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Degenerating conic Kähler–Einstein metrics to the normal cone

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Let X be a Fano manifold of dimension at least 2 and D be a smooth divisor in a multiple of the anticanonical class, $\frac{1}{\alpha}(-K_X)$ with $\alpha > 1$. It is well known that Kähler–Einstein metrics on X with conic singularities along D may exist only if the angle $2\pi\beta$ is bigger than some positive limit value $2\pi\beta_*$. Under the hypothesis that the automorphisms of D are induced by the automorphisms of the pair (X, D) , we prove that for $\beta > \beta_*$ close enough to β_* , such Kähler–Einstein metrics do exist. We identify the limits at various scales when $\beta \rightarrow \beta_*$ and, in particular, we exhibit the appearance of the Tian–Yau metric of $X \setminus D$.

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1 Introduction

1.1 Kähler–Einstein metrics with cone singularities along a divisor

Let X be a compact Kähler manifold of dimension n and let D be a smooth divisor on X . Given $\beta \in (0, 1)$, we say that a Kähler metric ω on $X \setminus D$ is a Kähler–Einstein metric with cone angle $2\pi\beta$ along D (KE_β for short) if

- $\text{Ric } \omega = \mu \omega$ on $X \setminus D$ for some $\mu \in \mathbb{R}$,
- ω has cone singularities with cone angle $2\pi\beta$ along D .

In this paper, we will only focus on the case $\mu > 0$, so that a KE_β metric will be assumed to have positive scalar curvature unless stated otherwise.

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The second item in the definition above means that if (z_1, \dots, z_n) is any holomorphic system of coordinates on a neighborhood U of a point in D such that $D = (z_1 = 0)$, then $\omega|_{U \setminus D}$ is quasi-isometric to the model flat cone metric

$$\frac{i dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{j \geq 2} i dz_j \wedge d\bar{z}_j.$$

In particular, the Kähler metric ω extends canonically to a Kähler current on X in the cohomology class $-\mu^{-1}c_1(K_X + (1 - \beta)D)$ which satisfies

$$\text{Ric } \omega = \omega + (1 - \beta)[D].$$

A natural and important question is, given a pair (X, D) , to determine the set

$$I := \{\beta \in (0, 1) : \exists \text{ KE}_\beta \text{ metric } \omega_\beta\}$$

and understand what happens to ω_β when β approaches a critical value $\beta_* \in \partial I$.

A necessary condition for the existence of a KE_β metric is that $-(K_X + (1 - \beta)D)$ be ample, but just as when $D = 0$, this condition is not sufficient in general. We refer to [2; 3; 14; 16; 29] for an account of the problem.

Determining the admissible cone angles and understanding degeneration of conic Kähler–Einstein metrics has drawn a lot of attention in the last decade; see, e.g., [8; 18; 21; 30; 35; 38].

1.2 Statement of the main theorem

Let us start by introducing the objects involved in the main theorem below.

1.2.1 Varieties Let X be a Fano manifold of dimension $n \geq 2$ such that $-K_X \sim_{\mathbb{Q}} \alpha D$ for some positive rational number $\alpha \in \mathbb{Q}$ satisfying $\alpha > 1$. The case $\alpha = 1$ has already been fully treated in [8]. The case $\alpha \in (0, 1)$ is quite different and involves the incomplete Ricci flat cone metrics constructed in [12; 22; 25], but we will not discuss it further here.

From the assumptions above, D is a connected Fano manifold and we will further assume that it admits a Kähler–Einstein metric ω_D , i.e., $\text{Ric } \omega_D = \omega_D$. Let $L := N_D \simeq \mathcal{O}_X(D)|_D$ be the normal bundle of D and set v to be the fiber coordinate so that $L = \{(x, v) : x \in D, v \in L_x\}$. We consider the hermitian metric h on L (unique up to a multiplicative constant) such that $i\Theta(L, h) = \frac{1}{\alpha-1}\omega_D$.

Next let $\bar{L} = L \sqcup \{x_\infty\}$ be the one-point compactification of L ; it can be endowed with the structure of a normal projective variety. Moreover, $\bar{L} \setminus D$ is an affine cone over D with apex x_∞ ; see Section 2 below.

Small neighborhoods $\mathcal{U}_L \subset X$ of D (resp. $\Delta_L \subset L$ of the zero section $\mathcal{O}_L \simeq D \subset L$), which are in general nonisomorphic, are pseudoconcave and carry a multivalued meromorphic n -form Ω (resp. Ω_L) with a pole of order α along D ; it is unique up to constant scaling by pseudoconcavity. One can define

$$(1-1) \quad j_0 \in \mathbb{N}_{>0} \cup \{+\infty\}$$

to be the largest integer such that there exists a diffeomorphism $\Upsilon : \Delta_L \rightarrow \mathcal{U}_L$ such that $\Upsilon^*\Omega$ and Ω_L coincide up to order $j_0 - 1$; see Proposition 5.3. Observe that since the n -form determines the complex

structure, j_0 coincides with the largest integer such that the formal neighborhoods of $D \subset X$ (resp. $D \subset L$) of order $j_0 - 1$ are isomorphic. By definition, this means that j_0 is the largest integer such that the ringed spaces $(D, \mathcal{O}_X/\mathcal{I}_D^{j_0})$ and $(D, \mathcal{O}_L/\mathcal{I}_D^{j_0})$ are isomorphic.

1.2.2 Metrics An important numerical factor in the following is the positive number

$$(1-2) \quad \beta_* := \frac{\alpha - 1}{n}.$$

Let us recall that one can always assume that $\alpha < n + 1$ (see Section 2). That is, we have $\beta_* \in (0, 1)$.

The Calabi ansatz enables one to construct on $\bar{L} \setminus D$ two canonical Kähler–Einstein metrics (of positive and zero curvature, respectively) having a conical singularity at the point $x_\infty = \bar{L} \setminus L$:

- the Ricci flat Tian–Yau cone metric $\omega_{\text{TY},L} := dd^c |v|_h^{-2\beta_*}$ which is complete near $D \subset L$;
- the KE_{β_*} metric $\omega_{\beta_*,L} := dd^c \log(1 + |v|_h^{-2\beta_*})$. It satisfies $\text{Ric } \omega_{\beta_*,L} = \omega_{\beta_*,L}$ on $L \setminus D$, has cone singularities of angle $2\pi\beta_*$ along $D \subset L$ and it is asymptotic to $\omega_{\text{TY},L}$ near the conical point.

We will explain in Section 2.3 that the Calabi ansatz metric $\omega_{\beta_*,L}$ is rigid in the sense that it is the only Kähler–Einstein metric on \bar{L} with cone singularities along D .

Finally, Tian and Yau [40] proved that under our assumptions, there exists

- a complete Ricci flat Kähler metric ω_{TY} on $X \setminus D$ which is asymptotic to $\omega_{\text{TY},L}$ at infinity,

and we refer to Section 6 for more details.

1.2.3 Results For $\beta \in (0, 1)$, we consider the Kähler–Einstein equation

$$(\text{KE}_\beta) \quad \text{Ric } \omega_\beta = \mu \omega_\beta + (1 - \beta)[D],$$

where $\mu = \mu(\beta) > 0$ is given by $\mu\alpha = \alpha + \beta - 1$, so that $\omega_\beta \in c_1(X) = \alpha c_1(D)$ is fixed, independent of β . Since $\mu(\beta_*) > 0$ the difference between normalizing the Einstein constant to be 1 or μ_β is geometrically irrelevant. Consider the interval

$$I := \{\beta \in (0, 1] : \exists \omega_\beta \text{ solution of } (\text{KE}_\beta)\}.$$

It is known that I is either empty or of the form (β_*, β^*) (or $(\beta_*, \beta^*]$) where β_* is defined in (1-2); see Proposition 2.1. The identification of β^* is a complicated problem in general and depends very much on the geometry of the pair (X, D) ; see [38]. However, I is never empty (i.e., $\beta^* > \beta_*$) as a byproduct of our main result below.

Main Theorem *Let X be a Fano manifold of dimension $n \geq 2$ and let D be a smooth divisor such that $-K_X \sim_{\mathbb{Q}} \alpha D$ for some $\alpha \in \mathbb{Q}_{>1}$. Assume that D is Kähler–Einstein and that the restriction map $\text{Aut}^\circ(X, D) \rightarrow \text{Aut}^\circ(D)$ is onto. Then there exists $\delta > 0$ such that:*

- (1) For any $\beta \in (\beta_*, \beta_* + \delta)$, there exists a KE_β metric ω_β .
- (2) There is convergence

$$(X, \omega_\beta) \xrightarrow{\beta \rightarrow \beta_*} (\bar{L}, \omega_{\beta_*,L})$$

in the Gromov–Hausdorff topology.

(3) We have

$$\varepsilon_\beta^{-1} \omega_\beta \xrightarrow{\beta \rightarrow \beta_*} \omega_{\text{TY}} \quad \text{in } C_{\text{loc}}^\infty(X \setminus D),$$

where ε_β is defined by

$$\varepsilon_\beta = \begin{cases} (\beta - \beta_*)^{\frac{\alpha-1}{nj_0}} & \text{if } j_0 < \alpha - 1, \\ \left(\frac{\beta - \beta_*}{-\log(\beta - \beta_*)} \right)^{\frac{1}{n}} & \text{if } j_0 = \alpha - 1, \\ (\beta - \beta_*)^{\frac{1}{n}} & \text{if } j_0 > \alpha - 1, \end{cases}$$

where j_0 is defined in (1-1).

Remark 1.1 (1) As a corollary and under the assumption that $\text{Aut}^\circ(X, D) \rightarrow \text{Aut}^\circ(D)$ is onto, we get that the Tian–Yau metric ω_{TY} is *canonically* attached to the pair (X, D) , modulo the obvious scaling and the action of $\text{Aut}^\circ(X, D)$.

(2) The condition that $\text{Aut}^\circ(X, D) \rightarrow \text{Aut}^\circ(D)$ is onto is satisfied automatically if X is toric; see Lemma 4.2. We are not aware of pairs (X, D) as in our setup for which there are holomorphic vector fields on D which do not lift to X . If X is allowed to be singular, such examples can be constructed; see Example 4.3.

(3) In Section 10 we provide examples where the three possible values for ε_β are achieved.

(4) Thanks to [4], any two KE_β metrics are related by an element in $\text{Aut}^\circ(X, D)$. In particular the isometry class of the metric space (X, ω_β) does not depend on the particular KE_β metric.

In the case of (\mathbb{P}^2, Q) where Q is the smooth quadric, the result above was conjectured by Li and Sun [30] and shortly after numerical evidence to that conjecture was provided by Li [27]. Recently, Delcroix [18] settled the conjecture in full — he actually considers the more general situation of a rank-one horosymmetric Fano manifold with the divisor being a codimension-one orbit.

1.3 Strategy of proof

The proof involves several steps.

1.3.1 Construction of the model metric near D We work on the normal bundle L and solve the conic Calabi ansatz to construct a semiexplicit Kähler–Einstein metric $\omega_{\beta,L} = dd^c \phi_{\beta,L}$ on L with cone singularities of angle $2\pi\beta$ along the zero section $D \subset L$; we refer to Section 3 for details relative to the conic Calabi ansatz in this setting. The ansatz can be used for every angle $\beta \in (0, 1)$ but the angle β_* is critical in the following sense.

For $\beta < \beta_*$, $\omega_{\beta,L}$ acquires a cone singularity along the divisor at infinity $D_\infty = \mathbb{P}(0 \oplus \mathcal{O}_D)$ in the smooth compactification $\mathbb{P}(L \oplus \mathcal{O}_D)$ of L .

At $\beta = \beta_*$, D_∞ is contracted, and $\omega_{\beta_*,L}$ has a conical singularity at the singular point at infinity of the compactification \bar{L} of L .

Finally, for $\beta > \beta_*$, $\omega_{\beta,L}$ is only defined away from some euclidean neighborhood D_∞ (or, equivalently, x_∞). More precisely, the domain of definition of $\omega_{\beta,L}$ is larger and larger as $\beta \searrow \beta_*$ and one can identify a zone (escaping to infinity in L) where the potential has the asymptotics

$$\phi_{\beta,L} = \phi_{\beta_*,L} + \frac{\beta - \beta_*}{r^{2n-2}} + \text{l.o.t.},$$

where r is the radius for the Ricci flat cone metric g_L on L (to which $\omega_{\beta_*,L}$ is asymptotic near the conical point).

1.3.2 Gluing with the Tian–Yau metric Fix two parameters $\varepsilon > 0$ and $\beta > \beta_*$. It can be useful to think of ε as the (modulus of the) base parameter in the deformation $\pi : \mathcal{X} \rightarrow \mathbb{C}$ of X to the normal cone of D . The ultimate goal is to produce for each $\varepsilon > 0$ small enough an angle $\beta = \beta(\varepsilon)$ and a KE_β metric $\omega_{\text{KE},\beta(\varepsilon)}$ on X (thought of as $X_\varepsilon = \pi^{-1}(\varepsilon)$). Moreover one wants $\beta(\varepsilon) \rightarrow \beta_*$ as $\varepsilon \rightarrow 0$.

Using a diffeomorphism identifying a neighborhood of $D \subset X$ with a neighborhood of $D \subset L$, one can cook up a Kähler metric $\omega_{\beta,\varepsilon}$ on X obtained by gluing $\omega_{\beta,L}$ and $\varepsilon\omega_{\text{TY}}$ closer and closer to D as $\varepsilon \rightarrow 0$. Constructing a KE_β metric $\omega_{\beta,\varepsilon} + dd^c f$ amounts to solving a Monge–Ampère equation

$$P_{\beta,\varepsilon}(f) = 0.$$

The relevant properties of $P_{\beta,\varepsilon}$ are as follows.

- $P_{\beta,\varepsilon}(0)$ measures the “Kähler–Einstein” defect of $\omega_{\beta,\varepsilon}$. It arises from three factors: the difference between the complex structures of X and L , the cut-offs performed to glue the two model metrics, the fact that ω_{TY} is Ricci-flat rather than having positive Einstein constant.
- $dP_{\beta,\varepsilon}(0) = \Delta_{\omega_{\beta,\varepsilon}} + \mu_\beta =: L_{\beta,\varepsilon}$ where μ_β is our choice of Einstein constant (it depends linearly on β with $\mu_{\beta_*} > 0$, so one can think of it as being constant). Small eigenvalues of $L_{\beta,\varepsilon}$ represent the obstructions to deform $\omega_{\beta,\varepsilon}$ to a KE_β metric.

1.3.3 Resolution modulo the obstruction The obstructions arise at least conceptually from three sources: harmonic functions on $(X \setminus D, g_{\text{TY}})$, harmonic functions on the cone (L, g_L) and functions on L in the kernel of $\Delta_{\omega_{\beta_*,L}} + \mu_{\beta_*}$. One can relatively easily kill the first two using suitable weighted Hölder spaces, and hence we are left to dealing with the third kind of obstruction.

Since $\omega_{\beta_*,L}$ is Kähler–Einstein, there is a one-to-one correspondence between $\ker(\Delta_{\omega_{\beta_*,L}} + \mu_{\beta_*})$ and $H^0(L, T_L) \simeq H^0(D, T_D) \oplus \mathbb{C}\xi$ where ξ is the radial vector field generating the \mathbb{C}^* action. The assumption that vector fields on D lift to X combined with the fact that $\text{Aut}^\circ(D) = G^\mathbb{C}$ is reductive (since D is KE) allows us to work equivariantly with respect to a compact group $G \subset \text{Aut}^\circ(X, D)$ and kill the obstruction induced by $H^0(D, T_D)$.

One can find an explicit function τ_β solving $i_\xi \omega_{\beta,L} = \bar{\partial} \tau_\beta$ which we then transplant to a function $\tau_{\beta,\varepsilon}$ on the whole X satisfying $|L_{\beta,\varepsilon} \tau_{\beta,\varepsilon}| \ll 1$. Therefore $\tau_{\beta,\varepsilon}$ is asymptotically in the cokernel of $L_{\beta,\varepsilon}$ and represents the one obstruction remaining in order to solve $P_{\beta,\varepsilon}(f) = 0$. In particular, one can solve modulo obstruction, i.e., one can find a (unique) function $f = f_{\beta,\varepsilon}$ such that

$$P_{\beta,\varepsilon}(f_{\beta,\varepsilon})^\perp = 0,$$

that is, $P_{\beta,\varepsilon}(f_{\beta,\varepsilon}) = a(f_{\beta,\varepsilon}) \cdot \tau_{\beta,\varepsilon}$ for a constant $a(f_{\beta,\varepsilon}) \in \mathbb{R}$. Actually, even that equation cannot be solved readily because $P_{\beta,\varepsilon}(0)$ is not small enough to apply the implicit function theorem; see Remark 9.2. This has to do with the fact that the Tian–Yau metric does not provide a good enough approximation of the solution (e.g., since it is Ricci flat). So it is necessary to construct a formal perturbation of ω_{TY} at a large order and use that metric instead of ω_{TY} in the gluing.

1.3.4 Deforming the cone angle to kill the obstruction We have two parameters ε , β and one obstruction $a(\beta, \varepsilon) := a(f_{\beta,\varepsilon})$ to kill. A signification fraction of the present paper is devoted to showing that the obstruction has an expansion which looks like

$$a(\beta, \varepsilon) = F(\varepsilon) - (\beta - \beta_*) + \text{l.o.t.},$$

where $F(\varepsilon)$ is a *positive* quantity which has either the form

$$F(\varepsilon) = \varepsilon^\nu \quad (\text{for some } 1 \leq \nu \leq n) \quad \text{or} \quad F(\varepsilon) = \varepsilon^n \log \frac{1}{\varepsilon}$$

depending on the convergence rate of J_X to J_L near D . The dominant positive contribution to the obstruction, that is, $F(\varepsilon)$, comes from either the complex structure change (between X and L), or cutting off the ω_{TY} , or both. The dominant negative contribution to the obstruction, that is, $-(\beta - \beta_*)$, comes from cutting off $\omega_{\beta,L}$ and its particular shape is due to the “Green’s function like” term $(\beta - \beta_*)/r^{2n-2}$ in the expansion of $\phi_{\beta,L}$.

Ensuring that $F(\varepsilon)$ is positive is absolutely crucial since it allows us to fix ε and deform the cone angle as $\beta = \beta_* + F(\varepsilon) + \gamma$ where $\text{l.o.t.} \ll |\gamma| \ll F(\varepsilon)$ will vary to ensure that $a(\beta, \varepsilon)$ takes positive and negative values hence vanishes for some value $\beta = \beta(\varepsilon)$. For technical reasons, we will actually fix β and vary ε to the same effect, in order to circumvent the troubles due to the functional spaces changing as β varies.

The computation of $F(\varepsilon)$ relies on two things:

- (i) the (delicate) construction of a tubular neighborhood of D where the difference between the holomorphic volume forms on L and X has a sign and
- (ii) the computation of the subleading term *and its sign* in the asymptotic expansion of the Tian–Yau potential. This result (Theorem 6.1) may be of independent interest.

1.4 Comparison with [8]

In [8], we had previously treated the case where $\alpha = 1$ and we would like to briefly single out the main differences between the techniques involved in that paper and in the present one.

In [8], the main difficulties stemmed from the collapsing of the KE_β metrics at stake coupled with the fact that the cone angle β goes to zero; that made the Schauder estimates extremely delicate to establish. The MA equation was obstructed as well (again because of the radial vector field in the normal bundle) but killing the obstruction with the cone angle turned out to be rather easy.

In our present situation, there is no collapsing and β remains bounded away from 0 and 1. However, the Calabi ansatz is much trickier to analyze since we cannot use a scaling argument anymore. More importantly, it is much harder to kill the obstruction as it relies on a very fine understanding of the geometry of the pair (X, D) and its associated canonical KE metrics, as we have explained in the previous paragraph.

2 Geometric setup

Let X be a Fano manifold of dimension $n \geq 2$, and let D be a smooth divisor such that

- $-K_X \sim_{\mathbb{Q}} \alpha D$ for some rational number $\alpha > 1$,
- D admits a Kähler–Einstein metric ω_D ; that is, $\text{Ric } \omega_D = \omega_D$.

Recall that $-K_X \sim_{\mathbb{Q}} \alpha D$ means that there exists an integer $m \geq 1$ such that $m\alpha \in \mathbb{N}$ and that the line bundles $K_X^{\otimes m}$ and $\mathcal{O}_X(-m\alpha D)$ are isomorphic.

Let us observe that $\alpha \leq n + 1$. Indeed, by Mori’s bend and break, there is always a (rational) curve C such that $(-K_X \cdot C) \leq n + 1$. In particular, this implies that $\alpha \leq \frac{n+1}{(D \cdot C)} \leq n + 1$. By Kobayashi–Ochiai, equality occurs if and only if $X = \mathbb{P}^n$ and D is a hyperplane. The latter case is irrelevant for our purposes, so we will assume that $\alpha < n + 1$ in the following.

2.1 Deformation to the normal cone \bar{L}

Let $L := N_D$ be the normal bundle of D , which is an ample divisor and set v to be the fiber coordinate so that $L = \{(x, v) : x \in D, v \in L_x\}$. We consider the hermitian metric h on L (unique up to a multiplicative constant) such that $i\Theta(L, h) = \frac{1}{\alpha-1}\omega_D$.

Next let \bar{L} be the one-point compactification of L ; it can be endowed with a structure of normal projective variety in two equivalent ways. First, one can consider the affine cone $C(D, L^{-1})$ obtained, e.g., as the contraction of the zero section in the total space of the negative bundle L^{-1} and define \bar{L} to be the projectivization of $C(D, L^{-1})$. Or we can view \bar{L} as the blow-down of $\mathbb{P}(N_D) \subset \mathbb{P}(N_D \oplus \mathcal{O}_D)$. It is a classical fact that \bar{L} can be achieved as the central fiber of a \mathbb{C}^* degeneration of X using the so-called degeneration to the normal cone. We recall the construction below.

Consider the family $\mathfrak{X} := \text{Bl}_{D \times \{0\}}(X \times \mathbb{C}) \rightarrow \mathbb{C}$. The fibers \mathfrak{X}_t for $t \neq 0$ are isomorphic to X while $\mathfrak{X}_0 = \text{Bl}_D X \cup \mathbb{P}(N_D \oplus \mathcal{O}_D) \simeq X \cup \mathbb{P}(N_D \oplus \mathcal{O}_D)$ and the two n -dimensional varieties meet along D and $\mathbb{P}(N_D \oplus 0) \simeq \mathbb{P}(N_D)$, the section at infinity. One can contract $\text{Bl}_D X \subset \mathfrak{X}$ via a map $\mathfrak{X} \rightarrow \mathcal{X}$; the resulting map is called $\pi : \mathcal{X} \rightarrow \mathbb{C}$. Its fibers outside the origin are isomorphic to X and identified via a \mathbb{C}^* action that lifts from \mathbb{C} , while the central fiber is the projective cone $\mathbb{P}(N_D \oplus \mathcal{O}_D)$ with the section at infinity contracted to a point, or said otherwise it is the one-point compactification \bar{L} of N_D mentioned above; see, for example, [28, §4.1] combined with the fact that for $m \geq 0$, one has $H^1(X, mD) = H^1(X, K_X + (-K_X + mD)) = 0$ by Kodaira vanishing since $-K_X + mD$ is ample.

On the singular cone \bar{L} , the Calabi-ansatz enables us to construct a KE_{β^*} metric

$$(2-1) \quad \omega_{\beta^*, L} := dd^c \log(1 + |v|_h^{-2\beta^*});$$

see (3-24). The metric is smooth and satisfies $\text{Ric } \omega_{\beta_*,L} = \omega_{\beta_*,L}$ on $L \setminus D$, has cone singularities of angle $2\pi\beta_*$ along $D \subset L$ and it is asymptotic to the Tian–Yau cone metric $\omega_{\text{TY},L} = dd^c|v|^{-2\beta_*}$ near the conical point. The existence of $\omega_{\beta_*,L}$ has important consequences as we now explain.

2.2 Smallest possible cone angle

For a given value $\beta \in (0, 1)$, we consider the Kähler–Einstein equation

$$(KE_\beta) \quad \text{Ric } \omega_\beta = \mu\omega_\beta + (1 - \beta)[D],$$

where $\mu = \mu(\beta) > 0$ is given by $\mu\alpha = \alpha + \beta - 1$, so that $\omega_\beta \in c_1(X) = \alpha c_1(D)$ is fixed, independent of β . Consider the interval

$$I := \{\beta \in (0, 1] : \exists \omega_\beta \text{ solution of } (KE_\beta)\}.$$

Recall that if $\text{Aut}^\circ(X, D) = \{1\}$, then I is open by [19]. Actually, much more can be said even without the assumption on holomorphic vector fields.

Proposition 2.1 [30] *There exists $\beta^* \in (0, 1]$ such that*

$$(2-2) \quad I = \emptyset, (\beta_*, \beta^*) \text{ or } (\beta_*, \beta^*],$$

where β_* is defined in (1-2).

Sketch of proof There are three main steps which we sketch below.

Step 1. The existence of the KE_β metric $\omega_{\beta_*,L}$ on the central fiber of a nontrivial test configuration shows that $\beta_* \notin I$. Actually, one can show $I \subset (\beta_*, 1)$. One way to do is to directly compute the Futaki invariant of $(X, (1 - \beta)D)$ associated to that test configuration [30] and show that when $\beta \leq \beta_*$, it has the “wrong” sign. Alternatively, one can use Fujita valuative criterion [20] with respect to the prime divisor D , and polarization $L = -(K_X + (1 - \beta)D) = -\mu K_X$ with $\mu = \frac{1}{\alpha}(\alpha - 1 + \beta)$. Then, one has $A_{(X,(1-\beta)D)}(D) := 1 + \text{ord}_D(K_X - (K_X + (1 - \beta)D)) = \beta$ and $\text{vol}(L - xD) = (\mu - \frac{x}{\alpha})^n (-K_X^n)$; hence the β invariant

$$\beta(D) = A_{(X,(1-\beta)D)}(D) \cdot (L^n) - \int_0^{+\infty} \text{vol}(L - xD) dx$$

satisfies $\beta(D) = \mu^n (-K_X^n) [\beta - \frac{\mu\alpha}{n+1}] = \frac{n}{n+1} \mu^n (-K_X^n) [\beta - \beta_*]$. In particular, $\beta(D) \geq 0$ if and only if $\beta \geq \beta_*$.

