in particular that $\overline{v}(\mathscr{E}_1)^2 + \overline{v}(\mathscr{E}_2)^2 = 0$, where $\overline{v}(\mathscr{E}_i)$ is the normalized Mukai vector of $\mathscr{E}_i$, i.e., the multiple of the Mukai vector with first entry equal to 1, and $\overline{v}(\mathscr{E}_2)^2$ is the square of $\overline{v}(\mathscr{E}_2)$ in the (rational) Mukai lattice of S. It follows that there are two types of choices of $\mathscr{E}_1, \mathscr{E}_2$ that might produce a sheaf $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ which is modular and stable, corresponding to $\overline{v}(\mathscr{E}_1)^2 = \overline{v}(\mathscr{E}_2)^2 = 0$ and $\overline{v}(\mathscr{E}_1)^2 < 0 < \overline{v}(\mathscr{E}_2)^2$, respectively. The first choice gives $v(\mathscr{E}_1) = v(\mathscr{E}_2)$ isotropic. This has been considered by Markman [2024b] and has led to the proof of the analogue of the Shafarevich conjecture for pairs of HK varieties of type $K3^{[n]}$; see [Markman 2024a]. We consider the second choice, i.e., $v(\mathscr{E}_1)^2 = -2$ and $\mathscr{E}_1$ is a spherical vector bundle. Thus $v(\mathscr{E}_2)^2 > 0$, and in fact $v(\mathscr{E}_2)^2$ may be arbitrarily large. A key fact that holds with some mild assumptions is that all (nearby) deformations of $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ are given by $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2')$ where $\mathscr{E}_2'$ is a (nearby) deformation of $\mathscr{E}_2$. Now suppose that $\mathscr{E}_2$ is stable and $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is slope stable. Then we get that as $\mathscr{E}_2'$ varies among stable deformations of $\mathscr{E}_2$ the sheaves $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2')$ fill out an irreducible component of a moduli space of sheaves on $S^{[2]}$ (and $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2')$ is locally free for a general such $\mathscr{E}_2'$). Actually one checks easily that we get a component birational to the relevant moduli space of (semistable) sheaves on S. But we are getting ahead of ourselves: the proof that this idea works is in Section 6. Section 3 lists examples of pairs $\mathscr{E}_1, \mathscr{E}_2$ such that $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is modular. In that section we also show that with a suitable choice of $\mathscr{E}_1, \mathscr{E}_2$, the pair $(S^{[2]}, \mathscr{G}(\mathscr{E}_1,\mathscr{E}_2))$ deforms to the pair $(F(Y), \bigwedge^2 \mathscr{Q})$, where $F(Y)$ is the variety of lines on a general cubic fourfold $Y \subset \mathbb{P}^5$ and $\mathscr{Q}$ is the restriction to $F(Y)$ of the tautological rank 4 quotient vector bundle on $\operatorname{Gr}(1,\mathbb{P}^5)$. These examples of modular vector bundles were discovered by Fatighenti [2024]. The present work has been motivated by the desire to understand Fatighenti's example. In Section 4 we perform more computations in order to determine whether $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is atomic or not (of course we assume that it is modular). The answer is that it is atomic if and only if the Mukai vectors $v(\mathscr{E}_1)$, $v(\mathscr{E}_2)$ are both isotropic. Section 5 extends the results on variation of slope (semi)stability of modular sheaves on a HK variety with respect to ample classes proved in [O'Grady 2022] to variation with respect to Kähler classes. In the same section we give results on slope (semi)stability of modular sheaves on a HK variety with a Lagrangian fibration which go beyond those proved in [loc. cit.]. They are needed in order to deal with stability of a sheaf on a Lagrangian fibration which restricts to a strictly semistable sheaf on a general Lagrangian fiber. As mentioned above, in Section 6 we prove that under certain hypotheses the sheaf $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is slope stable by applying some of the results in Section 5 together with results on certain sheaves on elliptic $K3$ surfaces which are proved in the Appendix. Theorem 1.2 is proved in Section 7. We show that a general sheaf $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ in the irreducible component of the moduli space of sheaves on $S^{[2]}$ described above extends to a nearby deformation of $(S^{[2]}, c_1(\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)))$ by applying Verbitsky's fundamental results on projectively hyperholomorphic vector bundles together with the results of Section 5.

2 Simple modular sheaves with many moduli

2.1 A construction of sheaves on $S^{[2]}$

Let S be a (complex) smooth projective surface, and let $X_2(S)$ be the blowup of S^2 along the diagonal. We have a commutative diagram

(2.1.1)
$$\begin{array}{ccc} X_2(S) & \xrightarrow{\ \tau\ } & S^2 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ S^{[2]} & \xrightarrow{\ \gamma\ } & S^{(2)} \end{array}$$

where π is the quotient map and γ is the cycle (or Hilbert-to-Chow) map. The map ρ in (2.1.1) is finite, flat, of degree 2. Let $\mathrm{pr}_i\colon S^2 \to S$ be the i-th projection, and let $\tau_i\colon X_2(S) \to S$ be the composition $\tau_i := \mathrm{pr}_i \circ \tau$.

Given sheaves $\mathscr{E}_1,\mathscr{E}_2$ on S, let $\mathscr{F} = \mathscr{F}(\mathscr{E}_1,\mathscr{E}_2)$ be the sheaf on $X_2(S)$ defined by

(2.1.2)
$$\mathscr{F}(\mathscr{E}_1,\mathscr{E}_2) := \tau_1^*\mathscr{E}_1 \otimes \tau_2^*\mathscr{E}_2 \oplus \tau_1^*\mathscr{E}_2 \otimes \tau_2^*\mathscr{E}_1.$$

Let σ be the involution of $X_2(S)$ which lifts the involution of S^2 that exchanges the factors. Thus $\tau_{3-i} \circ \sigma = \tau_i$ for $i \in \{1,2\}$. The obvious isomorphism $\sigma^*\mathscr{F}(\mathscr{E}_1,\mathscr{E}_2) \cong \mathscr{F}(\mathscr{E}_1,\mathscr{E}_2)$ defines an action of the symmetric group $\mathscr{S}_2$ on $\mathscr{F}(\mathscr{E}_1,\mathscr{E}_2)$, which is compatible with its action on $X_2(S)$. Since the action of $\mathscr{S}_2$ on $X_2(S)$ maps any fiber of ρ to itself, we get an action of $\mathscr{S}_2$ on $\rho_*(\mathscr{F})$ (i.e., an action lifting the trivial action on $S^{[2]}$).

Definition 2.1 Let $\mathscr{G} = \mathscr{G}(\mathscr{E}_1,\mathscr{E}_2) = \rho_*(\mathscr{F})^{\mathscr{S}_2}$ be the sheaf of $\mathscr{S}_2$-invariants for the action of $\mathscr{S}_2$ on $\rho_*(\mathscr{F})$.

By definition of the $\mathscr{S}_2$-action we have

(2.1.3)
$$\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2) \cong \rho_*(\tau_1^*\mathscr{E}_1 \otimes \tau_2^*\mathscr{E}_2).$$

The remark below explains why we define $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ as a sheaf of $\mathscr{S}_2$-invariants.

Remark 2.2 Suppose that S is a $K3$ surface. The Bridgeland–King–Reid (BKR) McKay correspondence [Bridgeland et al. 2001] applied to the category $D_{\mathscr{S}_2}(S^2)$ of $\mathscr{S}_2$-equivariant (coherent) sheaves on S^2 gives an equivalence between $D_{\mathscr{S}_2}(S^2)$ and the category of (coherent) sheaves on $S^{[2]}$. Let $\overline{\mathscr{F}}(\mathscr{E}_1,\mathscr{E}_2)$ be the $\mathscr{S}_2$-equivariant sheaf on S^2 defined by

$$\overline{\mathscr{F}}=\overline{\mathscr{F}}(\mathscr{E}_1,\mathscr{E}_2):=\mathrm{pr}_1^*\,\mathscr{E}_1\otimes\mathrm{pr}_2^*\,\mathscr{E}_2\oplus\mathrm{pr}_1^*\,\mathscr{E}_2\otimes\mathrm{pr}_2^*\,\mathscr{E}_1. \tag{2.1.4}$$

If we adopt the definition in [Krug 2018, Section 2.4], then $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ corresponds to $\overline{\mathscr{F}}(\mathscr{E}_1,\mathscr{E}_2)$ via the BKR McKay correspondence.

We discuss a few properties of the above construction. First note that

$$\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2'\oplus\mathscr{E}_2'')\cong\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2')\oplus\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2''). \tag{2.1.5}$$

Secondly we discuss the case $\mathscr{E}_1=\mathscr{E}_2=\mathscr{A}$ where $\mathscr{A}$ is locally free. Following [O'Grady 2022] we associate to $\mathscr{A}$ locally free sheaves $\mathscr{A}[2]^{\pm}$ on $S^{[2]}$ as follows. Let

$$\phi\colon\sigma^*\big(\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\big)\xrightarrow{\sim}\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A}) \tag{2.1.6}$$

be the isomorphism switching the factors of the tensor products. Then ϕ defines an action of $\mathscr{S}_2$ on $\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})$, which is compatible with its action on $X_2(S)$. Hence we get an $\mathscr{S}_2$-action on $\rho_*\big(\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\big)^{\mathscr{S}_2}$. The sheaf of $\mathscr{S}_2$ invariants of $\rho_*\big(\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\big)^{\mathscr{S}_2}$ for this action is $\mathscr{A}[2]^+$. One may define another action of $\mathscr{S}_2$ multiplying ϕ by -1. For this second action, The sheaf of $\mathscr{S}_2$ invariants of $\rho_*\big(\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\big)^{\mathscr{S}_2}$ (i.e., anti-invariants of the first action) is $\mathscr{A}[2]^-$.

Proposition 2.3 *Let $\mathscr{A}$ be a locally free sheaf on S. Then*

$$\mathscr{G}(\mathscr{A},\mathscr{A})\cong\mathscr{A}[2]^+\oplus\mathscr{A}[2]^-. \tag{2.1.7}$$

Proof We have injections

$$\begin{aligned}\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})&\hookrightarrow\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\oplus\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})=\mathscr{F}(\mathscr{A},\mathscr{A}),\quad\xi\mapsto(\xi,\xi),\\ \tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})&\hookrightarrow\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})\oplus\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})=\mathscr{F}(\mathscr{A},\mathscr{A}),\quad\xi\mapsto(\xi,-\xi).\end{aligned}$$

Moreover, $\mathscr{F}(\mathscr{A},\mathscr{A})$ splits as the direct sum of the images of the two injections. The first of the above maps is $\mathscr{S}_2$-equivariant if the action on $\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})$ is the first one defined above, and the second one is $\mathscr{S}_2$-equivariant if the action on $\tau_1^*(\mathscr{A})\otimes\tau_2^*(\mathscr{A})$ is the second one defined above. Taking $\mathscr{S}_2$ invariants of the direct images for ρ_* one gets the isomorphism in (2.1.7). □

Proposition 2.4 *With notation as above, the following hold*:

(1) *If $\mathscr{E}_1,\mathscr{E}_2$ are locally free, then $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is locally free.*

(2) *If $\mathscr{E}_1$ is locally free, and $\mathscr{E}_2$ is torsion free, then $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ is torsion free.*

Proof (1) Since $\mathscr{F} = \mathscr{F}(\mathscr{E}_1, \mathscr{E}_2)$ is a tensor product of locally free sheaves, it is locally free. Since the map ρ in (2.1.1) is finite and flat, it follows that $\rho_*(\mathscr{F})$ is locally free. Thus $\mathscr{G} = \rho_*(\mathscr{F})^{\mathscr{S}_2}$ is locally free.

(2) Since $\mathscr{E}_2$ is a torsion-free sheaf on a (smooth) surface it has a two-step locally free resolution which is an injection of vector bundles away from a subset of codimension 2. Pulling back via τ_2 to $X_2(S)$ and tensoring by $\tau_1^*\mathscr{E}_1$ we get that $\mathscr{F}(\mathscr{E}_1, \mathscr{E}_2)$ has a two-step locally free resolution which is an injection of vector bundles away from a subset of codimension 2. It follows that $\mathscr{F}(\mathscr{E}_1, \mathscr{E}_2)$ is torsion free. Hence $\rho_*\mathscr{F}(\mathscr{E}_1, \mathscr{E}_2)$ is torsion free, and a fortiori the subsheaf $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is torsion free. □

Next we do the construction in families. Let B be a scheme, and let $\mathsf{E}_1, \mathsf{E}_2$ be sheaves on $S \times B$. We define a sheaf $\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2)$ on $X_2(S) \times B$ by letting

$$\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2) := (\tau_1 \times \mathrm{Id}_B)^*\mathsf{E}_1 \otimes (\tau_2 \times \mathrm{Id}_B)^*\mathsf{E}_2 \oplus (\tau_1 \times \mathrm{Id}_B)^*\mathsf{E}_2 \otimes (\tau_2 \times \mathrm{Id}_B)^*\mathsf{E}_1.$$

The symmetric group $\mathscr{S}_2$ acts on $(\rho \times B)_*\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2)$: we let $\mathscr{G}(\mathsf{E}_1, \mathsf{E}_2)$ be the subsheaf of $\mathscr{S}_2$-invariants.

Proposition 2.5 *With notation as above, suppose that E_1 is a B-flat family of locally free sheaves on S, and that E_2 is a B-flat family of torsion-free sheaves on S. Then $\mathscr{G}(\mathsf{E}_1, \mathsf{E}_2)$ is a B-flat family of torsion free sheaves on $S^{[2]}$.*

Proof Since E_2 is a B-flat family of torsion-free sheaves on S, it has a two-step locally free resolution

$$0 \to \mathsf{E}_2^1 \to \mathsf{E}_2^0 \to \mathsf{E}_2 \to 0, \tag{2.1.8}$$

which restricts to a locally free resolution of $\mathsf{E}_{2|S\times\{b\}}$ for every (schematic) point of B. The pullback to $X_2(S) \times B$ of the exact sequence in (2.1.8) remains exact. It follows that $\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2)$ is the direct sum of two sheaves, each of which has a two-step locally free resolution which remains exact when restricted to each fiber of the projection $X_2(S) \times B \to B$. This implies that $\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2)$ is B-flat; see Theorem 22.5 in [Matsumura 1986]. Since ρ is flat we get that $(\rho \times \mathrm{Id})_*\mathscr{F}(\mathsf{E}_1, \mathsf{E}_2)$ is B-flat, and hence also its sheaf of $\mathscr{S}_2$-invariants. □

2.2 Main results

Notation is as in Section 2.1. Below are the main results of the present section.

Proposition 2.6 *Suppose that S is a $K3$ surface, that $\mathscr{E}_1, \mathscr{E}_2$ are simple sheaves on S, that $\mathscr{E}_1$ is locally free, and that $\mathscr{E}_2$ is torsion free. Then the following hold*:

(1) *The sheaf $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is simple if and only if*

$$\mathrm{ext}^0_S(\mathscr{E}_1, \mathscr{E}_2) \cdot \mathrm{ext}^0_S(\mathscr{E}_2, \mathscr{E}_1) = 0. \tag{2.2.1}$$

(2) *If $\mathscr{G}$ is simple, then*

$$\mathrm{ext}^1_{S^{[2]}}(\mathscr{G}, \mathscr{G}) = \mathrm{ext}^1_S(\mathscr{E}_1, \mathscr{E}_1) + \mathrm{ext}^1_S(\mathscr{E}_2, \mathscr{E}_2) + (\mathrm{ext}^0_S(\mathscr{E}_1, \mathscr{E}_2) + \mathrm{ext}^0_S(\mathscr{E}_2, \mathscr{E}_1)) \cdot \mathrm{ext}^1(\mathscr{E}_1, \mathscr{E}_2).$$

(3) *If*

$$\operatorname{ext}^0_S(\mathscr{E}_1,\mathscr{E}_2)=\operatorname{ext}^0_S(\mathscr{E}_2,\mathscr{E}_1)=0 \tag{2.2.2}$$

(in particular, $\mathscr{G}$ is simple), then

$$\operatorname{ext}^2_{S^{[2]}}(\mathscr{G},\mathscr{G})=2+\operatorname{ext}^1_S(\mathscr{E}_1,\mathscr{E}_2)^2+\operatorname{ext}^1_S(\mathscr{E}_1,\mathscr{E}_1)\cdot\operatorname{ext}^1_S(\mathscr{E}_2,\mathscr{E}_2). \tag{2.2.3}$$

(4) *If the equality* (2.2.2) *holds, then deformations of $\mathscr{G}$ are unobstructed, and* $\operatorname{Def}(\mathscr{G})$ *is identified with* $\operatorname{Def}(\mathscr{E}_1)\times\operatorname{Def}(\mathscr{E}_2)$ *via the map*

$$\operatorname{Def}(\mathscr{E}_1)\times\operatorname{Def}(\mathscr{E}_2)\xrightarrow{\Phi}\operatorname{Def}(\mathscr{G}) \tag{2.2.4}$$

that associates to deformations $\mathscr{E}_1(s)$ and $\mathscr{E}_2(t)$ of $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively, the sheaf $\mathscr{G}(\mathscr{E}_1(s),\mathscr{E}_2(t))$ (this makes sense by Proposition 2.5).

In order to state the other main result of the present section, we recall the description of the second cohomology of $S^{[2]}$. Let $E\subset X_2(S)$ be the exceptional divisor of the blowup map $\tau\colon X_2(S)\to S^2$, and let

$$e:=\operatorname{cl}(E)\in H^2(X_2(S),\mathbb{Z}). \tag{2.2.5}$$

There exist a homomorphism

$$\mu\colon H^2(S;\mathbb{Z})\to H^2(S^{[2]};\mathbb{Z}) \tag{2.2.6}$$

and a class $\delta\in H^2(S^{[2]};\mathbb{Z})$ such that

$$\rho^*\mu(\alpha)=\tau_1^*\alpha+\tau_2^*\alpha,\quad \rho^*\delta=e, \tag{2.2.7}$$

where α is an arbitrary class in $H^2(S;\mathbb{Z})$. We have a direct-sum decomposition

$$H^2(S^{[2]};\mathbb{Z})=\operatorname{Im}(\mu)\oplus\mathbb{Z}\delta, \tag{2.2.8}$$

whose addends are orthogonal for the Beauville–Bogomolov–Fujiki (BBF) symmetric bilinear form $q_{S^{[2]}}$. Moreover $q_{S^{[2]}}(\mu(\alpha))=\alpha^2$ for $\alpha\in H^2(S;\mathbb{Z})$, and $q_{S^{[2]}}(e)=-2$. As a matter of notation we denote by the same symbol μ the extension of μ to a linear map $H^2(S;\mathbb{C})\to H^2(S^{[2]};\mathbb{C})$.

Proposition 2.7 *Let S be a $K3$ surface. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be torsion-free sheaves on S of ranks r_1 and r_2, respectively, with $\mathscr{E}_1$ locally free. Let $\mathscr{G}=\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$.*

(1) *We have*

$$\operatorname{rk}(\mathscr{G})=2r_1r_2,\quad c_1(\mathscr{G})=r_2\mu(c_1(\mathscr{E}_1))+r_1\mu(c_1(\mathscr{E}_2))-r_1r_2\delta. \tag{2.2.9}$$

(2) *Suppose in addition that*

$$r_2\cdot c_1(\mathscr{E}_1)=r_1\cdot c_1(\mathscr{E}_2), \tag{2.2.10}$$

and that, letting $v(\mathscr{E}_i)$ be the Mukai vector of $\mathscr{E}_i$, we have

$$r_2^2\cdot v(\mathscr{E}_1)^2+r_1^2\cdot v(\mathscr{E}_2)^2=0, \tag{2.2.11}$$

where $v(\mathscr{E}_i)^2:=\langle v(\mathscr{E}_i),v(\mathscr{E}_i)\rangle$ is the square of the Mukai pairing $\langle\cdot,\cdot\rangle$. Then

$$\Delta(\mathscr{G})=\tfrac{1}{3}r_1^2r_2^2c_2(S^{[2]}). \tag{2.2.12}$$

Remark 2.8 The most important result in Proposition 2.7 is the assertion that, under suitable hypotheses, the discriminant of $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is a multiple of $c_2(S^{[2]})$, i.e., that $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is a modular sheaf. In Section 2.6 we recall the definition of modular sheaf. The key features of modular sheaves are the following. First, variation of stability behaves as if the hyperkähler variety were a surface. Secondly, one may relate stability of a modular sheaf on a hyperkähler with a Lagrangian fibration and (semi)stability of its restriction to a general Lagrangian fiber, provided the polarization is sufficiently close to the boundary of the ample cone which corresponds to the Lagrangian fibration. These results, which are presented in Section 5, provide the theoretical basis of our proof of the main result, i.e., Theorem 1.2.

Remark 2.9 Assume that the equality in (2.2.10) holds. Then the equality in (2.2.11) holds if and only if

$$\langle v(\mathscr{E}_1), v(\mathscr{E}_2)\rangle = 0. \tag{2.2.13}$$

Note that this is equivalent to the condition $\chi_S(\mathscr{E}_1, \mathscr{E}_2) = 0$.

