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A Weiss–Williams theorem for spaces of embeddings and the homotopy type of spaces of long knots

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We establish a pseudoisotopy result for embedding spaces in the line of that of Weiss and Williams for diffeomorphism groups. In other words, for $P \subset M$ an embedding of codimension at least 3, we describe the difference in a range of homotopical degrees between the spaces of block and ordinary embeddings of P into M as a certain infinite loop space involving the relative algebraic K-theory of the pair $(M, M-P)$. This range of degrees is the so-called concordance embedding stable range, which, by recent developments of Goodwillie, Krannich, and Kupers, is far beyond that of the aforementioned theorem of Weiss and Williams.

We use this result to obtain split fibre sequences in the concordance embedding stable range, with explicit analysable base and fibre, which determine the homotopy type of spaces of long knots of codimension at least 3. This leads to explicit computations of homotopy groups, including torsion information, in that range. In doing so, we carry out an extensive analysis of certain geometric involutions in algebraic K-theory that may be of independent interest.

1 Introduction

The classical approach to study the homotopy type of the diffeomorphism group $\mathrm{Diff}_\partial(M)$ of a compact, possibly with boundary, high-dimensional manifold M^d (i.e., $d \geq 5$) is based on the so-called *surgery-pseudoisotopy* program, which focuses on the homotopy fibre sequence

$$(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \to B\mathrm{Diff}_\partial(M) \xrightarrow{i} B\widetilde{\mathrm{Diff}}_\partial(M). \tag{1-1}$$

The right-hand term is the classifying space of the simplicial group $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$ of *block diffeomorphisms* of M (see Definition 2.8), an approximation to the ordinary diffeomorphism group of M that closely

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resembles the behaviour of the topological monoid $h\mathrm{Aut}_\partial(M)$ of homotopy automorphisms of M; for instance, one of the defining properties of $\widetilde{\mathrm{Diff}}_\partial(-)$ is that it satisfies a natural equivalence $\widetilde{\mathrm{Diff}}_\partial(M\times I)\simeq\Omega\widetilde{\mathrm{Diff}}_\partial(M)$, which also holds for $h\mathrm{Aut}_\partial(-)$ — but is very much not true for $\mathrm{Diff}_\partial(-)$. *Surgery theory*, as developed by Browder, Novikov, Ranicki, Sullivan, Wall, et al., roughly studies the difference between $\widetilde{\mathrm{Diff}}_\partial(M)$ and $h\mathrm{Aut}_\partial(M)$ in terms of the relative mapping space $(G/O)_*^{(M,\partial M)}$ and the quadratic L-theory of the integral group ring $\mathbb{Z}[\pi_1(M)]$ of the fundamental group(oid) of M, making the homotopy type of $\widetilde{\mathrm{Diff}}_\partial(M)$ theoretically accessible via homotopy theory and L-theory.

It is in understanding the homotopy type of $(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M)$, the homotopy fibre of the map i, where *pseudoisotopy theory* [Igusa 1988; Hatcher and Wagoner 1973] comes into play. Originally, this theory was concerned with the study of the topological group $C(M)$ of *concordances* or *pseudoisotopies* of M consisting of diffeomorphisms of $M\times I$ that restrict to the identity on a neighbourhood of $M\times\{0\}\cup\partial M\times I$; in other words, pseudoisotopies of M are precisely the automorphisms of the trivial h-cobordism starting at M. The spaces $C(M)$ and $\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)$ are intimately related, as was first made precise by Hatcher [1978, Proposition 2.1] through a spectral sequence. Hatcher's realisation eventually evolved into the following celebrated theorem of Weiss and Williams [1988, Theorem A], which will be of central importance all throughout this paper.

Theorem 1.1 (Weiss and Williams) *Let M^d be a compact smooth d-manifold. There exists a map*

$$\Phi^{\mathrm{Diff}}\colon(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M)\to\Omega^\infty(\boldsymbol{H}^s(M)_{hC_2})$$

which is $(\phi(d)+1)$-connected,[1] *where $\phi(d)$ denotes the concordance stable range of dimension d (which by Igusa's theorem* [1988] *is at least* $\min\left(\frac{1}{3}(d-4),\frac{1}{2}(d-7)\right)$*).*

The C_2-spectrum $\boldsymbol{H}^s(M)$ is the 1-connective cover of a spectrum $\boldsymbol{H}(M)$ which is roughly built out of deloopings of spaces of smooth h-cobordisms (see Notation 2.6 and Section 5.2), and whose involution corresponds (up to a minus sign) to "reversing the direction of an h-cobordism". This latter spectrum has a close connection to algebraic K-theory: its infinite loop space $\Omega^\infty\boldsymbol{H}(M)$ is equivalent to the (smooth) *stable h-cobordism space* $\mathcal{H}(M)$ — see Remark 3.10 — which fits in a fibre sequence of spaces [Waldhausen et al. 2013]

$$\mathcal{H}(M)\to Q_+M:=\Omega^\infty\Sigma^\infty_+M\xrightarrow{\nu}A(M),\tag{1-2}$$

where $A(M)$ denotes Waldhausen's [1985] A-theory space of M. This sequence is natural in codimension-zero embeddings, and the map ν is (naturally) a split injection. Even though the homotopy type of $A(M)$ and its involutions are difficult to understand in general, much can be said when M is homotopy equivalent to a point [Rognes 2002; 2003; Blumberg and Mandell 2019] or the circle [Hesselholt 2009], or when working rationally [Burghelea and Fiedorowicz 1986].

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[1] Recall that a map of spaces is said to be *n-connected* if, for every choice of basepoint in the domain, it induces isomorphisms on homotopy groups in degrees $*<n$ and a surjection in degree $*=n$.

1.1 The surgery-pseudoisotopy program for spaces of embeddings

The homotopy type of embedding spaces is intrinsically tied to that of diffeomorphism groups, as seen via the *isotopy extension theorem*. More precisely, let M^d be as before and let $\iota\colon P \hookrightarrow M$ be a compact submanifold that meets ∂M transversely. Write $\mathrm{Emb}_{\partial_0}(P, M)$ for the space of smooth embeddings of P into M which agree with ι in a neighbourhood of $\partial_0 P := P \cap \partial M$ and send $\partial P - \partial_0 P$ to the interior of M. Then there is a homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P, M) \to B\mathrm{Diff}_\partial(M - \nu P) \to B\mathrm{Diff}_\partial(M), \tag{1-3}$$

where νP is a small tubular neighbourhood of the standard embedding $\iota\colon P \hookrightarrow M$, and the subscript $\langle\iota\rangle$ stands for the collection of components of $\mathrm{Emb}_{\partial_0}(P, M)$ that are hit by the restriction map $\iota^*\colon \mathrm{Diff}_\partial(M) \to \mathrm{Emb}_{\partial_0}(P, M)$ that sends a diffeomorphism ϕ to $\phi\circ\iota$. In this sense, embedding spaces are the corresponding "relative analogues" of diffeomorphism groups, and often their homotopy types become easier to study.

We would like to advertise a direct approach for studying the homotopy type of embedding spaces (in a range of degrees) which is analogous to the one for diffeomorphism groups previously surveyed. As before, one first analyses the space of block embeddings $\widetilde{\mathrm{Emb}}_{\partial_0}(P, M)$ via relative surgery methods; the main result in this direction is due to Browder, Casson, Sullivan, and Wall (see [Goodwillie et al. 2001, Theorem 2.2.1]), and asserts that, as long as the codimension of $P \subset M$ is at least 3, then the space of block embeddings is the homotopy pullback of a diagram involving so-called *Poincaré block embeddings* and *immersions*, and ordinary block immersions. Due to the Smale–Hirsch immersion theorem, all the ingredients that come into the mix are accessible through homotopy theory and thus, up to extensions, so are block embeddings.

Following the same strategy as for the classical surgery-pseudoisotopy program, it remains to understand the difference between ordinary and block embeddings, i.e., the homotopy fibre

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) := \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P, M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P, M)), \tag{1-4}$$

by means of pseudoisotopy theory. This space also fits in another homotopy fibre sequence,

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \to (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M - \nu P) \xrightarrow{\mu} (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M), \tag{1-5}$$

obtained as the fibre of the map from (1-3) to its block analogue; see (3-4). What was previously fulfilled by Theorem 1.1 for the pseudoisotopy part of diffeomorphism groups seems to be missing in the case of embedding spaces; the best result known in this direction is *Morlet's lemma of disjunction* [Burghelea et al. 1975, Theorem 3.1] which, in that reformulation, determines the connectivity of the map μ.

Our first main result fills in this gap in the surgery-pseudoisotopy program for embedding spaces, and describes the homotopy type of $\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M)$ in a range outside of the connectivity of μ.

Theorem A *There exists a map*

$$\Phi^{\mathrm{Emb}}\colon \mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \to \Omega^\infty(\mathbf{CE}(P, M)_{hC_2})$$

which is $\phi_{C\mathrm{Emb}}(d,p)$*-connected if* $d \geq 4$ *and the handle dimension* p *of* P *relative to* $\partial_0 P$ *satisfies* $p \leq d-3$*. Here* $\phi_{C\mathrm{Emb}}$ *is the concordance embedding stable range* (see (1-6)) *and*

$$\mathbf{CE}(P,M) := \mathrm{hofib}(\boldsymbol{H}(M-\nu P) \to \boldsymbol{H}(M)).$$

That is, under the assumptions of the statement, the homotopy type (e.g., homotopy/homology groups) of $\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M)$ in degrees $* < \phi_{C\mathrm{Emb}}(d,p)$ agrees with that of the infinite loop space $\Omega^\infty(\mathbf{CE}(P,M)_{hC_2})$.

Remark 1.2 The involutions in the h-cobordism spectra involved in the statement of Theorem A are exactly those of Theorem 1.1, which naturally arise from Weiss' orthogonal calculus (see Sections 2.1 and A.2). When M is stably parallelisable (and localising away from 2), we relate these involutions to well-known algebraic ones in Theorem 5.13 and Corollary 5.17 (see Notation 5.1 for conventions). See also Corollary 5.9 for the effect of these involutions on $\pi_0^s(\boldsymbol{H}(M)) = \mathrm{Wh}(\pi_1(M))$ in terms of Milnor's involution [1966].

Remark 1.3 (splitting results and the Gromoll filtration) As a consequence of Theorem A, we establish a splitting result (see Theorem 4.3) for embedding spaces of manifolds containing interval factors, reminiscent of work of Burghelea and Lashof [1982, Corollary E]. This has remarkable consequences for the Gromoll filtration of embedding spaces (see Definition 4.5 and Corollary 4.6).

Remark 1.4 (topological version of Theorem A) Theorem 1.1 and the results of [Weiss and Williams 1988] admit topological analogues. Likewise, a topological version of Theorem A holds after some adjustments. First, smooth embedding spaces must be replaced by spaces of locally flat topological embeddings, and the h-cobordism spectra by their topological versions. Second, the result is only valid when P has geometric codimension zero in M, since a topological analogue of Proposition 3.3 cannot hold (see Remark 3.4). Third, when $d=4$ and $\mathrm{CAT}=\mathrm{Top}$, we additionally require M to be 1-connected (see Remark 3.2). One should also bear in mind, however, that the bound (1-7) is, a priori, only valid for smoothable topological manifolds. See Remarks 3.2, 3.6, and B.4 for modified arguments in the topological setting. A PL analogue of Theorem A should also exist after similar adjustments, though we leave the details to the reader (see also Warning 2.9).

We highlight two remarkable features of Theorem A that make it especially well-suited for computations:

1.1.1 The concordance embedding stable range Fix $\iota\colon P \hookrightarrow M$ as before. A *concordance embedding* of P into M is an embedding $\varphi\colon P \times I \hookrightarrow M \times I$ such that

(a) $\varphi^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$, and

(b) φ agrees with the inclusion $\iota \times \mathrm{Id}_I$ on a neighbourhood of $P \times \{0\} \cup \partial_0 P \times I$.

Denote by $C\mathrm{Emb}(P,M)$ the space of all such embeddings, topologised as a subspace of $\mathrm{Emb}(P \times I, M \times I)$. There are stabilisation maps

$$\Sigma\colon C\mathrm{Emb}(P,M) \to C\mathrm{Emb}(P \times I, M \times I)$$

given by taking the product of an embedding with I and unbending corners appropriately (see [Goodwillie et al. 2024, Figure 1]), and the *concordance embedding stable range* of the pair (M, P), denoted by $\phi_{C\mathrm{Emb}}(M, P)$, is the largest integer k such that all the stabilisations in

$$C\mathrm{Emb}(P, M) \xrightarrow{\Sigma} C\mathrm{Emb}(P \times I, M \times I) \xrightarrow{\Sigma} C\mathrm{Emb}(P \times I^2, M \times I^2) \xrightarrow{\Sigma} \cdots$$

are k-connected. Then the *concordance stable range* for a tuple (d, p) is

$$\phi_{C\mathrm{Emb}}(d, p) := \min\{\phi_{C\mathrm{Emb}}(M, P) : \dim M = d \text{ and } h\text{-}\dim(P, \partial_0 P) = p\}. \tag{1-6}$$

Here h-$\dim(P, \partial_0 P)$ denotes the *handle dimension* of the pair $(P, \partial_0 P)$, which is, by definition, the smallest number p such that P can be built from a closed collar on $\partial_0 P$ by attaching handles of index at most p. Goodwillie, Krannich, and Kupers [Goodwillie et al. 2024] have recently shown that if $p \le d - 3$, then

$$\phi_{C\mathrm{Emb}}(d, p) \ge 2d - p - 5, \tag{1-7}$$

which is far beyond Igusa's lower bound for the concordance stable range $\phi(d)$. In the concordance stable range $\phi(d)$, Theorem A is a consequence of Theorem 1.1 and the isotopy extension sequence (1-5), so our main contribution is improving the connectivity of the map Φ^{Emb} to the concordance embedding stable range $\phi_{C\mathrm{Emb}}(d, p)$. We will also see in Remark 6.6 that the bound (1-7) is sharp; this was apparent in [Goodwillie et al. 2024].

Remark 1.5 In fact, our proof will show that the map Φ^{Emb} of Theorem A is $\phi_{C\mathrm{Emb}}(M, P)$-connected under the codimension assumption $p \le d - 3$.

1.1.2 Relative algebraic K-theory via trace methods Given a map of spaces $Y \to X$, let $A(Y \to X)$ denote the homotopy fibre of the induced map $A(Y) \to A(X)$. By (1-2), there is an equivalence of spaces

$$\Omega A(M - P \to M) \simeq \Omega^\infty \mathbf{CE}(P, M) \times \Omega^2 Q(M/M - P).$$

The codimension assumption on the embedding $\iota: P \subset M$ in the statement of Theorem A guarantees that the inclusion $M - P \to M$ is 2-connected. This can be used to our advantage, as firstly it ensures that the spectrum $\mathbf{CE}(P, M)$ is connective (see Lemma 3.12), but more importantly that, via *trace methods*, the homotopy type of $A(M - P \to M)$ is far more accessible than those of $A(M - P)$ and $A(M)$ on their own.

Trace methods are concerned with the study of *topological cyclic homology* [Bökstedt et al. 1993], denoted by $TC(-)$, and related invariants as an approximation to algebraic K-theory. This is something sensible to do by the seminal work of Dundas, Goodwillie, and McCarthy [Dundas and McCarthy 1994; Dundas 1997; Dundas et al. 2013], who showed that the cyclotomic trace map provides an equivalence of relative theories $A(Y \to X) \simeq TC(Y \to X)$, so long as $Y \to X$ is 2-connected. The treatment of TC by Nikolaus and Scholze [2018] provides even further computational control of this invariant. In the cases we are concerned with (spherical group rings), the homotopy type of TC was fully described by Bökstedt, Hsiang, and Madsen [Bökstedt et al. 1993] in terms of the stable homotopy of the free loop space $L(-) := \mathrm{Map}(S^1, -)$ together with its natural S^1-action and cyclotomic structure. When working over the field of rational numbers, this whole story simplifies even further by Goodwillie's isomorphism [1986]

$$\pi_*(A(Y \to X)) \otimes \mathbb{Q} \cong \mathrm{HC}_*(\Omega X, \Omega Y; \mathbb{Q}) \cong H_*^{S^1}(LX, LY; \mathbb{Q}), \tag{1-8}$$

where HC_* denotes Connes' cyclic homology, and $H_*^{S^1}$ stands for the S^1-equivariant homology.

So as we have just seen, the homotopy type of (the infinite loop space of) the connective spectrum $\mathbf{CE}(P, M)$ of Theorem A is pretty accessible in general. However, one still needs to deal with the involution appearing in the statement in order to apply the result, which is a rather technical task that, so far, had only been carried out rationally by Bustamante, Farrell, and Jiang [Bustamante et al. 2020] — they relate this involution to one on the right-hand side of (1-8). Integrally, one has to proceed with more care; our analysis in Section 5 deals with this issue localised away from 2 and when M is stably parallelisable.

1.2 The homotopy type of spaces of long knots

The homology and homotopy of *spaces of long knots* $\mathrm{Emb}_\partial(D^p, D^d)$ has been subject to extensive research in recent years, especially through the lens of embedding calculus and its relation to the little disks operad and graph complexes. See for instance Volić [2006], Watanabe [2007], Scannel and Sinha [2002; Sinha 2009], and Budney and Cohen [Budney 2008; Budney and Cohen 2009] for when $p = 1$ and $d = 3, 4$ mainly, or more modern treatments as in Arone and Turchin [2014; 2015], Dwyer and Hess [2012], and Boavida de Brito and Weiss [2018], at last culminating in the work of Fresse, Turchin, and Willwacher [Fresse et al. 2017], where a complete description of $\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \otimes \mathbb{Q}$ is given in terms of the homology of the *hairy graph complex*. See also Boavida de Brito and Horel [2021] for some torsion computations in the homotopy groups of spaces of long knots when $p = 1$.

Our second main result is a full description of the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d - p \geq 3$, roughly in the concordance embedding stable range (1-6) and localised away from 2. This is done by analysing the homotopy fibre sequence

$$\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \to \mathrm{Emb}_\partial(D^p, D^d) \to \widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \tag{1-9}$$

following the surgery-pseudoisotopy program for embedding spaces surveyed in Section 1.1, a crucial step of which is Theorem A. Given a finite-dimensional virtual G-representation ρ over $\mathbb{R}$, denote by $\mathbb{S}^\rho$ the representation sphere spectrum associated to it; we will consider its homotopy orbit spectrum $\mathbb{S}^\rho_{hG}$, which is equivalent to the Thom spectrum of the associated virtual vector bundle $EG \times_G \rho \to BG$. Let ψ_m denote the real m-dimensional representation of the dihedral group D_m (seen as a subgroup of the symmetric group Σ_m) given by permuting the factors of $\mathbb{R}^m$, and let $\sigma\colon C_2 = \{\pm 1\} \hookrightarrow \mathbb{R}^\times$ be the sign representation (also regarded as a D_m-representation by restriction along the determinant $D_m \hookrightarrow O(2) \xrightarrow{\det} \{\pm 1\} = C_2$).

Theorem B *For $p \leq d - 3$ and $d \geq 5$, consider the virtual D_m-representations*

$$\rho_m := (d+1)(\sigma - 1) + \psi_m \otimes (d - p - 3 + \sigma).$$

Then the homotopy fibre sequence (1-9), *upon localising away from* 2 *and taking* $(\phi_{C\mathrm{Emb}}(d,p)-1)$*-th Postnikov sections, takes the form*

$$\prod_{m\geq 2}\Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m})\to \mathrm{Emb}_\partial(D^p,D^d)\to\Omega^p\,\mathrm{hofib}(G(d-p)/O(d-p)\to G/O). \tag{1-10}$$

The resulting sequence is split if $p\geq 2$, *and splits after being looped once if* $p=1$.

Remark 1.6 (i) The spaces $G(n)/O(n)$ and G/O appearing in (1-10) denote the homotopy fibres of the natural maps $BO(n)\to BG(n)$ and $BO\to BG$, respectively, where $G(n)$ is the topological grouplike monoid of self-homotopy equivalences of S^{n-1}, and G is the homotopy colimit of the suspension maps $G(n)\to G(n+1)$. Understanding the homotopy groups of these spaces roughly amounts to understanding unstable and stable homotopy groups of spheres.

(ii) The space $\mathrm{Emb}_\partial(D^p,D^d)$ is an $\mathbb{E}_p$-algebra, and so it can indeed be localised. When $d-p\geq 3$, it is exactly $(2d-3p-4)$-connected by work of Budney [2008, Proposition 3.9]. So when $2d-3p-4<0$, by *localising* $\mathrm{Emb}_\partial(D^p,D^d)$ we really mean localising each of its connected components one at a time (and *not* localising the abelian group [2] $\pi_0(\mathrm{Emb}_\partial(D^p,D^d))$ directly). The same applies to the outer terms in (1-9).

(iii) When $p=1$ and $d=4$ (i.e., the lowest dimensional case of interest if $d-p\geq 3$), Theorem B holds after looping both (1-9) and (1-10); see Remark 6.1. Using this to study the homotopy groups of $\mathrm{Emb}_\partial(D^1,D^4)$, however, yields weaker results than the ones of Budney [2008, Proposition 3.9]. See Remark 6.4 for further comparison of our work with Budney's, and for consequences of Theorem B for the Gromoll filtration of spaces of long knots.

For A an abelian group and ℓ a prime, write $A_{(\ell)}:=A\otimes\mathbb{Z}_{(\ell)}$, i.e., the localisation of A at the prime ℓ. We will compute some of the homotopy groups of the fibre in (1-10) in Section 6.2, and hence deduce new torsion information about the homotopy groups of $\mathrm{Emb}_\partial(D^p,D^d)$ in high dimensions (i.e., $d\geq 5$).

