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Shane Kelly, Shuji Saito and Georg Tamme

Grothendieck's formal functions theorem states that the coherent cohomology of a Noetherian scheme can be recovered from that of a blowup, and the infinitesimal thickenings of the center and of the exceptional divisor of the blowup. We prove an analogous descent result, called "pro-cdh descent", for certain cohomological invariants of arbitrary quasicompact, quasiseparated derived schemes. Our results in particular apply to algebraic K-theory, topological Hochschild and cyclic homology, and the cotangent complex.

As an application, we deduce that $K_n(X) = 0$ when $n < -d$ for quasicompact, quasiseparated derived schemes X of valuative dimension d. This generalizes Weibel's conjecture, which was originally stated for Noetherian (nonderived) X of Krull dimension d, and proved in this form in 2018 by Kerz, Strunk, and the third author.

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1 Introduction

An example of an *abstract blowup square of schemes* is a cartesian square

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ \downarrow & & \downarrow p \\ Z & \overset{i}{\hookrightarrow} & X \end{array} \tag{1}$$

where $Y = \mathrm{Bl}_Z(X)$ is the blowup of a Noetherian scheme X in a closed subscheme Z. More generally, an abstract blowup square of schemes is a cartesian square (1) of (potentially non-Noetherian schemes) where i is a finitely presented closed immersion and the morphism p is finitely presented, proper, and

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an isomorphism over the open complement of Z in X. Coverings $\{Z \hookrightarrow X, Y \to X\}$ for all abstract blowup squares as above together with the Nisnevich coverings generate Suslin and Voevodsky's [2000] *cdh* topology, which plays an important role in the study of motives. It is an interesting question how cohomological invariants behave under such abstract blowup squares.

Thomason [1993] (see also [Cortiñas et al. 2008, Proposition 1.5]) proved that for X quasicompact and quasiseparated (qcqs, for short), $Z \hookrightarrow X$ a *regular* closed immersion, and Y the blowup of X in Z, the square (1) gives rise to a (homotopy) cartesian square of algebraic K-theory spectra and hence to a long exact sequence of algebraic K-groups

$$\cdots \to K_i(X) \to K_i(Z) \oplus K_i(Y) \to K_i(E) \to K_{i-1}(X) \to \cdots.$$

For general abstract blowup squares (1), however, this will fail. If X is Noetherian, one can repair this defect by taking infinitesimal information into account. More precisely, if $Z(n)$ and $E(n)$ denote the n-th infinitesimal thickenings of Z in X and E in Y, respectively, Theorem A of [Kerz et al. 2018] states that the square of pro-spectra

$$\begin{array}{ccc} K(X) & \longrightarrow & \{K(Z(n))\}_n \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & \{K(E(n))\}_n \end{array}$$

is cartesian in a certain weak sense (see Section 4 for a definition). In particular, it induces a long exact sequence of pro-abelian groups. This result can be viewed as a K-theoretic version of Grothendieck's formal functions theorem [1961, Theorem 4.1.5]. It had several important precursors [Krishna and Srinivas 2002; Cortiñas 2006; Krishna 2010; Geisser and Hesselholt 2006; 2011; Morrow 2016b; 2018]. Because of the role of abstract blowup squares in the cdh topology, such a result is often called "pro-cdh descent."[1]

This pro-cdh descent result plays a central role in the resolution [Kerz et al. 2018] of Weibel's K-dimension conjecture, which states that the K-groups of a Noetherian scheme X vanish below the negative of its Krull dimension, and in the development of a continuous K-theory of rigid spaces [Morrow 2016a; Kerz et al. 2019; 2023].

Given the rising interest in non-Noetherian schemes, which appear naturally, for example, when working over a perfectoid base ring such as $\mathcal{O}_{\mathbb{C}_p}$, it is an obvious question, whether, or in which form, pro-cdh descent holds in this general setting. The following example, a variant of which was constructed by Dahlhausen and the third author [Dahlhausen and Tamme 2022] precisely for that purpose and which was independently studied by the first two authors [Kelly 2024, footnote 2] with respect to the pro-cdh topology, shows that pro-cdh descent as formulated above does not hold for general abstract blowup squares of non-Noetherian schemes.

[1] We remark that it would be more appropriate to call this "pro-cdh excision" since it is not precisely a descent statement for some topology. However, together with Nisnevich excision, it does imply actual (Čech) descent for the pro-cdh topology of [Kelly and Saito 2024]; see Theorem 6.1 there. For this reason, and to be in line with the existing literature, we stick to the term "pro-cdh descent."

Example 1.1 Let R be a valuation ring of dimension at least 2. Let $0 \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ be prime ideals in R, and choose $x \in \mathfrak{m} \setminus \mathfrak{p}$ and $y \in \mathfrak{p} \setminus \{0\}$. Then x^n divides y for all $n \geq 1$ and for each n the square

$$\begin{array}{ccc} \mathrm{Spec}(R/(x^n)) & \hookrightarrow & \mathrm{Spec}(R/(y)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \mathrm{Spec}(R/(x^n)) & \hookrightarrow & \mathrm{Spec}(R/(xy)) \end{array}$$

is an abstract blowup square. If the induced square of K-theory pro-spectra were weakly cartesian, the map $K(R/(xy)) \to K(R/(y))$ would be an equivalence. However, as $y^2 = 0$ in $R/(xy)$, $1 + y$ is a unit and defines a nontrivial element in the kernel of $K_1(R/(xy)) \to K_1(R/(y))$.

It turns out that the reason for this failure is that we took classical quotients when forming the infinitesimal thickenings, which discards torsion information. More concretely, let A be a ring and f an element of A. Derived algebraic geometry allows us to form the derived quotients $A/\!\!/f^n$, which are certain simplicial commutative rings. Then $\pi_0(A/\!\!/f^n)$ is the classical quotient A/f^n, whereas $\pi_1(A/\!\!/f^n) \cong \ker(f^n\colon A \to A)$ still carries the information about the f^n-torsion. We can then consider the pro-system $\{A/\!\!/f^n\}_n$ as a (derived) formal completion of A. For a Noetherian ring A, this formal completion is in fact equivalent to the usual (pro-)completion $\{A/f^n\}_n$. Our main theorem shows that with this modification, replacing infinitesimal neighborhoods by "derived infinitesimal neighborhoods", pro-cdh descent does in fact hold in general. It is natural to formulate it in the generality of derived schemes. For a precise definition and the involved finiteness conditions, we refer to Section 2.

Theorem A *Let $f\colon Y \to X$ be a proper, locally almost finitely presented morphism of quasicompact and quasiseparated (qcqs for short) derived schemes which is an isomorphism outside a closed subset $Z \subseteq |X|$ whose open complement is quasicompact. Denote by $X^\wedge_Z$ and $Y^\wedge_Z$ the formal completion of X (respectively Y) along Z (respectively $f^{-1}(Z)$) viewed as ind-derived schemes. Then the square of pro-spectra*

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X^\wedge_Z) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & K(Y^\wedge_Z) \end{array}$$

is weakly cartesian.

This result is not specific to K-theory. It holds more generally for every localizing invariant which is k-connective in the sense of [Land and Tamme 2019] for some integer k; see Theorem 4.6 for the general statement. For instance, it also applies to topological Hochschild homology THH, topological cyclic homology TC, and rational negative cyclic homology.

Remark 1.2 One may ask whether the same statement also holds for spectral schemes. As our proof makes essential use of the theory of derived blowups of derived schemes developed by Khan and Rydh [2025], we cannot treat this more general case with our methods.

We also prove the analogous result for the cotangent complex; see Theorem 5.4. By the arguments of [Elmanto and Morrow 2023] one then obtains pro-cdh descent in the above form for motivic cohomology (Corollary 5.5). We thank Matthew Morrow for suggesting this result for the cotangent complex and indicating the application to motivic cohomology.

In [Kerz et al. 2018], pro-cdh descent of K-theory was used to derive Weibel's conjecture on the vanishing of negative K-groups of a Noetherian scheme X below $-\dim(X)$. It is obvious to ask what we can say about negative K-groups if we merely assume X to be qcqs; see, e.g., Morrow's Oberwolfach talk [2022]. As the Krull dimension doesn't have good properties for general non-Noetherian schemes, one might expect that instead the valuative dimension introduced by Jaffard [1960] (see [Elmanto et al. 2021] or [Kelly and Saito 2024, §7.1] for accounts) gives the correct bounds. It coincides with the Krull dimension if X is Noetherian. Indeed, we prove the following:

Theorem B (Theorem 6.5) *Let X be a qcqs spectral scheme. Then the following hold.*

(1) $K_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.

(2) *For all* $i \geq \mathrm{vdim}(X)$ *and any integer* $r \geq 0$, *the pullback map* $K_{-i}(X) \to K_{-i}(\mathbb{A}^r_X)$ *is an isomorphism. In other words*, X *is* $K_{-\mathrm{vdim}(X)}$*-regular.*

Here by a spectral scheme we mean more precisely one with connective structure sheaf, and in (2), $\mathbb{A}^r$ denotes either the flat or the smooth affine space.

Remark 1.3 We don't know an example of a scheme X with $\mathrm{vdim}(X) > \dim(X)$ and $K_{-\mathrm{vdim}(X)}(X) \neq 0$.

This theorem immediately implies the same vanishing for Weibel's homotopy K-theory $KH(X)$. In other words, we get a new proof of assertion (1) in the following theorem, which we include here for the sake of completeness. As we discuss below, this theorem is well known to the experts. We write L_{cdh} for the cdh sheafification functor on presheaves of spectra and $K_{\geq 0}$ for the presheaf of connective algebraic K-theory.

Theorem 1.4 *Let X be a qcqs scheme of finite valuative dimension.*

(1) $KH_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.

(2) *The natural maps* $L_{\mathrm{cdh}} K_{\geq 0} \to L_{\mathrm{cdh}} K \to KH$ *are equivalences.*

For schemes essentially of finite type over a field of characteristic 0, this was first proven in [Haesemeyer 2004]. For general Noetherian X, (1) was first proven in [Kerz and Strunk 2017] and (2) in [Kerz et al. 2018]. The fact that KH is a cdh sheaf is [Cisinski 2013]. In the general case, the theorem follows easily from recent results on the cdh-topology [Elmanto et al. 2021] and the K-theory of valuation rings [Kelly and Morrow 2021; Kerz et al. 2021]. In fact, the proof given under Noetherian assumptions in [Kelly and Morrow 2021] still works, and for the reader's convenience we reproduce this proof at the end of the paper. Alternatively, the proof of [Kerz and Strunk 2017] (respectively [Kerz et al. 2018]) works mutatis mutandis, using the fact that a blowup does not increase the valuative dimension (this is not necessarily true for the Krull dimension).

Remark 1.5 For arbitrary qcqs schemes X, Theorem 1.4 together with the cartesian square

$$\begin{array}{ccc} K & \longrightarrow & KH \\ \downarrow & & \downarrow \\ \mathrm{TC} & \longrightarrow & L_{\mathrm{cdh}}\mathrm{TC} \end{array}$$

(-1)-connectivity of topological cyclic homology on affines, and [Elmanto et al. 2021] imply the slightly weaker vanishing $K_{-i}(X) = 0$ for $i > \mathrm{vdim}(X) + 2$. Elmanto and Morrow (over fields) [2023, Proof of Theorem 4.12] and Bouis (in general) [2024, Proposition 5.5.4] prove refinements of this for the motivic filtration.

Structure of the arguments and the paper

The proof of Theorem A follows the outline of [Kerz et al. 2018], but requires substantially more input from derived geometry. It is reduced to the two special cases of derived blowups and finite morphisms, respectively. This reduction is achieved by a structural result about modifications of derived schemes, Theorem 3.2, which we view as our main contribution here. In its proof, we need several preliminary results from derived algebraic geometry which we discuss in Section 2.

The case of Theorem A for derived blowups could essentially be proved as in [Kerz et al. 2018]. We here present a stronger result with a simplified proof due to Antieau [2018]; we reproduce his proof in Proposition 4.1. For the case of finite morphisms in Theorem A, compared to [Kerz et al. 2018] we give a simple proof of a more general result (Proposition 4.2).

To prove Theorem B in Section 6, we roughly follow Kerz's approach [2018] combined with our pro-cdh-descent for finite morphisms. Another new input is a generalization of results of Swanson and Huneke [2006] on reductions of ideals to the non-Noetherian setting in Section 6.2.

Related work

Another approach to pro-cdh descent statements has been proposed by Clausen and Scholze. It is based on condensed mathematics [Scholze and Clausen 2019] and Efimov's theory [2024] of localizing invariants of large categories.

In forthcoming work, the first two authors introduce a "pro-cdh topology" on qcqs derived schemes. It will follow from Theorem A (or Theorem 4.6) that every localizing invariant which is k-connective for some integer k satisfies descent for that topology.

2 Preliminaries on derived algebraic geometry

We freely use the language of derived algebraic geometry as developed by Toën and Vezzosi [2008; 2005], Lurie [2018], and others. The following subsections mainly serve to fix some notation and recall some notions and facts that will be used later on. Readers familiar with derived algebraic geometry may safely skip this section and only come back when needed.

2.1 Derived rings and schemes

We write CAlg^{Δ} for the ∞-category of simplicial commutative rings [Lurie 2018, §25.1] and refer to its objects as *derived rings*. There is a forgetful functor $\mathrm{CAlg}^{\Delta} \to \mathrm{CAlg}^{\mathrm{cn}}$, where $\mathrm{CAlg}^{\mathrm{cn}}$ denotes the ∞-category of connective $\mathbb{E}_\infty$-algebras in spectra. This functor preserves small limits and colimits and is conservative.

A *derived scheme* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of derived rings on $|X|$ such that ${}^{\mathrm{cl}}X := (|X|, \pi_0\mathcal{O}_X)$ is a classical scheme and the higher homotopy sheaves $\pi_i\mathcal{O}_X$ are quasicoherent $\pi_0\mathcal{O}_X$-modules. We call ${}^{\mathrm{cl}}X$ the *underlying classical scheme* or the *classical truncation* of X. Derived schemes form the objects of an ∞-category dSch, and similarly as above there is a forgetful functor from derived schemes to spectral schemes. There is also a functor of points approach to derived schemes. In other words, there is a fully faithful functor

$$\mathrm{dSch} \to \mathrm{Fun}(\mathrm{CAlg}^{\Delta}, \mathrm{Spc}),$$

which coincides with the Yoneda embedding on affine derived schemes [Lurie 2018, §1.6].