Remark 2.10 Let S be a $K3$ surface. The construction in Section 2.1 extends to $S^{[n]}$ and it does give modular sheaves under suitable hypotheses. Let $\mathscr{E}_1, \dots, \mathscr{E}_n$ be sheaves on S, locally free, with the possible exception of one which is torsion-free. Let $X_n(S)$ be the n-th isospectral scheme of S (see Definition 3.2.4 in [Haiman 2001]), with maps $\tau\colon X_n(S) \to S^n$ (the blowup of the big diagonal) and $\rho\colon X_n(S) \to S^{[n]}$. For $i \in \{1, \dots, n\}$ let $\tau_i\colon X_n(S) \to S$ be τ followed by the i-th projection. Let

$$\mathscr{F} := \bigoplus_{\sigma \in \mathscr{S}_n} \tau_1^*\mathscr{E}_{\sigma(1)} \otimes \cdots \otimes \tau_i^*\mathscr{E}_{\sigma(i)} \otimes \cdots \otimes \tau_n^*\mathscr{E}_{\sigma(n)}.$$

The pushforward $\rho_*(\mathscr{F})$ is torsion-free because $\mathscr{F}$ is torsion-free. If all the $\mathscr{E}_i$ are locally free then $\rho_*(\mathscr{F})$ is locally free because ρ is finite and flat (the latter is a highly nontrivial result of Haiman; see [loc. cit.]). The symmetric group $\mathscr{S}_n$ acts on $X_n(S)$ compatibly with its permutation action on S^n, and hence we get an $\mathscr{S}_n$-action on $\mathscr{F}$. Thus we also get an $\mathscr{S}_n$-action on $\rho_*(\mathscr{F})$: we let $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \dots, \mathscr{E}_n) \subset \rho_*(\mathscr{F})$ be the sheaf of $\mathscr{S}_n$-invariants. Let $r_i := \mathrm{rk}(\mathscr{E}_i)$ and $\overline{r} := r_1 \cdots r_n$. Then we have

$$\mathrm{rk}(\mathscr{G}) = n!\overline{r} \quad \text{and} \quad c_1(\mathscr{G}) = (n-1)!\overline{r}\left[\sum_{i=1}^{n} \mu\left(\frac{c_1(\mathscr{E}_i)}{r_i}\right) - \frac{n}{2}\delta_n\right],$$

where $\mu_n\colon H^2(S) \to H^2(S^{[n]})$ is the analogue of the homomorphism $\mu\colon H^2(S) \to H^2(S^{[2]})$, and $\delta_n \in H^2(S^{[n]}; \mathbb{Z})$ is the unique class such that $\rho^*\delta_n$ is the class of the exceptional divisor of τ. Next assume that each sheaf $\mathscr{E}_i$ is simple, that

$$r_j c_1(\mathscr{E}_i) = r_i c_1(\mathscr{E}_j) \tag{2.2.14}$$

for all i, j, and that

$$\sum_{i=1}^{n} \frac{v(\mathscr{E}_i)^2}{r_i^2} = 0. \tag{2.2.15}$$

Then

$$\Delta(\mathscr{G}) = \tfrac{1}{12}(n!)^2\overline{r}^2 c_2(S^{[n]}). \tag{2.2.16}$$

2.3 Proof of Proposition 2.6

Since S is a $K3$ surface, the Bridgeland–King–Reid (BKR) McKay correspondence gives an equivalence between the derived categories of $\mathscr{S}_2$ equivariant coherent sheaves on S^2 and of coherent sheaves on $S^{[2]}$. In particular, since $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is the sheaf on $S^{[2]}$ corresponding to the $\mathscr{S}_2$ equivariant sheaf $\mathscr{F} = \mathscr{F}(\mathscr{E}_1, \mathscr{E}_2)$, we have an isomorphism

$$\text{(2.3.1)} \qquad \operatorname{Ext}^p_{S^{[2]}}(\mathscr{G}, \mathscr{G}) \cong \operatorname{Ext}^p_{S^2}(\mathscr{F}, \mathscr{F})^{\mathscr{S}_2}.$$

By the Künneth decomposition we have

$$\text{(2.3.2)} \quad \operatorname{Ext}^p_{S^2}(\mathscr{F}, \mathscr{F}) \cong \bigoplus_{i=1}^{2} \bigoplus_{a+b=p} \operatorname{Ext}^a_S(\mathscr{E}_i, \mathscr{E}_i) \otimes \operatorname{Ext}^b_S(\mathscr{E}_{3-i}, \mathscr{E}_{3-i}) \oplus \bigoplus_{i=1}^{2} \bigoplus_{a+b=p} \operatorname{Ext}^a_S(\mathscr{E}_i, \mathscr{E}_{3-i}) \otimes \operatorname{Ext}^b_S(\mathscr{E}_{3-i}, \mathscr{E}_i).$$

Since $\operatorname{ext}^0_S(\mathscr{E}_i, \mathscr{E}_i) = 1$ for $i \in \{1, 2\}$, we get that $\operatorname{ext}^0_{S^2}(\mathscr{F}, \mathscr{F}) \geq 2$ and that the invariant subspace of $\operatorname{Ext}^0_{S^2}(\mathscr{F}, \mathscr{F})$ has dimension 1 if and only if (2.2.1) holds. This shows that item (1) holds.

A similar argument proves items (2) and (3).

Lastly we prove that item (4) holds. For $i \in \{1, 2\}$ let U_i be a ball with center 0 and let $\mathscr{A}_i$ be a sheaf on $S \times U_i$, flat over U_i, such that $\mathscr{A}_i(0) := \mathscr{A}_{i|S\times\{0\}}$ is isomorphic to $\mathscr{E}_i$, and the associated map $(U_i, 0) \to \operatorname{Def}(\mathscr{E}_i)$ is an isomorphism. Let $\mathscr{G}(\mathscr{A}_1, \mathscr{A}_2)$ be the sheaf on $S^{[2]} \times U_1 \times U_2$ that one gets by working in families; see Section 2.1. Then $\mathscr{G}(\mathscr{A}_1, \mathscr{A}_2)$ is flat over $U_1 \times U_2$ by Proposition 2.5. Thus we have a morphism of schemes

$$\text{(2.3.3)} \qquad U_1 \times U_2 \xrightarrow{\Phi} \operatorname{Def}(\mathscr{G})$$

which maps the point $s = (s_1, s_2)$ to the unique $\Phi(s) \in \operatorname{Def}(\mathscr{G})$ such that the corresponding sheaf on $S^{[2]}$ is isomorphic to $\mathscr{G}(\mathscr{A}_1(s_1), \mathscr{A}_n(s_2))$ (here $\operatorname{Def}(\mathscr{G})$ is a representative of the deformation space of $\mathscr{G}$, which is a universal deformation space because $\mathscr{G}$ is simple). Let $i \in \{1, 2\}$. Since $\mathscr{E}_i$ is a simple sheaf on a $K3$ surface, its deformation space is unobstructed, i.e., $\dim U_i = \operatorname{ext}^1_S(\mathscr{E}_i, \mathscr{E}_i)$. By item (2) it follows that

$$\text{(2.3.4)} \qquad \dim(U_1 \times U_2) = \operatorname{ext}^1_{S^{[2]}}(\mathscr{G}, \mathscr{G}).$$

Thus it suffices to prove that Φ is injective. By shrinking the U_i around 0 we may assume the following: if $s, t \in U_1 \times U_2$ and $\alpha, \beta \in \mathscr{S}_2$ are such that

$$\text{(2.3.5)} \qquad \operatorname{Hom}(\mathscr{A}_{\alpha(i)}(s_{\alpha(i)}), \mathscr{B}_{\beta(i)}(t_{\beta(i)})) \neq 0 \quad \text{for all } i \in \{1, 2\},$$

where $\mathscr{A}_{\alpha(i)}(s)$ equals $\mathscr{A}_{\alpha(i)|S\times\{t\}}$ and similarly for $\mathscr{B}_{\beta(i)}(t)$), then $s = t$ (and $\alpha = \beta$). In fact this follows from the hypothesis that each $\mathscr{E}_i$ is simple and from the vanishing $\operatorname{ext}^0_S(\mathscr{E}_1, \mathscr{E}_2) = \operatorname{ext}^0_S(\mathscr{E}_2, \mathscr{E}_1) = 0$. By an argument similar to those described above, this implies that if $\operatorname{Hom}(\mathscr{G}(\mathscr{A}_1(s_1), \mathscr{A}_2(s_2)), \mathscr{G}(\mathscr{A}_1(t_1), \mathscr{A}_2(t_2)))$ is nonzero, then $s = t$. This shows that Φ is injective, and concludes the proof of item (4). □

2.4 Pullback of $\mathscr{G}$ to $X_2(S)$

The sheaf $\rho^*\mathscr{G}$ is obtained from $\mathscr{F}$ via an elementary modification along E, where E is the exceptional divisor of the blowup map $\tau\colon X_2(S) \to S^2$. In order to explain this we introduce some notation. Let $D \subset S^2$ be the diagonal of S^2. Let

$$\overline{\epsilon}\colon D \xrightarrow{\sim} S \tag{2.4.1}$$

be the isomorphism given by restriction of either one of the projections. Let $\tau_E\colon E \to D$ be the restriction of τ to E, and let

$$\epsilon := \overline{\epsilon} \circ \tau_E\colon E \to S. \tag{2.4.2}$$

Let $\mathscr{R}$ be the locally free sheaf on E defined by

$$\mathscr{R} := \epsilon^*(\mathscr{E}_1 \otimes \mathscr{E}_2). \tag{2.4.3}$$

One can choose an isomorphism

$$\mathscr{F}_{|E} \cong \mathscr{R} \oplus \mathscr{R} \tag{2.4.4}$$

such that the eigensheaves of the action of the involution σ on $\mathscr{F}_{|E}$ (this makes sense because σ is the identity on E) are given by

$$\mathscr{F}_{|E}(U)^{\pm} = \{(s, \pm s) \mid s \in \mathscr{R}(U)\}. \tag{2.4.5}$$

(Here $U \subset E$ is an open subset.) Let

$$\mathscr{F}_{|E} \xrightarrow{\overline{\varphi}} \mathscr{R}, \quad (a,b) \mapsto a-b, \tag{2.4.6}$$

where the notation makes sense because of the isomorphism in (2.4.4) — we assume that it has been chosen so that the equalities in (2.4.5) hold. Let $\iota\colon E \hookrightarrow X_2(S)$ be the inclusion map, and let $\varphi\colon \mathscr{F} \to \iota_*\mathscr{R}$ be the morphism defined by the morphism $\overline{\varphi}$ in (2.4.6). Arguing as in the proof of [O'Grady 2022, Proposition 5.6] one gets that $\rho^*\mathscr{G}$ fits into the exact sequence

$$0 \to \rho^*\mathscr{G} \to \mathscr{F} \xrightarrow{\varphi} \iota_*\mathscr{R} \to 0. \tag{2.4.7}$$

2.5 Proof of Proposition 2.7

The rank of $\mathscr{G}$ can be computed away from the branch locus of ρ, and it is equal to the rank of $\mathscr{F}$, i.e., $2r_1r_2$. By the exact sequence in (2.4.7) we can express $\mathrm{ch}(\rho^*\mathscr{G})$ via $\mathrm{ch}(\mathscr{F})$ and the Chern character of the sheaf $\iota_*\mathscr{R}$. Applying the GRR theorem we get that *modulo* $H^6(X_2(S), \mathbb{Q})$ we have

$$\mathrm{ch}(\iota_*\mathscr{R}) = r_1r_2e + \tfrac{1}{2}c_1(\mathscr{F}) \cdot e - \tfrac{1}{2}r_1r_2e^2. \tag{2.5.1}$$

Hence we get that

$$c_1(\rho^*\mathscr{G}) = c_1(\mathscr{F}) - c_1(\iota_*(\mathscr{R})) = \biggl(\sum_{1 \le a,b \le 2}^{2} \frac{r_1r_2}{r_a} \tau_b^* c_1(\mathscr{E}_a) \biggr) - r_1r_2e. \tag{2.5.2}$$

By the equalities in (2.2.7) we get that

$$\rho^* c_1(\mathscr{G}) = \rho^*(r_2\mu(c_1(\mathscr{E}_1)) + r_1\mu(c_1(\mathscr{E}_2)) - r_1r_2\delta). \tag{2.5.3}$$

The pullback homomorphism $\rho^*\colon H^2(S^{[n]}) \to H^2(X_n(S))$ is injective because ρ is a finite map. Hence the second equality in (2.2.9) follows from the equality in (2.5.3). This finishes the proof of item (1) of Proposition 2.7.

Next we prove item (2). It suffices to show that

$$\rho^*\Delta(\mathscr{G}) = \tfrac{1}{3} r_1^2 r_2^2 \rho^* c_2(S^{[2]}). \tag{2.5.4}$$

By the equality in (2.5.1) we have

$$\begin{aligned}\rho^* \operatorname{ch}_2(\mathscr{G}) &= \operatorname{ch}_2(\mathscr{F}) - \operatorname{ch}_2\big(\iota_*(\mathscr{R})\big) \\ &= \sum_{1\le a,b\le 2} \left(\frac{r_1r_2}{r_a}\tau_b^* \operatorname{ch}_2(\mathscr{E}_a) + \frac{r_1r_2}{r_ar_b}\tau_1^* \operatorname{ch}_1(\mathscr{E}_a)\tau_2^* \operatorname{ch}_1(\mathscr{E}_b)\right)\Big|_{X_2(S)} \\ &\qquad - \frac{1}{2}e\cdot\left(\sum_{1\le a,b\le 2}\frac{r_1r_2}{r_a}\tau_b^* c_1(\mathscr{E}_a)\right) + \frac{r_1r_2}{2}e^2.\end{aligned} \tag{2.5.5}$$

We recall that

$$\rho^*\Delta(\mathscr{G}) = -4\overline{r}\rho^* \operatorname{ch}_2(\mathscr{G}) + \rho^* \operatorname{ch}_1(\mathscr{G})^2. \tag{2.5.6}$$

Let

$$\lambda = \frac{c_1(\mathscr{E}_1)}{r_1} = \frac{c_1(\mathscr{E}_2)}{r_2}. \tag{2.5.7}$$

(Recall the equality (2.2.10).) The equalities in (2.5.5) and (2.5.6), together with a few computations, give that

$$\rho^*\Delta(\mathscr{G}) = -4r_1^2r_2^2 \sum_{1\le a,b\le 2} \tau_b^*\left(\frac{\operatorname{ch}_2(\mathscr{E}_a)}{r_a}\right) + 4r_1^2r_2^2\left(\sum_{a=1}^{2}\tau_a^*\lambda^2\right) - \frac{r_1^2r_2^2}{2}e^2. \tag{2.5.8}$$

Let $\eta \in H^4(S)$ be the orientation class. By definition of Mukai pairing we have

$$\frac{\operatorname{ch}_2(\mathscr{E}_a)}{r_a} = -\frac{v(\mathscr{E}_a)^2}{2r_a^2}\eta - \eta + \frac{\lambda^2}{2}. \tag{2.5.9}$$

Replacing in the right hand side of (2.5.8) the above expression for $\operatorname{ch}_2(\mathscr{E}_a)/r_a$, and recalling the equality in (2.2.11), we get that

$$\rho^*\Delta(\mathscr{G}) = 8r_1^2r_2^2(\tau_1^*\eta + \tau_2^*\eta) - \tfrac{1}{2}r_1^2r_2^2e^2. \tag{2.5.10}$$

On the other hand we have (see [O'Grady 2024, Proposition 2.1])

$$\rho^* c_2(S^{[2]}) = 24(\tau_1^*(\eta) + \tau_2^*(\eta)) - 3e^2. \tag{2.5.11}$$

The validity of the equality in (2.5.4) follows from the equalities in (2.5.10) and in (2.5.11). □

2.6 Modular sheaves

Let X be a HK manifold of dimension $2n$. The *discriminant* of a torsion-free sheaf $\mathscr{F}$ on X is

$$\Delta(\mathscr{F}) := 2\,\mathrm{rk}(\mathscr{F})c_2(\mathscr{F}) - (\mathrm{rk}(\mathscr{F})-1)c_1(\mathscr{F})^2 = -2\,\mathrm{rk}(\mathscr{F})\,\mathrm{ch}_2(\mathscr{F}) + \mathrm{ch}_1(\mathscr{F})^2. \tag{2.6.1}$$

The sheaf $\mathscr{F}$ is modular (see [O'Grady 2022]) if there exists $d(\mathscr{F}) \in \mathbb{Q}$ such that

$$\int_X \Delta(\mathscr{F})\alpha^{2n-2} = d(\mathscr{F})(2n-3)!!\,q_X(\alpha)^{n-1} \tag{2.6.2}$$

for all $\alpha \in H^2(X)$, where q_X is the Beauville–Bogomolov–Fujiki quadratic form of X.

Example 2.11 If X is a $K3$ surface, then every torsion-free sheaf $\mathscr{F}$ on X is modular and $d(\mathscr{F}) = \int_X \Delta(\mathscr{F}) = v(\mathscr{F})^2 + 2\,\mathrm{rk}(\mathscr{F})^2$.

Example 2.12 Let $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \dots, \mathscr{E}_n)$ be the sheaf on $S^{[n]}$ appearing in Proposition 2.7 for $n = 2$ and in Remark 2.10 for general n. Then $\mathscr{G}$ is modular. In fact this follows from the equality in (2.2.12) and the formula $\int_{S^{[n]}} c_2(S^{[n]})\alpha^{2n-2} = 6(n+3)(2n-3)!!\,q_{S^{[n]}}(\alpha)^{n-1}$. Thus $d(\mathscr{G}) = (n+3)(n!)^2\overline{r}^2/2$.

We recall that the Fujiki constant of X (sometimes called the small Fujiki constant) is characterized by the validity of the equality

$$\int_X \alpha^{2n} = (2n-1)!!\,c_X q_X(\alpha)^n \tag{2.6.3}$$

for all $\alpha \in H^2(X)$.

Definition 2.13 Let X be a HK manifold, and let $\mathscr{F}$ be a modular torsion-free sheaf on X. Then

$$\mathsf{a}(\mathscr{F}) := \frac{\mathrm{rk}(\mathscr{F})^2 \cdot d(\mathscr{F})}{4c_X}, \tag{2.6.4}$$

where $d(\mathscr{F})$ is as in equation (2.6.2) and c_X is the Fujiki constant of X.

Example 2.14 Let $\mathscr{G}(\mathscr{E}_1, \dots, \mathscr{E}_n)$ be as in Example 2.12. Then

$$\mathsf{a}(\mathscr{G}(\mathscr{E}_1, \dots, \mathscr{E}_n)) = (n+3)(n!)^4\overline{r}^4/8. \tag{2.6.5}$$

In fact this equality follows from the formula for $d(\mathscr{G})$ given in Example 2.12 and the equality $c_{S^{[n]}} = 1$.

Definition 2.15 Let S be a $K3$ surface, and let $v = (r, l, s)$ be a Mukai vector on S. We set $\mathsf{a}(v) := (r^2(v^2 + 2r^2)/4$.

Definition 2.16 Let X be a HK manifold of dimension $2n > 2$. Let $w = (r, l, s) \in \mathbb{N}_+ \times \mathrm{NS}(X) \times H^{2,2}_{\mathbb{Z}}(X)$, and assume that there exists $d \in \mathbb{Q}$ such that

$$\int_X s \cdot \alpha^{2n-2} = d(2n-3)!!\,q_X(\alpha)^{n-1} \tag{2.6.6}$$

for all $\alpha \in H^2(X)$. We set $\mathsf{a}(w) := r^2 d/4c_X$.

Remark 2.17 Let $v=(r,l,s)$ be a Mukai vector on a $K3$ surface S. If $\mathscr{F}$ is a sheaf on S such that $v(\mathscr{F})=v$, then $\mathsf{a}(\mathscr{F})=\mathsf{a}(v)$; see Example 2.11. Let X be a HK manifold of dimension $2n>2$, and let $w=(r,l,s)$ be as in Definition 2.16. If $\mathscr{F}$ is a sheaf on X such that $w(\mathscr{F})=w$, then $\mathsf{a}(\mathscr{F})=\mathsf{a}(w)$.

3 Examples

3.1 Preliminaries

Let S be a $K3$ surface, and let $\mathscr{E}_1,\mathscr{E}_2$ be sheaves on S. For $i\in\{1,2\}$ let

(3.1.1) $$v(\mathscr{E}_i)=(r_i,l_i,s_i)$$

be the Mukai vector of $\mathscr{E}_i$.

Lemma 3.1 *If the hypotheses of Proposition 2.7* (*including* (2.2.10) *and* (2.2.11)) *hold*, *then*

(1) $v(\mathscr{E}_i)^2=0$ *for* $i\in\{1,2\}$ *and* $v(\mathscr{E}_1)$, $v(\mathscr{E}_2)$ *are proportional*, *or*

(2) *up to reindexing we have* $v(\mathscr{E}_1)^2=-2$ *and* $v(\mathscr{E}_2)^2>0$, *and there exists* $a\in\mathbb{N}_+$ *such that* $r_2=ar_1$ *and* $l_2=al_1$.

Proof Suppose that $v(\mathscr{E}_1)^2=v(\mathscr{E}_2)^2=0$. Then by the equality $r_2l_1=r_1l_2$ (see equation (2.2.10)) we may write $v(\mathscr{E}_1)=tv(\mathscr{E}_2)+(0,0,s)$ for some $t,s\in\mathbb{Q}$. Note that $t\neq 0$. Since $v(\mathscr{E}_1)^2=v(\mathscr{E}_2)^2=0$ it follows that $s=0$, and hence $v(\mathscr{E}_1)$, $v(\mathscr{E}_2)$ are proportional.

