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Ben Davison and Naoki Koseki

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Degree two Gopakumar–Vafa invariants of local curves

BEN DAVISON AND NAOKI KOSEKI

We investigate the Gopakumar–Vafa (GV) theory of local curves, namely, the total spaces of rank two vector bundles with canonical determinant on smooth projective curves. Under a certain genericity condition on the rank two bundles, we propose a general mechanism to compute the degree two GV invariants of local curves. In particular, we determine all the degree two GV invariants when the base curve has genus two. Combined with previous work by Bryan and Pandharipande, we obtain the GV/GW correspondence in this case. When the base curve has genus greater than two, we calculate GV invariants for some extremal genera, providing evidence for the GV/GW conjecture for curves of higher genus.

1 Introduction

1.1 Motivation and results

Throughout the paper, we work over the complex number field $\mathbb{C}$. We investigate local enumerative invariants of a Calabi–Yau (CY) 3-fold around a smooth curve. A model for this enumerative geometry is provided by *local curves*. Here, a local curve is a CY 3-fold of the form $\mathrm{Tot}_C(N)$, where C is a smooth projective curve and N is a rank two vector bundle on C with $\det(N) \cong \omega_C$. Local curves are a fundamental object of study in enumerative geometry [Bryan and Pandharipande 2001; 2006; 2008; Monavari 2022; Okounkov and Pandharipande 2010]. In particular, they have played a key role in the recent proof of the Gromov–Witten (GW)/Donaldson–Thomas (DT) correspondence for *any* smooth projective CY 3-fold [Pardon 2023]. Pardon reduced the problem to the case of (equivariant) local curves, and the GW/DT correspondence for the latter has been proved by directly computing the GW and DT invariants of local curves [Bryan and Pandharipande 2008; Okounkov and Pandharipande 2010]. Moreover, if we impose a certain rigidity condition on the bundle N, the local GW invariants agree with those on a projective CY 3-fold with curve class $r[C]$, where $C \subset X$ has normal bundle N [Bryan and Pandharipande 2001; 2005].

We study another curve counting theory on a CY 3-fold, that is, *Gopakumar–Vafa* (*GV*) *theory*. GV theory was originally introduced in physics [Gopakumar and Vafa 1998] as a way to count curves in CY 3-folds. Recently, Maulik and Toda [2018] proposed a mathematically rigorous definition of GV

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invariants, building on the previous works by Hosono, Saito, and Takahashi [Hosono et al. 2001] and Kiem and Li [2012]. See the next subsection for the review of Maulik and Toda's work. GV theory is also expected to be equivalent to GW theory (the *GV/GW correspondence conjecture*):

Conjecture 1.1 Let X be a smooth projective CY 3-fold. For a curve class $\beta \in H_2(X,\mathbb{Z})$ and a nonnegative integer $g \in \mathbb{Z}_{\geq 0}$, let $\mathrm{GW}_{g,\beta} \in \mathbb{Q}$ (resp. $n_{g,\beta} \in \mathbb{Z}$) denote the GW (resp. GV) invariant of X. Then we have the equation

$$\sum_{g,\beta} \mathrm{GW}_{g,\beta}\, \lambda^{2g-2} t^{\beta} = \sum_{g,\beta,k\geq 0} \frac{n_{g,\beta}}{k} \bigl(2\sin\bigl(\tfrac{1}{2}k\lambda\bigr)\bigr)^{2g-2} t^{k\beta}. \tag{1-1}$$

If X is a generic local curve, and we set $\beta = r[C]$, then in [Maulik and Toda 2018] the GV/PT correspondence (and thus the GV/DT correspondence, via [Bridgeland 2011; Toda 2010]) is proved for $r = 1$ or $g(C) \leq 1$ [Hosono et al. 2001; Maulik and Toda 2018]. The number r is called the *degree* of the invariant. As a first step toward a proof of the GV/GW correspondence conjecture, we investigate the *degree two* GV invariants of a local curve. The following are our main results:

Theorem 1.2 *Let C be a smooth projective curve of genus two, and let $L \in \mathrm{Pic}^3(C)$ be a generic line bundle. Let $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ be a generic rank two vector bundle on C, and let $X = \mathrm{Tot}_C(N)$. Then we have*

$$n_{g,2[C]}(X) = \begin{cases} 8 & \text{if } g = 3, \\ -2 & \text{if } g = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, Conjecture 1.1 holds in this case.

Theorem 1.3 *Let C be a smooth projective curve of arbitrary genus, and let $L \in \mathrm{Pic}^{2g(C)-1}(C)$ be a generic line bundle. Let $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ be a generic rank two vector bundle on C, and let $X = \mathrm{Tot}_C(N)$. Then we have*

$$n_{g,2[C]}(X) = \begin{cases} 0 & \text{if } g \geq 4g(C)-2, \\ -2^{2g(C)-3} & \text{if } g = g(C), \\ 0 & \text{if } g \leq g(C)-1. \end{cases}$$

In particular, the GV/GW correspondence holds for the above range of g.

1.2 GV invariants of a local curve and strategy of the proofs

First, we review the proposed definition of Maulik and Toda. Let X be a smooth quasiprojective CY 3-fold. We fix a curve class $\beta \in H_2(X,\mathbb{Z})$. Let $M_X(\beta,1)$ be the moduli space of slope stable (with respect to a fixed ample divisor) one-dimensional sheaves E on X with $[E] = \beta$ and $\chi(E) = 1$. We have the Hilbert–Chow morphism:

$$\pi\colon M_X(\beta,\chi) \to \mathrm{Chow}_\beta(X).$$

Then Maulik and Toda [2018] defined the invariants $n_{g,\beta}$ via the identity

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\boldsymbol{R}\pi_*\phi_M))q^i = \sum_{g\geq 0} n_{g,\beta}(q^{1/2}+q^{-1/2})^{2g}, \tag{1-2}$$

where ϕ_M on the left-hand side denotes *Joyce's perverse sheaf* on $M_X(\beta,1)$. Here, and throughout, if $\mathcal{P}$ is a constructible complex of sheaves on a space T, we define $\chi(\mathcal{P}) := \sum_{i\in\mathbb{Z}}(-1)^i \dim(\mathrm{H}^i(T,\mathcal{P}))$. One of the main difficulties in this approach to GV theory is the complicated gluing procedure of the construction of ϕ_M: it is defined as a gluing of locally defined perverse sheaves of vanishing cycles.

Let us focus on the case of a local curve $X = \mathrm{Tot}_C(N)$. In this case, the curve class β can be written as $\beta = r[C]$ for some integer $r > 0$, where $[C]$ is the class of the zero section. We define $M_N(r,1) := M_X(r[C],1)$. In this case, Kinjo and Masuda's result [2024] simplifies the situation substantially. They proved that $M_N(r,1)$ can be *globally* written as a critical locus (see also [Kinjo and Koseki 2021])

$$M_N(r,1) \cong \{dj = 0\} \subset M_L(r,1) \xrightarrow{j} \mathbb{C},$$

where L is a line bundle of $\deg(L) \gg 0$, $M_L(r,1)$ denotes the moduli space of slope stable sheaves on $\mathrm{Tot}_C(L)$, and the function j is constructed as follows: The Hilbert–Chow morphism is identified with the *Hitchin fibration*

$$h\colon M_L(r,1) \to \widetilde{B}_L = \bigoplus_{i=1}^{r} \mathrm{H}^0(C, L^{\otimes i}).$$

Then j is the composition of h and a linear function $\tilde{f}\colon \widetilde{B}_L \to \mathbb{C}$. Moreover $\phi_M \cong \phi_j(\mathrm{IC}_{M_L(r,1)})$, and since h is projective, the left-hand side of (1-2) is equal to

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\phi_{\tilde{f}}(\boldsymbol{R}h_*\,\mathrm{IC}_{M_L(r,1)})))q^i.$$

Here and throughout the paper, if Y is a smooth equidimensional variety we denote by $\mathrm{IC}_Y := \mathbb{Q}_Y[\dim(Y)]$ the constant perverse sheaf on Y. Using the equality above, and the distinguished triangle relating vanishing and nearby cycles, we will divide the computations of $n_{g,r[C]}(X)$ into the following two parts (see Proposition 2.15):

(1) GV type invariants of $h\colon M_L(r,1) \to \widetilde{B}_L$,

(2) GV type invariants of $h|_{U_\varepsilon}\colon h^{-1}(U_\varepsilon) \to U_\varepsilon$, where $U_\varepsilon := \{\tilde{f} = \varepsilon\} \subset \widetilde{B}_L$ (for $0 < \varepsilon \ll 1$).

When $r = 2$, we will give a general strategy to compute both (1) and (2). For (1), we will use the explicit description of $\mathrm{H}^*(M_L(2,1),\mathbb{Q})$ by generators and relations [de Cataldo et al. 2012; Hausel and Thaddeus 2003; 2004].

For (2), we will calculate the fibrewise contributions and then integrate. Namely, we have a stratification of U_ε given by the types of singularities of the spectral curves. Since the nearby hyperplane U_ε avoids the origin and we assume $r = 2$, the spectral curves $\widetilde{C}$ over points in U_ε are all *reduced* (more precisely, we first need to remove the trace part $\mathrm{H}^0(L)$). Hence, by [Migliorini and Shende 2013; Maulik and Yun 2014], the fibrewise contribution can be computed by the Euler characteristics of the Hilbert schemes $\mathrm{Hilb}^n(\widetilde{C})$. The latter is then computed by certain HOMFLY polynomials [Maulik 2016; Oblomkov and Shende 2012]. It remains to compute the Euler characteristic of each stratum $S \subset U_\varepsilon$. Our main idea is to realise (a resolution of) the closure of S as a degeneracy locus inside a product of copies of C. These varieties are closely related to generalised de Jonquières divisors [Farkas 2023; Ungureanu 2021].

In principal, the above strategy enables us to compute the GV invariants $n_{g,2[C]}(\mathrm{Tot}_C(N))$ for an arbitrary curve C. However, the difficulty of the calculations in each step grows rapidly when we increase the genus $g(C)$. The authors currently do not know how to resolve this issue.

Remark 1.4 To generalise our strategy to the case $r > 2$, there are at least two difficulties:

(1) For the first part, we do not have an explicit description of $\mathrm{H}^*(M_L(r,1))$ for $r \geq 3$. Instead, we may first need to prove the GV/PT correspondence for $\mathrm{Tot}_C(L \oplus L^{-1} \otimes \omega_C)$, and then use the result of [Monavari 2022].

(2) For the second part, a recursion formula determining the HOMFLY polynomials becomes complicated for $r \geq 3$. However, the serious difficulty is that we cannot assume the spectral curves over U_ε to be reduced.

1.3 Relation to previous work

The GV/PT (and hence GV/GW) correspondence was previously proved in the following cases:

- $X = \mathrm{Tot}_C(N)$ with $g(C) \leq 1$, N is superrigid, and β is arbitrary [Hosono et al. 2001],
- $X = \mathrm{Tot}_C(\omega_C \oplus \mathcal{O}_C)$ and β is arbitrary [Chuang et al. 2014; Hausel et al. 2022; Maulik and Shen 2024],
- $X = \mathrm{Tot}_S(\omega_S)$ and β is arbitrary, where S is a K3 surface [Shen and Yin 2022],
- $X = \mathrm{Tot}_S(\omega_S)$ and β is an irreducible curve class, where S is any smooth projective surface [Maulik and Toda 2018].

For superrigid rational and elliptic curves in the first case, the key fact proved in [Hosono et al. 2001] is that the moduli space $M_X(\beta, 1)$ is smooth and projective.

For $X = \mathrm{Tot}_C(\omega_C \oplus \mathcal{O}_C)$, as observed in [Chuang et al. 2014], the GV/PT correspondence follows from the $P = W$ conjecture, which has recently been proved [Hausel et al. 2022; Maulik and Shen 2024].

Shen and Yin [2022] investigated a compact analogue of the $P = W$ phenomenon for Lagrangian fibrations. As an application, they established the GV/PT correspondence for a local K3 surface.

For an arbitrary local surface $\mathrm{Tot}_S(\omega_S)$, Maulik and Toda [2018] used a similar idea as in this paper, namely, they described the moduli space $M_X(\beta, 1)$ as an explicit critical locus, locally over $\mathrm{Chow}_\beta(X)$, so that they can use the results of [Migliorini and Shende 2013; Maulik and Yun 2014].