A map $f : X \to Y$ of derived schemes is called *proper*, *a closed immersion*, *affine*, *or finite*, respectively, if the underlying map of classical schemes ${}^{\mathrm{cl}}f$ has the corresponding property. If $U \subseteq |X|$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is itself a derived scheme. Such a derived scheme is called an *open subscheme of* X, which we simply denote by U.

2.2 Quasicoherent modules

If A is a derived ring, we write $\mathrm{Mod}(A)$ for the symmetric monoidal ∞-category of A-modules (in spectra). For a derived scheme X, we denote by $\mathrm{QCoh}(X)$ the category of quasicoherent sheaves on X. These are stable ∞-categories with canonical t-structures, and we denote their connective parts by $\mathrm{Mod}(A)^{\mathrm{cn}}$ and $\mathrm{QCoh}(X)^{\mathrm{cn}}$, respectively. Moreover, they only depend on the underlying spectrum or spectral scheme, respectively. If $X = \mathrm{Spec}(A)$ is affine, we have an equivalence $\mathrm{QCoh}(X) = \mathrm{Mod}(A)$.

If $Z \subseteq |X|$ is a closed subset, we denote by $\mathrm{QCoh}(X \text{ on } Z)$ the full subcategory of $\mathrm{QCoh}(X)$ spanned by those quasicoherent sheaves which are supported on Z. If X is quasicompact and quasiseparated (qcqs, for short) and the open complement of Z is quasicompact, then $\mathrm{QCoh}(X \text{ on } Z)$ is compactly generated and its compact objects coincide with the perfect ones, [Lurie 2018, Proposition 9.6.1.1; Clausen et al. 2020, Proposition A.9]. Here a quasicoherent sheaf on X is called *perfect* if and only if its restriction to each affine open subscheme $U = \mathrm{Spec}(A) \subseteq X$ belongs to the smallest thick stable subcategory of $\mathrm{QCoh}(U) = \mathrm{Mod}(A)$ containing A. We write $\mathrm{Perf}(X)$ and $\mathrm{Perf}(X \text{ on } Z)$ for the corresponding subcategories. The arguments used to prove this also apply to the case of connective sheaves:

Lemma 2.1 *Let X be a qcqs derived scheme, and let $Z \subseteq |X|$ be a closed subset with quasicompact open complement. Then* $\mathrm{QCoh}(X \text{ on } Z)^{\mathrm{cn}}$ *is compactly generated and the inclusion* $\mathrm{QCoh}(X \text{ on } Z)^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(X)^{\mathrm{cn}}$ *preserves compact objects. An object of* $\mathrm{QCoh}(X \text{ on } Z)^{\mathrm{cn}}$ *is compact if and only if it is perfect* (*as an object of* $\mathrm{QCoh}(X)$).

Proof The same argument as in the proof of [Clausen et al. 2020, Proposition A.9] proves the first two claims: In case X is affine, these follow from [Lurie 2018, Proposition 7.1.1.12(e)] and the fact that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is compactly generated by [Lurie 2018, Proposition 9.6.1.2]. The reduction of the global case to the local case is done by [Lurie 2018, Example 10.3.0.2(4), Proposition 10.3.0.3, and Theorem 10.3.2.1(b)]. It then follows that an object of $\mathrm{QCoh}(X \text{ on } Z)^{\mathrm{cn}}$ is compact if and only if its image in $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is compact. The compact objects in the latter category coincide with the perfect, connective $\mathcal{O}_X$-modules by [Lurie 2018, Proposition 9.6.1.2] again. □

2.3 Free algebras

If A is a derived ring, we denote the category of derived A-algebras by CAlg^{Δ}_A. Its objects are derived rings B together with a map of derived rings $A \to B$. There is an obvious forgetful functor $\mathrm{CAlg}^{\Delta}_A \to \mathrm{Mod}(A)^{\mathrm{cn}}$ which admits a left adjoint, the free algebra functor, $\mathrm{LSym}^*_A\colon \mathrm{Mod}(A)^{\mathrm{cn}} \to \mathrm{CAlg}^{\Delta}_A$. If M is a connective A-module, the underlying A-module of $\mathrm{LSym}^*_A(M)$ is the direct sum $\bigoplus_{n\geq 0} \mathrm{LSym}^n_A(M)$, where $\mathrm{LSym}^n_A(M)$ is the n-th derived symmetric power of M as studied for instance by Quillen [Lurie 2018, Construction 25.2.2.6], whence the notation.

These constructions globalize: For a derived scheme X, write $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ for the ∞-category of sheaves of derived rings on $|X|$ equipped with a map from $\mathcal{O}_X$ such that the underlying sheaf of $\mathcal{O}_X$-modules is quasicoherent. There is an adjunction

$$\mathrm{LSym}^*_{\mathcal{O}_X}\colon \mathrm{QCoh}(X)^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X} : \mathrm{forget};$$

the forgetful functor is conservative and preserves sifted colimits. Consequently, $\mathrm{LSym}^*_{\mathcal{O}_X}$ sends compact objects to compact objects.

For $\mathcal{A} \in \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ one can form the relative spectrum $\mathrm{Spec}(\mathcal{A})$, which comes with an affine morphism $\mathrm{Spec}(\mathcal{A}) \to X$.

2.4 Finiteness conditions

A map of derived rings $A \to B$ is called *locally of finite presentation* if B is a compact object of CAlg^{Δ}_A. It is called *almost of finite presentation* if B is an almost compact object CAlg^{Δ}_A, i.e., each truncation $\tau_{\leq n} B$ is a compact object of $\tau_{\leq n}\mathrm{CAlg}^{\Delta}_A$; see [Lurie 2004, §3.1; 2018, §4.1] for the analog notions for $\mathbb{E}_\infty$-algebras. It turns out that $A \to B$ is almost of finite presentation if and only if the underlying map of $\mathbb{E}_\infty$-algebras is almost of finite presentation; the analog for being locally of finite presentation is wrong.

For completeness, we also mention that for an integer $n \geq 0$, there is a notion of *finite generation to order n*, see [Lurie 2018, Definition 4.1.1.1], and a morphism is almost finitely presented if and only if it is of finite generation to order n for all n.

These finiteness conditions are stable under base change: if $A \to B$ is locally or almost of finite presentation or of finite generation to order n and $A \to A'$ is an arbitrary map, then also $A' \to B \otimes_A A'$ is locally or almost of finite presentation or of finite generation to order n, respectively [Lurie 2017, Remark 7.2.4.28; 2018, Proposition 4.1.3.2]. They are also stable under composition [Lurie 2017, Remark 7.2.4.29 and Corollary 7.4.3.19; 2018, Proposition 4.1.3.1].

If A is Noetherian, i.e., $\pi_0(A)$ is Noetherian in the classical sense and all higher homotopy groups are finitely generated $\pi_0(A)$-modules, then a derived A-algebra B is almost of finite presentation if and only if B is Noetherian and $\pi_0(B)$ is a classically finitely generated $\pi_0(A)$-algebra [Lurie 2004, Proposition 3.1.5] or [Lurie 2017, Proposition 7.2.4.31].

A map $f\colon Y \to X$ of derived schemes is *called locally of finite presentation* or *locally almost of finite presentation* (*lafp*, for short) if for all affine open subschemes $U = \mathrm{Spec}(A) \subseteq X$ and $V = \mathrm{Spec}(B) \subseteq Y$ with $f(V) \subseteq U$, the induced morphism $A \to B$ is locally of finite presentation or almost of finite presentation, respectively. As in the affine case, these notions are stable under base change and composition, and there is a characterization in the Noetherian case.

For example, if $\mathcal{F}$ is a perfect, connective $\mathcal{O}_X$-module, i.e., a compact object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, then $\mathrm{Spec}(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{F}))$ is locally of finite presentation over X. Using this observation, we prove the following lemma:

Lemma 2.2 *Let X be a qcqs derived scheme, and let $U \subseteq X$ be a qc open subset with complement Z. Let $i\colon Y \to X$ be a closed immersion which is an isomorphism over U and such that the underlying map of classical schemes ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is finitely presented. Then there exists a factorization $Y \to Y' \to X$ of i such that*

(1) *the morphism $Y \to Y'$ is an isomorphism on underlying classical schemes,*

(2) *the morphism $Y' \to X$ is a closed immersion locally of finite presentation and an isomorphism over U.*

Proof Let $\mathcal{I} = \mathrm{fib}(\mathcal{O}_X \to i_*\mathcal{O}_Y)$. As i is a closed immersion and an isomorphism over U, we have $\mathcal{I} \in \mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$. As the morphism of classical schemes ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is classically of finite presentation, it follows that the image $\mathcal{J}$ of $\pi_0(\mathcal{I})$ in $\pi_0(\mathcal{O}_X)$ is of finite type. By Lemma 2.1, we can write $\mathcal{I}$ as a filtered colimit $\mathcal{I} = \mathrm{colim}_\alpha \mathcal{I}_\alpha$ where each I_α is in $\mathrm{Perf}(X \text{ on } Z)^{\mathrm{cn}}$. As $\pi_0(-)$ commutes with filtered colimits, we have $\mathrm{colim}_\alpha \pi_0(\mathcal{I}_\alpha) = \pi_0(\mathcal{I})$. As $\pi_0(\mathcal{I}) \to \mathcal{J}$ is surjective and $\mathcal{J}$ is of finite type, there exists an index α such that the induced map $\pi_0(\mathcal{I}_\alpha) \to \mathcal{J}$ is surjective. By construction, we have the following commutative diagram in $\mathrm{QCoh}(X)^{\mathrm{cn}}$:

$$\begin{array}{ccc} \mathcal{I}_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Y \end{array}$$

By adjunction, this induces a commutative diagram in $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$:

$$\begin{array}{ccc} \mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha) & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Y \end{array}$$

We define $\mathcal{A} \in \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ to be the tensor product $\mathcal{O}_X \otimes_{\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha)} \mathcal{O}_X$. The above diagram classifies a morphism $\mathcal{A} \to i_*\mathcal{O}_Y$ in $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$. We claim that it induces an isomorphism on π_0. Indeed, we compute[2]

$$\begin{aligned}\pi_0(\mathcal{A}) &\cong \pi_0(\mathcal{O}_X) \otimes^{\heartsuit}_{\pi_0(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha))} \pi_0(\mathcal{O}_X) \cong \pi_0(\mathcal{O}_X) \otimes^{\heartsuit}_{\mathrm{Sym}_{\pi_0(\mathcal{O}_X)}(\pi_0(\mathcal{I}_\alpha))} \pi_0(\mathcal{O}_X) \\ &\cong \pi_0(\mathcal{O}_X)/\operatorname{im}(\pi_0(\mathcal{I}_\alpha) \to \pi_0(\mathcal{O}_X)) = \pi_0(\mathcal{O}_X)/\mathcal{J} \cong \pi_0(i_*\mathcal{O}_Y).\end{aligned}$$

We set $Y' = \mathrm{Spec}(\mathcal{A})$. By construction, we get the factorization $Y \to Y' \to X$ of i. The above computation shows that $Y \to Y'$ is an isomorphism on underlying classical schemes. Moreover, as I_α is supported on Z, $Y' \to X$ is an isomorphism over U. Finally, as I_α is perfect, it follows that $Y' \to X$ is locally of finite presentation, as desired. □

2.5 Formal completion

Let X be a qcqs derived scheme and let $Z \subseteq |X|$ be a closed subset whose open complement $|X| \setminus Z$ is quasicompact. The *formal completion* $X^{\wedge}_Z$ is an ind-object of derived schemes with a map $i: X^{\wedge}_Z \to X$. It is determined by the following universal property: for any derived ring R, composition with i induces an equivalence of $\mathrm{Map}_{\mathrm{Ind(dSch)}}(\mathrm{Spec}(R), X^{\wedge}_Z)$ with the union of components of $\mathrm{Map}_{\mathrm{dSch}}(\mathrm{Spec}(R), X)$ consisting of those morphisms $\mathrm{Spec}(R) \to X$ that set-theoretically factor through Z; see [Gaitsgory and Rozenblyum 2014, Proposition 6.5.5] for the existence of $X^{\wedge}_Z$ as an ind-derived scheme. More concretely, we can write $X^{\wedge}_Z$ as the ind-system

$$X^{\wedge}_Z = \{Z'\}_{Z' \hookrightarrow X} \tag{2}$$

of (a small cofinal subsystem of) all closed immersions of derived schemes $Z' \hookrightarrow X$ with $|Z'| = Z$.

If $X = \mathrm{Spec}(A)$ is affine, there exist finitely many elements $f_1, \dots, f_r \in \pi_0(A)$ whose zero locus is Z. In this case, $X^{\wedge}_Z$ can also be represented by

$$X^{\wedge}_Z = \{\mathrm{Spec}(A /\!\!/ f_1^\alpha, \dots, f_r^\alpha)\}_{\alpha \geq 1}, \tag{3}$$

where $/\!\!/$ indicates the derived quotient, i.e., the (derived) tensor product $A \otimes_{\mathbb{Z}[t_1,\dots,t_r]} \mathbb{Z}$ where the maps send t_i to f_i^α and 0, respectively; see [Lurie 2004, Proposition 6.1.1] or [Lurie 2018, Lemma 8.1.2.2].

It follows immediately from the universal property that the formation of the derived completion commutes with base change: If $f: Y \to X$ is a quasicompact map, then $Y^{\wedge}_{f^{-1}(Z)} \simeq X^{\wedge}_Z \times_X Y$. We therefore also write $Y^{\wedge}_Z$ instead of $Y^{\wedge}_{f^{-1}(Z)}$.

If X is a Noetherian classical scheme, then the formal completion is itself classical, equal to the classical formal completion. This follows for example from [Lurie 2018, Lemma 17.3.5.7].

2.6 Ample line bundles

Ample line bundles on Noetherian derived schemes have been studied by Annala [2022]. We need some of the results in the more general setting of qcqs derived schemes. These are certainly well-known; the proofs are essentially the same as for classical schemes.

[2] Here the $\otimes^{\heartsuit}$ indicates the underived tensor product and Sym denotes the classical symmetric algebra.

Let X be a qcqs derived scheme. A line bundle $\mathcal{L}$ on X is called *ample* if for any point $x \in X$ there exists an $n \geq 1$ and a global section $s \in \pi_0\Gamma(X, \mathcal{L}^{\otimes n})$ such that the nonvanishing locus X_s of s is affine and contains x. If $f\colon X \to Y$ is a morphism of derived schemes, then $\mathcal{L}$ is called *f-ample* if for every affine open subscheme $U \subseteq Y$ the restriction of $\mathcal{L}$ to $f^{-1}(U)$ is ample.