Corollary C (Propositions 6.5 and 6.7) *For* $d-p\geq 3$ *and* ℓ *an odd prime, there are isomorphisms*

$$\pi_*(\mathrm{Emb}_\partial(D^p,D^d))_{(\ell)}\cong\pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p)\to G/O))_{(\ell)}\oplus\bigoplus_{m\geq 2}\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$$

in degrees $*\leq\phi_{C\mathrm{Emb}}(d,p)-1$. *When* $m\geq 2$ *and* $\ell\nmid 2m$:

- *If* d *is even and* p *is even, then*

$$\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}\cong\begin{cases}\pi^s_{*-m(d-p-2)}\otimes\mathbb{Z}_{(\ell)} & \textit{if } m=3,5,7,\dots,\\ 0 & \textit{otherwise.}\end{cases}$$

---

[2] If $d-p\geq 3$, the monoid $\pi_0(\mathrm{Emb}_\partial(D^p,D^d))$ is indeed an abelian group: that it is a group follows from Hudson's theorem (Theorem 3.1), which says that it is isomorphic to $\pi_0(\widetilde{\mathrm{Emb}}_\partial(D^p,D^d))\cong\pi_p(\widetilde{\mathrm{Emb}}_\partial(*,D^{d-p}))$. That it is abelian when $p=1$ is well-known, and can be proved by the "pulling one knot through another" trick, as illustrated in [Budney 2007, Figure 2].

- *If d is odd and p is odd, then*

$$\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \textit{if } m = 2, 4, 6, \dots, \\ 0 & \textit{otherwise.} \end{cases}$$

- *If d is even and p is odd, then*

$$\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \textit{if } m = 5, 9, 13, \dots, \\ 0 & \textit{otherwise.} \end{cases}$$

- *If d is odd and p is even, then*

$$\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \textit{if } m = 3, 7, 11, \dots, \\ 0 & \textit{otherwise.} \end{cases}$$

If ℓ divides m, the computation of $\pi^s_*(\mathbb{S}^{\rho_m}_{D_m})_{(\ell)}$ must be treated case by case. When $m = \ell = 3$ and $d - p = 3$, the first few such homotopy groups are given in Table 1 (see Proposition 6.7 for notation).

Rationally, this computation roughly recovers the homology of the 0- and 1-loop order part of the hairy graph complex appearing in [Fresse et al. 2017, (2)]; see Remark 6.6 for more details.

Structure of the paper

Sections 2 and 3 will be devoted to the proof of Theorem A. We start by briefly reviewing Weiss' theory of orthogonal calculus and then, in Section 2.2, we present the orthogonal functors that will play a role in the proof of Theorem A. In doing this, we will have to carefully describe the topology on spaces of bounded diffeomorphisms in such a way that we can employ the machinery of orthogonal calculus in this setting. After reducing to the codimension-zero case in Section 3.1, we use the results in the preceding section to define the map Φ^{Emb} in Section 3.3 and analyse its connectivity in Section 3.4.

Section 4 concerns the splitting result (Theorem 4.3) for embedding spaces mentioned in Remark 1.3, and its consequences for the Gromoll filtration (see Corollary 4.6).

Section 5 deals with the analysis of the C_2-spectra involved in the statements of Theorems 1.1 and A. The main results in this direction are Theorem 5.13 and Corollary 5.17, where the involutions on these spectra are expressed (up to homotopy) in terms of the standard involution in algebraic K-theory.

Section 6 is devoted to Theorem B, whose proof is a formal consequence of the results in the preceding sections. We then draw some conclusions on the homotopy groups of spaces of long knots in Section 6.2.

Appendix A deals with some subtleties regarding the definition of the first derivative of an orthogonal functor as an $O(1)$-spectrum, and with a technical argument in the proof of Proposition 2.2.

In Appendix B we explore certain aspects related to spaces of bounded diffeomorphisms and embeddings. Namely in Section B.1 we show that the topological models for these spaces introduced in Section 2.2 coincide (up to weak equivalence) with the simplicial ones of Definition 2.8. In Section B.2 we give a "moduli space of manifolds" description for the classifying space of the bounded diffeomorphism group.

In Appendix C we show that the h-cobordism stabilisation map anticommutes with the involutions in these spaces. This is analogous to a result of Hatcher [1978, Appendix I, Lemma] and Burghelea and Lashof [1982, Corollary A7] for spaces of concordance diffeomorphisms.

2 Orthogonal calculus and spaces of bounded diffeomorphisms

Much of the proof of Theorem 1.1 in [Weiss and Williams 1988] is an application of Weiss' *orthogonal calculus* but in disguise, as this theory was not yet formalised at the time. In this section we briefly review the main aspects of this theory and develop some necessary tools required for the proof of Theorem A.

2.1 A quick tour through orthogonal calculus

Weiss' *orthogonal calculus* [1995] is a calculus of functors useful to understand objects of geometric flavour. It studies *continuous* functors from the category $\mathcal{J}$ of real finite-dimensional inner product vector spaces and linear isometries to the category of (compactly generated weakly Hausdorff) spaces Top. Such a functor $F\colon \mathcal{J} \to \mathrm{Top}$ is said to be *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}}(U, V) \times F(U) \to F(V)$$

is continuous for all $U, V \in \mathcal{J}$. Here $\mathrm{mor}_{\mathcal{J}}(U, V)$ denotes the *Stiefel manifold* of linear isometries from U to V, so that $\mathcal{J}$ is enriched over Top. We will work in a slightly different setup, where Top is replaced by the category Top_* of pointed spaces and $\mathcal{J}$ is replaced by the pointed topological category $\mathcal{J}_0$ with the same objects and with

$$\mathrm{mor}_{\mathcal{J}_0}(U, V) := \mathrm{mor}_{\mathcal{J}}(U, V)_+$$

as morphism spaces. Similarly, a functor $F\colon \mathcal{J}_0 \to \mathrm{Top}_*$ is *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}_0}(U, V) \wedge F(U) \to F(V)$$

is continuous for all $U, V \in \mathcal{J}_0$. Such a functor $F(-)$ is also sometimes called an *orthogonal functor*.

The machinery of orthogonal calculus associates to each such orthogonal functor $F(-)$ a sequence of (naïve) $O(k)$-spectra[3] $\Theta F^{(k)}$ for $k \geq 1$ — the *derivatives of* F — which fit in a tower

(2-1)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & T_2F(-) \arrow[d] & \Omega^\infty((S^{2\cdot(-)} \wedge \Theta F^{(2)})_{hO(2)}) \arrow[l] \\
 & T_1F(-) \arrow[d] & \Omega^\infty((S^{1\cdot(-)} \wedge \Theta F^{(1)})_{hO(1)}) \arrow[l] \\
F(-) \arrow[r, "\eta_0"'] \arrow[ru] \arrow[ruu] \arrow[ruuu] & T_0F(-) = F(\mathbb{R}^\infty) &
\end{tikzcd}$$

of orthogonal functors — the *Taylor tower*. Here:

- $S^{k\cdot V}$ is the one-point compactification of $k \cdot V := \mathbb{R}^k \otimes V$, which is acted upon by $O(k)$ in the $\mathbb{R}^k$ component, and diagonally on the smash $S^{k\cdot V} \wedge \Theta F^{(k)}$.

---

[3] By *spectrum*, we will always mean prespectrum. For us, a G*-spectrum* X will be a sequence of based G-spaces X_n together with G-equivariant maps $S^1 \wedge X_n \to X_{n+1}$, where G acts trivially on S^1.

- The right-hand horizontal maps — the *layers* — indicate the inclusions of the homotopy fibres of the subsequent vertical maps between the *stages* $T_k F(-)$ of the tower.
- The 0-th stage $T_0 F(-)$ is given by $T_0 F(V) := \operatorname{hocolim}_k F(V \oplus \mathbb{R}^k)$, and thus admits a canonical equivalence from the constant orthogonal functor with *value at infinity* $F(\mathbb{R}^\infty) := \operatorname{hocolim}_k F(\mathbb{R}^k)$. The map $\eta_0 \colon F(V) \to T_0 F(V)$ is simply the inclusion map.

In the proof of Theorem A we will analyse the Taylor tower (2-1) only up to the first layer, so we shall now describe the spectrum $\Theta F^{(1)}$ in detail. For $V \in \mathcal{J}_0$, consider

$$F^{(1)}(V) := \operatorname{hofib}(F(V) \to F(V \oplus \mathbb{R})),$$

the homotopy fibre of the map induced by the standard inclusion $V \to V \oplus \mathbb{R}$. These spaces inherit an $O(1)$-action by declaring $-1 \in O(1)$ to act on V and $V \oplus \mathbb{R}$ by -1 on *all* coordinates. There are $O(1)$-equivariant maps

$$s_V \colon S^1 \wedge F^{(1)}(V) \to F^{(1)}(V \oplus \mathbb{R}), \tag{2-2}$$

given, roughly, by performing a 180° rotation of $V \oplus \mathbb{R}^2$ about the 2-plane $0 \oplus \mathbb{R}^2$; we give an explicit model of this map in Section A.1. As notation suggests in (2-2), $O(1)$ acts trivially on the suspension coordinate. (In general, we adopt the convention that S^n denotes the n-sphere with the trivial $O(1)$-action.) Then the $O(1)$-spectrum $\Theta F^{(1)}$ has $F^{(1)}(\mathbb{R}^n)$ as its n-th space, and $s_{\mathbb{R}^n}$ as the structure map.

Remark 2.1 This is not quite the $O(1)$-action described in [Weiss 1995, Proposition 3.1]; rather, it more closely follows the convention adopted in [Weiss and Williams 1988, p. 601]. We justify our convention choice in Section A.2.

For $V \in \mathcal{J}_0$, let $S(V)$ denote the unit sphere of V, seen as an unbased $O(1)$-space by the antipodal action. The following proposition will be the main ingredient for the construction of the map Φ^{Emb} of Theorem A:

Proposition 2.2 *Let $F \colon \mathcal{J}_0 \to \mathrm{Top}_*$ be an orthogonal functor. For each $n \geq 0$, there are maps*

$$\Phi^F_n \colon \operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \xrightarrow{\eta_1} \operatorname{hofib}(T_1 F(0) \to T_1 F(\mathbb{R}^n)) \simeq \Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \tag{2-3}$$

giving rise to a map of homotopy fibre sequences

$$\begin{tikzcd}
\operatorname{hofib}(F(0) \to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\Phi^F_n"] & \operatorname{hofib}(F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\Phi^F_{n+1}"] & \operatorname{hofib}(F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1})) =: \Theta F^{(1)}_n \arrow[d, "\mathrm{stab.}"] \\
\Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(S(\mathbb{R}^{n+1})_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(\Sigma^n \Theta F^{(1)})
\end{tikzcd}$$

where the vertical map "stab." *is $\Theta F^{(1)}_n \hookrightarrow \operatorname{hocolim}_k \Omega^k(\Theta F^{(1)}_{n+k})$. Letting $n \to \infty$ in* (2-3)*, we get*

$$\Phi^F_\infty \colon \operatorname{hofib}(F(0) \to F(\mathbb{R}^\infty)) \to \Omega^\infty(EO(1)_+ \wedge_{O(1)} \Theta F^{(1)}) =: \Omega^\infty(\Theta F^{(1)}_{hO(1)}). \tag{2-4}$$

Proof Given $F\colon \mathcal{J}_0 \to \mathrm{Top}_*$, write $L_1F := \mathrm{hofib}(T_1F \to T_0F)$ for the first homogeneous layer of F. Consider also the orthogonal functor

$$E(-) := \Omega^\infty((S^{(-)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}).$$

By the classification of homogeneous functors of [Weiss 1995, Theorem 9.1], there is a natural equivalence[4] of functors $L_1F \simeq E$, and hence a zigzag of orthogonal functors

$$F \xrightarrow{\eta_1} T_1F \leftarrow L_1F \simeq E.$$

This zigzag gives rise to the following commutative diagram, in which the vertical arrows induce maps of homotopy fibre sequences:

$$\begin{tikzcd}
\mathrm{hofib}(F(0) \to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\eta_1"] & \mathrm{hofib}(F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\eta_1"] & \Theta F_n^{(1)} \arrow[d, "\eta_1"] \\
\mathrm{hofib}(T_1F(0) \to T_1F(\mathbb{R}^n)) \arrow[r] & \mathrm{hofib}(T_1F(0) \to T_1F(\mathbb{R}^{n+1})) \arrow[r] & \Theta(T_1F)_n^{(1)} \\
\mathrm{hofib}(L_1F(0) \to L_1F(\mathbb{R}^n)) \arrow[r] \arrow[u, "\wr"'] \arrow[d, dash, "\wr"'] & \mathrm{hofib}(L_1F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[u, "\wr"'] \arrow[d, dash, "\wr"'] & \Theta(L_1F)_n^{(1)} \arrow[u, "\wr"'] \arrow[d, dash, "\wr"'] \\
\mathrm{hofib}(E(0) \to E(\mathbb{R}^n)) \arrow[r] & \mathrm{hofib}(E(0) \to E(\mathbb{R}^{n+1})) \arrow[r] & \Theta E_n^{(1)}
\end{tikzcd}$$

We claim that the bottom fibre sequence in this diagram is naturally equivalent to the bottom fibre sequence in the map of fibre sequences of the statement; indeed, for $n \geq 0$, write $\mathrm{Ind}_e^{O(1)} S^n$ for the wedge $S^n \vee S^n$ with the flip action. Then, for $V \in \mathcal{J}_0$ with $\dim V = v$, there is a diagram of $O(1)$-spaces

$$\begin{tikzcd}
S(V)_+ \arrow[r] \arrow[d, equal] & S(V \oplus \mathbb{R})_+ \arrow[r] \arrow[d] & \mathrm{Ind}_e^{O(1)} S^v \arrow[d, "a"] \\
S(V)_+ \arrow[r] \arrow[d] & S^0 \arrow[r] \arrow[d] & S^{V\cdot\sigma} \arrow[d] \\
* \arrow[r] & S^{(V\oplus\mathbb{R})\cdot\sigma} \arrow[r, equal] & S^{(V\oplus\mathbb{R})\cdot\sigma}
\end{tikzcd} \tag{2-5}$$

where every row and column is an $O(1)$-equivariant (homotopy) cofibre sequence. Here the map

$$a\colon \mathrm{Ind}_e^{O(1)} S^v \cong S_0^V \vee S_1^V \to S^{V\cdot\sigma}$$

---

[4] Theorem 9.1 in [Weiss 1995] states that L_1F is naturally equivalent to the functor $E' = \Omega^\infty((S^{(-)\cdot\sigma} \wedge \Theta^\# F^{(1)})_{hO(1)})$, where $\Theta^\# F^{(1)}$ is defined in (A-2). Since this $O(1)$-spectrum is naturally equivalent to $\Theta F^{(1)}$ by (A-3), it follows that E' is naturally equivalent to E.

sends $x \in V \subset S_i^V$ to $(-1)^i x$ for $i = 0, 1$. Applying $\Omega^\infty((- \wedge \Theta F^{(1)})_{hO(1)})$ to (2-5), we obtain a diagram for $V = \mathbb{R}^n$

$$\begin{tikzcd}
\Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] \arrow[d, "\wr"'] & \Omega^\infty(S(\mathbb{R}^{n+1})_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] \arrow[d] & \Omega^\infty(\Sigma^n \Theta F^{(1)}) \arrow[d] \\
\mathrm{hofib}(E(0) \to E(\mathbb{R}^n)) \arrow[r] \arrow[d] & E(0) \arrow[r] \arrow[d] & E(\mathbb{R}^n) \arrow[d] \\
* \arrow[r] & E(\mathbb{R}^{n+1}) \arrow[r, equal] & E(\mathbb{R}^{n+1})
\end{tikzcd}$$

where every row and column is a homotopy fibre sequence. Observe that the top horizontal cofibre sequence of (2-5) consists of free $O(1)$-spaces, which explains the underived balanced smash product $\wedge_{O(1)}$; note also that $\mathrm{Ind}_e^{O(1)} S^n \wedge_{O(1)} X \simeq S^n \wedge X \cong \Sigma^n X$ for any $O(1)$-spectrum X.

Consequently, we obtain a natural map of homotopy fibre sequences

$$\begin{tikzcd}
\mathrm{hofib}(F(0) \to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\Phi_n^F"] & \mathrm{hofib}(F(0) \to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\Phi_{n+1}^F"] & \Theta F_n^{(1)} \arrow[d, "s_n^F"] \\
\Omega^\infty(S(\mathbb{R}^n)_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(S(\mathbb{R}^{n+1})_+ \wedge_{O(1)} \Theta F^{(1)}) \arrow[r] & \Omega^\infty(\Sigma^n \Theta F^{(1)})
\end{tikzcd} \tag{2-6}$$

It remains to argue that s_n^F is naturally homotopic to the map $\mathrm{stab}_n^F \colon \Theta F_n^{(1)} \hookrightarrow \mathrm{hocolim}_k\, \Omega^k(\Theta F_{n+k}^{(1)})$. Indeed, there is a commutative diagram

$$\begin{tikzcd}
 & & & \Omega^\infty(\Sigma^n \Theta F^{(1)}) \arrow[d, dash, "\wr"'] \\
\Theta F_n^{(1)} \arrow[urrr, bend left=10, "s_n^F"] \arrow[r] \arrow[d, hook, "\mathrm{stab}_n^F"] & \Theta(T_1 F)_n^{(1)} \arrow[d, hook, "\mathrm{stab}_n^{T_1 F}", "\wr"'] & \Theta(L_1 F)_n^{(1)} \arrow[l, "\sim"'] \arrow[r, dash, "\sim"] \arrow[d, hook, "\mathrm{stab}_n^{L_1 F}", "\wr"'] & \Theta E_n^{(1)} \arrow[d, hook, "\mathrm{stab}_n^E", "\wr"'] \\
\Omega^\infty(\Sigma^n \Theta F^{(1)}) \arrow[r, "\sim"] & \Omega^\infty(\Sigma^n \Theta(T_1 F)^{(1)}) & \Omega^\infty(\Sigma^n \Theta(L_1 F)^{(1)}) \arrow[l, "\sim"'] \arrow[r, dash, "\sim"] & \Omega^\infty(\Sigma^n \Theta E^{(1)})
\end{tikzcd} \tag{2-7}$$

where the leftmost horizontal is an equivalence since $\eta_1 \colon \Theta F^{(1)} \to \Theta(T_1 F)^{(1)}$ is an equivalence by [Weiss 1995, Theorem 6.3(bis)], whilst all but the leftmost vertical stabilisation maps are equivalences because the spectra involved are Ω-spectra; indeed this is the case for $\Theta(T_1 F)^{(1)}$ and $\Theta(L_1 F)^{(1)}$ by [Weiss 1995, Corollary 5.12], whilst $\Theta E^{(1)}$ is the *spectrification* of the prespectrum $\Theta F^{(1)}$. (This is the case for any functor of the form $H_X(-) = \Omega^\infty((S^{(-)\cdot\sigma} \wedge X)_{hO(1)})$, for X an $O(1)$-(pre)spectrum; the first derivative of H_X is, nonequivariantly, the spectrification of X by [Weiss 1995, Section 7].) From this last observation, it also follows that the right vertical composite $\Omega^\infty(\Sigma^n \Theta F^{(1)}) \xrightarrow{\sim} \Omega^\infty(\Sigma^n \Theta E^{(1)})$ is induced by the spectrification map $\Theta F^{(1)} \xrightarrow{\sim} \Theta E^{(1)}$. Thus the sequence of equivalences in the lower row and rightmost column are all induced by equivalences of spectra

$$\Theta F^{(1)} \xrightarrow{\sim} \Theta(T_1 F)^{(1)} \xleftarrow{\sim} \Theta(L_1 F)^{(1)} \overset{\sim}{-\!-} \Theta E^{(1)} \xleftarrow{\sim} \Theta F^{(1)}. \tag{2-8}$$

To argue that stab_n^F and s_n^F are homotopic, we need to show that (2-8) lies in the homotopy class of the identity map of $\Theta F^{(1)}$. For this, we need to be slightly more explicit about the equivalence $L_1F \simeq E$, which factors as

$$L_1F \overset{(!)}{\simeq} H_{L_1F} := \Omega^\infty((S^{(-)\cdot\sigma} \wedge \Theta(L_1F)^{(1)})_{hO(1)}) \xleftarrow{\simeq} \Omega^\infty((S^{(-)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}) = E.$$

The right map in this zigzag is induced by the natural map $\Theta F^{(1)} \xrightarrow{\eta_1} \Theta(T_1F)^{(1)} \simeq \Theta(L_1F)^{(1)}$, whereas the equivalence (!) is provided in [Weiss 1995, Theorem 7.3]; this equivalence satisfies that the induced one on first derivatives $(!)\colon \Theta(L_1F)^{(1)} \simeq \Theta(H_{L_1F})^{(1)}$ becomes the identity when composed with the natural equivalence $\Theta(H_{L_1F})^{(1)} \xleftarrow{\simeq} \Theta(L_1F)^{(1)}$. It now easily follows that the zigzag (2-8) represents the homotopy class of $\mathrm{Id}_{\Theta F^{(1)}}$, and hence stab_n^F and s_n^F are indeed homotopic, as desired. □

Remark 2.3 In the proof of Theorem A, we will only need that the map s_n^F in (2-6) is as connected as the stabilisation map stab_n^F. This already follows from the commutative diagram (2-7), which shows that s_n^F and $\varphi_n^F \circ \mathrm{stab}_n^F$ are naturally homotopic, for some natural automorphism φ_n^F of the codomain $\Omega^\infty(\Sigma^n \Theta F^{(1)})$. Thus the last part of the proof above is not strictly necessary for our purposes. Nevertheless, we believe it is useful to have this technical point clarified.

2.2 The orthogonal functors of bounded diffeomorphisms

All throughout, let $\iota\colon P \hookrightarrow M$ be as in the statement of Theorem A. In this section we present the orthogonal functors that will play a role in the proof of Theorem A. These are built out of spaces of bounded diffeomorphisms, for which we will present point–set topological models that agree up to weak equivalence with the more classical simplicial ones.