1.4 Plan of the paper

The paper is organised as follows. In Section 2, we collect some preliminary results. In particular, we define the GV invariants for local curves and prove that they are computed by the contributions from twisted Higgs bundles and the nearby hyperplane (Proposition 2.15). In Section 3, we prove Theorems 1.2 and 1.3. In Section 4, we prove Theorem 1.3. In the Appendix, we recall some known results in GW theory that we use in the main text for verifying cases of the GV/GW correspondence.

2 Preliminaries

2.1 Twisted Higgs bundles

Let C be a smooth projective curve.

Definition 2.1 Let $\mathcal{V}$ be a vector bundle on C.

(1) A *$\mathcal{V}$-twisted Higgs bundle* is a pair (E, ϕ) consisting of a vector bundle E on C and a homomorphism $\phi\colon E \to E \otimes \mathcal{V}$ such that $\phi \wedge \phi = 0$.

(2) A $\mathcal{V}$-twisted Higgs bundle (E, ϕ) is *μ-stable* if the following condition holds: for any subsheaf $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$ and $\phi(F) \subset F \otimes \mathcal{V}$, we have

$$\frac{\deg(F)}{\mathrm{rk}(F)} < \frac{\deg(E)}{\mathrm{rk}(E)}.$$

We denote by $\mathcal{M}_{\mathcal{V}}(r,d)$ (resp. $M^{st}_{\mathcal{V}}(r,d)$) the moduli stack of $\mathcal{V}$-twisted Higgs bundles (resp. the moduli space of μ-stable $\mathcal{V}$-twisted Higgs bundles) of rank r and degree d.

Theorem 2.2 [Kinjo and Masuda 2024, Theorem 5.6 and Proposition 5.7] *Let N be a rank two vector bundle on C with $\det(N) \cong \omega_C$. Choose an exact sequence*

$$0 \to L^{-1} \otimes \omega_C \to N \to L \to 0$$

for some line bundle L with $\deg(L) \geq 2g(C)-1$. Let $f\colon \mathrm{H}^0(L^{\otimes 2}) \to \mathbb{C}$ be a linear function corresponding to the class $[N] \in \mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ via Serre duality: $\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C) \cong \mathrm{H}^0(L^{\otimes 2})^\vee$. Then we have an isomorphism

$$\mathcal{M}_N(r,d) \cong \mathrm{Crit}(j) \subset \mathcal{M}_L(r,d),$$

where $j\colon \mathcal{M}_L(r,d) \to \mathbb{C}$ is the function defined by $j((E,\phi)) = f(\mathrm{tr}(\phi^2))$.

Corollary 2.3 *In the setting of Theorem 2.2, assume further that $r = 2$. Then we have*

$$M^{st}_N(2,d) \cong \mathrm{Crit}(j) \subset M^{st}_L(2,d). \tag{2-1}$$

Proof We need to prove that the inclusion $i\colon \mathcal{M}_N(2,d) \subset \mathcal{M}_L(2,d)$ preserves μ-stability. Suppose for a contradiction that there exists $(E,\phi) \in M^{st}_N(2,d)$ such that $i(E,\phi)$ is not μ-stable. Then by the proof of [Kinjo and Koseki 2021, Lemma 3.4], the maximal destabilising subsheaf $F \subset E$ satisfies $\mathrm{Hom}(F, E/F \otimes L^{-1} \otimes \omega_C) \neq 0$. Under the current assumption of $\mathrm{rk}(E) = 2$, F and E/F are line bundles with $\deg(F) > \deg(E/F)$. Moreover $\deg(L^{-1} \otimes \omega_C) \leq 2g(C) - 2 - (2g(C)-1) < 0$. This leads to a contradiction. □

Remark 2.4 The isomorphism (2-1) also preserves d-critical structures; see [loc. cit., Proposition 3.6].

2.2 Rigidity of vector bundles

Definition 2.5 Let N be a rank two vector bundle on C with $\det(N) \cong \omega_C$. Then N is 2-*rigid* if $\mathrm{H}^0(\widetilde{C}, l^*N) = 0$ for any stable map $l\colon \widetilde{C} \to C$ of degree two.

Lemma 2.6 *Fix a line bundle L on C with degree $d \geq 2g(C)-1$. Assume that L is a generic element in $\mathrm{Pic}^d(C)$. Then a general $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ is 2-rigid.*

Proof By the arguments in [Bryan and Pandharipande 2006, Section 3], stable bundles in the complement of $\Delta(d) \subset \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ for $d \geq 0$, where

$$\Delta(d) := \{[N] : \text{there exists a saturated subbundle } S \subset N \text{ with } \deg(S) = d \text{ and } \mathrm{H}^0(S^{\otimes 2}) \neq 0\},$$

are 2-rigid. The proof of [loc. cit., Lemma 3.1] shows that

$$\dim \Delta(d) < 3g(C) - 3 < 3g(C) - 2 \leq \dim \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$$

for $0 \leq d \leq g(C) - 1$.

It remains to show that a generic $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ is slope semistable, which follows from Lemma 2.7 below. □

An earlier version of this paper omitted the check that a generic $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ is slope semistable; we thank the referee for pointing this out, as well as suggesting the proof of the following lemma:

Lemma 2.7 *Let L be a line bundle of degree $d \geq g(C) - 1$. Assume that L is a generic element in $\mathrm{Pic}^d(C)$. Then a general $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ is μ-semistable.*

Proof Since μ-semistability is an open condition, it is enough to find a line bundle $L \in \mathrm{Pic}^d(C)$ and a point $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ such that N is μ-semistable.

We may find a line bundle $L_0 \in \mathrm{Pic}^d(C)$ and line bundles M_0 and M_1 of degree $g(C) - 1$ such that there is a surjection $M_0 \oplus M_1 \to L_0$. We may then take a line bundle $L' \in \mathrm{Pic}^0(C)$ such that $M_0 \otimes M_1 \otimes (L')^{\otimes 2} \cong \omega_C$. Now, set $N := (M_0 \oplus M_1) \otimes L'$ and $L := L_0 \otimes L'$. Since M_0 and M_1 have the same degree, N is μ-semistable. Furthermore $L \in \mathrm{Pic}^d(C)$ and there exists a short exact sequence

$$0 \to L^{-1} \otimes \omega_C \to N \to L \to 0,$$

as required. □

2.3 Hitchin fibrations and Gopakumar–Vafa invariants

In this subsection, we fix a line bundle L on C with $\deg(L) \geq 2g(C) - 1$ and an extension class $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$. By the Serre duality $\mathrm{Ext}^1(L, L^{-1} \otimes \omega_C) \cong \mathrm{H}^0(L^{\otimes 2})^\vee$, the class $[N]$ defines a linear function $f : \mathrm{H}^0(L^{\otimes 2}) \to \mathbb{C}$. To simplify the notation, we put $M_L(r,1) := M_L^{\mathrm{st}}(r,1)$. We will also consider the moduli space of traceless Higgs bundles:

$$M_L^0(r,1) := \{(E,\phi) \in M_L(r,1) : \mathrm{tr}(\phi) = 0\}.$$

As before, we have the Hitchin fibration

$$h : M_L(r,1) \to \widetilde{B}_L := \bigoplus_{i=1,2} \mathrm{H}^0(L^{\otimes i}).$$

We denote the restriction of h to the traceless part $M_L^0(r,1)$ by

$$h^0\colon M_L^0(r,1) \to B_L := \mathrm{H}^0(L^{\otimes 2}).$$

We denote by $\tilde{f}\colon \widetilde{B}_L \to \mathbb{C}$ the composition

$$\tilde{f}\colon \widetilde{B}_L \to \mathrm{H}^0(L^{\otimes 2}) \xrightarrow{f} \mathbb{C},$$

where the first map is the projection to the direct summand.

Definition 2.8 (1) Let $X = \mathrm{Tot}_C(N)$ be a local curve. We define the *Gopakumar–Vafa invariants* $n_{g,r[C]}(X) \in \mathbb{Z}$ of X for the curve class $r[C]$ by the following identity:

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\phi_{\tilde{f}}(\boldsymbol{R}h_*\,\mathrm{IC}_{M_L(r,1)})))q^i = \sum_{g\geq 0} n_{g,r[C]}(X)(q^{-1/2}+q^{1/2})^{2g}. \tag{2-2}$$

(2) We define the integers $n_{g,r[C]}(\mathrm{Tot}_C(L))$ by the same equation (2-2) but removing the vanishing cycle functor $\phi_{\tilde{f}}$ on the left-hand side.

Remark 2.9 (1) If one takes $\deg(L) \gg 0$, the above definition agrees with the one in [Maulik and Toda 2018]; see [Kinjo and Koseki 2021, Proposition 3.10, (3.13)]. When $r = 2$, we can take an arbitrary L with $\deg(L) \geq 2g(C)-1$ due to Corollary 2.3.

(2) One can replace the moduli space $M(r,1)$ with $M(r,d)$ for an arbitrary $d \in \mathbb{Z}$ to define similar invariants $n_{g,r[C],d}(X)$, called the generalised GV invariants [Toda 2023]. By the χ-independence proved in [Kinjo and Koseki 2021], we have $n_{g,r[C],d}(X) = n_{g,r[C],1}(X)$ for all d.

Lemma 2.10 *We have*

$$\chi({}^p\mathcal{H}^i(\phi_{\tilde{f}}(\boldsymbol{R}h_*\,\mathrm{IC}_{M_L(r,1)}))) = (-1)^{\deg(L)-g+1}\chi({}^p\mathcal{H}^i(\phi_f(\boldsymbol{R}h^0_*\,\mathrm{IC}_{M_L^0(r,1)}))).$$

Proof We have an isomorphism

$$M_L(r,1) \xrightarrow{\sim} \mathrm{H}^0(L) \times M_L^0(r,1), \quad (E,\phi) \mapsto \big(\mathrm{tr}(\phi), \big(E, \phi - \big(\tfrac{1}{2}\,\mathrm{tr}(\phi)\big)\,\mathrm{id}\big)\big).$$

Under the above isomorphism, the Hitchin map is written as $h = \mathrm{id}_{\mathrm{H}^0(L)} \times h^0$. Then

$$\phi_{\tilde{f}}(\boldsymbol{R}h_*\,\mathrm{IC}_{M_L(r,1)}) = \phi_{\tilde{f}}(\mathrm{IC}_{\mathrm{H}^0(L)} \boxtimes \boldsymbol{R}h^0_*(\mathrm{IC}_{M_L^0(r,1)})) \cong \mathrm{IC}_{\mathrm{H}^0(L)} \boxtimes \phi_f(\boldsymbol{R}h^0_*(\mathrm{IC}_{M_L^0(r,1)})),$$

where the second isomorphism is a special case of the Thom–Sebastiani isomorphism [Massey 2001]. Noting that $\dim \mathrm{H}^0(L) = \deg(L) - g + 1$, we obtain the desired result. □

Lemma 2.11 *Let $P \in \mathrm{Perv}(B_L)$ be an $\mathbb{R}_{>0}$-equivariant perverse sheaf, and denote by $P_0 = i^*P$ the stalk at the origin, where $i\colon 0 \hookrightarrow B_L$ is the inclusion. Then $\chi(P) = \chi(P_0)$.*

Proof By [Kashiwara and Schapira 1994, Proposition 3.7.5], the adjunction morphism $\mathrm{H}(B_L, P) \to \mathrm{H}(B_L, P_0)$ is an isomorphism, and the result follows. □

Lemma 2.12 *Let $P \in \mathrm{Perv}(B_L)$ be a $\mathbb{R}_{>0}$-equivariant perverse sheaf. Then we have*

$$\chi(\phi_f(P)) = \chi(P_0) + \chi(P|_{U_\varepsilon}[-1]),$$

where P_0 denotes the stalk at the origin, and $U_\varepsilon := f^{-1}(\varepsilon)$ (for $0 < \varepsilon \ll 1$) is the nearby hyperplane.

Proof By Lemma 2.11, and from the long exact sequence in cohomology relating the vanishing cycles functor with the nearby cycles functor ψ_f

$$\mathrm{H}(B_L, \phi_f(P)) \to \mathrm{H}(B_L, P|_{f^{-1}(0)}) \to \mathrm{H}(B_L, \psi_f(P)) \to,$$

it is sufficient to prove that $\chi(\psi_f(P)) = \chi(P|_{U_\varepsilon})$. By Lemma 2.11 again, this is equivalent to showing that

$$\chi((\psi_f(P))_0) = \chi(P|_{U_\varepsilon}).$$

Let $s\colon f^{-1}(\mathbb{R}_{>0}) \hookrightarrow B_L$ and $z\colon f^{-1}(0) \hookrightarrow B_L$ be the inclusions. Recall also that $i\colon 0 \hookrightarrow B_L$ denotes the inclusion. Then by the definition of the nearby cycles functor, and one more application of Lemma 2.11,

$$\mathrm{H}(B_L, i_*i^*\psi_f(P)) = \mathrm{H}(B_L, i_*i^*z_*z^*s_*s^*P) \cong \mathrm{H}(B_L, s_*s^*P) \cong \mathrm{H}(f^{-1}(\mathbb{R}_{>0}), s^*P).$$

Finally, since s^*P is $\mathbb{R}_{>0}$-equivariant, the restriction map

$$\mathrm{H}(f^{-1}(\mathbb{R}_{>0}), s^*P) \to \mathrm{H}(U_\varepsilon, P|_{U_\varepsilon})$$

is an isomorphism. □

Remark 2.13 Since we assume that P is $\mathbb{C}^*$-equivariant in the above lemma, $P|_{U_\varepsilon}[-1]$ is perverse.