Suppose that $v(\mathscr{E}_1)^2$, $v(\mathscr{E}_2)^2$ are not both zero. Since $r_2^2v(\mathscr{E}_1)^2+r_1^2v(\mathscr{E}_2)^2=0$ there exists $i\in\{1,2\}$ such that $v(\mathscr{E}_i)^2<0$. Reindexing we may assume that $i=1$. By simplicity of $\mathscr{E}_1$ we get that $v(\mathscr{E}_1)^2=-2$, i.e., that

(3.1.2) $$r_1s_1-\tfrac{1}{2}l_1^2=1.$$

Hence $\operatorname{div}(l_1)$ and r_1 are coprime. The relation $r_2l_1=r_1l_2$ gives that there exists $a\in\mathbb{N}_+$ such that $r_2=ar_1$ and $l_2=al_1$ (because $\operatorname{div}(l_1)$ and r_1 are coprime). □

Remark 3.2 Let $\mathscr{E}_1,\dots,\mathscr{E}_n$ be sheaves on a $K3$ surface S as in Remark 2.10, and suppose that the equalities in (2.2.14) and (2.2.15) hold. For $j\in\{1,\dots,n\}$ let $v(\mathscr{E}_j)=(r_j,l_j,s_j)$. Then either $v(\mathscr{E}_j)^2=0$ for all $j\in\{1,\dots,n\}$ and the Mukai vectors $v(\mathscr{E}_1),\dots,v(\mathscr{E}_n)$ are proportional or else, up to reindexing, $v(\mathscr{E}_1)^2=-2$ and for all $j>1$ we have $v(\mathscr{E}_j)^2\geq 0$ as well as $r_j=a_jr_1$ and $l_j=a_jl_1$, where $a_j\in\mathbb{N}_+$.

Remark 3.3 The modular sheaves given by $\mathscr{G}(\mathscr{E}_1,\dots,\mathscr{E}_n)$, where the Mukai vectors $v(\mathscr{E}_1),\dots,v(\mathscr{E}_n)$ are isotropic (primitive) and all equal are studied by Markman [2024b, Section 11].

3.2 A series of examples

We discuss examples of $\mathscr{E}_1,\mathscr{E}_2$ such that item (2) of Lemma 3.1 holds.

Lemma 3.4 *Let (S,h) be a polarized $K3$ surface. Let $\mathscr{E}_1, \mathscr{E}_2$ be Gieseker–Maruyama stable torsion-free sheaves on S, with $\mathscr{E}_1$ spherical (and hence locally free). Assume also that there exists $a \in \mathbb{N}_+$ such that*

$$v(\mathscr{E}_2) = av(\mathscr{E}_1) - \frac{2a}{r_1}(0,0,1). \tag{3.2.1}$$

Then $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is a torsion-free simple sheaf on $S^{[2]}$, and

$$w(\mathscr{G}) = ar_1\big(2r_1, 2\mu(l_1) - r_1\delta, \tfrac{1}{3}ar_1^3 c_2(S^{[2]})\big). \tag{3.2.2}$$

If in addition

$$\mathrm{Hom}(\mathscr{E}_2, \mathscr{E}_1) = 0, \tag{3.2.3}$$

then

$$\mathrm{ext}^1_{S^{[2]}}(\mathscr{G},\mathscr{G}) = 2a^2+2, \quad \mathrm{ext}^2_{S^{[2]}}(\mathscr{G},\mathscr{G}) = 2, \tag{3.2.4}$$

$\mathscr{G}$ has unobstructed deformations, and $\mathrm{Def}(\mathscr{G})$ is identified with $\mathrm{Def}(\mathscr{E}_1) \times \mathrm{Def}(\mathscr{E}_2)$ via the map in (2.2.4).

Proof We claim that $\mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2) = 0$. In fact by the equality in (3.2.1) one gets that $\chi(S, \mathscr{E}_1(n))/r(\mathscr{E}_1) \succ \chi(S, \mathscr{E}_2(n))/r(\mathscr{E}_2)$ (here $\succ$ means that the left-hand side is greater than the right-hand side for $n \gg 0$, in fact for all n in this specific case), and hence $\mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2) = 0$ by stability. By Proposition 2.6 it follows that $\mathscr{G}$ is simple. The equalities in (2.2.10) and (2.2.11) hold because of the equality in (3.2.1), and thus the equalities in (3.2.2) hold by Proposition 2.7.

The validity of the remaining statements assuming the vanishing in (3.2.3) follows from Proposition 2.6, because $\mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2) = 0$. □

Remark 3.5 Let $\mathscr{E}_1, \mathscr{E}_2$ be as in Lemma 3.1, and let $\mathscr{G} := \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$. If item (1) of Lemma 3.1 holds, with $v(\mathscr{E}_1) = v(\mathscr{E}_2)$ and $\mathscr{E}_1, \mathscr{E}_2$ stable nonisomorphic vector bundles, then (2.2.3) gives that $\mathrm{ext}^2_{S^{[2]}}(\mathscr{G},\mathscr{G}) = 6$. If item (2) of Lemma 3.1 holds, with $\mathscr{E}_1, \mathscr{E}_2$ slope stable, then (2.2.3) gives that $\mathrm{ext}^2_{S^{[2]}}(\mathscr{G},\mathscr{G}) = 2$. There are examples with $\mathscr{E}_2$ Gieseker–Maruyama stable but not slope stable with $\mathrm{ext}^1_{S^{[2]}}(\mathscr{E}_1, \mathscr{E}_2)$ arbitrarily large, and hence $\mathrm{ext}^2_{S^{[2]}}(\mathscr{G},\mathscr{G})$ arbitrarily large by (2.2.3). However, in these cases $\mathscr{G}$ is unstable. This motivates our expectation that $\mathscr{G}$ belongs to a connected smooth component of the corresponding moduli space of (semi)stable sheaves on $S^{[2]}$. The main result in [Bottini 2024a] gives further evidence towards the expectation that moduli spaces of modular sheaves (or at least of projectivelt hyperholomorphic sheaves) are often smooth.

Remark 3.6 Set $r_1 = 2a$ or $r_1 = a$ with a odd in Lemma 3.4. In other words, suppose that

(1) $v(\mathscr{E}_1) = (2a, l_1, s_1)$ and $v(\mathscr{E}_2) = (2a^2, al_1, as_1 - 1)$, or

(2) $v(\mathscr{E}_1) = (a, l_1, s_1)$ and $v(\mathscr{E}_2) = (a^2, al_1, as_1 - 2)$ and a is odd.

Then the vector $v(\mathscr{E}_2)$ is primitive. Conversely, if in Lemma 3.4 the vector $v(\mathscr{E}_2)$ is primitive, then either item (1) or item (2) holds.

3.3 Fatighenti's example

Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface. Let $X \subset \mathbb{G}\mathrm{r}(1,\mathbb{P}^5)$ be the variety parametrizing lines in Y, and let h be the Plücker polarization of X. Then (X,h) is a general HK of type $K3^{[2]}$ with polarization of square 6 and divisibility 2. Let $\mathscr{Q}$ be the restriction to X of the tautological quotient rank 4 vector bundle on $\mathbb{G}\mathrm{r}(1,\mathbb{P}^5)$. Then $\mathscr{Q}$ is a rigid modular vector bundle which is stable if Y is general, and belongs to the class of vector bundles studied in [O'Grady 2022; 2024]. In [Fatighenti 2024] it is shown that $\bigwedge^2\mathscr{Q}$ is stable, and that

$$h^1(X,\mathrm{End}(\textstyle\bigwedge^2\mathscr{Q}))=20,\quad h^2(X,\mathrm{End}(\textstyle\bigwedge^2\mathscr{Q}))=2. \tag{3.3.1}$$

Let

$$w:=3(2,h,c_2(X))=\big(\mathrm{rk}(\textstyle\bigwedge^2\mathscr{Q}),c_1(\textstyle\bigwedge^2\mathscr{Q}),\Delta(\textstyle\bigwedge^2\mathscr{Q})\big).$$

A computation gives that $\big[\bigwedge^2\mathscr{Q}\big]\in M_w(X,h)$. Here we show that $\bigwedge^2\mathscr{Q}$ is a deformation of $\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$ for suitable $\mathscr{E}_1,\mathscr{E}_2$. More precisely, let (S,D) be a polarized $K3$ surface with $D\cdot D\equiv 2 \pmod 4$, and let $\mathscr{F}$ be the stable spherical vector bundle on S with Mukai vector

$$v(\mathscr{F})=\big(2,D,\tfrac14(D\cdot D+2)\big). \tag{3.3.2}$$

(Abusing notation we denote by the same symbol D and its Poincaré dual.) A straightforward computation gives that

$$v(\mathrm{Sym}^2\mathscr{F})=\big(3,3D,\tfrac32 D\cdot D-3\big)=3v\big(\textstyle\bigwedge^2\mathscr{F}\big)-(0,0,6). \tag{3.3.3}$$

Hence the equality in (3.2.1) is satisfied (with $r_1=1$ and $a=3$) by

$$\mathscr{E}_1:=\textstyle\bigwedge^2\mathscr{F},\quad \mathscr{E}_2:=\mathrm{Sym}^2\mathscr{F}. \tag{3.3.4}$$

Here we prove that the remaining hypotheses of Lemma 3.4 (i.e., stability of $\mathrm{Sym}^2\mathscr{F}$ and the validity of (3.2.3)) hold under additional hypotheses on (S,D). The result below follows from surjectivity of the period map for $K3$ surfaces.

Claim 3.7 *Let m_0,d_0 be positive natural numbers. There exist $K3$ surfaces S with an elliptic fibration $\varepsilon\colon S\to\mathbb{P}^1$ and elliptic fiber $C\subset S$ such that*

$$\mathrm{NS}(S)=\mathbb{Z}[D]\oplus\mathbb{Z}[C],\quad D\cdot D=2m_0,\quad D\cdot C=d_0. \tag{3.3.5}$$

Lemma 3.8 *Let S be an elliptic $K3$ surface as in Claim 3.7, and suppose d_0 is odd and $d_0>3(2m_0+1)$. Then*

(1) $\mathrm{Sym}^2\mathscr{F}$ *is a slope stable vector bundle*, *and*

(2) *there is no nonzero map* $\mathrm{Sym}^2\mathscr{F}\longrightarrow\bigwedge^2\mathscr{F}$.

Proof By Proposition 6.2 in [O'Grady 2022], the vector bundle $\mathscr{F}$ is slope stable for any polarization, in particular for D.

(1) Let $t \in \mathbb{P}^1$ and let $C_t := \varepsilon^{-1}(t)$ be the corresponding elliptic fiber. The restriction $\mathscr{F}_t := \mathscr{F}_{|C_t}$ is (slope) stable by [loc. cit., Proposition 6.2]. If C_t is smooth it follows that

$$\operatorname{Sym}^2 \mathscr{F}_t \cong \bigoplus_{0 \neq \alpha \in C_t[2]} \mathscr{O}_{C_t}(D+\alpha), \tag{3.3.6}$$

where $C_t[2] < \operatorname{Pic}^0(C_t)$ is the 2-torsion subgroup. In fact this can be proved as follows. By general results, $\mathscr{F}_t \otimes \mathscr{F}_t \cong L_0 \oplus L_1 \oplus L_2 \oplus L_3$, where each L_i is a line bundle. We may assume that $L_0 = \bigwedge^2 \mathscr{F}_t \cong \mathscr{O}_{C_t}(D)$. Let $\alpha \in C_t[2]$ and let $\tau_\alpha \colon C_t \to C_t$ be translation by α. Then $\tau_\alpha^*(\mathscr{F}_t \otimes \mathscr{F}_t) \cong \mathscr{F}_t \otimes \mathscr{F}_t$. Hence the translation action of $C_t[2]$ on $\operatorname{Pic}(C_t)$ permutes the isomorphism classes $[L_0], [L_1], [L_2], [L_3]$. This forces

$$\{[L_1], [L_2], [L_3]\} = \{[\mathscr{O}_{C_t}(D+\alpha)], [\mathscr{O}_{C_t}(D+\beta)], [\mathscr{O}_{C_t}(D+\alpha+\beta)]\},$$

where $\alpha, \beta \in C_t[2]$ are nonzero and distinct. We have proved the validity of (3.3.6).

Suppose that $\operatorname{Sym}^2 \mathscr{F}$ is not slope stable. Since it is slope polystable by general results (see for example [Huybrechts and Lehn 2010, Theorem 3.2.11]) and it has rank 3, it follows that

$$\operatorname{Sym}^2 \mathscr{F} \cong \mathscr{O}_S(D) \oplus \mathscr{V}, \tag{3.3.7}$$

where $\mathscr{V}$ is a rank 2 vector bundle. The above decomposition is incompatible with the direct sum decomposition in (3.3.6) because there is no nonzero map

$$\mathscr{O}_S(D)_{|C_t} = \mathscr{O}_{C_t}(D) \to \bigoplus_{0 \neq \alpha \in C_t[2]} \mathscr{O}_{C_t}(D+\alpha). \tag{3.3.8}$$

It follows that $\operatorname{Sym}^2 \mathscr{F}$ is slope stable.

(2) Restricting a map $\varphi \colon \operatorname{Sym}^2 \mathscr{F} \to \bigwedge^2 \mathscr{F}$ to a smooth fiber C_t and recalling the decomposition in (3.3.6), we get that the restriction of φ to C_t is zero. Since $\operatorname{Sym}^2 \mathscr{F}$ is locally free it follows that $\varphi = 0$. □

Let hypotheses be as in Lemma 3.8. By Lemma 3.4 the vector bundle

$$\mathscr{G}_0 := \mathscr{G}\big(\textstyle\bigwedge^2 \mathscr{F}, \operatorname{Sym}^2 \mathscr{F}\big) \tag{3.3.9}$$

is modular, one has

$$(\operatorname{rk}(\mathscr{G}_0), c_1(\mathscr{G}_0), \Delta(\mathscr{G}_0)) = 3(2, (2\mu(D)-\delta), 3c_2(S^{[2]}),$$

and $\mathscr{G}_0$ has unobstructed deformations given by $\mathscr{G}\big(\bigwedge^2 \mathscr{F}, \mathscr{A}\big)$, where $\mathscr{A}$ is a (nearby) deformation of $\operatorname{Sym}^2 \mathscr{F}$. Now notice that if $D \cdot D = 2$, then $2\mu(D)-\delta$ has square 6 and divisibility 2. It follows that if (X, h) is a general deformation of $(S^{[2]}, 2\mu(D)-\delta)$, then (X, h) is isomorphic to the variety of lines on a general cubic hypersurface in $\mathbb{P}^5$ polarized by the Plücker line bundle (note: $2\mu(D)-\delta$) is not ample).

Proposition 3.9 *Let hypotheses be as in Lemma 3.8. Assume in addition that $D \cdot D = 2$, and let (X, h) be a general deformation of $(S^{[2]}, 2\mu(D)-\delta)$. Then the pair $\big(S^{[2]}, \mathscr{G}(\bigwedge^2 \mathscr{F}, \operatorname{Sym}^2 \mathscr{F})\big)$ deforms to the pair $\big(X, \bigwedge^2 \mathscr{Q}\big)$, where $\mathscr{Q}$ is the restriction to X of the tautological quotient rank 4 vector bundle on $\mathbb{G}\mathrm{r}(1, \mathbb{P}^5)$.*

Proof Let $\mathscr{F}[2]^+$ be the (modular) vector bundle on $S^{[2]}$ associated to $\mathscr{F}$ according to Definition 5.1 in [O'Grady 2022]; see Section 2.1. Then

$$\big(\mathrm{rk}(\mathscr{F}[2]^+), c_1(\mathscr{F}[2]^+), \Delta(\mathscr{F}[2]^+)\big) = \big(4, 2\mu(D)-\delta, c_2(S^{[2]})\big). \tag{3.3.10}$$

(See Proposition 5.2 in [loc. cit.].) Moreover, the pair $(S^{[2]}, \mathscr{F}[2]^+)$ deforms to $(X, \mathscr{Q})$, where (X, h) is a general deformation of $(S^{[2]}, 2\mu(D)-\delta)$. Thus it suffices to prove that there is an isomorphism

$$\textstyle\bigwedge^2 \mathscr{F}[2]^+ \cong \mathscr{G}\big(\bigwedge^2 \mathscr{F}, \mathrm{Sym}^2 \mathscr{F}\big). \tag{3.3.11}$$

Let notation be as in Section 2.4. We have the exact sequence

$$0 \to \rho^* \mathscr{F}[2]^+ \to \tau_1^*(\mathscr{F}) \otimes \tau_2^*(\mathscr{F}) \to \iota_*(\epsilon^* \textstyle\bigwedge^2 \mathscr{F}) \to 0; \tag{3.3.12}$$

see equation (5.2.2) in [loc. cit.]. Taking the second exterior product we get an exact sequence described as follows. If V, W are (complex) vector spaces we may define an isomorphism

$$f\colon \mathrm{Sym}^2 V \otimes \textstyle\bigwedge^2 W \oplus \bigwedge^2 V \otimes \mathrm{Sym}^2 W \xrightarrow{\sim} \bigwedge^2 (V \otimes W)$$

by letting

$$f(v_1 v_2 \otimes w_1 \wedge w_2, 0) := v_1 \otimes w_1 \wedge v_2 \otimes w_2 - v_1 \otimes w_2 \wedge v_2 \otimes w_1,$$

and similarly for $f(0, v_1' v_2' \otimes w_1' \wedge w_2')$.

It follows that by taking the second exterior product of the terms in the exact sequence (3.3.12) we get the exact sequence

$$\begin{aligned} 0 \to \rho^*\big(\textstyle\bigwedge^2 \mathscr{F}[2]^+\big) \to \tau_1^*(\mathrm{Sym}^2 \mathscr{F}) \otimes \tau_2^*\big(\bigwedge^2 \mathscr{F}\big) \oplus \tau_1^*\big(\bigwedge^2 \mathscr{F}\big) \otimes \tau_2^*(\mathrm{Sym}^2 \mathscr{F}) \\ \xrightarrow{\psi} \iota_*\big(\epsilon^* \mathrm{Sym}^2 \mathscr{F} \otimes \textstyle\bigwedge^2 \mathscr{F}\big) \to 0. \end{aligned} \tag{3.3.13}$$

Now compare the above exact sequence with the one in (2.4.7) for $\mathscr{G} = \mathscr{G}\big(\bigwedge^2 \mathscr{F}, \mathrm{Sym}^2 \mathscr{F}\big)$. The middle terms are equal, and the quotient map ψ in (3.3.13) is identified with the quotient map φ in (2.4.7). Hence we get an isomorphism between $\rho^*\big(\bigwedge^2 \mathscr{F}[2]^+\big)$ and $\rho^* \mathscr{G}\big(\bigwedge^2 \mathscr{F}, \mathrm{Sym}^2 \mathscr{F}\big)$. The isomorphism descends to an isomorphism (3.3.11). □

4 Atomicity/nonatomicity

4.1 The main result

Let $\mathscr{E}_1, \mathscr{E}_2$ be sheaves on a $K3$ surface S satisfying the hypotheses of Proposition 2.7. Hence the sheaf $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is modular. The main result of the present section is the following.

Proposition 4.1 *The sheaf $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is atomic (see Section 4.2) if and only if*

$$v(\mathscr{E}_1)^2 = v(\mathscr{E}_2)^2 = 0. \tag{4.1.1}$$

Remark 4.2 Markman [2024b] showed that $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is numerically 1-obstructed (i.e., atomic) if the equalities in (4.1.1) hold (or, more generally, that $\mathscr{G}(\mathscr{E}_1, \dots, \mathscr{E}_n)$ is atomic if $v(\mathscr{E}_1)^2 = \cdots = v(\mathscr{E}_n)^2 = 0$). The point of our computation is to show the reverse implication: if $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is atomic then the equalities in (4.1.1) hold.

4.2 Recap of work by Taelman, Markman and Beckmann

In the present subsection we recall the notion, introduced by Beckmann [2023; 2025], of "extended Mukai vector" of a sheaf on a HK manifold X of dimension $2n$. Additional references are [Taelman 2023] and [Markman 2024b].