Step 2. The Ding functional F_β for $(X, (1 - \beta)D)$ is bounded below for $\beta = \beta_*$ by a classical argument relying on convexity of Mabuchi along (weak) geodesics [30].

Step 3. The functional F_β is affine in β and existence of a KE_β metric is equivalent to a suitable notion of properness by [16]. Hence the existence of a KE_{β_0} metric implies properness for all $\beta \in (\beta_*, \beta_0]$, and hence the existence of a KE_β metric for each of these angles. That is, I is connected and $\inf I \in \{\beta_*, +\infty\}$. \square

2.3 Rigidity of $\omega_{\beta_*, L}$

The aim of this short subsection is to explain that the cone angle $2\pi\beta_*$ is the only one for which (\bar{L}, D) admits a Kähler–Einstein cone metric. The rigidity is related to the existence of holomorphic vector field tangent to D (namely, the radial vector field induced by the \mathbb{C}^* action), since in the absence of such vector fields, the set of possible cone angles is open [19]. Such a rigidity phenomenon had already been observed for surfaces in [13, Example 2.8].

Proposition 2.2 *The pair (\bar{L}, D) admits a KE_β metric if and only if $\beta = \beta_*$.*

We provide below only a rough sketch of proof based on the computation of the log Futaki invariant associated to the radial vector field ξ . Proposition 2.2 is also a direct consequence of the main theorem (i.e., the existence of KE_β metrics on (X, D) for $\beta \in (\beta_*, \beta_* + \delta)$) and the nonexistence of KE_β metrics on (X, D) for $\beta \leq \beta_*$; see Proposition 2.1.

Proof We claim that there exists on L a Kähler metric $\omega := dd^c\varphi(u)$ with $u = \log|v|_h^2$ such that

- (i) φ is nonincreasing, convex and coincides with $-u + e^u$ near $-\infty$ and e^{-u} near $+\infty$,
- (ii) φ' is a potential for ξ , i.e., $i_\xi\omega = \bar{\partial}\varphi'$.

Indeed, with the help of a drawing one can easily convince oneself that such a function φ exists. It induces a Kähler metric on L thanks to the Calabi ansatz, see (3-4) below, and the metric actually extends smoothly to a Kähler metric on the singular cone \bar{L} . The second item is also a direct consequence of (3-4) and the identity $\bar{\partial}\varphi' = \varphi''\bar{\partial}u$.

The two properties above imply that $f := -\varphi' \geq 0$ is identically 1 on D and decreases with u . Moreover, we have $\omega = \varphi''(u)du \wedge d^c u + p^*\omega|_D$; hence we deduce

$$\begin{aligned} \frac{1}{\text{vol}(\bar{L}, \omega)} \int_{\bar{L}} f \frac{\omega^n}{n!} &= \frac{1}{\text{vol}(\bar{L}, \omega)} \int_{-\infty}^{+\infty} f(u)\varphi''(u) du \wedge d^c u \int_D \frac{\omega^{n-1}}{(n-1)!} \\ &< \frac{1}{\text{vol}(\bar{L}, \omega)} \int_{-\infty}^{+\infty} \varphi''(u) du \wedge d^c u \int_D \frac{\omega^{n-1}}{(n-1)!} \\ &= 1 \\ &= \frac{1}{\text{vol}(D, \omega)} \int_D f \frac{\omega^{n-1}}{(n-1)!}. \end{aligned}$$

Thanks to (the singular version of) [23], the log Futaki invariant associated to $(\bar{L}, D, \beta, -\xi, f)$ satisfies

$$\text{Fut}_{\xi, \beta}([\omega]) = \text{Fut}_{\xi, 1}([\omega]) - a(1 - \beta)$$

for some constant $a > 0$. In particular, there is at most one value of β such that the Futaki invariant vanishes, that is, $\text{Fut}_{\xi, \beta}([\omega]) = 0$. Of course, this is none other than $\beta = \beta_*$. □

3 Conic Calabi ansatz: construction and asymptotics

In this section, we rely on the Calabi ansatz to construct (possibly incomplete) KE_β metrics $\omega_{\beta,L}$ on the normal bundle L (with cone singularities along the zero section $D \subset L$) for each angle $\beta \in (0, 1)$ — the value $\beta = \beta_*$ being critical. We then analyze their asymptotic behavior as $\beta \searrow \beta_*$.

3.1 Reduction to an ODE

Let us record notation that will be used throughout the section:

$$\beta_* = \frac{\alpha - 1}{n}, \quad \mu = \frac{1}{\alpha}(\alpha + \beta - 1), \quad \lambda = \frac{\alpha - 1}{\mu},$$

$$a = \frac{\mu}{n + 1}, \quad b = \beta_* = \frac{\alpha - 1}{n}, \quad \psi = -\phi - \lambda u, \quad C_\beta = \frac{\alpha^n}{n + 1}(\beta - \beta_*).$$

The last constant C_β is recorded here for clarity but it will only be determined a few lines below.

On $L = N_D$, we want to solve

$$(3-1) \quad \text{Ric } \omega_\beta = \mu \omega_\beta + (1 - \beta)[D]$$

with

- (i) $\omega_\beta = dd^c \phi(u)$ where $u = \log |v|_h^2$, and h is a smooth hermitian metric on L such that $i \Theta(L, h) = \frac{1}{\alpha - 1} \omega_D$,
- (ii) ω_β has conic singularities with cone angle $2\pi\beta$ along D ,
- (iii) $\phi(u) \sim -\alpha u$ when $u \rightarrow -\infty$.

The last condition just means that the current $\omega_\beta|_D$ belongs to $\alpha c_1(D)|_D$. It is an arbitrary normalization that imposes the value of the Einstein constant to satisfy

$$\mu\alpha = \alpha + \beta - 1.$$

The fact that ω_β is a metric is equivalent to having

$$(3-2) \quad -\phi' > 0, \quad \phi'' > 0.$$

It is easy to check that

$$(3-3) \quad \Xi := e^{-(\alpha-1)u} du \wedge d^c u \wedge \omega_D^{n-1}$$

defines a Ricci flat volume form on L with a pole of order α along the zero section. The metric

$$(3-4) \quad \omega_\beta = dd^c \phi(u) = \phi''(u) du \wedge d^c u - \frac{2}{\alpha - 1} \phi'(u) \omega_D$$

satisfies

$$(3-5) \quad \omega_\beta^n = a_{n,\alpha} \phi''(u) (-\phi'(u))^{n-1} du \wedge d^c u \wedge \omega_D^{n-1}, \quad a_{n,\alpha} = \frac{2^{n-1} n}{(\alpha - 1)^{n-1}}.$$

Therefore ω_β is a solution of

$$(3-6) \quad \omega_\beta^n = a_{n,\alpha} e^{-\mu\phi} \Xi,$$

and therefore a solution of (3-1), if ϕ solves

$$(3-7) \quad (-\phi')^{n-1} \phi'' = e^{-\mu\phi - (\alpha-1)u}.$$

Multiplying each side by $-\mu\phi' - (\alpha - 1)$ and integrating, we get

$$(3-8) \quad (-\phi')^n (b + a\phi') = e^{-\mu(\phi + \lambda u)} - C_\beta$$

for some constant C_β to determine.

By setting $\psi := -\phi - \lambda u$, we get the equivalent autonomous equation

$$(3-9) \quad (\psi' + \lambda)^n (b - a(\psi' + \lambda)) = e^{\mu\psi} - C_\beta.$$

In order to choose the right constant C_β , let us remember that when $u \rightarrow -\infty$, we want our solution ψ to satisfy $\psi(u) \sim (\alpha - \lambda)u = \frac{\beta}{\mu}u$, so that $\psi' + \lambda \sim \alpha$. Therefore we want the constant C_β to satisfy

$$(3-10) \quad \alpha^n (b - a\alpha) = \frac{\alpha^n}{n + 1} \left(\frac{\alpha - 1}{n} - \beta \right) = -C_\beta.$$

From now on, we impose the constant C_β from (3-9) to be given by (3-10) above, i.e.,

$$C_\beta = \frac{\alpha^n}{n + 1} \cdot (\beta - \beta_*).$$

Observe that

$$C_{\beta_*} = 0 \quad \text{and} \quad C_\beta > 0 \quad \text{for} \quad \beta > \beta_*.$$

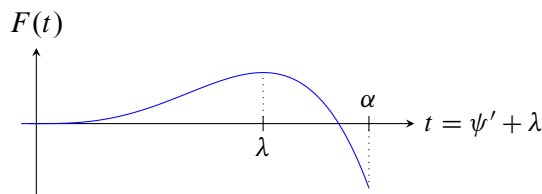
In the sections that follow, we investigate the existence and the behavior of solutions of (3-9).

From now on, we will assume that $\beta > \beta_$ until the end of Section 3 with the exception of Section 3.5.*

3.2 Construction of ψ near $-\infty$

In order for the problem to be well posed, we fix the initial value $\psi(\hat{u}_0) = \hat{\psi}_0$ for some couple $(\hat{u}_0, \hat{\psi}_0)$ that will be determined later.

Let $F(t) = t^n (b - at)$, defined on $[0, \alpha]$:



We have $F(0) = 0$, and $F(\alpha) \leq 0$ with equality if and only if $\beta = \beta_*$. Also F is increasing up until $t = \lambda$, and then decreasing. The value

$$(3-11) \quad F(\lambda) = \frac{\beta_* \lambda^n}{n + 1} > 0$$

is explicit but does not play any role. Also F admits two inverses

$$G_1 : [0, F(\lambda)] \rightarrow [0, \lambda] \quad \text{and} \quad G_2 : [F(\alpha) = -C_\beta, F(\lambda)] \rightarrow [\lambda, \alpha].$$

The equation (3-9) can be reformulated as

$$(3-12) \quad F(\psi' + \lambda) = e^{\mu\psi} - C_\beta,$$

which can be expressed as $\psi' = G_i(e^{\mu\psi} - C_\beta) - \lambda$, that is,

$$(3-13) \quad \frac{d\psi}{G_i(e^{\mu\psi} - C_\beta) - \lambda} = du$$

for either $i = 1, 2$. Neither G_1 nor G_2 is differentiable at the nexus point $F(\lambda)$.

The equation

$$(3-14) \quad \int_{\hat{\psi}_0}^{\psi} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda} = u - \hat{u}_0$$

uniquely defines a function $\psi = \psi(u)$ on a neighborhood of $u = \hat{u}_0$ solving (3-12). Here, we need $\hat{\psi}_0 \ll 0$ to make sure that $e^{\mu\hat{\psi}_0} \in (F(\alpha), F(\lambda))$ so that $G_2(e^{\mu x} - C_\beta)$ is well defined near $x = \hat{\psi}_0$. Let us define

$$(3-15) \quad \psi_0 := \frac{1}{\mu} \log(F(\lambda) + C_\beta), \quad \text{i.e., } e^{\mu\psi_0} - C_\beta = F(\lambda).$$

Since $G_2 \geq \lambda$, ψ is nondecreasing. Clearly, ψ reaches any value less than ψ_0 . Moreover, we claim that $\psi(u)$ is defined for any negative values of u . This follows from the fact that

$$\int_{\hat{\psi}_0}^{-\infty} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda} = -\infty.$$

Indeed,

$$G_2(e^{\mu x} - C_\beta) \underset{x \rightarrow -\infty}{\sim} \alpha \quad \text{and} \quad \alpha > \lambda.$$

In particular, one has $\psi(u) \rightarrow -\infty$ when $u \rightarrow -\infty$. One can actually say more. Indeed,

$$G_2(-C_\beta + s) = \alpha + \frac{s}{F'(\alpha)} + O(s^2)$$

when $s \rightarrow 0^+$. Equivalently, one finds $G_2(e^{\mu x} - C_\beta) - \lambda = (\alpha - \lambda) + \frac{e^{\mu x}}{F'(\alpha)} + O(e^{2\mu x})$ when $x \rightarrow -\infty$. Expanding the integral defining ψ , one finds near $u = -\infty$:

$$\psi(u) = (\alpha - \lambda)(u - \hat{u}_0) + \hat{\psi}_0 + \frac{\alpha - \lambda}{F'(\alpha)} e^{\mu\psi} + O(e^{2\mu\psi}).$$

Writing $\alpha - \lambda = \frac{\beta}{\mu}$, one infers that

$$\phi(u) = -\alpha u + u_\infty - \frac{\beta}{\mu F'(\alpha)} e^{\beta u} + O(e^{2\beta u})$$

when $u \rightarrow -\infty$, for some $u_\infty \in \mathbb{R}$. Since $F'(\alpha) = -\beta\alpha^{n-1} < 0$, ϕ satisfies the requirements (3-2) at least near $-\infty$ and $dd^c\phi(u)$ has indeed a cone singularity of angle $2\pi\beta$ along $D = (u = -\infty)$. Moreover, thanks to (3-8), we obtain the following expansions for ϕ' and ϕ'' :

$$(3-16) \quad \phi'(u) = -\alpha + c_1 e^{\beta u} + O(e^{2\beta u}), \quad \phi''(u) = c_2 e^{\beta u} + O(e^{2\beta u}),$$

where $c_1 = -\frac{\beta^2}{\mu F'(\alpha)}$ and $c_2 = \beta c_1$.

3.3 Extension of ψ past ψ_0

We are going to show that ψ can be smoothly extended past the value ψ_0 defined in (3-15). First, we claim that

$$\int_{\hat{\psi}_0}^{\psi_0} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda}$$

is convergent. Indeed, perform the change of variable $s := e^{\mu x} - C_\beta$ and set $\hat{s}_0 := e^{\mu \hat{\psi}_0} - C_\beta$ so that our integral becomes

$$\frac{1}{\mu} \int_{\hat{s}_0}^{F(\lambda)} \frac{ds}{(G_2(s) - \lambda)(s + C_\beta)} = \frac{1}{\mu} \int_\lambda^{G(\hat{s}_0)} \frac{-F'(t) dt}{(t - \lambda)(F(t) + C_\beta)},$$

where we have used another change of variable $t = G(s)$ to obtain the right-hand side. Now an easy computation shows that $F'(t) = \mu t^{n-1}(\lambda - t)$ and the integral becomes

$$\int_\lambda^{G(\hat{s}_0)} \frac{t^{n-1} dt}{F(t) + C_\beta},$$

which is obviously convergent. Now,

$$(3-17) \quad \int_{\psi_0}^{\psi} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda} = u - u_0$$

shows that ψ is defined on $(-\infty, u_0)$ and satisfies $\psi(u) \rightarrow \psi_0$ when $u \rightarrow u_0$. We set

$$(3-18) \quad u_0 := \hat{u}_0 + \int_{\hat{\psi}_0}^{\psi_0} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda} = \hat{u}_0 + \int_\lambda^{G(\hat{s}_0)} \frac{t^{n-1} dt}{F(t) + C_\beta}.$$

By the same arguments, one sees that

$$\int_{\psi_0}^{\psi} \frac{dx}{G_1(e^{\mu x} - C_\beta) - \lambda}$$

is convergent, and hence

$$(3-19) \quad \int_{\psi_0}^{\psi} \frac{dx}{G_1(e^{\mu x} - C_\beta) - \lambda} = u - u_0$$

defines an extension of ψ for $u > u_0$ close enough to u_0 which is continuous across u_0 and satisfies

$$\psi(u) \xrightarrow{u \rightarrow u_0} \psi_0, \quad \psi'(u) \xrightarrow{u \rightarrow u_0} 0.$$

Let us now show that ψ is smooth at u_0 . Differentiating (3-12) we get $\psi'' F'(\psi' + \lambda) = \mu \psi' e^{\mu \psi}$ away from u_0 . Since $F'(t) = -\mu t^{n-1}(t - \lambda)$, this implies that

$$(3-20) \quad \psi'' = -\frac{e^{\mu \psi}}{(\psi' + \lambda)^{n-1}},$$

which converges to the value $\frac{e^{\mu \psi_0}}{\lambda^{n-1}}$ when $u \rightarrow u_0$. In particular, ψ is C^2 near u_0 . We get smoothness iteratively thanks to (3-20).

Finally, since $G_1 \leq \lambda$, ψ is nonincreasing after u_0 . The expression (3-19) yields a function ψ satisfying $\psi \geq \psi_\beta$ for $\psi_\beta \in \mathbb{R}$ which is a solution of

$$e^{\mu \psi_\beta} = C_\beta.$$

We refer to Figure 1 below for the qualitative behavior of ψ .

We can actually provide an integral expression (or relation) for ψ which does not involve u_0 . Indeed, from (3-18), one gets

$$\begin{aligned} \lambda(u_0 - \hat{u}_0) &= \hat{\psi}_0 - \psi_0 + \int_{\hat{\psi}_0}^{\psi_0} \frac{G_2(e^{\mu x} - C_\beta) dx}{G_2(e^{\mu x} - C_\beta) - \lambda} \\ &= \hat{\psi}_0 - \psi_0 + \int_{\lambda}^{G_2(\hat{s}_0)} \frac{t^n dt}{F(t) + C_\beta} \end{aligned}$$

and similarly, one gets from (3-19)

$$\lambda(u - u_0) = \psi_0 - \psi + \int_{G_1(s(\psi))}^{\lambda} \frac{t^n dt}{F(t) + C_\beta}.$$

Adding the latter identities, we get

$$(3-21) \quad \phi(\hat{u}_0) - \phi(u) = \int_{G_1(s(\psi))}^{G_2(\hat{s}_0)} \frac{t^n dt}{F(t) + C_\beta},$$

where $s(\psi) = e^{\mu \psi} - C_\beta$. Recall that by construction, we have $\phi(\hat{u}_0) = -\hat{\psi}_0 - \lambda \hat{u}_0$, and that $\hat{\psi}_0$ is constrained to $\hat{s}_0 = s(\hat{\psi}_0) \in (F(\alpha), F(\lambda))$ where $s(x) = e^{\mu x} - C_\beta$. We fix such a $\hat{\psi}_0$ once and for all. At this point, \hat{u}_0 is arbitrary, but from now on we will choose \hat{u}_0 so that

$$\hat{\psi}_0 + \lambda \hat{u}_0 + \int_0^{G_2(\hat{s}_0)} \frac{t^n dt}{F(t) + C_\beta} = 0.$$

As a result, we obtain from (3-21)

$$(3-22) \quad \phi(u) = \int_0^{G_1(s(\psi))} \frac{t^n dt}{F(t) + C_\beta}.$$

3.4 Long time existence of ψ

There are two cases to consider depending on the sign of C_β .

- If $C_\beta > 0$ (i.e., $\beta > \beta_*$),

$$\int_{\psi_0}^{\psi_\beta} \frac{dx}{G_1(e^{\mu x} - C_\beta) - \lambda} = \int_0^\lambda \frac{t^{n-1} dt}{F(t) + C_\beta}$$

is convergent. In other words, there exists a finite time u_β such that $\psi(u_\beta) = \psi_\beta$. In particular, $dd^c \phi$ cannot be extended as a metric past u_β . Moreover, the previous integral goes to $+\infty$ as $\beta \rightarrow \beta_*$ which shows that $u_\beta \rightarrow +\infty$ when $\beta \rightarrow \beta_*$.

- If $C_\beta = 0$ (i.e., $\beta = \beta_*$), then ψ is defined for all times and $\psi(u) \rightarrow -\infty$ as $u \rightarrow +\infty$ since

$$\int_{\psi_0}^{-\infty} \frac{dx}{G_1(e^{\mu x}) - \lambda} = \int_0^\lambda \frac{dt}{t(b - at)} = +\infty$$

and

$$\int_{\psi_0}^{\psi_2} \frac{dx}{G_1(e^{\mu x}) - \lambda} < +\infty$$

for any finite $\psi_2 < \psi_0$. One can easily get from (3-21) the successive asymptotics $\psi(u) = -\lambda u + C_1 + o(1)$ and then $\psi(u) = -\lambda u + C_1 - C_2 e^{-\frac{\lambda \mu}{n} u} + O(e^{-\frac{2\lambda \mu}{n} u})$ when $u \rightarrow +\infty$, for some $C_2 > 0$. In terms of ϕ ,

$$(3-23) \quad \phi(u) = -C_1 + C_2 e^{-\beta_* u} + O(e^{-2\beta_* u}),$$

where we have used that $\frac{\lambda \mu}{n} = \beta_*$.

One can actually recover the qualitative behavior of ϕ_{β_*} from an explicit formula. Indeed, $\phi := \phi_{\beta_*}$ is a solution of

$$\phi'' = -\phi' \left(1 + \frac{\phi'}{\alpha} \right) \beta_*,$$

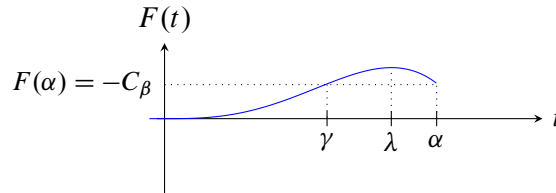
which can be integrated in

$$(3-24) \quad \begin{aligned} \phi_{\beta_*}(u) &= c_0 - \alpha u + \frac{\alpha}{\beta_*} \log(1 + c_1 e^{\beta_* u}) \\ &= c_2 + \frac{\alpha}{\beta_*} \log(1 + c_1^{-1} e^{-\beta_* u}), \end{aligned}$$

where $c_0, c_2 \in \mathbb{R}$ and $c_1 > 0$ are constant.

3.5 Analysis of the case $\beta < \beta_*$

Again, we fix the initial values $(\hat{u}_0, \hat{\psi}_0)$ such that $e^{\mu\hat{\psi}_0} \in (F(\alpha), F(\lambda))$. Under the assumption that $\beta < \beta_*$, we have $C_\beta < 0$, and hence the graph of F has the shape below:



Here we have set $\gamma := G_1(F(\alpha)) = G_1(-C_\beta)$. We can construct a solution ψ near u_0 solving $\psi(\hat{u}_0) = \hat{\psi}_0$.

Claim 3.1 *The solution $\psi = \psi(u)$ is defined for all $u \in (-\infty, +\infty)$.*

Proof The existence of $\psi(u)$ for u near $-\infty$ is justified by the divergence of

$$\int_{-\infty}^{\hat{\psi}_0} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda},$$

which is clear since $G_2(e^{\mu x} - C_\beta) \rightarrow \alpha$ when $x \rightarrow -\infty$. As in the case $\beta > \beta_0$, one see that there is a value u_0 such that $\psi(u) \rightarrow \psi_0$ when $u \nearrow u_0$ where $e^{\mu\psi_0} - C_\beta = F(\lambda)$, and that ψ can be extended smoothly past u_0 . The existence of $\psi(u)$ for u near $+\infty$ is justified by the divergence of

$$\int_{-\infty}^{\psi_0} \frac{dx}{G_1(e^{\mu x} - C_\beta) - \lambda},$$

which in turn is a consequence of $G_1(e^{\mu x} - C_\beta) \rightarrow \gamma \in (0, \lambda)$ when $x \rightarrow -\infty$. □

Claim 3.2 *The KE metric $\omega_\beta = d^d c \phi_\beta$ extends to a metric on the compactification $\mathbb{P}(L \oplus \mathcal{O}_D)$ with cone singularities along both D and D_∞ . The cone angle along D is $2\pi\beta$ and the cone angle along D_∞ is $2\pi\mu(\lambda - \gamma)$ which ranges in $(0, 2\pi(\alpha - 1))$ as β ranges in $(0, \beta_*)$.*

Proof Let us start with the behavior near $u = -\infty$. One writes $G_2(e^{\mu x} - C_\beta) = \alpha - c e^{\mu x} + O(e^{2\mu x})$ with $c > 0$. Plugging that in

$$\int_{\psi_0}^{\psi} \frac{dx}{G_2(e^{\mu x} - C_\beta) - \lambda} = u - u_0,$$

we infer iteratively $\psi - \psi_0 \approx (\alpha - \lambda)(u - u_0)$ and then $\psi - \psi_0 = (\alpha - \lambda)(u - u_0) - c' e^{\beta(u - u_0)} + O(e^{2\beta u})$ when $u \rightarrow -\infty$, where we have used $\alpha - \lambda = \frac{\beta}{\mu}$. This shows that

$$\phi(u) - \phi(u_0) = -\alpha(u - u_0) + c'' e^{\beta u} + O(e^{2\beta u})$$

with $c'' > 0$, which holds for the derivatives of ϕ as well thanks to the MA equation that the latter solves.

Near $u = +\infty$, the analysis is similar but we use instead $G_1(e^{\mu x} - C_\beta) = \gamma + c e^{\mu x} + O(e^{2\mu x})$ for some other $c > 0$. Then

$$\psi - \psi_0 \simeq (\gamma - \lambda)(u - u_0) \quad \text{and} \quad \phi(u) - \phi(u_0) = -\gamma(u - u_0) + c'' e^{-\mu(\lambda - \gamma)u} + O(e^{-2\mu(\lambda - \gamma)u}). \quad \square$$

Degenerating β to zero In the remaining part of this section, we will explain how ω_β degenerates when $\beta \rightarrow 0$.

We fix $\hat{u}_0 = 0$ and choose any $\hat{\psi}_0 < \psi_0$, which yields a value of u_0 by (3-18). Replacing ψ by $\psi(\cdot + u_0)$, one can assume without loss of generality that $u_0 = 0$. Since $\alpha - \lambda = \frac{\beta}{\mu} \rightarrow 0$, the values γ and λ converge to α . Moreover, one can check that

$$(3-25) \quad e^{\mu\psi_0} = F(\lambda) + C_\beta \sim \frac{\alpha^n}{2(\alpha - 1)} \cdot \beta^2.$$

When coupled with $F(t) = F(\lambda) + \frac{1}{2}(\lambda - t)^2 \cdot F''(\lambda) + O((\lambda - t)^3)$ evaluated at $t = \gamma$, we obtain

$$(3-26) \quad \lambda - \gamma \sim \frac{\alpha}{\alpha - 1} \cdot \beta$$

since $F''(\lambda) = -\mu\lambda^{n-1} \approx -\mu\alpha^{n-1}$.