Let $V \in \mathcal{J}$ be an inner product finite-dimensional real vector space with associated norm $\|-\|_V$, and let Q and Q' be smooth (possibly noncompact) manifolds equipped with proper maps $\pi\colon Q \to V$ and $\pi'\colon Q' \to V$. For $t \geq 0$, a smooth map $f\colon Q \to Q'$ is said to be *t-bounded* if the set

$$\{\|\pi'(f(q)) - \pi(q)\|_V : q \in Q\} \subset \mathbb{R}$$

is bounded by t. More generally, f is *bounded* if it is t-bounded for some $t \geq 0$. If $Q = N \times V$ for some compact manifold N, π will be assumed to be the projection to V.

Definition 2.4 Let $V \in \mathcal{J}$. The *space of bounded diffeomorphisms* of $M \times V$ relative to $\partial M \times V$ is

$$\mathrm{Diff}^b_\partial(M \times V) := \{(t,\phi) \in [0,\infty) \times \mathrm{Diff}_\partial(M \times V) : \phi \text{ is } t\text{-bounded}\},$$

endowed with the subspace topology inherited from the product $[0,\infty) \times \mathrm{Diff}_\partial(M \times V)$. Here $\mathrm{Diff}_\partial(M \times V)$ is endowed with the weak Whitney C^∞-topology. It is a group-like topological monoid under the rule

$$(t,\phi) \cdot (t',\phi') := (t + t', \phi \circ \phi').$$

Similarly, we define the space $\mathrm{Homeo}^b_\partial(M \times V)$ of *bounded homeomorphisms* of $M \times V$ as a subspace of the product $\mathrm{Homeo}_\partial(M \times V) \times [0,+\infty)$, where $\mathrm{Homeo}_\partial(M \times V)$ is endowed with the compact–open topology.

Example 2.5 Orthogonal calculus was largely inspired by the work of Weiss and Williams [1988], as can be seen in [loc. cit., Diagram 3.8]. For $U \in \mathcal{J}_0$, let $B\mathrm{Diff}^b_\partial(M \times U)$ be the classifying space of the group-like topological monoid just introduced. Denote by $B(-)$ the orthogonal functor given by $B(U) := B\mathrm{Diff}^b_\partial(M \times U)$ and, for $i\colon U \to V$ a morphism in $\mathcal{J}_0$, write $V = U \oplus U^\perp$ and let $B(i)$ be induced by the monoid homomorphism sending $(t,\phi) \in \mathrm{Diff}^b_\partial(M \times U)$ to $(t, \phi \oplus \mathrm{Id}_{U^\perp}) \in \mathrm{Diff}^b_\partial(M \times U \oplus U^\perp)$. Then

$$\Phi^B_\infty\colon \mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty)/\mathrm{Diff}_\partial(M) \to \Omega^\infty(\Theta B^{(1)})$$

should be[5] the map from [loc. cit., Theorem C], and Proposition 2.2 recovers [loc. cit., Proposition 3.1] in this case.

Notation 2.6 In the remainder of Section 2, we will denote by $E(-)$ and $B(-)$ the orthogonal functors given on objects by

$$E(V) := B\mathrm{Diff}^b_\partial((M - \nu P) \times V), \quad B(V) := B\mathrm{Diff}^b_\partial(M \times V), \tag{2-9}$$

where νP is an open tubular neighbourhood of the embedding $\iota\colon P \subset M$, and on morphisms as in Example 2.5. There is a natural transformation $E(-) \to B(-)$ given by extending a diffeomorphism by the identity on $\nu P \times (-)$, and the orthogonal functor $F(-) := \mathrm{hofib}(E(-) \to B(-))$ will play an especially important role in the proof of Theorem A. We will often use the following notation for the derivatives of these functors:

$$\mathbf{CE}(P, M) := \Theta F^{(1)}, \quad \boldsymbol{H}(M - \nu P) := \Theta E^{(1)}, \quad \boldsymbol{H}(M) := \Theta B^{(1)}. \tag{2-10}$$

Thus $\mathbf{CE}(P, M)$ is, by definition, the homotopy fibre of the map $\boldsymbol{H}(M - \nu P) \to \boldsymbol{H}(M)$.

Remark 2.7 Let us comment on the notation in (2-10). Write $H(M)$ for the space of smooth h-cobordisms starting at M (see Section 5.2). We will see in Remark 3.10 that there is an equivalence (of spaces)

$$\Omega^\infty \boldsymbol{H}(M) \simeq \mathcal{H}(M) := \operatorname*{hocolim}_k H(M \times D^k), \tag{2-11}$$

where the colimit on the right-hand side — the *stable h-cobordism space* — is induced by the h-cobordism stabilisation maps of Appendix C. The equivalence (2-11) is natural in codimension-zero embeddings, provided that we restrict to basepoint components. It should also be natural when considering all components, but proving this seems more tedious; see Remark 3.10 for further discussion of this naturality.

By the stable parametrised h-cobordism theorem of Waldhausen, Jahren, and Rognes [Waldhausen et al. 2013], the infinite loop space of the desuspension of the *smooth Whitehead spectrum* $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ is also equivalent to $\mathcal{H}(M)$ (as ordinary spaces). Moreover, Weiss and Williams [1988, Corollary 5.6] showed that the spectra $\Theta B^{(1)} = \boldsymbol{H}(M)$ and $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$ also share the same negative homotopy groups, which led them to rename the former as the latter. This, though conjecturally true, was not fully justified since no equivalence between these two spectra was given.

---

[5] Though Φ^B_∞ is not visibly the same map as the one appearing in [Weiss and Williams 1988], they share the same formal properties by Proposition 2.2.

We hope that the homotopy fibre sequence $C\mathrm{Emb}(P,M) \to H(M-\nu P) \to H(M)$ and Proposition 3.11 together explain why we denote $\Theta F^{(1)}$ by $\mathbf{CE}(P,M)$.

Spaces of bounded diffeomorphisms are usually defined as the geometric realisation of certain simplicial groups/sets. Before we recall these simplicial models in Definition 2.8 below, let us fix some notation. For a subset $S \subset \mathbb{R}^{p+1}$ and $\epsilon > 0$, let $B_\epsilon(S) \subset \mathbb{R}^{p+1}$ denote the open ϵ-ball around S. For $0 < \epsilon \le \frac{1}{2}$ and for any face $\sigma \subset \Delta^p$, we fix radial identifications $\rho_\sigma\colon \partial\sigma(\epsilon) := B_\epsilon(\partial\sigma) \cap \sigma \cong \partial\sigma \times [0,\epsilon)$; let us first do it for $\sigma = \Delta^p$. Given $x = (t_0,\dots,t_p) \in \partial\Delta^p(\epsilon)$, let $j \in [p]$ be such that $t_j \le t_i$ for every $i \in [p]$. Note that since x cannot be the barycenter $b_p = (1/(p+1),\dots,1/(p+1))$ of Δ^p (since this lies at distance greater than $\frac{1}{2} \ge \epsilon$ from $\partial\Delta^p$), we must have that t_j is strictly smaller than $1/(p+1)$. Then set

$$\rho_p\colon \partial\Delta^p(\epsilon) \xrightarrow{\cong} \partial\Delta^p \times [0,\epsilon), \quad x = (t_0,\dots,t_p) \mapsto \left(x - \frac{t_j}{1/(p+1) - t_j}(b_p - x), d(x,\partial\Delta^p)\right),$$

where $j = j(x)$ is as above, and $d(x,\partial\Delta^p)$ stands for the (Euclidean) distance between x and $\partial\Delta^p$. For a general face $\sigma \subset \Delta^p$, fix the standard order-preserving identification $\eta_\sigma\colon \sigma \cong \Delta^{|\sigma|}$; then the radial identification $\rho_\sigma\colon \partial\sigma(\epsilon) \cong \partial\sigma \times [0,\epsilon)$ is

$$\rho_\sigma\colon \partial\sigma(\epsilon) \xrightarrow{\eta_\sigma} \partial\Delta^{|\sigma|}(\epsilon) \xrightarrow{\rho_{|\sigma|}} \partial\Delta^{|\sigma|} \times [0,\epsilon) \xrightarrow{\eta_\sigma^{-1} \times \mathrm{Id}_{[0,\epsilon)}} \partial\sigma \times [0,\epsilon).$$

We will say that a continuous map $f\colon X \times \Delta^p \to Y \times \Delta^p$ over Δ^p (i.e., such that $\mathrm{proj}_{\Delta^p} = \mathrm{proj}_{\Delta^p} \circ f$) satisfies the *$\epsilon$-collaring condition* if for every face $\sigma \subset \Delta^p$,

$$f|_{X \times \partial\sigma(\epsilon)} \equiv f|_{X \times \partial\sigma} \times \mathrm{Id}_{[0,\epsilon)}$$

under the identifications $\rho_\sigma\colon \partial\sigma(\epsilon) \cong \partial\sigma \times [0,\epsilon)$ fixed above.

Definition 2.8 Let $V \in \mathcal{J}$. The *semisimplicial group* $\mathrm{Diff}^b_\partial(M \times V)_\bullet$ *of bounded diffeomorphisms* of $M \times V$ relative to $\partial M \times V$ has as p-simplices the set of diffeomorphisms of $\Delta^p \times M \times V$ over Δ^p which are bounded (with respect to V), that are the identity in a neighbourhood of $\Delta^p \times \partial M \times V$, and that satisfy the ϵ-collaring condition for some $0 < \epsilon \le \frac{1}{2}$. Face maps are determined by the coface maps of the cosimplicial space $\Delta^\bullet$. If we relax the condition on diffeomorphisms to be over Δ^p to only face-preserving (i.e., diffeomorphisms that send $\sigma \times M \times V$ to itself for every face $\sigma \subset \Delta^p$), we obtain the semisimplicial group $\widetilde{\mathrm{Diff}}{}^b_\partial(M \times V)_\bullet$ of *bounded block diffeomorphisms* of $M \times V$.

Warning 2.9 One could have defined the orthogonal functor $B(-)$ of Notation 2.6, for instance, to be

$$\mathcal{J}_0 \to \mathrm{Top}_*, \quad U \mapsto B|\mathrm{Diff}^b_\partial(M \times U)_\bullet|.$$

This latter rule, however, does not give rise to a continuous functor in the sense of orthogonal calculus, i.e., it is not enriched over Top_*. A way to fix this is to replace Top_* by sSet_*, $\mathcal{J}_0$ by a category $\mathcal{J}_0^\Delta$ enriched now over sSet_*, and doing orthogonal calculus for sSet_*-enriched functors $\mathcal{J}_0^\Delta \to \mathrm{sSet}_*$. This is morally the point of view taken by Weiss and Williams [1988], but orthogonal calculus for simplicially enriched functors has not yet been carried out rigorously, so we prefer to not pursue this approach. This also appears to be the main technicality in the PL case: we do not know how to define an actual *topological* (as opposed to simplicial) orthogonal functor out of spaces of bounded PL-homeomorphisms.

The simplicial models of Definition 2.8 are more convenient to work with than the point–set topological ones of Definition 2.4. Moreover, we will need some results from [Weiss and Williams 1988] that are stated in the simplicial setting, so we will have to argue that both models share the same weak homotopy type.

Proposition B.1 *There is a zigzag of weak equivalences of semisimplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M\times V)_\bullet \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M\times V)).$$

In particular, there is a zigzag of weak equivalences of group-like topological monoids connecting $|\mathrm{Diff}^b_\partial(M\times V)_\bullet|$ *and* $\mathrm{Diff}^b_\partial(M\times V)$.

We defer the proof of this proposition to Section B.1 in the appendix.

3 Proof of Theorem A

We now prove Theorem A. Section 3.1 will first reduce it to the case when P is a codimension-zero submanifold of M. Some necessary preliminaries will be presented in Section 3.2. Finally the map Φ^{Emb} of Theorem A and its connectivity will be analysed in Sections 3.3 and 3.4.

Before we move on to the next section, let us record a disjunction result for concordance embeddings known as *Hudson's concordance-implies-isotopy theorem* [1970, Theorem 2.1 and Addendum 2.1.2].

Theorem 3.1 (Hudson) *The space* $C\mathrm{Emb}(P,M)$ *is connected if* $p\le d-3$. *Equivalently, the natural map* $\pi_0(\mathrm{Emb}_{\partial_0}(P,M))\to\pi_0(\widetilde{\mathrm{Emb}}_{\partial_0}(P,M))$ *is an isomorphism.*

Remark 3.2 Hudson's theorem also holds in the PL setting [1970, Theorem 1.5]. As long as M is 1-connected if $d=\dim M=4$, it also holds in the topological setting [Pedersen 1976].

3.1 Reduction to geometric codimension-zero embeddings

Let $\iota\colon P\hookrightarrow M$ be as in the statement of Theorem A. It will be convenient to be able to assume that $P\subset M$ is a codimension-zero submanifold (though of handle codimension at least 3). The following result deals with this technicality, and shows that the difference between block and ordinary *smooth* embeddings is insensitive to the geometric codimension.

Proposition 3.3 *Let* M^d *be a compact smooth Riemannian manifold and* $\iota\colon P^p\hookrightarrow M^d$ *be a neat submanifold that is closed as a subspace. Let* $\bar\nu P$ *be the closed disk bundle of the normal bundle* ν_ι *of the embedding* ι, *and let* $\hat\iota\colon\bar\nu P\hookrightarrow M$ *be the induced embedding. Then the square*

(3-1)
$$\begin{tikzcd}
\mathrm{Emb}_{\partial_0,\hat\iota}(\bar\nu P,M) \arrow[r, "\mathrm{res}_P"] \arrow[d, hook] & \mathrm{Emb}_{\partial_0,\iota}(P,M) \arrow[d, hook] \\
\widetilde{\mathrm{Emb}}_{\partial_0,\hat\iota}(\bar\nu P,M) \arrow[r, "\widetilde{\mathrm{res}}_P"] & \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)
\end{tikzcd}$$

is homotopy cartesian. Here the subscripts ι or $\hat{\iota}$ in the embedding spaces stand for the path component consisting of embeddings isotopic to ι or $\hat{\iota}$ (relative to $\partial_0 P$).

Equivalently, by taking vertical homotopy fibres in (3-1) *and noting Hudson's theorem (Theorem 3.1) and that* res_P *and* $\widetilde{\mathrm{res}}_P$ *are surjective, there is a weak equivalence*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) \simeq \mathrm{hofib}_{\hat{\iota}}(\mathrm{Emb}_{\partial_0}(\bar{\nu}P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(\bar{\nu}P,M)).$$

Proof We will show that the horizontal homotopy fibre of the vertical inclusions in (3-1) can be identified, up to equivalence, with the identity map of the topological group $\mathrm{Aut}_{\partial_0}(\nu_\iota)$ of bundle automorphisms of ν_ι which are standard near $\partial_0 P$. In particular, the total homotopy fibre of (3-1) will be weakly contractible.

We first deal with the top horizontal homotopy fibre. Consider the fibration

$$E := \left\{ \begin{array}{ccc} \nu_\iota & \xhookrightarrow{G} & \tau_M \\ \downarrow & & \downarrow \\ P & \xhookrightarrow{\varphi} & M \end{array} \;\middle|\; \begin{array}{c} \varphi \in \mathrm{Emb}_{\partial_0,\iota}(P,M), \\ G \in \mathrm{BunInj}_{\partial_0}(\nu_\iota,\tau_M), \\ D\varphi \oplus G\colon \tau_P \oplus \nu_\iota \cong \varphi^*\tau_M \end{array} \right\} \xrightarrow{r} \mathrm{Emb}_{\partial_0,\iota}(P,M), \quad (G,\varphi) \mapsto \varphi.$$

Taking derivatives at the zero section of $\bar{\nu}P$ defines a map $D\colon \mathrm{Emb}_{\partial_0,\hat{\iota}}(\bar{\nu}P,M) \to E$ over $\mathrm{Emb}_{\partial_0,\iota}(P,M)$. A homotopy inverse $E \to \mathrm{Emb}_{\partial_0,\hat{\iota}}(\bar{\nu}P,M)$ to D can be defined using the exponential map. Therefore the homotopy fibre of res_P is equivalent to the fibre of r (observe that r is a fibration). Now $\iota^*\tau_M$ is already identified with $\tau_P \oplus \nu_\iota$, so the fibre $F := r^{-1}(\iota)$ can be described as the subspace of bundle automorphisms of $\tau_P \oplus \nu_\iota$ over P which are the identity on the tangent summand τ_P (and near $\partial_0 P$). As the space of bundle maps $\nu_\iota \to \tau_P$ over P is contractible, it follows that the inclusion $\mathrm{Aut}_{\partial_0}(\nu_\iota) \hookrightarrow F$ is a homotopy equivalence.

The argument for the bottom map of (3-1) is similar but trickier; we work with the simplicial model of block embeddings of Definition 2.8. First let ξ and π be vector bundles over spaces B and B', respectively, and fix some bundle map $I\colon \xi \to \pi$. For any closed subset $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi,\pi)_\bullet$ denote the semisimplicial set whose n-simplices consist of bundle maps $G\colon \Delta^n \times \xi \to \tau_{\Delta^n} \boxplus \pi := (\tau_{\Delta^n} \times B') \oplus (\Delta^n \times \pi)$ such that

- G agrees with $\mathbf{0}_{\Delta^n} \boxplus I$ near $\Delta^n \times \partial_0$, where $\mathbf{0}_{\Delta^n}\colon \epsilon^0_{\Delta^n} \cong \Delta^n \to \tau_{\Delta^n}$ is the inclusion as the zero section, and
- for every face $\sigma \subset \Delta^n$, we have $G(\sigma \times \xi) \subset \tau_\sigma \boxplus \pi \subset \tau_{\Delta^n} \boxplus \pi$.

Given a map $i\colon B \to B'$ which agrees with the underlying map of I on $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi,\pi;i)_\bullet$ be the semisimplicial subset consisting of those bundle maps G whose underlying map on the base spaces $\Delta^n \times B \to \Delta^n \times B'$ is $\mathrm{Id}_{\Delta^n} \times i$. Let $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi,\pi)_\bullet$ and $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi,\pi;i)_\bullet$ be the semisimplicial subsets of those bundle maps that are fibrewise injective. Then again, taking derivatives at the zero section of $\Delta^\bullet \times \bar{\nu}P$ yields a simplicial map $\widetilde{D}_\bullet$ from $\widetilde{\mathrm{Emb}}_{\partial_0,\hat{\iota}}(\bar{\nu}P,M)_\bullet$ to a semisimplicial set $\widetilde{E}_\bullet$ whose n-simplices are

$$\widetilde{E}_n := \left\{ \begin{array}{ccc} \Delta^n \times \nu_\iota & \xhookrightarrow{G} & \tau_{\Delta^n} \boxplus \tau_M \\ \downarrow & & \downarrow \\ \Delta^n \times P & \xhookrightarrow{\varphi} & \Delta^n \times M \end{array} \;\middle|\; \begin{array}{c} \varphi \in \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)_n, \\ G \in \widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota,\tau_M)_n, \\ D\varphi \oplus G\colon \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) \cong \varphi^*(\tau_{\Delta^n} \boxplus \tau_M) \end{array} \right\},$$

and whose face maps are given by restriction to face strata. The map $\tilde{r}_\bullet\colon \tilde{E}_\bullet \to \widetilde{\mathrm{Emb}}_{\partial_0}(P, M)_\bullet$ given by $\tilde{r}(G, \varphi) := \varphi$ is now a Kan fibration, and $\tilde{r}_\bullet \circ \tilde{D}_\bullet = \widetilde{\mathrm{res}}_P$. By a similar argument as in the previous case, the homotopy fibre of $\widetilde{\mathrm{res}}_P$ is equivalent to the fibre of $\tilde{r}_\bullet$. Using the canonical identification

$$(\mathrm{Id}_{\Delta^n} \times \iota)^*(\tau_{\Delta^n} \boxplus \tau_M) = \tau_{\Delta^n} \boxplus \iota^*\tau_M \cong \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota),$$

the fibre $\tilde{F}_\bullet := \tilde{r}_\bullet^{-1}(\iota)$ is isomorphic to the semisimplicial subset of $\widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota, \iota^*\tau_M; \mathrm{Id}_P)_\bullet$ of bundle maps

$$G = G_{\Delta^n} \oplus G_\tau \oplus G_\nu\colon \Delta^n \times \nu_\iota \to \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) = (\tau_{\Delta^n} \times P) \oplus (\Delta^n \times \tau_P) \oplus (\Delta^n \times \nu_\iota)$$

for which G_ν is an isomorphism. Thus

$$\tilde{F}_\bullet = \widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet \times \mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet,$$

where the boundary condition on $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ forces bundle maps to be zero near $\Delta^\bullet \times \partial_0 P$. Clearly $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ is weakly contractible; indeed given an n-cycle G in this semisimplicial set, a nullhomotopy of G is roughly given by regarding Δ^{n+1} as $(\Delta^n \times [0, 1], \Delta^n \times \{0\})$ and applying $t \cdot G$ on $\Delta^n \times \{t\}$, for $0 \le t \le 1$. Therefore $|\tilde{F}_\bullet| \simeq |\,\mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet| = \mathrm{Aut}_{\partial_0}(\nu_\iota)$, as required. □

Remark 3.4 Proposition 3.3 is false in the topological (and PL) setting. First, a locally flat embedding $\iota\colon P^p \hookrightarrow M^d$ does not always admit a normal microbundle (see [Rourke and Sanderson 1967]; they do admit one stably though [Hirsch 1966, Theorem B]). But even if it did, the statement would still not hold in general: the homotopy fibre of $\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(\bar{\nu} P, M) \to \mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(P, M)$ is a section space of a bundle over P whose fibre is the topological group $\mathrm{Top}(d, p)$ of homeomorphisms of $\mathbb{R}^d$ that fix pointwise the subspace $\mathbb{R}^p \times \{0\}$, whereas the homotopy fibre of $\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(\bar{\nu} P, M) \to \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(P, M)$ is a similar section space, but of a bundle whose fibre is the colimit

$$\widetilde{\mathrm{Top}}(d-p) := \mathrm{colim}(\mathrm{Top}(d, p) \xrightarrow{-\times\mathrm{Id}_{\mathbb{R}}} \mathrm{Top}(d+1, p+1) \xrightarrow{-\times\mathrm{Id}_{\mathbb{R}}} \mathrm{Top}(d+2, p+2) \xrightarrow{-\times\mathrm{Id}_{\mathbb{R}}} \cdots).$$

The map $\mathrm{Top}(d, p) \to \widetilde{\mathrm{Top}}(d-p)$ is *not* an equivalence. (Crucially, though, the smooth analogues $O(d-p) = O(d-p, 0) \to \cdots \to O(d+n, p+n)$ are indeed equivalences.)