Motivated by the above lemma, we define the following invariants:

Definition 2.14 Let $U_\varepsilon = f^{-1}(\varepsilon) \subset B_L$ be a nearby hyperplane, and let $h|_{U_\varepsilon}\colon (h^0)^{-1}(U_\varepsilon) \to U_\varepsilon$ be the restriction of the Hitchin fibration. We define the integers $n_g(U_\varepsilon)$ by the following identity:

$$\sum_{i\in\mathbb{Z}} \chi\big({}^p\mathcal{H}^i((\boldsymbol{R}(h^0|_{U_\varepsilon})_*\,\mathrm{IC}_{(h^0)^{-1}(U_\varepsilon)}))\big)q^i = \sum_{g\geq 0} n_g(U_\varepsilon)(q^{-1/2}+q^{1/2})^{2g}.$$

Proposition 2.15 *For $[N] \in \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1}\otimes\omega_C)$, we have*

$$n_{g,r[C]}(\mathrm{Tot}_C(N)) = n_{g,r[C]}(\mathrm{Tot}_C(L)) + n_g(U_\varepsilon).$$

Proof By the decomposition theorem,

$$\boldsymbol{R}h^0_*(\mathrm{IC}_{M^0_L(r,1)}) \cong \bigoplus_i P_i[i]$$

for some perverse sheaves $P_i \in \mathrm{Perv}(B_L)$. By Lemma 2.12,

$$\chi(\phi_f(P_i)) = \chi((P_i)_0) + \chi((P_i)|_{U_\varepsilon}[-1]) \tag{2-3}$$

for each i. By the $\mathbb{C}^*$-equivariance of the Hitchin fibration, $\chi((P_i)_0) = \chi(P_i)$ for each i by Lemma 2.11. Furthermore, $(P_i)|_{U_\varepsilon}[-1]$ is again a perverse sheaf on U_ε, and we have

$$\boldsymbol{R}(h^0|_{U_\varepsilon})_*(\mathrm{IC}_{(h^0)^{-1}(U_\varepsilon)}) \cong \bigoplus_i (P_i)|_{U_\varepsilon}[-1].$$

In summary, the first term (resp. the second term) of the right-hand side of (2-3) computes the invariants $n_{g,r[C]}(\mathrm{Tot}_C(L))$ (resp. $n_g(U_\varepsilon)$). □

Remark 2.16 Assume that $r = 2$. By Proposition 2.15, the computation of $n_{g,2[C]}(\mathrm{Tot}_C(N))$ is divided into two parts:

(1) Computation of $n_{g,2[C]}(\mathrm{Tot}_C(L))$: This can be done by using the explicit description of the global cohomology $\mathrm{H}^*(M_L(2,1),\mathbb{Q})$ by generators and relations [Hausel and Thaddeus 2003; 2004]. The cohomological and the perverse degree of each generator is determined in [de Cataldo et al. 2012].

(2) Computation of $n_g(U_\varepsilon)$: This part can be done by determining the fibrewise contributions. Since U_ε avoids the origin, the spectral curve C_a is *reduced* at every point of $a \in U_\varepsilon$. Hence, the fibrewise contributions are computed by certain HOMFLY polynomials [Migliorini and Shende 2013; Maulik 2016; Maulik and Yun 2014; Oblomkov and Shende 2012]. We further need to calculate the stratification of U_ε induced by the types of singularities of the spectral curves. This last step requires a careful analysis of the geometry of certain degeneracy loci. This analysis becomes increasingly complicated as we allow the genus to increase, and for this reason in this paper we only give the full result in the case $g(C) = 2$.

3 Genus two

Let C be a smooth projective curve of genus two. Let L be a line bundle on C of degree $3 = 2g(C) - 1$. Let $f : \mathrm{H}^0(L^{\otimes 2}) \to \mathbb{C}$ be a linear function. The goal of this section is to compute the GV invariants associated to L and f. We apply the strategy explained in Remark 2.16.

3.1 Stratification of the Hitchin base

We have the following diagram, in which the square is Cartesian:

$$\begin{array}{ccccc}
U_\varepsilon = f^{-1}(\varepsilon) & \hookrightarrow & B_L^* := \mathrm{H}^0(L^{\otimes 2}) \setminus \{0\} & & \\
& \searrow & \downarrow & & \\
& & \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) & \longrightarrow & \{L^{\otimes 2}\} \\
& & \big\downarrow & & \big\downarrow \\
& & \mathrm{Sym}^6(C) & \xrightarrow{\mathrm{AJ}^6} & \mathrm{Pic}^6(C)
\end{array}$$

We define $H_\infty := \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) \setminus U_\varepsilon$. By a partition $\lambda \vdash n$ of a number $n \in \mathbb{Z}_{\geq 1}$ we mean a weakly descending sequence $(\lambda_1, \dots, \lambda_{r(\lambda)})$ of integers $\lambda_i \in \mathbb{Z}_{\geq 1}$ such that $\sum_{i=1}^{r(\lambda)} \lambda_i = n$. We write $\mu \vdash \lambda$ to mean that the partition μ is a refinement of the partition λ, i.e., there is a partition of the set $\{1, \dots, r(\mu)\}$ such that λ is the partition of $n = |\mu|$ induced by μ along with this partition of sets. We have a natural stratification $\{S_\lambda\}_{\lambda \vdash 6}$ of $\mathrm{Sym}^6(C)$, indexed by partitions of 6:

$$S_\lambda := \{\lambda_1 x_1 + \dots + \lambda_m x_m : x_i \in C,\ x_i \neq x_j \text{ for } i \neq j\}.$$

The closures of the strata are given by

$$\overline{S}_\lambda = \{\lambda_1 x_1 + \cdots + \lambda_m x_m : x_i \in C\} = \coprod_{\lambda \vdash \mu} S_\mu.$$

Overview Here we explain a general strategy for computing the Euler characteristic of the stratum $U_\varepsilon \cap S_\lambda$, which also works for $g(C) > 2$.

(1) We first construct a map $\Psi\colon Z \to \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) \cap \overline{S}_\lambda$ from a smooth variety Z, as follows: Consider the diagram

$$\begin{array}{ccc} & C \times C^{\times m} & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\mu} \\ C & & C^{\times m} \end{array}$$

On $C^{\times m}$, we have a natural morphism of vector bundles

$$\alpha\colon \mathrm{H}^0(L^{\otimes 2}) \otimes \mathcal{O}_{C^{\times m}} \to \mu_*(\sigma^* L^{\otimes 2} \otimes \mathcal{O}_{\lambda_1 \Delta_1 + \cdots + \lambda_m \Delta_m}), \tag{3-1}$$

where

$$\Delta_i := \{(x, (y_j)_{1 \le j \le m}) \in C \times C^{\times m} : x = y_i\}.$$

The restriction of α to a point $(y_j)_{1 \le j \le m} \in C^{\times m}$ is the natural map

$$\mathrm{H}^0(L^{\otimes 2}) \to L^{\otimes 2}|_{\lambda_1 y_1 + \cdots + \lambda_m y_m}.$$

We define Z to be the $(\dim \mathrm{H}^0(L^{\otimes 2}) - 1)$-th degeneracy locus of α (This variety Z is called a *generalised de Jonquières divisor* in the literature [Farkas 2023; Ungureanu 2021].) By construction, we have a natural map $\Psi\colon Z \to \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) \cap \overline{S}_\lambda$.

(2) Next, we remove the pullback D of H_∞ from Z. D can be similarly described as a degeneracy locus, by replacing $\mathrm{H}^0(L^{\otimes 2})$ with its hyperplane $\{f = 0\}$ in (3-1).

(3) We then remove the deeper strata S_μ, $\lambda \vdash \mu$. We have that

$$\Psi^{-1}(S_\mu \cap U_\varepsilon) \to S_\mu \cap U_\varepsilon$$

is unramified of degree $d_{\mu,\lambda}$, where $d_{\mu,\lambda}$ is the number of ways to refine μ to λ. Finally, we obtain

$$d_\lambda \chi(S_\lambda \cap U_\varepsilon) = \chi(Z) - \chi(D) - \sum_{\mu \vdash \lambda} d_{\mu,\lambda} \chi(S_\mu \cap U_\varepsilon),$$

where d_λ is the size of the automorphism group of λ, i.e.,

$$d_\lambda = \prod_{n \ge 1} |\{i \ge 1 : \lambda_i = n\}|!.$$

We have used the fact that if $S' \to S$ is an unramified $n{:}1$ cover of a variety S, then $\chi(S') = n\chi(S)$. In what follows we will denote such S' by $S^{[n:1]}$; although there may be many such covers of S, the ambiguity in the notation will be irrelevant as long as we are only interested in the Euler characteristic of S'.

Example 3.1 (1) Suppose that $\mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})\cap S_\lambda$ is 0-dimensional. Since $\mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})\subset \mathrm{Sym}^{2g(C)-1}(C)$ has codimension $g=\dim J(C)$ and the dimension of S_λ is equal to the number of rows in λ, this happens precisely when

$$\#\{\text{rows in } \lambda\} = g(C).$$

In this case,

$$\chi(U_\varepsilon \cap S_\lambda) = \chi(Z)/d_\lambda$$

is given by the classical de Jonquières formula [1866].

(2) Suppose that

$$k := \#\{i : \lambda_i = 1\} > g(C).$$

In this case, slightly modifying the above construction, we define Z' via the Cartesian diagram

$$\begin{array}{ccc} Z' & \longrightarrow & \mathrm{Sym}^k(C) \\ \big\downarrow & & \big\downarrow \\ C^{\times(m-k)} & \xrightarrow{\ \eta\ } & \mathrm{Pic}^k(C) \end{array}$$

where $\eta\colon C^{\times(m-k)} \to \mathrm{Pic}^k(C)$ is defined by

$$\eta((x_i)) = L^{\otimes 2}(-\lambda_1 x_1 - \cdots - \lambda_{m-k}x_{m-k}).$$

Since $k > g(C)$, the Abel–Jacobi map $\mathrm{Sym}^k(C)\to \mathrm{Pic}^k(C)$ is a projective space bundle; hence so is $Z' \to C^{\times(m-k)}$. It is easier to compute $\chi(Z')$ than to compute $\chi(Z)$ for a degeneracy locus Z.

As in the strategy explained above, we then remove $D'\subset Z'$, the pullback of H_∞. It will turn out that D' is also a projective space bundle over $C^{\times(m-k)}$, of one dimension less than Z'. Finally, we will remove the deeper strata from Z' by the same formula as above.

Proposition 3.2 *Let L and f be general. Then we have*

$$\chi(U_\varepsilon \cap S_\lambda) = \begin{cases} 0 & \text{if } \lambda = (6), \\ 50 & \text{if } \lambda = (5,1), \\ 128 & \text{if } \lambda = (4,2), \\ -216 & \text{if } \lambda = (4,1,1), \\ 81 & \text{if } \lambda = (3,3), \\ -668 & \text{if } \lambda = (3,2,1), \\ 542 & \text{if } \lambda = (3,1,1,1), \\ -128 & \text{if } \lambda = (2,2,2), \\ 968 & \text{if } \lambda = (2,2,1,1), \\ -1012 & \text{if } \lambda = (2,1,1,1,1). \end{cases} \tag{3-2}$$

3.1.1 $\lambda = (6)$ We want to find $x \in C$ such that $\mathcal{O}(6x) \cong L^{\otimes 2}$. The locus $\{\mathcal{O}(6x) : x \in C\} \subset \mathrm{Pic}^6(C)$ is the image of the composition

$$C \cong \Theta \subset \mathrm{Pic}^1(C) \xrightarrow{(-)^{\otimes 6}} \mathrm{Pic}^6(C),$$

i.e., a divisor in $\mathrm{Pic}^6(C)$. Hence, by choosing $L \in \mathrm{Pic}^3(C)$ general, we have $L^{\otimes 2} \notin \{\mathcal{O}(6x) : x \in C\} \subset \mathrm{Pic}^6(C)$, that is, $S_{(6)} \cap \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) = \varnothing$.