Behavior near D . Let us first focus on what happens near D . For $t \in (\lambda, \alpha)$, one writes

$$F(t) - F(\alpha) = \alpha^{n-1}(\alpha - t)\left(\beta - \frac{1}{2}\mu(\alpha - t) + O(\beta^2)\right).$$

Neglecting the $O(\beta^2)$ term, one can solve for t and find

$$t - \lambda = \frac{\beta}{\mu} \sqrt{1 - \frac{2\mu}{\alpha^{n-1}} \frac{F(t) - F(\alpha)}{\beta^2}}.$$

Setting $t := G_2(e^{\mu x} - C_\beta)$ and using (3-25) this yields $G_2(e^{\mu x} - C_\beta) - \lambda \approx \frac{\beta}{\mu} \sqrt{1 - e^{\mu(x - \psi_0)}}$; hence

$$\begin{aligned} -\frac{\beta}{\mu}u &= \int_{\psi - \psi_0}^0 \frac{dx}{\sqrt{1 - e^{\mu x}}} \\ &= -(\psi - \psi_0) + c - \frac{1}{2\mu} e^{\mu(\psi - \psi_0)} + O(e^{2\mu(\psi - \psi_0)}), \end{aligned}$$

where $c > 0$ is a positive harmless constant (it might depend on β but does not blow up when $\beta \rightarrow 0$).

In terms of ϕ , we have

$$\phi_\beta(u) = -\psi_0 - c - \alpha u + \frac{1}{2\mu} e^{\beta u} + O(e^{2\beta u}).$$

In particular, if one sets $r := \sqrt{2\mu^{-1}} e^{\frac{1}{2}\beta u} \in (0, \sqrt{2\mu^{-1}})$, one sees that near D , the Riemannian metric g_β associated to ω_β is asymptotic (uniformly in β) to the Riemannian metric $dr^2 + \beta^2 \eta^2 + \frac{2\alpha}{\alpha - 1} g_D$ which collapses the circles. Here, g_D is the Riemannian metric associated to ω_D .

Behavior near D_∞ . Let us now discuss what happens near D_∞ . Since $F'(t) = -\mu t^{n-1}(t - \lambda)$, we derive from (3-26) that $F'(\gamma) \sim \frac{\mu\alpha^n}{\alpha - 1}\beta$ and $F''(\gamma) \approx -\mu\alpha^{n-1}$. So for $t \in (\gamma, \lambda)$, one has

$$F(t) - F(\gamma) = \mu\alpha^{n-1}(t - \gamma) \left(\frac{\alpha}{\alpha - 1}\beta - \frac{1}{2}(t - \gamma) + O(\beta^2) \right).$$

Neglecting the $O(\beta^2)$ term, one can complete the square and find

$$t - \lambda = -\frac{\alpha}{\alpha - 1} \beta \sqrt{1 - \frac{2(\alpha - 1)^2}{\mu \alpha^{n+1} \beta^2} e^{\mu x}}.$$

Setting $t = G_1(e^{\mu x} - C_\beta)$ and using (3-25), this yields $G_1(e^{\mu x} - C_\beta) - \lambda \approx -\frac{\alpha}{\alpha - 1} \beta \sqrt{1 - e^{\mu(x - \psi_0)}}$; hence

$$\begin{aligned} \frac{\alpha \beta}{\alpha - 1} u &= \int_{\psi - \psi_0}^0 \frac{dx}{\sqrt{1 - e^{\mu x}}} \\ &= -(\psi - \psi_0) + c - \frac{1}{2\mu} e^{\mu(\psi - \psi_0)} + O(e^{2\mu(\psi - \psi_0)}). \end{aligned}$$

Therefore

$$\phi_\beta(u) = -\psi_0 - c - \gamma u + \frac{1}{2\mu} e^{-\frac{\mu \alpha}{\alpha - 1} \beta u} + O(e^{-2\frac{\mu \alpha}{\alpha - 1} \beta u})$$

and we conclude similarly to earlier for the case of D . The actual expansion has additional corrective terms, e.g., the cone singularity has angle $2\pi\mu(\lambda - \gamma)$ which differs from $2\pi\mu\frac{\alpha}{\alpha - 1}\beta$ by a $O(\beta^2)$.

From the analysis above, one sees that the Kähler metric ω_β collapses to $\frac{2\alpha}{\alpha - 1}\omega_D$ on the interior of $L \setminus D = \mathbb{P}(L \oplus \mathcal{O}_D) \setminus (D \cup D_\infty)$. In conclusion the Gromov–Hausdorff limit of $(\mathbb{P}(L \oplus \mathcal{O}_D), g_\beta)$ is the union of two copies of $([0, 1] \times D, \frac{2\alpha}{\alpha - 1}(dr^2 + g_D))$ glued along $(r = 1) \simeq D$ or, equivalently, $([0, 2] \times D, \frac{2\alpha}{\alpha - 1}(dr^2 + g_D))$.

3.6 Asymptotic expansion of ϕ near $+\infty$

Now we want to investigate the behavior of $\phi(u) = \phi_\beta(u)$ both as β decreases to β_* and when $u \rightarrow +\infty$.

First, we know that $\psi \rightarrow -\infty$ so that $s(\psi) \rightarrow 0$. Since $F(t) \sim bt^n$ when $t \rightarrow 0$ and $\frac{1}{F(t) + C_\beta} \leq \frac{1}{F(t)}$, (3-22) tells us that

$$0 \leq \phi(u) \leq G_1(s(\psi))(b^{-1} + o(1)),$$

which tends to zero. Recall that $G(s) \sim b^{-\frac{1}{n}} s^{\frac{1}{n}}$ and $s(\psi) \leq e^{\mu\psi} = e^{-n\beta_*u - \mu\phi} \leq e^{-n\beta_*u}$ since ϕ is nonnegative. In short, we get

$$(3-27) \quad \phi(u) = O(e^{-\beta_*u}),$$

where $O(\cdot)$ is uniform in β , when $u \rightarrow +\infty$.

Since $\psi(u) = -\lambda u - \phi(u)$, the estimate (3-27) above is already enough to see that the extinction time u_β of ψ_β — i.e., when the latter reaches the value $\frac{1}{\mu} \log C_\beta$ — happens when $u_\beta \simeq -\frac{1}{\lambda\mu} \log C_\beta$, or more precisely

$$(3-28) \quad u_\beta = -\frac{1}{n\beta_*} \log C_\beta + O(C_\beta^{\frac{1}{n}}).$$

Therefore we have Figure 1.

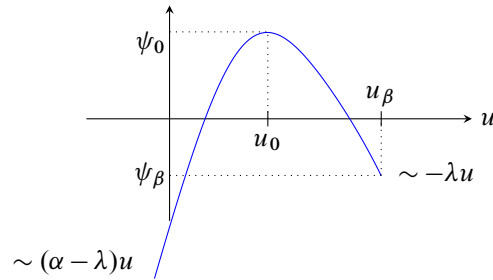


Figure 1: Graph of ψ .

To refine (3-27), we need to further analyze the integral $\phi = \int_0^{G_1(s(\psi))} \frac{t^n dt}{F(t) + C_\beta}$ as $s(\psi) \rightarrow 0$. Despite its simple form, it is actually slightly simpler to reverse the change of variable and work with the expression

$$(3-29) \quad \phi = \frac{1}{\lambda\mu} \int_0^{s(\psi)} \frac{G(s) ds}{(1 - \lambda^{-1}G(s))(s + C_\beta)}.$$

First, we claim that the function $G(s)$ has a polynomial expansion in $s^{\frac{1}{n}}$ at any (finite) order. We prove this by induction, given that this is clear at order one. Set $\gamma(s) = G(s^n)$, which satisfies

$$(3-30) \quad \gamma(s)^n (b - a\gamma(s)) = s^n.$$

Fix an integer $N \geq 0$, and assume that one can write $\gamma(s) = b^{-\frac{1}{n}}s + s^2(P_N(s) + R_N(s))$ where P_N is a polynomial of degree at most N while $R_N(s) = O(s^{N+1})$. By (3-30), we get

$$\begin{aligned} b - a(P_N + R_N) &= \left(\frac{s}{\gamma(s)}\right)^n = b^{\frac{1}{n}} \left(1 + \sum_{k=1}^{N+1} (-1)^k b^{\frac{k}{n}} s^k (P_N + R_N)^k\right) + O(s^{N+2}) \\ &= b^{\frac{1}{n}} \left(1 + \sum_{k=1}^{N+1} (-1)^k b^{\frac{k}{n}} s^k P_N^k\right) + O(s^{N+2}) \end{aligned}$$

from which one deduces that $R_N(s) = c_{N+1}s^{N+1} + O(s^{N+2})$, as desired.

The expansion of $G(s)$ induces an expansion of $G(s)/(1 - \lambda^{-1}G(s))$, which we can combine with

$$\frac{1}{s + C_\beta} = s^{-1} \left(1 - \frac{C_\beta}{s} + O\left(\left(\frac{C_\beta}{s}\right)^2\right)\right)$$

and (3-29) to obtain

$$(3-31) \quad \phi = P_N(s^{\frac{1}{n}}) + O(s^{\frac{N+1}{n}}) + \frac{C_\beta}{s^{1-\frac{1}{n}}} \left(c_0 + O(s^{\frac{1}{n}}) + O\left(\frac{C_\beta}{s}\right)\right),$$

where P_N is a polynomial of degree at most N vanishing at 0, $c_0 = 1/((n-1)b^{1+\frac{1}{n}})$ while

$$(3-32) \quad s := s(\psi) = e^{\mu\psi} - C_\beta.$$

This asymptotic expansion is valid in the zone $C_\beta \ll s \ll 1$ or, equivalently by (3-27), in

$$(3-33) \quad (\beta - \beta_*)^{\frac{1}{n}} \ll e^{-\beta_* u} \ll 1.$$

It will be convenient to introduce the variables

$$(3-34) \quad \varrho := e^{-\frac{1}{2}\beta_* u} \quad \text{and} \quad r = 2\varrho$$

(r will correspond to a Riemannian radius but ϱ is more convenient to write expansions). We expand ϕ_β in powers of ϱ^2 and $\delta := C_\beta/\varrho^{2n}$ (the latter is a $o(1)$ from (3-33)). Therefore, we need to expand s in powers of ϱ and ε .

First, one infers from (3-27)

$$(3-35) \quad e^{\mu\psi} = \varrho^{2n} (1 + O(\varrho^{2n})).$$

In particular, we find

$$(3-36) \quad s = e^{\mu\psi} (1 - C_\beta e^{-\mu\psi}) = \varrho^{2n} (1 + O(\varrho^{2n}) + \delta(-1 + O(\varrho^{2n}))).$$

We claim that we can improve (3-36) as follows. Given any order $N \geq 0$, one can find a polynomial Q_N of degree at most N such that

$$(3-37) \quad s^{\frac{1}{n}} = h\varrho^2 \quad \text{with} \quad h(\varrho) = Q_N(\varrho) + O(\varrho^{N+1}) + \delta(c_1 + O(\varrho^2) + O(\delta)),$$

for some (negative) constant c_1 . From (3-36), we have $Q_N(0) = 1$. Let us now prove the claim (3-37). In what follows g_\bullet will denote a function of the form $cst + O(\varrho^2) + O(\delta)$. This class of functions is stable under addition and multiplication. We assume that the claim is known at some level $N \geq 1$ and we write $h(\varrho) = Q_N(\varrho) + R_N + \delta g_N$. We need to show that $R_N = c_{N+1}\varrho^{N+1} + O(\varrho^{N+2})$.

An easy computation shows that for any $k \in \mathbb{N}$, we have

$$(3-38) \quad h(\varrho)^k = \text{polynomial} + k C_0^{\frac{k-1}{n}} R_N(\varrho) + \delta g_{h,k}.$$

In particular, we have $P_N(h\varrho^2) = \text{polynomial} + \delta g_N + O(\varrho^{N+3})$, and hence for any $k \in \mathbb{N}$, we have

$$(3-39) \quad (P_N(h\varrho^2) + O((h\varrho^2)^{N+1}))^k = \text{polynomial} + \delta g_{N,k} + O(\varrho^{N+3}).$$

Next, we have

$$\frac{C_\beta}{s^{1-\frac{1}{n}}} = \delta \varrho^2 h^{1-n} = \delta g_0$$

so that for any $k \geq 1$, (3-31) and (3-39) yield $\phi_\beta(\varrho)^k = \text{polynomial} + \delta g_{\phi,k} + O(\varrho^{N+3})$. Therefore,

$$\begin{aligned} h(\varrho)^n &= e^{-\mu\phi} - \delta \\ &= e^{-\mu a_0} \left(1 + \sum_{k=1}^N \frac{(-\mu^k)}{k!} \phi^k \right) + O(\varrho^{2N+2}) - \delta \\ &= \text{polynomial} + \delta g_{h,n} + O(\varrho^{N+3}) \end{aligned}$$

and (3-38) with $k = n$ yields the expected result (3-37) (we actually get the improved result that Q_N is even). It is now straightforward to infer the expansion of ϕ_β below:

$$(3-40) \quad \phi_\beta(\varrho) = \sum_{k=0}^N a_k \varrho^{2k} + O(\varrho^{2N+2}) + a_L \frac{\beta - \beta_*}{\varrho^{2n-2}} \cdot \left(1 + O(\varrho^2) + O\left(\frac{\beta - \beta_*}{\varrho^{2n}}\right) \right).$$

Moreover, one can check that coefficients $a_i = a_i(\beta)$ are smooth in β and the first ones are given by

$$a_0 = 0, \quad a_1 = \frac{1}{b^{1+\frac{1}{n}}}, \quad a_2 = \frac{1}{2\lambda b^{1+\frac{2}{n}}} \cdot \left(1 + \frac{1}{n+1} \right), \quad a_L = \frac{\alpha^n}{(n-1)n(n+1)b^{1+\frac{1}{n}}}.$$

At this point, it is convenient to (slightly) rescale L , replacing u by $u + (1 + \frac{1}{n})\frac{\log b}{b}$. We therefore consider $\tilde{\phi}(u) = \phi(u - (1 + \frac{1}{n})\frac{\log b}{b})$: this changes (3-7) and (3-6) to

$$(3-41) \quad \tilde{\phi}''(-\tilde{\phi}')^{n-1} = b^{n+1} e^{-\mu\tilde{\phi} - (\alpha-1)u},$$

$$(3-42) \quad \omega^n = e^{-\mu\tilde{\phi}} b_{n,\alpha} \Xi,$$

where now

$$(3-43) \quad b_{n,\alpha} = \beta_*^{n+1} a_{n,\alpha}.$$

As a result, the expansion (3-40) of $\tilde{\phi}$ has $a_1 = 1$.

We now consider that we have done the rescaling and we replace the previous ϕ by $\tilde{\phi}$, so ϕ is a solution of (3-41). In this way, ϕ_β coincides up to order 2 in ϱ with the unique potential ϕ_{β_*} defined in (3-24) with $c_1 = 1, c_2 = 0$, i.e.,

$$(3-44) \quad \phi_{\beta_*}(\varrho) = \log(1 + \varrho^2) = \log\left(1 + \frac{1}{4}r^2\right) = \frac{1}{4}r^2 - \frac{1}{32}r^4 + \dots.$$

Since the $a_i(\beta)$ are differentiable in β , the polynomial expansion of ϕ_β (that is, the first line in the expansion above) differs from that of ϕ_{β_*} by a factor $O((\beta - \beta_*)r^4)$. Therefore, we have:

Theorem 3.3 *Given any $N > 0$, the potential ϕ_β admits this expansion, valid in $(\beta - \beta_*)^{\frac{1}{n}} \ll r^2 \ll 1$:*

$$(3-45) \quad \phi_\beta(r) = \phi_{\beta_*}(r) + a_L \frac{\beta - \beta_*}{r^{2n-2}} \cdot \left(1 + O(r^2) + O\left(\frac{\beta - \beta_*}{r^{2n}}\right) \right) + O(r^{2N}),$$

where ϕ_{β_*} is given by (3-44) and $a_L > 0$ is a positive constant.

In the remainder of this section, let us now explain how to get expansions similar to (3-40) for ϕ'_β and ϕ''_β . Since $\varrho = e^{-\frac{1}{2}\beta_*u}$, one has $\partial_u = -\frac{1}{2}\beta_*\varrho\partial_\varrho$; hence one can rewrite (3-7) as $F(-\frac{1}{2}\beta_*\varrho\partial_\varrho\phi) = \varrho^{2n}e^{-\mu\phi} - C_\beta$. Recall that we introduced the function $\delta = C_\beta/\varrho^{2n}$ which goes to zero but at a rate which is “independent” of ϱ . The above identity becomes

$$\varrho\partial_\varrho\phi = -\frac{2}{\beta_*}G(\varrho^{2n}(e^{-\mu\phi} - \delta)).$$

Since $G(s)$ has finite expansions in powers of $s^{\frac{1}{n}}$ near 0 (see paragraph around (3-30)), any expansion of ϕ in powers of ϱ and δ will yield an expansion of $\varrho\partial_\varrho\phi$. Moreover, the coefficients in the expansion are formally determined by the equation above. More precisely, if one combines the expansion of G and the following expansion induced by (3-40)

$$\begin{aligned} \varrho^2(e^{-\mu\phi} - \delta)^{\frac{1}{n}} &= \left(\sum_{k=1}^N * \varrho^{2k} + O(\varrho^{2N+2}) \right) \cdot \left(1 - \frac{1}{n} \delta e^{\mu b_0} + O(\delta\varrho^2) + O(\delta^2) \right) \\ &= \sum_{k=1}^N * \varrho^{2k} + O(\varrho^{2N+2}) + \delta\varrho^2(* + O(\varrho^2) + O(\delta)), \end{aligned}$$

we get an expansion $\varrho\partial_\varrho\phi = \sum_{k=1}^N * \varrho^{2k} + O(\varrho^{2N+2}) + \delta\varrho^2(* + O(\varrho^2) + O(\delta))$ and we recover the coefficients from (3-40) by integration, i.e.,

$$(3-46) \quad \varrho\partial_\varrho\phi_\beta(r) = \sum_{k=1}^N 2ka_k\varrho^{2k} + O(\varrho^{2N+2}) + (2n-2)a_L \frac{\beta - \beta_*}{\varrho^{2n-2}} \cdot \left(1 + O(\varrho^2) + O\left(\frac{\beta - \beta_*}{\varrho^{2n}}\right) \right).$$

Feeding this into (3-7), we obtain

$$(3-47) \quad (\varrho\partial_\varrho)^2\phi_\beta(\varrho) = \sum_{k=1}^N (2k)^2 a_k \varrho^{2k} + O(\varrho^{2N+2}) + (2n-2)^2 a_L \frac{\beta - \beta_*}{\varrho^{2n-2}} \cdot \left(1 + O(\varrho^2) + O\left(\frac{\beta - \beta_*}{\varrho^{2n}}\right) \right).$$

Combining (3-45) and (3-47) and coming back to the variable r , this leads to

$$(3-48) \quad \omega_\beta = \omega_{\beta_*} + \left(O\left(\frac{\beta - \beta_*}{r^{2n}}\right) + O(r^{2N-2}) \right) \omega_{\beta_*}.$$

3.7 Asymptotics of g_β

In order to analyze the Riemannian metric g_β associated to ω_β , we introduce the variable $s := -\phi'$ which decreases from α (on D , i.e., at $u = -\infty$) to 0 (at the extinction time $u = u_\beta$ of the solution) corresponding to the moment map of the S^1 action on L and write g_β as

$$\begin{aligned} g_\beta &= \phi''(du^2 + 4\eta^2) + \frac{2s}{\alpha - 1} g_D \\ &= \frac{ds^2}{\phi''} + 4\phi''\eta^2 + \frac{2s}{\alpha - 1} g_D, \end{aligned}$$

where η is the connection 1-form, satisfying $d\eta = -\frac{1}{\alpha-1}\omega_D$. Using (3-9)–(3-8), we see that

$$\phi'' = \frac{1}{s^{n-1}}(s^n(b - as) + C_\beta).$$

Given (3-10), we have $\phi''(\alpha) = 0$; after factorizing, we get

$$\phi''(s) = s(\alpha - s) \left(a - (b - a\alpha) \frac{\alpha^n - s^n}{s^n(\alpha - s)} \right).$$

Using $\alpha a = \beta_* + \frac{1}{n+1}(\beta - \beta_*)$ and $b - a\alpha = \frac{1}{n+1}(\beta - \beta_*)$, we eventually find

$$(3-49) \quad \phi'' = s \left(1 - \frac{s}{\alpha} \right) \beta_* \left(1 + \frac{1}{(n+1)\beta_*} (\beta - \beta_*) \cdot \frac{\alpha^{n+1} - s^{n+1}}{s^n(\alpha - s)} \right).$$

Setting $V_\beta(s) = \phi''$, we can rewrite g_β as

$$(3-50) \quad g_\beta = \frac{ds^2}{V_\beta(s)} + V_\beta(s)4\eta^2 + \frac{2s}{\alpha - 1}g_D.$$

Using (3-49), we find:

- When $s \rightarrow \alpha$, $V_\beta(s) \simeq (\alpha - s)\beta$. Setting $r := 2\sqrt{(\alpha - s)/\beta}$, we see that

$$g_\beta \simeq dr^2 + \beta^2 r^2 \eta^2 + \frac{2s}{\alpha - 1}g_D,$$

i.e., g_β has a cone singularity of angle $2\pi\beta$ along D .

- When $s \rightarrow 0$,

$$(3-51) \quad V_\beta(s) \simeq \begin{cases} \frac{\alpha^n}{n+1}(\beta - \beta_*) \frac{1}{s^{n-1}} & \text{if } \beta > \beta_*, \\ s\beta_* & \text{if } \beta = \beta_*. \end{cases}$$

In the case where $\beta > \beta_*$, g_β is an incomplete metric where the circles blow up and the divisor gets contracted. In the case where $\beta = \beta_*$, g_{β_*} is incomplete again and contracts the divisor at infinity D_∞ in L ; that is, $D_\infty = \mathbb{P}(L) \subset \mathbb{P}(L \oplus \mathcal{O}_D)$. The metric g_{β_*} is equivalent near zero to the cone metric

$$g_{\text{cone}} = \frac{ds^2}{s\beta_*} + 4s\beta_*\eta^2 + \frac{2s}{\alpha - 1}g_D = dr^2 + r^2\beta_*^2\eta^2 + \frac{\beta_*r^2}{2(\alpha - 1)}g_D$$

with $r := 2\sqrt{s/\beta_*}$, which is the asymptotic cone of the Tian–Yau metric.

Let us now obtain an expansion in $\beta - \beta_*$ of the metric g_β , which we will be mostly interested in near $s = 0$ where we will glue it to the Tian–Yau metric. So we set

$$\begin{aligned} g_{\beta_*} &:= \frac{1}{s(1 - \frac{s}{\alpha})\beta_*} ds^2 + 4s \left(1 - \frac{s}{\alpha} \right) \beta_* \eta^2 + \frac{2s}{\alpha - 1} g_D \\ &= g_{\text{cone}} + \frac{s}{\alpha} \left(\frac{ds^2}{s\beta_*} - 4s\beta_*\eta^2 \right) + O(s^2) \\ &= g_{\text{cone}} + \frac{\beta_*}{4\alpha} r^2 dr^2 + O(r^4). \end{aligned}$$

The expansion of g_β is given by

$$g_\beta = g_{\beta_*} + (\beta - \beta_*)g_1 + O((\beta - \beta_*)^2),$$

where

$$g_1 = \frac{\alpha^{n+1} - s^{n+1}}{s^n(\alpha - s)} \cdot \frac{1}{(n+1)\beta_*} \left(\frac{-ds^2}{s(1 - \frac{s}{\alpha})\beta_*} + 4s \left(1 - \frac{s}{\alpha} \right) \beta_* \eta^2 \right).$$

3.8 Finding the scaling factor and the gluing region

In order to perform a gluing construction, we need to ensure that most of the domain of the potential ϕ_β be included in an infinitesimally small (i.e., as $\beta \rightarrow \beta_*$) neighborhood of the zero section. This is certainly not the case as such but it can be achieved by using the radial automorphisms of L . More precisely, we use the scaling \mathbb{C}^* action on L by $t \cdot (x, v) = (x, tv)$ to get a new potential $\phi_\beta^t(v) = \phi_\beta(tv)$. In terms of the function u , we have $\phi_\beta^t(u) = \phi_\beta(u - \log |t|^2)$.

From now on we use a diffeomorphism of a small neighborhood of fixed euclidean size of $D \subset X$ with a small neighborhood of the zero section in L to identify both sets. If we glue near u , we have to make sure that the size of D measured with ω_β and $\varepsilon\omega_{TY}$ is the same (here $\varepsilon = \varepsilon_\beta$), which yields

$$(3-52) \quad (-\phi_\beta^t)'(u) \approx \varepsilon e^{-\beta_* u},$$

where \approx means that the log of the quotient is bounded. Given (3-40), we find that in $(\beta - \beta_*)^{\frac{1}{n}} \ll e^{-\beta_* u} \ll 1$ we have $\phi'_\beta(u) \approx e^{-\beta_* u}$; hence (3-52) is satisfied provided

$$\varepsilon \approx |t|^{2\beta_*}, \quad \text{i.e., } \log |t|^2 = \frac{1}{\beta_*} \log \varepsilon + O(1).$$

It will be convenient to introduce the scaling (shrinking) map $\lambda_\varepsilon : L \rightarrow L$ defined by

$$(3-53) \quad \lambda_\varepsilon(v) = \varepsilon^{\frac{1}{2\beta_*}} v.$$

From now on we set

$$\widehat{u}_\beta := -\frac{1}{n\beta_*} \log(\beta - \beta_*).$$

Finally, we set

$$\psi_\beta := \phi_\beta \left(\cdot - \frac{1}{\beta_*} \log \varepsilon \right) = (\lambda_\varepsilon)_* \phi_\beta,$$

which is defined on $(u \leq \widehat{u}_\beta + \frac{1}{\beta_*} \log \varepsilon)$. The asymptotics of $\phi_\beta(v)$ match those of the conic Tian–Yau potential in the region where $1 \ll v \ll -\frac{1}{n\beta_*} \log(\beta - \beta_*)$; see (3-40). Therefore we need to glue $dd^c \psi_\beta(u)$ to $\varepsilon_\beta \omega_{TY}$ near $u = u_\beta$ (i.e., in a zone like $|u - u_\beta| \leq 1$) satisfying

$$\frac{1}{\beta_*} \log \varepsilon_\beta \ll u_\beta \ll \frac{1}{\beta_*} \log \frac{\varepsilon}{(\beta - \beta_*)^{\frac{1}{n}}}.$$

In terms of the Tian–Yau radius $R = 2|v|^{-\beta_*}$, this means

$$(3-54) \quad (\beta - \beta_*)^{\frac{1}{n}} \ll \varepsilon R^2 \ll 1.$$

In the gluing zone, (3-27) guarantees that $s(u) = -\phi'(u - \frac{1}{\beta_*} \log \varepsilon) \simeq \varepsilon R^2$; hence $s^n \gg \beta - \beta_*$ by (3-54). Combined with (3-49), this shows that

$$(3-55) \quad g_\beta \approx g_{\text{cone}} = \frac{ds^2}{s\beta_*} + 4s\beta_*\eta^2 + \frac{2s}{\alpha - 1} g_D.$$

4 Automorphisms and related objects

4.1 Relations between $\text{Aut}^\circ(X, D)$ and $\text{Aut}^\circ(D)$

Let $Y := \mathbb{P}(L \oplus \mathcal{O}_D)$, let $p : Y \rightarrow \bar{L}$ be the contraction of the divisor at infinity $D_\infty := \mathbb{P}(L \oplus 0) \subset Y$ and let $x_\infty = p(D_\infty) \in \bar{L}$; see Section 2 for the definition of \bar{L} . The zero section $D := \mathbb{P}(L \oplus 1)$ can be equivalently seen inside \bar{L} or Y . We have a natural sequence

$$(4-1) \quad 1 \rightarrow \mathbb{C}^* \rightarrow \text{Aut}^\circ(\bar{L}, D) \rightarrow \text{Aut}^\circ(D) \rightarrow 1,$$

where the action of \mathbb{C}^* is given on Y by $\lambda \cdot [v : z] := [\lambda v : z]$; it obviously preserves $D = (v = 0)$ and $D_\infty = (z = 0)$, and hence descends to an action of \mathbb{C}^* on \bar{L} preserving D . The second arrow is simply the restriction to D .