To see this, consider the case $(M, P) = (D^d, D^p)$ for $p \le d-3$. Both $\mathrm{Emb}^{\mathrm{Top}}_{\partial}(D^p, D^d)$ and $\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial}(D^p, D^d)$ are contractible by the Alexander trick. However, using the topological version of Theorem A (see Remark 1.4), we will see in Remark 6.3 that the homotopy fibre of the map

$$\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \to \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^d \times D^{d-p}, D^d)$$

is not contractible. In particular, the topological analogue of the square (3-1) cannot possibly be homotopy cartesian in this case.

3.2 Last ingredients

From now on, let $\iota\colon P^d \hookrightarrow M^d$ be a codimension-zero closed embedding that meets ∂M transversely in $\partial_0 P$, and denote by p the handle dimension of P relative to $\partial_0 P$; we will write $\overline{M - P}$ instead of the isotopy equivalent manifold $M - \nu P$ to emphasise that P has codimension zero in M. It suffices to prove Theorem A in this case by Proposition 3.3. We now present the last necessary preliminary results.

3.2.1 Parametrised isotopy extension theorem The *parametrised isotopy extension theorem* states that for $\varphi_t\colon P \hookrightarrow M$ any continuous family of embeddings parametrised by $t \in \Delta^k$ (with P *compact*), there exists a continuous family of diffeomorphisms $\{\phi_t\}_{t \in \Delta^k}$ of M (which are the identity away from a compact set of M) such that $\phi_0 = \mathrm{Id}_M$ and $\phi_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times \Delta^k$. Moreover, if $K \subset \Delta^k$ is some contractible subcomplex containing the 0-th vertex and $\{\phi'_t\}_{t \in K}$ is another continuous family of diffeomorphisms of M parametrised by K such that $\phi'_0 = \mathrm{Id}_M$ and $\phi'_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times K$, then we can arrange $\{\phi_t\}_{t \in \Delta^k}$ as above to agree with $\{\phi'_t\}_{t \in K}$ on K. A consequence of this fact due to Palais [1960] (see [Lima 1964] for a simple proof) is that the restriction map $\mathrm{Diff}_\partial(M) \to \mathrm{Emb}_{\partial_0}(P, M)$ is a locally trivial fibre bundle with $\mathrm{Diff}_\partial(\overline{M - P})$ as fibre. Such a fibration can be delooped to the homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P, M) \to B\mathrm{Diff}_\partial(\overline{M - P}) \to B\mathrm{Diff}_\partial(M), \tag{3-2}$$

where the subscript $\langle\iota\rangle$ stands for the union of all the components in $\mathrm{Emb}_{\partial_0}(P, M)$ that contain embeddings of the form $\phi \circ \iota$ for $\phi \in \mathrm{Diff}_\partial(M)$. By replacing P and M in (3-2) by $P \times I$ and $M \times I$, and modifying the boundary conditions, we get a similar homotopy fibre sequence

$$C\mathrm{Emb}(P, M) \to BC(\overline{M - P}) \to BC(M). \tag{3-3}$$

Note that $C\mathrm{Emb}(M, P)$ is connected by Hudson's theorem (Theorem 3.1). Finally, there is a block analogue of (3-2).

Proposition 3.5 *There is a homotopy fibre sequence*

$$\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P, M) \to B\widetilde{\mathrm{Diff}}_\partial(\overline{M - P}) \to B\widetilde{\mathrm{Diff}}_\partial(M). \tag{3-4}$$

Proof There is a right action of the simplicial group $\widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet$ on $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$; we will write $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet$ for the simplicial set of (levelwise) cosets of this right action. The geometric realisation $|\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet|$ of this simplicial set is homotopy equivalent to the homotopy fibre of the right map of (3-4), so it suffices to show that the action map

$$a\colon \widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet \to \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P, M)_\bullet, \quad [\phi] \mapsto \phi \circ \iota,$$

is an isomorphism. It is visibly injective, for if $\phi \circ \iota = \psi \circ \iota$ for $\phi, \psi \in \widetilde{\mathrm{Diff}}_\partial(M)_\bullet$, then $\psi^{-1} \circ \phi \in \widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet$ and hence $[\psi] = [\psi \circ \psi^{-1} \circ \phi] = [\phi]$ in $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M - P})_\bullet$.

For surjectivity, let φ be some k-simplex in $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P, M)_\bullet$. Then there exists some $\phi \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ for which φ and $\phi \circ \iota$ lie in the same component in $\widetilde{\mathrm{Emb}}_{\partial_0}(P, M)_\bullet$. Then $\varphi' := \phi^{-1} \circ \varphi \in \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P, M)_k$ and, in fact, we can arrange that its restriction to the 0-th vertex φ'_0 is ι by rechoosing ϕ (if necessary) using the isotopy extension theorem. Applying the isotopy extension theorem to φ' restricted to each of the faces that contains the 0-th vertex, inductively on the dimension of the face, we obtain some $\Phi' \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ such that $\Phi'|_{P \times \Delta^k} \equiv \varphi'$. Then $\Phi := \phi \circ \Phi' \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ is such that $\Phi|_{P \times \Delta^k} \equiv \varphi$, as desired. □

Remark 3.6 There also exist topological and PL versions of the isotopy extension theorem (see [Edwards and Kirby 1971, Corollary 1.4] and [Hudson 1966], respectively). The same proof as above also works in the topological or PL setting.

Remark 3.7 (speculative) Weiss and Williams [1988, Section 1] point out that an analogue of the (parametrised) isotopy extension theorem in the bounded setting does not hold (see [Hirsch 1976, Chapter 8, Exercise 9] for a counterexample in codimension 2). However, we believe that a weaker version of the theorem should still hold: namely, for $V \in \mathcal{J}_0$ define the bounded embedding space $\mathrm{Emb}^b_{\partial_0}(P \times V, M \times V)$ as in Definition 2.4. Then there should be a homotopy fibre sequence

$$\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times V, M \times V) \to E(V) = B\mathrm{Diff}^b_\partial(\overline{M-P} \times V) \to B(V) = B\mathrm{Diff}^b_\partial(M \times V),$$

where $E(-)$ and $B(-)$ are as in Notation 2.6. We will not give a proof of this claim, as it seems rather technical and we will not need it for the argument of Theorem A. The reader may however find it useful to think of the orthogonal functor $F(-) := \mathrm{hofib}(E(-) \to B(-))$ as $\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times (-), M \times (-))$.

3.2.2 Alexander trick-like equivalences For $V \in \mathcal{J}_0$, let $D(V) \subset V$ denote the corresponding closed unit disk (so that $D^k = D(\mathbb{R}^k)$). The following is proved in Propositions 1.8, 1.10, and 1.12 of [Weiss and Williams 1988]. Even though we state it for the orthogonal functor $B(-)$ of Notation 2.6, it of course holds for $E(-)$ too.

Proposition 3.8 *For $V \in \mathcal{J}_0$, the Alexander trick-like map*

$$\mathrm{alex}\colon C(M \times D(V)) \xrightarrow{\sim} \Omega^{V \oplus \mathbb{R}} B^{(1)}(V) = \Omega^{V \oplus \mathbb{R}}(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V))$$

is a weak equivalence. Moreover, there is a homotopy commutative diagram

$$\begin{tikzcd}
C(M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{alex}", "\wr"'] & C(M \times D(V) \times D^1) \arrow[r, equal, "\sim"] & C(M \times D(V \oplus \mathbb{R})) \arrow[d, "\mathrm{alex}", "\wr"'] \\
\Omega^{V \oplus \mathbb{R}} B^{(1)}(V) \arrow[rr, "s_V^\vee"] & & \Omega^{V \oplus \mathbb{R}^2} B^{(1)}(V \oplus \mathbb{R})
\end{tikzcd} \tag{3-5}$$

where Σ denotes the usual concordance stabilisation map and $s_V^\vee$ is the adjoint of the structure map (2-2) *for the orthogonal spectrum $\Theta B^{(1)} = \boldsymbol{H}(M)$.*

We will describe the map "alex" below, but we first discuss this statement and its consequences:

Remark 3.9 Both the domain and codomain of the map "alex" of Proposition 3.8 are group-like $\mathbb{E}_1$-spaces; the former by composition of concordance diffeomorphisms, and the latter by the loop space structure induced by $\Omega^{\mathbb{R}}(-)$. In Section 5.2, we construct a (nonconnected) delooping of this map; see (5-3).

It seems likely that the homotopy commutative square (3-5) can also be delooped in a similar manner. Proving this, however, is quite technical and we will not need it in any case. What we will need instead is the observation that if M is replaced by $M \times I$ in Proposition 3.8, the whole statement can be delooped once with respect to the $\mathbb{E}_1$-structures induced by *stacking in the I-direction*. This is straightforward to check from the proofs in [Weiss and Williams 1988].

Remark 3.10 By Proposition 3.8 there is a natural (for codimension-zero embeddings) equivalence

$$\Omega^{\infty+1}\boldsymbol{H}(M) \simeq \mathcal{C}(M) := \operatorname*{hocolim}_k C(M \times D^k).$$

As pointed out in Remark 3.9, this equivalence can be delooped once if we replace M by $M \times I$. Moreover, the (nonequivariant) homotopy types of both $\boldsymbol{H}(-)$ and $\mathcal{C}(-)$ are invariant under crossing with I, namely, there are natural equivalences $\boldsymbol{H}(M \times I) \simeq \boldsymbol{H}(M)$ (by Lemma 5.15 below) and $\mathcal{C}(M \times I) \simeq \mathcal{C}(M)$ (by definition). By this line of reasoning, we obtain natural equivalences

$$\Omega_0^\infty \boldsymbol{H}(M) \simeq \Omega_0^\infty \boldsymbol{H}(M \times I) \simeq B\mathcal{C}(M \times I) \simeq B\mathcal{C}(M).$$

By [Vogell 1985, Proposition 2.1], $B\mathcal{C}(M)$ is also naturally equivalent to the basepoint component of the space of stable h-cobordisms $\mathcal{H}(M)$ of Remark 2.7, and by [Weiss and Williams 1988, Corollary 5.6] and the s-cobordism theorem, the groups $\pi_0^s(\boldsymbol{H}(M))$ and $\pi_0(\mathcal{H}(M))$ are both isomorphic to the Whitehead group $\mathrm{Wh}(\pi_1 M)$. Since there is a (nonnatural) equivalence *of spaces* $\Omega^\infty X \simeq \Omega_0^\infty X \times \pi_0^s(X)$, we obtain the promised equivalence (2-11)

$$\Omega^\infty \boldsymbol{H}(M) \simeq \mathcal{H}(M).$$

This, of course, ought to be an equivalence of infinite loop spaces, but that seems to be more difficult to see. Making the above equivalence natural for codimension-zero embeddings requires establishing the (nonconnected) delooped analogues of (3-5), using the delooped Alexander trick-like maps (5-3) constructed in Section 5.2 (which satisfy this naturality by construction). However, this is a rather tedious task that we do not undertake.

Proof of Proposition 3.8 This is proved in Propositions 1.8 and 1.10 and Lemma 1.12 of [Weiss and Williams 1988]. Let us just explain how the Alexander trick-like map

$$\text{alex}\colon C(M \times D(V)) \to \Omega^{V \oplus \mathbb{R}}(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V))$$

is defined: given a concordance diffeomorphism $\phi\colon M \times D(V) \times I \cong M \times D(V) \times I$, extend it by $\phi|_{M \times D(V) \times \{1\}} \times \mathrm{Id}_{[1,+\infty)}$ on $M \times D(V) \times [1,+\infty)$ and by the identity elsewhere to obtain a bounded self-diffeomorphism $\hat{\phi}$ of $M \times V \oplus \mathbb{R}$; then shift it along $V \oplus \mathbb{R}$ to obtain a $(V \oplus \mathbb{R})$-fold loop in $\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)$. We refer to [loc. cit.] for the rest of the proofs. □

Taking fibres of Proposition 3.8 for $E(-)$ and $B(-)$ yields the first part of the analogous result for $F(-)$.

Proposition 3.11 *For $V \in \mathcal{J}_0$, there are weak equivalences*

$$\text{alex}\colon \Omega\, C\mathrm{Emb}(P \times D(V), M \times D(V)) \xrightarrow{\sim} \Omega^{1+V} F^{(1)}(V),$$

making the diagram

(3-6)

$$\begin{tikzcd}
\Omega\, C\mathrm{Emb}(P \times D(V), M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\wr"', "\text{alex}"] & \Omega\, C\mathrm{Emb}(P \times D(V \oplus \mathbb{R}), M \times D(V \oplus \mathbb{R})) \arrow[d, "\wr"', "\text{alex}"] \\
\Omega^{1+V} F^{(1)}(V) \arrow[r, "s_V^\vee"] & \Omega^{1+V\oplus\mathbb{R}} F^{(1)}(V \oplus \mathbb{R})
\end{tikzcd}$$

commute up to homotopy, where Σ is the concordance embedding stabilisation map of Section 1.1.1. Moreover, if $p \le d-3$, there is a natural equivalence

$$\Omega^\infty(\mathbf{CE}(P,M)) := \Omega^\infty(\Theta F^{(1)}) \simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P,M) := \operatorname*{hocolim}_k C\mathrm{Emb}(P\times D^k, M\times D^k). \tag{3-7}$$

To establish (3-7), we will need the following result, which was suggested to us by Manuel Krannich:

Lemma 3.12 *For $p \le d-3$ and $d+n \ge 5$, the space $\Theta F_n^{(1)} = F^{(1)}(\mathbb{R}^n)$ is n-connected.*

Proof It suffices to show that the map $\Theta E_n^{(1)} \to \Theta B_n^{(1)}$, call it λ, is such that $\pi_*(\lambda)$ is

(a) surjective if $* = n+1$,

(b) injective if $* = 0$, and

(c) an isomorphism if $1 \le * \le n$.

For (a), observe that by Proposition 3.8 $\Omega^{n+1}\lambda$ is, up to equivalence, the natural map of concordance spaces $C(\overline{M-P}\times D^n) \to C(M\times D^n)$. By exactness of

$$\pi_0(C(\overline{M-P}\times D^n)) \xrightarrow{\pi_{n+1}(\lambda)} \pi_0(C(M\times D^n)) \to \pi_0(C\mathrm{Emb}(P\times D^n, M\times D^n)) = *,$$

where the equality on the right is the statement of Hudson's theorem (Theorem 3.1), it follows that $\pi_{n+1}(\lambda)$ is surjective.

For (b) and (c), consider the commutative diagram

$$\begin{tikzcd}
\Theta E_n^{(1)} \arrow[r, "\lambda"] \arrow[d, "\text{stab.}"] & \Theta B_n^{(1)} \arrow[d, "\text{stab.}"] \\
\Omega^\infty(\Sigma^n \Theta E^{(1)}) \arrow[r, "\eta"] & \Omega^\infty(\Sigma^n \Theta B^{(1)})
\end{tikzcd}$$

We claim that the map of (nonconnective) spectra $\boldsymbol{H}(\overline{M-P}) \to \boldsymbol{H}(M)$ underlying η is an isomorphism in π_*^s for $* \le 0$; indeed, the inclusion $\overline{M-P} \hookrightarrow M$ is 2-connected and $\pi_*^s(\boldsymbol{H}(-))$ for $* \le 0$ (see (3-9)) only depends on $\pi_1(-)$ by [Weiss and Williams 1988, Corollary 5.6]. So η itself satisfies (b) and (c). By [loc. cit., Corollary 5.8], both vertical maps are injective in π_0 and isomorphisms in $\pi_{1\le *\le n}$ if $d+n \ge 5$. □

Proof of Proposition 3.11 It remains to deloop the natural equivalence

$$\Omega^{\infty+1}(\mathbf{CE}(P,M)) \simeq \Omega\,\mathcal{C}\mathcal{E}\mathrm{mb}(P,M)$$

obtained from the squares (3-6), so as to yield (3-7). We do this as in Remark 3.10.

Both $\Omega^\infty\,\mathbf{CE}(P,M)$ and $\mathcal{C}\mathcal{E}\mathrm{mb}(P,M)$ are connected under the codimension assumption — the former by Lemma 3.12 and the latter by Hudson's theorem (Theorem 3.1). Just like in Remark 3.9, the homotopy commutative square (3-6) can be delooped if we replace $\iota\colon P \hookrightarrow M$ by $\iota\times\mathrm{Id}_I\colon P\times I \hookrightarrow M\times I$. Finally, there are natural equivalences $\mathbf{CE}(P\times I, M\times I) \simeq \mathbf{CE}(P,M)$ (by Lemma 5.15) and $\mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I) \simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P,M)$ (by definition). We thus obtain the desired chain of natural equivalences

$$\Omega^\infty\,\mathbf{CE}(P,M) \simeq \Omega^\infty\,\mathbf{CE}(P\times I, M\times I) \simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I) \simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P,M). \qquad \square$$

3.3 The map Φ^{Emb} of Theorem A

Recall the map Φ^F_∞ of Proposition 2.2 for $F(-)$. Noting that $F(0) \simeq \mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M)$ by the isotopy extension sequence (3-2), this map, up to equivalence, takes the form

$$\Phi^F_\infty\colon \mathrm{hofib}(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \to F(\mathbb{R}^\infty)) \to \Omega^\infty(\mathbf{CE}(P,M)_{hC_2}).$$

To obtain Φ^{Emb}, we need to replace $F(\mathbb{R}^\infty)$ by $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)$ above (and deal with some path-component considerations). This turns out to be possible by a principle similar to that of [Weiss and Williams 1988, Remark 3.5].

Proposition 3.13 *There is a map*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) \to \mathrm{hofib}(F(0) \to F(\mathbb{R}^\infty))$$

which is an equivalence if $d \geq 5$ and $d-p \geq 3$. If $d=4$ and $d-p \leq 3$, the map becomes an equivalence upon looping once.

Proof First observe that the map

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)) \to \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M))$$

is an inclusion of path components, and it is an equivalence if $d-p \leq 3$ by Hudson's theorem (Theorem 3.1). We obtain a map in the opposite direction by sending every component that is not hit to the basepoint. Therefore it suffices to construct a homotopy commutative diagram

$$\begin{tikzcd}
F(0) \arrow[r] & F(\mathbb{R}^\infty) \arrow[r, "i", "\sim"'] & \widetilde{F}(\mathbb{R}^\infty) \\
\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[u, "\wr"', "(3\text{-}2)"] \arrow[rr, hook] & & \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[u, hook, "j"']
\end{tikzcd} \tag{3-8}$$

where the map j will be an inclusion of path components if $d-p \leq 3$ and $d \geq 5$. (This will be the case when $d=4$ after looping.)

To that end, let $\mathcal{J}_0^\delta$ denote the underlying ordinary category of the topological category $\mathcal{J}_0$, and write $\widetilde{E}(-)$ and $\widetilde{B}(-)$ for the functors $\mathcal{J}_0^\delta \to \mathrm{Top}_*$ given by

$$\widetilde{E}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V)_\bullet|, \quad \widetilde{B}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(M \times V)_\bullet|.$$

Set $\widetilde{F}(-) := \mathrm{hofib}(\widetilde{E}(-) \to \widetilde{B}(-))$. Then the map i of (3-8) arises as the map on homotopy fibres in

$$\begin{tikzcd}
F(\mathbb{R}^\infty) \arrow[r] \arrow[d, dashed, "i"] & E(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] \arrow[d, hook, "\wr"] & B(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty) \arrow[d, hook, "\wr"] \\
\widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] & \widetilde{B}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(M \times \mathbb{R}^\infty)
\end{tikzcd}$$

The middle and right vertical maps are equivalences by [Weiss and Williams 1988, Theorem B], so i is too by the five lemma.

The map j of (3-8) arises as the map on homotopy fibres in

$$\begin{tikzcd}
\widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) \arrow[r] & \widetilde{B}(\mathbb{R}^\infty) \\
\widetilde{F}(0) \simeq \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[r] \arrow[u, dashed, "j"] & \widetilde{E}(0) = B\widetilde{\mathrm{Diff}}_\partial(\overline{M-P}) \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "(\dagger)"] & \widetilde{B}(0) = B\widetilde{\mathrm{Diff}}_\partial(M) \arrow[u, hook]
\end{tikzcd}$$

Then the square (3-8) is the homotopy fibre of the map between the similar (strictly commutative) squares associated to $E(-)$ and $B(-)$, and so it is homotopy commutative by construction.