3.1.2 $\lambda = (5, 1)$ Consider the following composition:

$$\eta\colon \mathrm{Pic}^1(C) \xrightarrow{(-)^{\otimes 5}} \mathrm{Pic}^5(C) \xrightarrow{L^{\otimes 2} \otimes (-)^{-1}} \mathrm{Pic}^1(C).$$

We have

$$S_{(5,1)} \cap \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) = \{(x, y) \in \Theta \times \Theta : \eta(x) = y\} = \{50 \text{ points}\},$$

where $50 = \eta_*\Theta.\Theta = 5^2\Theta^2$. Note also that we have chosen a general L so that $\eta(\Theta)$ and Θ intersect properly. By taking the linear function f to be generic, the divisor H_∞ does not intersect with the finite set of points $S_{(5,1)} \cap \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$.

3.1.3 $\lambda = (4, 2)$ Similarly to $\lambda = (5, 1)$, we get $S_{(2,4)} \cap U_\varepsilon = \{128 \text{ points}\}$, where $128 = 4^2 \cdot 2^2 \cdot \Theta^2$.

3.1.4 $\lambda = (4, 1, 1)$ Let $Z \subset C \times \mathrm{Sym}^2(C)$ be the subvariety defined by $\mathcal{O}_C(4x + y + z) \cong L^{\otimes 2}$. We claim that there is an isomorphism $\psi\colon C \to Z$ sending x to the inverse under Ψ of the unique section of $L^{\otimes 2}$ vanishing to order (at least) 4 at x. This follows from the fact that for all $x \in C$ there is an equality $h^0(L^{\otimes 2}(-4x)) = 1$. This in turn follows from genericity of L and the Riemann–Roch theorem.

We calculate

$$Z \cap H_\infty = 36$$

as follows. The divisor H_∞ is linearly equivalent to the divisor D_p defined by the vanishing of the section s at p, for some fixed $p \in C$. If a section $s = \psi(q)$ in Z vanishes at p, either $p = x$, in which case the intersection multiplicity is 4, since the regular function defining D_p vanishes to order 4, or p is one of the other two zeros of $\Psi(q)$. Counting these points corresponds to counting solutions to $L^{\otimes 2}(-p) \cong \mathcal{O}_C(4x + y)$. But now this is the same theta divisor calculation as we have seen before, and we find that there are $4^2 \cdot 1^2 \cdot 2$ such points. So $Z \cap H_\infty = 32 + 4 = 36$.

We also need to remove deeper strata from Z. We find that it contains one copy each of $S_{(5,1)}$ and $S_{(4,2)}$. So finally we calculate

$$\chi(S_{(4,1,1)} \cap U_\varepsilon) = -2 - 36 - 50 - 128 = -216.$$

3.1.5 $\lambda = (3, 3)$ As in the case of $\lambda = (5, 1)$, the number of solutions to $\mathcal{O}_C(3x + 3y) \cong L^{\otimes 2}$ is

$$3^2 \cdot 3^2 \cdot \Theta^2 = 162.$$

Here we are double counting: two solutions (x, y) and (x', y') with $x = y'$ and $y = x'$ should only count for one point. So we find

$$\chi(S_{(3,3)} \cap U_\varepsilon) = 81.$$

3.1.6 $\lambda = (3, 2, 1)$ Let $Z \subset C \times C \times C$ be the subvariety defined by $\mathcal{O}_C(3x + 2y + z) \cong L^{\otimes 2}$. We claim that $\chi(Z) = -196$. We postpone the proof until the next subsection; see Proposition 3.6.

Fixing some $p \in C$, the preimage D of the divisor H_∞ is equivalent to the divisor $3\{p\} \times C \times C + 2C \times \{p\} \times C + C \times C \times \{p\}$. So D consists of $132 = 3 \cdot 8 + 2 \cdot 18 + 72$ points. These numbers come from the same calculation of intersections of theta divisors as before, so for example 72 appears via $72 = 2 \cdot 2^2 \cdot 3^2$. Z also contains some copies of the deeper strata: it contains one copy of $S_{(4,2)}$ (corresponding to $z = x$), one copy of $S_{(5,1)}$ given by setting $x = y$, and a variety $S^{[2:1]}_{(3,3)}$ (corresponding to $y = z$). So putting everything together we find

$$\chi(S_{(3,2,1)} \cap U_\varepsilon) = -196 - 132 - 128 - 50 - 162 = -668.$$

3.1.7 $\lambda = (3, 1^3)$ Consider Z, the subvariety of pairs $(x, (y, z, w)) \in C \times \mathrm{Sym}^3(C)$ giving solutions to $\mathcal{O}_C(3x + y + z + w) \cong L^{\otimes 2}$. Projecting along the morphism that forgets y, z, and w, this is a $\mathbb{P}^1$-bundle $\pi\colon Z \to C$, and so has Euler characteristic $-4 = 2(-2)$.

We claim that the pullback D of H_∞ to Z is isomorphic to C. To show this, it is enough to prove that $D.F = 1$, where F is the fibre class. The fibre of π over a point $x_0 \in C$ is $\pi^{-1}(x_0) = \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}(-3x_0))$. The map $\pi^{-1}(x_0) \to \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ is induced by a natural embedding $u\colon \mathrm{H}^0(L^{\otimes 2}(-3x_0)) \hookrightarrow \mathrm{H}^0(L^{\otimes 2})$ and hence is linear, and meets the hyperplane H_ε at a single point. Since f is generic, and C is one-dimensional, the image of u intersects H_ε transversally for all x_0.

Next we remove deeper strata: one copy of $S_{(3,2,1)}$, one copy of $S_{(4,1,1)}$, a variety $S^{[2:1]}_{(3,3)}$, one copy of $S_{(4,2)}$, and one copy of $S_{(5,1)}$. Putting everything together we find

$$\chi(S_{(3,1,1,1)} \cap U_\varepsilon) = -4 - (-2) - (-668) - (-216) - 2 \cdot 81 - 128 - 50 = 542.$$

3.1.8 $\lambda = (2^3)$ Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^3(C) & \xhookrightarrow{2(-)} & \mathrm{Sym}^6(C) \\ \big\downarrow{\scriptstyle \mathrm{AJ}^3} & & \big\downarrow{\scriptstyle \mathrm{AJ}^6} \\ \mathrm{Pic}^3(C) & \xrightarrow{(-)^{\otimes 2}} & \mathrm{Pic}^6(C) \end{array}$$

The closure $\overline{S}_{(2^3)} \cap \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ is the fibre $\eta^{-1}(L^{\otimes 2})$, where $\eta = \mathrm{AJ}^6 \circ 2(-) = (-)^{\otimes 2} \circ \mathrm{AJ}^3$. The morphism AJ^3 is a $\mathbb{P}^1$-fibration. Hence

$$\overline{S}_{(2^3)} \cap \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2}) = \coprod_{16} \mathbb{P}^1.$$

The embeddings $\mathbb{P}^1 \hookrightarrow \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ are quadric, induced by

$$\mathrm{H}^0(M) \hookrightarrow \mathrm{H}^0(L^{\otimes 2}), \quad s \mapsto s^{\otimes 2},$$

where M is a choice of a square root of $L^{\otimes 2}$. Hence $\mathbb{P}^1 \cap H_\infty = \{2 \text{ points}\}$.

We conclude that

$$\bar{S}_{(2^3)} \cap U_\varepsilon = \coprod_{16} (\mathbb{P}^1 \setminus \{2 \text{ points}\}).$$

By removing the deeper stratum $S_{(4,2)}$, we obtain

$$\chi(U_\varepsilon \cap S_{(2^3)}) = 16(\chi(\mathbb{P}^1) - 2) - 128 = -128.$$

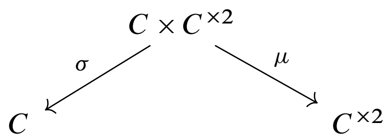
3.1.9 $\boldsymbol{\lambda = (2,2,1,1)}$ Let $Z \subset C \times C \times \mathrm{Sym}^2(C)$ be the subvariety defined by $\mathcal{O}_C(2x+2y+z+w) \cong L^{\otimes 2}$. The projection $\pi\colon Z \to C \times C$ is given by forgetting z and w, and is the blowup of $C^{\times 2}$ at 32 points corresponding to solutions to $L^{\otimes 2}(-2x-2y) \cong \omega_C$. So

$$\chi(Z) = 4 + 32 = 36.$$

Let D be the pullback of H_∞ via $Z \to \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$.

Lemma 3.3 *We have $\chi(D) = -176$.*

Proof Consider the rational map $\phi\colon C \times C \dashrightarrow \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ which sends generic $(x,y) \in C \times C$ to the one-dimensional subspace $\mathrm{H}^0(L^{\otimes 2}(-2x-2y)) \subset \mathrm{H}^0(L^{\otimes 2})$. As discussed above, the indeterminacy locus of this rational map is 32 points in $C \times C$. In the following, we calculate the line bundle inducing the rational map ϕ. Consider the diagram



where σ is the first projection and μ is the projection forgetting the first factor. Then the line bundle inducing the rational map ϕ is the double dual of $\mathcal{F} := \mu_*(\sigma^* L^{\otimes 2}(-2\Delta_1 - 2\Delta_2))$, where

$$\Delta_i := \{(x,(y_1,y_2)) \in C \times C^{\times 2} : x = y_i\} \quad \text{for } i = 1,2.$$

We calculate the first Chern character of $\mathcal{F}$ by the Grothendieck–Riemann–Roch formula. Let $\eta \in \mathrm{H}^2(C \times C^{\times 2}, \mathbb{Q})$ denote the dual class of $\mathrm{pt} \times C^{\times 2}$. From now on we freely refer to cohomology classes by using their dual homology classes. We have

$$\begin{aligned} \mathrm{ch}(\sigma^* L^{\otimes 2}(-2\Delta_1 - 2\Delta_2)).\,\mathrm{td}_\mu &= \big(1, -2\Delta_1 - 2\Delta_2, \tfrac{1}{2}4(\Delta_1+\Delta_2)^2, *\big).(1, 6\eta, 0, 0).(1, -\eta, 0, 0) \\ &= (1, 5\eta - 2\Delta_1 - 2\Delta_2, *_2, *_3), \end{aligned}$$

where

$$*_2 = 2(\Delta_1+\Delta_2)^2 - 10\eta.(\Delta_1+\Delta_2) = 2(-2C_3 + 2\delta - 2C_2) - 10(C_3 + C_2) = -14C_3 - 14C_2 + 4\delta,$$

and δ denotes the class of the small diagonal, C_2 denotes the class of $\mathrm{pt} \times C \times \mathrm{pt}$, and C_3 is the class of $\mathrm{pt} \times \mathrm{pt} \times C$. The intersection $\Delta_1^2 = -2C_3$ can be calculated as

$$\Delta_1^2 = p_{12}^*(\Delta^2) = -2p_{12}^*(\mathrm{pt}),$$

where $p_{12}\colon C \times (C^{\times 2}) \to C \times C$ is given by forgetting the third factor, and $\Delta \subset C \times C$ is the diagonal. The other parts of the calculations for $*_2$ are similar.

Hence, we obtain $\mathrm{ch}_1(\mathcal{F}) = -14(f_1 + f_2) + 4\Delta$, and

$$\phi^*\mathcal{O}(1) = \mathcal{O}_{C\times C}(14(f_1 + f_2) - 4\Delta),$$

where f_1 is the class of $C \times \mathrm{pt}$ and f_2 is the class of $\mathrm{pt} \times C$. Now, the map $Z \to \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ is induced by the strict transform of $\pi^*(14(f_1 + f_2) - 4\Delta)$, and hence

$$D = \pi^*(14(f_1 + f_2) - 4\Delta) - e,$$

where e denotes the exceptional divisor of π. We obtain:

$$\begin{aligned} -\chi(D) &= \deg K_D = (K_Z + D).D \\ &= ((\pi^*(2f_1 + 2f_2) + e) + (\pi^*(14(f_1 + f_2) - 4\Delta) - e)).(\pi^*(14(f_1 + f_2) - 4\Delta) - e) \\ &= (16(f_1 + f_2) - 4\Delta).(14(f_1 + f_2) - 4\Delta) = 2 \cdot 16 \cdot 14 - 2(16 + 14)4 + 4^2(-2) = 176. \quad \square \end{aligned}$$

Z contains points corresponding to deeper strata; for example it contains three copies of $S_{(4,2)}$ given by either setting $z = w = x$, or $z = w = y$, or $x = y$ and $z = w$.