Lemma 4.1 *The short sequence (4-1) is exact.*

Proof The rightmost arrow is exact since any automorphism of D induces a linear automorphism of $L = (\alpha - 1)K_{\bar{D}}^{-1}$. It remains to see that we have exactness in the middle. So let $f \in \text{Aut}^\circ(\bar{L}, D)$ acting trivially on D ; we need to show that f comes from the \mathbb{C}^* action. First, f obviously preserves the singular locus $\bar{L}^{\text{sing}} = \{x_\infty\}$. Since p is simply the blow up of $x_\infty \in \bar{L}$, f lifts to an automorphism $\tilde{f} \in \text{Aut}^\circ(Y)$ acting trivially on D . A classical argument due to Blanchard [11] shows that any element in the identity component of the automorphism group of a fiber bundle with compact fibers fixing the base has to preserve the fiber bundle structure. In our case it means that \tilde{f} preserves the \mathbb{P}^1 fibers of the projection $\rho : Y \rightarrow D$. Given $x \in D$, \tilde{f} acts on the fiber $\rho^{-1}(x) \simeq \mathbb{P}^1$ preserving 0 (i.e., $D \cap \rho^{-1}(x)$) and ∞ (i.e., $D_\infty \cap \rho^{-1}(x)$), and hence it can be written as an homothety $[u : v] \mapsto [a(x)u, v]$. The function a defines a holomorphic map $D \rightarrow \mathbb{C}^*$, and hence it is constant. \square

Lemma 4.2 *Assume $(X, D) \neq (\mathbb{P}^n, H)$ for H a hyperplane and consider the morphism ρ_D induced by restriction to D*

$$(4-2) \quad \rho_D : \text{Aut}^\circ(X, D) \rightarrow \text{Aut}^\circ(D).$$

The following hold.

- (i) ρ_D has finite kernel.
- (ii) If X is toric, then ρ_D is a surjective, finite étale cover.

Proof Since the groups at stake are algebraic, the lemma can be rephrased by saying that

$$H^0(X, T_X(-\log D)) \rightarrow H^0(D, T_D)$$

is injective (first item) and surjective when X is toric (second item). We have the exact sequence

$$(4-3) \quad 0 \rightarrow T_X(-D) \rightarrow T_X(-\log D) \rightarrow T_D \rightarrow 0.$$

By a result of J. Wahl [42], we have $H^0(X, T_X \otimes \mathcal{O}_X(-D)) = 0$ since $(X, D) \neq (\mathbb{P}^n, H)$; this shows (i).

Using (4-3) again, (ii) reduces to showing that $H^1(X, T_X(-D)) = 0$. Since $T_X \simeq \Omega_X^{n-1} \otimes K_X^{-1}$, we have to show that

$$H^1(X, \Omega_X^{n-1} \otimes L) = 0,$$

where $L = -K_X - D \simeq (\alpha - 1)D$ is an ample line bundle. But this follows precisely from Bott’s vanishing theorem for toric varieties; see [34, §3.3]. □

If D is allowed to be singular, then ρ_D need not be isomorphic as shown by the following example, which was communicated to us by Thibaut Delcroix.

Example 4.3 Let $X = Q_4 = \{\sum_{i=0}^5 z_i^2 = 0\} \subset \mathbb{P}^5$ be the four-dimensional smooth quadric and let $D = \{z_0^2 + z_1^2 + z_2^2 = 0 = z_3^2 + z_4^2 + z_5^2\} \subset X$. Then D is irreducible and reduced, with an action of $SO_3(\mathbb{C}) \times SO_3(\mathbb{C})$ with dense orbit, and satisfies $-K_X \sim_{\mathbb{Z}} 2D$. But the one-parameter family of automorphisms

$$t \cdot [z_0 : \cdots : z_5] = [tz_0 : tz_1 : tz_2 : t^{-1}z_3 : t^{-1}z_4 : t^{-1}z_5]$$

does not extend to $PSO_6(\mathbb{C})$. Moreover, one can check that D admits a Kähler–Einstein metric combining [17; 29].

4.2 About the normal bundle

In this section, we investigate whether the formal neighborhood of D in X of order one (that is, the ringed space $(D, \mathcal{O}_X/\mathcal{I}_D^2)$) coincides with the formal neighborhood of D in L (that is, the ringed space $(D, \mathcal{O}_L/\mathcal{I}_D^2)$). From [1, Proposition 1.5], this is equivalent to the normal short exact sequence

$$(4-4) \quad 0 \rightarrow T_D \rightarrow T_X|_D \rightarrow N_D \rightarrow 0$$

being split. A sufficient condition for the splitting of the sequence (4-4) to happen is to have the vanishing $H^1(D, T_D \otimes N_D^{-1}) = 0$ as we see by tensorizing the above sequence with N_D^{-1} and observing that extensions of \mathcal{O}_D by $T_D \otimes N_D^{-1}$ are parametrized by the image of $1 \in H^0(D, \mathcal{O}_D)$ in $H^1(D, T_D \otimes N_D^{-1})$ under the connecting homomorphism in the long exact sequence in cohomology associated to (4-4). We give a couple of situations where (4-4) is split or nonsplit.

Example 4.4 If $X = \mathbb{P}^n$, then a theorem of van de Ven [41] shows that (4-4) is split if and only if D is a linear subspace of \mathbb{P}^n . Alternatively, if (4-4) is split, then T_D is a quotient of the ample bundle $T_{\mathbb{P}^n}|_D$; hence it is ample, too. By Mori’s theorem [33] (or Siu and Yau’s proof of Frankel’s conjecture [36]), we have $D \simeq \mathbb{P}^{n-1}$.

Example 4.5 If X is toric and $\alpha > 2$, then (4-4) is split if and only if $(X, D) = (\mathbb{P}^n, H)$ where H is a hyperplane. Indeed, we have

$$H^1(X, T_X(-2D)) \simeq H^1(X, \Omega_X^{n-1} \otimes K_X^{-1}(-2D)) = H^1(X, \Omega_X^{n-1} \otimes \mathcal{O}_X((\alpha - 2)D)) = 0$$

by Bott’s vanishing theorem. In particular, the exact sequence

$$0 \rightarrow T_X(-2D) \rightarrow T_X(-D) \rightarrow T_X|_D \otimes N_D^{-1} \rightarrow 0$$

shows that any section of $T_X|_D \otimes N_D^{-1}$ extends to a section of $T_X(-D)$. Now, if (4-4) splits, we precisely get a nonzero section $H^0(D, T_X|_D \otimes N_D^{-1})$, and hence we obtain a nonzero section $H^0(X, T_X(-D))$. By [42], this implies that (X, D) is isomorphic to (\mathbb{P}^n, H) .

Example 4.6 If D is toric and $\alpha > 2$, then (4-4) is always split. Indeed, we have

$$H^1(D, T_D \otimes N_D^{-1}) \simeq H^1(D, \Omega_D^{n-2} \otimes \mathcal{O}_D((\alpha - 2)D)) = 0$$

by Bott’s vanishing theorem.

Example 4.7 If $X = Q_n \subset \mathbb{P}^{n+1}$ is the n -dimensional quadric and $D = Q_{n-1} = Q_n \cap H$ for a hyperplane H , then we have $K_X^{-1} = \mathcal{O}_X(n) \sim nD$, i.e., $\alpha = n$, and D is Kähler–Einstein. Moreover, for any $m \geq 1$, we have the vanishing $H^1(Q_m, T_{Q_m}(-1)) \simeq H^1(Q_m, \Omega_{Q_m}^{m-1}(m-1)) = 0$ by [37, p. 174]. Apply this to $m = n - 1$ to get that (4-4) is split. Conversely, if $D \subset Q_n$ is a hypersurface such that (4-4) splits, then $D \simeq Q_{n-1}$ by [24, Theorem 4.7].

5 The tubular neighborhood of D

Since $-K_X \sim \alpha D$, the normal bundle $L = N_D$ of D in X has a canonical volume form, which in local coordinates (z^0, \dots, z^{n-1}) such that $D = \{z^0 = 0\}$ can be written as

$$(5-1) \quad \Omega_L = v(z^1, \dots, z^{n-1}) \frac{dz^0}{(z^0)^\alpha} \wedge dz^1 \wedge \dots \wedge dz^{n-1}.$$

This is not completely correct, since $(z^0)^\alpha$ a priori does not make sense: if we write $\alpha = \frac{p}{q}$ then we only have a section Ω^q of $K_X^q(pD)$ on X . But since at the end we need only to consider the real volume form $\Omega \wedge \bar{\Omega}$ which is well defined as $(\Omega^q \wedge \bar{\Omega}^q)^{\frac{1}{q}}$, we will simplify the notation and formulas by writing Ω (and Ω_L on L).

5.1 The complex structure

Let $\Delta_L \subset L$ be the disc normal bundle of D . The connection 1-form η on $L \setminus D$ gives a horizontal space $H \subset T\Delta_L$ which is transverse to each disc. If we have another complex structure J on Δ_L , we compare it to the complex structure J_L of L by a tensor $\phi \in \Omega_{J_L}^{0,1} \otimes T_{J_L}^{1,0}$ so that

$$T_J^{0,1} = \{X + \phi X, X \in T_{J_L}^{0,1}\}.$$

The integrability of J gives the equation $\bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0$, where the bracket is constructed from the exterior product of forms and from the bracket of vector fields.

We will use this canonical parametrization of a small tubular neighborhood of D by Δ_L [6, Theorem 4.1]:

Proposition 5.1 *There exists a small neighborhood U_L of D in X and a diffeomorphism $\Upsilon : \Delta_L \rightarrow U_L$ such that $\phi = \Upsilon^* J_X - J_L \in \Omega^{0,1}(T^{1,0})$ is a section of $H_{0,1} \otimes H^{1,0}$ (that is, ϕ is purely horizontal) which satisfies $\phi \lrcorner d\eta = 0$, $\phi|_D = 0$ and is holomorphic along the discs of Δ_L .*

Moreover $\Upsilon^* \Omega = v(1 - \phi)^* \Omega_L$ where v is a function on Δ_L which is holomorphic along the discs and satisfies $v|_D = 1$. □

5.2 The complex volume form

The meaning of Proposition 5.1 is that any neighborhood of D which is close enough to a disc bundle Δ_L carries a unique fibration by holomorphic discs satisfying the conclusions of the proposition. But there is no particular choice of U_L . Therefore we get an additional freedom by perturbing U_L . Using this flexibility, one should be able to obtain $v \equiv 1$. As we need only finite developments, it will be sufficient to prove the following proposition. We denote by ζ the variable in the total space of L .

We decompose $T_{\mathbb{C}}L = \mathbb{C}\zeta\partial_{\zeta} \oplus T_{\mathbb{C}}D$. In the following proposition we will measure the norm of an endomorphism of TL with respect to this decomposition (that is, we consider $\zeta\partial_{\zeta}$ as having norm 1).

Proposition 5.2 *There are purely horizontal tensors $\phi_i \in \Omega_D^{0,1}(T^{1,0}D \otimes L^{-i})$ for $i > 0$ such that $\phi_i \lrcorner d\eta = 0$ and the following is true.*

Given any $j > 0$, one can choose a diffeomorphism $\Upsilon : \Delta_L \rightarrow U_L$ such that:

- (1) $\phi = \Upsilon^*J_X - J_L$ satisfies $\phi = \sum_1^{j-1} \zeta^i \phi_i + O(\zeta^j)$;
- (2) $\Upsilon^*\Omega = v(1 - \phi)^*\Omega_L$ with $v = 1 + O(\zeta^j)$,

where $O(\cdot)$ is meant with derivatives.

This statement is weaker than Proposition 5.1 for the complex structure ϕ but gives more information on v . It is likely that one can obtain both the gauge of Proposition 5.1 and $v \equiv 1$, but this problem may involve a loss of derivatives which makes it difficult, so we will prove only Proposition 5.2.

The rest of this subsection is devoted to the proof of the proposition. So we are on the disc bundle Δ_L , with a complex structure J whose difference with J_L is given by the tensor ϕ as described in Proposition 5.1.

We consider the infinitesimal action of the real part of the $(1, 0)$ vector field (for J_L)

$$(5-2) \quad \xi = fi\zeta\partial_{\zeta} + X,$$

where $X \in H^{1,0}$ is horizontal, and $i\zeta\partial_{\zeta} = R^{1,0}$ is the $(1, 0)$ part of the Reeb vector field $R = \frac{d}{d\theta}$. The infinitesimal action on the complex structure J_L is

$$(5-3) \quad \bar{\partial}\xi = 2\eta^{0,1} \otimes (\bar{\zeta}\partial_{\bar{\zeta}}f\zeta\partial_{\zeta} + \bar{\zeta}\partial_{\bar{\zeta}}X) + (\bar{\partial}_H f + X \lrcorner d\eta) \otimes i\zeta\partial_{\zeta} + \bar{\partial}_H X.$$

Observe that if $X = -\sharp\bar{\partial}_H f$ and f is holomorphic along the fibers ($\partial_{\bar{\zeta}}f = 0$), then $\bar{\partial}\xi = -\bar{\partial}_H\sharp\bar{\partial}_H f$ is purely horizontal, that is, the vector field

$$(5-4) \quad \xi_f = fi\zeta\partial_{\zeta} - \sharp\bar{\partial}_H f$$

preserves infinitesimally the gauge of ϕ (of Proposition 5.1) if f is holomorphic along the fibers. This is the flexibility alluded to on U_L since f is determined by its value on the boundary of U_L and corresponds to perturbing this boundary. (This is actually the complexification of the action of the contactomorphisms of $\partial\Delta_L$ which was studied in [7, §7]).

Let us now investigate the action of such deformations on the function v . The model holomorphic volume form (5-1) can be written as

$$(5-5) \quad \Omega_L = \eta^{1,0} \wedge \Omega_D, \quad \Omega_D \in H^0(K_D L^{\alpha-1}).$$

Recall that $-K_D = (\alpha - 1)L$ so the section Ω_D is canonical up to a multiplicative constant that we can choose so that it satisfies

$$(5-6) \quad i^{(n-1)^2} \Omega_D \wedge \overline{\Omega_D} = 8b_{n,\alpha} e^{-n\beta_* u} \omega_D^{n-1},$$

where $b_{n,\alpha}$ was defined in (3-43). The holomorphic volume form for the complex structure J can be written as

$$(5-7) \quad \Omega = v(1 - \phi)^*(\eta^{1,0} \wedge \Omega_D)$$

for some function v . Then the infinitesimal action of ξ_f is given by

$$(5-8) \quad \begin{aligned} (\mathcal{L}_{\xi_f} \Omega_L)^{1,0} &= \partial(\xi_f \lrcorner \Omega_L) \\ &= \partial\left(\frac{1}{2} f \Omega_D + \eta^{1,0} \wedge \sharp \bar{\partial}_H f \lrcorner \Omega_D\right) \\ &= \eta^{1,0} \wedge (i \zeta \partial_\zeta (f \Omega_D) - \partial_H (\sharp \bar{\partial}_H f \lrcorner \Omega_D)) \\ &= i \eta^{1,0} \wedge (\zeta \partial_\zeta (f \Omega_D) - 2 \Delta_\partial (f \Omega_D)). \end{aligned}$$

We now prove the proposition by induction on j . From Proposition 5.1 we know that Proposition 5.2 is true for $j = 1$. (One of course does not need the full strength of Proposition 5.1, it is sufficient to construct by hand a good enough Υ). So suppose it is true for j and let us prove it for $j + 1$.

We begin by rectifying ϕ and constructing ϕ_j . Let ϕ' contain the order- j terms of ϕ , that is, $\phi = \sum_1^{j-1} \phi_i \zeta^i + \phi' + O(\zeta^{j+1})$. We first consider the infinitesimal problem, so we consider the infinitesimal action of for some vector field $\xi = i f \zeta \partial_\zeta + X$ of order j . Denote by $\phi'_{HH}, \phi'_{HV}, \phi'_{VH}$ and ϕ'_{VV} the various components of ϕ' in the decomposition $T_{\mathbb{C}} = \mathbb{C} \zeta \partial_\zeta \oplus H_{\mathbb{C}}$. From the first term in (5-3) we see that we can find $\xi = f i \zeta \partial_\zeta + X$ of order j so that $(\bar{\partial} \xi)_{VH} = \phi'_{VH}$ and $(\bar{\partial} \xi)_{VV} = \phi'_{VV}$; note that f and X are defined up to fiberwise holomorphic terms, that is, up to terms $\zeta^j (f_j i \zeta \partial_\zeta + X_j)$. Thanks to the action of ξ we are now reduced to the case where $\phi'_{VH} = 0$ and $\phi'_{VV} = 0$, that is, ϕ' has the form

$$(5-9) \quad \phi' = \tilde{\phi} + \alpha \otimes i \zeta \partial_\zeta + \phi'_{HH},$$

where $\alpha \in H_{0,1}$ and ϕ'_{HH} are terms of order j . The integrability equation $\bar{\partial} \phi' + \frac{1}{2} [\phi', \phi'] = 0$ implies

$$(5-10) \quad \bar{\zeta} \partial_{\bar{\zeta}} \phi' = O(\zeta^{j+1})$$

and in particular $\partial_{\bar{\zeta}} \alpha = 0$ and $\partial_{\bar{\zeta}} \phi'_{HH} = 0$. Defining a horizontal vector field X by $X \lrcorner d\eta = \alpha$ we have from (5-3)

$$\bar{\partial} X = \alpha \otimes i \zeta \partial_\zeta + \bar{\partial}_H X$$

and therefore the infinitesimal action of X kills the residual term α .

Summarizing, we get a vector field ξ of order j which infinitesimally brings ϕ' to a purely horizontal $\tilde{\phi}_j$. Choosing some smooth Kähler metric near D and its exponential, we can now apply the diffeomorphism $\exp(\Re \xi)$ to (J, Ω) : we consider $\Upsilon' = \Upsilon \circ \exp(\Re \xi)$ and the corresponding pair (ϕ', v') . Since the dominant term of the action is given by the infinitesimal action of ξ_f (the exponential gives only higher-order terms) this implies that:

- $\phi' = \sum_1^{j-1} \phi_j \zeta^j + \tilde{\phi}_j + O(\zeta^{j+1})$ with $\tilde{\phi}_j$ purely horizontal;
- we still have $v' = 1 + O(\zeta^j)$ since ξ has order j .

Also note that (5-10) implies that $\tilde{\phi}_j$ is holomorphic with respect to ζ , that is, has the required form $\zeta^j \tilde{\phi}_j$. So we have proved the induction for ϕ , and there remains to rectify the function v .

Since ξ is obtained by solving a problem disk per disk, it a priori has the same regularity as ϕ' , when one would expect a better regularity (one more derivative on X and two more on f). This explains the loss of derivatives alluded to above.

We now must kill the order- j term of v' . From $\bar{\partial} \Omega = 0$ and the fact that ϕ is purely horizontal up to order j , we deduce that the j -th order term of v' is holomorphic along the fiber (that is, with respect to ζ), which means that it can be written as $v'_j \zeta^j$ for a section v'_j of L^{-j} . From the Weitzenböck formula $2\Delta_\partial = \nabla^* \nabla + j$ on sections of $K_D L^{\alpha-j-1}$, we deduce, for a section $f \Omega_D$ of $K_D L^{\alpha-j-1}$,

$$(5-11) \quad -\zeta \partial_\zeta (f \Omega_D) + 2\Delta_\partial (f \Omega_D) = (\nabla^* \nabla + \alpha - 1)(f \Omega_D).$$

Since $\alpha > 1$ the operator $\nabla^* \nabla + \alpha - 1$ is an isomorphism, and therefore the term $v'_j \zeta^j$ can be killed infinitesimally by a vector ξ_f for some $f = f_j \zeta^j$ and $f_j \Omega_D$ is a section of L^{-j} . We now apply as above the diffeomorphism $\exp(\Re \xi_f)$ to (J, Ω) : we obtain a new pair (ϕ'', v'') . Since the dominant term of the action is given by the infinitesimal action of ξ_f , this implies that

- the order- j term of v' , that is, $v'_j \zeta^j$, is killed by $\exp(\Re \xi_f)$, so we have now $v'' = 1 + O(\zeta^{j+1})$;
- as observed above, the vector field ξ_f infinitesimally preserves the horizontal gauge, it actually modifies the order- j term of ϕ' by $-\bar{\partial}_H \# \bar{\partial}_H f$ so we get an order- j term $\zeta^j (\tilde{\phi}_j + \bar{\partial}_H \# \bar{\partial}_H f_j)$.

So by choosing $\phi_j = \tilde{\phi}_j + \bar{\partial}_H \# \bar{\partial}_H f_j$ we have finished to prove the proposition for $j + 1$. □

Proposition 5.3 *Given the normal form from Proposition 5.2, the smallest integer j_0 such that $\phi_{j_0} \neq 0$ is intrinsic. Actually j_0 is the largest integer such that there exists a diffeomorphism $\Upsilon : \Delta_L \rightarrow \mathcal{U}_L$ such that $\Upsilon^* \Omega$ and Ω_L coincide up to order $j_0 - 1$.*

Proof Let ℓ be the order of coincidence of Ω and Ω_L as defined in the proposition, and j_0 the first integer so that $\phi_{j_0} \neq 0$. By definition we have $\ell \geq j_0 - 1$. On the other hand, if Ω and Ω_L coincide up to order ℓ , then the complex structures J and J_L also coincide up to order ℓ , since they are determined by the holomorphic forms. If $j_0 < \ell + 1$ it therefore means that our term $\zeta^{j_0} \phi_{j_0}$ can be killed by a diffeomorphism: as it is the first nonvanishing term, it means that it can be killed by an infinitesimal diffeomorphism which also preserves the condition $v = 1$. The infinitesimal action of diffeomorphisms was calculated in the proof of Proposition 5.2, and it turns out that at each step the vector field is unique

(to preserve the normal form of ϕ one needs $\xi = \xi_f$ with f holomorphic along the fibers; then the condition $v = 1$ kills this remaining degree of freedom). Therefore it is impossible to kill the first term by an infinitesimal diffeomorphism, so we must have $\ell = j_0 - 1$. \square

Remark 5.4 The $T_D^{1,0}$ -valued $(0, 1)$ -form $\phi_1 \zeta$ is $\bar{\partial}$ -closed and represent the extension class κ of N_D by T_D in $H^1(D, T_D \otimes N_D^{-1})$. In particular, we have $j_0 = 1$ if and only if $\kappa \neq 0$, or equivalently, if and only if the normal exact sequence (4-4) does not split; see the first paragraph of Section 4.2.

5.3 The real volume form

Using Propositions 5.1 and 5.2 we can now give an optimal development of the volume form $\Omega \wedge \bar{\Omega}$ by comparison with $\Omega_L \wedge \bar{\Omega}_L$. We develop the tensor ϕ from Proposition 5.1 in powers of ζ as

$$(5-12) \quad \phi = \phi_1 \zeta + \phi_2 \zeta^2 + \dots .$$

One can identify ϕ_j with a section on D of $\Omega_D^{0,1}(T_D^{1,0} \otimes L^{-j})$. This is valid up to some order which we can take large enough for what follows.

Suppose for simplicity that $\phi_1 \neq 0$. The linearization with respect to ϕ of $(1 - \phi)^* \Omega_L$ is $\phi \lrcorner \Omega_L$: here $\phi \in \Omega^{0,1} \otimes T^{1,0}$, $\Omega_L \in \Omega^{n,0}$ and $\phi \lrcorner \Omega_L \in \Omega^{n-1,1}$. It follows from Proposition 5.2 that

$$\Upsilon^* \Omega = \Omega_L - \phi_1 \zeta \lrcorner \Omega_L + O(\zeta^2).$$

Therefore

$$\begin{aligned} \Upsilon^*(\Omega \wedge \bar{\Omega}) &= \Omega_L \wedge \bar{\Omega}_L + (\phi_1 \zeta \lrcorner \Omega_L) \wedge \overline{(\phi_1 \zeta \lrcorner \Omega_L)} + O(\zeta^3) \Omega_L \wedge \bar{\Omega}_L \\ &= \Omega_L \wedge \bar{\Omega}_L (1 - |\phi_1 \zeta|^2 + O(\zeta^3)). \end{aligned}$$

The essential feature in this formula is that no linear term in ϕ_1 appears because $\phi_1 \lrcorner \Omega_L$ is of type $(n - 1, 1)$. This makes possible for the correction term to exhibit a sign which will be important for us.