It remains to show that j is an equivalence or, equivalently, that the square (†) is homotopy cartesian. Write $\widetilde{F}^{(1)}(V) := \mathrm{hofib}(\widetilde{F}(V) \to \widetilde{F}(V \oplus \mathbb{R}))$, and similarly for $\widetilde{E}^{(1)}(V)$ and $\widetilde{B}^{(1)}(V)$. In other words,

$$\widetilde{E}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}{}^b_\partial(\overline{M-P} \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}{}^b_\partial(\overline{M-P} \times V)}, \quad \widetilde{B}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}{}^b_\partial(M \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}{}^b_\partial(M \times V)}.$$

For a group π and an integer $j \le 1$, set $\kappa_j(\pi) := \pi_j^s(\mathbf{Wh}^{\mathrm{Diff}}(B\pi))$. More explicitly,

$$\kappa_j(\pi) = \begin{cases} \mathrm{Wh}_1(\pi) & \text{if } j = 1, \\ \widetilde{K}_0(\mathbb{Z}\pi) & \text{if } j = 0, \\ K_j(\mathbb{Z}\pi) & \text{if } j \le -1. \end{cases} \tag{3-9}$$

It was shown in [Weiss and Williams 1988, Corollary 5.5] — see also [Anderson and Pedersen 1983] — that, for a certain C_2-action on $\kappa_j(\pi)$, there are maps for $n \ge 0$

$$\beta\colon \pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \to H_*(C_2; \kappa_{1-n}(\pi_1(M-P))), \quad \beta\colon \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \to H_*(C_2; \kappa_{1-n}(\pi_1(M))),$$

which, as long as $d + n \ge 5$, are injective if $* = 0$ and isomorphisms if $* \ge 1$. Moreover, it is not difficult to see from its proof that these are compatible, in the sense that the square

$$\begin{tikzcd}
\pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \arrow[r] \arrow[d, "\beta"] & \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \arrow[d, "\beta"] \\
H_*(C_2; \kappa_{1-n}(\pi_1(M-P))) \arrow[r, "\cong"] & H_*(C_2; \kappa_{1-n}(\pi_1(M)))
\end{tikzcd}$$

is commutative. The lower horizontal map is an isomorphism because the fundamental groups of $M - P$ and M can be identified under the obvious inclusion by the assumption that $p \le d - 3$. Hence, as $\widetilde{F}^{(1)}(V) \to \widetilde{E}^{(1)}(V) \to \widetilde{B}^{(1)}(V)$ is a homotopy fibre sequence for all V, it follows that $\widetilde{F}^{(1)}(\mathbb{R}^n)$ is weakly contractible for all $n \ge 0$ with $d + n \ge 5$. If $d \ge 5$, using the homotopy fibre sequences

$$\mathrm{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^n)) \to \mathrm{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^{n+1})) \to \widetilde{F}^{(1)}(\mathbb{R}^n) \simeq *$$

for $n \ge 0$, we must have by induction that $\mathrm{hofib}(j\colon \widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^\infty))$ is contractible, i.e., that j is an inclusion of path components, as desired.

If $d = 4$, the argument above shows that $\widetilde{F}^{(1)}(\mathbb{R}^n) \simeq *$ for $n \geq 1$. We claim further that $\Omega \widetilde{F}^{(1)}(0)$ is also contractible — if so, looping (3-8) once, we obtain a commutative diagram

$$\begin{tikzcd}
\Omega F(0) \arrow[r] & \Omega F(\mathbb{R}^\infty) \arrow[r, "i", "\sim"'] & \Omega \widetilde{F}(\mathbb{R}^\infty) \\
\Omega \operatorname{Emb}_{\partial_0,\langle\iota\rangle}(P, M) \arrow[u, "\wr"', "(3\text{-}2)"] \arrow[rr, hook] & & \Omega \widetilde{\operatorname{Emb}}_{\partial_0,\langle\iota\rangle}(P, M) \arrow[u, hook, "\Omega j"']
\end{tikzcd}$$

where the rightmost vertical map Ωj is an inclusion of path components, and hence the conclusion of the statement would hold after looping once if $d - p \leq 3$.

To show that $\Omega \widetilde{F}^{(1)}(0) \simeq *$, it suffices to prove that both $\Omega \widetilde{E}^{(1)}(0)$ and $\Omega \widetilde{B}^{(1)}(0)$ are contractible. This follows from the block analogue of Proposition 3.8 (whose proof in [Weiss and Williams 1988, Section 1] applies verbatim in the block setting), which makes the identifications $\Omega \widetilde{E}^{(1)}(0) \simeq \widetilde{C}(\overline{M - P})$ and $\Omega \widetilde{B}^{(1)}(0) \simeq \widetilde{C}(M)$. As the space of block concordances of a manifold $\widetilde{C}(X)$ is well-known to be contractible, the claim follows. □

Definition 3.14 The map Φ^{Emb} of Theorem A is the zigzag

$$\operatorname{hofib}_\iota(\operatorname{Emb}_{\partial_0}(P, M) \hookrightarrow \widetilde{\operatorname{Emb}}_{\partial_0}(P, M)) \to \operatorname{hofib}(F(0) \to F(\mathbb{R}^\infty)) \qquad \text{by Proposition 3.13,}$$

$$\xrightarrow{\Phi^F_\infty} \Omega^\infty(\Theta F^{(1)}_{hO(1)}) = \Omega^\infty(\mathbf{CE}(P, M)_{hC_2}) \qquad \text{by (2-4) and (2-10).}$$

By Proposition 3.13, the first map is an equivalence if $d - p \geq 3$ and $d \geq 5$ (or $d = 4$ after looping once).

3.4 Connectivity of the map Φ^{Emb}

Let $d - p \geq 3$. In this section we show that the map Φ^{Emb} just defined is $\phi_{C\mathrm{Emb}}(M, P)$-connected, at last establishing Theorem A (modulo the proof of Proposition B.1). First assume that $d \geq 5$, so that by Proposition 3.13, the connectivity of Φ^{Emb} is that of Φ^F_∞; we show by induction on $n \geq 0$ that the maps Φ^F_n of Proposition 2.2 are at least $\phi_{C\mathrm{Emb}}(M, P)$-connected. Note that this is clear for $n = 0$, as both the domain and codomain are contractible.

Suppose now that Φ^F_n is $\phi_{C\mathrm{Emb}}(M, P)$-connected for some $n \geq 0$. To show that Φ^F_{n+1} has this connectivity, it suffices to show that the map $\text{stab.}\colon \Theta F^{(1)}_n \to \Omega^\infty(\Sigma^n \Theta F^{(1)})$ of Proposition 2.2 is $(\phi_{C\mathrm{Emb}}(M, P){+}n)$-connected. But $\Theta F^{(1)}_n$ is n-connected by Lemma 3.12 and $\Sigma^n \Theta F^{(1)}$ is $(n{+}1)$-connective. So it suffices to show that $\Omega^{n+1}(\text{stab.})$ is $(\phi_{C\mathrm{Emb}}(M, P){-}1)$-connected. This follows from the homotopy commutative diagram

$$\begin{tikzcd}
\Omega\, C\mathrm{Emb}(P \times D^n, M \times D^n) \arrow[r, hook] \arrow[d, "\wr"', "\text{alex}"] & \Omega\, C\mathcal{E}\mathrm{mb}(P, M) \arrow[d, "\wr"', "(3\text{-}7)"] \\
\Omega^{n+1}\Theta F^{(1)}_n \arrow[r, "\Omega^{n+1}(\text{stab.})"] & \Omega^{\infty+1}(\Theta F^{(1)})
\end{tikzcd} \tag{3-10}$$

By definition, the connectivity of the top horizontal map is

$$\phi_{C\mathrm{Emb}}(M \times D^n, P \times D^n) - 1 \geq \phi_{C\mathrm{Emb}}(M, P) - 1.$$

One obtains the above homotopy commutative diagram from stacking together squares of the form (3-6) with $V=\mathbb{R}^k$ for $k\geq n$. This establishes Theorem A when $d\geq 5$.

Now if $d=4$, we shall show that $\Omega\Phi^{\mathrm{Emb}}$ is $(\phi_{C\mathrm{Emb}}(M,P)-1)$-connected — this would show that Φ^{Emb} is indeed $\phi_{C\mathrm{Emb}}(M,P)$-connected, as both the domain and codomain of the map are connected (the former by Hudson's theorem (Theorem 3.1), and the latter as $\mathbf{CE}(P,M)$ is 1-connective by Lemma 3.12, and $(-)_{hC_2}$ preserves connectivity). By (3-8), the connectivity of $\Omega\Phi^{\mathrm{Emb}}$ is that of $\Omega\Phi^F_\infty$ so, just like before, we need only show that the maps $\Omega\,\mathrm{stab.}\colon\Omega\Theta F_n^{(1)}\to\Omega^{\infty+1}(\Sigma^n\Theta F^{(1)})$ are $(\phi_{C\mathrm{Emb}}(M,P)+n-1)$-connected for $n\geq 0$. When $n=0$, this follows from the homotopy commutative diagram (3-10) (for $n=0$), and for $n\geq 1$, the same argument as before applies since $\Theta F_n^{(1)}$ is indeed n-connected for $n\geq 1$ when $d\geq 4$. This concludes the proof of Theorem A. □

4 A splitting result for embedding spaces and the Gromoll filtration

In this section we derive, as a consequence of Theorem A, a general splitting result[6] for embedding spaces of manifolds with interval factors. This will be used for the splitting part of Theorem B. Later we discuss consequences for the Gromoll filtration of embedding spaces (see Definition 4.5). Throughout, let $\iota\colon P^p\subset M^d$ be as in the statement of Theorem A.

For $D(-)$ any of $\mathrm{Diff}^b_\partial(-\times V)$ with $V\in\mathcal{J}_0$, $\widetilde{\mathrm{Diff}}_\partial(-)$ or $C(-)$, there are *graphing maps*

$$\Gamma\colon\Omega D(M)\to D(M\times I)$$

given (roughly) by regarding a 1-parameter family of automorphisms of M as an automorphism of $M\times I$ itself. These are natural with respect to codimension-zero embeddings. Moreover, these maps can be delooped as, up to homotopy, they intertwine the (group-like) $\mathbb{E}_1$-structures of concatenating loops for the domain, and stacking automorphisms in the I-direction for the codomain. There are similar maps

$$\Gamma\colon\Omega E(P,M)\to E(P\times I,M\times I) \tag{4-1}$$

for $E(-,-)$ denoting any of $\mathrm{Emb}_{\partial_0}(-,-)$, $\widetilde{\mathrm{Emb}}_{\partial_0}(-,-)$, $\mathrm{Emb}^{(\sim)}_{\partial_0}(-,-)$ (see (1-4)), or $C\mathrm{Emb}(-,-)$. In what follows, we will write Γ for any map of this same nature.

Remark 4.1 Most of the functors $D(-)$ and $E(-,-)$ above either admit a point–set topological model or a simplicial model. In the first case, the graphing maps just introduced are really zigzags of maps

$$\Omega E(P,M)\xleftarrow{\simeq}\Omega^{\mathrm{col,sm}}E(P,M)\xrightarrow{\Gamma}E(P\times I,M\times I),$$

where, for X a pointed (Fréchet) manifold, here $\Omega^{\mathrm{col,sm}}X$ stands for the space of smooth loops $\gamma\colon S^1\to X$ which are *collared* in the sense that there exists some neighbourhood of $1\in S^1$ which is sent by γ to the basepoint in X. The inclusion $\Omega^{\mathrm{col,sm}}E(P,M)\hookrightarrow\Omega E(P,M)$ is an equivalence by smooth approximation of continuous functions.

---

[6] This should be compared with the analogous result of Burghelea and Lashof [1982, Corollary E].

In the simplicial case, the graphing maps Γ are the geometric realisations of the simplicial maps

$$\Gamma_\bullet \colon (\Omega E(P,M))_\bullet \to E(P\times I, M\times I)_\bullet$$

that send a q-simplex in $(\Omega E(P,M))_\bullet$ (seen as a $(q+1)$-simplex $g\in E(P,M)_\bullet$ whose 0-th face and vertex are the basepoint $*\in E(P,M)_\bullet$) to the q-simplex in $E(P\times I, M\times I)_\bullet$ obtained from g by expanding out the 0-th vertex of Δ^{q+1} to a q-dimensional simplex (i.e., regarding Δ^{q+1} as $(\Delta^q\times I, \Delta^q\times\{0\})$).

In the cases when $D(-)$ or $E(-,-)$ admit both models, one verifies that these two graphing maps agree up to homotopy. We ignore both of these technicalities in most of what follows.

4.1 Splitting results

Observe that there is a graphing map

$$\Gamma^{(\sim)}\colon \Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) \to \mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I) \tag{4-2}$$

obtained as the homotopy fibre of the ordinary and block graphing maps.

Proposition 4.2 *If $d-p\geq 3$, then the pseudoisotopy graphing map $\Gamma^{(\sim)}$ of* (4-2) *is nullhomotopic after localising away from* 2 *and taking $(\phi_{C\mathrm{Emb}}(d+1,p+1)-1)$-th Postnikov sections.*

Proof By Proposition 3.3, we may assume that $\dim P = \dim M = d$. Then, resembling Notation 2.6, let $\Omega F(-)$ and $FI(-)$ denote the orthogonal functors given by

$$\begin{aligned}\Omega F(V) &:= \Omega\,\mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M-P}\times V)\to B\mathrm{Diff}^b_\partial(M\times V)),\\ FI(V) &:= \mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M-P}\times I\times V)\to B\mathrm{Diff}^b_\partial(M\times I\times V)).\end{aligned}$$

By taking fibres of the (delooped) graphing maps introduced at the beginning of the section, we obtain a natural transformation $\Gamma\colon \Omega F(-)\to FI(-)$ of orthogonal functors, giving rise to a map of $O(1)$-spectra $\Gamma\colon \Theta(\Omega F)^{(1)}\to \Theta FI^{(1)}$ and a commutative diagram

$$\begin{tikzcd}
\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I) \arrow[r, "\Phi^{FI}_\infty"] & \Omega^\infty(\Theta FI^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] & \Omega^\infty\Theta FI^{(1)} \\
\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) \arrow[r, "\Phi^{\Omega F}_\infty"] \arrow[u, "\Gamma"] & \Omega^\infty(\Theta(\Omega F)^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] \arrow[u, "\Gamma"] & \Omega^\infty\Theta(\Omega F)^{(1)} \arrow[u, "\Gamma"]
\end{tikzcd} \tag{4-3}$$

where $\mathrm{Trf}_{O(1)}$ is the $O(1)$-*transfer map*. This map is injective in the homotopy category of infinite loop spaces at odd primes (it splits the quotient map $X\to X_{hC_2}$). So since Φ^{FI}_∞ is $\phi_{C\mathrm{Emb}}(d+1,p+1)$-connected by Section 3.4 (and thus becomes an equivalence after taking $(\phi_{C\mathrm{Emb}}(d+1,p+1)-1)$-th Postnikov sections), it will suffice to show that the rightmost vertical map in (4-3) is nullhomotopic. By (3-7), we have that $\Omega^\infty\Theta(\Omega F)^{(1)}\simeq \Omega\,\mathcal{C}\mathcal{E}\mathrm{mb}(P,M)$ and $\Omega^\infty\Theta FI^{(1)}\simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I)$; under these equivalences, the right vertical map in (4-3) then becomes the graphing map

$$\Gamma\colon \Omega\,\mathcal{C}\mathcal{E}\mathrm{mb}(P,M)\to \mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I). \tag{4-4}$$

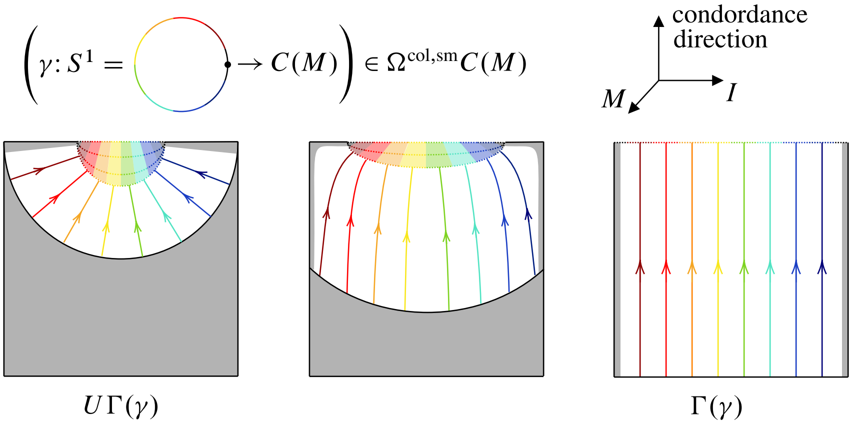

Figure 1: Images of $\gamma \in \Omega C(M)$ under the graphing maps Γ and $U\Gamma$, and the homotopy between them. The concordances are equal to the identity on grey shaded regions.

This is because both the concordance stabilisation map and the Alexander trick-like map of Proposition 3.11 that give rise to the previous equivalences commute on the nose with the graphing maps, i.e., the following diagrams commute:

$$\begin{tikzcd}
\Omega\, C\mathrm{Emb}(P \times D^{n+1}, M \times D^{n+1}) \arrow[r, "\Gamma"] & C\mathrm{Emb}(P \times I \times D^{n+1}, M \times I \times D^{n+1}) \\
\Omega\, C\mathrm{Emb}(P \times D^{n}, M \times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\Sigma"] & C\mathrm{Emb}(P \times I \times D^{n}, M \times I \times D^{n}) \arrow[u, "\Sigma"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^n \Theta(\Omega F)^{(1)}_n \arrow[r, "\Gamma"] & \Omega^n \Theta F I^{(1)}_n \\
\Omega\, C\mathrm{Emb}(P \times D^{n}, M \times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\mathrm{alex}", "\wr"'] & C\mathrm{Emb}(P \times I \times D^{n}, M \times I \times D^{n}) \arrow[u, "\mathrm{alex}", "\wr"']
\end{tikzcd}$$

So in order to show that (4-4) is nullhomotopic, it suffices to argue that it is so unstably, i.e., that the graphing maps

$$\Gamma: \Omega\, C\mathrm{Emb}(P \times D^n, M \times D^n) \to C\mathrm{Emb}(P \times I \times D^n, M \times I \times D^n)$$

are nullhomotopic for all $n \geq 0$. Replacing $M \times D^n$ by M, we may assume $n = 0$. This claim is a consequence of the following trick, due to Oscar Randal-Williams: there is a "U-shaped graphing map"

$$U\Gamma: \Omega\, C\mathrm{Emb}(P, M) \to C\mathrm{Emb}(P \times I, M \times I),$$

which is homotopic to the standard Γ by pulling down the U-shape to the base of the concordance. This homotopy is illustrated in Figure 1, where we replace $C\mathrm{Emb}(-,-)$ by standard concordances $C(-)$ because it is easier to depict, but the idea is the same. Observe that, throughout the homotopy, there are no issues about smoothness in the upper corners because the concordances are equal to the identity near these. Here we are explicitly using the collared condition imposed by the functor $\Omega^{\mathrm{col,sm}}(-)$; see Remark 4.1.

But clearly $U\Gamma$ factors through the path space $\mathrm{Map}(I, C\mathrm{Emb}(P,M))$, and hence there is a homotopy commutative diagram

$$\begin{tikzcd}
C\mathrm{Emb}(P\times I, M\times I) \arrow[r, equal] & C\mathrm{Emb}(P\times I, M\times I) & & \\
 & & \mathrm{Map}(I, C\mathrm{Emb}(P,M)) \arrow[ul, "U\Gamma"'] \arrow[r, "\mathrm{ev}_0", "\sim"'] & C\mathrm{Emb}(P,M) \\
\Omega\, C\mathrm{Emb}(P,M) \arrow[uu, "\Gamma"] \arrow[r, equal] & \Omega\, C\mathrm{Emb}(P,M) \arrow[uu, "U\Gamma"] \arrow[ur, hook] \arrow[urr, bend right=10, "\mathrm{ev}_0=*"'] & &
\end{tikzcd}$$

which exhibits the leftmost vertical map as nullhomotopic, as desired. □

As a consequence of this proposition, we obtain the following splitting result for embedding spaces:

Theorem 4.3 *Let I and J both denote closed intervals. For $p \le d-3$, $N := \phi_{C\mathrm{Emb}}(d+1, p+1)-2$ and $N' := \phi_{C\mathrm{Emb}}(d+2, p+2)-1$, there are equivalences away from 2*

$$\tau_{\le N}(\Omega\,\mathrm{Emb}_{\partial_0}(P\times I, M\times I)) \simeq_{[\frac{1}{2}]} \tau_{\le N}(\Omega\,\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I, M\times I)\times\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I)),$$

$$\tau_{\le N'}\mathrm{Emb}_{\partial_0}(P\times I\times J, M\times I\times J) \simeq_{[\frac{1}{2}]} \tau_{\le N'}(\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I\times J, M\times I\times J)\times\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I\times J, M\times I\times J)).$$

These splittings are compatible with graphing maps, in the sense that

$$\begin{tikzcd}
\tau_{\le N}(\Omega\,\mathrm{Emb}_{\partial_0}(P\times I, M\times I)) \arrow[r, phantom, "\simeq_{[\frac{1}{2}]}"] \arrow[d, "\Gamma"] & \tau_{\le N}(\Omega\,\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I, M\times I)\times\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I)) \arrow[d, "\tilde{\Gamma}\times\Gamma^{(\sim)}\sim\tilde{\Gamma}\times *"] \\
\tau_{\le N}\,\mathrm{Emb}_{\partial_0}(P\times I\times J, M\times I\times J) \arrow[r, phantom, "\simeq_{[\frac{1}{2}]}"] & \tau_{\le N}(\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I\times J, M\times I\times J)\times\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I\times J, M\times I\times J))
\end{tikzcd}$$

is homotopy commutative. (*The homotopy in the rightmost vertical map follows from Proposition 4.2.*)

Proof Suppose given a map of fibration sequences

$$\begin{tikzcd}
F' \arrow[r] & E' \arrow[r, "p'"] & B' \\
F \arrow[u, "f\simeq *"] \arrow[r] & E \arrow[u, "e"] \arrow[r, "p"] & B \arrow[u, "b", "\wr"']
\end{tikzcd}$$

such that f is nullhomotopic and b is an equivalence. If $\delta\colon \Omega B \to F$ (and similarly for δ') denotes the connecting map, then it follows that $\delta' \circ \Omega b \simeq f \circ \delta \simeq *$, and thus Ωb lifts, up to homotopy, to a map $\tilde{\sigma}\colon \Omega B \to \Omega E'$. Then for $(\Omega b)^{-1}$ any homotopy inverse to Ωb, the map $\sigma := \tilde{\sigma}\circ(\Omega b)^{-1}\colon \Omega B' \to \Omega E'$ is a homotopy section of the fibration $\Omega F' \to \Omega E' \to \Omega B'$ and so provides a splitting $\Omega E' \simeq \Omega B' \times \Omega F'$. This observation, applied to the map of fibre sequences obtained from

$$\begin{tikzcd}
\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I) \arrow[r] & \mathrm{Emb}_{\partial_0}(P\times I, M\times I) \arrow[r] & \widetilde{\mathrm{Emb}}_{\partial_0}(P\times I, M\times I) \\
\Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \arrow[u, "\Gamma^{(\sim)}"] \arrow[r] & \Omega\,\mathrm{Emb}_{\partial_0}(P, M) \arrow[u, "\Gamma"] \arrow[r] & \Omega\,\widetilde{\mathrm{Emb}}_{\partial_0}(P, M) \arrow[u, "\tilde{\Gamma}", "\wr"']
\end{tikzcd}$$

by localising away from 2 and taking Postnikov $(N+1)$-th sections, yields the first equivalence in the statement by Proposition 4.2. (Note that, for any space X, we have $\tau_{\leq N}(\Omega X) \simeq \Omega \tau_{\leq N+1} X$.)