Let $\mathcal{L}$ be any line bundle on X and $s \in \pi_0\Gamma(X, \mathcal{L})$ a global section. We view the latter as a map $s\colon \mathcal{O}_X \to \mathcal{L}$. If $\mathcal{F}$ is any quasicoherent sheaf on X, we get a diagram

$$\mathcal{F} \xrightarrow{\otimes s} \mathcal{F} \otimes \mathcal{L} \xrightarrow{\otimes s} \mathcal{F} \otimes \mathcal{L}^{\otimes 2} \to \cdots. \tag{4}$$

In the case of classical schemes, the following lemma is standard. For Noetherian derived schemes, it is [Annala 2022, Lemma 2.6].

Lemma 2.3 *In the above situation, there is a canonical equivalence*

$$\operatorname*{colim}_n \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \simeq \Gamma(X_s, \mathcal{F}).$$

Proof If we take sections over X_s in (4), then all maps become equivalences. Thus the restriction maps induce a map

$$\operatorname*{colim}_n \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \operatorname*{colim}_n \Gamma(X_s, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \simeq \Gamma(X_s, \mathcal{F}).$$

We claim that this map is an equivalence. As X is qcqs, a standard induction reduces us to the case where $X = \operatorname{Spec}(A)$ is affine and $\mathcal{L}$ is trivial. So we may assume $X = \operatorname{Spec}(A)$, $\Gamma(X, \mathcal{L}) = A$, and $\mathcal{F}$ corresponds to the A-module $M = \Gamma(X, \mathcal{F})$. In this case, the colimit in question identifies with $M[s^{-1}]$, which also identifies with $\Gamma(X_s, \mathcal{F})$ as $\mathcal{F}$ is quasicoherent. □

Exactly as in [Annala 2022, Lemma 2.11], this can be used to prove the following lemma:

Lemma 2.4 *Let $f\colon X \to Y$ be a morphism of qcqs derived schemes, $\mathcal{L}$ a line bundle on X, and $(U_i)_{i \in I}$ an open covering of Y. Write f_i for the restricted morphism $f^{-1}(U_i) \to U_i$. Then the following are equivalent*:

(1) *$\mathcal{L}$ is f-ample.*

(2) *For every $i \in I$, the restriction $\mathcal{L}|_{f^{-1}(U_i)}$ of $\mathcal{L}$ is f_i-ample.* □

The existence of an ample line bundle implies the resolution property in the following form:

Lemma 2.5 *Let X be a qcqs derived scheme which carries an ample line bundle, and let $\mathcal{F}$ be a connective, quasicoherent $\mathcal{O}_X$-module such that $\pi_0\mathcal{F}$ is of finite type as a $\pi_0\mathcal{O}_X$-module. Then there exists a vector bundle, i.e., a locally free $\mathcal{O}_X$-module of finite rank, $\mathcal{E}$ on X together with a map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ which is surjective on π_0.*

Proof Choose an ample line bundle $\mathcal{L}$. Replacing $\mathcal{L}$ by an appropriate tensor power, we may assume that there exist finitely many global sections $s_i \in \pi_0\Gamma(X, \mathcal{L})$, whose nonvanishing loci X_{s_i} form an affine covering of X. As each X_{s_i} is affine and $\mathcal{F}$ is quasicoherent, we have isomorphisms $\pi_0\Gamma(X_{s_i}, \mathcal{F}) = \pi_0\Gamma(X_{s_i}, \pi_0\mathcal{F})$ and similarly for $\mathcal{O}_X$ in place of $\mathcal{F}$. The assumption that $\pi_0\mathcal{F}$ is a $\pi_0\mathcal{O}_X$-module of finite type then implies that each $\pi_0\Gamma(X_{s_i}, \mathcal{F})$ is a finitely generated $\pi_0\Gamma(X_{s_i}, \mathcal{O}_X)$-module. Choose finitely many generators $m_{ij} \in \pi_0\Gamma(X_{s_i}, \mathcal{F})$. By Lemma 2.3 there exists an integer N such that all the m_{ij} extend to global sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes N}$. These give rise to a map $\mathcal{E} := \bigoplus_{ij} \mathcal{L}^{\otimes(-N)} \to \mathcal{F}$ which is surjective on π_0 by construction. □

2.7 Quasismooth closed immersions and derived blowups

Derived blowups were first introduced in [Kerz et al. 2018] for affine schemes in order to prove pro-cdh descent of algebraic K-theory on Noetherian schemes. They were then systematically studied and developed much further by Khan and Rydh [2025] and Hekking [2021].

Let X be a derived scheme, and let $Z \hookrightarrow X$ be a *quasismooth closed immersion* [Khan and Rydh 2025, 2.3.6], i.e., Zariski locally on X, $Z \hookrightarrow X$ is the derived pullback of the map $\{0\} \hookrightarrow \mathbb{A}^r_{\mathbb{Z}}$ for some r and some morphism $X \to \mathbb{A}^r_{\mathbb{Z}}$. Equivalently, Zariski locally on X, $Z \hookrightarrow X$ is given by $\mathrm{Spec}(A /\!\!/ f_1, \dots, f_r) \hookrightarrow \mathrm{Spec}(A)$ for suitable elements $f_i \in \pi_0(A)$. The number r is called the *virtual codimension* of the closed immersion. Then one can form the *derived blowup* $p\colon \mathrm{dBl}_Z(X) \to X$ *of* X *in* Z (or *with center* Z) which is characterized by a universal property: it classifies virtual Cartier divisors on derived X-schemes; see [Khan and Rydh 2025, §4.1] for details. The construction of the derived blowup commutes with arbitrary base change. Locally, if Z is the derived pullback of $\{0\} \hookrightarrow \mathbb{A}^r_{\mathbb{Z}}$ along a map $X \to \mathbb{A}^r_{\mathbb{Z}}$, then $\mathrm{dBl}_Z(X)$ is the derived pullback of the classical blowup $\mathrm{Bl}_{\{0\}}(\mathbb{A}^r_{\mathbb{Z}}) \to \mathbb{A}^r_{\mathbb{Z}}$. By [Khan and Rydh 2025, Theorem 4.1.5(v)], the morphism $p\colon \mathrm{dBl}_Z(X) \to X$ is quasismooth, i.e., Zariski locally factors as a quasismooth closed immersion followed by a smooth morphism, and in particular is locally of finite presentation. Clearly, p is an isomorphism outside Z and proper.

It follows from the description of the underlying classical scheme of $\mathrm{dBl}_Z(X)$ in [Khan and Rydh 2025, Theorem 4.1.5(vii)] that there is always a closed immersion $\mathrm{Bl}_{^{\mathrm{cl}}Z}(^{\mathrm{cl}}X) \hookrightarrow {}^{\mathrm{cl}}\mathrm{dBl}_Z(X)$ and this is an isomorphism over $^{\mathrm{cl}}U$ where U is the open complement of Z.

The derived blowup $\mathrm{dBl}_Z(X)$ carries a canonical line bundle $\mathcal{O}(1)$ (the ideal sheaf defining the universal virtual Cartier divisor) and this line bundle is p-ample. Indeed, by Lemma 2.4 we may work affine-locally on X and hence assume that $p\colon \mathrm{dBl}_Z(X) \to X$ is the pullback of the classical blowup $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_{\mathbb{Z}}) \to \mathbb{A}^n_{\mathbb{Z}}$, and the line bundle $\mathcal{O}(1)$ is the pullback of the classical canonical line bundle on $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ which is ample. As being relatively ample is stable under base change (the proof in [Annala 2022, Proposition 2.12] works in general), it follows that $\mathcal{O}(1)$ is p-ample.

In the presence of an ample line bundle, every closed subset with quasicompact complement is the support of a quasismooth closed subscheme. More precisely, we have the following lemma.

Lemma 2.6 *Let X be a qcqs derived scheme which carries an ample line bundle. Let $Z_0 \hookrightarrow {}^{\mathrm{cl}}X$ be a classically finitely presented closed subscheme. Then there exists a quasismooth closed subscheme $Z \hookrightarrow X$ whose classical truncation is Z_0.*

Proof This is very similar to [Bachmann et al. 2022, Construction A.2.2]. Let $J = \mathrm{fib}(\mathcal{O}_X \to \mathcal{O}_{Z_0})$. As $\mathcal{O}_{Z_0}$ is a discrete quasicoherent sheaf and $\mathcal{O}_X \to \mathcal{O}_{Z_0}$ is surjective on π_0, the sheaf J is connective, quasicoherent, and $\pi_0 J$ is the ideal sheaf defining Z_0 in ${}^{\mathrm{cl}}X$. As $Z_0 \hookrightarrow {}^{\mathrm{cl}}X$ is classically of finite presentation, $\pi_0 J$ is a $\pi_0\mathcal{O}_X$-module of finite type. Hence Lemma 2.5 implies the existence of a vector bundle $\mathcal{E}$ on X and a map $\mathcal{E} \to \mathcal{J}$ which is surjective on π_0. Let $V(\mathcal{E})$ be the geometric vector bundle $V(\mathcal{E}) := \mathrm{Spec}(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{E}))$ over X. The composition $\mathcal{E} \to \mathcal{J} \to \mathcal{O}_X$ defines a section $s\colon X \to V(\mathcal{E})$, and we form the fiber product

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle 0} \\ X & \xrightarrow{\ s\ } & V(\mathcal{E}) \end{array}$$

Equivalently, we have

$$Z = \mathrm{Spec}(\mathcal{O}_X \otimes_{\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{E})} \mathcal{O}_X).$$

By construction, $i\colon Z \hookrightarrow X$ is quasismooth. The same computation as in the proof of Lemma 2.2 shows that ${}^{\mathrm{cl}}Z = Z_0$, as desired. □

2.8 Pushouts of derived schemes

We also need certain pushouts of derived schemes. These have been studied in [Gaitsgory and Rozenblyum 2017]:

Lemma 2.7 *Let $i\colon Y_1 \to Y_1'$ be a closed immersion of derived schemes which is an isomorphism on underlying topological spaces, and let $f\colon Y_1 \to Y_2$ be an affine map in* dSch. *Then the following hold.*

(1) *The pushout square*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\ i\ } & Y_1' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y_2 & \xrightarrow{\ i'\ } & Y_2 \sqcup_{Y_1} Y_1' \end{array}$$

exists in dSch. *Write $Y_2' := Y_2 \sqcup_{Y_1} Y_1'$. The map $Y_2 \to Y_2'$ is a closed immersion and an isomorphism on underlying topological spaces. In particular, if Y_2 is qcqs, then so is Y_2'.*

(2) *For an affine open subscheme $U_2 \subseteq Y_2$ with $U_1 := f^{-1}(U_2) \subseteq Y_1$, and the corresponding open subschemes $U_i' \subseteq Y_i'$ (for $i = 1, 2$), the map*

$$U_2 \sqcup_{U_1} U_1' \to U_2'$$

is an isomorphism.

(3) *If f is an open immersion, then so is f'.*

(4) *Assume i exhibits Y_1 as the underlying classical scheme of Y_1'. Then $i'\colon Y_2 \to Y_2'$ is an isomorphism on underlying classical schemes.*

Proof Except for (4), this is [Gaitsgory and Rozenblyum 2017, Chapter 1, Corollary 1.3.5]. By (2), we may assume $Y_i = \mathrm{Spec}(A_i)$ (for $i = 1, 2$) and $Y_1' = \mathrm{Spec}(A_1')$ are affine, and $A_1 = \pi_0(A_1')$. By the construction of the pushout, $Y_2' = \mathrm{Spec}(A_2')$ where $A_2' = A_2 \times_{A_1} A_1'$ is the pullback in derived rings. We thus have an exact sequence of homotopy groups

$$\pi_1(A_1) \to \pi_0(A_2') \to \pi_0(A_2) \oplus \pi_0(A_1') \to \pi_0(A_1),$$

which implies (4) as $\pi_1(A_1) = 0$ and $\pi_0(A_1') \cong \pi_0(A_1)$. □

3 Modifications of derived schemes

In the following, X always denotes a qcqs derived scheme.

Definition 3.1 Let $U \subseteq X$ be a quasicompact open subscheme. A *U-modification* of X is a proper morphism $f: Y \to X$ which is an isomorphism over U. A *closed U-modification* is a U-modification which is a closed immersion.

Note that we do not assume that a U-modification induces a bijection of the set of generic points. For example, if $Z \hookrightarrow X$ is a quasismooth closed immersion with $|Z| \cap |U| = \varnothing$, then the derived blowup $\mathrm{Bl}_Z(X) \to X$ is a U-modification which is moreover lafp (see Section 2.7). In fact, derived blowups and lafp closed U-modifications generate all lafp U-modifications in the following sense:

Theorem 3.2 *Assume that X carries an ample line bundle. Let $U \subseteq X$ be a quasicompact open subscheme. Let $f: Y \to X$ be a U-modification of X whose underlying map of classical schemes ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is classically finitely presented. Then there exists a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{p} & X \end{array}$$

where g is U-modification of Y, h is a closed U-modification, p is a derived blowup with center set-theoretically contained in $X \setminus U$, and the induced map $(h, g): Y' \to \widetilde{X} \times_X Y$ is locally of finite presentation. In particular, g is locally of finite presentation, and, if f is locally of finite presentation (or lafp, or locally of finite generation to order n, respectively), then h is locally of finite presentation (or lafp, or locally of finite generation to order n).

Moreover, Y' carries a $(p \circ h)$-ample line bundle.

Proof By assumption, the morphism of classical schemes ${}^{\mathrm{cl}}f: {}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is proper and classically of finite presentation. It is also an isomorphism over ${}^{\mathrm{cl}}U$. By [Raynaud and Gruson 1971, Corollary 5.7.12] (or [Stacks 2005–, Tag 081T]), there exists a ${}^{\mathrm{cl}}U$-admissible blowup[3] $Y_0 \to {}^{\mathrm{cl}}X$ such that the morphism $Y_0 \to {}^{\mathrm{cl}}X$ factors through ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$. For the constructions to come, we need the open immersion ${}^{\mathrm{cl}}U \to Y_0$ to be affine. As this is not necessarily the case, we make a further blowup: As ${}^{\mathrm{cl}}U$ is quasicompact, there exists a finitely presented closed subscheme $T_0 \hookrightarrow Y_0$ whose underlying topological space is $Y_0 \setminus {}^{\mathrm{cl}}U$. This follows for instance from [Grothendieck and Dieudonné 1971, Corollary 6.9.15], which allows us to write

[3] A blowup in a finitely presented closed subscheme of ${}^{\mathrm{cl}}X$ which is set-theoretically contained in the complement of ${}^{\mathrm{cl}}U \subseteq {}^{\mathrm{cl}}X$.

the ideal sheaf of some closed subscheme of Y_0 with support $Y_0 \setminus {}^{\mathrm{cl}}U$ as a filtered colimit of quasicoherent subideal sheaves of finite type. By quasicompactness of ${}^{\mathrm{cl}}U$, the closed subscheme defined by one of them will have support $Y_0 \setminus {}^{\mathrm{cl}}U$. Let $Y_1 \to Y_0$ be the blowup of Y_0 in T_0. Then the canonical open immersion ${}^{\mathrm{cl}}U \to Y_1$ is affine as its complement is a Cartier divisor. As the composite of two ${}^{\mathrm{cl}}U$-admissible blowups is a ${}^{\mathrm{cl}}U$-admissible blowup [Stacks 2005–, Tag 080L], the composition $Y_1 \to Y_0 \to {}^{\mathrm{cl}}X$ is a ${}^{\mathrm{cl}}U$-admissible blowup, say $Y_1 = \mathrm{Bl}_{S_0}({}^{\mathrm{cl}}X)$ for some classically finitely presented closed subscheme $S_0 \hookrightarrow {}^{\mathrm{cl}}X$.