The same calculation when the first nonzero term is ϕ_{j_0} leads to:

Proposition 5.5 *Suppose Υ is given by Proposition 5.2 and that the first nonzero term in (5-12) is of order j_0 . Then*

$$\Upsilon^*(\Omega \wedge \bar{\Omega}) = \Omega_L \wedge \bar{\Omega}_L (1 - |\phi_{j_0} \zeta^{j_0}|^2 + O(\zeta^{2j_0+1})). \quad \square$$

6 The Tian–Yau metric and its formal perturbation

Fix a Kähler–Einstein metric ω_D on D such that $\text{Ric } \omega_D = \omega_D$ and a Hermitian metric h on $L = N_D$ such that $i \Theta(L, h) = \frac{1}{\alpha-1} \omega_D$. Let σ be the section of $[D]$ vanishing on D , and Ω the holomorphic volume form on $X \setminus D$, normalized as in Section 5.2 (again, only some power is well defined). The Tian–Yau metric [40] of (X, D) is defined in the following way: there is a unique Ricci flat Kähler metric $\omega_{\text{TY}} = dd^c \Phi_{\text{TY}}$ on $X \setminus D$, solving the Monge–Ampère equation

$$(6-1) \quad (dd^c \Phi_{\text{TY}})^n = i^{n^2} \Omega \wedge \bar{\Omega}$$

and such that near D ,

$$(6-2) \quad \Phi_{\text{TY}} \sim |\sigma|^{-2\beta_*}.$$

Here it may be necessary to adjust the metric on L by a multiplicative constant in order to obtain exactly (6-1).

The asymptotic Calabi–Yau cone of the Tian–Yau metric is easily described using the formalism of Section 3.1: it is the bundle L equipped with the cone metric

$$(6-3) \quad \omega_{\text{TY},L} = dd^c e^{-\beta_* u}.$$

Indeed from (5-5) and (5-6) the holomorphic volume form satisfies

$$(6-4) \quad \begin{aligned} i^{n^2} \Omega_L \wedge \overline{\Omega}_L &= 8b_{n,\alpha} i e^{-n\beta_* u} \eta^{1,0} \wedge \eta^{0,1} \wedge \omega_D^{n-1} \\ &= b_{n,\alpha} e^{-n\beta_* u} du \wedge d^c u \wedge \omega_D^{n-1} \\ &= b_{n,\alpha} \Xi. \end{aligned}$$

Comparing with (3-5), where $-\phi'_{\text{TY},L}(u) = \beta_* e^{-\beta_* u}$, we obtain exactly

$$(6-5) \quad \omega_{\text{TY},L}^n = i^{n^2} \Omega_L \wedge \overline{\Omega}_L,$$

which finishes the description of the Calabi–Yau cone. In particular the Calabi–Yau cone has radius $R = 2e^{-\frac{1}{2}\beta_* u} = 2|\sigma_L|^{-\beta_*}$, where σ_L is the tautological section of L over L .

6.1 Asymptotic expansion of the Tian–Yau potential

We now use the diffeomorphism $\Upsilon : \Delta_L \rightarrow U_L$ of Section 5 to give an asymptotic expansion of Φ_{TY} . It will be convenient to distinguish objects on Δ_L from objects pulled-back by Υ by an index (for example, d^c is for the complex structure J on U_L and $d^c_{J_L}$ for J_L on L). To simplify notation, we will use Proposition 5.2 as if $j = \infty$, that is, as if ϕ was purely horizontal at all orders: indeed taking the order j large enough in the proposition leads only to small error terms which do not play any role; see Section 8 where the error terms are given.

On L we have the exact potential $\Phi_0 = |\sigma_L|^{-2\beta_*} = R^2/4$. Because the tensor ϕ expressing the difference $\Upsilon^* J - J_L$ is horizontal and Φ_0 depends only on R (or $|\sigma_L|$), it follows that

$$dd^c \Phi_0 = dd^c_{J_L} \Phi_0.$$

If we use Φ_0 in (6-1), the first error therefore comes from the difference between $\Omega \wedge \overline{\Omega}$ and $\Omega_L \wedge \overline{\Omega}_L$:

$$(6-6) \quad \frac{(dd^c \Phi_0)^n}{i^{n^2} \Omega \wedge \overline{\Omega}} = 1 + |\phi_{j_0}|^2 + O(|\sigma|^{2j_0+1}).$$

Extend Φ_0 as a potential on $X \setminus D$ and define

$$\omega_0 = dd^c \Phi_0 \quad \text{and} \quad g = \frac{(dd^c \Phi_0)^n}{i^{n^2} \Omega \wedge \overline{\Omega}},$$

so that the expansion of g at infinity is given by the right-hand side of (6-6). To find the exact solution Φ_{TY} we must solve the Monge–Ampère equation on $X \setminus D$

$$(6-7) \quad (\omega_0 + dd^c f)^n = g\omega_0^n.$$

Observe that $|\phi_{j_0}|^2 = O(|\sigma|^{2j_0}) = O(R^{-\frac{2j_0}{\beta_*}})$, and that since $\alpha < n + 1$, the decay rate satisfies

$$(6-8) \quad \frac{2j_0}{\beta_*} > 2j_0 \geq 2.$$

We can now apply [15, Theorems 2.1 and 2.4]. There are two cases, depending whether the decay rate $\frac{2j_0}{\beta_*}$ allows the Green’s function (R^{-2n+2}) to be the first perturbation of Φ_0 , and an additional limit case:

- If $\frac{2j_0}{\beta_*} < 2n$, that is, if $\alpha \geq 1 + j_0$, then one can solve (6-7) with $f = O(R^{-\frac{2j_0}{\beta_*} + 2})$ (and the corresponding weighted decays for the derivatives).
- If $\frac{2j_0}{\beta_*} > 2n$, that is, if $\alpha < 1 + j_0$, then one can solve (6-7) with

$$(6-9) \quad f = \frac{a}{R^{2n-2}} + O(R^{-2n+2-\varepsilon}).$$

- If $\frac{2j_0}{\beta_*} = 2n$, we have an indicial root of the Laplacian (corresponding to the decay R^{-2n+2} of the Green function) so we need to first kill by hand the term of order R^{-2n} of g : we correct the potential by a function $f_0 = aR^{-2n+2} \log R + O(R^{-2n+2})$ so that $\omega_0 + dd^c f_0$ now satisfies (6-7) with $g - 1 = O(R^{-2n-\varepsilon})$, then we can proceed as in the previous case.

In each case the solution is unique. We will introduce the notation

$$v_0 := \frac{j_0}{\beta_*}$$

so that the qualitative asymptotics of the solution f depend on the value of v_0 in relation to n . This gives the first part of the following result:

Theorem 6.1 *The potential of the Tian–Yau metric on $X \setminus D$ has the expansion*

$$(6-10) \quad \Upsilon^* \Phi_{\text{TY}} = \frac{1}{4} R^2 + \begin{cases} O(R^{-2v_0+2}) & \text{if } v_0 < n, \\ a \frac{\log R}{R^{2n-2}} + O(R^{-2n+2}) & \text{if } v_0 = n, \\ \frac{a}{R^{2n-2}} + O(R^{-2n+2-\varepsilon}) & \text{if } v_0 > n, \end{cases}$$

where $\varepsilon > 0$ is small. Moreover in the last two cases we have $a > 0$.

Proof Here we work with the Tian–Yau metric as background metric. There remains to prove the statement on the sign of a . The two cases $v_0 = n$ and $v_0 > n$ are different in nature, since the argument is local in the first case, and global in the second case.

In the first case ($\nu_0 = n$) the coefficient a is formally determined by the asymptotic terms of (6-7): the linearization of the left-hand side of (6-7) is $-\Delta f$, and on the asymptotic cone

$$-\Delta(R^{-2n+2} \log R) = (2n - 2)R^{-2n+2};$$

since $g = 1 + bR^{-2n+2} + O(R^{-2n+2-\varepsilon})$ with $b > 0$ (see (6-6)) we have $a > 0$.

The second case ($\nu_0 > n$) requires more work. For a one-form u on $X \setminus D$ denote by $\nabla^- u$ the J -anti-invariant part of ∇u :

$$\begin{aligned} (\nabla^- u)_{X,Y} &= \frac{1}{2}((\nabla_X u)_Y - (\nabla_{JX} u)_{JY}) \\ &= -\frac{1}{2}\omega((\mathcal{L}_{\sharp u} J)X, Y). \end{aligned}$$

Here \sharp denotes the Riemannian duality between TX and T^*X . In particular $\nabla^- u = 0$ if and only if $\sharp u$ is a holomorphic vector field. Given a function f the $(0, 2)$ part of $\nabla^- df$ is the familiar operator $\bar{\partial}\sharp\bar{\partial}f$. Moreover one has the Weitzenböck type formula [5, (2.51) and (2.53)]

$$(6-11) \quad \delta \nabla^- u = \frac{1}{2} \Delta u - \text{Ric } u.$$

The Tian–Yau potential $\Phi = \Phi_{\text{TY}}$ satisfies $\Delta \Phi = -4n$ and we deduce from (6-11) that

$$(6-12) \quad \delta \nabla^- d\Phi = \frac{1}{2} \Delta d\Phi = \frac{1}{2} d\Delta \Phi = 0.$$

By integration by parts, denoting by \vec{n} the outward normal vector to the hypersurface S_R of level R :

$$\begin{aligned} (6-13) \quad 0 &= \int_{X \setminus D} \langle \delta \nabla^- d\Phi, d\Phi \rangle \\ &= \int_{X \setminus D} |\nabla^- d\Phi|^2 - \lim_{R \rightarrow \infty} \int_{S_R} \nabla^- d\Phi(\vec{n}, \nabla \Phi) \vec{n} \lrcorner d\text{vol}. \end{aligned}$$

From the development (5-12) of the complex structure and the development (6-10) of the potential, and defining $p = \inf(2n, \frac{j_0}{\beta_*})$, we have

$$\begin{aligned} g &= g_{\text{cone}} + O(R^{-p}), \\ \vec{n} &= \partial_R + O(R^{-p}), \\ \nabla \Phi &= 2R\partial_R + O(R^{-p+1}), \\ \nabla^- dR^2 &= O(R^{-p}). \end{aligned}$$

Under our hypothesis that $\frac{2j_0}{\beta_*} > 2n$, we have that $p > n$ and therefore

$$(6-14) \quad \lim_{R \rightarrow \infty} \int_{S_R} \nabla^- dR^2(\vec{n}, \nabla \Phi) \vec{n} \lrcorner d\text{vol} = \lim_{R \rightarrow \infty} \int_{S_R} \nabla^- dR^2(\partial_R, 2R\partial_R) \partial_R \lrcorner d\text{vol}$$

since the other (quadratic) terms are too small to give a nonzero limit. But the term at order R^{-j_0/β_*} of $\nabla^- dR^2$ is linear in ϕ_{j_0} and $\bar{\phi}_{j_0}$, so has Fourier coefficients of orders only $\pm j_0$ so its integral on S_R against constant terms vanishes.

Therefore our boundary term reduces to

$$(6-15) \quad \lim_{R \rightarrow \infty} \int_{S_R} a \nabla^- dR^{-2n+2} (\partial_R, 2R\partial_R) \partial_R \lrcorner d \text{vol}.$$

But

$$\nabla^- dR^{-2n+2} = 4n(n-1) \frac{((dR)^2)^-}{R^{2n}} + \text{l.o.t.}$$

and it follows that the limit in (6-15) is $a\ell$ for some $\ell > 0$. Coming back to (6-13) we finally obtain

$$(6-16) \quad \int_{X \setminus D} |\nabla^- d\Phi|^2 = a\ell$$

and it follows that $a \geq 0$, with $a = 0$ if and only if $\nabla^- d\Phi = 0$, that is, if $\nabla\Phi$ is a (real) holomorphic vector field and an homothety of the metric. Since $\nabla\Phi \sim 2R\partial_R$ we have a cone and $\nabla\Phi$ is the dilation vector field. Considering the asymptotics at infinity, the cone can only be the asymptotic cone of the Tian–Yau metric, that is, $\bar{L} \setminus D$, where \bar{L} is the one point compactification of L considered in Section 4. This can be smooth only in the case where $X = \mathbb{P}^n$ and D is a hyperplane, which was excluded. \square

Remark 6.2 The result on the sign of a in this theorem is opposite to that obtained on crepant resolutions of a finite quotient of \mathbb{C}^n ; see [26, Theorem 8.2.3].

Remark 6.3 The proof of Theorem 6.1 is reminiscent of the argument in [9, §3.2]. The difference is that the argument in [9] is in real Riemannian geometry, so (6-11) is replaced by $\nabla\Delta f = \Delta\nabla f + \text{Ric}\nabla f$, and the complex Hessian $\nabla^- df$ by the real Hessian of f .

6.2 The formal perturbation

The Tian–Yau metric is Ricci flat, but we want to glue it with a metric with $\text{Ric} = \mu$. Given the scale ε , this means we will construct a perturbation $\omega_{\text{TY},\varepsilon}$ of ω_{TY} such that $\text{Ric}(\omega_{\text{TY},\varepsilon}) = \varepsilon\mu\omega_{\text{TY},\varepsilon}$. This perturbation will match better the metric constructed on the normal bundle L and will be needed to successfully perform the gluing later; see Remark 9.2.

Since $X \setminus D$ is not compact, this is of course not possible, but it can be done formally up to any order. Denote the development of the potential $\phi_{\beta_*,L}$ near the cone singularity by

$$(6-17) \quad \phi_{\beta_*,L}(x) = \frac{1}{4}r^2 + a_2r^4 + a_3r^6 + \dots$$

We want to solve the Kähler–Einstein equation on $X \setminus D$:

$$(6-18) \quad P_\varepsilon(\varphi) := \log \frac{(\omega_{\text{TY}} + dd^c\varphi)^n}{i^{n^2}\Omega \wedge \bar{\Omega}} + \varepsilon\mu(\Phi_{\text{TY}} + \varphi) = 0.$$

Proposition 6.4 For any integer k there is a development $\varphi = \varepsilon\varphi_1 + \dots + \varepsilon^k\varphi_k$ such that:

- (1) There is a formal development $P_\varepsilon(\varphi) = \sum_{i>k} \varepsilon^i p_i$ with $p_i = O(R^{2i})$ (and the same for the weighted derivatives $R^j\nabla^j$); moreover, on any domain $R \leq C\varepsilon^{-\theta/2}$ the series for $P_\varepsilon(\varphi)$ converges if ε is

small enough, and one has

$$|R^j \nabla^j P_\varepsilon(\varphi)| \leq C_{k,j} \varepsilon^{k+1} R^{2k+2}.$$

(2) At infinity one has $\varphi_k(y) = a_{k+1} R^{2k+2} + O(R^{2k+2-2\nu_0}) + O(R^{-2n+2})$; again the same estimates hold on the weighted derivatives $R^j \nabla^j$.

Proof We solve inductively (6-18) in powers of ε . The first equation is

$$-\Delta\varphi_1 + \mu\Phi_{TY} = 0.$$

Because $\phi_{\beta_*,L}$ is an exact solution on the cone, the function $a_2 R^4$ is a solution of $\Delta(a_2 R^4) = \frac{1}{4}\mu R^2$ up to the order where the equations for J and J_L differ, that is, up to the order where $\Omega \wedge \bar{\Omega}$ and $\Omega_L \wedge \bar{\Omega}_L$ differ: so given (6-10) we obtain

$$-\Delta(a_2 R^4) + \mu\Phi_{TY} = O(R^{2-2\nu_0}).$$

Therefore we can find $\varphi_1 = \chi a_2 R^4 + \psi_1$ (where χ is a cutoff function near infinity) by solving $\Delta\psi_1 = -\Delta(\chi a_2 R^4) + \mu\Phi_{TY}$, which is possible with $\psi_1 = O(R^{4-2\nu_0}) + O(R^{-2n+2})$. (The second term is present for the case where $4 - 2\nu_0 < -2n + 2$). The error term in $P_\varepsilon(\varepsilon\varphi_1)$ is

- the nonlinear terms in Monge–Ampère: $dd^c(\varepsilon\varphi_1) = O(\varepsilon R^2)$ so the nonlinear terms are $O((\varepsilon R^2)^i)$ with $i > 1$; applying the log gives the formal development in ε ;
- $\varepsilon^2\mu\varphi_1$ which is also $O(\varepsilon^2 R^4)$.

On the domain $R \leq C\varepsilon^{-\theta/2}$, we have $\varepsilon R^2 \leq C\varepsilon^{1-\theta}$ which is small when $\varepsilon \rightarrow 0$, and it follows that the series of the logarithm converges, which implies the estimate.

Continuing this process inductively we get the proposition. □

7 Gluing and linear analysis

7.1 Gluing with the conic Calabi ansatz and the Tian–Yau metric

Recall that on L , we have the coordinate $u := \log |v|_h^2$, and a Kähler–Einstein potential $\phi_{\beta,L} = \phi_{\beta,L}(u)$ defined for larger and larger values of u as β approaches β_* ; more precisely it is defined in the zone $(u \leq u_\beta)$ where $u_\beta = -\frac{1}{n\beta_*} \log(\beta - \beta_*)$ up to a constant; see (3-28).

In order to glue the potential $\phi_{\beta,L}$ to the Tian–Yau potential via Υ , one first needs to rescale L so that most of the domain of $\phi_{\beta,L}$ is included in the domain of Υ . We have already identified the correct rescaling $\lambda_\varepsilon : L \rightarrow L$ in (3-53). It will be convenient to introduce the notation

$$(7-1) \quad \phi_{\beta,L,\varepsilon} := (\lambda_\varepsilon)_* \phi_{\beta,L}$$

as well as

$$\Upsilon_\varepsilon := \Upsilon \circ \lambda_\varepsilon,$$

which is defined on larger and larger regions in L . Finally, if ζ is the variable in L , we will write

$$(7-2) \quad \zeta_\varepsilon := \lambda_\varepsilon^* \zeta = \varepsilon^{\frac{1}{2\beta_*}} \zeta.$$

Moreover, in order for the potential to match the Tian–Yau potential, the gluing needs to be done near u_ε (i.e., on a zone of the form $|u - u_\varepsilon| \leq 1$) satisfying

$$1 \ll u_\varepsilon \ll -\frac{1}{n\beta_*} \log(\beta - \beta_*),$$

or equivalently in terms of the variable $r = 2e^{-\frac{1}{2}\beta_* u}$,

$$(7-3) \quad (\beta - \beta_*)^{\frac{1}{n}} \ll r_\varepsilon^2 \ll 1,$$

where $r_\varepsilon := 2e^{-\frac{1}{2}\beta_* u_\varepsilon}$. We will choose r_ε of the form $r_\varepsilon = \varepsilon^{\frac{1}{2}(1-\theta)}$ for some $\theta \in (0, 1)$ to be determined later so that

$$(7-4) \quad \frac{\varepsilon}{r_\varepsilon^2} = \varepsilon^\theta.$$

We fix once and for all a point $x_0 \in X \setminus U_L$ and we set $R = d_{g_{TY}}(\cdot, x_0)$. Next, we introduce a positive function $\rho = \rho(\beta, \varepsilon)$ on X by

$$\rho^2 := \begin{cases} -\frac{4}{\beta_*} \phi'_{\beta,L} \circ \Upsilon_\varepsilon^{-1} & \text{on } \Upsilon_\varepsilon(\{u \leq u_\varepsilon\}), \\ \varepsilon R^2 & \text{on } \Upsilon_\varepsilon(\{u \geq u_\varepsilon\} \cap \Delta_L), \\ \varepsilon & \text{on } X \setminus U_L. \end{cases}$$

The function ρ has values in $[\sqrt{\varepsilon}, 2\sqrt{\alpha/\beta_*}]$. More precisely, it is equal to $2\sqrt{\alpha/\beta_*}$ on D , then “decreases” to reach the value r_ε near the gluing zone, and from there keeps decreasing to $\sqrt{\varepsilon}$.

We fix two positive integers j, k and we consider the diffeomorphism Υ from Proposition 5.2 as well as the k -th order formal perturbation of the Tian–Yau potential $\Phi_{TY,\varepsilon} = \Phi_{TY} + \varphi$ constructed in Section 6.2. We will determine suitable values for j and k further in the text.

Let $\chi : [0, +\infty) \rightarrow [0, 1]$ be a nonincreasing function which is identically 0 on $[0, \frac{1}{2}]$ and identically 1 on $[2, +\infty)$, and let $\chi_\varepsilon : X \rightarrow [0, 1]$ be defined by $\chi_\varepsilon = \chi(\rho/r_\varepsilon)$, so that

$$\chi_\varepsilon = \begin{cases} 1 & \text{on } \rho \geq 2r_\varepsilon, \\ 0 & \text{on } \rho \leq \frac{1}{2}r_\varepsilon. \end{cases}$$

Finally, we introduce the potential

$$\varphi_{\beta,\varepsilon} := \chi_\varepsilon(\Upsilon_\varepsilon)_* \phi_{\beta,L} + (1 - \chi_\varepsilon)\varepsilon \Phi_{TY,\varepsilon},$$

which is well defined globally on X . In the gluing zone, we have

$$(7-5) \quad \Upsilon_\varepsilon^* \varphi_{\beta,\varepsilon} = \phi_{\beta,L} + (1 - \chi_\varepsilon)(\Upsilon_\varepsilon^*(\varepsilon \Phi_{TY,\varepsilon}) - \phi_{\beta,L}).$$

We set $\omega_{\beta,\varepsilon} := dd^c \varphi_{\beta,\varepsilon}$, which is a Kähler metric on $X \setminus D$ with cone singularities of angle $2\pi\beta$ along D . We denote by $g_{\beta,\varepsilon}$ the corresponding Riemannian metric.

We will solve the Kähler–Einstein equation $\text{Ric}(\omega_{\beta,\varepsilon} + dd^c\varphi) = \mu\omega_{\beta,\varepsilon} + dd^c\varphi$ under the form

$$(7-6) \quad P_{\beta,\varepsilon}(\varphi) := \log \frac{(\omega_{\beta,\varepsilon} + dd^c\varphi)^n}{\varepsilon^n i^{n^2} \Omega \wedge \bar{\Omega}} + \mu(\varphi_\beta + \varphi) = 0,$$

where the normalization factor ε^n comes from the rescaling λ_ε . Indeed, on L the constants were chosen so that one has exactly $(dd^c\phi_{\beta,L})^n = e^{-\mu\phi_{\beta,L}} i^{n^2} \Omega_L \wedge \bar{\Omega}_L$; see (3-42) and (6-4). Therefore, we get

$$(7-7) \quad (dd^c\phi_{\beta,L,\varepsilon})^n = \varepsilon^n e^{-\mu\phi_{\beta,L,\varepsilon}} i^{n^2} \Omega_L \wedge \bar{\Omega}_L$$

with the notation (7-1); see also (8-1) further below. We introduce the notation

$$(7-8) \quad L_{\beta,\varepsilon} := \Delta_{\omega_{\beta,\varepsilon}} + \mu_\beta \quad \text{and} \quad Q_{\beta,\varepsilon} := P_{\beta,\varepsilon} - P_{\beta,\varepsilon}(0) - L_{\beta,\varepsilon}.$$

An elementary computation shows that if $|dd^c f|_{\omega_{\beta,\varepsilon}} \leq 1$, then

$$(7-9) \quad |Q_{\beta,\varepsilon}(f)| \leq C |dd^c f|_{\omega_{\beta,\varepsilon}}^2$$

for some $C > 0$ independent of f, β, ε .