To obtain the second equivalence, observe that for $E(-,-)$ any of the mapping spaces involved in the proof of Proposition 4.2, the space $E(P \times J, M \times J)$ is a group-like topological monoid with respect to stacking in the J-direction. Then replacing (M, P) by $(M \times J, P \times J)$, one checks that each of the steps in the argument of Proposition 4.2 can be delooped with respect to this $\mathbb{E}_1$-structure. This results in getting rid of the loopings in the first equivalence of the statement, thus yielding the second one.

For the compatibility part of the statement, note that given a diagram of fibre sequences

$$\begin{tikzcd}
F'' \arrow[r] & E'' \arrow[r, "p''"] & B'' \\
F' \arrow[r] \arrow[u, "f'"] & E' \arrow[r, "p'"] \arrow[u, "e'"] & B \arrow[u, "b'", "\wr"'] \\
F \arrow[r] \arrow[u, "f \simeq *"] & E \arrow[r, "p"] \arrow[u, "e"] & B \arrow[u, "b", "\wr"']
\end{tikzcd}$$

the splitting of the top fibre sequence (upon looping) provided by the nullhomotopy $f' \circ f \simeq f' \circ * = *$ is compatible with that of the middle fibre sequence. If, moreover, f' is nullhomotopic and we have a homotopy of nullhomotopies of $f' \circ f$ from $f' \circ f \simeq f' \circ * = *$ to $f' \circ f \simeq * \circ f = *$, then the splitting of the top fibre sequence induced by $f' \simeq *$ is compatible with that induced by $f' \circ f \simeq f' \circ * = *$. The compatibility part in the statement of the theorem follows from this observation, since it is clear from construction that the two nullhomotopies of the composition

$$\Omega^2\, C\mathrm{Emb}(P, M) \to \Omega\, C\mathrm{Emb}(P \times J, M \times J) \to C\mathrm{Emb}(P \times I \times J, M \times I \times J)$$

are themselves homotopic (both come from different deformation retractions of the space $I \times J$). □

The following result will be used to establish the splitting part of Theorem B:

Corollary 4.4 *For* $2 \leq p \leq d-3$ *and* $N := \phi_{C\mathrm{Emb}}(d, p) - 1$, *there is an equivalence away from* 2

$$\tau_{\leq N} \operatorname{Emb}_\partial(D^p, D^d) \simeq_{[\frac{1}{2}]} \tau_{\leq N}(\widetilde{\operatorname{Emb}}_\partial(D^p, D^d) \times \operatorname{Emb}_\partial^{(\sim)}(D^p, D^d)).$$

For $p = 1$, *this equivalence exists only after looping*, *i.e.*,

$$\Omega\tau_{\leq N} \operatorname{Emb}_\partial(D^1, D^d) \simeq_{[\frac{1}{2}]} \Omega\tau_{\leq N}(\widetilde{\operatorname{Emb}}_\partial(D^1, D^d) \times \operatorname{Emb}_\partial^{(\sim)}(D^1, D^d)).$$

Proof When $p \geq 2$, set $(M, P) = (D^{d-2}, D^{p-2})$ in the second equivalence of Theorem 4.3. For $p = 1$, set $(M, P) = (D^{d-1}, D^0)$ in the first equivalence of the same theorem. □

4.2 The Gromoll filtration

For $n \geq 5$, Gromoll [1966] introduced a descending filtration on the group of exotic $(n+1)$-spheres $\Theta_{n+1} \cong \pi_0(\mathrm{Diff}_\partial(D^n))$ given by

$$0 = \Gamma^{n+1}\Theta_{n+1} \subset \Gamma^n\Theta_{n+1} \subset \cdots \subset \Gamma^1\Theta_{n+1} \subset \Gamma^0\Theta_{n+1} = \Theta_{n+1},$$

where $\Gamma^j\Theta_{n+1}$ is, by definition, the image of the j-th graphing homomorphism

$$\pi_0(\Gamma^j)\colon \pi_j(\mathrm{Diff}_\partial(D^{n-j})) \to \pi_0(\mathrm{Diff}_\partial(D^n)) \cong \Theta_{n+1}.$$

When $n-m \geq 3$, Budney [2008, Definition 3.7] considered an analogous filtration on the abelian group $\pi_0(\mathrm{Emb}_\partial(D^m, D^n))$ of isotopy classes of long knots. We will consider an even more general case.

Definition 4.5 Fix $\iota\colon P^p \subset M^d$ as in Theorem A. A homotopy class $x \in \pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))$ is said to have *Gromoll degree* j, for $0 \leq j \leq n$, if it is in the image of the j-th graphing homomorphism

$$\pi_k(\Gamma^j)\colon \pi_{k+j}(\mathrm{Emb}_{\partial_0}(P \times D^{n-j}, M \times D^{n-j})) \to \pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))$$

but, if $j < n$, not in the image of the $(j+1)$-st graphing homomorphism $\pi_k(\Gamma^{j+1})$. We will say the same for homotopy classes localised away from 2, that is, in the group $\pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\frac{1}{2}\bigr]$.

As a consequence of Theorem 4.3, we can determine the Gromoll degree of many homotopy classes away from 2 of the embedding spaces $\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n)$.

Corollary 4.6 *Let $d-p \geq 3$, let k be an integer such that $k \leq 2d-p-6+n$, and set*

$$j := \min\bigl(n-1, \bigl\lfloor \tfrac{1}{2}(2d-p-6+n-k)\bigr\rfloor\bigr).$$

Then any k-th homotopy class $x \in \pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\frac{1}{2}\bigr]$ has Gromoll degree either 0 or $\geq j$. Moreover, letting $x = (a, b)$ under the splitting[7]

$$\begin{aligned}\pi_k(\mathrm{Emb}_{\partial_0}&(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr] \\ &\cong \pi_k(\widetilde{\mathrm{Emb}}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr] \oplus \pi_k(\mathrm{Emb}^{(\sim)}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr]\end{aligned}$$

of Theorem 4.3:

(i) *If $a = 0$, i.e., x is in the image of*

$$\pi_k(\mathrm{Emb}^{(\sim)}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr] \to \pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr],$$

then the Gromoll degree of x is 0.

(ii) *If $b = 0$, then the Gromoll degree of x is $\geq j$.*

Proof First note that, by the bound (1-7) of [Goodwillie et al. 2024], we have $\phi_{C\mathrm{Emb}}(d+n, p+n) - 1 \geq 2d-p-6+n$ and that j is the biggest integer $\leq n-1$ such that

$$k + j \leq 2d - p - 6 + n - j \leq \phi_{C\mathrm{Emb}}(d+n-j, p+n-j) - 1.$$

Thus, if $0 < \ell \leq j$, it follows by the last part of Theorem 4.3 that the ℓ-th graphing homomorphism

$$\pi_k(\Gamma^\ell)\colon \pi_{k+\ell}(\mathrm{Emb}_{\partial_0}(P \times D^{n-\ell}, M \times D^{n-\ell}))\bigl[\tfrac{1}{2}\bigr] \to \pi_k(\mathrm{Emb}_{\partial_0}(P \times D^n, M \times D^n))\bigl[\tfrac{1}{2}\bigr]$$

---

[7] The splitting of homotopy groups also holds when $n = 1$ and $k = 0$ since $\mathrm{Emb}^{(\sim)}_{\partial_0}(P \times D^n, M \times D^n)$ is connected by Hudson's theorem (Theorem 3.1).

is of the form

$$\begin{aligned}\mathrm{Id}\oplus 0: \pi_{k+\ell}(\widetilde{\mathrm{Emb}}_{\partial_0}(P\times D^{n-j}, M\times D^{n-j}))\big[\tfrac{1}{2}\big]\oplus \pi_{k+\ell}(\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times D^{n-\ell}, M\times D^{n-\ell}))\big[\tfrac{1}{2}\big]\\ \to \pi_k(\widetilde{\mathrm{Emb}}_{\partial_0}(P\times D^{n}, M\times D^{n}))\big[\tfrac{1}{2}\big]\oplus \pi_k(\mathrm{Emb}^{(\sim)}_{\partial_0}(P\times D^{n}, M\times D^{n}))\big[\tfrac{1}{2}\big],\\ \begin{pmatrix}a\\ b\end{pmatrix}\mapsto\begin{pmatrix}a\\ 0\end{pmatrix}.\end{aligned}$$

It follows that $x=(a,b)$ is in the image of $\pi_k(\Gamma^{\ell})$ for some $0<\ell\leq j$ if and only if this is the case for $\ell=j$ and $x=(a,0)$. This proves the first assertion of the statement and part (ii).

Part (i) follows from the elementary claim that, given a commutative diagram of groups

$$\begin{tikzcd}0 \arrow[r] & A \arrow[r, hook, "i"] & B \arrow[r, two heads] & C \arrow[r] & 0\\ & A' \arrow[u, "0"] \arrow[r] & B' \arrow[u, "\beta"] \arrow[r] & C \arrow[u, equal] &\end{tikzcd}$$

where the top is a short exact sequence and the bottom is only exact at B', we have that $\mathrm{Im}\, i\cap \mathrm{Im}\,\beta=0$. □

Remark 4.7 We will compare this result to those of Budney [2008] on the Gromoll filtration of $\pi_0(\mathrm{Emb}_{\partial}(D^p, D^d))$ in Remark 6.4.

5 Involutions in algebraic K-theory

The aim of this section is to explore the involutions of the C_2-spectra involved in the statements of Theorems 1.1 and A, and to express them in terms of simpler and more computable involutions coming from algebraic K-theory — the main result in this direction is Theorem 5.13, which is further simplified by Proposition 5.22 in the case of a suspension. This will then be used in Section 6 to study the case $(M,P)=(D^d,D^p)$. As we will shortly see in Section 5.1, it will be significantly helpful to invert the prime 2 in the analysis of these involutions. Let us now introduce the notation that will be relevant in this section.

Notation 5.1 (i) For M a compact (smooth) manifold and $\iota\colon P\subset M$ a compact submanifold, recall from Notation 2.6 the definitions of the C_2-spectra $\boldsymbol{H}(M)$, the *h-cobordism spectrum* of M, and $\mathbf{CE}(P,M)$, the *concordance embedding spectrum* of $\iota\colon P\hookrightarrow M$. We refer to their involutions by τ_{WW}, for Weiss and Williams.

(ii) Given a space X and a spherical fibration ξ over X equipped with a section, Vogell [1985, p. 300] defined an involution τ_ξ on[8] $\boldsymbol{A}(X)$ by means of Spanier–Whitehead duality with respect to the Thom spectrum of ξ; we will write $\boldsymbol{A}(X;\xi)$ for the corresponding C_2-spectrum. When $\xi=\epsilon:=X\times S^0$ is the

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[8] Vogell defined τ_ξ on the A-theory space $A(X)$, but this involution can be upgraded to $\boldsymbol{A}(X)$ by specifying it on the Waldhausen category of retractive spaces over X "with ξ-duality" and appealing to the definition of algebraic K-theory via the $S_\bullet$-construction.

trivial 0-dimensional sphere bundle, τ_ϵ fits in a commutative square

$$\begin{tikzcd}
\Sigma^\infty_+ X \arrow[r, "\nu"] \arrow[d, equal] & \boldsymbol{A}(X) \arrow[r, two heads] \arrow[d, "\tau_\epsilon"] & \mathbf{Wh}^{\mathrm{Diff}}(X) \arrow[d, dashed, "\tau_\epsilon"] \\
\Sigma^\infty_+ X \arrow[r, "\nu"] & \boldsymbol{A}(X) \arrow[r, two heads] & \mathbf{Wh}^{\mathrm{Diff}}(X)
\end{tikzcd}$$

and hence, on cofibres, induces the dashed vertical arrow — this is an involution on $\mathbf{Wh}^{\mathrm{Diff}}(X)$, which we shall also denote by τ_ϵ. We will refer to τ_ϵ as the *canonical involution* of K-theory, and sometimes write $\boldsymbol{A}(X)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X)$ for $\boldsymbol{A}(X;\epsilon)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X;\epsilon)$. We will recall a construction of τ_ϵ in terms of Spanier–Whitehead duality in Section 5.4.

5.1 Homotopy involutions

A *homotopy involution* τ on a space or infinite loop space or spectrum X is a self-map $\tau\colon X \to X$ whose square τ^2 is homotopic to the identity Id_X. In this section we explain why, in the stable setting and once the prime 2 is inverted, an involution carries the same amount of information as its underlying homotopy involution. This will be very useful when comparing the C_2-spectra $\boldsymbol{H}(M)$ and $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)$; see Corollary 5.17. Let us fix some notation first.

Notation 5.2 (i) Let C denote any of Top_*, Ω^∞-Top or Sp, and let $X, X' \in C$ be equipped with homotopy involutions τ and τ', respectively. A map $f\colon X \to X'$ will be said to be *homotopy C_2-equivariant*, or *C_2-equivariant up to homotopy*, if $f\tau \simeq \tau' f$. If X and X' can be connected by a zigzag of homotopy C_2-equivariant weak equivalences, we will say that X and X' are *homotopy C_2-equivariantly equivalent* and write

$$X \approx X'.$$

A *C_2-equivariant equivalence* will always mean a zigzag of weak equivalences which are C_2-equivariant.

(ii) An *H-group* (X,μ) is a group-like $\mathbb{A}_3$-space (i.e., a homotopy associative H-space such that $\pi_0(X)$ is a group with respect to μ). Given H-spaces (X,μ) and (X',μ'), a based map $f\colon X \to X'$ will be said to be *monoidal up to homotopy*, or simply an *H-map*, if the following diagram is homotopy commutative:

$$\begin{tikzcd}
X \times X \arrow[r, "f\times f"] \arrow[d, "\mu"] & X' \times X' \arrow[d, "\mu'"] \\
X \arrow[r, "f"] & X'
\end{tikzcd}$$

An *equivalence of H-groups* will mean a zigzag of H-maps that are additionally weak equivalences. In practice, all H-groups we will consider are actually $\mathbb{E}_1$-groups (i.e., group-like $\mathbb{E}_1$-spaces), and all H-maps can be upgraded to $\mathbb{E}_1$-maps even though we will not need this.

(iii) Given a C_2-object X in an appropriate category, the symbol X_{hC_2} will stand for $\mathrm{hocolim}_{BC_2} X$.

In the cases of interest to us and once the prime 2 is inverted, taking homotopy C_2-orbits with respect to a homotopy involution turns out to make sense.

Proposition 5.3 *Let X denote a spectrum or infinite loop space (a.k.a. connective spectrum), and let τ be a homotopy involution on X. Suppose that multiplication by 2 is invertible on X, i.e., $2 : X \xrightarrow{\sim} X$ is an equivalence, and define*

$$E(X,\tau) := \operatorname{hocolim}(X \xrightarrow{\frac{1}{2}(1+\tau)} X \xrightarrow{\frac{1}{2}(1+\tau)} \cdots),$$

where $\frac{1}{2}(1+\tau)$ really stands for the zigzag $X \xrightarrow{1+\tau} X \xleftarrow[\sim]{2} X$. Then if τ is an actual involution on X, there is a natural equivalence away from 2,

$$X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau).$$

Proof Let us assume that X is a C_2-spectrum (the other case is completely analogous). We also assume that 2 is inverted. Observe now that as $t \cdot \frac{1}{2}((1+t)) = \frac{1}{2}(1+t)$ in $\mathbb{Z}[C_2]$, the diagram

$$\begin{tikzcd}
X \arrow[r, "\frac{1}{2}(1+\tau)"] \arrow[dr, "q"'] & X \arrow[r, "\frac{1}{2}(1+\tau)"] \arrow[d, "q"] & \cdots \arrow[dl, "q"] \\
& X_{hC_2} &
\end{tikzcd}$$

commutes up to homotopy, where $q\colon X \to X_{hC_2} = \operatorname{hocolim}_{BC_2} X$ is the map on colimits induced by the inclusion of categories $\{*\} \hookrightarrow BC_2$. We thus obtain a map $\eta_{(X,\tau)}\colon E(X,\tau) \to X_{hC_2}$. The homotopy orbits spectral sequence for X, together with the assumption that 2 is inverted, gives a natural isomorphism $\pi_*(X_{hC_2}) \cong H_0(C_2;\pi_*(X)) \cong \pi_*(X)_{C_2}$. Also by definition, $\pi_*(E(X,\tau)) \cong \operatorname{Im}\bigl(\frac{1}{2}(1+\tau)\colon \pi_*(X) \to \pi_*(X)\bigr)$. Under these identifications, $\pi_*(\eta_{(X,\tau)})$ is the natural isomorphism (away from 2) sending an element $\beta = \frac{1}{2}(1+\tau)\alpha \in \pi_*(E(X,\tau))$ to $[\beta] = [\alpha] \in \pi_*(X)_{C_2}$. So $\eta_{(X,\tau)}$ is the desired equivalence $X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau)$. □

Corollary 5.4 *Let X and X' be C_2-spectra and let the prime 2 be inverted.*

(i) *If there is a homotopy C_2-equivariant equivalence $X \approx X'$, then there is an equivalence of spectra*

$$X_{hC_2} \simeq_{[\frac{1}{2}]} X'_{hC_2}.$$

(ii) *If there is only a homotopy C_2-equivariant equivalence $\Omega^\infty X \approx \Omega^\infty X'$ of* ***H*-spaces***, then we still have an equivalence of spaces*

$$\Omega^\infty(X_{hC_2}) \simeq_{[\frac{1}{2}]} \Omega^\infty(X'_{hC_2}).$$

Proof Let us only deal with (ii) (as (i) is analogous and easier). Assume without loss of generality that the equivalence $\Omega^\infty X \approx \Omega^\infty X'$ of H-spaces is induced by a single homotopy C_2-equivariant H-map $g\colon \Omega^\infty X \xrightarrow{\sim} \Omega^\infty X'$. Then the diagram *of spaces*

$$\begin{tikzcd}
\Omega^\infty X \arrow[r, "\frac{1}{2}(1+\tau)"] \arrow[d, "g", "\wr"'] & \Omega^\infty X \arrow[r, "\frac{1}{2}(1+\tau)"] \arrow[d, "g", "\wr"'] & \cdots \\
\Omega^\infty X' \arrow[r, "\frac{1}{2}(1+\tau')"] & \Omega^\infty X' \arrow[r, "\frac{1}{2}(1+\tau')"] & \cdots
\end{tikzcd}$$

commutes up to homotopy, where τ and τ' are the involutions of X and X', respectively. Since the forgetful map from infinite loop spaces to spaces preserves directed colimits, the diagram above (upon taking horizontal colimits) induces an equivalence of spaces

$$E(\Omega^\infty X,\tau)\simeq E(\Omega^\infty X',\tau').$$

The claim now follows from Proposition 5.3 and because the natural map $(\Omega^\infty X)_{hC_2}\to\Omega^\infty(X_{hC_2})$ is an equivalence away from 2 (this is a consequence of the homotopy orbits spectral sequence). □

Remark 5.5 The upshot of part (ii) of the previous corollary is that, given a C_2-spectrum X that is local away from 2, the homotopy type of $\Omega^\infty(X_{hC_2})$ *as a space* is completely determined by the homotopy type of the space $\Omega^\infty X$ and the homotopy classes of the maps $\tau\colon\Omega^\infty X\to\Omega^\infty X$ and $+\colon\Omega^\infty X\times\Omega^\infty X\to\Omega^\infty X$.

The following result, though unrelated to what has been discussed so far in this section, will be useful later on. Given an $\mathbb{E}_1$-space X, we will write X^{op} for X equipped with the opposite $\mathbb{E}_1$-structure. An *anti-involution* τ on an $\mathbb{E}_1$-space X is an $\mathbb{E}_1$-map $\tau\colon X\to X^{\mathrm{op}}$ whose square equals the identity of X (noting that $(X^{\mathrm{op}})^{\mathrm{op}}\cong X$). Up to equivalence, there is a standard way of delooping such an anti-involution.

Lemma 5.6 *Let X be an $\mathbb{E}_1$-space. There is a natural equivalence*

$$\iota\colon B(X^{\mathrm{op}})\simeq BX$$

such that, for any anti-involution τ on X, the composition

$$\bar{B}\tau\colon BX\xrightarrow{B\tau}B(X^{\mathrm{op}})\overset{\iota}{\simeq}BX$$

is an involution on BX.

Proof For each $k\ge 0$, the map $\mathbb{E}_1(k)\to\pi_0(\mathbb{E}_1(k))$ is an equivalence, and hence there is a natural zigzag of equivalences of $\mathbb{E}_1$-algebras

$$B(\pi_0(\mathbb{E}_1),\mathbb{E}_1,X)\xleftarrow{\simeq}B(\mathbb{E}_1,\mathbb{E}_1,X)\xrightarrow{\simeq}X.$$

But the $\mathbb{E}_1$-structure on the left-hand side factors through the associative operad $\mathcal{A}\mathrm{ss}:=\pi_0(\mathbb{E}_1)$, so for simplicity, we may assume that X is strictly associative. The equivalence ι is then induced on the realisation of the nerve $N_\bullet X$ by the maps

$$X^q\times\Delta^q\to X^q\times\Delta^q,\quad (x_1,\dots,x_q,r)\mapsto(x_q,\dots,x_1,\Phi_q(r)),$$

where $\Phi_q\colon\Delta^q\cong\Delta^q$ is the linear homeomorphism induced by reversing the order of the vertices. It is easy to check that the map $\bar{B}\tau$ indeed defines an involution on BX. □

5.2 From the *h*-cobordism spectrum to spaces of *h*-cobordisms

All throughout this section, assume that $d=\dim M\ge 5$; this condition will not be a problem later, as all of the results in this section will be used only once our original manifold M has been stabilised sufficiently many times.