As X carries an ample line bundle, Lemma 2.6 implies the existence of a quasismooth closed immersion of derived schemes $S \hookrightarrow X$ whose classical truncation is $S_0 \hookrightarrow {}^{\mathrm{cl}}X$. Let $p\colon \widetilde{X} \to X$ be the derived blowup of X in S. Then there is a canonical closed immersion $Y_1 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$, which is an isomorphism over ${}^{\mathrm{cl}}U$. Note that Y_1 need not be classically of finite presentation over X. However, using [Grothendieck and Dieudonné 1971, Corollary 6.9.15] again we see that the closed immersion $Y_1 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$ can be written as a cofiltered limit of classically finitely presented closed immersions $Y_\alpha \to {}^{\mathrm{cl}}\widetilde{X}$ all of which are isomorphisms over ${}^{\mathrm{cl}}U$. As ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is classically of finite presentation, there exists an α such that the ${}^{\mathrm{cl}}X$-morphism $Y_1 \to {}^{\mathrm{cl}}Y$ factors through a morphism $Y_\alpha \to {}^{\mathrm{cl}}Y$; see [Grothendieck 1966, Proposition 8.14.2]. Thus, so far, we have constructed a commutative diagram of classical schemes finitely presented over ${}^{\mathrm{cl}}X$

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & {}^{\mathrm{cl}}Y \\ \downarrow & & \downarrow \\ {}^{\mathrm{cl}}\widetilde{X} & \longrightarrow & {}^{\mathrm{cl}}X \end{array}$$

in which all morphisms are isomorphisms over ${}^{\mathrm{cl}}U$. The lower-right corner is the classical truncation of the cospan $\widetilde{X} \xrightarrow{p} X \xleftarrow{f} Y$.

As the composite of the affine open immersion ${}^{\mathrm{cl}}U \to Y_1$ with the closed immersion $Y_1 \hookrightarrow Y_\alpha$, the open immersion ${}^{\mathrm{cl}}U \to Y_\alpha$ is affine, too. By Lemma 2.7 we may hence form the pushout $Y_2 = Y_\alpha \sqcup_{{}^{\mathrm{cl}}U} U$ of derived schemes, for which we have ${}^{\mathrm{cl}}Y_2 = Y_\alpha$. As p and f are isomorphisms over U, we get induced morphisms $h_2\colon Y_2 \to \widetilde{X}$ and $g_2\colon Y_2 \to Y$, and a commutative diagram

$$\begin{array}{ccc} Y_2 & \xrightarrow{g_2} & Y \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{p} & X \end{array}$$

Note that h_2 is a closed immersion, as this only depends on the underlying map of classical schemes. Moreover, all morphisms in the above diagram are U-modifications. However, h_2 and g_2 need not satisfy the desired finiteness condition. In order to remedy this, we consider the induced morphism $k_2 = (h_2, g_2)\colon Y_2 \to \widetilde{X} \times_X Y$. As ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is separated, and as h_2 is a closed immersion, k_2 is a closed immersion too. It is also an isomorphism over U. By construction, the map of underlying classical schemes ${}^{\mathrm{cl}}k_2\colon Y_\alpha = {}^{\mathrm{cl}}Y_2 \to {}^{\mathrm{cl}}(\widetilde{X} \times_X Y)$ is classically of finite presentation. We can hence apply Lemma 2.2 to obtain a factorization of k_2 through a closed derived subscheme $k'\colon Y' \hookrightarrow \widetilde{X} \times_X Y$ such that k' is locally of finite presentation and an isomorphism over U, and ${}^{\mathrm{cl}}Y' \cong {}^{\mathrm{cl}}Y_2$. Define g and h

to be the composites of k' with the two projections from $\widetilde{X} \times_X Y$ to Y and $\widetilde{X}$, respectively. Both are U-modifications. As $p\colon \widetilde{X} \to X$ is locally of finite presentation, so is the first projection, and hence g. If f is locally of finite presentation (or lafp, or locally of finite generation to order n), then so is the second projection, and hence also h. As the underlying map of classical schemes ${}^{\mathrm{cl}}h$ identifies with ${}^{\mathrm{cl}}h_2\colon {}^{\mathrm{cl}}Y_2 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$, it is a closed immersion. This finishes the construction of the asserted commutative diagram, and the proof of the required finiteness conditions.

It remains to prove the claim about ample line bundles. As discussed in Section 2.7, the canonical line bundle $\mathcal{O}(1)$ on the derived blowup $\widetilde{X}$ is p-ample. As h is a closed immersion and thus in particular affine, the pullback $h^*\mathcal{O}(1)$ is then $(p \circ h)$-ample. □

4 Pro-cdh descent for connective localizing invariants

In this section, we prove our main results on pro-descent for localizing invariants. The strategy is the same as in [Kerz et al. 2018]: we first prove the result for the special cases of derived blowups and finite modifications and then use the geometric input from Theorem 3.2 to handle the general case.

We begin by fixing some notation. Let k be a fixed commutative base ring (e.g., $k = \mathbb{Z}$). If E is an additive invariant of small k-linear ∞-categories with values in a stable presentable ∞-category $\mathcal{C}$, e.g., the ∞-category of spectra, and X is a qcqs derived k-scheme, we write $E(X)$ for $E(\mathrm{Perf}(X))$. Let $Z \subseteq |X|$ be a closed subset with quasicompact open complement. Recall from Section 2.5 that the formal completion $X^\wedge_Z$ is an ind-derived scheme. Applying E we thus obtain a pro-object $E(X^\wedge_Z)$. We write $E(X, X^\wedge_Z)$ for the relative term $\mathrm{fib}(E(X) \to E(X^\wedge_Z))$ in $\mathrm{Pro}(\mathcal{C})$.

A version of the following proposition was first proven in [Kerz et al. 2018]. There all schemes were assumed to be classical Noetherian schemes, E was K-theory, and the result only gave a weakly cartesian square of pro-spectra; see below for this notion. In a letter to Kerz, Antieau [2018] described a simplification of the proof which at the same time gives a cartesian square. We thank Ben Antieau for allowing us to include his argument in our paper.

Proposition 4.1 *Let X, Z, and E be as above. Let $\widetilde{X} \to X$ be a derived blowup in a quasismooth closed immersion $S \hookrightarrow X$ with S set-theoretically contained in Z. Then we have a cartesian square in* $\mathrm{Pro}(\mathcal{C})$:

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(\widetilde{X}) & \longrightarrow & E(\widetilde{X}^\wedge_Z) \end{array}$$

Proof Let $r \geq 1$ be the virtual codimension of the derived blowup, and let D be its exceptional divisor, i.e., the universal virtual Cartier divisor on the derived blowup, so that there is a commutative diagram

$$\begin{array}{ccc} D & \xhookrightarrow{j} & \widetilde{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ S & \xhookrightarrow{i} & X \end{array}$$

Recall from [Khan 2020, Theorem C] that $\mathrm{Perf}(\widetilde{X})$ has a semiorthogonal decomposition as follows. The functor $p^*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(\widetilde{X})$ is fully faithful; denote its essential image by $\mathcal{B}(0)$. For $1 \le k \le r-1$, the composed functor $j_*(q^*(-) \otimes_{\mathcal{O}_D} \mathcal{O}_D(-k))\colon \mathrm{Perf}(S) \to \mathrm{Perf}(\widetilde{X})$ is fully faithful; denote its essential image by $\mathcal{B}(-k)$. Then the sequence of full subcategories $\mathcal{B}(0), \mathcal{B}(-1), \ldots, \mathcal{B}(-r+1)$ forms a semiorthogonal decomposition of $\mathrm{Perf}(\widetilde{X})$. In particular, there is a decomposition

$$E(\widetilde{X}) \simeq E(\mathcal{B}(0)) \oplus \bigoplus_{k=1}^{r-1} E(\mathcal{B}(-k)) \simeq E(X) \oplus \bigoplus_{k=1}^{r-1} E(S). \tag{5}$$

Now let $Z' \hookrightarrow X$ be any closed derived subscheme with $|Z'| = Z$. Note that all ∞-categories appearing above are in fact $\mathrm{Perf}(X)$-linear, as are the functors between them. In particular, we can base change the semiorthogonal decomposition of $\mathrm{Perf}(\widetilde{X})$ along $\mathrm{Perf}(X) \to \mathrm{Perf}(Z')$. As there are canonical equivalences (as follows from [Lurie 2018, Corollary 9.4.3.8] by passing to compact objects; see also [Ben-Zvi et al. 2010, Theorem 4.7] with slightly different hypotheses)

$$\mathrm{Perf}(\widetilde{X}) \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Z') \simeq \mathrm{Perf}(\widetilde{X} \times_X Z'), \quad \mathrm{Perf}(S) \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Z') \simeq \mathrm{Perf}(S \times_X Z'),$$

we conclude that $\mathrm{Perf}(\widetilde{X} \times_X Z')$ admits a semiorthogonal decomposition

$$(\mathcal{B}(0)_{Z'}, \mathcal{B}(-1)_{Z'}, \ldots, \mathcal{B}(-r+1)_{Z'})$$

with $\mathcal{B}(0)_{Z'} \simeq \mathrm{Perf}(Z')$ and $\mathcal{B}(-k)_{Z'} \simeq \mathrm{Perf}(S \times_X Z')$ for $1 \le k \le r-1$. In particular,

$$E(\widetilde{X} \times_X Z') \simeq E(Z') \oplus \bigoplus_{k=1}^{r-1} E(S \times_X Z'). \tag{6}$$

Recall from (2) that $E(X^\wedge_Z) \in \mathrm{Pro}(\mathcal{C})$ is given concretely as the pro-object $\{E(Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$ where Z' runs through all closed derived subschemes of X with $|Z'| = Z$. As formal completion is compatible with base change (see Section 2.5), we similarly have $E(\widetilde{X}^\wedge_Z) = \{E(\widetilde{X} \times_X Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$. Comparing the decompositions (5) and (6) it thus suffices to prove that the functor induced by pullback

$$\mathrm{Perf}(S) \to \{\mathrm{Perf}(S \times_X Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$$

is an equivalence of pro-∞-categories. For this, it suffices to check that the map of ind-derived schemes $\{S \times_X Z'\}_{Z' \hookrightarrow X, |Z'|=Z} \to S$ is an equivalence. But this is clear: By Section 2.5 again, the source represents the Z-completion $S^\wedge_Z$ of the target. As S is set-theoretically contained in Z, we clearly have $S^\wedge_Z = S$. □

Let l be an integer. Recall from [Land and Tamme 2019, Definition 2.5] that a spectra-valued localizing invariant E is called *l-connective* if, for any n-connective map (for $n \ge 1$) of connective $\mathbb{E}_1$-ring spectra $A \to B$, the induced map $E(A) \to E(B)$ is $(n+l)$-connective. For example, K-theory, topological cyclic homology TC, and rational negative cyclic homology $\mathrm{HN}(- \otimes \mathbb{Q}/\mathbb{Q})$ are 1-connective. THH is 0-connective; see [Land and Tamme 2019, Example 2.6].

Recall also that a map of pro-spectra $\{C_\alpha\}_\alpha \to \{D_\alpha\}_\alpha$ is called a *weak equivalence* if each truncation $\{\tau_{\leq n} C_\alpha\}_\alpha \to \{\tau_{\leq n} D_\alpha\}_\alpha$ is an equivalence in Pro(Sp) and there are similar notions of being weakly cartesian, weakly contractible, and so on; see [Land and Tamme 2019, Definition 2.27].

Proposition 4.2 *Let E be a localizing invariant of small k-linear ∞-categories that is l-connective for some integer l. Let $f\colon Y \to X$ be a finite, lafp morphism of qcqs derived k-schemes which is an isomorphism outside the closed subset Z with $|X| \setminus Z$ quasicompact. Then the commutative square of pro-spectra*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(Y) & \longrightarrow & E(Y^\wedge_Z) \end{array} \tag{7}$$

is weakly cartesian.

Proof We first reduce to the case that X is affine: As X is qcqs, we can write X as the colimit of a finite diagram of open affine subschemes $V_i \hookrightarrow X$. As any localizing invariant satisfies Zariski descent, we get $E(X) = \lim_i E(V_i)$ and $E(Y) = \lim_i E(Y \times_X V_i)$. If $Z' \hookrightarrow X$ is a closed subscheme with $|Z'| = Z$, we also have $E(Z') = \lim_i E(V_i \times_X Z')$. As finite limits in $\mathrm{Pro}(\mathcal{C})$ are computed levelwise and formal completion is compatible with base change, this implies $E(X^\wedge_Z) = \lim_i E((V_i)^\wedge_Z)$ and similarly for $E(Y^\wedge_Z)$. Replacing X by V_i and Z by $V_i \cap Z$ we thus reduce to the case that X is affine.