7.2 Functional spaces

Fix a real number δ . For any function f on X , we define the weighted norms (which depend on β and ε)

$$(7-10) \quad \|f\|_{C_\delta^0} = \sup_X \rho^\delta |f|, \quad \|f\|_{C_\delta^{0,\alpha}} = \|f\|_{C_\delta^0} + [\rho^\delta f]_\alpha,$$

where the seminorm $[f]_\alpha$ is also weighted:

$$(7-11) \quad [f]_\alpha = \sup_{d_{g_{\beta,\varepsilon}}(x,y) < \text{inj}} \min\{\rho^\alpha(x), \rho^\alpha(y)\} \cdot \frac{|f(x) - f(y)|}{d_{g_{\beta,\varepsilon}}(x,y)^\alpha}.$$

As for the $C^{2,\alpha}$ norm, since we will in general be working with cone angles $2\pi\beta > \pi$, we cannot require to control all the second-order derivatives but rather only the mixed ones (normal to D), as observed by Donaldson [19]. Therefore we need to distinguish two regions $X = A \cup B$ where $A = (r \geq 1)$ and $B = (r \leq 1)$. On B , i.e., away from D , we set

$$(7-12) \quad \|f\|_{C_\delta^{2,\alpha}(B)} = \sup \sum_{0 \leq j \leq 2} \rho^{\delta+j} |\nabla^j f|_{g_{\beta,\varepsilon}} + [\rho^{\delta+2} \nabla^2 f]_\alpha.$$

Now we need to define $\|f\|_{C_\delta^{2,\alpha}(A)}$; on A the weight function is essentially one so it is irrelevant. The function $\Upsilon^* f|_A$ lives on a smaller and smaller neighborhood of the zero section in Δ_L as ε decreases to 0. Consider the dilation $\lambda_\varepsilon : L \rightarrow L, (x, v) \mapsto (x, \varepsilon^{1/(2\beta^*)} v)$. Then $f_\varepsilon := \lambda_\varepsilon^* \Upsilon^* f|_A$ lives on Δ_L , and we are going to set

$$\|f\|_{C^{2,\alpha}(A)} := \|f_\varepsilon\|_{C^{2,\alpha}(\Delta_L)},$$

where the $C^{2,\alpha}(\Delta_L)$ norm is the one defined by Donaldson, whose construction we recall in the paragraph below. Finally, we set

$$(7-13) \quad \|f\|_{C_{\delta}^{2,\alpha}} = \|f\|_{C^{2,\alpha}(A)} + \|f\|_{C_{\delta}^{2,\alpha}(B)}.$$

In this paragraph, let h be a function on Δ_L . Let V be a chart V with a holomorphic system of coordinates (z_1, \dots, z_n) where the zero section D is given by $(z_1 = 0)$; set $z' = (z_2, \dots, z_n)$. The flat cone Kähler metric is given by $dd^c(|z_1|^{2\beta} + |z'|^2)$ and the associated Riemannian metric is

$$\bar{g}_{\beta} := (dr^2 + \beta^2 r^2 d\theta^2) + g_{\mathbb{C}^{n-1}},$$

where $r := |z_1|^{\beta}$ and $\theta := \arg(z_1)$. We consider the derivatives

$$D_1 := \partial_r, \quad D_2 := \frac{1}{\beta r} \partial_{\theta}, \quad D_{2j-1} = \partial_{x_j}, \quad D_{2j} = \partial_{y_j},$$

where $z_j = x_j + iy_j$ for $2 \leq j \leq n$. The $C^{0,\alpha}$ norm of h is simply

$$\|h\|_{C^{0,\alpha}(V)} = \sup_V |h| + \sup_{x,y \in V} \frac{|h(x) - h(y)|}{d_{\bar{g}_k}(x,y)^{\alpha}},$$

and we define the $C^{2,\alpha}$ norm of h on V as

$$\|h\|_{C^{2,\alpha}(V)} = \sup_V (|h| + |\nabla^{\bar{g}_{\beta}} h|) + \sum_{i=1}^{2n} \sum_{j=3}^{2n} \|D_i D_j h\|_{C^{0,\alpha}(V)} + \|\Delta_{\bar{g}_{\beta}} h\|_{C^{0,\alpha}(V)}.$$

Finally, one can cover Δ_L with finitely many coordinate charts V_1, \dots, V_N , and we set

$$\|h\|_{C^{2,\alpha}(\Delta_L)} = \sum_{k=1}^N \|h\|_{C^{2,\alpha}(V_k)}.$$

Proposition 7.1 *Let $L_{\beta,\varepsilon} := \Delta_{g_{\beta,\varepsilon}} + \mu_{\beta}$. Then there exists a constant $C > 0$ independent of β, ε such that for any $f \in C_{\delta}^{2,\alpha}(X)$, we have*

$$\|f\|_{C_{\delta}^{2,\alpha}} \leq C(\|f\|_{C_{\delta}^0} + \|L_{\beta,\varepsilon} f\|_{C_{\delta+2}^{0,\alpha}}).$$

Proof We first look at the zone $A = (\rho \geq 1)$ and we need to bound $\|f\|_{C^{2,\alpha}(A)}$. Given the definition of the norms involved, it amounts to proving the equivalent Schauder estimate for the operator $\Delta_{g_{\beta}} + \mu_{\beta}$ on Δ_L , uniformly in β . This has been proved for the flat cone metric (i.e., for $\Delta_{\bar{g}_{\beta}} + \mu_{\beta}$) in [19] for a given fixed β , and the uniformity in the cone angle (even when $\beta \rightarrow 0$, which is irrelevant here though) has been proved in [8, Theorem 6.1]. It is classical (see, e.g., [8, Remark 6.8]) to reduce the estimate for the operator $\Delta_{g_{\beta}} + \mu_{\beta}$ to the one for the flat cone metric by proving that g_{β} is C^{α} -close to \bar{g}_{β} . This, in turn, follows from the explicit description of ω_{β} in (3-4) and the expansion (3-16) (use $\rho = e^{\frac{1}{2}\beta(u+\phi_n)}$ where ϕ_h is a local weight of the hermitian metric h on L).

Next, in the zone $(1 \geq \rho \geq \frac{1}{2}r_\varepsilon)$, one can use as before the diffeomorphism Υ_ε and the problem is equivalent to that with the Calabi metric $dd^c\phi_{\beta,L}$ on a compact subset $K_\varepsilon \Subset L \setminus D$. Over K_ε , the metric is uniformly comparable to the cone metric (3-55), so weighted Schauder estimates follow.

In the zone $(\rho \leq \frac{1}{2}\rho_\varepsilon)$, we have $g_{\beta,\varepsilon} = \varepsilon g_{TY}$. The Tian–Yau metric is asymptotically conical hence satisfies the desired Schauder estimate with weight function R . Since $\rho \approx \sqrt{\varepsilon}R$ globally on the zone $(\rho \geq \frac{1}{2}\rho_\varepsilon)$, the desired estimate follows from the classical one for the Tian–Yau metric by the definition of our norms. □

7.3 Vector fields and eigenfunctions of the Laplacian on the cone L

We begin with the following elementary computation relating eigenfunctions of the Laplacian for g_D and g_β .

Lemma 7.2 *Let $g \in C^\infty(D)$ such that $(\Delta_{\omega_D} + 1)g = 0$. Define for $\beta \geq \beta_*$ the function $f_\beta = \phi'_\beta g$ on an open subset of L . Then we have*

$$(\Delta_{\omega_\beta} + \mu_\beta)f_\beta = 0.$$

Proof We omit the index β to lighten notation. Let $\psi = \psi(u)$ be a smooth function of u . We have

$$dd^c(\psi g) = g \left(\psi'' du \wedge d^c u - \frac{2\psi'}{\alpha - 1} \omega_D \right) + \psi dd^c g.$$

Given (3-4), we deduce

$$(7-14) \quad \Delta_{\omega_\beta}(\psi g) = \frac{\psi''}{\phi''} g + (n - 1) \frac{\psi'}{\phi'} g - (\alpha - 1) \frac{\psi}{\phi'} \Delta_{\omega_D} g = \left(\frac{\psi''}{\psi \phi''} + (n - 1) \frac{\psi'}{\psi \phi'} + (\alpha - 1) \frac{1}{\phi'} \right) \psi g.$$

Now, (3-7) can be rewritten as

$$\log \phi'' + (n - 1) \log \phi' + \mu \phi + (\alpha - 1)u = 0,$$

and after differentiation, we obtain

$$(7-15) \quad \frac{\phi'''}{\phi''} + (n - 1) \frac{\phi''}{\phi'} + \mu \phi' + (\alpha - 1) = 0.$$

Plugging $\psi := \phi'$ into (7-14) and using (7-15), we obtain

$$\Delta_{\omega_\beta}(\phi' g) = -\mu \phi' g. \quad \square$$

Remark 7.3 The \mathbb{C}^* action on L yields another eigenvector of Δ_{ω_β} , which is $\phi'_\beta + \lambda_\beta$. More precisely, the computation above shows that $(\Delta_{\omega_\beta} + \mu_\beta)(\phi'_\beta + \lambda_\beta) = 0$.

Definition 7.4 Let (f_1, \dots, f_ℓ) be a basis of $\{f \in C^\infty(D) : (\Delta_{g_D} + 1)f = 0\}$. We define $E \subset C^0(L)$ to be the finite-dimensional real vector space

$$E := \text{Span}\{\phi'_{\beta_*} + \lambda_{\beta_*}, \phi'_{\beta_*} f_1, \dots, \phi'_{\beta_*} f_\ell\}.$$

Proposition 7.5 *Let $h \in C^\infty(L \setminus D)$ be a solution of $(\Delta_{g_{\beta_*}} + \mu_{\beta_*})h = 0$ such that $\|h\|_{C_\delta^{2,\alpha}} < +\infty$ for some $\delta \in (0, 1)$ small enough and let $v \in H^0(\bar{L}, T_{\bar{L}}(-\log D))$. Then*

- (i) *The vector field $v_h := \text{grad}_{g_{\beta_*}}^{1,0} h$ is holomorphic and tangent to D .*
- (ii) *There exists a unique h_v such that $v = \text{grad}_{g_{\beta_*}}^{1,0} h_v$ and $(\Delta_{g_{\beta_*}} + \mu_{\beta_*})h_v = 0$. Moreover, $h_v \in E$.*

The two constructions are inverse to each other, in the sense that $h_{v_h} = h$.

Proof We start by proving the first assertion. It turns out that we'll need to refine the condition $h \in C_\delta^{2,\alpha}(L)$ in the following computations. This is the aim of the following:

Claim 7.6 *One has*

$$\sup_{0 < r < 1} |\nabla h|_{g_{\beta_*}} + r |\nabla^2 h|_{g_{\beta_*}} < +\infty,$$

where $r = e^{-\frac{1}{2}\beta_* u}$.

Proof of Claim 7.6 By Remark 7.3, $\phi'_\beta + \lambda_\beta$ satisfies the same linear equation as h and clearly it satisfies the bounds in the claim. Therefore, up to replacing h with $h - (\phi'_\beta + \lambda_\beta)$, one can assume that $h(0) = 0$. Then the behavior of h near $r = 0$ is governed by the first positive exceptional weight δ_1 of Δ , which satisfies $\delta_1 > 1$ (see, for example, [15, Remark 2.10]). It follows that $h \in C_{\delta'}^{2,\alpha}$ for any $\delta' < \delta_1$ and in particular for $\delta' = 1$. So $\nabla h \in C_0^{1,\alpha}$ which gives the claim. \square

To lighten notation, we will drop the index β_* in the following and simply write v for v_h . Since $\text{Ric } \omega = \mu\omega$, a classical computation shows that

$$(7-16) \quad \bar{\partial}^* \bar{\partial} v = \text{grad}^{1,0}[(\Delta_g + \mu)h] = 0.$$

Moreover,

$$(7-17) \quad (\langle \bar{\partial} v, \bar{\partial} v \rangle_g - \langle \bar{\partial}^* \bar{\partial} v, v \rangle_g) d\text{vol}_g = d(\langle * \bar{\partial} v, v \rangle_g),$$

where $*$: $\Omega_X^{0,1} \otimes T_X^{1,0} \rightarrow \Omega_X^{n,n-1} \otimes T_X^{1,0}$ is the Hodge star operator.

For $\varepsilon > 0$, we set $L_\varepsilon := \{\log \varepsilon \leq u \leq -\log \varepsilon\}$. Integrating (7-17) over L_ε and using (7-16) yields

$$\begin{aligned} \int_{L_\varepsilon} \|\bar{\partial} v\|^2 d\text{vol}_g &= \int_{\partial L_\varepsilon} \langle * \bar{\partial} v, v \rangle_g \\ &= \int_{u=\log \varepsilon} \langle * \bar{\partial} v, v \rangle_g + \int_{u=-\log \varepsilon} \langle * \bar{\partial} v, v \rangle_g \\ &=: I(\varepsilon) + J(\varepsilon) \end{aligned}$$

and we now have to show that $I(\varepsilon)$ and $J(\varepsilon)$ go to zero when $\varepsilon \rightarrow 0$. Since g is has conic singularities along D and h belongs to Donaldson's C^{2,α,β_*} space, the claim for I has already been showed in the proof of [19, Proposition 8]. So we now focus on $J(\varepsilon)$. At infinity in L , recall from (3-51) that we have

$$g \approx dr^2 + r^2(\eta^2 + g_D),$$

where $r = e^{-\frac{1}{2}\beta_* u}$. Next, we have

$$\|\langle * \bar{\partial} v, v \rangle\|_g \leq \|\bar{\partial} v\|_g \cdot \|v\|_g \leq \|\nabla^{0,2} h\|_g \cdot \|\nabla h\|_g \leq r^{-1}$$

by Claim 7.6. On $\{u = -\log \varepsilon\}$, we have $r \simeq r_\varepsilon := \varepsilon^{\frac{1}{2}\beta_*}$; hence

$$|J(\varepsilon)| \lesssim r_\varepsilon^{2n-2}$$

goes to zero as desired.

We now move on to proving the second assertion. Let ξ be the radial holomorphic vector field. By Lemma 4.1, we can decompose v as

$$v = a\xi + w,$$

where $a \in \mathbb{C}$ and w is induced by vector field $v_D \in H^0(D, T_D)$; of course we have $w|_D = v_D$. We want to find a function h such that $v = \text{grad}^{1,0} h$ or, equivalently, $i_v \omega = \bar{\partial} h$.

Thanks to Remark 7.3, we have $i_\xi \omega = \bar{\partial} \phi'$. Therefore, it is enough to show the claim when $a = 0$, i.e., when $v = w$ is induced by a vector field v_D on D . By the classical case, one can write $i_{v_D} \omega_D = \bar{\partial} h_D$ for some smooth function h_D on D satisfying $(\Delta_{g_D} + 1)(h_D) = 0$. Set $h := -\frac{1}{\alpha-1} \phi' h_D$; since $(\Delta_{g_{\beta_*}} + \mu_{\beta_*})(h) = 0$ by Lemma 7.2, all we have left to prove is

$$(7-18) \quad i_v \omega = \bar{\partial} h.$$

In order to prove (7-18), let $x \in D$ and let (w_1, \dots, w_{n-1}) be a system of coordinates defined on a open set $U \subset D$. On U , $L = -\frac{1}{\alpha-1} K_D$ is trivialized by $\sigma := (dw_1 \wedge \dots \wedge dw_{n-1})^{\otimes -\frac{1}{\alpha-1}}$ and we get another coordinate z on $p^{-1}(U)$ given by $v = z\sigma$ if $v \in L_x$. On U , one can write $v_D = \sum a_i \frac{\partial}{\partial w_i}$. We have $L_{v_D} \sigma = \frac{\text{div}(v_D)}{\alpha-1} \cdot \sigma$ so that the lift of v_D to $p^{-1}(U)$ is

$$v = \frac{\text{div}(v_D)}{\alpha-1} \cdot z \frac{\partial}{\partial z} + \sum_{i=1}^{n-1} a_i \frac{\partial}{\partial w_i}.$$

At x , we can always assume up to doing a linear change of coordinates that the local weight φ of the hermitian metric on $-K_D$ satisfies $\varphi(x) = 0$ and $d\varphi(x) = 0$ so that $\partial u = \frac{dz}{z}$ at x . In particular, we have at any point in $p^{-1}(x)$

$$\omega = \phi'' i \frac{dz}{z} \wedge \frac{d\bar{z}}{\bar{z}} - \frac{\phi'}{\alpha-1} \omega_D \quad \text{and} \quad \bar{\partial}(\phi' h_D) = \phi'' h_D \frac{d\bar{z}}{\bar{z}} + \phi' \bar{\partial} h_D.$$

Since $i_{v_D} \omega_D = \bar{\partial} h_D$, we have $\text{div}(v_D) = -i \Delta_{g_D} h_D = i h_D$. This implies that

$$i_v \omega = \frac{\text{div}(v_D)}{\alpha-1} \phi'' i \frac{d\bar{z}}{\bar{z}} - \frac{\phi'}{\alpha-1} i_{v_D} \omega_D = \frac{-1}{\alpha-1} \left(\phi'' h_D \frac{d\bar{z}}{\bar{z}} + \phi' \bar{\partial} h_D \right) = \bar{\partial} h,$$

which shows (7-18) at any point in the fiber of x , and hence everywhere on L .

As for the last assertion, since h_{v_h} and h satisfy $i_{v_h} \omega = \bar{\partial} h = \bar{\partial} h_{v_h}$, we have that $h - h_{v_h}$ is constant, and hence zero since it lies in the kernel of $\Delta_g + \mu$. □

7.4 Reduction to a one-dimensional problem

Let $G_D = \text{Isom}^\circ(D, \omega_D)$ be the connected component of the group of holomorphic isometries of the Kähler–Einstein manifold (D, ω_D) . By Matsushima theorem [32], G_D is a compact connected Lie group such that $G_D^{\mathbb{C}} = \text{Aut}^\circ(D)$.

From now on, we assume that the restriction morphism (4-2) is surjective. Thanks to Lemma 4.2, there exists a surjective, finite étale map $G \rightarrow G_D$ of algebraic groups where G is a (compact) subgroup of $\text{Aut}^\circ(X, D)$. We define the space of G -invariant functions

$$C_\delta^{2,\alpha}(X)^G := \{f \in C_\delta^{2,\alpha}(X) : \forall \sigma \in G, \sigma^* f = f\}.$$

Let (h_1, \dots, h_ℓ) be an orthonormal basis of $\{h \in C^\infty(D) : (\Delta_{g_D} + 1)h = 0\}$, and let $\xi_i := \text{grad}_{g_D}^{1,0} h_i$ be the corresponding holomorphic vector field on D . By our assumption, there exists a unique vector field on X tangent to D extending ξ_i ; see proof of Lemma 4.2. We will abusively still denote it by ξ_i .

Next, recall from Remark 7.3 that $\phi'_\beta + \lambda_\beta$ is the potential of the scaling \mathbb{C}^* action on $(L, \omega_{\beta,L})$. Borrowing the notation from the beginning of Section 7, we set

$$(7-19) \quad \tau = \tau_{\beta,\varepsilon} := \chi_\varepsilon(\Upsilon_\varepsilon)_*(\phi'_{\beta,L}) + \lambda_\beta,$$

which is well defined globally on X , and satisfies $\|\tau_{\beta,\varepsilon}\|_{C_\delta^0} = \alpha(\alpha + \lambda)$. It is asymptotic in the kernel of $L_{\beta,\varepsilon}$ in the following sense:

Lemma 7.7 *Let $f \in C_\delta^{2,\alpha}(X)$ and let $\delta \in (0, 2n)$. Then we have*

$$\langle L_{\beta,\varepsilon} f, \tau_{\beta,\varepsilon} \rangle = O(r_\varepsilon^{2n-\delta} \|f\|_{C_\delta^0}).$$

If, moreover, one has $\delta < n - 2$ and $|dd^c g|_{\omega_{\beta,\varepsilon}} \leq 1$ then we have

$$\langle Q_{\beta,\varepsilon}(f), \tau_{\beta,\varepsilon} \rangle = \begin{cases} O(\|f\|_{C_\delta^{2,\alpha}}^2) & \text{if } n > 2, \\ O(\varepsilon^{-\delta} \|f\|_{C_\delta^{2,\alpha}}^2) & \text{if } n = 2, \end{cases}$$

where $O(\cdot)$ are uniform in f, g, β, ε .

Proof Let us start with the first estimate. On $(\rho \geq 2\rho_\varepsilon)$, we have $L_{\beta,\varepsilon} \tau_{\beta,\varepsilon} \equiv 0$ thanks to Remark 7.3.

On $(\rho \leq \frac{1}{2}\rho_\varepsilon)$, we have $L_{\beta,\varepsilon} \tau_{\beta,\varepsilon} = \mu_\beta \lambda_\beta = O(1)$. Since $f \leq \rho^{-\delta} \|f\|_{C_\delta^0}$ and

$$\int_{\rho \leq \frac{1}{2}\rho_\varepsilon} \rho^{-\delta} d\text{vol}_{\varepsilon g_{TY,\varepsilon}} \approx \int_{\frac{1}{2}\rho_\varepsilon}^{\rho_\varepsilon} t^{2n-1-\delta} dt = O(\rho_\varepsilon^{2n-\delta}),$$

we are done.

It remains to analyze the contribution of the gluing zone $(\frac{1}{2}\rho_\varepsilon \leq \rho \leq 2\rho_\varepsilon)$. In that zone, we know from (3-27) and (3-8) that $\phi_{\beta,\varepsilon,L}^{(k)} = O(r^2)$ for any $k \geq 0$, where the derivative is to take with respect to u and $r = e^{-\frac{1}{2}\beta_* u} \approx \rho$. In particular, $d\phi'_{\beta,\varepsilon,L} = O(r^2)du$ and $dd^c \phi'_{\beta,\varepsilon,L} = O(r^2) \cdot (du \wedge d^c u + \omega_D)$. Since $g_{\beta,\varepsilon,L}$ is asymptotic to $dr^2 + r^2 g_D$, we find $|\nabla \phi'_{\beta,\varepsilon,L}| = O(r)$ and $|\Delta \phi'_{\beta,\varepsilon,L}| = O(1)$. Finally, we have $|\nabla^k \chi_\varepsilon| = O(r_\varepsilon^{-k})$ so that in the end, $|L_{\beta,\varepsilon} \tau_{\beta,\varepsilon}| = O(1)$. One can now conclude with the same arguments as for the previous zone.

Let us now consider the integral involving $Q_{\beta,\varepsilon}$; unlike the previous one we will need to look at the normal bundle zone as well. From (7-9), we have $|Q(f)| \leq \rho^{-2\delta-4} \|f\|_{C_\delta^{2,\alpha}}^2$ pointwise. The desired estimate now follows from the same computations as in the previous step combined with the fact that the integral $\int_X \rho^{-\kappa} \omega_{\beta,\varepsilon}^n$ is uniformly bounded when $\kappa < 2n$ while $\int_{\rho \leq 2\rho_\varepsilon} \rho^{-2\delta} \omega_{\beta,\varepsilon}^2 = O(\varepsilon^{-\delta})$ when $n = 2$. \square

Finally, we introduce

$$C_\delta^{2,\alpha}(X)^\perp := \left\{ f \in C_\delta^{2,\alpha}(X) : \int_X f \tau_{\beta,\varepsilon} \, d\text{vol}_{g_{\beta,\varepsilon}} = 0 \right\},$$

set $C_\delta^{2,\alpha}(X)^{G,\perp} := C_\delta^{2,\alpha}(X)^G \cap C_\delta^{2,\alpha}(X)^\perp$ and write $\perp : C_\delta^{2,\alpha}(X) \rightarrow C_\delta^{2,\alpha}(X)^\perp$ the orthogonal projection, that is,

$$f^\perp = f - \frac{\langle f, \tau_{\beta,\varepsilon} \rangle}{|\tau_{\beta,\varepsilon}|^2} \cdot \tau_{\beta,\varepsilon}.$$

Proposition 7.8 Fix $0 < \delta < 1$. There is a constant $C > 0$ independent of β, ε (with $|\beta - \beta_*| + \varepsilon \ll 1$) such that for every $f \in C_\delta^{2,\alpha}(X)^{G,\perp}$, we have

$$\|f\|_{C_\delta^{2,\alpha}} \leq C \|(L_{\beta,\varepsilon} f)^\perp\|_{C_{\delta+2}^{0,\alpha}}.$$

Proof We proceed by contradiction. We can extract a sequence $(\beta_j, \varepsilon_j) \rightarrow 0$ and find functions $f_j \in C_\delta^{2,\alpha}(X)^G \cap C_\delta^{2,\alpha}(X)^\perp$ such that $\|f_j\|_{C_\delta^{2,\alpha}} = 1$ but satisfying $\|(L_{\beta_j,\varepsilon_j} f_j)^\perp\|_{C_{\delta+2}^{0,\alpha}} \rightarrow 0$ when $j \rightarrow +\infty$.

Thanks to the first item in Lemma 7.7, we have $\langle L_{\beta_j,\varepsilon_j} f_j, \tau_{\beta_j,\varepsilon_j} \rangle \rightarrow 0$, and hence $\|L_{\beta_j,\varepsilon_j} f_j\|_{C_{\delta+2}^{0,\alpha}} \rightarrow 0$ since $L_{\beta_j,\varepsilon_j} f_j = (L_{\beta_j,\varepsilon_j} f_j)^\perp + (\langle f_j, L\tau_{\beta_j,\varepsilon_j} \rangle / |\tau_{\beta_j,\varepsilon_j}|) \tau_{\beta_j,\varepsilon_j}$. By Proposition 7.1, $\|f\|_{C_\delta^0}$ does not converge to zero. For simplicity, one will assume that $\|f\|_{C_\delta^0} = 1$ and we choose $x_j \in X$ such that $r^\delta |f_j|(x_j) = 1$.

Case 1: $\rho(x_j) > \eta > 0$. Using Υ_ε , one can view the f_j as functions on larger and larger open subsets of L , with controlled $C^{2,\alpha}$ bound with respect to $g_{\beta,L}$. Since $\rho(x_j) \geq \eta$, the points x_j belong to a compact set of L and one can assume that (x_j) converges to $x_\infty \in L$. Thanks to the Arzelà–Ascoli theorem, one can extract a limit function $f \in C_\delta^{2,\alpha}(L)$ with respect to the cone metric g_{β_*} satisfying

$$f(x_\infty) \neq 0 \quad \text{and} \quad (\Delta_{g_{\beta_*}} + \mu_{\beta_*})(f) = 0.$$

From Proposition 7.5, it follows that there are constants $a_0, \dots, a_\ell \in \mathbb{R}$ such that

$$f = a_0(\phi'_{\beta_*} + \lambda_{\beta_*}) + \sum_{i=1}^{\ell} a_i h_i.$$

Since f_j is G -invariant, we have $L_{\xi_i} f_j = 0$ on X for any $i = 1, \dots, \ell$. Since ξ_i is tangent to D , one can restrict the previous identity to D and let $j \rightarrow +\infty$. It yields $L_{\xi_i}(f|_D) = 0$ on D . Since $\xi_i = \text{grad}^{1,0} h_i$,

$$\int_D L_{\xi_i} f \, d\text{vol}_{g_D} = \sum_{k=1}^{\ell} a_k \langle \text{grad}^{1,0} h_i, \text{grad}^{1,0} h_k \rangle = - \sum_{k=1}^{\ell} a_k \langle h_i, \Delta h_k \rangle = a_i$$

since $\Delta h_k = -h_k$. In particular, we get $a_i = 0$ for all $i = 1, \dots, \ell$ so that $f = a_0(\phi'_{\beta_*} + \lambda_{\beta_*})$.

Next, the same computations as in the proof of (7-19) show that

$$\left| \int_{r \leq 2r_\varepsilon} f_j \tau_{\beta_j, \varepsilon_j} d\text{vol}_{g_{\beta_j, \varepsilon_j}} \right| \leq C \int_{r \leq 2r_\varepsilon} r^{-\delta} d\text{vol}_{g_{\beta_j, \varepsilon_j}} = O(r_\varepsilon^{2n-\delta}).$$

Since f_j is orthogonal to $\tau_{\beta_j, \varepsilon_j}$, it follows from the dominated convergence theorem that

$$\int_L f(\phi'_{\beta_*} + \lambda_{\beta_*}) d\text{vol}_{g_{\beta_*}} = 0.$$

Therefore, we get $a_0 = 0$, and hence $f \equiv 0$, a contradiction.