Let $H^2(X) := H^2(X; \mathbb{Q})$. The extended rational Mukai lattice of X is given by the rational vector space

$$\widetilde{H}(X) := \mathbb{Q}\alpha \oplus H^2(X) \oplus \mathbb{Q}\beta \tag{4.2.1}$$

with the bilinear symmetric form $\widetilde{b}$ defined as follows. The direct sum decomposition in (4.2.1) is orthogonal for $\widetilde{b}$, the restriction to $H^2(X)$ equals the BBF bilinear symmetric form, and

$$\widetilde{b}(\alpha, \alpha) = \widetilde{b}(\beta, \beta) = 0, \quad \widetilde{b}(\alpha, \beta) = -1. \tag{4.2.2}$$

For $v \in \widetilde{H}(X)$ we let $\widetilde{q}(v) = \widetilde{b}(v, v)$. Let $\mathfrak{g}_{\mathbb{Q}}(X)$ be the rational Looijenga–Lunts–Verbitsky algebra of X, and let $\mathrm{SH}(X) \subset H(X)$ be the Verbitsky subalgebra, generated over $\mathbb{Q}$ by $H^2(X)$. One has an isomorphism of Lie algebras $\mathfrak{g}_{\mathbb{Q}}(X) \cong \mathfrak{so}(\widetilde{H}(X))$ such that there is an embedding of $\mathfrak{so}(\widetilde{H}(X))$-modules

$$\mathrm{SH}(X) \xrightarrow{\Psi} \mathrm{Sym}^n \widetilde{H}(X) \tag{4.2.3}$$

described as follows. Associate to $\lambda \in H^2(X)$ the element $e_\lambda \in \mathfrak{so}(\widetilde{H}(X))$ defined by

$$e_\lambda(\alpha) = \lambda, \qquad e_\lambda(\mu) = \widetilde{b}(\lambda, \mu)\beta \quad \text{for all } \mu \in H^2(X), \qquad e_\lambda(\beta) = 0. \tag{4.2.4}$$

Note that e_λ and e_μ commute for any $\lambda, \mu \in H^2(X)$. One defines Ψ by letting

$$\Psi(\lambda_1 \cdots \lambda_k) := e_{\lambda_1} \cdots e_{\lambda_k}\left(\frac{\alpha^n}{n!}\right). \tag{4.2.5}$$

The map Ψ is an isometric embedding with respect to the nondegenerate bilinear forms on $\mathrm{SH}(X)$ and $\mathrm{Sym}^n \widetilde{H}(X)$ defined as follows. The Mukai pairing $(\ ,\ )_{\mathrm{M}}$ on $\mathrm{SH}(X)$ is defined by requiring that

$$(\xi, \eta)_{\mathrm{M}} := (-1)^p \int_X \xi \cdot \eta \tag{4.2.6}$$

if $\xi \in \mathrm{SH}^{2p}(X), \eta \in \mathrm{SH}^{4n-2p}(X)$ (we let $\mathrm{SH}^{2d}(X) := \mathrm{SH}(X) \cap H^{2d}(X)$), and that

$$\mathrm{SH}^{2p}(X) \perp \mathrm{SH}^{2q}(X) \quad \text{if } p + q \neq 2n. \tag{4.2.7}$$

Note: if X is a $K3$ surface, then $(\ ,\ )_{\mathrm{M}}$ is the opposite of the classical Mukai pairing. The bilinear form $\widetilde{b}_{[n]}$ on $\mathrm{Sym}^n\,\widetilde{H}(X)$ is defined (following Beckmann [2023, p. 119]) by

$$\widetilde{b}_{[n]}(x_1\cdots x_n, y_1\cdots y_n) := c_X(-1)^n \sum_{\sigma\in\mathscr{S}_n}\prod_{i=1}^{n}\widetilde{b}(x_i, y_{\sigma(i)}). \tag{4.2.8}$$

As stated above, one has

$$\widetilde{b}_{[n]}(\Psi(\xi),\Psi(\eta)) = (\xi,\eta)_{\mathrm{M}} \quad \text{for all } \xi,\eta\in \mathrm{SH}(X). \tag{4.2.9}$$

Since $(\xi,\eta)_{\mathrm{M}}$ and $\widetilde{b}_{[n]}$ are nondegenerate, the orthogonal projection

$$\mathrm{Sym}^n\,\widetilde{H}(X) \xrightarrow{T} \mathrm{SH}(X) \tag{4.2.10}$$

is well defined.

There is also a well-defined orthogonal projection

$$H(X)\to \mathrm{SH}(X), \quad \eta\mapsto\overline{\eta}, \tag{4.2.11}$$

because the intersection form on $H(X)$ and its restriction to $\mathrm{SH}(X)$ are nondegenerate. Let $\mathscr{F}$ be a sheaf on X (or an object of $D^b(X)$). The associated Mukai vector is given by

$$v(\mathscr{F}) := \mathrm{ch}(\mathscr{F})\sqrt{\mathrm{Td}(X)}. \tag{4.2.12}$$

Note that $v(\mathscr{F})\in H(X)$, i.e., it's a rational cohomology class. Note that $\overline{v(\mathscr{F})}\in \mathrm{SH}(X)$.

Definition 4.3 [Beckmann 2023, Definition 4.15] Let $v\in\widetilde{X}$. Then v is an extended Mukai vector of $\mathscr{F}$ if

$$\mathrm{span}(\overline{v(\mathscr{F})}) = \mathrm{span}\{T(v^n)\}. \tag{4.2.13}$$

Abusing notation, if an extended Mukai vector of $\mathscr{F}$ exists we denote it by $\widetilde{v}(\mathscr{F})$.

Remark 4.4 If $\mathscr{F}$ is atomic [Beckmann 2025], i.e., numerically 1-obstructed [Markman 2024b], then it has an extended Mukai vector. For HK manifolds of type $K3^{[2]}$ the two notions are equivalent, but in general there is no reason why having an extended Mukai vector should imply atomic.

4.3 An "extended Mukai vector" of $\mathscr{F}$ is determined by $r(\mathscr{F})$, $c_1(\mathscr{F})$ and $\Delta(\mathscr{F})$

Let $\mathscr{F}$ be a sheaf with positive rank r. Suppose that an extended Mukai vector of $\mathscr{F}$. Markman and Beckmann showed that one may assume $\widetilde{v}(\mathscr{F}) = r\alpha + c_1(\mathscr{F}) + s\beta$. They also showed $\mathscr{F}$ is modular. Bottini [2024b, Corollary 3.10] showed how to determine s from r, $c_1(\mathscr{F})$ and $\Delta(\mathscr{F})$ if $\dim X = 4$.

In this subsection we extend Bottini's computation to arbitrary X. First we recall a few definitions. Following Beckmann we let $\mathsf{q}_{2i}\in \mathrm{SH}^{4i}(X)$ be the element characterized by the requirement that

$$\int_X \mathsf{q}_{2i}\cdot\xi^{2n-2i} = c_X(2n-2i-1)!!\,q_X(\xi)^{n-i} \quad \text{for all } \xi\in H^2(X). \tag{4.3.1}$$

In [Bottini 2024b] classes $\mathfrak{g}_{2i}$ are defined: one has the relation $\mathsf{q}_{2i} = c_X \mathfrak{g}_{2i}$. We let $C(c_2(X))$ be the rational number such that

$$\int_X c_2(X) \cdot \xi^{2n-2} = C(c_2(X))\, q_X(\xi)^{n-1} \tag{4.3.2}$$

for all $\xi \in H^2(X)$. A simple argument shows that

$$\overline{c_2(X)} = \frac{C(c_2(X))}{c_X (2n-3)!!} \mathsf{q}_2. \tag{4.3.3}$$

Proposition 4.5 (Beckmann, Markman, Bottini + ϵ) *Suppose that $\mathscr{F}$ is a sheaf with positive rank r which has an extended Mukai vector. Then $\mathscr{F}$ is modular, i.e., $\overline{\Delta(\mathscr{F})}$ is a multiple of q_2, and we may assume that $\widetilde{v}(\mathscr{F}) = r\alpha + c_1(\mathscr{F}) + s\beta$ where s is determined by the equality*

$$\overline{\Delta(\mathscr{F})} = \left(\frac{C(c_2(X)) r^2}{12 c_X (2n-3)!!} - 2rs + q_X(c_1(\mathscr{F})) \right) \mathsf{q}_2. \tag{4.3.4}$$

Proof By hypothesis there exist $x, s, \rho \in \mathbb{Q}$ and $\lambda \in H^2(X)$ such that

$$\overline{v(\mathscr{F})} = \rho T\big((x\alpha + \lambda + s\beta)^n\big). \tag{4.3.5}$$

We define a grading

$$\operatorname{Sym}^n \widetilde{H}(X) = \bigoplus_d [\operatorname{Sym}^n \widetilde{H}(X)]_{2d} \tag{4.3.6}$$

by letting $\alpha^i \omega_1 \cdots \omega_k \beta^j \in [\operatorname{Sym}^n \widetilde{H}(X)]_{2k+4j}$. The inclusion Ψ maps $\mathrm{SH}^{2d}(X)$ into $[\operatorname{Sym}^n \widetilde{H}(X)]_{2d}$ and the orthogonal projection maps $[\operatorname{Sym}^n \widetilde{H}(X)]_{2d}$ onto $\mathrm{SH}^{2d}(X)$. Hence the equality matches homogeneous elements of the same degrees. Developing up to degree 4 we get that

$$\begin{aligned} r + c_1(\mathscr{F}) + \overline{\mathrm{ch}_2(\mathscr{F})} + \frac{r C(c_2(X))}{24 c_X (2n-3)!!} \mathsf{q}_2 \\ = \rho \cdot \left(x^n T(\alpha^n) + n x^{n-1} T(\alpha^{n-1}\lambda) + n x^{n-1} s T(\alpha^{n-1}\beta) + \binom{n}{2} x^{n-2} T(\alpha^{n-2}\lambda^2) \right). \end{aligned}$$

(Recall the equality in (4.3.3).) We have

$$T(\alpha^n) = n! \quad \text{and} \quad T(\alpha^{n-1}\lambda) = (n-1)!\, \lambda \tag{4.3.7}$$

because $\Psi(n!) = \alpha^n$ and $\Psi((n-1)!\, \lambda) = \alpha^{n-1}\lambda$. On the other hand by [Beckmann 2023, Lemma 3.5] and [Bottini 2024b, Lemma 3.8] (note that $c_X \mathfrak{g}_2 = \mathsf{q}_2$) we have

$$T(\alpha^{n-1}\beta) = (n-1)!\, \mathsf{q}_2 \quad \text{and} \quad T(\alpha^{n-2}\lambda^2) = (n-2)!\, (\lambda^2 - q_X(\lambda) \mathsf{q}_2). \tag{4.3.8}$$

Hence we get the equality

$$r + c_1(\mathscr{F}) + \overline{\mathrm{ch}_2(\mathscr{F})} + \frac{rC(c_2(X))}{24c_X(2n-3)!!}\mathsf{q}_2 = \rho\cdot\bigl(n!\,x^n + n!\,x^{n-1}\lambda + n!\,x^{n-1}s\mathsf{q}_2 + \tfrac{1}{2}n!\,x^{n-2}(\lambda^2 - q_X(\lambda)\mathsf{q}_2)\bigr).$$

Since $r>0$ we may choose $x=r$ and

$$\rho = \frac{1}{n!\,r^{n-1}}. \tag{4.3.9}$$

The proposition follows. □

Example 4.6 Let X be of type $K3^{[2]}$, and hence $\mathrm{SH}(X)=H(X)$. Then $c_X=1$ and $C(c_2(X))=30$, i.e., $c_2(X)=30\mathsf{q}_2$. Hence $\tilde{v}(\mathscr{F}) = r\alpha + c_1(\mathscr{F}) + s\beta$, where s is the solution of the equation

$$\Delta(\mathscr{F}) = \left(\frac{r^2}{12} - \frac{rs}{15} + \frac{q_X(c_1(\mathscr{F}))}{30}\right)c_2(X). \tag{4.3.10}$$

4.4 The four-dimensional case

In the present subsection we assume that $\dim X = 4$. If $\lambda\in H^2(X)$ we let $\lambda^\vee\in H^6(X)\cong H^2(X)^\vee$ be the linear form associated to λ by q_X, i.e., such that for all $\mu\in H^2(X)$ we have

$$\int_X \lambda^\vee\cdot\mu = q_X(\lambda,\mu). \tag{4.4.1}$$

Let $\mathscr{F}$ be a sheaf of positive rank r with an extended Mukai vector $\tilde{v}(\mathscr{F})$. Thus by Proposition 4.5 we may assume that $\tilde{v}(\mathscr{F}) = r\alpha + c_1(\mathscr{F}) + s\beta$, where s is the solution of the linear equation in (4.3.4). The proposition below is essentially contained in [Bottini 2024b, Corollary 3.10]. More precisely, the addend $s\lambda^\vee/r$ on the right-hand side of the equation in the statement of Bottini's corollary should be multiplied by c_X.

Proposition 4.7 *Keep assumptions and notation as above. Then*

$$\overline{\mathrm{ch}_3(\mathscr{F})} = \left(\frac{sc_X}{r} - \frac{C(c_2(X))}{24}\right)c_1(\mathscr{F})^\vee. \tag{4.4.2}$$

Proof Let $\lambda\in H^2(X)$. A straightforward computation gives that

$$T(\lambda\beta) = c_X\lambda^\vee \quad\text{and}\quad \overline{c_2(X)\lambda} = C(c_2(X))\lambda^\vee. \tag{4.4.3}$$

Since $r\alpha + c_1(\mathscr{F}) + s\beta$ is an extended Mukai vector of $\mathscr{F}$ with proportionality factor $1/2r$ (see (4.3.9)) we have

$$\begin{aligned}\overline{\mathrm{ch}_3(\mathscr{F})} + \frac{C(c_2(X))}{24}c_1(\mathscr{F})^\vee &= \overline{\mathrm{ch}_3(\mathscr{F})} + \frac{\overline{c_2(X)c_1(\mathscr{F})}}{24}c_1(\mathscr{F})^\vee \\ &= \overline{v_3(\mathscr{F})} = \frac{1}{2r}T\bigl([(r\alpha + c_1(\mathscr{F}) + s\beta)^2]_6\bigr) = \frac{c_Xs}{r}c_1(\mathscr{F})^\vee.\end{aligned} \tag{4.4.4}$$

The proposition follows. □

Example 4.8 Let X be of type $K3^{[2]}$; in particular, $\mathrm{SH}(X) = H(X)$. Then the equation in Proposition 4.7 reads

$$\mathrm{ch}_3(\mathscr{F}) = \left(\frac{s}{r} - \frac{5}{4}\right) c_1(\mathscr{F})^\vee. \tag{4.4.5}$$

4.5 Computation of $\mathrm{ch}_3(\mathscr{G})$

Let $\mathscr{E}_1, \mathscr{E}_2$ be sheaves on a $K3$ surface S satisfying the hypotheses of Proposition 2.7. Let $\lambda \in H^2(S;\mathbb{Q})$ be given by

$$\lambda := \frac{c_1(\mathscr{E}_1)}{r_1} = \frac{c_1(\mathscr{E}_2)}{r_2} \in H^2(S;\mathbb{Q}), \quad \text{and set} \quad d := \int_S \lambda^2. \tag{4.5.1}$$

Proposition 4.9 *Let* $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$. *Then*

$$\mathrm{ch}_3(\mathscr{G}) = \frac{d-3}{2} c_1(\mathscr{G})^\vee. \tag{4.5.2}$$

Proof We adopt the notation introduced in Section 2. In particular, r_i is the rank of $\mathscr{E}_i$, $\eta \in H^4(S)$ is the fundamental class of S, and $e \in H^2(X_2(S);\mathbb{Z})$ is the Poincaré dual of the exceptional divisor of the blowup map $\tau\colon X_2(S) \to S^2$. We let $\overline{r} = r_1 r_2$ (as in Remark 2.10).

The exact sequence in (2.4.7) and the GRR theorem applied to $\iota_{1,2,*}(\mathscr{R}_{1,2})$ give that

$$\begin{aligned}\rho^* \mathrm{ch}_3(\mathscr{G}) = \overline{r}\big[\tau_1^*\lambda \cdot \tau_2^*(\lambda^2 - 2\eta) + \tau_1^*(\lambda^2 - 2\eta) \cdot \tau_2^*\lambda + e \cdot (\tau_1^*\eta + \tau_2^*\eta) \\ - \tfrac{1}{2} e \cdot (\tau_1^*\lambda + \tau_2^*\lambda)^2 + \tfrac{1}{2} e^2 \cdot (\tau_1^*\lambda + \tau_2^*\lambda) - \tfrac{1}{6} e^3\big].\end{aligned} \tag{4.5.3}$$

(Note: one must use the equality in (2.5.9).) Next we note that we have the following relations in the cohomology of $X_2(S)$:

$$\tfrac{1}{2} e^2 \cdot (\tau_1^*\lambda + \tau_2^*\lambda) = -\tau_1^*\eta \cdot \tau_2^*\lambda - \tau_1^*\lambda \cdot \tau_2^*\eta, \quad e^3 = -12 e (\tau_1^*\eta + \tau_2^*\eta). \tag{4.5.4}$$

(To prove them, intersect both sides of the equalities with generators of $H^2(X_2(S))$.) Feeding the equalities in (4.5.4) into the equality in (4.5.3) one gets that

$$\rho^* \mathrm{ch}_3(\mathscr{G}) = (d-3)\overline{r}\big[\tau_1^*\lambda \cdot \tau_2^*\eta + \tau_1^*\eta \cdot \tau_2^*\lambda - e \cdot (\tau_1^*\eta + \tau_2^*\eta)\big]. \tag{4.5.5}$$

On the other hand, using the equalities in (4.5.4) one gets that

$$\begin{aligned}2\int_{S^{[2]}} \mathrm{ch}_3(\mathscr{G}) \cdot (\mu(\alpha) + t\delta) &= \int_{X_2(S)} \rho^* \mathrm{ch}_3(\mathscr{G}) \cdot \rho^*(\mu(\alpha) + t\delta) \\ &= (d-3) q_{S^{[2]}}(c_1(\mathscr{G}), \mu(\alpha) + t\delta).\end{aligned} \tag{4.5.6}$$

Proposition 4.9 follows. □

Remark 4.10 Let us assume that the sheaf $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ has an extended Mukai vector $\widetilde{v}(\mathscr{G})$. By Example 4.6 we may set $\widetilde{v}(\mathscr{G}) := (2\overline{r}\alpha + \overline{r}(2\mu(\lambda) - \delta) + s_{\mathscr{G}}\beta)$, where $s = s_{\mathscr{G}}$ is the solution of the equation in (4.3.10) with $\mathscr{F}=\mathscr{G}$. We claim that the equality in (4.4.5) holds for $\mathscr{F}=\mathscr{G}$. In fact by Proposition 2.7 we get that

$$s_{\mathscr{G}}=\tfrac{1}{2}\overline{r}(2d-1), \tag{4.5.7}$$

and hence

$$\widetilde{v}(\mathscr{G})=2\overline{r}\alpha+2\overline{r}\mu(\lambda)-\overline{r}\delta+\tfrac{1}{2}\overline{r}(2d-1)\beta. \tag{4.5.8}$$

A straightforward computation shows that the equality in (4.4.5) holds.

4.6 Proof of Proposition 4.1

In order to prove Proposition 4.1 we compute $\mathrm{ch}_4(\mathscr{G})$. We adopt the notation of Section 4.5, and for $i\in\{1,2\}$ we let $v(\mathscr{E}_i)=(r_i,l_i,s_i)$ be the Mukai vector of $\mathscr{E}_i$.

Proposition 4.11 *Let $\mathscr{G}=\mathscr{G}(\mathscr{E}_1,\mathscr{E}_2)$. Then*

$$\int_{S^{[2]}}\mathrm{ch}_4(\mathscr{G})=s_1s_2-\tfrac{1}{2}\overline{r}(3d-4). \tag{4.6.1}$$

Proof The exact sequence in (2.4.7) gives that $\rho^*\,\mathrm{ch}_4(\mathscr{G})=\mathrm{ch}_4(\mathscr{F})-\mathrm{ch}_4(\iota_{1,2,*}\mathscr{R}_{1,2})$ A straightforward computation gives that

$$\int_{X_2(S)}\mathrm{ch}_4(\mathscr{F})=2(s_1-r_1)(s_2-r_2)=2s_1s_2-2\overline{r}d+2\overline{r}. \tag{4.6.2}$$

The last equality holds because (2.2.11) reads

$$\frac{s_1}{r_1}+\frac{s_2}{r_2}=d. \tag{4.6.3}$$

The GRR theorem applied to $\iota_{1,2,*}(\mathscr{R}_{1,2})$ gives that

$$\int_{X_2(S)}\mathrm{ch}_4(\iota_*\mathscr{R})=\int_{X_2(S)}\left(-\frac{\overline{r}e^4}{24}+\frac{r_2(s_1-r_1)}{2}+\frac{r_1(s_2-r_2)}{2}+\frac{\overline{r}d}{2}\right)=\overline{r}(d-2).$$

It follows that

$$\int_{X_2(S)}\rho^*\,\mathrm{ch}_4(\mathscr{G})=2s_1s_2-\overline{r}(3d-4). \tag{4.6.4}$$

This proves the proposition because ρ has degree 2. □

Proof of Proposition 4.1 Since the Verbitsky subalgebra of $S^{[2]}$ is equal to the whole cohomology algebra, $\mathscr{G}$ is atomic if and only if $v(\mathscr{G})=(4\overline{r})^{-1}T(\widetilde{v}^2)$, where $\widetilde{v}$ is given by (4.5.8); see (4.3.9) for the factor $(4\overline{r})^{-1}$. By Remark 4.10 the equality holds except possibly for the degree 8 components. Hence $\mathscr{G}$ is atomic if and only if

$$\int_{S^{[2]}}v_4(\mathscr{G})=\frac{\overline{r}(2d-1)^2}{16}\int_{S^{[2]}}T(\beta^2)=\frac{\overline{r}(2d-1)^2}{16}, \tag{4.6.5}$$

where $v_4(\mathscr{G})$ is the component of degree 8 of the Mukai vector of $\mathscr{G}$. We compute $v_4(\mathscr{G})$, the component of degree 8 of the Mukai vector of $\mathscr{G}$. We have

$$\sqrt{\mathrm{Td}(S^{[2]})} = 1 + \tfrac{5}{4}\mathrm{q}_2 + \tfrac{25}{32}\mathrm{q}_4. \tag{4.6.6}$$

The equalities in (2.2.9) and (2.2.12) give that

$$\mathrm{ch}_2(\mathscr{G}) = \tfrac{1}{4}\bar{r}\big(2\mu(\lambda)-\delta\big)^2 - \tfrac{1}{12}\bar{r}c_2(S^{[2]}). \tag{4.6.7}$$

Thus we have computed $\mathrm{ch}(\mathscr{G})$, and we get that

$$\int_{S^{[2]}} v_4(\mathscr{G}) = s_1s_2 - \frac{\bar{r}(3d-4)}{2} + \frac{25\bar{r}}{16} + \frac{5\bar{r}}{48}\int_{S^{[2]}} \mathrm{q}_2 \cdot \big(3(2\mu(\lambda)-\delta)^2 - c_2(S^{[2]})\big).$$

The last integral is computed by invoking the defining property of q_2, the equalities $c_2(X) = 30\mathrm{q}_2$ (see Example 4.6) and $\int_{S^{[2]}} c_2(X)^2 = 828$. Summing up, one gets that

$$\int_{S^{[2]}} v_4(\mathscr{G}) = s_1s_2 - \tfrac{1}{16}\bar{r}(4d-1). \tag{4.6.8}$$

Thus $\mathscr{G}$ is atomic if and only if both the equalities in (4.6.5) and (4.6.8) hold, i.e., if and only if

$$s_1s_2 = \tfrac{1}{4}\bar{r}d^2. \tag{4.6.9}$$

Since the equality in (4.6.3) holds, it follows that $\mathscr{G}$ is atomic if and only if $s_1/r_1 = s_2/r_2$. The latter equation holds if and only if $v(\mathscr{E}_1)^2 = v(\mathscr{E}_2)^2 = 0$. □

5 Modular sheaves and stability

5.1 Main results

In the present section we note that the results in [O'Grady 2022] on variation of slope semistability of modular sheaves with respect to polarizations hold also when considering slope semistability with respect to Kähler classes. We also extend the results in [loc. cit.] on suitable polarizations of Lagrangian fibrations in order to deal with sheaves whose restriction to a general Lagrangian fiber is slope semistable but not stable.