So assume now that X is affine, say $X = \mathrm{Spec}(A)$. As f is finite, Y is also affine, say $Y = \mathrm{Spec}(B)$. Let $\phi\colon A \to B$ be the corresponding morphism of derived rings and write $J = \mathrm{fib}(A \to B)$. As A and B are connective, J is (-1)-connective. By [Lurie 2018, Corollary 5.2.2.2] the A-algebra B is almost perfect as an A-module; hence also J is almost perfect as an A-module, i.e., $\tau_{\leq n} J$ is a compact object in $\tau_{\leq n+1} \mathrm{Mod}(A)_{\geq -1} = \mathrm{Mod}(A)_{[-1,n]}$ for every n.[4]

Choose $f_1, \dots, f_r \in \pi_0(A)$ whose zero set is Z. Recall from (3) that $X^\wedge_Z$ is then represented by the ind-derived scheme $\{\mathrm{Spec}(A /\!\!/ f_1^\alpha, \dots, f_r^\alpha)\}_{\alpha \geq 1}$. Consider the commutative diagram of derived rings

$$\begin{array}{ccc} A & \longrightarrow & A /\!\!/ f_1^\alpha, \dots, f_r^\alpha \\ \downarrow & & \downarrow \\ B & \longrightarrow & B /\!\!/ f_1^\alpha, \dots, f_r^\alpha \end{array} \tag{8}$$

We claim that as pro-system in α, this square is weakly cartesian. The map of vertical fibers (in A-modules) is the canonical map

$$J \to J /\!\!/ f_1^\alpha, \dots, f_r^\alpha,$$

[4] The $(n+1)$-truncated objects in $\mathrm{Mod}(A)_{\geq -1}$ are precisely the n-truncated, (-1)-connective objects in $\mathrm{Mod}(A)$ with respect to the standard t-structure. Hence the usual truncation $\tau_{\leq n} J$ coming from the t-structure is the categorical $(n+1)$-truncation $\mathrm{Mod}(A)_{\geq -1} \to \tau_{\leq n+1} \mathrm{Mod}(A)_{\geq -1}$.

so we have to prove that this map is a weak equivalence as a pro-system in α. For any $i = 1, \dots, r$, the fiber of the map of pro-systems $J \to \{J /\!\!/ f_i^\alpha\}_\alpha$ is the pro-system

$$\{J \xleftarrow{f_i} J \xleftarrow{f_i} \cdots\}. \tag{9}$$

We show below that this system is weakly contractible. We then get weak equivalences $J \xrightarrow{\simeq} \{J /\!\!/ f_1^\alpha\}_\alpha$ and, taking derived quotients by powers of f_2, $\{J /\!\!/ f_2^\alpha\}_\alpha \xrightarrow{\simeq} \{J /\!\!/ f_1^\alpha, f_2^\alpha\}_\alpha$. Composing with $J \xrightarrow{\simeq} \{J /\!\!/ f_2^\alpha\}_\alpha$ we get the weak equivalence $J \xrightarrow{\simeq} \{J /\!\!/ f_1^\alpha, f_2^\alpha\}_\alpha$, and continuing like this we finally arrive at the weak equivalence $J \xrightarrow{\simeq} \{J /\!\!/ f_1^\alpha, \dots, f_r^\alpha\}_\alpha$. So $\{(8)\}_\alpha$ is indeed weakly cartesian.

The assumption that f is an isomorphism outside Z implies that $J[f_i^{-1}] = 0$ for $i = 1, \dots, r$. Note that

$$J[f_i^{-1}] = \operatorname{colim}(J \xrightarrow{f_i} J \xrightarrow{f_i} J \xrightarrow{f_i} \cdots).$$

As the standard t-structure on $\mathrm{Mod}(A)$ is compatible with filtered colimits, we have

$$0 = \tau_{\le n} J[f_i^{-1}] = \operatorname{colim}(\tau_{\le n} J \xrightarrow{f_i} \tau_{\le n} J \xrightarrow{f_i} \tau_{\le n} J \xrightarrow{f_i} \cdots).$$

As $\tau_{\le n} J$ is compact in $\mathrm{Mod}(A)_{[-1,n]}$, we have

$$0 = \pi_0(\mathrm{map}(\tau_{\le n} J, \tau_{\le n} J[f_i^{-1}])) \cong \operatorname{colim} \pi_0(\mathrm{map}(\tau_{\le n} J, \tau_{\le n} J)),$$

which means that there is an N such that the power f_i^N acts nullhomotopically on $\tau_{\le n} J$. It follows that (9) is weakly contractible, and hence the pro-system of squares $\{(8)\}_\alpha$ is indeed weakly cartesian.

Note that for every α the canonical map $B \otimes_A (A /\!\!/ f_1^\alpha, \dots, f_r^\alpha) \to B /\!\!/ f_1^\alpha, \dots, f_r^\alpha$ is an equivalence. Thus we may apply the variant of [Land and Tamme 2019, Theorem 2.32] for l-connective localizing invariants to deduce that $\{(8)\}_\alpha$ induces a weakly cartesian square of E-theory pro-spectra. □

Remark 4.3 In Proposition 4.2 one can actually relax the finiteness assumption if one adds other hypotheses: As the proof shows, we only need that the pro-systems (9) are weakly contractible for each i (using notation of the proof). This is satisfied if, on each truncation $\tau_{\le n} J$, some power of each f_i acts nullhomotopically.

If X and Y are n-truncated, then J is also n-truncated. It is then enough to assume that J is perfect to order n in the sense of [Lurie 2018, Definition 2.7.0.1] in order to conclude that the pro-systems (9) are weakly contractible.

As a special case, if the map f in Proposition 4.2 is a closed immersion of classical qcqs schemes which is classically finitely presented, then the conclusion of the proposition holds, i.e., (7) is weakly cartesian. On the other hand, we cannot drop the finite presentation assumption, as the following example shows:

Example 4.4 Let A be a discrete commutative ring containing an element f and a nontrivial ideal J such that $J[f^{-1}] = 0$ and J is f-divisible. Note that this implies that $J^2 = 0$. For example, take $A = \mathbb{Z} \oplus \mathbb{Q}_p/\mathbb{Z}_p$, $f = p$, and $J = \mathbb{Q}_p/\mathbb{Z}_p$.

The closed immersion $Y = \mathrm{Spec}(A/J) \to X = \mathrm{Spec}(A)$ is then an isomorphism outside $Z = V(f)$. We claim that the square

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X^\wedge_Z) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & K(Y^\wedge_Z) \end{array} \tag{10}$$

is not weakly cartesian.

Aiming at a contradiction, assume (10) is weakly cartesian. Fix some positive integer n. As J is f-divisible,

$$J /\!\!/ f^n = \mathrm{fib}(A /\!\!/ f^n \to (A/J) /\!\!/ f^n)$$

is 1-connective. As a consequence, the map of K-theory spectra

$$K(A /\!\!/ f^n) \to K((A/J) /\!\!/ f^n)$$

is 2-connective. So also the right vertical map, and hence the left vertical map in (10) is 2-connective. In particular, the map $K_1(A) \to K_1(A/J)$ is an isomorphism. However, its retract $A^* \to (A/J)^*$ is not injective, as every $1 + x$ with $x \in J$ is an element of its kernel.

Remark 4.5 There is a version of the above proposition for stacks: Let X be a qcqs ANS derived algebraic stack [Bachmann et al. 2022, A.1], $Y \to X$ a finite, locally almost finitely presented morphism of derived algebraic stacks, and $Z \hookrightarrow X$ a closed immersion with quasicompact open complement. Let E be any connected localizing invariant in the sense of [Bachmann et al. 2022, Definition C.1.3] (e.g., a 2-connective or a finitary 1-connective localizing [Bachmann et al. 2022, Remark C.1.5]). Then the square (7) of pro-spectra is weakly cartesian.

Indeed, the proof of [Bachmann et al. 2022, Theorem 4.2.1] works with the following changes: As in the proof of Lemma 2.3.2 in op. cit., the proof of our Proposition 4.2 shows that the formally completed square

$$\begin{array}{ccc} Y^\wedge_Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^\wedge_Z & \longrightarrow & X \end{array} \tag{11}$$

is weakly cocartesian. As in the proof of Theorem 2.4.1 of op. cit., this implies that the square of derived (pro-)categories induced by (11) is weak pro-Milnor and satisfies weak pro-base change in the sense of Definitions C.2.4 and C.2.6 there. Hence by Theorem C.3.1 there, the square (7) is weakly cartesian.

We now come to our main descent result, which in particular includes Theorem A:

Theorem 4.6 *Let E be a localizing invariant of small k-linear ∞-categories which is l-connective for some integer l. Let X be a qcqs derived k-scheme, $U \subseteq X$ a quasicompact open subscheme, and denote* *by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then the square of pro-spectra*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(Y) & \longrightarrow & E(Y^\wedge_Z) \end{array} \tag{12}$$

is weakly cartesian.

If X and Y are Noetherian classical schemes, and f is classically of finite type, then f is lafp (see Section 2.4). Moreover, the formal derived completions are equivalent to the classical formal completions (see Section 2.5). We thus recover the classical pro-cdh descent statement as formulated (for K-theory) for example in [Kerz et al. 2018, Theorem A].

Proof of Theorem 4.6 We first prove the theorem under the additional assumption that Y carries an f-ample line bundle. Exactly as in the proof of Proposition 4.2 we reduce to the case that X is affine. Then Y carries an ample line bundle.

By Theorem 3.2, we find an lafp U-modification $g\colon Y' \to Y$ such that $f \circ g$ factors as an lafp, closed U-modification $h\colon Y' \hookrightarrow \widetilde{X}$ followed by a derived blowup $p\colon \widetilde{X} \to X$ with center set-theoretically contained in Z. Both of these are isomorphisms over U. By Propositions 4.1 and 4.2, the maps of relative E-theory pro-spectra

$$E(X, X^\wedge_Z) \to E(\widetilde{X}, \widetilde{X}^\wedge_Z) \to E(Y', Y'^\wedge_Z)$$

are weak equivalences. Hence also the composite

$$E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z) \to E(Y', Y'^\wedge_Z) \tag{13}$$

is a weak equivalence. It follows that $E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z)$ is the inclusion of a direct summand (in the weak sense). As Y carries an ample line bundle, we can repeat this argument for the U-modification $Y' \to Y$. So the second map in (13) is also the inclusion of a direct summand. It now follows that both maps are in fact weak equivalences.

We now prove the theorem for general Y. As before, we may assume that X is affine. By Theorem 3.2 again, there exists an lafp U-modification $g\colon Y' \to Y$ such that Y' carries an ample line bundle relative to X. As X is affine, this line bundle is in fact ample and it is also ample relative to Y. Hence, by step 1, the maps of pro-spectra $E(X, X^\wedge_Z) \to E(Y', Y'^\wedge_Z)$ and $E(Y, Y^\wedge_Z) \to E(Y', Y'^\wedge_Z)$ both are weak equivalences. By 3-for-2, $E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z)$ is also a weak equivalence, as desired. □

5 Pro-cdh descent for the cotangent complex and motivic cohomology

We fix some base ring k (e.g., $k = \mathbb{Z}$). For X a qcqs derived k-scheme, we denote by $L_X \in \mathrm{QCoh}(X)$ its (algebraic) cotangent complex relative to k, and by L^i_X its i-th derived exterior power (for $i \geq 0$). If $X = \mathrm{Spec}(A)$ is affine, we also write L^i_A for the A-module $\Gamma(X, L^i_X)$ corresponding to L^i_X. If $Y \to X$

is a morphism of qcqs derived k-schemes, we denote by $L_{Y/X}$ its relative cotangent complex and by $L^i_{Y/X}$ its derived exterior powers. Similarly for a morphism of derived k-algebras $A \to B$.

If $Z \subseteq |X|$ is a closed subset with quasicompact open complement, we obtain a *pro-completion along Z* functor

$$(-)^\wedge_Z \colon \mathrm{QCoh}(X) \to \mathrm{Pro}(\mathrm{QCoh}(X))$$

by pulling back to the ind-derived scheme $X^\wedge_Z$ and pushing forward. The usual formal completion along Z is given by composing the above functor with $\lim\colon \mathrm{Pro}(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X)$.

Lemma 5.1 *Let $X = \mathrm{Spec}(A)$ be an affine derived k-scheme, and $Z \subseteq |X|$ a closed subset with quasicompact open complement. Let $M \in \mathrm{Mod}(A \text{ on } Z)^{\mathrm{aperf}}$ be an almost perfect A-module supported on Z. Then the canonical map*

$$M \to M^\wedge_Z$$

is a weak equivalence in $\mathrm{Pro}(\mathrm{Mod}(A))$.

Proof This was established during the proof of Proposition 4.2 (replace J there by M). □

For a pro-system of derived rings $\{A(\alpha)\}_\alpha$ we denote by $\mathrm{Pro}(\mathrm{Mod})(\{A(\alpha)\})$ the ∞-category of pro-systems of modules over the pro-ring $\{A(\alpha)\}_\alpha$ (see [Land and Tamme 2019, §2.4] for a precise definition).

Lemma 5.2 *In the situation of the previous lemma, choose $f_1, \dots, f_r \in \pi_0(A)$ defining Z, and write $A(\alpha) = A /\!\!/ f_1^\alpha, \dots, f_r^\alpha$. Then the pro-system*

$$\{L_{A(\alpha)/A}\}_\alpha \in \mathrm{Pro}(\mathrm{Mod})(\{A(\alpha)\})$$

vanishes. In fact, all transition maps in this pro-system are nullhomotopic.

Proof This follows by base change from the universal case: Let $R = k[T_1, \dots, T_r]$ be the polynomial ring over k, let $R \to k$ be the map sending all T_i to 0, and let $g_\alpha\colon R \to A$ be the map sending T_i to f_i^α. Then $A(\alpha) = k \otimes_{R, g_\alpha} A$, and consequently

$$L_{A(\alpha)/A} \simeq L_{k/R} \otimes_{R, g_\alpha} A \simeq L_{k/R} \otimes_k A(\alpha).$$

Consider the pro-system $\{L_{k/R}\}_\alpha \in \mathrm{Pro}(\mathrm{Mod}(k))$ whose transition maps are induced by the maps $R \to R$ sending the T_i to T_i^β. Then

$$\{L_{A(\alpha)/A}\}_\alpha \simeq \{L_{k/R}\}_\alpha \otimes_k \{A(\alpha)\}_\alpha.$$

Hence it suffices to prove that all transition maps $L_{k/R} \to L_{k/R}$ are nullhomotopic in $\mathrm{Mod}(k)$. As $R \to k$ has a section, the transitivity triangle yields an equivalence $L_{k/R} \simeq \Sigma L_{R/k} \otimes_R k$. As R is a polynomial ring, $L_{R/k}$ is discrete, given by the module of Kähler differentials $\Omega^1_{R/k} = \bigoplus_i R\,dT_i$. The map $T_i \mapsto T_i^\beta$ induces $dT_i \mapsto \beta T_i^{\beta-1} dT_i$ in $\Omega^1_{R/k}$, and hence the zero map in $\Omega^1_{R/k} \otimes_R k \simeq \Sigma^{-1} L_{k/R}$ for $\beta > 1$. □

Lemma 5.3 *In the situation of Lemma 5.2, the canonical map*

$$\{L^i_A \otimes_A A(\alpha)\}_\alpha \to \{L^i_{A(\alpha)}\}_\alpha$$

is an equivalence in $\mathrm{Pro}(\mathrm{Mod})(\{A(\alpha)\})$.