Case 2: $1 \gg \rho(x_j) \gg \sqrt{\varepsilon}$. The set $\Upsilon_\varepsilon^{-1}(U_L)$ is of the form $\{+\infty \geq r \geq \sqrt{\varepsilon}\}$ with the notation of (3-34). The set $\Upsilon_\varepsilon^{-1}(1 \gg \rho \gg \sqrt{\varepsilon})$ in L is simply $(1 \gg r \gg \sqrt{\varepsilon})$. We claim that in that zone, $\Upsilon_\varepsilon^* g_{\beta, \varepsilon}$ is asymptotic to $g_{\text{TY}, L}$ whose Kähler form is $\omega_{\text{TY}, L} = \frac{1}{4} d d^c r^2$. Let us briefly justify the claim in each of the three relevant zones. In the Tian–Yau part (i.e., when $g_{\beta, \varepsilon} = \varepsilon^2 g_{\text{TY}, \varepsilon}$), it follows from Theorem 6.1 since the first-order asymptotics of the formal perturbation are the same as the ones for the genuine Tian–Yau metric by Proposition 6.4. On the normal bundle part (i.e., before the gluing) the claim follows from (3-48) since $d d^c \phi_{\beta_*, L}$ has the same asymptotic cone as $\omega_{\text{TY}, L}$. Finally, in the gluing zone, it follows from (8-9) below.

Now, set $r_j := \rho(x_j)$, $\hat{r} := r/r_j$ and consider the function $\hat{f}_j := \Upsilon_\varepsilon^*(r_j^\delta f_j|_{U_L})$ on $(1/r_j \gg \hat{r} \gg \sqrt{\varepsilon}/r_j)$. It is bounded in the usual weighted $C^{2, \alpha}$ norm with respect to the metric $\Upsilon_\varepsilon^*(r_j^{-2} g_{\beta, \varepsilon})$ — whose Kähler form is asymptotic to $\frac{1}{4} d d^c \hat{r}^2$ — and the weight \hat{r} . Moreover, it satisfies

$$\Delta_{\Upsilon_\varepsilon^*(r_j^{-2} g_{\beta, \varepsilon})} \hat{f}_j + r_j^2 \mu_\beta \hat{f}_j \rightarrow 0$$

in $C_{\text{loc}}^\alpha(L^\times)$ in the usual sense. By Arzelà–Ascoli, \hat{f}_j converges (up to extracting a subsequence) in the usual C_{loc}^2 topology on L^\times to a nonzero function f satisfying

$$\Delta_{g_{\text{TY}, L}} f = 0 \quad \text{and} \quad \sup_{L^\times} r^\delta |f| < \infty.$$

Let $\varepsilon > 0$; since $\delta \in (0, 1)$ and $n > 1$, the function $\pm f - \varepsilon/r^{2n-2}$ is harmonic and tends to $-\infty$ near the apex $r = 0$ and 0 when $r \rightarrow +\infty$. By the maximum principle, it is nonpositive; hence $\pm f \leq \varepsilon/r^{2n-2}$. Since this holds for any $\varepsilon > 0$, we infer that $f \equiv 0$, a contradiction.

Case 3: $\rho(x_j) = O(\sqrt{\varepsilon})$. In this case, x_j belongs to a fixed compact subset of $X \setminus D$ and $\rho(x_j) \approx \sqrt{\varepsilon} j$. Thanks to the Arzelà–Ascoli theorem, a diagonal extraction argument allows one to find a subsequential nonzero limit f of the sequence of functions $(\varepsilon_j^{\delta/2} f_j)$ on $X \setminus D$ for the $C_{\text{loc}}^{2, \alpha}$ topology. Moreover, f satisfies

$$\sup_{X \setminus D} \rho^\delta |f| < +\infty \quad \text{and} \quad \Delta_{g_{\text{TY}}} f = 0.$$

By the maximum principle, we must have $f \equiv 0$, a contradiction. □

8 Estimating the error and the obstruction

8.1 The error

Let us start by estimating $P_{\beta,\varepsilon}(0)$, where $P_{\beta,\varepsilon}$ is defined in (7-6). There are three zones:

Zone ($\rho \geq 2r_\varepsilon$). Since $\Upsilon^* J_X - J_L$ is horizontal up to order j and $\phi_{\beta,L}$ only depends on u , an elementary computation shows that the $(1, 1)$ -component of $\Upsilon_\varepsilon^* dd^c \varphi_{\beta,\varepsilon} - dd^c \varphi_{\beta,L}$ is dominated by a multiple of

$$|\zeta_\varepsilon|^j [(\phi''_{\beta,L} - \phi'_{\beta,L})du \wedge d^c u - \phi'_{\beta,L} \omega_D],$$

where ζ_ε is defined in (7-2). Given (3-4), we have

$$\Upsilon_\varepsilon^*(dd^c \varphi_{\beta,\varepsilon})^n = (dd^c \phi_{\beta,L})^n \left(1 + O \left(|\zeta_\varepsilon|^j \left(\left| \frac{\phi'_{\beta,L}}{\phi''_{\beta,L}} \right| + \left| \frac{\phi''_{\beta,L}}{\phi'_{\beta,L}} \right| \right) \right) \right).$$

Now, the ratio $\phi''_{\beta,L}/\phi'_{\beta,L}$ is globally bounded while $-\phi'_{\beta,L}/\phi''_{\beta,L}$ is uniformly bounded away from $u = -\infty$ and behaves like $e^{-\beta u} = |\zeta|^{-2\beta}$ near D ; see (3-16). Given (6-4) and (3-3), we have

$$(8-1) \quad \lambda_\varepsilon^*(\Omega_L \wedge \bar{\Omega}_L) = \varepsilon^{-n} \Omega_L \wedge \bar{\Omega}_L$$

so that Proposition 5.5 yields

$$(8-2) \quad \Upsilon_\varepsilon^*(\varepsilon^n \Omega \wedge \bar{\Omega}) = \Omega_L \wedge \bar{\Omega}_L (1 - |\phi_{j_0} \zeta_\varepsilon^{j_0}|^2 + O(\zeta_\varepsilon^{2j_0+1}))$$

Plugging (8-2) into (7-7), we obtain

$$(8-3) \quad \Upsilon_\varepsilon^* P_{\beta,\varepsilon}(0) = |\phi_{j_0} \zeta_\varepsilon^{j_0}|^2 + O(\zeta_\varepsilon^{2j_0+1}) + O(|\zeta_\varepsilon|^{j-2}) = |\phi_{j_0}|^2 \cdot |\zeta_\varepsilon|^{2j_0} + O(\zeta_\varepsilon^{2j_0+1})$$

for $j > 2(j_0 + 1)$. We choose such a j from now on. In terms of the coordinate u , we have

$$\Upsilon_\varepsilon^* P_{\beta,\varepsilon}(0) = O(\varepsilon^{\frac{j_0}{\beta^*}} e^{j_0 u}) = O\left(\left(\frac{\varepsilon}{r^2}\right)^{\nu_0}\right).$$

For any $\delta < 2\nu_0$, the weighed supremum of the latter quantity is attained at r_ε , i.e.,

$$(8-4) \quad \sup_{\rho \geq 2r_\varepsilon} \rho^\delta |P_{\beta,\varepsilon}(0)| \lesssim r_\varepsilon^\delta \varepsilon^{\theta \nu_0}.$$

and the same bound holds for the $C_\delta^{0,\alpha}$ norm.

Zone ($2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon$). Given (7-5), we have to evaluate $\Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY},\varepsilon}) - \phi_{\beta,L}$. Since $\lambda_\varepsilon^*(\varepsilon R^2) = r^2$, the second item of Proposition 6.4 yields

$$(8-5) \quad \begin{aligned} \Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY},\varepsilon}) &= \Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY}}) + \sum_{j=2}^k \left[a_j r^{2j} + O\left(r^{2j} \left(\frac{\varepsilon}{r^2}\right)^{\nu_0}\right) + O\left(\varepsilon^j \left(\frac{\varepsilon}{r^2}\right)^{n-1}\right) + O(r^{2k+2}) \right] \\ &= \Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY}}) + \left(\sum_{j=2}^k a_j r^{2j} \right) + O\left(r^4 \left(\frac{\varepsilon}{r^2}\right)^{\nu_0}\right) + O\left(\varepsilon r^2 \left(\frac{\varepsilon}{r^2}\right)^n\right) + O(r^{2k+2}) \\ &= (\Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY}}) - \frac{1}{4}r^2) + \phi_{\beta_*}(r) + O\left(r^4 \left(\frac{\varepsilon}{r^2}\right)^{\nu_0}\right) + O\left(\varepsilon r^2 \left(\frac{\varepsilon}{r^2}\right)^n\right) + O(r^{2k+2}). \end{aligned}$$

Now, Theorem 6.1 says that

$$(8-6) \quad \Upsilon_\varepsilon^*(\varepsilon\Phi_{TY}) = \frac{1}{4}r^2 + \begin{cases} O\left(r^2\left(\frac{\varepsilon}{r^2}\right)^{\nu_0}\right) & \text{if } \nu_0 < n, \\ ar^2\left(\frac{\varepsilon}{r^2}\right)^n \left(\log \frac{r}{\sqrt{\varepsilon}} + O(1)\right) & \text{if } \nu_0 = n, \\ ar^2\left(\frac{\varepsilon}{r^2}\right)^n + O\left(r^2\left(\frac{\varepsilon}{r^2}\right)^{n+\sigma}\right) & \text{if } \nu_0 > n, \end{cases}$$

where $\sigma > 0$ is small. The error terms in (8-6) are larger than the ones in (8-5), at least in the gluing zone and for k large enough.

Combining (8-6) with (3-45), one infers that in the gluing zone

$$(8-7) \quad \Upsilon_\varepsilon^*(\varepsilon\Phi_{TY,\varepsilon}) - \phi_{\beta,L} = \begin{cases} -a_L \frac{\beta - \beta_*}{r^{2n-2}} + O\left(r^2\left(\frac{\varepsilon}{r^2}\right)^{\nu_0} + \frac{\beta - \beta_*}{r^{2n-2}} \cdot \delta(r)\right) & \text{if } \nu_0 < n, \\ \frac{a\varepsilon^n \log\left(\frac{r}{\sqrt{\varepsilon}}\right) - a_L(\beta - \beta_*)}{r^{2n-2}} + O\left(\frac{\varepsilon^n}{r^{2n-2}} + \frac{\beta - \beta_*}{r^{2n-2}} \cdot \delta(r)\right) & \text{if } \nu_0 = n, \\ \frac{a\varepsilon^n - a_L(\beta - \beta_*)}{r^{2n-2}} + O\left(r^2\left(\frac{\varepsilon}{r^2}\right)^{n+\sigma} + \frac{\beta - \beta_*}{r^{2n-2}} \cdot \delta(r)\right) & \text{if } \nu_0 > n, \end{cases}$$

where

$$(8-8) \quad \delta(r) := r^2 + \frac{\beta - \beta_*}{r^{2n}},$$

and the same estimates hold as well for the weighed derivatives $r^j \nabla^j$. Next, the tensors $r^j \nabla^j \chi_\varepsilon$ are bounded for any j . In light of (7-5) and relying again on the fact that $J - J_L$ can be chosen horizontal up to an arbitrary large order, we infer the coarser estimate

$$(8-9) \quad \|\Upsilon_\varepsilon^* \omega_{\beta,\varepsilon} - \omega_{\beta,L}\|_{\omega_{\beta,L}} = \begin{cases} O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta\nu_0}\right) & \text{if } \nu_0 < n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n} \log \frac{1}{\varepsilon}\right) & \text{if } \nu_0 = n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n}\right) & \text{if } \nu_0 > n, \end{cases}$$

Similarly, thanks to the estimates of $P_{\beta,\varepsilon}(0)$ obtained in the previous zone we also have

$$(8-10) \quad \sup_{2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon} |P_{\beta,\varepsilon}(0)| = \begin{cases} O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta\nu_0}\right) & \text{if } \nu_0 < n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n} \log \frac{1}{\varepsilon}\right) & \text{if } \nu_0 = n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n}\right) & \text{if } \nu_0 > n, \end{cases}$$

and the weighted $C_\delta^{0,\alpha}$ estimate on the considered zone follows immediately from (8-10) since $\rho \simeq r_\varepsilon$.

Zone ($\rho \leq \frac{1}{2}r_\varepsilon$). The first second item of Proposition 6.4 shows that

$$(8-11) \quad \sup_{\rho \leq \frac{1}{2}r_\varepsilon} \rho^\delta |P_{\beta,\varepsilon}(0)| \lesssim r_\varepsilon^{2k+2+\delta},$$

where k can be chosen arbitrary large and the same bound holds for the $C_\delta^{0,\alpha}$ norm. Said otherwise, the ‘‘Tian–Yau’’ zone will not contribute to the error terms.

In conclusion, we find that for any $\delta \in (0, 2\nu_0)$, we have

$$(8-12) \quad \|P_{\beta,\varepsilon}(0)\|_{C_\delta^{0,\alpha}} = \begin{cases} O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n-\delta}} + r_\varepsilon^\delta \varepsilon^{\theta\nu_0}\right) & \text{if } \nu_0 < n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n-\delta}} + r_\varepsilon^\delta \varepsilon^{\theta n} \log \frac{1}{\varepsilon}\right) & \text{if } \nu_0 = n, \\ O\left(\frac{\beta - \beta_*}{r_\varepsilon^{2n-\delta}} + r_\varepsilon^\delta \varepsilon^{\theta n}\right) & \text{if } \nu_0 > n. \end{cases}$$

8.2 The obstruction

In this section, we estimate

$$\langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle = \int_X P_{\beta,\varepsilon}(0) \tau_{\beta,\varepsilon} \, d\text{vol}_{g_{\beta,\varepsilon}},$$

where $\tau_{\beta,\varepsilon}$ has been defined in (7-19). Here again, we study each of the three zones separately.

Zone ($\rho \geq 2r_\varepsilon$). Here, we have $\Upsilon_\varepsilon^* \tau_{\beta,\varepsilon} = \phi'_{\beta,L} + \lambda_\beta$. Since we can work as if ϕ were horizontal, the estimate (8-3) shows that the main contribution to be computed is $\varepsilon^{\nu_0} \cdot I_{\varepsilon,\beta}$ where

$$\begin{aligned} I_{\varepsilon,\beta} &:= \int_{u \leq u_\varepsilon - \log 4} |\phi_{j_0}|^2 e^{j_0 u} (\phi'_{\beta,L} + \lambda_\beta) \cdot \phi''_{\beta,L} (-\phi'_{\beta,L})^{n-1} \, du \wedge d^c u \wedge \omega_D^{n-1} \\ &= \int_{u \leq u_\varepsilon - \log 4} |\phi_{j_0}|^2 e^{\beta_*(\nu_0-n)u} (\phi'_{\beta,L} + \lambda_\beta) \cdot e^{-\mu\phi_{\beta,L}} \, du \wedge d^c u \wedge \omega_D^{n-1}. \end{aligned}$$

Performing fiberwise integral we are reduced to studying the one-dimensional integral

$$J_{\beta,\varepsilon} := \int_{-\infty}^{u_\varepsilon - \log 4} e^{\beta_*(\nu_0-n)u} (\phi'_{\beta,L} + \lambda_\beta) \cdot e^{-\mu\phi_{\beta,L}} \, du.$$

Near $u = -\infty$, the integral is indeed convergent since $e^{-\mu\phi_{\beta,L}} \approx e^{(n\beta_* + \beta)u}$. Near 0, the integrand behaves like $e^{\beta_*(\nu_0-n)u}$, and hence it will converge (when $r_\varepsilon \rightarrow 0$) if and only if $\nu_0 < n$. The sign of the integral at the limit is crucial but not obvious at this point since $\phi'_{\beta,L} + \lambda_\beta$ does not have a sign (it is negative near D and positive near the conical point). Let us work out this issue next and separate three cases.

Case $\nu_0 < n$. One writes

$$e^{\beta_*(\nu_0-n)u} (\phi'_{\beta,L} + \lambda_\beta) \cdot e^{-\mu\phi_{\beta,L}} = -\frac{1}{\mu} e^{\beta_*\nu_0 u} \cdot (e^{-\mu\phi_{\beta,L} - n\beta_* u})'$$

and integrates by parts to obtain

$$J_{\beta,\varepsilon} = \frac{\beta_* \nu_0}{\mu} \int_{-\infty}^{u_\varepsilon - \log 4} e^{\beta_*(\nu_0 - n)u - \mu \phi_{\beta,L}} du + O(e^{\beta_*(\nu_0 - n)u_\varepsilon});$$

hence there is a number $a_\beta > 0$ converging to a positive number a_{β_*} when $\beta \rightarrow \beta_*$ such that

$$(8-13) \quad I_{\beta,\varepsilon} = a_\beta + O(r_\varepsilon^{2(n-\nu_0)}).$$

Case $\nu_0 = n$. Thanks to (3-40), we find a positive number a'_β such that

$$(8-14) \quad I_{\beta,\varepsilon} = a'_\beta u_\varepsilon + O(1).$$

Case $\nu_0 > n$. The rough estimate $I_{\varepsilon,\beta} = O(r_\varepsilon^{2(n-\nu_0)})$ will suffice. All in all, we have (up to scaling a'_β)

$$(8-15) \quad \int_{\rho \geq 2r_\varepsilon} P_{\beta,\varepsilon}(0) \tau_{\beta,\varepsilon} d\text{vol}_{g_{\beta,\varepsilon}} = \begin{cases} a_\beta \varepsilon^{\nu_0} + O(r_\varepsilon^{2n} \varepsilon^{\theta \nu_0}) & \text{if } \nu_0 < n, \\ a'_\beta \varepsilon^n \log \frac{1}{r_\varepsilon} + O(\varepsilon^n) & \text{if } \nu_0 = n, \\ O(\varepsilon^{\theta \nu_0} r_\varepsilon^{2n}) & \text{if } \nu_0 > n. \end{cases}$$

Zone ($2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon$). To lighten the notation in this paragraph, let us set

$$A_\varepsilon := \Upsilon_\varepsilon^{-1}(\{2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon\}) \approx (2r_\varepsilon \geq r \geq \frac{1}{2}r_\varepsilon)$$

and

$$\psi_{\beta,\varepsilon} := (1 - \chi_\varepsilon)(\Upsilon_\varepsilon^*(\varepsilon \Phi_{\text{TY},\varepsilon}) - \phi_{\beta,L})$$

so that

$$(8-16) \quad \Upsilon_\varepsilon^* \omega_{\beta,\varepsilon} = dd^c \phi_{\beta,L} + dd^c \psi_{\beta,\varepsilon} + \text{h.o.t.}$$

by (7-5). If we additionally set

$$(8-17) \quad E := \begin{cases} \frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta \nu_0} & \text{if } \nu_0 < n, \\ \frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n} \log \frac{1}{\varepsilon} & \text{if } \nu_0 = n, \\ \frac{\beta - \beta_*}{r_\varepsilon^{2n}} + \varepsilon^{\theta n} & \text{if } \nu_0 > n, \end{cases}$$

we see from (8-7) and (8-10) that

$$(8-18) \quad \sup_{A_\varepsilon} |\psi_{\beta,\varepsilon}| = O(r_\varepsilon^2 E) \quad \text{and} \quad \sup_{A_\varepsilon} (|dd^c \psi_{\beta,\varepsilon}|_{\omega_{\beta,L}} + |\Upsilon_\varepsilon^* P_{\beta,\varepsilon}(0)|) = O(E).$$

Thanks to (8-16), the almost horizontality of ϕ and (8-18), we have

$$(8-19) \quad \Upsilon_\varepsilon^* \omega_{\beta,\varepsilon}^n = \omega_{\beta,L}^n (1 + O(E)).$$

The integral we need to estimate is

$$\begin{aligned} \int_{A_\varepsilon} \Upsilon_\varepsilon^*(P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon}\omega_{\beta,\varepsilon}^n) &= \int_{A_\varepsilon} \Upsilon_\varepsilon^*(P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon})\omega_{\beta,L}^n + O(r_\varepsilon^{2n} E \sup_{2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon} |P_{\beta,\varepsilon}(0)|) \\ &= \int_{A_\varepsilon} \Upsilon_\varepsilon^*(P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon})\omega_{\beta,L}^n + O(r_\varepsilon^{2n} E^2) \end{aligned}$$

thanks to (8-18). So from now on we focus on the integral

$$\int_{A_\varepsilon} \Upsilon_\varepsilon^*(P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon})\omega_{\beta,L}^n.$$

First, we write

$$(8-20) \quad \Upsilon_\varepsilon^* P_{\beta,\varepsilon}(0) = \log \frac{|\Omega_L|^2}{\Upsilon_\varepsilon^*(\varepsilon^n |\Omega|^2)} + L_\beta \psi_{\beta,\varepsilon} + Q(dd^c \psi_{\beta,\varepsilon}),$$

where $L_\beta = \Delta_{\omega_{\beta,L}} + \mu_\beta$ and $Q(dd^c \psi_{\beta,\varepsilon}) = O(E^2)$. In the gluing zone where we are working, the estimate (8-2) reads

$$(8-21) \quad \log \frac{|\Omega_L|^2}{\Upsilon_\varepsilon^*(\varepsilon^n |\Omega|^2)} = O(\varepsilon^{\theta v_0}).$$

Since $\Upsilon_\varepsilon^* \tau_{\beta,\varepsilon} \approx \lambda_\beta$, we have

$$(8-22) \quad (\mu_\beta \psi_{\beta,\varepsilon} + Q(dd^c \psi_{\beta,\varepsilon})) \cdot \Upsilon_\varepsilon^* \tau_{\beta,\varepsilon} = O(r_\varepsilon^2 E + E^2)$$

by (8-18). We are left to evaluating

$$I := \int_{A_\varepsilon} \Delta_{\omega_{\beta,L}} \psi_{\beta,\varepsilon} \cdot (\chi_\varepsilon \phi'_{\beta,L} + \lambda_\beta) \omega_{\beta,L}^n = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{A_\varepsilon} \psi_{\beta,\varepsilon} \cdot \Delta_{\omega_{\beta,L}} (\chi_\varepsilon \phi'_{\beta,L}) \omega_{\beta,L}^n, \quad I_2 = - \int_{r=\frac{1}{2}r_\varepsilon} \partial_v \psi_{\beta,\varepsilon} \cdot (\phi'_{\beta,L} + \lambda_\beta) d\sigma, \quad I_3 = \int_{r=\frac{1}{2}r_\varepsilon} \psi_{\beta,\varepsilon} \cdot \partial_v \phi'_{\beta,L} d\sigma$$

thanks to Stokes' theorem since $\psi_{\beta,\varepsilon} = 0$ near $r = 2r_\varepsilon$, where v is the outward normal vector and σ is the measure on $(r = cst)$ induced by $\omega_{\beta,L}$. From (3-4) and (3-47), we see that $\partial_v = (c_1 + O(\delta(r)))\partial_r$ and $d\sigma = (c_2 + O(\delta(r)))r^{2n-1}\eta \wedge d\text{vol}_{g_D}$ where c_1, c_2 are positive constants and $\delta(r)$ has been defined in (8-8).

Let us first deal with I_1 . We have $|\Delta_{\omega_{\beta,L}} \chi_\varepsilon| + |\nabla \chi_\varepsilon|^2 = O(r_\varepsilon^{-2})$. Moreover, the estimate $\phi'_{\beta,L} = O(r^2)$ derived from (3-45) can be iterated to higher orders by (3-7) to obtain $\Delta_{\omega_{\beta,L}} (\chi_\varepsilon \phi'_{\beta,L}) = O(1)$. Combined with (8-18), this shows that the integrand in I_1 is a $O(r_\varepsilon^2 E)$ so that

$$(8-23) \quad I_1 = O(r_\varepsilon^{2n+2} E).$$

Let us now get to I_2 . We have $\phi'_{\beta,L} + \lambda_\beta = \lambda_\beta + O(r_\varepsilon^2)$ and thanks to (8-7)

$$\frac{-1}{2n-2} \partial_r \psi_{\beta,\varepsilon} = \begin{cases} -a_L \frac{\beta - \beta_*}{r^{2n-1}} + O\left(r \varepsilon^{\theta\nu_0} + \frac{\beta - \beta_*}{r^{2n-1}} \cdot \delta(r)\right) & \text{if } \nu_0 < n, \\ \frac{a\varepsilon^n \log\left(\frac{r}{\sqrt{\varepsilon}}\right) - a_L(\beta - \beta_*)}{r^{2n-1}} + O\left(\frac{\varepsilon^n}{r^{2n-1}} + \frac{\beta - \beta_*}{r^{2n-1}} \cdot \delta(r)\right) & \text{if } \nu_0 = n, \\ \frac{a\varepsilon^n - a_L(\beta - \beta_*)}{r^{2n-1}} + O\left(r \varepsilon^{\theta(n+\sigma)} + \frac{\beta - \beta_*}{r^{2n-1}} \cdot \delta(r)\right) & \text{if } \nu_0 > n. \end{cases}$$

Therefore, there exists a positive constant $\kappa = \kappa(\nu_0, \beta) > 0$ such that

$$(8-24) \quad I_2 = \begin{cases} -\kappa(\beta - \beta_*) + O(r_\varepsilon^{2n} \varepsilon^{\theta\nu_0} + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 < n, \\ \kappa \left(a\varepsilon^n \log\left(\frac{r_\varepsilon}{\sqrt{\varepsilon}}\right) - a_L(\beta - \beta_*) \right) + O(\varepsilon^n + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 = n, \\ \kappa(a\varepsilon^n - a_L(\beta - \beta_*)) + O(r_\varepsilon^{2n} \varepsilon^{\theta(n+\sigma)} + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 > n. \end{cases}$$

Finally, we have

$$\psi_{\beta,\varepsilon} \cdot \partial_\nu \phi'_{\beta,L} = O(r_\varepsilon^3 E),$$

which shows that

$$(8-25) \quad I_3 = O(r_\varepsilon^{2n+2} E).$$

Given (8-20), (8-21), (8-22), (8-23) and (8-25), all the terms contributing to $\int_{A_\varepsilon} \Upsilon_\varepsilon^*(P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon}\omega_{\beta,\varepsilon}^n)$ except for I_2 are dominated by

$$r_\varepsilon^{2n} E^2 + r_\varepsilon^{2n+2} E + r^{2n} \varepsilon^{\theta\nu_0}$$

and it is straightforward to see that these terms are at most of the same order as the error term in the integral I_2 ; see (8-24). In conclusion,

$$(8-26) \quad \int_{2r_\varepsilon \geq \rho \geq \frac{1}{2}r_\varepsilon} P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon} d\text{vol}_{g_{\beta,\varepsilon}} = \begin{cases} -\kappa(\beta - \beta_*) + O(r_\varepsilon^{2n} \varepsilon^{\theta\nu_0} + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 < n, \\ \kappa \left(a\varepsilon^n \log\left(\frac{r_\varepsilon}{\sqrt{\varepsilon}}\right) - a_L(\beta - \beta_*) \right) + O(\varepsilon^n + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 = n, \\ \kappa(a\varepsilon^n - a_L(\beta - \beta_*)) + O(r_\varepsilon^{2n} \varepsilon^{\theta(n+\sigma)} + (\beta - \beta_*) \cdot \delta(r)) & \text{if } \nu_0 > n. \end{cases}$$

Zone ($\rho \leq \frac{1}{2}r_\varepsilon$). Since the volume of the zone is of order $\varepsilon^n R_\varepsilon^{2n} = r_\varepsilon^{2n}$, the second item of Proposition 6.4 shows that

$$(8-27) \quad \int_{\rho \leq \frac{1}{2}r_\varepsilon} P_{\beta,\varepsilon}(0)\tau_{\beta,\varepsilon} d\text{vol}_{g_{\beta,\varepsilon}} = O(r_\varepsilon^{2n+2k+2}),$$

where k can be chosen arbitrary large. As before, this zone will not contribute to the obstruction.