In all, we must remove three copies of $S_{(4,2)}$, a double cover $S^{[2:1]}_{(3,3)}$, two copies of $S_{(3,2,1)}$, a cover $S^{[6:1]}_{(2,2,2)}$, one copy of $S_{(5,1)}$, and one copy of $S_{(4,1,1)}$. The number six deserves some explanation. Let (a, b, c) be a point of $\mathrm{Sym}^6(C)$, lying in the image of $S_{(2,2,2)}$. Then we can lift it to a point of Z in six different ways, the first by setting $x = a$, $y = b$, and $z = w = c$, and the other five by permuting a, b, and c in these three equalities. Finally, note that after removing T, the union of all of these contributions from deeper strata, the cover $Z \setminus T \to S_{(2,2,1,1)}$ is $2:1$. Putting this all together, we find

$$\chi(S_{(2,2,1,1)}) = (36 - (-176) - 3 \cdot 128 - 2 \cdot 81 - 2(-668) - 6(-128) - 50 - (-216))/2 = 968.$$

3.1.10 $\boldsymbol{\lambda = (2, 1^4)}$ We consider solutions $Z \subset C \times \mathrm{Sym}^4(C)$ to $\mathcal{O}_C(2x + y + z + w + t) \cong L^{\otimes 2}$. This is a $\mathbb{P}^2$-bundle $r\colon Z \to C$, where r forgets y, z, w, and t. So

$$\chi(P) = 3(-2) = -6.$$

Let D be the pullback of H_∞ to Z, and F a fibre of r. As in the case $\lambda = (3, 1, 1, 1)$, we can see that $D \cap F$ is isomorphic to $\mathbb{P}^1$, linearly embedded into F. In other words, D is a $\mathbb{P}^1$-bundle over C and so $\chi(D) = -4 = (-2) * 2$.

We need to remove the deeper strata:

$$S^{[2:1]}_{(2,2,1,1)} + S_{(3,1,1,1)} + S^{[3:1]}_{(2,2,2)} + 2S_{(3,2,1)} + S_{(4,1,1)} + S^{[2:1]}_{(3,3)} + 2S_{(4,2)} + S_{(5,1)}.$$

The degree of a cover $mS^{[n:1]}_\pi \to S_\pi$ is mn. The above degrees are given as follows: a point in, say $S_{(3,2,1)}$, is represented by a tuple $(x_0, \dots, x_5) \in \mathrm{Sym}^6(C)$ with $x_0 = x_1 = x_2$ and $x_3 = x_4$. In order to lift to Z we need to pick one out of x_0 or x_3 to be the value of x. In general the degree of the cover over S_π is given by counting the number of rows in π of length at least 2.

Putting this all together we calculate

$$\chi(S_{(2,1^4)} \cap U_\varepsilon) = -6-(-4)-2\cdot 968-542-3(-128)-2(-668)-(-216)-2\cdot 81-2\cdot 128-50 = -1012.$$

3.2 A degeneracy locus calculation

Let $Z \subset C \times C \times C$ be the subvariety defined by $\mathcal{O}_C(3x+2y+z) \cong L^{\otimes 2}$. The goal of this subsection is to compute the Euler characteristic of Z. To do so, we first describe Z as a degeneracy locus. Consider the diagram

$$C \xleftarrow{\ \sigma\ } C \times C^{\times 3} \xrightarrow{\ \mu\ } C^{\times 3}$$

where σ denotes the first projection and μ denotes the projection forgetting the first factor. On $C \times C^{\times 3}$, we have a natural map of vector bundles

$$\sigma^* L^{\otimes 2} \to \sigma^* L^{\otimes 2} \otimes \mathcal{O}_{3\Delta_1+2\Delta_2+\Delta_3},$$

where

$$\Delta_i := \{(x,(y_1,y_2,y_3)) \in C \times C^{\times 3} : x = y_i\}$$

for $i = 1,2,3$. Pushing this down via μ, we obtain

$$\alpha \colon \mathrm{H}^0(C, L^{\otimes 2}) \otimes \mathcal{O}_{C^{\times 3}} \to \mathcal{F} := \mu_*(\sigma^* L^{\otimes 2} \otimes \mathcal{O}_{3\Delta_1+2\Delta_2+\Delta_3}).$$

The fibre of the vector bundle $\mathcal{F}$ at a point $(x,y,z) \in C^{\times 3}$ is the six-dimensional vector space $L^{\otimes 2}|_{3x+2y+z}$.

Hence, our variety $Z \subset C^{\times 3}$ is the 4-th degeneracy locus of α, or equivalently, of $\alpha^\vee$.

Lemma 3.4 *We have* $\chi(Z) = c_2(\mathcal{F})c_1(C^{\times 3}) - c_1(\mathcal{F})c_2(\mathcal{F}) - c_3(\mathcal{F})$.

Proof For simplicity, we put $V = \mathrm{H}^0(C, L^{\otimes 2})$. We will apply [Pragacz 1988, Example 5.8(i)] to the map

$$\alpha^\vee \colon \mathcal{F}^\vee \to V \otimes \mathcal{O}.$$

Note that $m = 6$, $n = 5$, and $r = 4$ in the notations of [loc. cit.]. We first fix some more notation. For a vector bundle E, we denote by $s_i(E) = (-1)^i c_i(-E)$ the i-th Segre class of E, and for two vector bundles E and E', we put

$$s_i(E - E') := \sum_{p=0}^{i} (-1)^{i-p} s_p(E) c_{i-p}(E').$$

Finally, we put

$$s_{(2,1)}(E-E') := s_2(E-E')s_1(E-E') - s_3(E-E')s_0(E-E').$$

Now, by [Pragacz 1988, Example 5.8(i)], we have

$$\begin{aligned}\chi(Z) &= s_2(V\otimes\mathcal{O}-\mathcal{F}^\vee)c_1(C^{\times 3}) - s_{(2,1)}(V\otimes\mathcal{O}-\mathcal{F}^\vee) - 2s_3(V\otimes\mathcal{O}-\mathcal{F}^\vee)\\ &= c_2(\mathcal{F}^\vee)c_1(C^{\times 3}) - (-c_2(\mathcal{F}^\vee)c_1(\mathcal{F}^\vee)+c_3(\mathcal{F}^\vee)) + 2c_3(\mathcal{F}^\vee)\\ &= c_2(\mathcal{F})c_1(C^{\times 3}) - c_1(\mathcal{F})c_2(\mathcal{F}) - c_3(\mathcal{F})\end{aligned}$$

as desired. □

We will use the following cohomology classes to express the Chern classes of $\mathcal{F}$:

- $\eta := \sigma^*\,\mathrm{pt}$,
- $\Delta_i := \{(x,(y_1,y_2,y_3)) \in C\times C^{\times 3} : x = y_i\}$ for $i = 1,2,3$,
- $\Delta_{ij} := \{(x,(y_1,y_3,y_3)) \in C\times C^{\times 3} : x = y_i = y_j\}$ for $1 \le i < j \le 3$,
- $C_{ij} := \tilde{p}_{ij}^*\,\mathrm{pt}$, where $\tilde{p}_{ij}\colon C\times C^{\times 3}\to C\times C$ denotes the projection *forgetting* the $(i+1)$-th and $(j+1)$-th factors for $1 \le i < j \le 3$,
- $C_i := \tilde{p}_i^*\,\mathrm{pt}$, where $\tilde{p}_i\colon C\times C^{\times 3}\to C^{\times 3}$ denotes the projection forgetting the $(i+1)$-th factor for $i = 1,2,3$,
- δ is the small diagonal in $C\times C^{\times 3}$.

By abuse of notation, we denote the pushforward via μ of the above classes by the same symbols. We first compute the Chern character of $\mathcal{F}$ by using the Grothendieck–Riemann–Roch formula:

Lemma 3.5 *We have*

(1) $\mathrm{ch}_0(\mathcal{F}) = 6$,

(2) $\mathrm{ch}_1(\mathcal{F}) = 24C_{23} + 14C_{13} + 6C_{12} - 6\Delta_{12} - 3\Delta_{13} - 2\Delta_{23}$,

(3) $\mathrm{ch}_2(\mathcal{F}) = -60C_3 - 27C_2 - 16C_1 + 6\delta$,

(4) $\mathrm{ch}_3(\mathcal{F}) = 66\,\mathrm{pt}$.

Proof We denote by $*_i$ the i-th component of $\mathrm{ch}(\sigma^*L^{\otimes 2}\otimes\mathcal{O}_{3\Delta_1+2\Delta_2+\Delta_3}).\,\mathrm{td}_\mu$. It is obvious that

$$*_0 = 0, \quad *_1 = 3\Delta_1 + 2\Delta_2 + \Delta_3.$$

Noting that $\mathrm{ch}(\sigma^*L^{\otimes 2}).\,\mathrm{td}_\mu = (1, 5\eta, 0, 0, 0)$, we obtain

$$\begin{aligned}*_2 &= -\tfrac{1}{2}(3\Delta_1+2\Delta_2+\Delta_3)^2 + 5\eta.(3\Delta_1+2\Delta_2+\Delta_3)\\ &= -\tfrac{1}{2}(-18C_{23} - 8C_{13} - 2C_{12} + 12\Delta_{12} + 6\Delta_{13} + 4\Delta_{23}) + 15C_{23} + 10C_{13} + 5C_{13}\\ &= 24C_{23} + 14C_{13} + 6C_{12} - 6\Delta_{12} - 3\Delta_{13} - 2\Delta_{23},\end{aligned}$$

$$
\begin{aligned}
*_3 &= \tfrac{1}{6}(3\Delta_1+2\Delta_2+\Delta_3)^3-\tfrac{5}{2}\eta.(3\Delta_1+2\Delta_2+\Delta_3)^2\\
&= \tfrac{1}{6}(-18C_{23}-8C_{13}-2C_{12}+12\Delta_{12}+6\Delta_{13}+4\Delta_{23}).(3\Delta_1+2\Delta_2+\Delta_3)\\
&\qquad\qquad -\tfrac{5}{2}\eta.(-18C_{23}-8C_{13}-2C_{12}+12\Delta_{12}+6\Delta_{13}+4\Delta_{23})\\
&= \tfrac{1}{6}\bigl(-24C_3-6C_2+36(-2)C_3+18(-2)C_2+12\delta-36C_3-4C_1+24(-2)C_3+12\delta+8(-2)C_1\\
&\qquad\qquad -18C_2-8C_1+12\delta+6(-2)C_2+4(-2)C_1\bigr)\\
&\qquad\qquad +5(-6C_3-3C_2-2C_1)\\
&= -30C_3-12C_2-6C_1+6\delta-30C_3-15C_2-10C_1\\
&= -60C_3-27C_2-16C_1+6\delta.\\
*_4 &= -\tfrac{1}{4}\cdot\tfrac{1}{6}(3\Delta_1+2\Delta_2+\Delta_3)^4+\tfrac{5}{6}\eta.(3\Delta_1+2\Delta_2+\Delta_3)^3\\
&= -\tfrac{1}{4}(-30C_3-12C_2-6C_1+6\delta).(3\Delta_1+2\Delta_2+\Delta_3)+5\eta.(-30C_3-12C_2-6C_1+6\delta)\\
&= -\tfrac{1}{4}(-18-36-24-24-30-12)+5\cdot 6\\
&= 66.
\end{aligned}
$$

Most of the above calculations are straightforward. Slightly nontrivial ones are, for example:

$$\Delta_1^2=-2C_{23},\quad \Delta_1^3=0,\quad \delta.\Delta_1=-2\,\mathrm{pt}.$$

The first two equalities follow by observing that $\Delta_1=p_{01}^*\Delta$, where p_{01} is the projection to the first two factors, and $\Delta\subset C\times C$ denotes the diagonal. Then the third one follows, as $\delta=\Delta_1.\Delta_{23}$.

By pushing forward $*_i$ via μ, we obtain the result. □

Proposition 3.6 *We have* $\chi(Z)=-196$.