Proof The transitivity triangle for the maps $A \to A(\alpha)$ and Lemma 5.2 imply the case $i = 1$. Passing to derived exterior powers over $\{A(\alpha)\}$ implies the general case. □

The following theorem was suggested by Matthew Morrow. It generalizes [Morrow 2016b, Theorem 2.4] (see also [Elmanto and Morrow 2023, Lemma 8.5]).

Theorem 5.4 *Let X be a qcqs derived k-scheme, $U \subseteq X$ a quasicompact open subscheme, and denote by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then for every $i \geq 0$, the square of pro-spectra (or pro-complexes)*

$$\begin{array}{ccc} \Gamma(X, L^i_X) & \longrightarrow & \Gamma(X^\wedge_Z, L^i_{X^\wedge_Z}) \\ \downarrow & & \downarrow \\ \Gamma(Y, L^i_Y) & \longrightarrow & \Gamma(Y^\wedge_Z, L^i_{Y^\wedge_Z}) \end{array} \tag{14}$$

is weakly cartesian.

Here, similarly to the previous section, $\Gamma(X^\wedge_Z, L^i_{X^\wedge_Z})$ denotes the pro-object $\{\Gamma(Z', L^i_{Z'})\}_{Z' \hookrightarrow X}$, where $Z' \hookrightarrow X$ runs through all closed immersions of derived schemes with $|Z'| = Z$.

Proof By Zariski descent we may assume that $X = \mathrm{Spec}(A)$ is affine. Choose $f_1, \dots, f_r \in \pi_0(A)$ defining Z, and write $A(\alpha) = A /\!\!/ f_1^\alpha, \dots, f_r^\alpha$ so that $X^\wedge_Z = \{\mathrm{Spec}(A(\alpha))\}_{\alpha \geq 1}$. Abusing notation slightly, we also write $f_* L^i_Y \in \mathrm{Mod}(A)$ for the module $\Gamma(X, f_* L^i_Y) = \Gamma(Y, L^i_Y)$ corresponding to $f_* L^i_Y \in \mathrm{QCoh}(X)$.

By Lemma 5.3 we have

$$\Gamma(X^\wedge_Z, L^i_{X^\wedge_Z}) \simeq L^i_A \otimes_A \{A(\alpha)\}.$$

Similarly, using affine coverings of Y and the usual induction, we get

$$\Gamma(Y^\wedge_Z, L^i_{Y^\wedge_Z}) \simeq (f_* L^i_Y) \otimes_A \{A(\alpha)\}.$$

So we have to prove that

$$\begin{array}{ccc} L^i_A & \longrightarrow & L^i_A \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_* L^i_Y & \longrightarrow & (f_* L^i_Y) \otimes_A \{A(\alpha)\} \end{array} \tag{15}$$

is weakly cartesian. The map $L^i_A \to f_* L^i_Y$ factors as $L^i_A \to f_* f^* L^i_A \to f_* L^i_Y$, so it suffices to show that the two squares

$$\begin{array}{ccc} L^i_A & \longrightarrow & L^i_A \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_* f^* L^i_A & \longrightarrow & (f_* f^* L^i_A) \otimes_A \{A(\alpha)\} \end{array} \quad \text{and} \quad \begin{array}{ccc} f_* f^* L^i_A & \longrightarrow & (f_* f^* L^i_A) \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_* L^i_Y & \longrightarrow & (f_* L^i_Y) \otimes_A \{A(\alpha)\} \end{array} \tag{16}$$

are weakly cartesian.

We first treat the left one. Using the projection formula, we rewrite that square as the tensor product of L^i_A with the square

$$\begin{array}{ccc} A & \longrightarrow & \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_* \mathcal{O}_Y & \longrightarrow & f_* \mathcal{O}_Y \otimes_A \{A(\alpha)\} \end{array} \tag{17}$$

As f is lafp, $f_* \mathcal{O}_Y$ is almost perfect [Lurie 2018, Theorem 5.6.0.2]. On the other hand, by assumption $A \to f_* \mathcal{O}_Y$ is an isomorphism outside Z. Hence the left vertical fiber in (17) lies in $\mathrm{Mod}(A \text{ on } Z)^{\mathrm{aperf}}$. It now follows from Lemma 5.1 that the map on vertical fibers is a weak equivalence, and hence that (17) is weakly cartesian. As L^i_A is connective, tensoring with L^i_A preserves weak equivalences and weakly cartesian squares (see, e.g., [Land and Tamme 2019, Lemma 2.29]). Thus the left-hand square in (16) is weakly cartesian.

We now consider the right square in (16). The transitivity triangle for f gives rise to a finite filtration on L^i_Y whose graded pieces are given by

$$f^* L^k_X \otimes_{\mathcal{O}_Y} L^{i-k}_{Y/X} \quad \text{for } k = 0, \dots, i.$$

It follows that the left vertical cofiber in our square has a finite filtration whose graded pieces are given by

$$f_*(f^* L^k_X \otimes_{\mathcal{O}_Y} L^{i-k}_{Y/X}) \simeq L^k_X \otimes_A f_* L^{i-k}_{Y/X} \quad \text{for } k = 0, \dots, i-1.$$

As f is lafp, the relative cotangent complex $L_{Y/X}$ is almost perfect [Lurie 2004, Proposition 3.2.14] and so are its wedge powers $L^{i-k}_{Y/X}$. Again because f is lafp, the direct images $f_* L^{i-k}_{Y/X}$ are almost perfect [Lurie 2018, Theorem 5.6.0.2]. As f is an isomorphism outside Z, the sheaf $f_* L^{i-k}_{Y/X}$ is supported on Z for $k < i$. So we may again apply Lemma 5.1 to deduce that the map $f_* L^{i-k}_{Y/X} \to (f_* L^{i-k}_{Y/X}) \otimes_A \{A(\alpha)\}_\alpha$ is a weak equivalence for $k = 0, \dots, i-1$, and hence so is the map

$$L^k_X \otimes_A f_* L^{i-k}_{Y/X} \to (L^k_X \otimes_A f_* L^{i-k}_{Y/X}) \otimes_A \{A(\alpha)\}_\alpha.$$

Using the above filtration, it follows that the map on vertical fibers in the right-hand square in (17) is a weak equivalence, and hence that square is also weakly cartesian. □

Combining Theorem 5.4 with the arguments of [Elmanto and Morrow 2023, §8.1], we obtain pro-cdh descent for Elmanto and Morrow's motivic cohomology, denoted by $\mathbb{Z}(j)^{\mathrm{mot}}(-)$.

Corollary 5.5 *Let $\mathbb{F}$ be a prime field. Let X be a qcqs derived $\mathbb{F}$-scheme, $U \subseteq X$ a quasicompact open subscheme, and denote by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then the square of pro-complexes*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(j)^{\mathrm{mot}}(X_Z^{\wedge}) \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{\mathrm{mot}}(Y) & \longrightarrow & \mathbb{Z}(j)^{\mathrm{mot}}(Y_Z^{\wedge}) \end{array}$$

is weakly cartesian.

Proof The proof of Elmanto and Morrow goes through verbatim once we replace their Lemma 8.5 by the above Theorem 5.4. □

6 Generalized Weibel vanishing

In this section we prove our generalized Weibel vanishing result. As its formulation involves the valuative dimension, we start by recalling the latter (Section 6.1). In Section 6.2 we generalize some results about reductions of ideals from [Huneke and Swanson 2006] to non-Noetherian rings. These will go into the proof of Weibel vanishing in Section 6.3.

6.1 Valuative dimension

The valuative dimension of a commutative ring was introduced and studied by Jaffard [1960]; we refer to [Elmanto et al. 2021, §2.3] for an account and to [Kelly and Saito 2024, Lemma 7.2], which reproves some of Jaffard's key results in modern language. For an integral domain A, it is defined as

$$\mathrm{vdim}(A) = \sup\{n \mid \exists A \subseteq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subseteq \mathrm{Frac}(A),\ V_i \text{ is a valuation ring}\},$$

and in general as $\mathrm{vdim}(A) = \sup\{\mathrm{vdim}(A/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}$. For a classical scheme X, one then sets

$$\mathrm{vdim}(X) = \sup\{\mathrm{vdim}(\mathcal{O}_X(U)) \mid U \subseteq X \text{ affine open}\}.$$

We remark that $\mathrm{vdim}(X) = \sup \dim \mathrm{RZ}(X_\lambda)$ is the supremum of the Krull dimensions of the Riemann–Zariski spaces as X_λ ranges over the irreducible components of X. That is, the supremum of the Krull dimensions of all blowups (of finite type) of all X_λ. We always have $\dim(X) \leq \mathrm{vdim}(X)$ and equality holds if X is Noetherian [Jaffard 1960, Chapter IV, Theorem 1, Corollary 2 of Theorem 5]. Note that the valuative dimension only depends on the underlying reduced scheme of a scheme. In particular, it makes sense to talk about the valuative dimension of a closed subset of a scheme, by equipping it with any closed subscheme structure.

Let now X be a derived or spectral scheme. The Krull dimension $\dim(X)$ of X is defined to be the Krull dimension of the underlying topological space of X. The valuative dimension $\mathrm{vdim}(X)$ is defined to be the valuative dimension of the underlying classical scheme ${}^{\mathrm{cl}}X$.

We will need the following result:

Lemma 6.1 *For a scheme X and a point $x \in X$ with closure $\overline{\{x\}}$ in X,*

$$\mathrm{vdim}(\mathcal{O}_{X,x}) + \mathrm{vdim}(\overline{\{x\}}) \leq \mathrm{vdim}(X).$$

In particular,

$$\mathrm{vdim}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) \leq \mathrm{vdim}(X).$$

Proof By definition it suffices to prove the lemma when $X = \mathrm{Spec}(A)$ with A integral. In this case, the first assertion is [Jaffard 1960, Chapter IV, Proposition 2]; indeed, this follows from the basic facts that for any field extension L/K any valuation $R \subseteq K$ admits at least one extension $S \subseteq L$, and $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is always surjective. The second assertion follows from the fact that $\dim(Y) \leq \mathrm{vdim}(Y)$ for every scheme Y. □

6.2 Reductions

Let R be a ring, and let $J \subseteq I$ be ideals in R. Then J is called a reduction of I if there is a positive integer n such that $JI^n = I^{n+1}$. Associated with an ideal I we have its Rees algebra

$$R[It] = \sum_{n\geq 0} I^n t^n \subseteq R[t], \quad R[It] \cong \bigoplus_{n\geq 0} I^n$$

and its extended Rees algebra

$$R[It, t^{-1}] = \sum_{n<0} Rt^n + \sum_{n\geq 0} I^n t^n \subseteq R[t, t^{-1}].$$

So $\mathrm{Proj}(R[It])$ is the blowup of $\mathrm{Spec}(R)$ in $\mathrm{Spec}(R/I)$, and $\mathrm{Spec}(R[It, t^{-1}])$ is a flat deformation of $\mathrm{Spec}(R)$ to the normal cone $\mathrm{Spec}\big(\bigoplus_{n\geq 0} I^n/I^{n+1}\big)$.

Lemma 6.2 *Assume that $J \subseteq I$ are ideals in R.*

(1) *If I is finitely generated*, *and J is a reduction of I*, *then the ring extension $R[Jt] \subseteq R[It]$ is finite.*

(2) *Conversely*, *if the ring extension $R[Jt] \subseteq R[It]$ is finite*, *then J is a reduction of I.*

Proof This is [Huneke and Swanson 2006, Theorem 8.2.1] which is formulated for ideals in Noetherian rings. The proof given there works verbatim in the asserted generality. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let $I \subset R$ be a finitely generated proper ideal. Then the fiber cone

$$\mathcal{F}_I(R) = R[It] \otimes_R^{\heartsuit} \kappa = \kappa \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots$$

is a finitely generated κ-algebra, and in particular Noetherian. Its Krull dimension $\ell(I) := \dim \mathcal{F}_I(R)$ is called the analytic spread of I.

Proposition 6.3 *Let R be a local ring*, *and $I \subseteq R$ a finitely generated proper ideal. Then the analytic spread of I is bounded by the valuative dimension of R*:

$$\ell(I) \leq \mathrm{vdim}(R).$$

Proof This proof follows [Huneke and Swanson 2006, Proposition 5.1.6], which treats the Noetherian case. We may assume that $\mathrm{vdim}(R)$ is finite. Let $\mathrm{gr}_I(R) = R[It] \otimes_R^{\heartsuit} R/I = R/I \oplus I/I^2 \oplus \cdots = R[It, t^{-1}]/(t^{-1})$. As $\mathcal{F}_I(R)$ is a quotient of $\mathrm{gr}_I(R)$ and $\mathcal{F}_I(R)$ is Noetherian, we have

$$\ell(I) = \mathrm{vdim}\, \mathcal{F}_I(R) \le \mathrm{vdim}\, \mathrm{gr}_I(R).$$

As t^{-1} is a nonzero divisor in $R[It, t^{-1}]$, it is not contained in any minimal prime ideal of $R[It, t^{-1}]$. Thus $\mathrm{vdim}\, \mathrm{gr}_I(R) \le \mathrm{vdim}\, R[It, t^{-1}] - 1$ by [Elmanto et al. 2021, Proposition 2.3.2(4)]. We claim that $\mathrm{vdim}\, R[It, t^{-1}] = \mathrm{vdim}\, R + 1$. This will finish the proof. To prove the claim, as the minimal prime ideals of $R[It, t^{-1}]$ are in bijection with those of R (see the general discussion in [Huneke and Swanson 2006] before Theorem 5.1.4), we may assume that R is a domain. As $\mathrm{Frac}(R[It, t^{-1}]) = \mathrm{Frac}(R)(t)$ has transcendence degree 1 over $\mathrm{Frac}(R)$, [Jaffard 1960, Chapter I, Theorem 2, p. 10] implies that if v' is any valuation of $\mathrm{Frac}(R[It, t^{-1}])$ extending a valuation v of $\mathrm{Frac}(R)$, then the rank $\mathrm{rank}(v')$ of v' is at most $\mathrm{rank}(v) + 1$. It follows that $\mathrm{vdim}\, R[It, t^{-1}] \le \mathrm{vdim}\, R + 1$. The kernel of the surjection $R[It, t^{-1}] \to R$, $t \mapsto 1$ is a nontrivial prime ideal, and hence $\mathrm{vdim}\, R[It, t^{-1}] > \mathrm{vdim}\, R$, for example by Lemma 6.1. □

Proposition 6.4 *Let R be a local ring and $I \subseteq R$ be a finitely generated proper ideal with analytic spread $\ell(I)$. Then there exists an $n \ge 1$ and a reduction J of I^n which is generated by $\ell(I)$ elements.*

Proof We follow the proof of [Huneke and Swanson 2006, Proposition 8.3.8]. Let $\mathfrak{m}$ denote the maximal ideal of R, and $\kappa = R/\mathfrak{m}$. By graded Noether normalization (see, e.g., [Huneke and Swanson 2006, Theorem 4.2.3]) there exists an integer n and homogeneous elements $\bar{a}_1, \dots, \bar{a}_l \in \mathcal{F}_I(R)_n = I^n/\mathfrak{m}I^n$ such that $A = \kappa[\bar{a}_1, \dots, \bar{a}_l] \subseteq \mathcal{F}_I(R)$ is a polynomial ring and the extension $A \subseteq \mathcal{F}_I(R)$ is finite. In particular, $l = \dim \mathcal{F}_I(R) = \ell(I)$. Moreover, the subextension $A \subseteq \bigoplus_{k \ge 0} I^{nk}/\mathfrak{m}I^{nk}$ is also finite (up to grading, the latter algebra is $\mathcal{F}_{I^n}(R)$). Let nd be the maximal degree of a finite set of homogeneous generators. Then $I^{n(d+1)}/\mathfrak{m}I^{n(d+1)} = \sum_{i=1}^{l} \bar{a}_i \cdot I^{nd}/\mathfrak{m}I^{nd}$. Choose lifts $a_i \in I^n$ of the $\bar{a}_i$ and let $J = (a_1, \dots, a_l) \subseteq I^n$. The previous equality implies $JI^{nd} + \mathfrak{m}I^{n(d+1)} = I^{n(d+1)}$. As I is finitely generated, the Nakayama lemma gives $JI^{nd} = I^{n(d+1)}$, i.e., J is a reduction of I^n. □

6.3 Weibel vanishing

The following theorem generalizes [Kerz et al. 2018, Theorem B], which treats the case of Noetherian classical schemes.