5.2 Variation of stability with respect to Kähler classes

Let X be a compact Kähler manifold of dimension m. Let $\mathscr{K}(X) \subset H^{1,1}_{\mathbb{R}}(X)$ be the Kähler cone (whose elements are the cohomology classes of Kähler metrics). Let $\omega \in \mathscr{K}(X)$. If $\mathscr{A}$ is a (nonzero) torsion-free sheaf on X the *ω slope of $\mathscr{A}$* is given by

$$\mu_\omega(\mathscr{A}) := \frac{c_1(\mathscr{A})\cdot\omega^{m-1}}{\mathrm{rk}(\mathscr{A})}. \tag{5.2.1}$$

A torsion-free sheaf $\mathscr{F}$ on X is called ω *slope semistable* if for all nonzero subsheaves $\mathscr{H} \subset \mathscr{F}$ with $0 < \mathrm{rk}(\mathscr{H}) < \mathrm{rk}(\mathscr{F})$ we have

$$\mu_\omega(\mathscr{H}) \le \mu_\omega(\mathscr{F}), \tag{5.2.2}$$

and it is ω *slope stable* if strict inequality holds for all such $\mathscr{H}$. If $\mathscr{H}, \mathscr{F}$ are sheaves on an irreducible smooth variety X we let

$$\lambda_{\mathscr{H},\mathscr{F}} := r(\mathscr{F})c_1(\mathscr{H}) - r(\mathscr{H})c_1(\mathscr{F}). \tag{5.2.3}$$

The proof of Lemma 3.7 in [O'Grady 2022] extends with no changes in the more general framework.

Lemma 5.1 *Let X be a HK manifold, and let ω be a Kähler class on X. Let $\mathscr{H}, \mathscr{F}$ be nonzero torsion-free sheaves on X. Then $\mu_\omega(\mathscr{H}) \ge \mu_\omega(\mathscr{F})$ if and only if*

$$q_X(\lambda_{\mathscr{H},\mathscr{F}}, \omega) \ge 0. \tag{5.2.4}$$

Moreover equality in (5.2.4) *holds if and only if $\mu_\omega(\mathscr{H}) = \mu_\omega(\mathscr{F})$.*

Let $\mathscr{F}$ be a torsion-free modular sheaf on a HK manifold X. We define a decomposition of $\mathscr{K}(X)$ into walls and chambers related to slope stability of $\mathscr{F}$.

Definition 5.2 Let a be a positive real number. An a-*wall* of $\mathscr{K}(X)$ is the intersection $\lambda^\perp \cap \mathscr{K}(X)$, where $\lambda \in H^{1,1}_{\mathbb{Z}}(X)$ is a class such that $-\mathsf{a} \le q_X(\lambda) < 0$ (orthogonality is with respect to the BBF quadratic form q_X).

As is well-known, the set of a-walls is locally finite; in particular, the union of all the a-walls is closed in $\mathscr{K}(X)$.

Definition 5.3 An *open* a-*chamber* of $\mathscr{K}(X)$ is a connected component of the complement of the union of all the a-walls of $\mathscr{K}(X)$. A Kähler class is a-generic if it belongs to an open a-chamber.

Proposition 5.4 *Let X be a HK manifold, and let $\omega_0, \omega_1 \in \mathscr{K}(X)$. Suppose that $\mathscr{F}$ is a torsion-free modular sheaf on X which is ω_0 slope stable and not ω_1 slope stable. Then there exists a real t with $0 < t \le 1$ such that $t\omega_0 + (1-t)\omega_1$ belongs to an $\mathsf{a}(\mathscr{F})$-wall.*

Proof One needs to prove the versions of Propositions 3.8 and 3.10 in [O'Grady 2022] that one gets upon replacing the ample cone by the Kähler cone. We show how to adapt the proofs in the present context. By Lemma 6.2 in [Greb and Toma 2017] (the proof is in [Greb and Toma 2013]) there exists a real t with $0 < t \le 1$ such that, letting $\omega_t := t\omega_0 + (1-t)\omega_1$, the sheaf $\mathscr{F}$ is strictly ω_t slope semistable, i.e., ω_t slope semistable but not ω_t slope stable. Hence there exists an exact sequence of torsion-free nonzero sheaves

$$0 \to \mathscr{A} \to \mathscr{F} \to \mathscr{B} \to 0 \tag{5.2.5}$$

with $0 < \mathrm{rk}(\mathscr{E}) < \mathrm{rk}(\mathscr{F})$ and $\mu_{\omega_t}(\mathscr{A}) = \mu_{\omega_t}(\mathscr{F})$, i.e., (by Lemma 5.1)

$$q_X(\lambda_{\mathscr{A},\mathscr{F}}, \omega_t) = 0. \tag{5.2.6}$$

We may assume that $\mathscr{B}$ is torsion free (note that $\mathscr{A}$ is torsion free because $\mathscr{F}$ is torsion free). Moreover, both $\mathscr{A}$ and $\mathscr{B}$ are ω_t slope semistable. By Theorem 1.1 in [Li et al. 2017] it follows that Bogomolov's inequality holds for $\mathscr{A}$ and $\mathscr{B}$, i.e., that $\Delta(\mathscr{A})\cdot\omega^{2n-2}\geq 0$ and $\Delta(\mathscr{B})\cdot\omega^{2n-2}\geq 0$, where $2n$ is the dimension of X. To be precise, Theorem 1.1 in [loc. cit.] states that Bogomolov's inequality holds for slope semistable reflexive sheaves. From this one gets Bogomolov's inequality for a torsion-free slope semistable sheaf $\mathscr{H}$ arguing as follows. We have a canonical exact sequence

$$0\to\mathscr{H}\to\mathscr{H}^{\vee\vee}\to\mathscr{Q}\to 0,$$

where $\mathscr{Q}$ is supported on an analytic subspace of codimension at least 2. Since $\mathscr{H}$ is slope semistable, so is the double dual $\mathscr{H}^{\vee\vee}$. Since the double dual is reflexive we have $\Delta(\mathscr{H}^{\vee\vee})\cdot\omega^{2n-2}\geq 0$ by Theorem 1.1 in [Li et al. 2017]. Since $c_1(\mathscr{H})=c_1(\mathscr{H}^{\vee\vee})$ and $c_2(\mathscr{H})=c_2(\mathscr{H}^{\vee\vee})+Z$, where Z is an effective codimension-2 cycle supported on the codimension-2 components of $\operatorname{supp}(\mathscr{Q})$, we get that $\Delta(\mathscr{H})\cdot\omega^{2n-2}\geq 0$. Since Bogomolov's inequality holds for $\mathscr{A}$ and $\mathscr{B}$, the proof of Proposition 3.10 in [O'Grady 2022] extends to our case and hence we get that

$$-\mathsf{a}(\mathscr{F})\leq q_X(\lambda_{\mathscr{A},\mathscr{F}})\leq 0. \tag{5.2.7}$$

Suppose that $q_X(\lambda_{\mathscr{A},\mathscr{F}})=0$. Then $\lambda_{\mathscr{A},\mathscr{F}}=0$ and it follows that $\mu_\omega(\mathscr{A})=\mu_\omega(\mathscr{F})$ for all Kähler classes ω. By the exact sequence in (5.2.5) this contradicts the assumption that $\mathscr{F}$ is ω_0 slope stable. Thus $q_X(\lambda_{\mathscr{A},\mathscr{F}})<0$, and hence $\lambda_{\mathscr{A},\mathscr{F}}^{\perp}\cap\mathscr{K}(X)$ is an $\mathsf{a}(\mathscr{F})$-wall. We are done by (5.2.6). □

Corollary 5.5 *Let X be a HK manifold, and let $\mathscr{F}$ be a torsion-free modular sheaf on X. Then the following hold:*

1. *Suppose that ω is a Kähler class on X which is $\mathsf{a}(\mathscr{F})$-generic. If $\mathscr{F}$ is strictly ω slope semistable, there exists an exact sequence of torsion-free nonzero sheaves*

$$0\to\mathscr{A}\to\mathscr{F}\to\mathscr{B}\to 0 \tag{5.2.8}$$

   such that $r(\mathscr{F})c_1(\mathscr{A})-r(\mathscr{A})c_1(\mathscr{F})=0$.

2. *Suppose that ω_0,ω_1 are Kähler classes on X belonging to the same open $\mathsf{a}(\mathscr{F})$-chamber. Then $\mathscr{F}$ is ω_0 slope-stable if and only if it is ω_1 slope-stable.*

Proof Item (1) follows from the proof of Proposition 5.4. In fact, let (5.2.8) be an exact sequence with $0<\operatorname{rk}(\mathscr{A})<\operatorname{rk}(\mathscr{F})$ and $\mu_\omega(\mathscr{A})=\mu_\omega(\mathscr{F})$, i.e., $q_X(\lambda_{\mathscr{A},\mathscr{F}},\omega)=0$. Then the inequalities in (5.2.7) hold, and hence $\lambda_{\mathscr{A},\mathscr{F}}=0$ because ω_0, ω_1 belong to the same open $a(\mathscr{F})$-chamber. Item (2) follows from the statement of Proposition 5.4 because ω_0, ω_1 belong to the same open $\mathsf{a}(\mathscr{F})$-chamber. □

5.3 Modular sheaves on Lagrangian fibrations

Let $\pi\colon X\to\mathbb{P}^n$ be a Lagrangian fibration of a HK manifold of dimension $2n$. We let

$$f:=\pi^*c_1(\mathscr{O}_{\mathbb{P}^n}(1)). \tag{5.3.1}$$

Let $\mathscr{F}$ be a sheaf on X. If $t \in \mathbb{P}^n$ we let $X_t = \pi^{-1}(t)$ and $\mathscr{F}_t := \mathscr{F}_{|X_t}$. Whenever we consider a "general" $t \in \mathbb{P}^n$ we may (and will) assume that X_t is smooth.

Remark 5.6 Suppose that X_t is smooth. Then the image of the restriction map $H^2(X;\mathbb{R}) \to H^2(X_t;\mathbb{R})$ is of dimension 1 and is generated by the class of an ample class $\theta_t \in H^{1,1}_{\mathbb{Z}}(X_t)$ [Wieneck 2016]; by slope (semi)stability of a sheaf on X_t we mean stability with respect to θ_t.

Definition 5.7 Let $\pi\colon X \to \mathbb{P}^n$ be a Lagrangian fibration of a HK manifold of dimension $2n$, and let $\mathsf{a} > 0$. A polarization h of X is a-*suitable with respect to* π (or simply a-suitable whenever there is no ambiguity regarding the fibration) if the following holds. If $\lambda \in H^{1,1}_{\mathbb{Z}}(X)$ is such that $-\mathsf{a} \le q_X(\lambda) < 0$, then

(1) $q_X(\lambda, h) > 0$ implies that $q_X(\lambda, f) \ge 0$,

(2) $q_X(\lambda, h) = 0$ implies that $q_X(\lambda, f) = 0$, and

(3) $q_X(\lambda, h) < 0$ implies that $q_X(\lambda, f) \le 0$.

Remark 5.8 In [O'Grady 2022, Definition 3.5], a polarization h is a-suitable if it is a-suitable according to Definition 5.7 and in addition $q_X(\lambda, f) = 0$ implies that $q_X(\lambda, h) = 0$ (where λ is as in Definition 5.7). To avoid confusion let us say that h is *strongly* a-*suitable* if it is a-suitable according to [O'Grady 2022, Definition 3.5]. This definition is useful only if the Picard rank $\rho(X)$ is 2. In fact if $\rho(X) > 2$ and $\mathsf{a} \gg 0$ then there is no strongly a-suitable polarization because the quadratic form q_X defines a negative definite quadratic form on the nonzero quotient $f^{\perp}/\mathbb{Z}f$, and hence a general $\lambda \in f^{\perp}$ has negative square and nonzero intersection with h.

Proposition 5.9 *Let $\pi\colon X \to \mathbb{P}^n$ be a Lagrangian fibration of a HK variety. Let $\mathsf{a} > 0$, and let $h \in \operatorname{Amp}(X)$ be an a-suitable polarization. Let $\mathscr{F}$ be a torsion-free modular sheaf on X such that $\mathsf{a}(\mathscr{F}) \le \mathsf{a}$.*

(a) *Suppose that $\mathscr{F}$ is not h-slope stable, and that $\mathscr{F}_t$ is slope semistable for general $t \in \mathbb{P}^n$; see Remark 5.6. Then there exists a subsheaf $\mathscr{H} \subset \mathscr{F}$ with $0 < r(\mathscr{H}) < r(\mathscr{F})$ such that $\mu_h(\mathscr{H}) \ge \mu_h(\mathscr{F})$ and $\mu(\mathscr{H}_t) = \mu(\mathscr{F}_t)$ for general $t \in \mathbb{P}^n$.*

(b) *If $\mathscr{F}$ is h-slope stable, then $\mathscr{F}_t$ is slope semistable for general $t \in \mathbb{P}^n$; see Remark 5.6.*

Corollary 5.10 *Let $\pi\colon X \to \mathbb{P}^n$ be a Lagrangian fibration of a HK variety. Let $\mathsf{a} > 0$, and let $h \in \operatorname{Amp}(X)$ be an a-suitable polarization. Let $\mathscr{F}$ be a torsion-free modular sheaf on X such that $\mathsf{a}(\mathscr{F}) \le \mathsf{a}$. If $\mathscr{F}_t$ is slope stable for general t, then $\mathscr{F}$ is h slope stable.*

Remark 5.11 Corollary 5.10 is the same as item (i) of [O'Grady 2022, Proposition 3.6], but with the "correct" definition of a-suitability.

Before proving Proposition 5.9 we go through a few preliminaries. The result below is somewhat technical.

Lemma 5.12 (K. Yoshioka) *Let X be a hyperkähler manifold. Let $h, f \in H^{1,1}_{\mathbb{Z}}(X)$ be such that $q_X(h)>0$ and $q_X(f)=0$. Suppose that $\lambda \in H^{1,1}_{\mathbb{Z}}(X)$ and that*

$$q_X(\lambda,h)=0, \quad q_X(\lambda,f)\neq 0. \tag{5.3.2}$$

Then

$$q_X(\lambda)\le -\frac{q_X(h)}{q_X(h,f)^2}. \tag{5.3.3}$$

Proof The proof is analogous to the proof of [Yoshioka 1999, Lemma 1.1]. Write $\lambda = ah+bf+\xi$, where $\xi \in \{h,f\}^{\perp}\cap H^{1,1}_{\mathbb{Z}}(X)$. Since q_X is negative definite on $h^{\perp}\cap H^{1,1}_{\mathbb{Z}}(X)$,

$$q_X(\lambda)\le q_X(ah+bf)=a^2q_X(h)+2abq_X(h,f)=-a^2q_X(h), \tag{5.3.4}$$

where the last equality follows from $0=q_X(\lambda,h)=aq_X(h)+bq_X(h,f)$. By hypothesis $q_X(\lambda,f)=aq_X(h,f)$ is nonzero, and since it is an integer we get that $a^2q_X(h,f)^2\ge 1$, i.e., $a^2\ge q_X(h,f)^{-2}$. Plugging this inequality in (5.3.4) we get the inequality in (5.3.3). □

The next results guarantee the existence of a-suitable polarizations.

Proposition 5.13 *Let $\mathsf{a}>0$. Let $\pi\colon X\to\mathbb{P}^n$ be a Lagrangian fibration of a HK manifold of dimension $2n$. Let h be a polarization of X. If $q_X(h)>\mathsf{a}\cdot q_X(h,f)^2$ then h is a-suitable.*

Proof Let $\lambda\in H^{1,1}_{\mathbb{Z}}(X)$ be as in Definition 5.7. Suppose that $q_X(\lambda,h)=0$. Then $q_X(\lambda,f)=0$ by Lemma 5.12. To finish the proof we assume that $q_X(\lambda,h)$, $q_X(\lambda,f)$ are nonzero of opposite signs and we get a contradiction. Let $m_0:=q_X(\lambda,h)$ and $n_0:=-q_X(\lambda,f)$. Let $h_0:=m_0f+n_0h$. Then $q_X(\lambda,h_0)=0$. Since m_0,n_0 are both nonzero of the same sign we have $q_X(h_0)>0$ and $q_X(h_0,f)\neq 0$. By Lemma 5.12 (with $h=h_0$) we get that

$$q_X(\lambda)\le -\frac{q_X(h_0)}{q_X(h_0,f)^2}=-\frac{q_X(h)}{q_X(h,f)^2}-\frac{2m_0}{n_0q_X(h,f)}<-\mathsf{a}. \tag{5.3.5}$$

This is a contradiction. □

Corollary 5.14 (R. Friedman) *Let $\pi\colon X\to\mathbb{P}^n$ be a Lagrangian fibration of a HK manifold of dimension $2n$. Let h_0 be a polarization of X and let $\mathsf{a}>0$. Let $N\in\mathbb{N}$ be such that $2N>\mathsf{a}\cdot q_X(h_0,f)$. Then h_0+Nf is an a-suitable polarization.*

Proof Clearly h_0+Nf is a polarization, and since

$$q_X(h_0+Nf)=q_X(h_0)+2N\cdot q_X(h_0,f)>\mathsf{a}\cdot q_X(h_0,f)^2=\mathsf{a}\cdot q_X(h_0+Nf,f)^2,$$

it is a-suitable by Proposition 5.13. □

Proof of Proposition 5.9 (a) Let $\mathscr{H}\subset\mathscr{F}$ be a subsheaf. We recall that

$$\mu_h(\mathscr{H})-\mu_h(\mathscr{F}) \text{ and } q_X(\lambda_{\mathscr{H},\mathscr{F}},h) \text{ have the same sign or are both 0.} \tag{5.3.6}$$

(See [O'Grady 2022, Lemma 3.7].) Moreover, if $t \in \mathbb{P}^n$ is general we have

(5.3.7) $\mu_h(\mathscr{H}_t)-\mu_h(\mathscr{F}_t)$ and $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)$ have the same sign or are both 0.

In fact, (5.3.7) follows from the equalities

$$r(\mathscr{H})r(\mathscr{F})(\mu(\mathscr{H}_t)-\mu(\mathscr{F}_t)) = \int_{X_t} \lambda_{\mathscr{H}_t,\mathscr{F}_t}\cdot h_t^{n-1} = \int_X \lambda_{\mathscr{H},\mathscr{F}}\cdot h^{n-1}\cdot f^n = n!\, c_X\cdot q_X(h,f)^{n-1}\cdot q_X(\lambda_{\mathscr{H},\mathscr{F}}, f), \tag{5.3.8}$$

and positivity of c_X and $q_X(h,f)$. By (5.3.6) and (5.3.7) it suffices to prove that there exists a subsheaf $\mathscr{H}\subset\mathscr{F}$ such that $0<r(\mathscr{H})<r(\mathscr{F})$ and $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)\geq 0$, $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$.

First assume $\mathscr{F}$ is h-slope semistable. Hence there exists a subsheaf $\mathscr{H}\subset\mathscr{F}$ such that $0<r(\mathscr{H})<r(\mathscr{F})$ and $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)=0$. By [O'Grady 2022, Proposition 3.10] we get that $-\mathsf{a}(\mathscr{F})\leq q_X(\lambda_{\mathscr{H},\mathscr{F}})\leq 0$. If $\lambda_{\mathscr{H},\mathscr{F}}=0$ then $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$ trivially; if $\lambda_{\mathscr{H},\mathscr{F}}\neq 0$ then $q_X(\lambda_{\mathscr{H},\mathscr{F}})<0$ because the restriction of q_X to $H^{1,1}_{\mathbb{Z}}(X)$ has signature $(1,\rho(X))$, and hence $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$ because h is a-suitable.