Proof We need to calculate the Chern classes of $\mathcal{F}$ from its Chern character:

$$c_2(\mathcal{F})=\tfrac{1}{2}\,\mathrm{ch}_1(\mathcal{F})^2-\mathrm{ch}_2(\mathcal{F}),\quad c_3(\mathcal{F})=\tfrac{1}{6}\,\mathrm{ch}_1(\mathcal{F})^3-\mathrm{ch}_1(\mathcal{F})\,\mathrm{ch}_2(\mathcal{F})+2\,\mathrm{ch}_3(\mathcal{F}).$$

Firstly,

$$
\begin{aligned}
\mathrm{ch}_1(\mathcal{F})^2 &= (24C_{23}+14C_{13}+6C_{12}-6\Delta_{12}-3\Delta_{13}-2\Delta_{23})^2\\
&= 24(14C_3+6C_2-6C_3-3C_2-2C_{23}.\Delta_{23})\\
&\qquad +14(24C_3+6C_1-6C_3-3C_{13}.\Delta_{13}-2C_1)\\
&\qquad +6(24C_2+14C_1-6C_{12}.\Delta_{12}-3C_2-2C_1)\\
&\qquad -6(24C_3+14C_3+6C_{12}.\Delta_{12}-6(-2)C_3-3\delta-2\delta)\\
&\qquad -3(24C_2+14C_{13}.\Delta_{13}+6C_2-6\delta-3(-2)C_2-2\delta)\\
&\qquad -2(24C_{23}.\Delta_{23}+14C_1+6C_1-6\delta-3\delta-2(-2)C_1)\\
&= 144C_3+90C_2+80C_1-96C_{23}.\Delta_{23}-84C_{13}.\Delta_{13}-72C_{12}.\Delta_{12}+72\delta.
\end{aligned}
$$

We then have

$$\begin{aligned}\mathrm{ch}_1(\mathcal{F})^3 &= (144C_3+90C_2+80C_1-96C_{23}.\Delta_{23}-84C_{13}.\Delta_{13}-72C_{12}.\Delta_{12}+72\delta)\\ &\qquad .(24C_{23}+14C_{13}+6C_{12}-6\Delta_{12}-3\Delta_{13}-2\Delta_{23})\\ &= 24(80-84-72+72)+14(90-96-72+72)+6(144-96-84+72)\\ &\quad -6(90+80-96-84-72(-2)+72(-2))-3(144+80-96-84(-2)-72+72(-2))\\ &\quad -2(144+90-96(-2)-84-72+72(-2))\\ &= -396\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}_1(\mathcal{F})\,\mathrm{ch}_2(\mathcal{F}) &= -60(6-3-2)-27(14-6-2)-16(24-6-3)+6(24+14+6-6(-2)-3(-2)-2(-2))\\ &= -66.\end{aligned}$$

We also have

$$\begin{aligned}c_2(\mathcal{F})c_1(C^{\times 3}) &= \big(\tfrac{1}{2}\,\mathrm{ch}_1(\mathcal{F})^2-\mathrm{ch}_2(\mathcal{F})\big)(-2)(C_{23}+C_{13}+C_{12})\\ &= (132C_3+72C_2+56C_1-48C_{23}.\Delta_{23}-42C_{13}.\Delta_{13}-36C_{12}.\Delta_{12}+30\delta)\\ &\qquad .(-2)(C_{23}+C_{13}+C_{12})\\ &= -2\{(56-42-36+30)+(72-48-36+30)+(132-48-42+30)\}\\ &= -196.\end{aligned}$$

Finally, by Lemma 3.4, we get

$$\begin{aligned}\chi(Z) &= c_2(\mathcal{F})c_1(C^{\times 3})-c_1(\mathcal{F})c_2(\mathcal{F})-c_3(\mathcal{F})\\ &= c_2(\mathcal{F})c_1(C^{\times 3})-\tfrac{2}{3}\,\mathrm{ch}_1(\mathcal{F})^3+2\,\mathrm{ch}_1(\mathcal{F})\,\mathrm{ch}_2(\mathcal{F})-2\,\mathrm{ch}_3(\mathcal{F}) = -196+264-132-132\\ &= -196.\end{aligned}$$

□

3.3 Fibrewise GV invariants

In this subsection, we determine the fibrewise GV invariants associated to the spectral curves over points in U_ε. Given a partition $\lambda \vdash 6$ and an integer $g \ge 0$ we define $n_g(\lambda)$ to be the local contribution of a spectral curve corresponding to a point in $S_\lambda \cap U_\varepsilon$ to the genus g Gopakumar–Vafa invariant; by the formulas recalled in the proof of Proposition 3.7, these contributions only depend on λ, so that these numbers are well-defined. In order to do so, we calculate certain HOMFLY polynomials: We define the Laurent polynomials $P(T_{2,n})$ for $n \ge 0$ as follows:

$$\begin{aligned}P(T_{2,0}) &= \frac{a-a^{-1}}{q-q^{-1}},\\ P(T_{2,1}) &= 1,\\ P(T_{2,n}) &= -a(q-q^{-1})P(T_{2,n-1})+a^2P(T_{2,n-2}) \quad \text{for } n \ge 2.\end{aligned} \tag{3-3}$$

λ	$g=3$	4	5	6	$\chi(U_\varepsilon\cap S_\lambda)$
$(5,1)$	0	3	-4	1	50
$(4,2)$	-1	4	-4	1	128
$(4,1,1)$	0	1	-3	1	-216
$(3,3)$	0	4	-4	1	81
$(3,2,1)$	0	2	-3	1	-668
$(3,1^3)$	0	0	-2	1	542
(2^3)	-1	3	-3	1	-128
$(2,2,1,1)$	0	1	-2	1	968
$(2,1^4)$	0	0	-1	1	-1012
(1^6)	0	0	0	1	256

Table 1

This last equation is the standard recursion relation for HOMFLY polynomials of torus knots, and so $P(T_{2,n})$ is the HOMFLY polynomial of the torus knot $T_{2,n}$; see [Oblomkov and Shende 2012, Section 2]. We will use the specialisation $(q/a)^{n-1}P(T_{2,n})|_{a=0}$ for $2\le n\le 5$:

$$\begin{aligned}(q/a)P(T_{2,2})|_{a=0}&=q\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\},\\ (q/a)^2P(T_{2,3})|_{a=0}&=q^2\{(q^{-1}-q)^2+2\},\\ (q/a)^3P(T_{2,4})|_{a=0}&=q^3\left\{(q^{-1}-q)^3+3(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\},\\ (q/a)^4P(T_{2,5})|_{a=0}&=q^4\{(q^{-1}-q)^4+4(q^{-1}-q)^2+3\}.\end{aligned}$$

Proposition 3.7 *The local contributions to the GV invariants satisfy $n_g(\lambda)=0$ for $g>6$ or $g<3$, and the remaining values are as in Table 1. We have also included the Euler characteristics of the corresponding strata of U_ε, given by Proposition 3.2, for reference.*

Proof Let $C_a\to C$ be the spectral cover corresponding to a point $a\in S_\lambda\cap\mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$. For any partition λ, the spectral curve C_a is reduced. Hence, by [Migliorini and Shende 2013; Maulik and Yun 2014], the fibrewise contribution $n_g(\lambda)$ is computed by the Euler characteristics of the Hilbert schemes of points on C_a. The latter is then computed by using certain HOMFLY polynomials. More precisely, [Migliorini and Shende 2013; Maulik and Yun 2014] gives the first, and [Oblomkov and Shende 2012; Maulik 2016], gives the second equality in the sequence of equalities

$$\begin{aligned}(3\text{-}4)\quad \sum_{g\ge 0}q^{2g(C_a)-2}(-1)^g n_g(\lambda)(q^{-1}-q)^{2g-2}&\\ =\int_{C_a^{[*]}}q^{2l}\,d\chi&=(1-q^2)^{-\chi(C_a)}\prod_{i=1}^{r(\lambda)}[(q/a)^{\lambda_i-1}P(T_{2,\lambda_i})]_{a=0}\\ &=q^{-\chi(C_a)}(q^{-1}-q)^{-\chi(C_a)}\prod_{i=1}^{r(\lambda)}[(q/a)^{\lambda_i-1}P(T_{2,\lambda_i})]_{a=0},\end{aligned}$$

where we denote by $r(\lambda)$ the number of rows of λ. We give a little more detail, firstly in order to explain the sign $(-1)^g$ appearing in the left side of (3-4). Note that $2g(C_a)-2=10$ for all a. By [Migliorini and Shende 2013; Maulik and Yun 2014], we have the following (cf. [Maulik and Toda 2018, (4.4)]):

$$\sum_l \chi(\mathrm{IC}_{C_a^{[l]}})q^l = q^{g(C_a)-1}\frac{1}{(q^{-1/2}+q^{1/2})^2}\sum_i \chi(\mathrm{Gr}_P^i\,\mathrm{IC}(\bar{J}))q^i.$$

After the variable change $q\mapsto q^2$, the left side is equal to

$$\int_{C_a^{[*]}}(-1)^l q^{2l}\,d\chi,$$

and the right side is equal to

$$q^{2g(C_a)-2}\sum_g n_g(\lambda)(q^{-1}+q)^{2g-2}.$$

By a further variable change $q^2\mapsto -q^2$, we obtain the first equation in (3-4).

Let us also explain the second equality. Let $\{p_i\}$ be the set of singular points of C_a. By [Oblomkov and Shende 2012, Conjecture 1] (which is proved in [Maulik 2016]), we have

$$\int_{C_a^{[*]}} q^{2l}\,d\chi = (1-q^2)^{-\chi(C_a)}\prod_i[(q/a)^{\mu_i}P(L_i)]_{a=0},$$

where μ_i denotes the Milnor number of the singularity p_i and L_i denotes the link of C_a at p_i. In our case, the singular points of C_a are parametrised by the rows of the partition $\lambda=(\lambda_i)$, and the type of the singular point corresponding to λ_i is $\{y^2=x^{\lambda_i}\}$. As a result, we have $\mu_i=\lambda_i-1$ and $L_i=T_{2,\lambda_i}$; these equalities imply the second equality in (3-4).

(1) $\underline{\lambda=(5,1)}$: In this case, the spectral curve C_a is a $2:1$ cover branched at two points, so $\chi(C_a)=2\chi(C)-2=-6$. It has a unique singular point of type $\{y^2=x^5\}$. Hence

$$\begin{aligned}\int_{C_a^{[*]}} q^{2l}\,d\chi &= q^6(q^{-1}-q)^6[(q/a)^4P(T_{2,5})]_{a=0} = q^6(q^{-1}-q)^6q^4\{(q^{-1}-q)^4+4(q^{-1}-q)^2+3\}\\ &= q^{10}\{(q^{-1}-q)^{10}+4(q^{-1}-q)^8+3(q^{-1}-q)^6\}.\end{aligned}$$

(2) $\underline{\lambda=(4,2)}$: We have $\chi(C_a)=-6$. C_a has two singular points, of types $\{y^2=x^2\}$ and $\{y^2=x^4\}$. We obtain

$$\begin{aligned}\int_{C_a^{[*]}} q^{2l}\,d\chi &= q^6(q^{-1}-q)^6[(q/a)^3P(T_{2,4})]_{a=0}[(q/a)P(T_{2,2})]_{a=0}\\ &= q^6(q^{-1}-q)^6q^3\left\{(q^{-1}-q)^3+3(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}q\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}\\ &= q^{10}\{(q^{-1}-q)^{10}+4(q^{-1}-q)^8+4(q^{-1}-q)^6+(q^{-1}-q)^4\}.\end{aligned}$$

(3) $\underline{\lambda = (4,1,1)}$: We have $\chi(C_a) = 2\chi(C) - 3 = -7$. C_a has a unique singular point of type $\{y^2 = x^4\}$. We obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^7(q^{-1}-q)^7[(q/a)^3 P(T_{2,4})]_{a=0} = q^7(q^{-1}-q)^7 q^3\left\{(q^{-1}-q)^3+3(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}$$
$$= q^{10}\{(q^{-1}-q)^{10}+3(q^{-1}-q)^8+(q^{-1}-q)^6\}.$$

(4) $\underline{\lambda = (3,3)}$: We have $\chi(C_a) = -6$. C_a has two singular points of type $\{y^2 = x^3\}$. So we obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^6(q^{-1}-q)^6([(q/a)^2 P(T_{2,3})]_{a=0})^2 = q^6(q^{-1}-q)^6 q^4\{(q^{-1}-q)^2+2\}^2$$
$$= q^{10}\{(q^{-1}-q)^{10}+4(q^{-1}-q)^8+4(q^{-1}-q)^6\}.$$

(5) $\underline{\lambda = (3,2,1)}$: We have $\chi(C_a) = -7$. C_a has two singular points, of types $\{y^2 = x^2\}$ and $\{y^2 = x^3\}$. Hence we obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^7(q^{-1}-q)^7[(q/a)^2 P(T_{2,3})]_{a=0}[(q/a) P(T_{2,2})]_{a=0}$$
$$= q^7(q^{-1}-q)^7 q^2\{(q^{-1}-q)^2+2\} q\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}$$
$$= q^{10}\{(q^{-1}-q)^{10}+3(q^{-1}-q)^8+2(q^{-1}-q)^6\}.$$