Theorem 6.5 *Let X be a qcqs spectral scheme. Then the following hold*:

(1) $K_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.

(2) *For all* $i \ge \mathrm{vdim}(X)$ *and any integer* $r \ge 0$, *the pullback map* $K_{-i}(X) \to K_{-i}(\mathbb{A}^r_X)$ *is an isomorphism.*

In the spectral setting, the affine space in (2) is either the flat one or the smooth one; the assertion holds for both. In fact, one reduces immediately to underlying classical schemes (see the beginning of the proof in Section 6.5), and in both cases this is just the classical affine space.

The proof of Theorem 6.5 is along the lines of Kerz's proof in the Noetherian case [2018], which is a bit more direct than the one in [Kerz et al. 2018, Theorem B]. It uses the theory of reductions presented in Section 6.2 in order to prove better descent results for blowups. The following lemma replaces [Kerz and Strunk 2017, Lemma 4] and is essentially due to Scheiderer [1992]:

Lemma 6.6 *Let X be a spectral space, and $\mathcal{F}$ a sheaf of abelian groups on X. Let $r \geq 0$ be an integer. Assume that $\mathcal{F}_y = 0$ for all points $y \in X$ with $\dim(\overline{\{y\}}) > r$. Then $H^n(X, \mathcal{F}) = 0$ for all integers $n > r$.*

Proof This is a direct consequence of results of Scheiderer [1992]; see the proof of Proposition 4.7 there. Let $\mathrm{sp}_\bullet(X) \to X$ be the quasiaugmented simplicial topological space defined in [Scheiderer 1992, §2]. Its set of n-simplices is given by chains of specializations

$$x_0 \succ \cdots \succ x_n$$

of points in X with coincidences between the x_i allowed. The topology is induced by the constructible topology on X^{n+1}, and the quasiaugmentation is given by the canonical map $\mathrm{sp}_0(X) \to X$. Let $\gamma_n : \mathrm{sp}_n(X) \to X$ be the map sending a chain as above to x_0. By [Scheiderer 1992, Remark 2.5 and Theorem 4.1], the cohomology groups $H^*(X, \mathcal{F})$ are computed by the complex

$$\Gamma(\mathrm{sp}_0(X), \gamma_0^*\mathcal{F}) \to \Gamma(\mathrm{sp}_1(X), \gamma_1^*\mathcal{F}) \to \Gamma(\mathrm{sp}_2(X), \gamma_2^*\mathcal{F}) \to \cdots,$$

which arises from the cosimplicial abelian group $[n] \to \Gamma(\mathrm{sp}_n(X), \gamma_n^*\mathcal{F})$. Let

$$B_0 \to B_1 \to B_2 \to \cdots$$

be the associated normalized subcomplex, which also computes $H^*(X, \mathcal{F})$. As in the proof of [Scheiderer 1992, Proposition 4.7], B_n coincides with the group of those sections $a \in \Gamma(\mathrm{sp}_n(X), \gamma_n^*\mathcal{F})$ whose supports consist only of nondegenerate simplices $x \in \mathrm{sp}_n(X)$. Hence it suffices to show $(\gamma_n^*\mathcal{F})_x = 0$ for such x if $n > r$. So let $x \in \mathrm{sp}_n(X)$ be given by a chain of specializations $x_0 \succ \cdots \succ x_n$ with pairwise different x_i. Then $\dim(\overline{\{x_0\}}) \geq n > r$. Thus the assumption implies $(\gamma_n^*\mathcal{F})_x = \mathcal{F}_{x_0} = 0$, as wanted. □

Lemma 6.7 *Let X be a qcqs spectral scheme of finite valuative dimension. Let F be a sheaf of spectra on X for the Zariski topology. Assume that for every $x \in X$, the homotopy groups of the stalks $\pi_{-i}(F_x)$ vanish for $i > \mathrm{vdim}(\mathcal{O}_{X,x})$. Then $\pi_{-i}(F(X)) = 0$ for all $i > \mathrm{vdim}(X)$.*

Note that if F is K-theory, then $\pi_{-i}(K_x) \cong K_{-i}(\mathcal{O}_{X,x})$ as K-theory commutes with filtered colimits.

Proof This is deduced from Lemma 6.6 by the same argument as the proof of [Kerz and Strunk 2017, Proposition 3]. From [Clausen and Mathew 2021, Theorem 3.12] we know that the homotopy dimension of X_{Zar} is bounded by the Krull dimension $\dim(X)$, and in particular the ∞-topos of sheaves of spaces on X_{Zar} is hypercomplete. Hence there is a convergent Zariski descent spectral sequence

$$E_2^{p,q} = H^p(X, \widetilde{F}_{-q}) \Rightarrow \pi_{-p-q}(F(X)),$$

where $\widetilde{F}_{-q}$ is the Zariski sheaf of abelian groups on X associated to the presheaf $U \mapsto \pi_{-q}(F(U))$, and $E_2^{p,*} = 0$ unless $0 \leq p \leq \dim(X)$. It suffices to show $E_2^{p,q} = 0$ for $p + q > \mathrm{vdim}(X)$. We may assume

$p \leq \dim(X)$, so that $q > 0$. Note that $\widetilde{F}_{-q,x} = \pi_{-q}(F_x)$. By Lemma 6.6 it now suffices to check that $\pi_{-q}(F_x) = 0$ for all points x with $\dim(\overline{\{x\}}) > \mathrm{vdim}(X) - q$. But for such points x we have

$$\mathrm{vdim}(\mathcal{O}_{X,x}) \leq \mathrm{vdim}(X) - \dim(\overline{\{x\}}) < q$$

by Lemma 6.1, and hence $\pi_{-q}(F_x) = 0$ by assumption. □

6.4 *K*-theoretic preparations

In the proof of Theorem 6.5, we use the following well-known facts about nonpositive K-theory on affine (derived) schemes:

Lemma 6.8 (1) *Let A be a derived ring. Then the canonical map $K(A) \to K(\pi_0(A))$ is* 2*-connective*, *i.e.*, *it induces an isomorphism on π_i for $i \leq 1$ and a surjection on π_2.*

(2) *Let A be a discrete commutative ring*, *and let $I \subseteq A$ be a locally nilpotent ideal* (*i.e.*, *every element of I is nilpotent*)*. Then the map $K(A) \to K(A/I)$ is* 1*-connective.*

(3) *Let A be a discrete commutative ring*, *$X = \mathrm{Spec}(A)$*, *and $X = X_1 \cup X_2$ be a closed covering of X. Write $X_{12} = X_1 \cap X_2$. Then there is a long exact sequence*

$$K_1(X) \to K_1(X_1) \oplus K_1(X_2) \to K_1(X_{12}) \to K_0(X) \to \cdots.$$

Proof (1) For connective K-theory, this is due to Waldhausen [1978, Proposition 1.1]. The general case follows from this together with [Blumberg et al. 2013, Theorem 9.53] (or [Kerz et al. 2018, Theorem 2.16]). Alternatively, see [Land and Tamme 2019, Lemma 2.4] for a slightly more general statement.

(2) Writing I as a filtered colimit of nilpotent ideals, we may assume that I itself is nilpotent. Then $K_0(I) = 0$ by [Weibel 2013, Exercise II.2.5]; hence $K_1(A) \to K_1(A/I)$ is surjective by Proposition III.2.3 there. Moreover, $K_0(A) \to K_0(A/I)$ is an isomorphism by [Weibel 2013, Lemma II.2.2]. As the ideal generated by I in any A-algebra is still nilpotent, it then follows from the definition of negative K-groups [Weibel 2013, Definition III.4.1] that $K_i(A) \to K_i(A/I)$ is an isomorphism for all $i \leq 0$.

(3) Let I and J be the ideals defining X_1 and X_2, respectively. Then $X_{12} = \mathrm{Spec}(A/I + J)$. Let $A' = A/I \cap J$. Then A' sits in a Milnor square

$$\begin{array}{ccc} A' & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A/J & \longrightarrow & A/I + J \end{array}$$

Hence, by [Weibel 2013, Theorem III.4.3], there is a long exact sequence

$$K_1(A') \to K_1(A/I) \oplus K_1(A/J) \to K_1(A/I + J) \to K_0(A') \to \cdots.$$

As $X_1 \cup X_2 = X$, the ideal $I \cap J$ is contained in the nilradical of A and so is locally nilpotent. Using (2), we may thus replace A' by A in the above sequence to get a long exact sequence of the statement. □

The previous lemma has as a simple consequence the following lemma, which will allow us to get some control on the negative K-theory of certain blowups in Proposition 6.12.

Lemma 6.9 *Let X be a separated derived scheme admitting a covering by l affine open subschemes. Then the canonical morphism*

$$K(X) \to K({}^{\mathrm{cl}}X'),$$

where ${}^{\mathrm{cl}}X'$ denotes ${}^{\mathrm{cl}}X$ with any closed subscheme structure between ${}^{\mathrm{cl}}X^{\mathrm{red}}$ and ${}^{\mathrm{cl}}X$, is $(-l+2)$-connective.

Proof We proceed by induction on l. If $l = 1$, this is Lemma 6.8(1)–(2). For $l > 1$ write $X = V \cup U$ with U affine and V admitting an open covering by $l-1$ affines. As X is separated, $V \cap U$ also has an open covering by $l-1$ affines. The claim follows from the inductive hypothesis by considering the map from the cartesian square

$$\begin{array}{ccc} K(X) & \longrightarrow & K(V) \\ \downarrow & & \downarrow \\ K(U) & \longrightarrow & K(V\cap U) \end{array}$$

to the similar cartesian square with the underlying classical schemes with the subscheme structure induced from X'. □

In the following, if X is a qcqs derived scheme and $Z \hookrightarrow X$ is a derived subscheme, we denote by $K(X, Z)$ the relative K-theory $\mathrm{fib}(K(X) \to K(Z))$.

Lemma 6.10 *Let X and Y be quasicompact, separated classical schemes, and $Z \hookrightarrow X$ a classically finitely presented closed subscheme. Let $f\colon Y \to X$ be a finite morphism which is an isomorphism outside Z. Assume that X has a covering by l affine open subschemes. Then the canonical map*

$$K(X,Z) \to K(Y, f^{-1}(Z))$$

is $(-l+1)$-connective.

Proof By induction on l, as in the proof of Lemma 6.9, it suffices to treat the case $l = 1$, i.e., when X, and hence Y, are affine. As K-theory commutes with filtered colimits, by Lemma 6.11 below we may assume that $f\colon Y \to X$ is finitely presented. By Remark 4.3 the square of pro-spectra

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X^\wedge_Z) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & K(Y^\wedge_Z) \end{array}$$

is then weakly cartesian. Hence the square of spectra obtained by applying $\lim\colon \mathrm{Pro(Sp)} \to \mathrm{Sp}$,

$$\begin{array}{ccc} K(X) & \longrightarrow & \lim K(X^\wedge_Z) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & \lim K(Y^\wedge_Z) \end{array}$$

is cartesian. By Lemma 6.8(1)–(2), each transition map in the pro-spectrum $K(X^\wedge_Z)$ is 1-connective. In particular, using the Milnor sequence we get isomorphisms $\pi_0(\lim K(X^\wedge_Z)) \cong \lim K_0(X^\wedge_Z) \cong K_0(Z)$ and surjections $\pi_1(\lim K(X^\wedge_Z)) \to \lim K_1(X^\wedge_Z) \to K_1(Z)$. In other words, $\lim K(X^\wedge_Z) \to K(Z)$ is 1-connective. Consider the following diagram, in which the left vertical map is an equivalence by the previous cartesian square:

$$\begin{array}{ccc} \mathrm{fib}(K(X) \to \lim K(X^\wedge_Z)) & \longrightarrow & K(X, Z) \\ \big\downarrow{\simeq} & & \big\downarrow \\ \mathrm{fib}(K(Y) \to \lim K(Y^\wedge_Z)) & \longrightarrow & K(Y, f^{-1}(Z)) \end{array}$$

The fiber of the top horizontal map identifies with $\Omega\,\mathrm{fib}(\lim K(X^\wedge_Z) \to K(Z))$ and is hence 0-connective. By the same arguments, the lower horizontal map is 0-connective. It follows that the right vertical map is also 0-connective. □

Lemma 6.11 *Let $\varphi\colon A \to B$ be a finite morphism of rings such that $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism outside $V(I)$, where $I = (f_1, \dots, f_r) \subseteq A$. Then we can write φ as a filtered colimit of finitely presented, finite maps $\varphi_\lambda\colon A \to B_\lambda$ such that each $\mathrm{Spec}(\varphi_\lambda)$ is an isomorphism outside $V(I)$.*

Proof Choose A-module generators $b_1, \dots, b_s$ of B. We get a surjection $A[T_1, \dots, T_s] \to B$, $T_i \mapsto b_i$. Let J denote its kernel. As b_i is integral over A, there exists a monic polynomial $p_i \in A[X]$ with $p_i(b_i) = 0$. Clearly $p_i(T_i) \in J$.