Next assume that $\mathscr{F}$ is not h-slope semistable. Thus there exists $\mathscr{G}\subset\mathscr{F}$ such that $0<r(\mathscr{G})<r(\mathscr{F})$ and $q_X(\lambda_{\mathscr{G},\mathscr{F}},h)>0$. If $q_X(\lambda_{\mathscr{G},\mathscr{F}}, f)=0$ we are done, hence we may assume that $q_X(\lambda_{\mathscr{G},\mathscr{F}}, f)\neq 0$. Then $q_X(\lambda_{\mathscr{G},\mathscr{F}}, f)<0$ by (5.3.7). Let S be the set of rational numbers $s\in(0,1)$ for which there exists a subsheaf $\mathscr{H}\subset\mathscr{F}$, with $0<r(\mathscr{H})<r(\mathscr{F})$, such that

$$q_X(\lambda_{\mathscr{H},\mathscr{F}}, (1-s)h+sf)=0. \tag{5.3.9}$$

Then S is nonempty because there exists a rational number $s\in(0,1)$ for which (5.3.9) holds with $\mathscr{H}=\mathscr{G}$. We claim that S is finite. In fact if (5.3.9) holds with $s\in(0,1)$ then $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)\geq 0$, because $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)\leq 0$ by (5.3.7). Since the set of subsheaves $\mathscr{H}\subset\mathscr{F}$ satisfying $\mu_h(\mathscr{H})\geq\mu_h(\mathscr{F})$ is bounded, it follows that S is finite. Hence S has a maximum s_*. Let $h_*:=(1-s_*)h+s_*f$.

Suppose that $\mathscr{F}$ is not h_* slope semistable. Then there exists a subsheaf $\mathscr{H}\subset\mathscr{F}$ with $0<r(\mathscr{H})<r(\mathscr{F})$ such that $q_X(\lambda_{\mathscr{H},\mathscr{F}},h_*)>0$. If $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)<0$ then there exists $s\in(s_*,1)$ such that (5.3.9) holds, and this is a contradiction because s_* is the maximum of S. Thus $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$ by (5.3.7). Since $q_X(\lambda_{\mathscr{H},\mathscr{F}},h_*)>0$ and $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$, we get $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)>0$, and we are done.

Suppose that $\mathscr{F}$ is h_* slope semistable. Then there exists a subsheaf $\mathscr{H}\subset\mathscr{F}$ with $0<r(\mathscr{H})<r(\mathscr{F})$ such that $q_X(\lambda_{\mathscr{H},\mathscr{F}},h_*)=0$. We claim that $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$. Granting this for the time being, we get that $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)=0$ because $q_X(\lambda_{\mathscr{H},\mathscr{F}},h_*)=0$, and hence we are done. We finish by proving that $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$. By [O'Grady 2022, Proposition 3.10] we get that $-\mathsf{a}(\mathscr{F})\leq q_X(\lambda_{\mathscr{H},\mathscr{F}})\leq 0$. If $\lambda_{\mathscr{H},\mathscr{F}}=0$, then $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)=0$ trivially. If $\lambda_{\mathscr{H},\mathscr{F}}\neq 0$ then $q_X(\lambda_{\mathscr{H},\mathscr{F}})<0$. Suppose that $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)\neq 0$. Then $q_X(\lambda_{\mathscr{H},\mathscr{F}}, f)<0$ by (5.3.7), and hence $q_X(\lambda_{\mathscr{H},\mathscr{F}},h)<0$ as h is a-suitable. This contradicts the equality $q_X(\lambda_{\mathscr{H},\mathscr{F}},h_*)=0$.

(b) Item (ii) of [O'Grady 2022, Proposition 3.6] is the same exact statement. The proof there works as well with our new definition of a-suitable polarization. □

If $\dim X=2$, then one can prove a stronger version of item (b) of Proposition 5.9.

Proposition 5.15 *Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration of a $K3$ surface. Let $\mathscr{F}$ be a torsion-free sheaf on X, and let $h \in H^{1,1}_{\mathbb{Z}}(X)$ be an ample class which is $\mathsf{a}(\mathscr{F})$-suitable. If $\mathscr{F}$ is h-slope semistable, then $\mathscr{F}_t$ is semistable for general $t \in \mathbb{P}^1$.*

Proof If $\mathscr{F}$ is h-slope stable, then $\mathscr{F}_t$ is slope semistable for general $t \in \mathbb{P}^1$ by Proposition 5.9 (every torsion-free sheaf on a $K3$ surface is modular). Now suppose that $\mathscr{F}$ is strictly h-slope semistable, and let

$$0 = \mathscr{G}_0 \subsetneq \mathscr{G}_1 \subsetneq \mathscr{G}_2 \subsetneq \cdots \subsetneq \mathscr{G}_m = \mathscr{F} \tag{5.3.10}$$

be a (slope) Jordan–Hölder filtration of $\mathscr{F}$ with $\mathscr{G}_i/\mathscr{G}_{i-1}$ torsion free for all i. This means that for $i \in \{1,\dots,m\}$ the quotient $\mathscr{G}_i/\mathscr{G}_{i-1}$ is h slope stable and $\mu_h(\mathscr{G}_i) = \mu_h(\mathscr{F})$. Let $i \in \{1,\dots,m-1\}$. Then $q_X(\lambda_{\mathscr{G}_i,\mathscr{F}}, h) = 0$ and, by Proposition 3.10 in [O'Grady 2022], we have $-\mathsf{a}(\mathscr{F}) \le q_X(\lambda_{\mathscr{G}_i,\mathscr{F}}) \le 0$ with $q_X(\lambda_{\mathscr{G}_i,\mathscr{F}}, h) = 0$ only if $\lambda_{\mathscr{G}_i,\mathscr{F}} = 0$. Since h is $\mathsf{a}(\mathscr{F})$-suitable, we get that $q_X(\lambda_{\mathscr{G}_i,\mathscr{F}}, f) = 0$.

Let $t \in \mathbb{P}^1$ be a general point. From equation (3.3.2) in [O'Grady 2022] (in the notation of that equation, we have $\lambda^2_{\mathscr{E},\mathscr{F}} \le 0$) one gets that $\mathsf{a}(\mathscr{G}_i/\mathscr{G}_{i-1}) \le \mathsf{a}(\mathscr{F})$, and hence h is $\mathsf{a}(\mathscr{G}_i/\mathscr{G}_{i-1})$-suitable. By Proposition 5.9 it follows that the restriction of $\mathscr{G}_i/\mathscr{G}_{i-1}$ to X_t is slope semistable. Moreover $\mu(\mathscr{G}_{i|X_t}) = \mu(\mathscr{F}_{|X_t})$ because $q_X(\lambda_{\mathscr{G}_i,\mathscr{F}}, f) = 0$. Hence we get a filtration

$$0 \ne \mathscr{G}_{1|X_t} \subsetneq \mathscr{G}_{2|X_t} \subsetneq \cdots \subsetneq \mathscr{G}_{m|X_t} = \mathscr{F}_t, \tag{5.3.11}$$

where each term has the same slope, and each successive quotient is semistable. It follows that $\mathscr{F}_t$ is slope-semistable. □

6 A component of $M_{w_0}(S^{[2]}, h_{S^{[2]}})$ birational to $\mathscr{M}_{v_2}(S, h_S)$

6.1 Main result

In the present section S is a $K3$ surface with an elliptic fibration $\varepsilon\colon S \to \mathbb{P}^1$ as in Claim 3.7. Recall that this means the following. Letting $C \subset S$ be a elliptic fiber, we have

$$\mathrm{NS}(S) = \mathbb{Z}[D] \oplus \mathbb{Z}[C], \quad D \cdot D = 2m_0, \quad D \cdot C = d_0, \tag{6.1.1}$$

where m_0 and d_0 are positive natural numbers. Moreover m_0 and d_0 can be assigned arbitrarily. The *associated Lagrangian fibration* of $S^{[2]}$ is given by

$$S^{[2]} \xrightarrow{\ \pi\ } (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2, \quad [Z] \mapsto \sum_{p \in S} \ell(\mathscr{O}_{Z,p})\varepsilon(p). \tag{6.1.2}$$

Assumption-Definition 6.1 Keeping assumptions as above, suppose that r_1, a are positive integers such that

$$r_1 \,|\, 2a, \quad r_1 \,|\, (m_0+1), \quad \gcd(r_1, d_0) = 1. \tag{6.1.3}$$

Let

$$v_1 := \left(r_1, D, \frac{m_0+1}{r_1}\right), \quad v_2 := av_1 - \frac{2a}{r_1}(0,0,1), \tag{6.1.4}$$

$$w_0 := ar_1\left(2r_1, 2\mu(D) - r_1\delta, \frac{ar_1^3 c_2(S^{[2]})}{3}\right). \tag{6.1.5}$$

Let h_S be a polarization of S which is $\mathsf{a}(v_2)$-suitable, and let $h_{S^{[2]}}$ be a polarization of $S^{[2]}$ which is $\mathsf{a}(w_0)$-suitable. In the present section we show that the moduli space $\mathscr{M}_{v_2}(S, h_S)$ is birational to an irreducible component of $M_w(S^{[2]}, h_{S^{[2]}})$ if $a \geq 2$. In order to formulate our result more precisely, we note the following. Since $v_1^2 = -2$ there is a unique h_S stable sheaf $\mathscr{E}_1$ with $v(\mathscr{E}_1) = v_1$, and it is locally free. Since $v_1^2 = 2a^2 + 2$ the moduli space $\mathscr{M}_{v_2}(S, h_S)$ is irreducible of dimension $2a^2 + 2$. Let $[\mathscr{E}_2] \in \mathscr{M}_{v_2}(S, h_S)$ be a general point. Then $\mathscr{E}_2$ is locally free and hence $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is locally free. By Assumption-Definition 6.1 and Lemma 3.4, we have

$$w(\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)) = w_0. \tag{6.1.6}$$

Theorem 6.2 *Suppose that Assumption-Definition 6.1 holds, and that $a \geq 2$. Let h_S be a polarization of S which is $\mathsf{a}(v_2)$-suitable, and let $h_{S^{[2]}}$ be a polarization of $S^{[2]}$ which is $\mathsf{a}(w_0)$-suitable. Let $[\mathscr{E}_2] \in \mathscr{M}_{v_2}(S, h_S)$ be a general point. Then the locally free sheaf $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is $h_{S^{[2]}}$ slope stable. The rational map*

$$\mathscr{M}_{v_2}(S, h_S) \overset{\varphi}{\dashrightarrow} M_w(S^{[2]}, h_{S^{[2]}}), \quad [\mathscr{E}_2] \mapsto [\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)], \tag{6.1.7}$$

is birational onto an irreducible component $M_{w_0}(S^{[2]}, h_{S^{[2]}})^{\bullet}$ of $M_{w_0}(S^{[2]}, h_{S^{[2]}})$.

6.2 Stability of $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$

Proposition 6.3 *Suppose that Assumption-Definition 6.1 holds. Let h_S be a polarization of S which is $\mathsf{a}(v_2)$-suitable. Then the following hold:*

(a) *The restriction of $\mathscr{E}_1$ to any fiber of the elliptic fibration with elliptic fiber C is slope stable.*

(b) *Suppose that $a \geq 2$. Let $[\mathscr{E}_2] \in M_{v_2}(S, h_S)$ be a general point and let C be a general elliptic fiber. The restriction $\mathscr{E}_{2|C}$ is semistable, with pairwise nonisomorphic Jordan–Hölder (JH) addends, none of which is isomorphic to $\mathscr{E}_{1|C}$.*

Proof The polarization h_S is $\mathsf{a}(v_1)$-suitable because $\mathsf{a}(v_1) < \mathsf{a}(v_2)$. Hence item (a) holds by [O'Grady 2022, Proposition 6.2]. Since $a \geq 2$ and $[\mathscr{E}_2] \in M_{v_2}(S, h_S)$ is a general point, the decomposition curve $\Lambda(\mathscr{E}_2)$ is integral and smooth by Corollary A.7. It follows that if $x \in \mathbb{P}^1$ is a general point the intersection $C_x \cap \Lambda(\mathscr{E}_2) = \Lambda(\mathscr{E}_2)_x$ consists of a distinct points, and hence the JH addends of $\mathscr{E}_{2|C_x}$ are pairwise nonisomorphic. Moreover, $C_x \cap \Lambda(\mathscr{E}_2) \neq C_x \cap \Sigma$ (for all $x \in \mathbb{P}^1$ because $\Lambda(\mathscr{E}_2) \cdot \Sigma = 0$), and hence none of the JH addends is isomorphic to $\mathscr{E}_{1|C}$. □

Proposition 6.4 *Let hypotheses be as in Theorem 6.2. If $[\mathscr{E}_2] \in M_{v_2}(S, h_S)$ is a general point, then $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is an $h_{S^{[2]}}$ slope stable locally free sheaf.*

Proof Let $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$. Let $x \neq y \in \mathbb{P}^1$, and let C_x, C_y be the corresponding fibers of the elliptic fibration $S \to \mathbb{P}^1$. Then we have an identification

$$\pi^{-1}(x+y) = C_x \times C_y. \tag{6.2.1}$$

Letting $t := x + y$, we have

$$\mathscr{G}_t := \mathscr{G}_{|\pi^{-1}(t)} = (\mathscr{E}_{1|C_x}) \boxtimes (\mathscr{E}_{2|C_y}) \oplus (\mathscr{E}_{2|C_x}) \boxtimes (\mathscr{E}_{1|C_y}). \tag{6.2.2}$$

By Proposition 6.3 both $\mathscr{E}_{1|C_x}$ and $\mathscr{E}_{1|C_y}$ are stable. Suppose in addition that $x \neq y$ are general. By Proposition 6.3 both $\mathscr{E}_{2|C_x}$ and $\mathscr{E}_{2|C_y}$ are semistable with pairwise nonisomorphic JH addends $V_1(x), \ldots, V_a(x)$ and $V_1(y), \ldots, V_a(y)$. Moreover, no $V_i(x)$ is isomorphic to $\mathscr{E}_{1|C_x}$, and no $V_j(y)$ is isomorphic to $\mathscr{E}_{1|C_y}$. The restriction of the polarization $h_{S^{[2]}}$ to $C_x \times C_y$ is of product type. It follows that for $i, j \in \{1, \ldots, a\}$, the tensor products

$$(\mathscr{E}_{1|C_x}) \boxtimes V_j(y) \quad \text{and} \quad V_i(x) \boxtimes (\mathscr{E}_{1|C_x}) \tag{6.2.3}$$

are slope stable [O'Grady 2022, Proposition 6.10], and they all have the same slope (with respect to the restriction of $h_{S^{[2]}}$). The upshot is that the left-hand side of (6.2.2) is slope semistable, with pairwise nonisomorphic JH addends given by the sheaves appearing in (6.2.3). It follows that any subsheaf $\mathscr{A} \subset \mathscr{G}_t$ such that $\mu(\mathscr{A}) = \mu(\mathscr{G}_t)$ (the slope is with respect to the restriction of $h_{S^{[2]}}$) is a direct sum $\mathscr{A} = \mathscr{A}' \oplus \mathscr{A}''$, where $\mathscr{A}'$ and $\mathscr{A}''$ are slope semistable subsheaves of the first and second addends of the decomposition in (6.2.2) respectively, and their JH addends are a subset of the JH addends appearing in (6.2.3).

We are ready to show that $\mathscr{G}$ is $h_{S^{[2]}}$ slope stable. Recall that $\mathscr{G}$ is modular; see Example 2.12. The polarization $h_{S^{[2]}}$ is $\mathsf{a}(\mathscr{G})$-suitable because it is $\mathsf{a}(w_0)$-suitable; recall (6.1.6). By Proposition 5.9 it suffices to show that there does not exists a subsheaf $\mathscr{H} \subset \mathscr{G}$ such that $0 < r(\mathscr{H}) < r(\mathscr{G})$ and $\mu(\mathscr{H}_t) = \mu(\mathscr{G}_t)$ for general $t = x + y$.

Suppose that such a subsheaf $\mathscr{H}$ exists. Since $\mathscr{G}_t$ is slope semistable, $\mathscr{H}_t$ is slope semistable. As shown above, we have $\mathscr{H}_t = \mathscr{H}_t' \oplus \mathscr{H}_t''$, where $\mathscr{H}_t'$ and $\mathscr{H}_t''$ are slope semistable subsheaves of the first and second addends of the decomposition in (6.2.2) respectively, and their JH addends are a subset of the JH addends appearing in (6.2.3). Of course the collection of JH addends is symmetric with respect to the involution exchanging the addends of the decomposition in (6.2.2). The set of JH addends of the first addend of the latter decomposition is in one-to-one correspondence with the points of $\Lambda(\mathscr{E}_2)_y$, and the set of JH addends of the second addend of the latter decomposition is in one-to-one correspondence with the points of $\Lambda(\mathscr{E}_2)_x$. These addends are invariant for the monodromy action. Since the decomposition curve $\Lambda(\mathscr{E}_2)$ is integral (by Corollary A.7) we get that the set of JH addends of $\mathscr{H}_t$ is the same as the set of JH addends of $\mathscr{G}_t$, and hence $\mathrm{rk}(\mathscr{H}) = \mathrm{rk}(\mathscr{G})$. That is a contradiction. □

6.3 Proof of Theorem 6.2

Let $[\mathscr{E}_2] \in \mathscr{M}_{v_2}(S, h_S)$ be a general point. Then $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ is locally free because both $\mathscr{E}_1$ and $\mathscr{E}_2$ are locally free, and it is $h_{S^{[2]}}$ slope stable by Proposition 6.4. Hence the rational map φ in (6.1.7) is defined. The image of φ, i.e., the closure of the image of the open dense subset on which φ is regular, is irreducible because $\mathscr{M}_{v_2}(S, h_S)$ is irreducible; we denote it by $M_w(S^{[2]}, h_{S^{[2]}})^{\bullet}$. By item (b) of Proposition 6.3 we have $\mathrm{Hom}(\mathscr{E}_2, \mathscr{E}_1) = 0$. Since $\mathscr{E}_2$ is stable (because $[\mathscr{E}_2]$ is a general point of $\mathscr{M}_{v_2}(S, h_S)$), the full set of hypotheses of Lemma 3.4 is satisfied. It follows that $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ has unobstructed deformations, and $\mathrm{Def}(\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2))$ is identified with $\mathrm{Def}(\mathscr{E}_2)$ via the map in (2.2.4) (recall that $\mathscr{E}_1$ is spherical, hence rigid). This proves that $M_{w_0}(S^{[2]}, h_{S^{[2]}})^{\bullet}$ has the same dimension as $\mathscr{M}_{v_2}(S, h_S)$ and is an irreducible component of $M_{w_0}(S^{[2]}, h_{S^{[2]}})$.

It remains to prove that the map

$$\mathscr{M}_{v_2}(S, h_S) \overset{\overline{\varphi}}{\dashrightarrow} M_{w_0}(S^{[2]}, h_{S^{[2]}})^{\bullet} \tag{6.3.1}$$

defined by φ is birational. Since domain and codomain have the same dimension it suffices to show that $\overline{\varphi}$ is generically injective. Let $[\mathscr{E}_2], [\mathscr{E}_2'] \in \mathscr{M}_{v_2}(S, h_S)$ be general distinct points. Then $\mathscr{E}_2$, $\mathscr{E}_2'$ are h_S stable and, by Proposition 6.4, both $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$ and $\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2')$ are $h_{S^{[2]}}$ stable. Since $\mathrm{Hom}(\mathscr{E}_2, \mathscr{E}_2') = 0$, we have

$$\mathrm{Hom}(\tau_1^*\mathscr{E}_1 \otimes \tau_2^*\mathscr{E}_2 \oplus \tau_1^*\mathscr{E}_2 \otimes \tau_2^*\mathscr{E}_1, \tau_1^*\mathscr{E}_1 \otimes \tau_2^*\mathscr{E}_2' \oplus \tau_1^*\mathscr{E}_2' \otimes \tau_2^*\mathscr{E}_1) = 0. \tag{6.3.2}$$

By the BKR McKay correspondence it follows that $\mathrm{Hom}(\mathscr{G}(\mathscr{E}_1, \mathscr{E}_2), \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2')) = 0$. This proves that $\overline{\varphi}$ is generically injective. □

7 The main result

7.1 Sideways nearby deformations

Let X be a HK manifold, and let $\mathscr{F}$ be a torsion-free sheaf on X. Suppose that $\Delta(\mathscr{F}) \in H^{2,2}_{\mathbb{Z}}(X)$ remains of type $(2,2)$ for all (nearby) deformations of X. Then $\mathscr{F}$ is a modular sheaf. In fact by Remark 1.2 in [O'Grady 2022] it suffices to show that the orthogonal projection of $\Delta(\mathscr{F})$ to the Verbitsky subalgebra $D(X) \subset H(X)$ generated by $H^2(X)$ is a multiple of the class $q_X^{\vee}$ dual to the BBF quadratic form q_X. This holds because $q_X^{\vee}$ generates the subspace of classes in $D(X) \subset H^4(X)$ which remain of type $(2,2)$ for all (nearby) deformations of X. The definition below is not standard.

Definition 7.1 Let X be a HK manifold, and let ω be a Kähler class on X. A vector bundle $\mathscr{F}$ on X is *strongly ω-projectively hyperholomorphic* if $\mathscr{F}$ is ω slope stable, $\Delta(\mathscr{F}) \in H^{2,2}_{\mathbb{Z}}(X)$ remains of type $(2,2)$ for all (nearby) deformations of X, and ω belongs to an open $\mathsf{a}(\mathscr{F})$-chamber in $\mathscr{K}(X)$ ($\mathsf{a}(\mathscr{F})$ is defined because $\mathscr{F}$ is modular).

This definition is motivated by Verbitsky's fundamental results [1996, Theorem 2.5 and Section 11].

Proposition 7.2 *Let X_0 be a HK manifold, and let ω_0 be a Kähler class on X_0. Suppose that $\mathscr{F}$ is a strongly ω_0-projectively hyperholomorphic vector bundle on X_0. Then the natural maps*

$$\mathrm{Def}(X_0, \mathbb{P}(\mathscr{F})) \to \mathrm{Def}(X_0) \quad \textit{and} \quad \mathrm{Def}(X_0, \mathscr{F}) \to \mathrm{Def}(X_0, c_1(\mathscr{F})) \tag{7.1.1}$$

are surjective.