We now recall Vogell's model [1985, p. 296] for spaces of h-cobordisms. A *partition* of a manifold M^d is a triple (W, F, V), where W is a codimension-zero submanifold of $M\times[-1,1]$, V is the closure of the complement of W, and $F^d := W\cap V$. For technical reasons, we require F to be standard near $\partial M\times[-1,1]$, and that it intersects it in $\partial M\times\{0\}$. Let $H(M)_\bullet$ denote the simplicial set whose p-simplices are (locally trivial smooth) families of partitions of M parametrised by Δ^p such that W is an h-cobordism from $M\times\{-1\}\times\Delta^p$ to F. Set $H(M) := |H(M)_\bullet|$ and write $H^s(M)\subset H(M)$ for the connected component containing the trivial partition $* = (M\times[-1,0], M\times\{0\}, M\times[0,1])$. There is a canonical involution ι_H given by turning partitions upside down. Namely

$$\iota_H\colon H(M)\to H(M),\quad \rho=(W,F,V)\to\rho^* := (V^*,F^*,W^*),$$

where W^*, F^*, and V^* are respectively the images of W, F, and V under the reflection $r=\mathrm{Id}_M\times -1$. For the smooth case, we will also need a small variant of this h-cobordism space, denoted by $H_{\mathrm{col}}(M)$, a point of which consists of a partition $\rho=(W,F,V)\in H(M)$ together with a bicollar of F for W and V which is standard near $\partial M\times[-1,1]$. The forgetful map $H_{\mathrm{col}}(M)\to H(M)$ is a weak equivalence by the contractibility of the space of collars. In this section we construct a homotopy C_2-equivariant $(\phi(d)+1)$-connected map

$$\mathrm{alex}\colon H(M)\to\Omega^\infty\boldsymbol{H}(M) \tag{5-1}$$

which generalises the map $B(\mathrm{alex})$ of Proposition 3.8. We first recall an important construction.

5.2.1 The geometric Eilenberg swindle An h-cobordism $W\colon M\overset{h}{\rightsquigarrow}M'$ induces a unique (up to contractible choice) bounded diffeomorphism

$$\mathrm{ES}_W\colon M\times\mathbb{R}\cong M'\times\mathbb{R} \tag{5-2}$$

as follows: choose embeddings $i_r\colon W\hookrightarrow M\times I$ rel $M\times\{0\}$ and $i_\ell\colon W\hookrightarrow M'\times I$ rel $M'\times\{1\}$. Both of these embeddings are unique up to isotopy, for given another such embedding $i_r'\colon W\hookrightarrow M\times I$ rel $M\times\{0\}$, the embeddings $\mathrm{Id}_{-W}\cup_M i_r$ and $\mathrm{Id}_{-W}\cup_M i_r'$, where $-W\colon M'\overset{h}{\rightsquigarrow}M$ is an inverse of W, are isotopic by the contractibility of the space of collars. But then, so are $\mathrm{Id}_{W\cup_{M'}-W}\cup_M i_r$ and $\mathrm{Id}_{W\cup_{M'}-W}\cup_M i_r'$, and these in turn are isotopic to i_r and i_r' (rel M) by choosing an identification $W\cup_{M'}-W\cong M\times I$. Similarly for i_ℓ.

Now write $V_r := \overline{M\times I - i_r(W)}$ and $V_\ell := \overline{M'\times I - i_\ell(W)}$; both of these manifolds are h-cobordisms $M'\overset{h}{\rightsquigarrow}M$, and in fact they are diffeomorphic relative to *both* ends since

$$V_\ell\cong V_\ell\cup_M M\times I\cong V_\ell\cup_M W\cup_{M'}V_r\cong M'\times I\cup_{M'}V_r\cong V_r\quad\text{rel }\ M'\sqcup M.$$

Then the *Eilenberg swindle* diffeomorphism ES_W is given by the composition

$$\mathrm{ES}_W\colon M\times\mathbb{R}=\cdots\cup_{M'}V_r\cup_M W\cup_{M'}V_r\cup_M\cdots\cong\cdots\cup_{M'}V_\ell\cup_M W\cup_{M'}V_\ell\cup_M\cdots=M'\times\mathbb{R}.$$

Clearly, by construction, ES_W is bounded by 1 and unique up to contractible choice.

5.2.2 The map (5-1) Fix an embedding $M^d \subset \mathbb{R}^N \subset \mathbb{R}^\infty$ and recall that $B\mathrm{Diff}_\partial(M)$ admits a model as the moduli space of manifolds embedded in $\mathbb{R}^\infty$ which are abstractly diffeomorphic to M relative to the boundary ∂M. Similarly $B\mathrm{Diff}^b_\partial(M \times \mathbb{R})$ is the moduli space of manifolds embedded in $\mathbb{R}^\infty \times \mathbb{R}$ which are abstractly diffeomorphic to $M \times \mathbb{R} \subset \mathbb{R}^\infty \times \mathbb{R}$ *boundedly* with respect to the $\mathbb{R}$-direction and relative to the boundary $\partial M \times \mathbb{R}$ (this is proved in Section B.2). For the remainder of this section, we will denote by $\underline{\mathbb{R}}$ the bounded direction, i.e., the last coordinate in $\mathbb{R}^\infty \times \mathbb{R} =: \mathbb{R}^\infty \times \underline{\mathbb{R}}$. There is a natural map $- \times \underline{\mathbb{R}}\colon B\mathrm{Diff}_\partial(M) \hookrightarrow B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ given by sending a manifold $N \subset \mathbb{R}^\infty$ to $N \times \underline{\mathbb{R}} \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$. In fact, in light of (5-2), this map extends to

$$- \times \underline{\mathbb{R}}\colon \coprod_{[M']} B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}}), \quad N \mapsto N \times \underline{\mathbb{R}},$$

where the coproduct in the domain runs over all diffeomorphism classes of manifolds M' with boundary ∂M that are h-cobordant to M rel ∂M. The map $- \times \underline{\mathbb{R}}$ is the value of the morphism $0 \to \underline{\mathbb{R}}$ in $\mathcal{J}_0$ under

$$\widehat{B}\colon \mathcal{J}_0 \to \mathrm{Top}_*, \qquad \widehat{B}(V) := \begin{cases} \coprod_{[M']} B\mathrm{Diff}_\partial(M') & \text{if } V = 0, \\ B(V) = B\mathrm{Diff}^b_\partial(M \times V) & \text{otherwise.} \end{cases}$$

Clearly $\widehat{B}$ is an orthogonal functor, and we will write $\widehat{\boldsymbol{H}}(M)$ for its first derivative, which is canonically equivalent to $\boldsymbol{H}(M)$. The Alexander trick-like map (5-1) will factor through $\widehat{\boldsymbol{H}}(M)_0 \hookrightarrow \Omega^\infty \widehat{\boldsymbol{H}}(M) \simeq \Omega^\infty \boldsymbol{H}(M)$.

Suppose we are given some partition $\rho = (W, F, V) \in H(M)$ of $M \times \underline{[-1,1]} \subset \mathbb{R}^N \times \underline{\mathbb{R}}$. Then W is an h-cobordism from M to F rel boundary, and so the manifold $F \subset \mathbb{R}^N \times \mathbb{R} \subset \mathbb{R}^\infty$ gives rise to a point in $B\mathrm{Diff}_\partial(F) \subset \widehat{B}(0)$; more precisely, the image of the embedding

$$i_\rho\colon F \subset M \times I \subset \mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1} \subset \mathbb{R}^\infty$$

is a point in $B\mathrm{Diff}_\partial(F)$, where the isomorphism $\mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1}$ identifies $\underline{\mathbb{R}}$ with the last coordinate in $\mathbb{R}^{N+1}$. We now construct a point in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ by *extending W towards infinity*. Consider the embedding of $F \times [0,1]$ into $\mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}$ given by

$$R\colon F \times [0,1] \hookrightarrow \mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}, \quad (x,t) \mapsto \underline{e} + (\mathrm{Id}_{\mathbb{R}^N} \times Q_{-\pi t/2})(x - \underline{e}),$$

where $Q_\theta\colon \mathbb{R} \times \underline{\mathbb{R}} \cong \mathbb{R} \times \underline{\mathbb{R}}$ is the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and $\underline{e}$ denotes the unit-length vector in $\underline{\mathbb{R}}$. Write $r := R|_{F \times \{1\}}\colon F \hookrightarrow \mathbb{R}^{N+1} \times \underline{\{1\}}$, and consider

$$S\colon F \times [1, +\infty) \hookrightarrow \mathbb{R}^{N+1} \times \underline{\mathbb{R}}, \qquad (x,t) \mapsto \begin{cases} r(x) + (1-t)\cdot e_{N+1} + (t-1)\cdot \underline{e} & \text{if } t \in [1,2], \\ r(x) - e_{N+1} + (t-1)\cdot \underline{e} & \text{if } t \geq 2. \end{cases}$$

Finally consider the region $D \subset \mathbb{R} \times \underline{\mathbb{R}}$ given by tuples (u,v) with

$$u \geq 0, \quad u \leq 2 - v, \quad \text{and if } 0 \leq u \leq 1, \text{ then } u \leq (1 - (v-1)^2))^{1/2}.$$

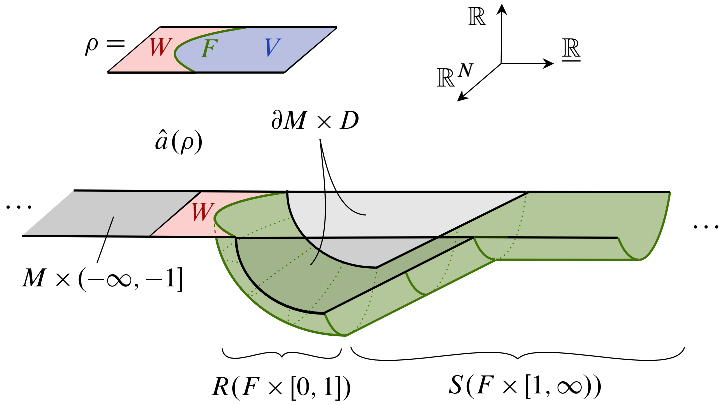

Figure 2: The topological manifold $\hat{a}(\rho)$ for $\rho \in H^s(M)$.

Then we define a topological manifold $\hat{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$, depicted in Figure 2, by

$$\hat{a}(\rho) := M \times \{0\} \times (-\infty \times -1] \cup W \cup R(F \times [0,1]) \cup S(F \times [1,+\infty)) \cup \partial M \times D.$$

Now if ρ is a collared partition, i.e., a point in $H^s_{\mathrm{col}}(M)$, one can use the collar of F to smooth out the corners of the topological manifold $\hat{a}(\rho)$, and thus obtain a smooth manifold $\tilde{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$ with the same boundary as $M \times \underline{\mathbb{R}}$ and which is boundedly diffeomorphic to $M \times \mathbb{R}$ relative to the boundary by a one-sided Eilenberg swindle argument. This construction can be done simplexwise in $H_{\mathrm{col}}(M)_\bullet \simeq H(M)_\bullet$, and so up to weak equivalence gives rise to an Alexander trick-like map

$$\begin{aligned} &\mathrm{alex}\colon H(M) \xrightarrow{\sim} \widehat{\boldsymbol{H}}(M)_0 := \mathrm{hofib}\Big(\coprod_{[M']} B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})\Big), \\ &\rho = (W,F,V) \mapsto \left(i_\rho(F), \gamma_W\colon [-\infty,\infty] \ni t \mapsto \begin{cases} M \times \underline{\mathbb{R}} & \text{if } t = -\infty, \\ \tilde{a}(\rho) - t \cdot \underline{e} & \text{if } -\infty < t < +\infty, \\ i_\rho(F) \times \underline{\mathbb{R}} & \text{if } t = +\infty, \end{cases} \right), \end{aligned} \tag{5-3}$$

where we can regard the path γ_W as a 1-simplex in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})_\bullet$ from $M \times \underline{\mathbb{R}}$ to $F \times \underline{\mathbb{R}}$. Then (5-1) is

$$\mathrm{alex}\colon H(M) \xrightarrow[\sim]{(5\text{-}3)} \widehat{\boldsymbol{H}}(M)_0 \hookrightarrow \Omega^\infty \widehat{\boldsymbol{H}}(M) \simeq \Omega^\infty \boldsymbol{H}(M)$$

Proposition 5.7 *The map* (5-3) *is indeed an equivalence. Therefore* (5-1) *is* $(\phi(d)+1)$*-connected.*

Proof Noting the equivalence $H^s(M) \simeq BC(M)$ — see [Vogell 1985, Proposition 2.1] — the map (5-3) is, up to homotopy, a (nonconnected) delooping of the Alexander trick-like equivalence $C(M) \simeq \Omega(\mathrm{Diff}^b_\partial(M \times \mathbb{R})/\mathrm{Diff}(M))$ of [Weiss and Williams 1988, Proposition 1.10], and therefore it is an equivalence on basepoint components.

Given a diffeomorphism class $[M']$ of manifolds h-cobordant to M (rel boundary), denote by $H(M, M')$ the collection of path components in $H(M)$ consisting of (collared) partitions $\rho = (W, F, V)$ with $F \in [M']$. A choice of basepoint $\rho_0 = (W_0, F_0, V_0) \in H(M, M')$, a bicollar $c_0\colon F_0 \times [-\epsilon, \epsilon] \hookrightarrow M \times [-1, 1]$, and a diffeomorphism $\phi_0\colon M' \cong F_0$ gives rise to an equivalence $H(M', M') \xrightarrow{\sim} H(M, M')$ which sends a partition $\rho' = (W', F', V')$ of $M' \times [-1,1]$ to the partition of $M \times [-1,1]$ whose F-part is the image of F' under

$$M' \times [-1,1] \xrightarrow{\phi_0 \times \cdot\epsilon} F_0 \times [-\epsilon, \epsilon] \xhookrightarrow{c_0} M \times [-1,1].$$

By the s-cobordism theorem, a homotopy inverse $H(M, M') \xrightarrow{\sim} H(M', M')$ is given by the same kind of map for a choice of basepoint $\rho_0' = (W_0', F_0', V_0') \in H(M', M')$ such that $(\phi_0 \times \mathrm{Id}_{[-1,1]})(W_0')$ is an h-cobordism starting at F_0 with the same torsion as V_0 (the inverse of W_0).

Observe also that the choices (ρ_0, c_0) and ϕ_0 above give a preferred path in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}}) = B\mathrm{Diff}^b_\partial(M' \times \underline{\mathbb{R}})$ from $M \times \underline{\mathbb{R}}$ to $M' \times \underline{\mathbb{R}}$; namely, it is the composition of γ_{W_0}, as defined in (5-3), with the mapping cylinder of $\phi_0 \times \mathrm{Id}_{\underline{\mathbb{R}}}$. These preferred paths give rise to the "change of basepoint" equivalences in the right column of the homotopy commutative diagram

$$\begin{tikzcd}
H(M) \arrow[rr, "\text{(5-3)}"] \arrow[d, equal] & & \mathrm{hofib}_{M \times \underline{\mathbb{R}}}\big(\coprod_{[M']} B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})\big) \arrow[d, phantom, sloped, "\simeq"] \\
\coprod_{[M']} H(M, M') \arrow[d, phantom, sloped, "\simeq"] & & \coprod_{[M']} \mathrm{hofib}_{M' \times \underline{\mathbb{R}}}(B\mathrm{Diff}_\partial(M') \to B\mathrm{Diff}^b_\partial(M' \times \underline{\mathbb{R}})) \arrow[d, equal] \\
\coprod_{[M']} H(M', M') \arrow[rr, "\coprod_{[M']} \mathrm{alex}"] & & \coprod_{[M']} \boldsymbol{H}(M')_0
\end{tikzcd}$$

where alex: $H(M', M') \to \boldsymbol{H}(M')_0 \subset \widehat{\boldsymbol{H}}(M')_0$ is the restriction to $H(M', M') \subset H(M')$ of the map (5-3) for $M = M'$. Moreover if $d \geq 5$, this map is an isomorphism in π_0 (in fact it is an equivalence by the argument above); indeed, the inverse

$$\pi_0(\mathrm{Diff}^b_\partial(M' \times \underline{\mathbb{R}})/\mathrm{Diff}_\partial(M')) \to \pi_0(H(M', M')) \subset \mathrm{Wh}(M)$$

sends the coset $[\phi]$ of some bounded diffeomorphism $\phi \in \mathrm{Diff}^b_\partial(M' \times \underline{\mathbb{R}})$ — say bounded by $\frac{1}{2}$ for simplicity — to a partition of $M' \times [-1,1]$ whose W-part is the h-cobordism obtained as the region in $M' \times \underline{\mathbb{R}}$ between $M' \times \{0\}$ and $\phi(M' \times \{1\})$; see [Weiss and Williams 1988, Corollary 5.4]. It follows that the lower horizontal map, and hence (5-3), is an isomorphism in π_0. This proves the first claim.

The second claim is a consequence of the fact that $\widehat{\boldsymbol{H}}(M)_0 \hookrightarrow \Omega^\infty \widehat{\boldsymbol{H}}(M)$ is $(\phi(d){+}1)$-connected; indeed this map is $\phi(d)$-connected upon looping once by [loc. cit., Lemma 1.12], and is an isomorphism in π_0 by the analysis above and [loc. cit., Proposition 1.8 and Corollary 5.3]. □

Proposition 5.8 *The map* (5-3) *is C_2-equivariant up to homotopy. Therefore so is* (5-1).

Proof We will give an argument only in the topological setting; in the smooth setting one works with $H_{\mathrm{col}}(M)$ instead to smooth out corners, and uses smooth approximations of the continuous functions that will appear in the proof below. We will however state the argument in the smooth setting to simplify notation. We will also assume at any point in the argument where it is necessary that a partition (W, F, V) is equipped with some collar of F in W and V. We adopt the convention that $\pm\infty + r \equiv \pm\infty$ for any real number $r \in \mathbb{R}$.

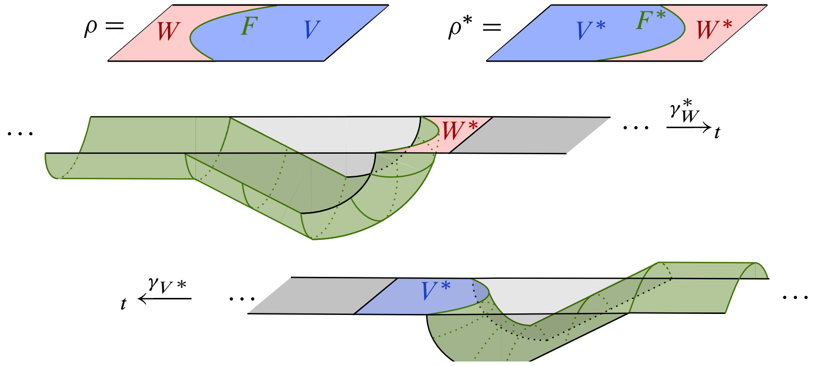

Figure 3: Paths γ_W^* and γ_{V^*} in $B\mathrm{Diff}^b_\partial(M\times\underline{\mathbb{R}})$. The arrow indicates the direction of the path as time increases.

The Weiss–Williams involution on $\widehat{\boldsymbol{H}}(M)_0$ is induced by the identity on $\coprod_{[M']} B\mathrm{Diff}_\partial(M')$ and the involution $U\mapsto U^* = (\mathrm{Id}_{\mathbb{R}^\infty}\times(-1)_{\underline{\mathbb{R}}})(U)$ on $B\mathrm{Diff}^b_\partial(M\times\underline{\mathbb{R}})$. Then for $\rho=(W,F,V)\in H^s(M)$,

$$\tau_{\mathrm{WW}}\circ\mathrm{alex}(W,F,V)=(i_\rho(F),\gamma_W^*),\quad \mathrm{alex}\circ\iota_H(W,F,V)=(i_{\rho^*}(F^*),\gamma_{V^*}),$$

where $\gamma_W^*(t):=(\gamma_W(t))^*$. We have depicted the paths γ_W^* and γ_{V^*} in Figure 3. We need to find a path $\eta\colon[-1,1]\to B\mathrm{Diff}_\partial(M)$ from $i_\rho^*(F^*)$ to $i_\rho(F)$ and a homotopy $\{H_s(-)\}_{-1\le s\le 1}$ from $\gamma_{V^*}(-)$ to $\gamma_W^*(-)$ such that $H_s(-\infty)=M\times\underline{\mathbb{R}}$ and $H_s(+\infty)=\eta(s)\times\underline{\mathbb{R}}$ for all $s\in[-1,1]$.

For η, we use the last two coordinates in $\mathbb{R}^{N+2}$ to do a half rotation of that plane. More explicitly, $\eta(s):=(\mathrm{Id}_{\mathbb{R}^N}\times Q_{\pi\cdot(s+1)/2})(i_\rho(F))$, where $Q_\theta\colon\mathbb{R}^2\cong\mathbb{R}^2$ is as before.

View $\mathbb{R}^\infty$ as $\mathbb{R}^\infty\times\underline{\{0\}}\subset\mathbb{R}^\infty\times\underline{\mathbb{R}}$ and write $N:=\bigcup_{s\in[-1,1]}\eta(s)+s\cdot\underline{e}$. For $X\subset\mathbb{R}^\infty\times\underline{\mathbb{R}}$, write $X|_{[a,b]}$ for $X\cap(\mathbb{R}^\infty\times\underline{[a,b]})$. Then consider the compact manifold

$$U_\rho:=(\hat{a}(\rho^*)|_{[-1,2]}-3\cdot\underline{e})\cup N\cup((\hat{a}(\rho)|_{[-1,2]})^*+3\cdot\underline{e})$$

depicted in Figure 4. Using the contractibility of $\mathrm{Emb}_\partial(F^*\times\underline{[-3,3]},\mathbb{R}^\infty\times\underline{[-3,3]})$, we obtain a path from $\overline{U_\rho\setminus(V^*\cup W^*)}$ (the green part in Figure 4) to a scaled (in the $\underline{\mathbb{R}}$-direction) version of the bicollar of F^* in W^* and V^*. Rescaling this bicollar back to normal whilst dragging V^* and W^* in the process, we obtain a path ψ from U_ρ to $M\times\underline{[-4,4]}=M\times\underline{[-4,-1]}\cup V^*\cup W^*\cup M\times\underline{[1,4]}$ in the moduli space of manifolds inside $\mathbb{R}^{N+2}\times\underline{[-4,4]}$ which are diffeomorphic to $M\times\underline{[-4,4]}$ relative to its boundary $M\times\underline{\{-4\}}\cup\partial M\times\underline{[-4,4]}\cup M\times\underline{\{4\}}$.