Let f be one of the generators f_j of I. As $\mathrm{Spec}(\varphi)$ is an isomorphism outside $V(I)$, we have $A[f^{-1}] \xrightarrow{\simeq} B[f^{-1}]$. In particular, $B[f^{-1}]$ is finitely presented as an $A[f^{-1}]$-algebra, and it follows that $J[f^{-1}] \subseteq A[f^{-1}][T_1, \dots, T_s]$ is a finitely generated ideal. We may pick finitely many generators q_{jk} which already live in J.

Now write J as an increasing union of finitely generated ideals J_λ, where each J_λ contains all the $p_i(T_i)$ and all the q_{jk}. By construction, $B_\lambda = A[T_1, \dots, T_s]/J_\lambda$ is then finitely presented over A, is finite (as in B_λ each T_i is integral over A), and $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is an isomorphism outside $V(I)$ (as $J_\lambda[f_j^{-1}] = J[f_j^{-1}]$ for all j). Clearly $\mathrm{colim}_\lambda B_\lambda = B$. □

Proposition 6.12 *Let R be a local ring, and $I \subseteq R$ be a finitely generated proper ideal of analytic spread $l = \ell(I)$. Let $Z = V(I)$, $f\colon \mathrm{Bl}_Z(X) \to X$ be the blowup, and $E = f^{-1}(Z)$ be the classical exceptional divisor. Then the pullback map*

$$K(X, Z) \to K(\mathrm{Bl}_Z(X), E)$$

is $(-l+1)$-connective. More generally, for any $r \geq 0$, the pullback map

$$K(\mathbb{A}^r \times X, \mathbb{A}^r \times Z) \to K(\mathbb{A}^r \times \mathrm{Bl}_Z(X), \mathbb{A}^r \times E)$$

is $(-l+1)$-connective.

Proof By Proposition 6.4, there exists a reduction J of some power I^n of I generated by l elements, say $J = (f_1, \dots, f_l)$. As $\mathrm{Bl}_{I^n}(R) = \mathrm{Bl}_I(R)$ canonically, Lemma 6.2 yields a finite X-morphism

$\mathrm{Bl}_Z(X) = \mathrm{Bl}_I(R) \to \mathrm{Bl}_J(R)$. Let $\widetilde{X}$ be the derived blowup of X in $f_1, \dots, f_l$. Then we have a closed immersion $\mathrm{Bl}_J(R) \hookrightarrow \widetilde{X}$ and in total a finite morphism $\mathrm{Bl}_I(X) \to \widetilde{X}$. Moreover, $\widetilde{X}$ has a covering by l affine open subschemes. Write $D = {}^{\mathrm{cl}}(\widetilde{X} \times_X Z)$. We now argue similarly as in the proof of Lemma 6.10. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{fib}(K(X) \to \lim K(X^{\wedge}_Z)) & \longrightarrow & K(X, Z) \\ \downarrow{\simeq} & & \downarrow \\ \mathrm{fib}(K(\widetilde{X}) \to \lim K(\widetilde{X}^{\wedge}_Z)) & \longrightarrow & K(\widetilde{X}, D) \end{array}$$

The left vertical map is an equivalence by Proposition 4.1 (note that $|Z| = |V(J)|$). The top horizontal map is 0-connective, and the lower horizontal map is $(-l+1)$-connective by Lemma 6.9. Hence the right vertical map is $(-l+1)$-connective. Using Lemma 6.9 again, $K(\widetilde{X}, D) \to K({}^{\mathrm{cl}}\widetilde{X}, D)$ is $(-l+2)$-connective. In total, $K(X, Z) \to K({}^{\mathrm{cl}}\widetilde{X}, D)$ is $(-l+1)$-connective.[5] By Lemma 6.10, the map $K({}^{\mathrm{cl}}\widetilde{X}, D) \to K(\mathrm{Bl}_Z(X), E)$ is $(-l+1)$-connective. Hence the composite is $(-l+1)$-connective, as asserted.

The more general claim can be proved by exactly the same argument, using that $\mathbb{A}^r$ is affine and flat over $\mathbb{Z}$ and derived blowups commute with arbitrary base change. □

6.5 Proof of Weibel vanishing

Proof of Theorem 6.5 We write $N^{(r)}K(X)$ for the cofiber of the canonical split inclusion $K(X) \to K(\mathbb{A}^r_X)$, so that we have $K(\mathbb{A}^r_X) \cong K(X) \oplus N^{(r)}K(X)$. Then assertion (2) of the theorem is equivalent to the statement that $N^{(r)}K_{-i}(X) = 0$ for all $i \geq \mathrm{vdim}(X)$.

We may assume that $d = \mathrm{vdim}(X)$ is finite and prove the theorem by induction on d. By Lemma 6.7, applied to K and $\Sigma N^{(r)}K$, respectively, we may assume that X is affine and local. In this case, $K_{-i}(\mathbb{A}^r_X) \cong K_{-i}(\mathbb{A}^r_{{}^{\mathrm{cl}}X^{\mathrm{red}}})$ for all $i \geq 0$ and all $r \geq 0$ by Lemma 6.8(1)–(2). So we can assume that X is a classical reduced affine local scheme.

If $d = 0$, then the Krull dimension $\dim(X)$ is also 0. As X is local and reduced, it is the spectrum of a field. Hence $K_{-i}(X) = 0$ for all $i > 0$, and $N^{(r)}K(X) = 0$ (by [Weibel 2013, Theorem II.7.8] and the definition of negative K-groups), as X is regular Noetherian.

Now assume that d is positive and the assertion is proven for all qcqs derived schemes that are of valuative dimension $< d$.

Claim *Theorem 6.5 holds for X as above, under the additional assumption that X is irreducible.*

Proof of the claim Write $X = \mathrm{Spec}(R)$. So R is a local integral domain. Let γ be an element in $K_{-i}(R)$ with $i > d$ or in $N^{(r)}K_{-i}(R) \subseteq K_{-i}(\mathbb{A}^r_R)$ with $i \geq d$ if $r > 0$. We have to show that $\gamma = 0$.

As K-theory commutes with filtered colimits, there exists a subring R_0 of R, finitely generated as a $\mathbb{Z}$-algebra, and $\gamma_0 \in K_{-i}(\mathbb{A}^r_{R_0})$ pulling back to γ. By [Kerz and Strunk 2017, Proposition 5], there

[5] In fact, using the above connectivity estimates, this follows more easily from the derived generalization of Thomason's theorem [1993] in [Kerz et al. 2018, Theorem 3.7], which holds without any Noetherianity assumptions (with the same proof as in loc. cit.)

exists a (finitely generated) ideal $I_0 \subseteq R_0$ such that γ_0 is annihilated in $K_{-i}(\mathbb{A}^r \times \mathrm{Bl}_{I_0}(R_0))$. Let $I = RI_0 \subseteq R$, and $Z = V(I)$. By functoriality of the blowup, γ is annihilated in $K_{-i}(\mathbb{A}^r \times \mathrm{Bl}_Z(X))$. Let $E \subseteq \mathrm{Bl}_Z(X)$ denote the exceptional divisor. By construction, $Z \subseteq X$ is a finitely presented, proper subscheme; hence $\mathrm{Bl}_Z(X) \to X$ is a modification and thus $\operatorname{vdim} \mathrm{Bl}_Z(X) = \operatorname{vdim} X = d$ [Elmanto et al. 2021, Proposition 2.3.2(6)]. Furthermore $\operatorname{vdim}(Z), \operatorname{vdim}(E) < d$ by loc. cit., Assertion 4.

We now treat the case of $\gamma \in K_{-i}(R)$ with $i > d$. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} K_{-i+1}(Z) & \longrightarrow & K_{-i}(X, Z) & \longrightarrow & K_{-i}(X) & \longrightarrow & K_{-i}(Z) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{-i+1}(E) & \longrightarrow & K_{-i}(\mathrm{Bl}_Z(X), E) & \longrightarrow & K_{-i}(\mathrm{Bl}_Z(X)) & \longrightarrow & K_{-i}(E) \end{array}$$

As $\operatorname{vdim}(Z), \operatorname{vdim}(E) < d$, the four outer groups vanish by induction. If $I = R$, then $\mathrm{Bl}_Z(X) = X$ and hence $\gamma = 0$. So we may now assume that $I \subset R$ is a proper ideal. By Proposition 6.3 we then have $\ell(I) \leq \operatorname{vdim}(R) = d$. As $i > d$, Proposition 6.12 implies that the second vertical map is an isomorphism. Hence $K_{-i}(X) \to K_{-i}(\mathrm{Bl}_Z(X))$ is an isomorphism too, and hence $\gamma = 0$.

The argument in the case $\gamma \in N^{(r)}K_{-i}(R)$ with $i \geq d$ is exactly the same with the K-groups above replaced by the $N^{(r)}K$-groups. □

We now consider the general case. So X is now a classical, reduced, affine, local scheme of valuative dimension $d > 0$. For ease of notation, we only consider assertion (1) of the theorem. The proof of (2) is completely parallel. To reduce to the integral case treated in the claim above, we apply an argument from [Elmanto et al. 2021, Theorem 2.4.15] as follows: Take an integer $i > d = \operatorname{vdim}(X)$ and an element $\gamma \in K_{-i}(X)$. We wish to show that $\gamma = 0$. Let

$$\mathcal{E} = \{Z \hookrightarrow X \text{ reduced, closed} \mid \gamma|_Z \neq 0 \in K_{-i}(Z)\}.$$

We need to prove that $\mathcal{E} = \varnothing$. Note that every $Z \in \mathcal{E}$ is itself a reduced local affine scheme and that $\mathcal{E}$ is ordered by inclusion. Let $(Z_\lambda)_{\lambda \in \Lambda}$ be a descending chain in $\mathcal{E}$ and put $Z = \lim_{\lambda \in \Lambda} Z_\lambda$. As K-theory commutes with filtered colimits of rings, $K_{-i}(Z) = \operatorname{colim}_{\lambda \in \Lambda} K_{-i}(Z_\lambda)$. So if $\gamma|_Z = 0$, then there exists a λ such that $\gamma|_{Z_\lambda} = 0$, which is a contradiction. Hence $\gamma|_Z \neq 0$, so that $Z \in \mathcal{E}$ and Z is a lower bound of $(Z_\lambda)_{\lambda \in \Lambda}$. If $\mathcal{E} \neq \varnothing$, we may apply Zorn's lemma to conclude that $\mathcal{E}$ has a minimal element Z. As $K_{-i}(Z) \neq 0$, Z must be reducible. Let Z^{gen} be the set of the generic points of Z equipped with the induced topology from the underlying topological space of Z. By [Henriksen and Jerison 1965, Corollary 2.4] (see also [Elmanto et al. 2021, Lemma 2.4.14]), there exists a decomposition $Z^{\mathrm{gen}} = S_1 \sqcup S_2$ with S_i closed and nonempty for $i = 1, 2$. Letting Z_i be the closure of S_i in Z with reduced scheme structure, we have $Z = Z_1 \cup Z_2$ and $Z_i \cap Z^{\mathrm{gen}} = S_i$ for $i = 1, 2$. In particular, $Z_1 \cap Z_2 \cap Z^{\mathrm{gen}} = \varnothing$, so that $\operatorname{vdim}(Z_1 \cap Z_2) < d$ by [Elmanto et al. 2021, Proposition 2.3.2(4)]. By excision in nonpositive K-theory for closed coverings of affine schemes (Lemma 6.8(3)), we have an exact sequence

$$K_{-i+1}(Z_1 \cap Z_2) \to K_{-i}(Z) \to K_{-i}(Z_1) \oplus K_{-i}(Z_2).$$

The group on the left-hand side vanishes by induction. As Z was minimal in $\mathcal{E}$, we must have $\gamma|_{Z_i} = 0$ for $i = 1, 2$. Thus we get $\gamma|_Z = 0$, which is a contradiction. So $\mathcal{E} = \varnothing$, which completes the proof of Theorem 6.5. □

We now give a proof of Theorem 1.4, reproduced from [Kelly and Morrow 2021, Remark 3.5]. We refer to [Elmanto et al. 2021, §2.1] for a discussion of the cdh topology in this generality but note that the definition of the cdh topology used in [Elmanto et al. 2021] is the one used by Suslin and Voevodsky [2000, Definition 5.7], and the proof that cdh Čech descent is equivalent to cdh excision is Voevodsky's proof [2010], rewritten in modern language in [Asok et al. 2017, Theorem 3.2.5]. Voevodsky's proof that cdh Čech descent is equivalent to cdh hyperdescent requires Noetherian hypotheses that were lifted in [Elmanto et al. 2021].

Proof of Theorem 1.4 We note that homotopy K-theory is a cdh sheaf. This was first proven by Cisinski [2013] for Noetherian schemes of finite dimension (and in characteristic 0 previously by Haesemeyer [2004]), which implies the general statement by absolute Noetherian approximation. Alternatively it follows from the fact that KH is truncating; see [Land and Tamme 2019, Corollary A.5]. In particular, we get the maps

$$L_{\mathrm{cdh}} K_{\geq 0} \to L_{\mathrm{cdh}} K \to KH, \tag{18}$$

which we want to show are equivalences. By [Elmanto et al. 2021, Theorem 2.4.15 and Corollary 2.3.3] the ∞-topos of cdh sheaves of spaces on finitely presented X-schemes is locally of finite homotopy dimension and of homotopy dimension $\leq \mathrm{vdim}(X)$. This implies that we get a convergent spectral sequence

$$E_2^{p,q} = H^p(X_{\mathrm{cdh}}, \widetilde{X}_{-q} KH) \Rightarrow KH_{-p-q}(X), \tag{19}$$

where $\widetilde{X}_q KH$ denotes the cdh sheafified homotopy groups of KH. A conservative family of points for the cdh topology is given by the spectra of henselian valuation rings [Goodwillie and Lichtenbaum 2001; Gabber and Kelly 2015, Theorems 2.3 and 2.6; Elmanto et al. 2021, Corollary 2.4.19]. As K-theory of valuation rings is connective and agrees with its homotopy K-theory [Kelly and Morrow 2021, Theorem 3.4; Kerz et al. 2021, Lemma 4.3] and K and KH commute with filtered colimits, the maps (18) are equivalences so we get Theorem 1.4(2). The vanishing

$$KH_{-i}(X) = 0 \quad \text{for all } i > \mathrm{vdim}(X)$$

claimed in part (1) follows from the spectral sequence (19). □

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