(6) $\underline{\lambda = (3,1,1,1)}$: We have $\chi(C_a) = -8$. C_a has a unique singular point of type $\{y^2 = x^3\}$. We obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^8(q^{-1}-q)^8[(q/a)^2 P(T_{2,3})]_{a=0} = q^8(q^{-1}-q)^8 q^2\{(q^{-1}-q)^2+2\}$$
$$= q^{10}\{(q^{-1}-q)^{10}+2(q^{-1}-q)^8\}.$$

(7) $\underline{\lambda = (2,2,2)}$: We have $\chi(C_a) = -7$. C_a has three singular points of type $\{y^2 = x^2\}$. We obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^7(q^{-1}-q)^7([(q/a) P(T_{2,2})]_{a=0})^3 = q^7(q^{-1}-q)^7 q^3\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}^3$$
$$= q^{10}\{(q^{-1}-q)^{10}+3(q^{-1}-q)^8+3(q^{-1}-q)^6+(q^{-1}-q)^4\}.$$

(8) $\underline{\lambda = (2,2,1,1)}$: We have $\chi(C_a) = -8$. C_a has two singular points of type $\{y^2 = x^2\}$. We obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^8(q^{-1}-q)^8([(q/a) P(T_{2,2})]_{a=0})^2 = q^8(q^{-1}-q)^8 q^2\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}^2$$
$$= q^{10}\{(q^{-1}-q)^{10}+2(q^{-1}-q)^8+(q^{-1}-q)^6\}.$$

(9) $\underline{\lambda = (2,1,1,1,1)}$: We have $\chi(C_a) = -9$. C_a has a unique singular point of type $\{y^2 = x^2\}$. We obtain

$$\int_{C_a^{[*]}} q^{2l}\, d\chi = q^9(q^{-1}-q)^9[(q/a) P(T_{2,2})]_{a=0} = q^9(q^{-1}-q)^9 q\left\{(q^{-1}-q)+\frac{1}{q^{-1}-q}\right\}$$
$$= q^{10}\{(q^{-1}-q)^{10}+(q^{-1}-q)^8\}.$$ □

3.4 Contributions from the nearby hyperplane

Putting the discussions in the previous subsections together, we obtain:

Proposition 3.8 *We have*

$$n_g(U_\varepsilon)=\begin{cases}1 & \text{if } g=6,\\ -8 & \text{if } g=5,\\ 18 & \text{if } g=4,\\ 0 & \text{otherwise.}\end{cases}$$

Proof We have

$$n_g(U_\varepsilon)=\sum_{\lambda\vdash 6} n_g(\lambda)\chi(U_\varepsilon\cap S_\lambda).$$

Since $n_6(\lambda)=1$ and $n_{g\le 2}(\lambda)=0=n_{g\ge 7}(\lambda)$ for all partitions $\lambda\vdash 6$, we have $n_6(U_\varepsilon)=1$ and $n_{g\le 2}(U_\varepsilon)=0=n_{g\ge 7}(U_\varepsilon)$.

For $g=3$, we have

$$n_3(U_\varepsilon)=-1\cdot\chi(U_\varepsilon\cap S_{(4,2)})-1\cdot\chi(U_\varepsilon\cap S_{(2,2,2)})=-(128-128)=0.$$

For $g=4$, we have

$$n_4(U_\varepsilon)=3\cdot 50+4\cdot 128+1(-216)+4\cdot 81+2(-668)+3(-128)+1\cdot 968=18.$$

For $g=5$, we have

$$n_5(U_\varepsilon)=-4\cdot 50-4\cdot 128-3(-216)-4\cdot 81-3(-668)-2\cdot 542-3(-128)-2\cdot 968-1(-1012)=-8. \qquad \square$$

3.5 GV invariants for twisted Higgs bundles

Proposition 3.9 *We have*

$$n_{g,2[C]}(\mathrm{Tot}_C(L))=\begin{cases}-1 & \text{if } g=6,\\ 8 & \text{if } g=5,\\ -18 & \text{if } g=4,\\ 8 & \text{if } g=3,\\ -2 & \text{if } g=2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof Let $\widehat{M}$ denote the moduli space of semistable L-twisted PGL_n-Higgs bundles.

We first describe a basis of $\mathrm{H}^i(\widehat{M})$ for each i. By [Mozgovoy 2012, Subsection 6.3], adopting the notation of [loc. cit.], the Poincaré polynomial of $\widehat{M}$ is

$$P_y(\widehat{M})=P^{(1)}_{2,2}(y)/(1+y)^4=2y^8+4y^7+8y^6+4y^5+2y^4+4y^3+y^2+1. \tag{3-5}$$

On the other hand, the cohomology ring $\mathrm{H}^*(\widehat{M})$ has the following multiplicative generators [de Cataldo et al. 2012; Hausel and Thaddeus 2003; 2004]:

$$\alpha\in\mathrm{H}^2(\widehat{M}),\qquad \beta\in\mathrm{H}^4(\widehat{M}),\qquad \psi_i\in\mathrm{H}^3(\widehat{M}),\quad \text{for } i=1,\dots,4.$$

Using the relations in [de Cataldo et al. 2012, Theorem 1.2.10, Proposition 4.2.1] and (3-5), we have that

$$\mathrm{H}^0(\widehat{M}) = \langle 1\rangle, \qquad \mathrm{H}^1(\widehat{M}) = 0, \qquad \mathrm{H}^2(\widehat{M}) = \langle \alpha\rangle,$$
$$\mathrm{H}^3(\widehat{M}) = \langle \psi_i\rangle_{1\le i\le 4}, \qquad \mathrm{H}^4(\widehat{M}) = \langle \alpha^2, \beta\rangle, \qquad \mathrm{H}^5(\widehat{M}) = \langle \alpha\psi_i\rangle_{1\le i\le 4},$$
$$\mathrm{H}^6(\widehat{M}) = \langle \psi_i\psi_j, \alpha^3, \alpha\beta\rangle_{1\le i<j\le 4}, \qquad \mathrm{H}^7(\widehat{M}) = \langle \alpha^2\psi_i\rangle_{1\le i\le 4}, \qquad \mathrm{H}^8(\widehat{M}) = \langle \alpha^4, \alpha\gamma\rangle,$$

where we put $\gamma := -2(\psi_1\psi_3 + \psi_2\psi_4)$. In the following, we only explain the above equalities for $\mathrm{H}^7(\widehat{M})$ and $\mathrm{H}^8(\widehat{M})$. The arguments are similar and simpler for lower cohomological degrees. For $\mathrm{H}^7(\widehat{M})$, we have

$$\mathrm{H}^7(\widehat{M}) = \langle \beta\psi_i, \alpha^2\psi_i\rangle_{1\le i\le 4} = \langle \alpha^2\psi_i\rangle_{1\le i\le 4},$$

where the second equality follows from [de Cataldo et al. 2012, Proposition 4.2.1]. On the other hand, by (3-5), we also know that $\dim \mathrm{H}^7(\widehat{M}) = 4$. Hence we can conclude that $\{\alpha^2\psi_i\}_{1\le i\le 4}$ is a basis of $\mathrm{H}^7(\widehat{M})$. For $\mathrm{H}^8(\widehat{M})$, we have

$$\begin{aligned}\text{(3-6)}\quad \mathrm{H}^8(\widehat{M}) &= \langle \alpha^4, \alpha^2\beta, \alpha\psi_i\psi_j, \beta^2\rangle_{1\le i<j\le 4} \\ &= \langle \alpha^4, \alpha^2\beta, \alpha\psi_1\psi_2, \alpha\psi_1\psi_4, \alpha\psi_2\psi_3, \alpha\psi_3\psi_4, \alpha(\psi_1\psi_3 - \psi_2\psi_4), \alpha\gamma, \beta^2\rangle = \langle \alpha^4, \alpha\gamma\rangle.\end{aligned}$$

To see the third equality, we claim that

$$\alpha \in I_3^0, \quad \beta^2 \in I_1^2, \quad \tfrac{1}{2}\alpha^2\beta + 2\alpha\gamma \in I_1^2,$$

where $I_b^a \subset \mathbb{Q}[\alpha, \beta, \gamma]$ is an ideal defined in [de Cataldo et al. 2012, Definition 1.2.8] for $a, b \ge 0$. Indeed

$$\alpha = \rho_{1,0,0}^0, \quad \beta^2 = \rho_{0,2,0}^3, \quad \tfrac{1}{2}\alpha^2\beta + 2\alpha\gamma = \rho_{2,1,0}^2,$$

where $\rho_{r,s,t}^c \in I_b^a$ are elements defined in [loc. cit.]. Hence $\beta^2 = 0$ and $\alpha^2\beta \in \langle \alpha^4, \alpha\gamma\rangle$. Furthermore, since we have

$$\text{(3-7)}\qquad \psi_1\psi_2, \psi_1\psi_4, \psi_2\psi_3, \psi_3\psi_4, (\psi_1\psi_3 - \psi_2\psi_4) \in \Lambda_0^2,$$

with Λ_0^2 as defined in [loc. cit.], we conclude that the elements in (3-7), multiplied by $\alpha \in I_3^0$, are all zero in $\mathrm{H}^*(\widehat{M})$. By the above arguments, the third equality in (3-6) holds.

Now, we rearrange the above basis of $\mathrm{H}^*(\widehat{M})$ using the perverse degree instead of the cohomological degree. Note that the generators α, β, and ψ_i all have perverse degree 2 by [de Cataldo et al. 2012, Theorem 4.2.2]. Hence we obtain

$$\begin{aligned}
&\mathrm{H}^*_{\le 0}(\widehat{M}) = \mathrm{H}^*_{\le 1}(\widehat{M}) = \langle 1\rangle,\\
&\mathrm{H}^*_{\le 2}(\widehat{M}) = \mathrm{H}^*_{\le 3}(\widehat{M}) = \langle \mathrm{H}^*_{\le 0}(\widehat{M}), \alpha, \beta, \psi_i\rangle_{1\le i\le 4},\\
&\mathrm{H}^*_{\le 4}(\widehat{M}) = \mathrm{H}^*_{\le 5}(\widehat{M}) = \langle \mathrm{H}^*_{\le 2}(\widehat{M}), \alpha^2, \alpha\beta, \alpha\psi_i, \psi_i\psi_j\rangle_{1\le i<j\le 4},\\
&\mathrm{H}^*_{\le 6}(\widehat{M}) = \mathrm{H}^*_{\le 7}(\widehat{M}) = \langle \mathrm{H}^*_{\le 4}(\widehat{M}), \alpha^3, \alpha^2\psi_i, \alpha\gamma\rangle_{1\le i\le 4},\\
&\mathrm{H}^*_{\le 8}(\widehat{M}) = \langle \mathrm{H}^*_{\le 6}(\widehat{M}), \alpha^4\rangle.
\end{aligned}$$

Here we have followed the notation and normalisation conventions of [de Cataldo et al. 2012], regarding the perverse filtration on $\mathrm{H}^*(\widehat{M})$. Precisely,

$$\mathrm{H}^*_{\leq i}(\widehat{M}) := \mathrm{H}^{*-a}(\widetilde{B}_L, {}^p\tau_{\leq i} h_* \mathbb{Q}[a]),$$

where $a = \dim(\widetilde{B}_L) = 4g(C) - 1$.

Combined with the isomorphism $\mathrm{H}^*(M) \cong \mathrm{H}^*(\widehat{M}) \otimes \mathrm{H}^*(\mathrm{Pic}^0(C))$, we obtain

$$\sum_i \chi({}^p\mathcal{H}^i(\boldsymbol{R}h_* \mathrm{IC}_M))q^i = -(q^{-4} - 2q^{-2} + 4 - 2q^2 + q^4)(q^{-1/2} + q^{1/2})^4,$$

from which we can read off the integers $n_{g,2[C]}(\mathrm{Tot}_C(L))$. The minus sign in the right-hand side comes from the fact that $\dim M = 8g(C) - 3$ is odd. □

3.6 GV invariants for Tot(N)

Theorem 3.10 *Let C be a genus two curve, and let L be a generic line bundle on C of degree 3. Let $f \in (H^0(L^{\otimes 2}))^\vee \cong \mathrm{Ext}^1(L, L^{-1} \otimes \omega_C)$ be a generic linear function, and N the corresponding rank two vector bundle. Then we have*

$$n_{g,2[C]}(\mathrm{Tot}_C(N)) = \begin{cases} 8 & \text{if } g = 3, \\ -2 & \text{if } g = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the GV/GW correspondence holds for $\mathrm{Tot}_C(N)$ and the curve class $2[C]$.