In conclusion, (8-15), (8-26) and (8-27) imply that there are positive constants $\kappa_i = \kappa_i(v_0, \beta)$ for $i = 1, \dots, 6$ (where the dependence in β is continuous up to $\beta = \beta_*$ and $\kappa_i(v_0, \beta_*) > 0$) such that

$$(8-28) \quad \langle P_{\beta, \varepsilon}(0), \tau_{\beta, \varepsilon} \rangle = \begin{cases} \kappa_1 \varepsilon^{v_0} - \kappa_2(\beta - \beta_*) + O(r_\varepsilon^{2n} \varepsilon^{\theta v_0} + (\beta - \beta_*) \cdot \delta(r_\varepsilon)) & \text{if } v_0 < n, \\ \kappa_3 \varepsilon^n \log \frac{1}{\varepsilon} - \kappa_4(\beta - \beta_*) + O(\varepsilon^n + (\beta - \beta_*) \cdot \delta(r_\varepsilon)) & \text{if } v_0 = n, \\ \kappa_5 \varepsilon^n - \kappa_6(\beta - \beta_*) + O(r_\varepsilon^{2n} \varepsilon^{n\theta + \sigma} + (\beta - \beta_*) \cdot \delta(r_\varepsilon)) & \text{if } v_0 > n. \end{cases}$$

This is clear except maybe in the case where $v_0 = n$, where the first term $\kappa_3 \varepsilon^n \log \frac{1}{\varepsilon}$ comes from the contributions of the obstruction from both the normal bundle — term $a'_\beta \varepsilon^n \log \frac{1}{r_\varepsilon} = \frac{1}{2}(1 - \theta)a'_\beta \varepsilon^n \log \frac{1}{\varepsilon}$ — and the gluing zone — term $\kappa a \varepsilon^n \log \frac{r_\varepsilon}{\sqrt{\varepsilon}} = \frac{1}{2}\theta \kappa a \varepsilon^n \log \frac{1}{\varepsilon}$ — which are both *positive*.

8.3 Choice of $\varepsilon = \varepsilon(\beta)$

Given (8-28), it is natural to introduce the scaling parameter ε_β satisfying

$$(8-29) \quad \beta - \beta_* = \begin{cases} \frac{\kappa_1}{\kappa_2} \cdot \varepsilon_\beta^{v_0} & \text{if } v_0 < n, \\ \frac{\kappa_3}{\kappa_4} \cdot \varepsilon_\beta^n \log \frac{1}{\varepsilon_\beta} & \text{if } v_0 = n, \\ \frac{\kappa_5}{\kappa_6} \cdot \varepsilon_\beta^n & \text{if } v_0 > n, \end{cases}$$

and then we choose from now on ε close to ε_β in the sense that

$$(8-30) \quad \varepsilon = \varepsilon_\beta(1 + \eta) \quad \text{with } |\eta| \ll 1,$$

where $\eta \in \mathbb{R}$ will be allowed to vary later for the purpose of solving out Monge–Ampère equation; in particular it will be important that η takes positive and negative values. Let us be more precise now and distinguish the three possible cases.

Case $v_0 < n$. Condition (7-3) translates into

$$(8-31) \quad \theta > 1 - \frac{v_0}{n},$$

which can be fulfilled. Next, we have

$$\kappa_1 \varepsilon^{v_0} - \kappa_2(\beta - \beta_*) = v_0 \kappa_1 \varepsilon_\beta^{v_0} \cdot (\eta + O(\eta^2)),$$

while the error term in (8-28) is of order

$$r_\varepsilon^{2n} \varepsilon^{\theta v_0} + (\beta - \beta_*) \cdot \delta(r_\varepsilon) \approx \varepsilon^{v_0} (\varepsilon^{(1-\theta)(n-v_0)} + \varepsilon^{1-\theta} + \varepsilon^{v_0-n(1-\theta)})$$

and the latter is a $O(\varepsilon^{v_0+(1-\theta)\min\{1, n-v_0\}})$ as soon as $\theta > 1 - \frac{v_0-1}{n}$, which we assume from now on.

Then we have

$$(8-32) \quad \langle P_{\beta, \varepsilon}(0), \tau_{\beta, \varepsilon} \rangle = v_0 \kappa_1 \varepsilon_\beta^{v_0} \cdot (\eta + O(\varepsilon^{(1-\theta)\min\{1, n-v_0\}}) + O(\eta^2)).$$

Moreover, E in (8-17) behaves like $\varepsilon_\beta^{v_0-n(1-\theta)}$.

Case $\nu_0 = n$. Condition (7-3) is automatic. Next, we have

$$\kappa_3 \varepsilon^n \log \frac{1}{\varepsilon} - \kappa_4 (\beta - \beta_*) = n \kappa_3 \varepsilon_\beta^n \log \frac{1}{\varepsilon_\beta} \cdot \left(\eta + O\left(\frac{\eta}{\log \frac{1}{\varepsilon_\beta}}\right) + O(\eta^2) \right),$$

while the order of the error term $O(\varepsilon^n + (\beta - \beta_*) \cdot \delta(r_\varepsilon))$ is simply $O(\varepsilon_\beta^n)$. We then have

$$(8-33) \quad \langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle = n \kappa_3 \varepsilon_\beta^n \log \frac{1}{\varepsilon_\beta} \cdot \left(\eta + O\left(\frac{1}{\log \frac{1}{\varepsilon_\beta}}\right) + O(\eta^2) \right).$$

Moreover, E behaves like $\varepsilon_\beta^{n\theta} \log \frac{1}{\varepsilon_\beta}$.

Case $\nu_0 > n$. Condition (7-3) is automatic too and

$$\kappa_5 \varepsilon^n - \kappa_6 (\beta - \beta_*) = n \kappa_5 \varepsilon_\beta^n \cdot (\eta + O(\eta^2)),$$

while the order of the error term $O(r_\varepsilon^{2n} \varepsilon^{n\theta+\sigma} + (\beta - \beta_*) \cdot \delta(r_\varepsilon))$ is $O(\varepsilon_\beta^{n+\sigma})$ as soon as $\theta > \frac{1}{n+1}$. Therefore, we get

$$(8-34) \quad \langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle = n \kappa_5 \varepsilon_\beta^n \cdot (\eta + O(\varepsilon_\beta^\sigma) + O(\eta^2)).$$

Moreover, E behaves like $\varepsilon_\beta^{n\theta}$.

In summary, we get from (8-12) and the case by case analysis just above:

Proposition 8.1 *If β, ε, η are chosen as in (8-29)–(8-30), then there exists $\kappa \in \mathbb{R}$ satisfying*

$$(8-35) \quad \langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle = (\beta - \beta_*) \cdot (\kappa \eta + O(F) + O(\eta^2))$$

and κ is such that $\kappa_0 \leq |\kappa| \leq \kappa_0^{-1}$ for some constant $\kappa_0 > 0$ independent of β, ε and

$$F := \begin{cases} \varepsilon_\beta^{(1-\theta) \min\{1, n-\nu_0\}} & \text{if } \nu_0 < n, \\ \frac{1}{\log \frac{1}{\varepsilon_\beta}} & \text{if } \nu_0 = n, \\ \varepsilon_\beta^\sigma & \text{if } \nu_0 > n. \end{cases}$$

Moreover, if $\delta \in (0, 2\nu_0 - 2)$, we have

$$(8-36) \quad \|P_{\beta,\varepsilon}(0)\|_{C_{\delta+2}^{0,\alpha}} = O(r_\varepsilon^{\delta+2} E),$$

where $O(\cdot)$ is uniform in β, ε, η and E is defined by

$$(8-37) \quad E := \begin{cases} \varepsilon^{\nu_0 - n(1-\theta)} & \text{if } \nu_0 < n, \\ \varepsilon^{n\theta} \log \frac{1}{\varepsilon} & \text{if } \nu_0 = n, \\ \varepsilon^{n\theta} & \text{if } \nu_0 > n. \end{cases}$$

9 Resolution of the Monge–Ampère equation

9.1 Resolution modulo the obstruction

The first step is to solve the Monge–Ampère equation up to the obstruction term. More precisely, we have:

Proposition 9.1 *Let $\delta \in (0, 1)$. Suppose ε close enough to 0 and $0 < \beta - \beta_* < c\varepsilon^{\inf(v_0, n)}$ (or $c\varepsilon^n \log \frac{1}{\varepsilon}$ is $v_0 = n$), where c is some fixed constant. Then there is a unique function $f_{\beta, \varepsilon} \in C_{\delta}^{2, \alpha}(X)^{G, \perp}$ such that*

$$P_{\beta, \varepsilon}(f_{\beta, \varepsilon})^{\perp} = 0 \quad \text{and} \quad \|f_{\beta, \varepsilon}\|_{C_{\delta}^{2, \alpha}} \leq C \|P_{\beta, \varepsilon}(0)^{\perp}\|_{C_{\delta+2}^{0, \alpha}}$$

for some constant $C > 0$ independent of β, ε .

Remark 9.2 We will see in the proof that it is crucial to have $\|P_{\beta, \varepsilon}(0)\|_{C_{2+\delta}^{\alpha}} \ll \varepsilon^{1+\frac{\delta}{2}}$. If one had not replaced the Tian–Yau metric by its formal perturbation (or had only done it up to, say, order R^4 in the potential so that $\text{Ric}(\varepsilon\omega_{\text{TY}, \varepsilon}) \approx \varepsilon\omega_{\text{TY}, \varepsilon}$), the Tian–Yau part would contribute a factor $r_{\varepsilon}^{4+\delta} = \varepsilon^{(1-\theta)(2+\frac{\delta}{2})}$ (resp. $r_{\varepsilon}^{6+\delta} = \varepsilon^{(1-\theta)(3+\frac{\delta}{2})}$) in the norm $\|P_{\beta, \varepsilon}(0)\|_{C_{2+\delta}^{\alpha}}$ and the latter need not be small compared to $\varepsilon^{1+\frac{\delta}{2}}$ since θ will have to be chosen arbitrarily close to 1 in the proof of Proposition 9.3 below.

Proof Given any $\nu' < \inf(v_0, n)$ (which we take very close to $\inf(v_0, n)$), Proposition 8.1 shows that for $\theta < 1$ close enough to 1 we have

$$(9-1) \quad \|P_{\beta, \varepsilon}(0)\|_{C_{2+\delta}^{0, \alpha}} = O(\varepsilon^{\nu'}).$$

The quadratic term $Q_{\beta, \varepsilon}$ of the equation, defined by (7-8), is controlled exactly as in [10, §1.4]: it satisfies

$$(9-2) \quad |Q_{\beta, \varepsilon}(f_1) - Q_{\beta, \varepsilon}(f_2)| \leq C |dd^c(f_1 - f_2)| (|dd^c f_1| + |dd^c f_2|)$$

as soon as $|dd^c f_j| \leq 1$ for $j = 1, 2$. Our weighted space $C_{\delta+2}^0$ carries a (small) weight $\varepsilon^{1+\frac{\delta}{2}}$ on the Tian–Yau part, and it follows that

$$(9-3) \quad \|Q_{\beta, \varepsilon}(f_1) - Q_{\beta, \varepsilon}(f_2)\|_{C_{2+\delta}^0} \leq C \varepsilon^{-1-\frac{\delta}{2}} \|dd^c(f_1 - f_2)\|_{C_{2+\delta}^0} (\|dd^c f_1\|_{C_{2+\delta}^0} + \|dd^c f_2\|_{C_{2+\delta}^0}).$$

This estimate extends to the Hölder spaces $C_{2+\delta}^{\alpha}$ by local change of scale.

Finally we have a uniform estimate for the inverse of the linearization provided by Proposition 7.8. One can then apply an inverse function theorem on $P_{\beta, \varepsilon}(f)^{\perp} = 0$, provided that $\|P_{\beta, \varepsilon}(0)\|_{C_{2+\delta}^{\alpha}} \ll \varepsilon^{1+\frac{\delta}{2}}$; see, for example, [10, Lemma 1.3]. Therefore from (9-1) it is sufficient to have $\nu' > 1 + \frac{\delta}{2}$, which can be achieved provided $\inf(v_0, n) > 1$, that is, provided that $v_0 > 1$ which is true since $\beta_* < 1$. \square

9.2 Killing the obstruction by varying ε

The outcome of Proposition 9.1 is that for fixed (ε, β) close enough to $(0, \beta_*)$, there exists a unique couple of a function and a scalar $(f_{\beta, \varepsilon}, a_{\beta, \varepsilon}) \in C_{\delta}^{2, \alpha}(X)^{G, \perp} \times \mathbb{R}$ such that

$$P_{\beta, \varepsilon}(f_{\beta, \varepsilon}) = a_{\beta, \varepsilon} \tau_{\beta, \varepsilon}.$$

The object of the next proposition is to precisely measure the effect that varying the cone angle (or the scale parameter ε) has on the obstruction $a_{\beta,\varepsilon}$. Most of the work has already been done in Proposition 8.1.

Proposition 9.3 *Let β, ε, η as in (8-29)–(8-30). For a fixed choice of $\delta, \theta \in (0, 1)$ such that $\delta + (1 - \theta) \ll 1$,*

$$\langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle = (\beta - \beta_*) \cdot (\kappa\eta + O(F) + O(\eta^2))$$

where κ, F are the ones from Proposition 8.1.

Proof Given (8-35), all we have left to prove is that

$$(9-4) \quad \langle L_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle + \langle Q_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle = o(\eta(\beta - \beta_*)).$$

Step 1: preliminaries. We first claim that

$$(9-5) \quad \|f_{\beta,\varepsilon}\|_{C_\delta^{2,\alpha}} = O(r_\varepsilon^{\delta+2} E),$$

where E is given in (8-37). Indeed, (8-35) shows that

$$|P_{\beta,\varepsilon}(0)^\perp| \leq |P_{\beta,\varepsilon}(0)| + \frac{1}{|\tau_{\beta,\varepsilon}|} \cdot |\langle P_{\beta,\varepsilon}(0), \tau_{\beta,\varepsilon} \rangle| = O(E + |\gamma|).$$

Now one can easily check that $|\gamma| \ll E$ since $\theta < 1$. The estimate (9-5) is then a consequence of Proposition 9.1 and (8-36).

Next, we claim that

$$(9-6) \quad |dd^c f_{\beta,\varepsilon}|_{\omega_{\beta,\varepsilon}} = o(1).$$

This is an immediate consequence of (9-5) on $\rho \geq \frac{1}{2}r_\varepsilon$ given the definition of our norms and the fact that $E = o(1)$. So it remains to check the claim on the Tian–Yau part. There, one can just write $|dd^c f_{\beta,\varepsilon}|_{\omega_{\beta,\varepsilon}} \leq \rho^{-\delta-2} \|f_{\beta,\varepsilon}\|_{C_\delta^{2,\alpha}} = O(\varepsilon^{-\theta(\frac{\delta}{2}+1)} E)$. If $\nu_0 < n$, the latter goes to zero provided that

$$\theta > \frac{n - \nu_0}{n - 1 - \frac{\delta}{2}},$$

which can always be achieved for some $\delta, \theta \in (0, 1)$ since $\nu_0 > 1$. If $\nu_0 \geq n$, it goes to zero provided $\delta < 2(n - 1)$ which can also be achieved since we have assumed that $n > 1$.

Step 2: the term $\langle L_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle$. By Lemma 7.7, the integral $I := \langle L_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle$ satisfies

$$I = O(r_\varepsilon^{2n-\delta} \|f\|_{C_\delta^0}) = O(r_\varepsilon^{2n+2} E),$$

where the last identity follows from (9-5). We have

$$r_\varepsilon^{2n+2} E = \begin{cases} \varepsilon^{\nu_0+(1-\theta)} & \text{if } \nu_0 < n, \\ \varepsilon^{n+1-\theta} \log \frac{1}{\varepsilon} & \text{if } \nu_0 = n, \\ \varepsilon^{n+1-\theta} & \text{if } \nu_0 > n. \end{cases}$$

and the latter is a $O((\beta - \beta_*)F)$ since $\varepsilon^{1-\theta} = O(F)$ — as soon as $\sigma < 1 - \theta$ in the case $\nu_0 > n$.

Step 3: the term $\langle Q_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle$. Thanks to (9-6), the second estimate from Lemma 7.7 is valid for the function $f = f_{\beta,\varepsilon}$, and hence $\langle Q_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle = O(\|f_{\beta,\varepsilon}\|_{C_\delta^{2,\alpha}}^2)$ if $n > 2$ and $\delta < n - 2$ and $\langle Q_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle = O(\varepsilon^{-\delta} \|f_{\beta,\varepsilon}\|_{C_\delta^{2,\alpha}}^2)$ if $n = 2$.

Assume that $n > 2$ for now. Given (9-5), we have to prove that

$$r_\varepsilon^{2\delta+2} E^2 = O((\beta - \beta_*)F).$$

We will show actually show that

$$(9-7) \quad r_\varepsilon^2 E^2 = \varepsilon^{1-\theta} E^2 = O((\beta - \beta_*)F)$$

for suitable θ . More precisely, if $\nu_0 < n$, we choose $\theta > 1 - \frac{\nu_0}{n}$ while if $\nu_0 \geq n$ we choose $\theta > \frac{n-1}{2n-1}$. In the first case, $(1 - \theta) + 2\nu_0 - 2(1 - \theta)n > (1 - \theta)$. In the second case, $1 - \theta + 2n\theta > n$. Since

$$E^2 = \begin{cases} \varepsilon^{2\nu_0-2(1-\theta)n} & \text{if } \nu_0 < n, \\ \varepsilon^{2n\theta} \log^2 \frac{1}{\varepsilon} & \text{if } \nu_0 = n, \\ \varepsilon^{2n\theta} & \text{if } \nu_0 > n, \end{cases}$$

the estimate (9-7) is satisfied.

If $n = 2$, then we have to show instead

$$(9-8) \quad \varepsilon^{-\delta\theta} r_\varepsilon^2 E^2 = \varepsilon^{1-(1+\delta)\theta} E^2 = O((\beta - \beta_*)F)$$

and we can apply the previous case to $(1 + \delta)\theta$ (in place of θ) for $\delta \ll 1$ and θ as before so that (9-8) follows from (9-7). □

9.3 Proof of the main theorem

We now get on to proving the main theorem. We need one last technical result to reach the conclusion.

Lemma 9.4 *There exists $\delta_0 > 0$ such that the functional $\Psi : (\beta, \varepsilon) \mapsto \langle P_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle$ is continuous on $(\beta_*, \beta_* + \delta_0) \times (0, \delta_0)$.*

Proof Let us choose $\sigma := (\beta, \varepsilon)$ close enough to $(\beta_*, 0)$ and let $\sigma_k \rightarrow \sigma$ be any sequence. By the Arzelà–Ascoli theorem and the bound (9-5) one can extract a subsequence such that the functions f_{σ_k} converge locally smoothly on $X \setminus D$ to a function $f_\infty \in C_\delta^{2,\alpha}(X)^G$. In particular,

$$(9-9) \quad P_{\sigma_k}(f_{\sigma_k}) \rightarrow P_\sigma(f_\infty)$$

on $X \setminus D$. Since $|dd^c f_{\sigma_k}|_{\omega_{\sigma_k}}$ is bounded, Lebesgue domination theorem shows that

$$(9-10) \quad \langle f_{\sigma_k}, \tau_{\sigma_k} \rangle \rightarrow \langle f_\infty, \tau_\sigma \rangle \quad \text{and} \quad \Psi(f_{\sigma_k}) \rightarrow \langle P_\sigma(f_\infty), \tau_\sigma \rangle =: a_\infty.$$

Since $P_{\sigma_k}(f_{\sigma_k})^\perp = 0$, it follows from (9-9) that

$$P_\sigma(f_\infty) = a_\infty \frac{\tau_\sigma}{|\tau_\sigma|^2} \quad \text{and} \quad \langle f_\infty, \tau_\sigma \rangle = 0.$$

Given the uniqueness property in Proposition 9.1, we infer that $f_\infty = f_\sigma$. By the second item in (9-10), Ψ is continuous at σ . □

Proof of Main Theorem We first prove the existence and then the convergence results.

(1) We fix β and vary ε as prescribed by (8-30). Proposition 9.3 then shows that $\langle P_{\beta,\varepsilon}(f_{\beta,\varepsilon}), \tau_{\beta,\varepsilon} \rangle$ achieves positive (resp. negative) values whenever η satisfies $1 \gg \eta \gg F$ (resp. $1 \gg -\eta \gg F$). By Lemma 9.4 above, this quantity is continuous with respect to η . Therefore, given any $\beta > 0$ close enough to β_* , there exists $\varepsilon = \varepsilon(\beta)$ as in (8-30) such that

$$(9-11) \quad P_{\beta,\varepsilon}(f_{\beta,\varepsilon}) = 0,$$

i.e., $\omega_{\beta,\varepsilon} + dd^c f_{\beta,\varepsilon}$ is Kähler–Einstein. Moreover, one has, by (9-6),

$$(9-12) \quad (1 - o(1)) \cdot \omega_{\beta,\varepsilon} \leq \omega_{\beta,\varepsilon} + dd^c f_{\beta,\varepsilon} \leq (1 + o(1)) \cdot \omega_{\beta,\varepsilon}.$$

(2) We need to identify the Gromov–Hausdorff limit of $(X, \omega_{\text{KE},\beta})$. By (9-12) above, it is enough to show the result for the model metric $\omega_{\beta,\varepsilon}$. To lighten notation, let us write $\omega_{\text{KE},\beta} := \omega_{\beta,\varepsilon} + dd^c f_{\beta,\varepsilon}$. Since the diameter of the both the gluing zone and the Tian–Yau part are a $O(r_\varepsilon)$, and since the diameter of $((r \leq r_\varepsilon), \omega_{\beta_*,L})$ is also a $O(r_\varepsilon)$, it is enough to show that

$$\sup_{r \geq r_\varepsilon} |\omega_{\beta,L} - \omega_{\beta_*,L}|_{\omega_{\beta_*,L}} \rightarrow 0$$

when $\varepsilon \rightarrow 0$ or, equivalently, when $\beta \rightarrow \beta_*$. Now the latter follows from (3-48) and the choice of r_ε which satisfies (7-3).

(3) Finally, the locally smooth convergence $\varepsilon_\beta^{-1} \omega_{\text{KE},\beta}$ to the Tian–Yau metric on $X \setminus D$ is an immediate consequence of (9-12). Indeed, the latter gives C^0 convergence of the tensor on each compact subset of $X \setminus D$, and the usual bootstrapping arguments allow us to improve it to smooth convergence thanks to the Monge–Ampère equation (7-6)–(9-11) solved by $\varepsilon_\beta^{-1} \omega_{\text{KE},\beta}$. \square

10 Examples

10.1 Projective hypersurfaces

10.1.1 \mathbb{P}^n and the quadric Here, we consider $X = \mathbb{P}^n$ and $D = Q_{n-1}$ for $n \geq 2$. We have $\alpha = \frac{1}{2}(n+1)$, D is homogeneous, and hence Kähler–Einstein, and the vector fields on D clearly lift to X . Moreover, we have $j_0 = 1$ by Remark 5.4 and Example 4.4. In particular, we get

$$\varepsilon_\beta = \begin{cases} (\beta - \beta_*)^{\frac{1}{2}} & \text{if } n = 2, \\ \left(\frac{\beta - \beta_*}{-\log(\beta - \beta_*)} \right)^{\frac{1}{3}} & \text{if } n = 3, \\ (\beta - \beta_*)^{\frac{n-1}{2n}} & \text{if } n \geq 4. \end{cases}$$

10.1.2 \mathbb{P}^n and a large-degree hypersurface We consider now the case where $X = \mathbb{P}^n$ and D is a general hypersurface of degree n (resp. $n - 1$). We may assume that $n \geq 3$ (resp. $n \geq 4$) so that we are not back to the case of the quadric above. We have $\alpha = \frac{n+1}{n}$ (resp. $\frac{n+1}{n-1}$), D is Kähler–Einstein by

[39, Theorem 4.3] and the condition on the automorphisms is fulfilled thanks to Lemma 4.2(ii). Actually, when $n \geq 3$ one has $\text{Aut}^\circ(D) = \{1\}$ by [31]. Moreover, we have $j_0 = 1$ by Remark 5.4 and Example 4.4 and $\alpha < 2$; hence

$$\varepsilon_\beta = (\beta - \beta_*)^{\frac{1}{n}}.$$

10.2 Homogeneous varieties

10.2.1 The quadric and a hyperplane section Let $X = Q_n \subset \mathbb{P}^{n+1}$ be the smooth quadric of dimension n for $n \geq 2$ and let $D \simeq Q_{n-1}$ be a general hyperplane section. We have $\alpha = n$ and all the assumptions of the main theorem are clearly met. The normal exact sequence is split by Example 4.7, and hence $j_0 \geq 2$. It would be interesting to determine the precise value of j_0 , but we will not pursue this here.

10.2.2 Some grassmannians and a hyperplane section Here, we let $X = \text{Gr}(2, 2n)$ for some $n \geq 2$ and D be a general hyperplane section under the Plücker embedding. Note that X has Picard number one. Since D parametrizes 2-planes in the kernel of a symplectic 2-form, D is homogeneous under $\text{Sp}(2n)$ and therefore the assumptions of the theorem are met.

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