Proof Abusing notation we denote by the same symbols representatives of the deformation spaces in (7.1.1). In particular, we identify $\mathrm{Def}(X_0)$ and $\mathrm{Def}(X_0, c_1(\mathscr{F}))$ with open neighborhoods of the origins in $H^{1,1}(X_0)$ and in $H^{1,1}(X_0) \cap c_1(\mathscr{F})^{\perp}$, respectively. Let $\mathscr{U} \subset \mathscr{K}(X_0)$ be the open $a(\mathscr{F})$-chamber containing ω_0. Then $\mathscr{F}$ is ω slope stable for all $\omega \in \mathscr{U}$ by Corollary 5.5. Let $\omega \in \mathscr{U}$, and let $\mathscr{X} \to T(X_0, \omega)$ be the twistor family of deformations of X_0 associated to ω. By Theorem 11.1 in [Verbitsky 1996] the projective bundle $\mathbb{P}(\mathscr{F}) \to X_0$ extends to a family of projective bundles over the fibers of $\mathscr{X} \to T(X, \omega)$. This proves that the image of the first map in (7.1.1) contains a neighborhood of 0 in the open cone $\mathscr{U} \subset H^{1,1}_{\mathbb{R}}(X_0)$; since the image is a complex-analytic subset of $H^{1,1}(X_0) = H^{1,1}_{\mathbb{R}}(X_0) \otimes_{\mathbb{R}} \mathbb{C}$, it follows that it contains a neighborhood of 0 in $H^{1,1}(X_0)$. This proves that the first map in (7.1.1) is surjective.

Next we prove that the second map in (7.1.1) is surjective. Let X be a (nearby) deformation of X_0, and let $g_X \colon \boldsymbol{P}_X \to X$ be a deformation of the projective bundle $g_0 \colon \mathbb{P}(\mathscr{F}) \to X_0$ (it exists by surjectivity of the first map in (7.1.1)). It suffices to show that if the deformation of X belongs to $H^{1,1}(X_0) \cap c_1(\mathscr{F})^{\perp}$, i.e., $c_1(\mathscr{F})$ remains of type $(1,1)$, then $\boldsymbol{P}_X$ is the projectivization of a vector bundle. Let r be the rank of $\mathscr{F}$, and let $\xi_0 = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(1))$. Then

$$c_1(\Theta_{\mathbb{P}(\mathscr{F})/X_0}) = (r+1)\xi_0 + g_0^* c_1(\mathscr{F}). \tag{7.1.2}$$

This shows that if the parallel transport to $H^2(X)$ of the class $c_1(\mathscr{F})$ is of type $(1,1)$, then also the parallel transport to $H^2(\boldsymbol{P}_X)$ of the class ξ_0 is of type $(1,1)$. If $\xi \in H^{1,1}(\boldsymbol{P}_X)$ is the parallel transport of ξ_0, and L is the corresponding holomorphic line bundle on $\boldsymbol{P}_X$, then the dual of the vector bundle $g_{X,*}(L)$ is an extension of $\mathscr{F}$ to X (we adopt the pre-Grothendieck convention for the projectivization of a vector bundle). This proves that the second map in (7.1.1) is surjective. □

Proposition 7.3 *Let notation and hypotheses be as in Theorem 6.2. Let $[\mathscr{E}_2] \in \mathscr{M}_{v_2}(S, h_S)$ be a general point and set $\mathscr{G} = \mathscr{G}(\mathscr{E}_1, \mathscr{E}_2)$. Then the map*

$$\mathrm{Def}(S^{[2]}, \mathscr{G}) \to \mathrm{Def}(S^{[2]}, c_1(\mathscr{G})) \tag{7.1.3}$$

is surjective.

Proof Since $[\mathscr{E}_2]$ is a general point of $\mathscr{M}_{v_2}(S, h_S)$, the sheaf $\mathscr{E}_2$ is locally free, and hence also $\mathscr{G}$ is locally free. Let $h_{S^{[2]}}$ be a polarization of $S^{[2]}$ as in Theorem 6.2, i.e., which is $\mathsf{a}(w_0)$-suitable with respect to the Lagrangian fibration π appearing in (6.1.2). Then $\mathscr{G}$ is $h_{S^{[2]}}$ slope stable by Proposition 6.4, and $\Delta(\mathscr{G})$ remains of type $(2,2)$ for all (nearby) deformations of $S^{[2]}$ because it is a multiple of $c_2(S^{[2]})$. Now let $\omega \in \mathscr{K}(S^{[2]})$ be a class belonging to an open $\mathsf{a}(\mathscr{G})$-chamber whose closure contains $h_{S^{[2]}}$. Then $\mathscr{G}$ is

ω slope stable by Proposition 5.4. Hence $\mathscr{G}$ is strongly ω-projectively hyperholomorphic, and therefore the proposition follows from Proposition 7.2. □

7.2 Proof of the main result

We recall the main result, i.e., Theorem 1.2. The hypotheses are the following: let r_1 be a positive integer, let (X, h) be a polarized HK variety of type $K3^{[2]}$ such that (1.2.4) and (1.2.5) hold, let a be a positive integer greater than 1 such that $2a$ is a multiple of r_1, and let w be the mock Mukai vector in (1.2.3). The thesis is that the moduli space $M_w(X, h)$ is nonempty, and for (X, h) general it has an irreducible component of dimension $2a^2 + 2$. The proof proper is at the end of the subsection.

Throughout the subsection we suppose that Assumption-Definition 6.1 holds. In addition we assume the following:

(7.2.1) $$\text{all singular elliptic fibers of } \epsilon\colon S \to \mathbb{P}^1 \text{ have simple node singularities.}$$

Note that this holds generically, and that if it holds then each singular fiber has exactly one node because S has Picard number 2. Let $\mathscr{L}$ be the line bundle on $S^{[2]}$ such that

(7.2.2) $$c_1(\mathscr{L}) = 2\mu(D) - r_1\delta.$$

Proposition 7.4 *Let π be the Lagrangian fibration in* (6.1.2)*. If $d_0 \geq 2r_1$, then $\mathscr{L}$ is π ample* (*recall that $d_0 = D \cdot C$, where C is a fiber of ϵ*)*.*

Proof It suffices to show that for every $(x_1 + x_2) \in (\mathbb{P}^1)^{(2)} = \mathbb{P}^2$, the restriction of $\mathscr{L}$ to every irreducible component of the reduced fiber $\pi^{-1}(x_1 + x_2)_{\text{red}}$ is ample. For $x \in \mathbb{P}^1$ we let $C_x^{[2]}, W_x \subset S^{[2]}$ be the subsets parametrizing subschemes of $C_x = \epsilon^{-1}(x)$ and nonreduced schemes Z supported on C_x, respectively. Let $(x_1 + x_2) \in (\mathbb{P}^1)^{(2)} = \mathbb{P}^2$. The irreducible decomposition of $\pi^{-1}(x_1 + x_2)_{\text{red}}$ is

(7.2.3) $$\pi^{-1}(x_1 + x_2)_{\text{red}} = \begin{cases} C_{x_1} \times C_{x_2} & \text{if } x_1 \neq x_2, \\ C_x^{[2]} \cup W_x & \text{if } x_1 = x_2 = x. \end{cases}$$

It follows that if $x_1 \neq x_2$ then $\mathscr{L}$ is ample on $\pi^{-1}(x_1 + x_2)_{\text{red}}$. Before continuing we introduce the following notation. Let Z be a smooth curve. We let

(7.2.4) $$\mathrm{NS}(Z) \xrightarrow{\mu_Z} \mathrm{NS}(Z^{(2)})$$

be the map associating to (the class of) a line bundle L the class of a line bundle whose pullback to Z^2 via the quotient map is isomorphic to $L \boxtimes L$. Let us prove that $\mathscr{L}$ is ample on $C_x^{[2]}$. Suppose first that C_x is smooth. Then $C_x^{[2]} = C_x^{(2)}$. By hypothesis we have $2d_0 = 3r_1 + r_1 + b$, where $b \geq 0$. Let A be a divisor on C_x such that $\deg A = r_1 + b$, and let L be a line bundle on C_x of degree 3. In $\mathrm{NS}(C_x^{(2)})$, we have

(7.2.5) $$\mathrm{cl}(\mathscr{L}_{C_x^{(2)}}) \cong r_1(\mu_{C_x}(L) - \delta_x) + r_1\mu_{C_x}(A), \quad \text{where } \delta_x := \delta_{|C_x^{(2)}}.$$

We claim that $\mu_{C_x}(L)-\delta_x$ is ample. Granting this for the moment being, it follows that $\mathscr{L}$ is ample on $C_x^{(2)}$ because $\mu_{C_x}(A)$ is nef. To prove that $\mu_{C_x}(L)-\delta_x$ is ample consider the map $\varphi\colon C_x^{(2)}\to |L|$ which associates to $p+q$ the unique divisor $E\in|L|$ such that $E-p-q$ is effective. A straightforward computation gives that $c_1(\varphi^*\mathscr{O}_{|L|}(1))=\mu_{C_x}(L)-\delta_x$. We are done because φ is finite, and hence $\varphi^*\mathscr{O}_{|L|}(1)$ is ample on $C_x^{(2)}$.

Now we prove that $\mathscr{L}$ is ample on $C_x^{[2]}$ if C_x is singular. Let

$$\mathbb{P}^1\cong\widetilde{C}_x\xrightarrow{\alpha}C_x \tag{7.2.6}$$

be the normalization map. By hypothesis C_x is nodal, with exactly one node p. Let $p',p''\in\widetilde{C}_x$ be the two points mapped to p by the normalization map. Let

$$\mathbb{F}_1\cong\mathrm{Bl}_{p'+p''}(\widetilde{C}_x^{(2)})\xrightarrow{\beta}\widetilde{C}_x^{(2)}\cong\mathbb{P}^2 \tag{7.2.7}$$

be the blowup map. The normalization of $C_x^{[2]}$ is naturally isomorphic to $\mathrm{Bl}_{p'+p''}(\widetilde{C}_x^{(2)})$. Let $\nu\colon\mathbb{F}_1\to C_x^{[2]}$ be the normalization map. It suffices to prove that $\nu^*(\mathscr{L}_{|C_x^{[2]}})$ is ample. Let $\Sigma\subset\mathbb{F}_1$ be the negative section of the $\mathbb{P}^1$-fibration $\mathbb{F}_1\to\mathbb{P}^1$, and let $\Omega\subset\mathbb{F}_1$ be the inverse image via the blowup map $\mathbb{F}_1\to\mathrm{Bl}_{p'+p''}(\widetilde{C}_x^{(2)})$ of the conic parametrizing nonreduced divisors. Then, letting $\Delta\subset S^{[2]}$ be the divisor parametrizing nonreduced subschemes, we have

$$\nu^*(\Delta_{|C_x^{[2]}})=2\Sigma+\Omega. \tag{7.2.8}$$

Since $2\delta=\mathrm{cl}(\Delta)$, it follows that

$$\nu^*(\mathscr{L}_{|C_x^{[2]}})\cong\mathscr{O}_{\mathbb{F}_1}((2d_0-r_1)\beta^*H-r_1\Sigma), \tag{7.2.9}$$

where $H\subset\mathbb{P}^2$ is a line. It follows that $\nu^*(\mathscr{L}_{|C_x^{[2]}})$ is ample.

It remains to show that $\mathscr{L}$ is ample on W_x. Note that we have a $\mathbb{P}^1$-fibration $W_x\to C_x$, in fact $W_x\cong\mathbb{P}(\Theta_{S|C_x})$. One applies the Kleiman–Nakai–Moishezon criterion. First, one computes

$$c_1(\mathscr{L})^2\cdot W_x=8r_1d_0>0 \tag{7.2.10}$$

by noting that the cycle W_x represents the cohomology class $\mu(C)\cdot\delta$. Now suppose that C_x is smooth, and let $\Gamma\subset W_x$ be an integral curve. If Γ is a fiber of the $\mathbb{P}^1$-fibration $W_x\to C_x$, then $\mathscr{L}\cdot\Gamma=r_1>0$. If the restriction to Γ of the $\mathbb{P}^1$-fibration $W_x\to C_x$ is dominant then $\delta\cdot\Gamma\le 0$ because δ is the class of the tautological (sub)line bundle on $W_x\cong\mathbb{P}(\Theta_{S|C_x})$ and $\Theta_{S|C_x}$ is an extension of trivial line bundles on C_x. From this it follows that $\mathscr{L}\cdot\Gamma\ge 2d_0>0$. This shows that $\mathscr{L}$ is ample on W_x if C_x is smooth.

Lastly, suppose C_x is singular. Let α be the normalization map in (7.2.6), and let $\widetilde{W}_x:=\mathbb{P}(\alpha^*\Theta_{S|C_x})$. The natural map $\psi\colon\widetilde{W}_x\to W_x$ is the normalization. It suffices to prove that if $\Gamma\subset W_x$ is an integral curve then $\psi^*\mathscr{L}\cdot\Gamma>0$. Note that $\widetilde{W}_x\cong\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(2)\oplus\mathscr{O}_{\mathbb{P}^1}(-2))$. If Γ is a fiber of the $\mathbb{P}^1$-fibration $\widetilde{W}_x\to\widetilde{C}_x\cong\mathbb{P}^1$ then $\psi^*\mathscr{L}\cdot\Gamma=r_1>0$. Next suppose that the restriction to Γ of the $\mathbb{P}^1$-fibration $W_x\to C_x$ is dominant, and let deg be its degree. Since $\psi^*(\delta_{|W_x})$ is the class of the tautological (sub)line bundle on $\widetilde{W}_x\cong\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(2)\oplus\mathscr{O}_{\mathbb{P}^1}(-2))$ one gets that $\psi^*(\delta_{|W_x})\cdot\Gamma\le 2\deg$. On the other hand $\psi^*(2\mu(D)_{|W_x})\cdot\Gamma\ge 2d_0\deg$. Thus $\psi^*(\mathscr{L}_{|W_x})\cdot\Gamma\ge 2(d_0-r_1)\deg>0$. □

Let π be the Lagrangian fibration in (6.1.2). We let

$$f := \pi^* c_1(\mathscr{O}_{\mathbb{P}^2}(1)) = \mu(C), \tag{7.2.11}$$

where C is a fiber of the elliptic fibration $\epsilon\colon S \to \mathbb{P}^1$. Letting $i \in \{1,2\}$ be the integer defined by the condition $i \equiv r_1 \pmod 2$, we let

$$g := \frac{1}{i}\bigl(2\mu(D) - r_1\delta\bigr). \tag{7.2.12}$$

Claim 7.5 *Let $\Lambda \subset \mathrm{NS}(S^{[2]})$ be the lattice generated by f and g.*

1. *Λ is saturated of rank 2.*
2. *Λ contains $c_1(\mathscr{G})$ for any sheaf $\mathscr{G}$ such that $w(\mathscr{G}) = w_0$.*
3. *The restriction of the BBF quadratic form to Λ has discriminant equal to $2d_0/i$.*
4. *If $d_0 \geq 2r_1$ then Λ contains ample classes which are $\mathsf{a}(w_0)$-suitable with respect to the Lagrangian fibration π, where w_0 is as in (6.1.5).*

Proof Item (1) holds because $\mathrm{NS}(S)$ is freely generated by the classes $[C]$ and $[D]$. Item (2) holds by definition of w_0; see (6.1.5). A straightforward computation gives (3). We prove (4). By Proposition 7.4 the class g is π ample, hence $g + Nf$ is ample for $N \gg 0$. By Corollary 5.14 we get that for an $M \gg N$, the class $g + Mf$ is ample and $\mathsf{a}(w_0)$-suitable. □

From now on we assume that $d_0 \geq 2r_1$. Let $\mathscr{X}(\Lambda) \to T(\Lambda)$ be a representative of the deformation space $\mathrm{Def}(S^{[2]}, \Lambda)$ of deformations of $S^{[2]}$ which keep the classes in Λ of type $(1,1)$. For $t \in T(\Lambda)$ we let X_t be the corresponding HK variety. We let $X_0 = S^{[2]}$. We assume that monodromy acts trivially on classes in Λ. For $t \in T(\Lambda)$ we let $\Lambda_t \subset \mathrm{NS}(X_t)$ be the rank 2 lattice obtained from Λ by parallel transport. For $t \in T(\Lambda)$ we let w_t be the mock Mukai vector for X_t obtained from w_0 by parallel transport.

Choose an $\mathsf{a}(w_0)$-suitable polarization $p_0 \in \Lambda_0 = \Lambda$, and for $t \in T(\Lambda)$ let $p_t \in \Lambda_t$ be the corresponding $\mathsf{a}(w_0)$-suitable polarization. Similarly, for $t \in T(\Lambda)$ let $g_t \in \Lambda_t$ be the class obtained from g by parallel transport. By Maruyama there exists a scheme of finite type $M(\mathscr{X}(\Lambda)/T(\Lambda))$ and a regular map $M(\mathscr{X}(\Lambda)/T(\Lambda)) \to T(\Lambda)$ whose fiber over t is isomorphic to the moduli space $M_{w_t}(X_t, p_t)$. (Recall item (2) of Claim 7.5.)

Claim 7.6 *There exists an open dense subset $\mathscr{U}(\Lambda) \subset T(\Lambda)$ such that for $t \in \mathscr{U}(\Lambda)$ the moduli space $M_{w_t}(X_t, p_t)$ has an irreducible component $M_{w_t}(X_t, p_t)^{\bullet}$ of dimension $2a^2 + 2$.*

Proof Follows from Theorem 6.2 and Proposition 7.3. □

Claim 7.7 *Assume that $4m_0 - r_1^2 > 0$. Then there exists $b(m_0)$ such that the following holds. Suppose that $D \cdot D = 2m_0$, $D \cdot C = d_0 > b(m_0)$, and $t \in T(\Lambda)$ is such that $\mathrm{NS}(X_t) = \Lambda_t$. Then g_t is ample, w_t-generic, and the open w_t-chamber containing it contains also p_t.*

Proof The lattice Λ is generated by f and g, and $q_{S^{[2]}}(g) = (8m_0 - 2r_1^2)/i^2 > 0$. Let $\gamma \in \Lambda$ be such that $q(\gamma) < 0$. Then

$$q_{S^{[2]}}(\gamma) \leq -\frac{4d_0^2}{i^2 + (8m_0 - 2r_1^2)} \tag{7.2.13}$$

by Lemma 4.3 in [O'Grady 2022]. In particular if $d_0 \gg 0$ we have $q_{S^{[2]}}(\gamma) < -10$. Since $q_{S^{[2]}}(g_t) > 0$ it follows that one of g_t, $-g_t$ is ample, and then it has to be g_t (note: we are showing that if $d_0 \gg 0$ the ample cone coincides with the positive cone). Similarly, if $d_0 \gg 0$ we have $q_{S^{[2]}}(\gamma) < -10a^4 r_1 8$, and it follows that there is a single open w_t-chamber; see Example 2.14. □

Now assume that $4m_0 - r_1^2 > 0$, and that $d_0 > b(m_0)$ (of course $d_0 \geq 2r_1$). Let $t_* \in \mathscr{U}(\Lambda)$ be such that $\mathrm{NS}(X_{t_*}) = \Lambda_{t_*}$. By Claim 7.7 the class g_{t_*} is ample, w_{t_*}-generic, and the moduli space $M_{w_{t_*}}(X_{t_*}, g_{t_*})$ is isomorphic to $M_{w_{t_*}}(X_{t_*}, p_{t_*})$. By Claim 7.6 it follows that $M_{w_{t_*}}(X_{t_*}, g_{t_*})$ has an irreducible component $M_{w_{t_*}}(X_t, g_{t_*})^\bullet$ of dimension $2a^2 + 2$.

Let $\mathscr{Y}(g_{t_*}) \to T(g_{t_*})$ be a complete family of polarized HK varieties of type $K3^{[2]}$ with irreducible parameter space, containing a polarized pair isomorphic to (X_{t_*}, g_{t_*}). For $t \in T(g_{t_*})$ we let (Y_t, g_t) be the corresponding polarized HK of type $K3^{[2]}$, and we let w_t be the mock Mukai vector for Y_t obtained from w_0 by parallel transport. By Maruyama there exist a scheme of finite type $M(\mathscr{Y}(g_{t_*})/T(g_{t_*}))$ and a regular map $M(\mathscr{Y}(g_{t_*})/T(g_{t_*})) \to T(g_{t_*})$ whose fiber over t is isomorphic to the moduli space $M_{w_t}(Y_t, g_t)$. By Proposition 7.2 we get that for $t \in T(g_{t_*})$ general the moduli space $M_{w_t}(Y_t, g_t)$ contains an irreducible component of dimension $2a^2 + 2$.

Next note that

$$\mathrm{div}(g_t) = \begin{cases} 1 & \text{if } r_1 \equiv 0 \pmod 2\text{, i.e., } i = 2, \\ 2 & \text{if } r_1 \equiv 1 \pmod 2\text{, i.e., } i = 1, \end{cases} \tag{7.2.14}$$

$$q_{Y_t}(g_t) = \begin{cases} 2m_0 - \frac{1}{2}r_1^2 & \text{if } r_1 \equiv 0 \pmod 2\text{, i.e., } i = 2, \\ 8m_0 - 2r_1^2 & \text{if } r_1 \equiv 1 \pmod 2\text{, i.e., } i = 1. \end{cases} \tag{7.2.15}$$

Now recall that $r_1 \mid (m_0 + 1)$; see (6.1.3). It follows that (1.2.4) and (1.2.5) hold and that, conversely, if (1.2.4) and (1.2.5) hold then (X, h) is isomorphic to (Y_t, g_t) for a suitable $t \in T(g_{t_*})$. This finishes the proof of Theorem 1.2.