Proof By Proposition 2.15,

$$n_{g,2[C]}(\mathrm{Tot}_C(N)) = n_g(U_\varepsilon) + n_{g,2[C]}(\mathrm{Tot}_C(L)).$$

Now the result follows from Propositions 3.8 and 3.9. □

4 Higher genus

In this section, we fix a smooth projective curve C of genus $g(C) \geq 3$ and a line bundle L of degree $2g(C) - 1$.

Proposition 4.1 *We have*

$$n_{g,2[C]}(\mathrm{Tot}_C(L)) = \begin{cases} 0 & \text{if } g \geq 4g(C) - 1, \\ -1 & \text{if } g = 4g(C) - 2, \\ -2^{2g(C)-3} & \text{if } g = g(C), \\ 0 & \text{if } g \leq g(C) - 1. \end{cases}$$

Proof For simplicity, we write $M = M_L(2,1)$ and $\widehat{M} = \widehat{M}_L(2,1)$. By [Chaudouard and Laumon 2016] (see also [Maulik and Shen 2023, Theorem 0.4]), we have

$${}^p\mathcal{H}^i(\boldsymbol{R}h_* \mathrm{IC}_M) \cong \mathrm{IC}\big(\textstyle\bigwedge^i \boldsymbol{R}^1\pi_* \mathrm{IC}_{\mathcal{C}}\big) \quad \text{for } i \in \mathbb{Z},$$

where $U \subset \widetilde{B}_L$ is the dense open subset parametrising smooth spectral curves and $\pi\colon \mathcal{C} \to U$ is the universal curve. Since a smooth spectral curve has genus $4g(C)-2$, the degree of the Laurent polynomial $\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\boldsymbol{R}h_*\,\mathrm{IC}_M))q^i$ is less than or equal to $4g(C)-2$. The coefficient of $q^{4g(C)-2}$ is

$$\chi(\mathrm{IC}_{\widetilde{B}_L}) = (-1)^{\dim \widetilde{B}_L}\chi(\mathbb{Q}_{\widetilde{B}_L}) = -1,$$

where $\dim \widetilde{B}_L = 4g(C)-1$. This proves that

$$n_{g,2[C]}(\mathrm{Tot}_C(L)) = \begin{cases} 0 & \text{if } g \ge 4g(C)-1, \\ -1 & \text{if } g = 4g(C)-2. \end{cases}$$

To determine the GV invariants for small g, recall that we have an isomorphism

$$\mathrm{H}^*(M) \cong \mathrm{H}^*(\mathrm{Pic}^0(C)) \otimes \mathrm{H}^*(\widehat{M}).$$

Moreover, the generators ε_i of $\mathrm{H}^*(\mathrm{Pic}^0(C))$ have cohomological degree one and perverse degree one. Hence we have

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\boldsymbol{R}h_*\,\mathrm{IC}_M))q^i = (q^{-1/2}+q^{1/2})^{2g(C)}\biggl(\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\boldsymbol{R}\hat{h}_*\,\mathrm{IC}_{\widehat{M}}))q^i\biggr).$$

From this, we see that $n_{g,2[C]}(\mathrm{Tot}_C(L)) = 0$ for $g \le g(C)-1$. Furthermore $n_{g,2[C]}(\mathrm{Tot}_C(L)) = -\chi(\widehat{M})$. This can be computed as follows: Let $P_t(M)$ denote the Poincaré polynomial of M. We can compute $P_{-t}(M)$ by substituting $u = v = -t$ in [Chuang et al. 2011, (A.7)]:

$$\begin{aligned} &P_{-t}(M) \\ &= (1-t)^{2g(C)}\biggl\{\frac{(1-t)^{2g(C)-2}(1+t+t^2)^{2g(C)}}{(1+t)^2(1+t^2)} + \frac{(1+t)^{2g(C)}}{4(1+t^2)} + \frac{t^{4g(C)-2}(1-t)^{2g(C)-2}}{2(1+t)}\Bigl(2g(C)-\frac{1}{1+t}\Bigr) \\ &\qquad + \frac{t^{4g(C)-2}(1-t)^{2g(C)-1}}{2(1+t)}\bigl(-2g(C)+1-\tfrac{1}{2}\bigr)\biggr\}. \end{aligned}$$

Substituting $t = 1$ in $P_{-t}(M)/(1-t)^{2g(C)}$, we obtain $\chi(\widehat{M}) = 2^{2g(C)-3}$. □

Proposition 4.2 *The following assertions hold*:

(1) $n_g(U_\varepsilon) = \begin{cases} 0 & \text{if } g \ge 4g(C)-1, \\ 1 & \text{if } g = 4g(C)-2, \\ 0 & \text{if } g \le 2g(C)-2. \end{cases}$

(2) $n_{2g(C)-1}(U_\varepsilon) = -\sum_\lambda \chi(S_\lambda \cap U_\varepsilon)$, *where* $\lambda = (\lambda_i)$ *runs over all partitions of* $4g(C)-2$ *such that* λ_i *is even for every* i.

Proof Given a partition λ of $4g(C)-2$ and an element $a \in S_\lambda \cap U_\varepsilon$, let $C_a \to C$ be the corresponding spectral curve. In view of the recursion formula (3-3) computing $P(T_{2,i})$ and the formula (3-4) for $\chi(\mathrm{Hilb}^n(C_a))$, we see that the polynomial $\int_{C_a^{[*]}} q^{2l}\,d\chi$ has degree

$$-2\chi(C) - \deg(L^{\otimes 2}) = 8g(C)-6$$

and the leading coefficient is 1. Hence we get

$$n_g(U_\varepsilon) = \begin{cases} 0 & \text{if } g \geq 4g(C)-1, \\ 1 & \text{if } g = 4g(C)-2. \end{cases}$$

On the other hand, if we write $\int_{C_a^{[*]}} q^{2l}\,d\chi$ as a polynomial in $(q^{-1}-q)$, the lowest term has degree

$$-2\chi(C) + \#\{i : \lambda_i \text{ is odd}\} \tag{4-1}$$

and coefficient 1. Among all the partitions $\lambda = (\lambda_i)$, (4-1) achieves its minimum if and only if every λ_i is even. Hence the remaining assertions also hold. □

Theorem 4.3 *Let* $L \in \mathrm{Pic}^{2g(C)-1}(C)$ *and* $[N] \in \mathbb{P}\,\mathrm{H}^0(L^{\otimes 2})$ *be arbitrary. Then we have*

$$n_{g,2[C]}(\mathrm{Tot}_C(N)) = \begin{cases} 0 & \text{if } g \geq 4g(C)-2, \\ -2^{2g(C)-3} & \text{if } g = g(C), \\ 0 & \text{if } g \leq g(C)-1. \end{cases}$$

In particular, the GV/GW correspondence holds for the above range of g.

Proof The proof is the same as that of Theorem 3.10, this time combining Propositions 4.1 and 4.2 with Proposition 2.15. The final assertion, regarding the GV/GW correspondence, follows by comparison with Corollary A.3. □

Appendix Some results from Gromov–Witten theory

Let $X = \mathrm{Tot}_C(N)$ be a local curve, with N a 2-rigid bundle. Consider the generating series

$$Z^{\mathrm{GW}}_{\leq 2}(X) = \sum_{g\geq 0} \mathrm{GW}_{g,1}(X)\lambda^{2g-2}t + \sum_{g\geq 0} \mathrm{GW}_{g,2}(X)\lambda^{2g-2}t^2,$$

where $\mathrm{GW}_{g,d}(X)$ denotes the GW invariants of $X = \mathrm{Tot}_C(N)$ for the curve class $d[C]$. We *define* $n^{\mathrm{GW}}_{g,d}(X)$ by (1-1). The goal of this appendix is to determine $n^{\mathrm{GW}}_{g,2}(X)$ for some g, using the following:

Theorem A.1 [Bryan and Pandharipande 2008, Section 8] *We have*

$$\exp(Z^{\mathrm{GW}}_{\leq 2}(X)) = 1 + \bigl(2\sin\bigl(\tfrac12\lambda\bigr)\bigr)^{2g(C)-2}t + \bigl(2\sin\bigl(\tfrac12\lambda\bigr)\bigr)^{4g(C)-4}\bigl\{\bigl(4-4\sin\bigl(\tfrac12\lambda\bigr)\bigr)^{g(C)-1} + \bigl(4+4\sin\bigl(\tfrac12\lambda\bigr)\bigr)^{g(C)-1}\bigr\}t^2 + \cdots.$$

Corollary A.2 *Assume that* $g(C)=2$. *Then we have*

$$n^{\mathrm{GW}}_{g,1}(X) = \begin{cases} 1 & \text{if } g=2, \\ 0 & \text{otherwise}, \end{cases} \qquad n^{\mathrm{GW}}_{g,2}(X) = \begin{cases} 8 & \text{if } g=3, \\ -2 & \text{if } g=2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof It is immediate to read off $n_{g,1}(X)$ from the formula in Theorem A.1 To compute $n_{g,2}(X)$, it is convenient to work with the variable $Q = e^{i\lambda}$. Then

$$(Q^{1/2} - Q^{-1/2})^2 = -\bigl(2\sin\bigl(\tfrac12\lambda\bigr)\bigr)^2.$$

Firstly, the coefficient of t^2 in the series $\exp(Z^{\mathrm{GW}}_{\leq 2}(X))$ is

$$-\tfrac{1}{2}(Q-Q^{-1})^2+\tfrac{1}{2}(Q^{1/2}-Q^{-1/2})^4+\sum_{g\geq 0}(-1)^{g-1}n^{\mathrm{GW}}_{g,2}(X)(Q^{1/2}-Q^{-1/2})^{2g-2}$$
$$=-2(Q^{1/2}-Q^{-1/2})^2+\sum_{g\geq 0}(-1)^{g-1}n^{\mathrm{GW}}_{g,2}(X)(Q^{1/2}-Q^{-1/2})^{2g-2}.$$

Secondly, the coefficient of t^2 in the right-hand side of the formula in Theorem A.1 is

$$8(Q^{1/2}-Q^{-1/2})^4.$$

We obtain the conclusion by comparing the above two expressions. □

Corollary A.3 *Let $g(C)$ be arbitrary. We have*

$$n^{\mathrm{GW}}_{g,1}(X)=\begin{cases}1 & \text{if } g=g(C),\\ 0 & \text{otherwise,}\end{cases}\qquad n^{\mathrm{GW}}_{g,2}(X)=\begin{cases}0 & \text{if } g\geq 4g(C)-2,\\ -2^{2g(C)-3} & \text{if } g=g(C),\\ 0 & \text{if } g\leq g(C)-1.\end{cases}$$

Proof Similarly to the above corollary, we work with the variable $Q=e^{i\lambda}$. The coefficient of t^2 in the series $\exp(Z^{\mathrm{GW}}_{\leq 2}(X))$ is

$$-\tfrac{1}{2}(Q-Q^{-1})^{2g(C)-2}+\tfrac{1}{2}(Q^{1/2}-Q^{-1/2})^{4g(C)-4}+\sum_{g\geq 0}(-1)^{g-1}n^{\mathrm{GW}}_{g,2}(X)(Q^{1/2}-Q^{-1/2})^{2g-2},$$

and

$$-\tfrac{1}{2}(Q-Q^{-1})^{2g(C)-2}+\tfrac{1}{2}(Q^{1/2}-Q^{-1/2})^{4g(C)-4}$$
$$=-\tfrac{1}{2}\{(Q^{1/2}-Q^{-1/2})^4+4(Q^{1/2}-Q^{-1/2})^2\}^{g(C)-1}+\tfrac{1}{2}(Q^{1/2}-Q^{-1/2})^{4g(C)-4}$$
$$=-2^{2g(C)-3}(Q^{1/2}-Q^{-1/2})^{2g(C)-2}+\cdots.$$

Comparing with the right-hand side of the formula in Theorem A.1, we obtain the results. □

Remark A.4 Similarly, one can check that the GV type invariants for $\mathrm{Tot}_C(L)$ that are computed in Propositions 3.9 and 4.1 agree with the GW invariants of $\mathrm{Tot}_C(L\oplus(L^{-1}\otimes\omega_C))$ computed in [Bryan and Pandharipande 2008, Corollary 7.2].

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BEN DAVISON ben.davison@ed.ac.uk
School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh
Edinburgh, United Kingdom

NAOKI KOSEKI koseki@liverpool.ac.uk
The University of Liverpool, Liverpool, United Kingdom

Proposed: Jim Bryan Received: September 23, 2024
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