Appendix Sheaves on elliptic K3 surfaces

A.1 Outline of the section

Let S be a $K3$ surface with an elliptic fibration $S \to \mathbb{P}^1$. Let $\mathscr{F}$ be a torsion-free sheaf on S, and let h_S be an ample divisor on S which is $\mathsf{a}(\mathscr{F})$-suitable. Suppose that $\mathscr{F}$ is h_S slope semistable. Let C_x be the elliptic fiber over $x \in \mathbb{P}^1$. If x is general then $\mathscr{F}_x = \mathscr{F}_{|C_x}$ is slope semistable by Proposition 5.15. The graded vector bundle associated to the Jordan–Hölder filtration of $\mathscr{F}_x$ is isomorphic to a direct sum $E_1(x) \oplus \cdots \oplus E_a(x)$ of vector bundles with equal ranks and degrees. Taking the determinants of the direct factors, and letting x vary in $\mathbb{P}^1$, one gets a curve $\Lambda(\mathscr{F})$ in a suitable Jacobian fibration $J^d(S/\mathbb{P}^1)$.

By associating to $[\mathscr{F}] \in \mathscr{M}_v(S, H_S)$ (here v is a Mukai vector, and h_S is v-suitable) the curve $\Lambda(\mathscr{F})$, one gets a regular map from $\mathscr{M}_v(S, H_S)$ to a linear system on $J^d(S/\mathbb{P}^1)$. We show that, under certain hypotheses, this map is surjective.

A.2 The decomposition curve

Let S be an elliptic $K3$ surface with an elliptic fibration $\varepsilon\colon S \to \mathbb{P}^1$ as in Claim 3.7. We recall that this means that

$$\text{(A.2.1)} \qquad \mathrm{NS}(S) = \mathbb{Z}[D] \oplus \mathbb{Z}[C], \quad D \cdot D = 2m_0, \quad D \cdot C = d_0,$$

where m_0, d_0 are positive integers and C is an elliptic fiber. Let $u := (0, C, d_0)$, and let $J^{d_0}(S/\mathbb{P}^1) := \mathscr{M}_u(S, H_S)$ be the relative degree-d_0 Jacobian of $S \to \mathbb{P}^1$. By Mukai's well-known results, $J^{d_0}(S/\mathbb{P}^1)$ is a $K3$ surface. Moreover there is the regular map $J^{d_0}(S/\mathbb{P}^1) \to \mathbb{P}^1$ associating to $[\xi] \in J^{d_0}(S/\mathbb{P}^1)$ the point $x \in \mathbb{P}^1$ such that $C_x := \varepsilon^{-1}(x)$ is the support of the sheaf ξ. This is an elliptic fibration with the section which associates to $x \in \mathbb{P}^1$ the class of the restriction of $\mathscr{O}_S(D)$ to C_x. Hence $J^{d_0}(S/\mathbb{P}^1)$ is a $K3$ elliptic surface with a section. Let Γ be an elliptic fiber of $J^{d_0}(S/\mathbb{P}^1) \to \mathbb{P}^1$, and let Σ be the image of the section defined above. We have

$$\text{(A.2.2)} \qquad \Sigma \cdot \Sigma = -2, \quad \Sigma \cdot \Gamma = 1, \quad \Gamma \cdot \Gamma = 0.$$

Now let r_1 be a positive integer as in Assumption-Definition 6.1, and let v_1 be as in (6.1.4), i.e.,

$$\text{(A.2.3)} \qquad v_1 := \left(r_1, D, \frac{m_0 + 1}{r_1}\right).$$

Let $a, b \in \mathbb{N}$ with $a > 0$, and let

$$\text{(A.2.4)} \qquad v_2 := a v_1 - b(0, 0, 1).$$

Let $\mathscr{F}$ be a torsion-free sheaf on S with $v(\mathscr{F}) = v_2$. We define the associated decomposition curve in $J^{d_0}(S/\mathbb{P}^1)$ as follows. Let $\mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1) \to \mathbb{P}^1$ be the relative (Simpson) moduli space parametrizing semistable sheaves on fibers C_x of rank r_1 and degree d_0 (they are all stable because $\gcd\{r_1, d_0\} = 1$). We have an isomorphism

$$\text{(A.2.5)} \qquad \mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1) \xrightarrow{\sim} J^{d_0}(S/\mathbb{P}^1), \quad [\mathscr{G}] \mapsto [\det \mathscr{G}].$$

Remark A.1 Let h_S be an ample divisor on S which is v_2-suitable (with respect to the elliptic fibration $\epsilon\colon S \to \mathbb{P}^1$). By Proposition 6.3 there exists a unique h_S-stable sheaf $\mathscr{E}_1$ on S such that $v(\mathscr{E}_1) = v_1$, and the restriction of $\mathscr{E}_1$ to every elliptic fiber is stable. Thus $\mathscr{E}_1$ determines a section $\sigma\colon \mathbb{P}^1 \to \mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1)$. The image of σ is equal to Σ because $\det(\mathscr{E}_1) \cong \mathscr{O}_S(D)$.

Let $U \subset \mathbb{P}^1$ be a sufficiently small open subset in the classical topology, and let $S_U := f^{-1}(U)$. Then there exists a tautological sheaf G_U on $S_U \times_{\mathbb{P}^1} \mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1)$. For $x \in \mathbb{P}^1$ and $[\mathscr{G}] \in \mathscr{M}_{r_1,d_0}(C_x)$ we have $\chi_{\mathscr{O}_{C_x}}(\mathscr{G},\mathscr{F}_{|C_x})=0$. It follows that there exists a determinant line bundle $\mathscr{L}_U$ on $\mathscr{M}_{r_1,d_0}(S_U)$ and a section $s_U\in\Gamma(\mathscr{M}_{r_1,d_0}(S_U),\mathscr{L}_U)$ whose zero-scheme $Z(s_U)$ is supported on the set

$$\{[\mathscr{G}]\in\mathscr{M}_{r_1,d_0}(S_U)\mid \mathscr{G}\text{ supported on } C_x \text{ and } \mathrm{Hom}(\mathscr{G}_x,\mathscr{F}_{|C_x})\neq 0\}.$$

The line bundles $\mathscr{L}_U$ and sections s_U for varying U glue to give a line bundle $\mathscr{L}(\mathscr{F})$ and section $s(\mathscr{F})$ on $\mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1)$. Let $\Lambda(\mathscr{F})\subset\mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1)$ be the zero-scheme of $s(\mathscr{F})$. Via the identification in (A.2.5) we view $\Lambda(\mathscr{F})$ as a subscheme of $J^{d_0}(S/\mathbb{P}^1)$. Restricting the elliptic fibration $J^{d_0}(S/\mathbb{P}^1)\to\mathbb{P}^1$ to $\Lambda(\mathscr{F})$ we get a regular map $\Lambda(\mathscr{F})\to\mathbb{P}^1$; we let $\Lambda(\mathscr{F})_x$ be the fiber of this map over $x\in\mathbb{P}^1$.

Remark A.2 Let T be irreducible and let $\mathscr{F}\to S\times T$ be a T-flat family of sheaves as above, i.e., for all $t\in T$ we have $v(\mathscr{F}_{|S\times\{t\}})=v_2$. Then the isomorphism class of the line bundle $\mathscr{L}(\mathscr{F}_{|S\times\{t\}})$ is independent of $t\in T$. Since the moduli space $\mathscr{M}_{v_2}(S,h_S)$ is irreducible it follows that the isomorphism class of $\mathscr{L}(\mathscr{F})$ is independent of the point $[\mathscr{F}]\in\mathscr{M}_{v_2}(S,h_S)$. We let $\mathscr{L}(v_2):=\mathscr{L}(\mathscr{F})$ for any $[\mathscr{F}]\in\mathscr{M}_{v_2}(S,h_S)$.

Remark A.3 Suppose that the restriction of $\mathscr{F}$ to a general elliptic fiber is semistable. Then $\Lambda(\mathscr{F})$ is a curve. Let $x\in\mathbb{P}^1$ be such that $\mathscr{F}_{|C_x}$ is semistable. Since r_1 and d_0 are coprime, the associated graded vector bundle of $\mathscr{F}_{|C_x}$ is isomorphic to $V_1(x)\oplus\cdots\oplus V_a(x)$, where $r(V_i(x))=r_1$ and $\deg(V_i(x))=d_0$ for all $i\in\{1,\dots,a\}$. Then, identifying codimension 1 subschemes of C_x with effective divisors, we have

$$\Lambda(\mathscr{F})_x=[\det V_1(x)]+\cdots+[\det V_a(x)]. \tag{A.2.6}$$

If $\mathscr{F}_{|C_x}$ is not semistable, then C_x appears in $\Lambda(\mathscr{F})$ (we identify $\Lambda(\mathscr{F})$ with an effective divisor on $J^{d_0}(S/\mathbb{P}^1)$) with positive multiplicity.

A.3 Decomposition curves and Lagrangian fibrations

Below is the main result of the present section.

Proposition A.4 *Keep notation and hypotheses as above; in particular, v_2 is given by* (A.2.4). *Then the following hold*:

(1) *We have an isomorphism*

$$\mathscr{L}(v_2)\cong\mathscr{O}_{J^{d_0}(S/\mathbb{P}^1)}(a\Sigma+br_1\Gamma). \tag{A.3.1}$$

(2) *Suppose that $br_1\geq 2a$, and that h_S is an ample divisor on S which is v_2-suitable. Then the map* (*see item* (1) *and Remark A.3*)

$$\mathscr{M}_{v_2}(S,H_S)\to|\mathscr{O}_{J^{d_0}(S/\mathbb{P}^1)}(a\Sigma+br_1\Gamma)|,\quad [\mathscr{F}]\mapsto\Lambda(\mathscr{F}), \tag{A.3.2}$$

is a Lagrangian fibration.

Before proving Proposition A.4 we go through a preliminary result. Let X be a smooth (irreducible) surface, and let $\pi\colon X\to T$ a projective map to a smooth curve. Let $\mathscr{E}$ be a torsion-free sheaf on X with the property that for all $t\in T$ we have $\chi(X_t,E_t)=0$, where $X_t:=\pi^{-1}(t)$, and $E_t:=\mathscr{E}_{|X_t}$. Then there exists a determinant line bundle $\mathscr{L}(\mathscr{E})$ on T and a section $s(\mathscr{E})$ of $\mathscr{L}(\mathscr{E})$ whose zero-scheme $Z(s(\mathscr{E}))$ is supported on the set of t such that $h^0(X_t, E_t) = h^1(X_t, E_t) > 0$. Note that the double dual $\mathscr{E}^{\vee\vee}$ satisfies the same hypotheses as $\mathscr{E}$, hence we have a determinant line bundle $\mathscr{L}(\mathscr{E}^{\vee\vee})$ on T and a section $s(\mathscr{E}^{\vee\vee})$ of $\mathscr{L}(\mathscr{E}^{\vee\vee})$. We let $Q(\mathscr{E}) := \mathscr{E}^{\vee\vee}/\mathscr{E}$, where $\mathscr{E} \hookrightarrow \mathscr{E}^{\vee\vee}$ by the canonical (injective) map.

Lemma A.5 *Keep hypotheses as above, and assume in addition that* $h^0(X_t, \mathscr{E}^{\vee\vee}_{|X_t}) = h^1(X_t, \mathscr{E}^{\vee\vee}_{|X_t}) = 0$ *for all* $t \in T$, *and hence* $Z(s(\mathscr{E}^{\vee\vee})) = 0$ (*we identify codimension* 1 *subschemes of* T *with effective divisors*). *Then*

$$Z(s(\mathscr{E})) = \sum_{p \in \operatorname{supp} Q(\mathscr{E})} \ell(\mathscr{O}_{Q(\mathscr{E}),p})\pi(p). \tag{A.3.3}$$

Proof Let A be a relative effective divisor on X such that $h^1(X_t, E_t \otimes (A_t)) = 0$ for all $t \in T$, where $A_t := A_{|X_t}$. Assume also that the supports of A and $Q(\mathscr{E})$ are disjoint. Consider the commutative diagram of sheaves on T:

$$\begin{array}{ccc} \pi_*(\mathscr{E} \otimes \mathscr{O}_X(A)) & \xrightarrow{\quad\gamma\quad} & \pi_*(\mathscr{E} \otimes \mathscr{O}_X(A)_{|A}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \pi_*(\mathscr{E}^{\vee\vee} \otimes \mathscr{O}_X(A)) & \xrightarrow{\quad\delta\quad} & \pi_*(\mathscr{E}^{\vee\vee} \otimes \mathscr{O}_X(A)_{|A}) \end{array} \tag{A.3.4}$$

All sheaves in the above diagram are locally free of the same rank, and $Z(s(\mathscr{E}))$ is the zero-scheme of the determinant of γ. The map β is an isomorphism (the supports of A and $Q(\mathscr{E})$ are disjoint), and δ is an isomorphism because $h^0(X_t, \mathscr{E}^{\vee\vee}_{|X_t}) = h^1(X_t, \mathscr{E}^{\vee\vee}_{|X_t}) = 0$ for all $t \in T$ by hypothesis. It follows that $Z(s(\mathscr{E}))$ equals the zero-scheme of $\det\alpha$. The lemma follows because we have an exact sequence

$$0 \to \pi_*(\mathscr{E} \otimes \mathscr{O}_X(A)) \xrightarrow{\alpha} \pi_*(\mathscr{E}^{\vee\vee} \otimes \mathscr{O}_X(A)) \to \pi_*(Q(\mathscr{E})) \to 0. \tag{A.3.5}$$ □

Proof of Proposition A.4 (1) Since $v_1^2 = -2$, the moduli space $\mathscr{M}_{v_1}(S, H_S)$ is a singleton parametrizing a vector bundle $\mathscr{E}$ on S whose restriction to every elliptic fiber C_x is the unique stable vector bundle of rank r_1 and determinant isomorphic to $\mathscr{O}_{C_x}(D)$. It follows that

$$\mathscr{M}_{(ar_1, aD, as_1)}(S, h_S) = \{[\mathscr{E}^{\oplus a}]\}. \tag{A.3.6}$$

Clearly $\Lambda(\mathscr{E}^{\oplus a}) = a\Sigma$, and this proves the validity of (A.3.1) if $b = 0$. Now assume that $b > 0$. Let $\mathscr{F}$ be a sheaf on S fitting into the exact sequence

$$0 \to \mathscr{F} \to \mathscr{E}^{\oplus a} \xrightarrow{\phi} \mathbb{C}_{y_1} \oplus \cdots \oplus \mathbb{C}_{y_b} \to 0, \tag{A.3.7}$$

where $y_1, \ldots, y_b \in S$ are general points, and ϕ is a general morphism. Let $x_i = f(y_i)$. Then $\mathscr{F}$ is an h_S slope stable torsion-free sheaf, and $v(\mathscr{F}) = v_2$. It is clear that there exists $m_1, \ldots, m_b \in \mathbb{N}$ such that

$$\Lambda(\mathscr{F}) = a\Sigma + m_1\Gamma_{x_1} + \cdots + m_b\Gamma_{x_b}. \tag{A.3.8}$$

Let $i \in \{1, \ldots, b\}$. Since $\mathscr{F}_{|C_{y_i}} \cong \overline{\mathscr{F}}_i \oplus \mathbb{C}_{y_i}$, where $\overline{\mathscr{F}}_i$ is a subsheaf of $(\mathscr{E}_{|C_{y_i}})^{\oplus a}$, it follows that $\dim \operatorname{Hom}(\mathscr{G}, \mathscr{F}_{|C_{y_i}}) = r_1$ for $[\mathscr{G}] \in (\mathscr{M}_{r_1,d_0}(C_{y_i}) \setminus \{[\mathscr{E}_{|C_{y_i}}]\})$. Thus $m_i \geq r_1$. One proves that equality holds by applying Lemma A.5. In fact, let $T \subset \mathscr{M}_{r_1,d_0}(S/\mathbb{P}^1)$ be a (nonprojective) smooth irreducible curve with the following properties: T meets $\mathscr{M}_{r_1,d_0}(C_{y_i})$ at a single point $[\mathscr{G}] \neq [\mathscr{E}_{|C_{y_i}}]$ and the intersection is transverse, all sheaves parametrized by T are pushforwards of *locally free* sheaves on curves of the elliptic fibration $f: S \to \mathbb{P}^1$ (i.e., T does not meet the critical set of f), and the surface $X := S \times_{\mathbb{P}^1} T$ is smooth. Let $\rho: X \to S$ and $\pi: X \to T$ be the projections. On X we have the sheaf $\rho^*\mathscr{F}$, and a tautological locally free sheaf $\mathscr{A}$ with the property that $\mathscr{A}|p_T^{-1}(t)$ is isomorphic to the vector bundle on $X_t \cong C_t$ corresponding to t (there exists such a tautological sheaf because $H^2(T, \mathscr{O}_T^*) = 0$). The pullback of the determinant line bundle $\mathscr{L}(F)$ to T is isomorphic to the determinant line bundle $\mathscr{L}(\mathscr{A}^\vee \otimes \rho^*\mathscr{F})$. The hypotheses of Lemma A.5 are satisfied by the sheaf $\mathscr{E} := \mathscr{A}^\vee \otimes \rho^*\mathscr{F}$ on the smooth surface X. By that lemma we get that the canonical section of $\mathscr{L}(\mathscr{A}^\vee \otimes \rho^*\mathscr{F})$ vanishes at y_i with multiplicity r_1, and hence $m_i = r_1$. This proves that $\mathscr{L}(\mathscr{F}) \cong \mathscr{O}_{J^{d_0}(S/\mathbb{P}^1)}(a\Sigma + br_1\Gamma)$.

(2) Let $J^{d_0} := J^{d_0}(S/\mathbb{P}^1)$. Then $H^p(J^{d_0}, \mathscr{O}_{J^{d_0}}(a\Sigma + br_1\Gamma) = 0$ for $p > 0$ because of the hypothesis $br_1 \geq 2a$ (for example because $a\Sigma + br_1\Gamma$ is big and nef). By Hirzebruch–Riemann–Roch it follows that

$$\dim |\mathscr{L}(v_2)| = \dim |a\Sigma + br_1\Gamma| = 1 + abr_1 - a^2.$$

The map in (A.3.2) is not constant because all the curves appearing in (A.3.8) belong to the image. By Matsushita's theorem the image of the map has dimension equal to

$$\tfrac{1}{2}\dim \mathscr{M}_{v_2}(S, h_S) = \tfrac{1}{2}(2 + v_2^2) = 1 + abr_1 - a^2. \tag{A.3.9}$$

This finishes the proof of (2). □

Proposition A.6 *Let $n \geq 2m \geq 2$ and let $A \in |m\Sigma + n\Gamma|$ be a general divisor. Let $A = A_{\mathrm{hor}} + A_{\mathrm{vert}}$ be the unique decomposition into effective divisors such that A_{vert} is a sum of elliptic fibers and the support of A_{hor} contains no elliptic fiber. Then A_{hor} is an integral divisor. If $m \geq 2$, then A itself is an integral smooth divisor.*

Proof We proceed by induction on m. If $m = 1$ the statement is trivially true because $A_{\mathrm{hor}} = \Sigma$ for any $A \in |\Sigma + n\Gamma|$. Now assume that $n \geq 2m \geq 4$. We claim that

$$H^1(J^{d_0}, \mathscr{O}_{J^{d_0}}((m-1)\Sigma + n\Gamma)) = 0. \tag{A.3.10}$$

In fact, let $B \in |(m-1)\Sigma + n\Gamma|$ be general. Since B_{hor} is an integral divisor, the divisor B is connected, i.e., $h^0(B, \mathscr{O}_B) = 1$. It follows that $H^1(J^{d_0}, \mathscr{O}_{J^{d_0}}(-B)) = 0$, and by Serre duality we get the vanishing in (A.3.10). The restriction of $\mathscr{O}_S(m\Sigma + n\Gamma)$ to Σ has nonnegative degree because $n \geq 2m \geq 4$, and hence (the restriction) has nonzero sections because Σ is rational. By the vanishing in (A.3.10) it follows that $|m\Sigma + n\Gamma|$ is globally generated at every point of Σ. Since $|m\Sigma + n\Gamma|$ is clearly globally generated away from Σ, it follows that it is globally generated. Let $A \in |m\Sigma + n\Gamma|$ be general. Then A is smooth because $|m\Sigma + n\Gamma|$ is globally generated. We claim that A is irreducible (and hence integral). Suppose the contrary. Then $A = A_1 + A_2$ where $A_i \in |m_i\Sigma + n_i\Gamma|$ are (nonzero) smooth divisors. Since A is smooth it follows that $A_1 \cdot A_2 = 0$. This leads to a contradiction. In fact, it implies right away that $m_1 > 0$ and $m_2 > 0$, and since $n_i \geq 2m_i > 0$ we get that $A_1 \cdot A_2 \geq 2m_1 m_2 > 0$. □

The above proposition gives the following result.

Corollary A.7 *Let hypotheses be as in item* (2) *of Proposition A.4*; *in particular*, v_2 *is given by* (A.2.4) *and* $br_1 \geq 2a$. *If* $a \geq 2$ *and* $[\mathscr{F}] \in M_{v_2}(S, h_S)$ *is a general point*, *then* $\Lambda(\mathscr{F})$ *is an integral and smooth divisor.*

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