We now describe the homotopy $\{H_s(-)\}_{-1\le s\le 1}$. Fix some homeomorphism $l\colon[-1,1]\cong[-\infty,+\infty]$, and assume that the path ψ from $M\times\underline{[-4,4]}$ to U_ρ just described is parametrised by $[-\infty,+\infty]$. Then $H_s(-)$ is the concatenation of two paths $H_s^{(1)}(-)$ and $H_s^{(2)}(-)$ in $B\mathrm{Diff}^b_\partial(M\times\underline{\mathbb{R}})$: The path $H_s^{(1)}(-)$ performs $\psi(-)$ on $M\times\underline{[-4,4]}+l(s)\cdot\underline{e}\subset M\times\underline{\mathbb{R}}$ (if $s=\pm1$, $H_s^{(1)}(-)$ is constant on $M\times\underline{\mathbb{R}}$). The path $H_s^{(2)}(-)$ starts at $H_s^{(1)}(+\infty)$, and sends $H_s^{(1)}(+\infty)|_{(-\infty,l(s)+s]}$ and $H_s^{(1)}(+\infty)|_{[l(s)+s,+\infty)}$ towards $\mathbb{R}^{N+2}\times\underline{\pm\infty}$, respectively, extending by $H_s^{(1)}(+\infty)|_{l(s)+s}$ times an interval of diverging length. The resulting paths $H_{\pm1}(-)=H_{\pm1}^{(1)}(-)\cdot H_{\pm1}^{(2)}(-)$ are reparametrisations of γ_{V^*} and γ_{W^*}

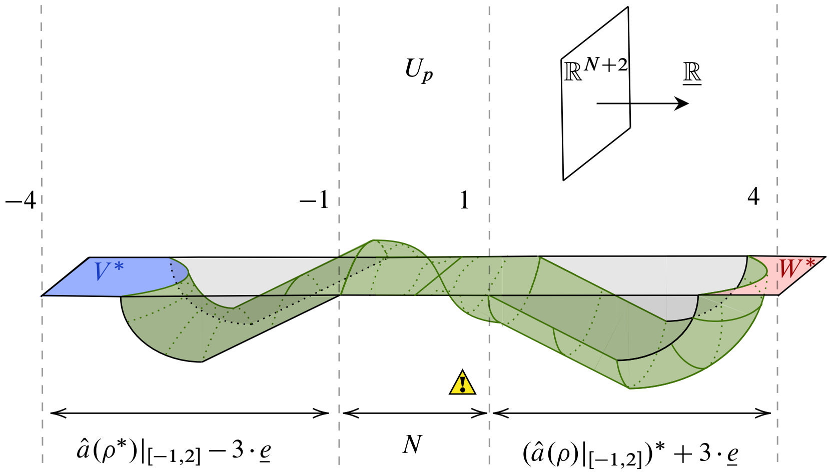

Figure 4: The manifold U_ρ. Proceed with caution: the part of the picture corresponding to N takes place in an extra dimension that we are unable to depict accurately.

(the reparametrisations only depend on our choice of homeomorphism $l\colon [-1,1] \cong [-\infty,+\infty]$ and the parametrisation of the path ψ). Thus η and H give rise to the required homotopy $\tau_{\mathrm{WW}} \circ \mathrm{alex} \simeq \mathrm{alex} \circ \iota_H$. $\square$

Corollary 5.9 *The Weiss–Williams involution on* $\pi_0^s(\boldsymbol{H}(M)) \cong \pi_0(H(M)) \cong \mathrm{Wh}(\pi_1 M)$ *corresponds to the rule* $\kappa \mapsto (-1)^{d-1}\bar{\kappa}$*, where* $\overline{(-)}$ *is Milnor's involution* [1966] *on* $\mathrm{Wh}(\pi_1(M))$ (see Warning 5.21).

Proof The isomorphism $\pi_0(H(M)) \cong \mathrm{Wh}(\pi_1 M)$ sends the class $[\rho]$ of a partition $\rho = (W, F, V)$ to the Whitehead torsion $\tau(W, M)$ of W with respect to M. The claim follows from Proposition 5.8, the *duality formula* of [Milnor 1966, Section 10], and the fact that, upon identifying $\pi_1 M \cong \pi_1 W \cong \pi_1 F$, we have $\tau(V, F) = -\tau(W, M)$; see [Milnor 1966, Lemma 7.8]. $\square$

Recall that $H^s(M) \simeq BC(M)$. Now if $P \subset M$ is a codimension-zero embedding and $p \le d-3$ (in the notation of Theorem A), then $C\mathrm{Emb}(P, M) \simeq \mathrm{hofib}(H^s(\overline{M-P}) \to H^s(M))$ by the isotopy extension sequence (3-3), and therefore $C\mathrm{Emb}(P, M)$ inherits an involution ι_H up to weak equivalence. The map (5-3) is functorial with respect to codimension-zero embeddings, so the following diagram is commutative:

(5-4)
$$\begin{tikzcd}
H^s(\overline{M-P}) \arrow[r, "\mathrm{alex}"] \arrow[d] & \boldsymbol{H}(\overline{M-P})_0 \arrow[r, hook] \arrow[d] & \Omega^\infty(\boldsymbol{H}(\overline{M-P})) \arrow[d] \\
H^s(M) \arrow[r, "\mathrm{alex}"] & \boldsymbol{H}(M)_0 \arrow[r, hook] & \Omega^\infty(\boldsymbol{H}(M))
\end{tikzcd}$$

Corollary 5.10 *The vertical homotopy fibre of the horizontal compositions in* (5-4) *gives a map*

$$\mathrm{alex}\colon C\mathrm{Emb}(P, M) \to \Omega^\infty(\mathbf{CE}(P, M))$$

which is $\phi_{C\mathrm{Emb}}(d, p)$*-connected and* C_2*-equivariant up to homotopy.*

Proof The connectivity of this map is the content of Proposition 3.11. It is homotopy C_2-equivariant since both $\mathrm{alex}\colon H(M) \to \Omega^\infty(\boldsymbol{H}(M))$ and $\mathrm{alex}\colon H(\overline{M-P}) \to \Omega^\infty(\boldsymbol{H}(\overline{M-P}))$ are by Proposition 5.8. $\square$

Warning 5.11 There is a canonical involution ι_C in the concordance space $C(M)$ given by turning a concordance upside down and precomposing by the inverse of the top diffeomorphism; see, e.g., [Vogell 1985, p. 296]. The restriction map $C(M) \to C\mathrm{Emb}(P, M)$ is *not* C_2-equivariant with respect to ι_C and ι_H — rather, it is antiequivariant. This may seem to contradict [Vogell 1985, Proposition 2.2], but what Vogell really proves there is that there is a homotopy C_2-equivariant equivalence $C(M) \approx \Omega^\sigma H(M) := \mathrm{Map}_*(S^\sigma, H(M))$, where we recall S^σ stands for the representation sphere of the 1-dimensional sign representation σ. This is due to an extra flip in the loop component that he introduces at the end of the proof of the proposition.

5.3 From h-cobordism spaces back to A-theory

Given a spherical fibration ξ over M equipped with a section, fibrewise smashing a retractive space over M with ξ gives rise to a functor $-\cdot\xi\colon \boldsymbol{A}(M) \to \boldsymbol{A}(M)$ which, by [Vogell 1985, Proposition 2.5], makes the diagram

$$\begin{tikzcd} \boldsymbol{A}(M) \arrow[r, "\tau_\epsilon"] \arrow[dr, "\tau_\xi"'] & \boldsymbol{A}(M) \arrow[d, "-\cdot\xi"] \\ & \boldsymbol{A}(M) \end{tikzcd} \tag{5-5}$$

homotopy commutative, where $\epsilon := \epsilon^0 = M \times S^0$ is the trivial 0-dimensional sphere bundle over M. When $\xi = \epsilon^d = M \times S^d$ is the trivial d-spherical fibration, $-\cdot\xi$ corresponds to $\Sigma^d_M(-)\colon \boldsymbol{A}(M) \to \boldsymbol{A}(M)$, the d-fold *fibrewise suspension over* M; see [Vogell 1985, p. 281]. By the *additivity theorem* of [Waldhausen 1985, Proposition 1.6.2] applied to the Waldhausen category of retractive spaces over M, it follows that Σ^d_M acts (up to homotopy) as $(-1)^d$ on $\boldsymbol{A}(M)$, and thus by (5-5),

$$\xi = M \times S^d \implies \boldsymbol{A}(M;\xi) \approx S^{d\cdot(\sigma-1)} \wedge \boldsymbol{A}(M;\epsilon). \tag{5-6}$$

Vogell [1985, p. 299] also introduced a model for the homotopy fibre sequence

$$\mathcal{H}(M) \to Q_+M \to A(M) := \Omega^\infty \boldsymbol{A}(M) \tag{5-7}$$

of the parametrised h-cobordism theorem of Waldhausen, Jahren, and Rognes [Waldhausen et al. 2013] (see Remark 5.12), and equipped each of the terms in the sequence with compatible homotopy involutions. The one on $\mathcal{H}(M)$ is compatible with ι_H on $H(M) \subset \mathcal{H}(M)$.

Remark 5.12 Vogell's work precedes (by more than 25 years) that of Waldhausen, Jahren, and Rognes, so let us explain how both fit together. Vogell [1985, p. 299] presents a commutative square with compatible homotopy involutions in each of the terms, and this square is equivalent to the one considered by Waldhausen [1982, p. 8] in his "manifold approach" paper. This latter square is, up to equivalence, of the form

$$\begin{tikzcd} \mathcal{H}(M) \arrow[r] \arrow[d] & Q \arrow[d] \\ * \arrow[r] & A \end{tikzcd}$$

for some spaces Q and A, and the goal of that paper was to argue that (i) the square becomes homotopy cartesian upon plus-constructing the vertical right map, and (ii) that $A^+ \simeq A(M)$; these are, respectively, Propositions 5.5 and 5.4 in [Waldhausen 1982]. While (ii) was fully proved there, only an outline of the argument for (i) was provided, with forward references to a preliminary version of [Waldhausen et al. 2013].

As stated at the very end of page 22 in [loc. cit.], the square considered by Waldhausen (after plus-constructing the vertical right arrow) is equivalent to the homotopy cartesian square of [loc. cit., Proposition 1.4.8], which in turn is equivalent to one giving rise to the fibre sequence (5-7).

All of the above takes place in the PL-setting, but as explained in [loc. cit., pages 15–16], these arguments also deal with the remaining categories Top and Diff.

Going back to Vogell's work, his homotopy involution[9] $\mathcal{T}$ on $A(M)$ is further showed in [Vogell 1985, Corollary 2.10] to agree up to equivalence with the involution τ_ξ when ξ is the d-spherical fibration associated to the once-stabilised tangent bundle $TM \oplus \epsilon^1$ of M^d.

The upshot of this discussion then is that when M^d is stably parallelisable, the Weiss–Williams involution is compatible with $(-1)^d \tau_\epsilon$ in the following sense:

Theorem 5.13 *If M is stably parallelisable, then there is an equivalence away from two*

$$\Omega^\infty(\boldsymbol{H}(M)_{hC_2}) \simeq_{[\frac{1}{2}]} \Omega^\infty\big((S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon))_{hC_2}\big).$$

Remark 5.14 Though it probably is, we do not claim the equivalence above is one of infinite loop spaces.

We will need the a few preliminary results for the proof of Theorem 5.13.

Lemma 5.15 *For each $k \geq 0$, there is a natural C_2-equivariant equivalence of spectra*

$$e_k\colon \boldsymbol{H}(M \times I^k) \simeq S^{k\cdot(\sigma-1)} \wedge \boldsymbol{H}(M) \tag{5-8}$$

such that the following square is homotopy commutative:

$$\begin{tikzcd}
\boldsymbol{H}(M \times I^k) \arrow[r, "\sim", "e_1"'] \arrow[d, "\wr"', "e_k"] & S^{\sigma-1} \wedge \boldsymbol{H}(M \times I^{k-1}) \arrow[d, "\wr"', "S^{\sigma-1}\wedge e_{k-1}"] \\
S^{k\cdot(\sigma-1)} \wedge \boldsymbol{H}(M) \arrow[r, equal] & S^{k\cdot(\sigma-1)} \wedge \boldsymbol{H}(M)
\end{tikzcd}$$

Proof Set $e_0 = \mathrm{Id}_{\boldsymbol{H}(M)}$. By inductively defining e_k to fit in the commutative square above, we may assume that $k = 1$. Recall $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b\cdot\sigma$, and for any orthogonal functor $F(-)$ let $C_2 = O(1)$ act on $F(\mathbb{R}^{a,b})$ by the induced action. Finally let $B(-) := B\mathrm{Diff}^b_\partial(M \times (-))$ and $BI(-) := B\mathrm{Diff}^b_\partial(M \times I \times (-))$. The Alexander trick-like map of [Weiss and Williams 1988, Proposition 1.5] is a C_2-equivariant map

$$\mathrm{alex}\colon \mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^{a,b}) \xrightarrow{\sim} \Omega\mathrm{Diff}^b_\partial(M \times \mathbb{R}^{a+1,b}).$$

---

[9] Vogell refers to $\mathcal{T}$ as a *weak* involution in the sense that it is a homotopy involution when restricted to "any compactum" (see [Vogell 1985, Lemma 2.4]) or, in better words, to each stage of the colimit in [loc. cit., p. 299] modelling $A(M)$.

Upon delooping, this gives a C_2-equivariant equivalence on basepoint components $B(\mathrm{alex})\colon BI(\mathbb{R}^{a,b}) \simeq_0 \Omega B(\mathbb{R}^{a+1,b})$. Writing Ξ for the C_2-spectrum whose n-th space is $B^{(1)}(\mathbb{R}^{1,n+1})$ and with stabilisation maps $s_{0,1}\colon S^1 \wedge B^{(1)}(\mathbb{R}^{1,n}) \to B^{(1)}(\mathbb{R}^{1,n+1})$, we obtain a C_2-equivariant equivalence of spectra

$$B(\mathrm{alex})\colon \boldsymbol{H}(M\times I) := \Theta(BI)^{(1)} \xrightarrow{\sim} \Xi := \{B^{(1)}(\mathbb{R}^{1,n+1})\}_{n\geq 0}. \tag{5-9}$$

But now the stabilisation map $s_{1,0}\colon S^{\sigma} \wedge B^{(1)}(\mathbb{R}^{0,n}) \to B^{(1)}(\mathbb{R}^{1,n}) = \Xi_{n-1}$ induces another C_2-equivariant equivalence of spectra

$$S^{\sigma-1} \wedge \boldsymbol{H}(M) \xrightarrow{\sim} \Xi.$$

Composing these two equivalences gives the one in the statement. □

Vogell [1985, p. 298] introduced the *lower* and *upper stabilisation maps* $\Sigma_\ell, \Sigma_u\colon H(M) \to H(M\times I)$. Roughly, the former sends a partition $\rho = (W, F, V)$ to $(U(W), W \cup_F W, \overline{M\times I\times I \setminus U(W)})$, where $U(W)$ is obtained from W by bending $W \times I$ into a U-shape, whilst Σ_u does the same to V instead of W (see Figure 7 for a pictorial representation of Σ_ℓ). We will only be interested in the lower stabilisation Σ_ℓ, which we will denote by Σ for simplicity. Here's how it interacts with the h-cobordism involution ι_H:

Lemma 5.16 *Let $+_I$ stand for the "stacking in the I-direction" $\mathbb{E}_1$-algebra structure*[10] *in $H(M\times I)$. Then if J denotes another copy of I,*

(a) $\iota_H\Sigma +_I \Sigma\iota_H \simeq *\colon H(M) \to H(M\times I)$,

(b) $\iota_H\Sigma^2 \simeq \Sigma^2\iota_H\colon H(M) \to H(M\times I\times J)$.

Proof We defer the proof of (a) to Lemma C.1 in Appendix C as it is a bit technical. Note that $+_I$ and $\Sigma\colon H(M\times I) \to H(M\times I\times J)$ are compatible in the sense that

$$\Sigma(\rho +_I \rho') = \Sigma\rho +_I \Sigma\rho', \quad \rho, \rho' \in H(M\times I).$$

Then (b) follows from

$$\iota_H\Sigma^2 \simeq \iota_H\Sigma^2 +_I \Sigma(\iota_H\Sigma +_I \Sigma\iota_H) \simeq (\iota_H\Sigma +_I \Sigma\iota_H)\Sigma +_I \Sigma^2\iota_H \simeq \Sigma^2\iota_H.$$ □

Proof of Theorem 5.13 We may assume without loss of generality that $\dim M \geq 5$, for if not replace it by $M \times J^{2k}$ for $k \geq 3$. The effect this has on both sides of the equivalence in the statement is rather mild: as there is a homotopy C_2-equivariant equivalence $S^2 \approx S^{2\sigma}$, it follows by Lemma 5.15 and Corollary 5.4(i) that there are equivalences of spectra

$$\boldsymbol{H}(M\times J^{2k})_{hC_2} \simeq_{[\frac{1}{2}]} \boldsymbol{H}(M)_{hC_2},$$
$$(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M\times J^{2k};\epsilon))_{hC_2} \simeq_{[\frac{1}{2}]} (S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon))_{hC_2}.$$

In the second equivalence we also use that $\mathbf{Wh}^{\mathrm{Diff}}(-)$ and τ_ϵ are homotopy invariants of $(-)$.

---

[10] The technical assumption we imposed on a partition $\rho = (W, F, V) \in H(M\times I)$ so that the intersection of F with $\partial(M\times I)\times[-1,1]$ is standard and happens exactly at $\partial(M\times I)\times\{0\}$ makes $+_I$ well-defined.

Let $k \geq 0$ and let ξ denote the $(d+k)$-spherical fibration corresponding to the stable tangent bundle $T(M \times I^k) \oplus \epsilon^1$. If M (and hence $M \times I^k$) is stably parallelisable, the involution $(-1)^{d+k}\tau_\epsilon$ is homotopic to τ_ξ by (5-6), and by [Vogell 1985, Corollary 2.10] it agrees with $\mathcal{T}$ (and hence extends ι_H). All in all, we obtain a zigzag of homotopy C_2-equivariant maps

$$\begin{tikzcd}
\Omega^\infty \boldsymbol{H}(M \times I^k) \arrow[d, dash, "\wr", "(5\text{-}8)"'] & H(M \times I^k) \arrow[l, "\mathrm{alex}", "(5\text{-}1)"'] \arrow[r, "(\dagger)"] & \Omega^\infty(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)) \arrow[d, dash, "\wr"] \\
\Omega^\infty(S^{k\cdot(\sigma-1)} \wedge \boldsymbol{H}(M)) & & \Omega^\infty(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M))
\end{tikzcd} \tag{5-10}$$

Every space involved in (5-10) is an $\mathbb{E}_1$-group if $k \geq 1$: both $H(M \times I^k)$ and $\Omega^\infty(\boldsymbol{H}(M \times I^k))$ by stacking in the first of the I^k-coordinates, and the others by their own infinite loop structures.

Claim 1 *All of the maps in* (5-10) *are* H*-maps if* $k \geq 1$.

Proof The $\mathbb{E}_1$-algebra structure $+_I$ on $\Omega^\infty \boldsymbol{H}(M \times I^k)$ is equivalent to any of the other ones coming from its infinite loop structure. As (5-8) and the right vertical equivalence are infinite loop maps, they are in particular H-maps. As for (5-1), it is an $\mathbb{E}_1$-map since (5-3): $H(M \times I^k) \to \widehat{\boldsymbol{H}}(M \times I^k)_0$ is (by construction).

Nonequivariantly, $(\dagger)$ is the composition $(\dagger)$: $H(M \times I^k) \hookrightarrow \mathcal{H}(M \times I^k) \simeq \Omega^{\infty+1}\mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)$, where the last equivalence is the stable parametrised h-cobordism theorem of Waldhausen, Jahren, and Rognes [Waldhausen et al. 2013, Theorem 0.1]. As communicated to us in private by Bjørn Jahren and John Rognes, such equivalence is only stated to hold in the category of spaces (and not of infinite loop spaces, though it should definitely also hold there). We now explain why this equivalence is one of H-groups (which is the general consensus, but we couldn't find it written down anywhere): again assume $k = 1$. One reduces to the PL-case as in [loc. cit., pages 15–16]. Then if X is a simplicial set such that $|X| \simeq M$, the equivalence in the PL-setting is induced by a zigzag of equivalences of simplicial sets (see the left vertical column of [loc. cit., (0.4)])

$$\mathcal{H}(M \times I)_\bullet \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} s\mathcal{C}^h(X \times I),$$

where $s\mathcal{C}^h(X \times I)$ is the category (seen as a simplicial set by taking its nerve) of finite[11] acyclic cofibrations $X \times I \hookrightarrow Y$ together with simple maps over $X \times I$. One can verify that it makes sense to stack in the I-direction in each of the simplicial sets involved in the zigzag, and that the maps between them respect this monoidal structure. Let $\mu_0\colon s\mathcal{C}^h(X \times I) \times s\mathcal{C}^h(X \times I) \to s\mathcal{C}^h(X \times I)$ stand for this monoidal structure, given explicitly by $\mu_0(Y, Z) := Y_{X\times 1} \cup_{X \times 0} Z$. There is another monoidal structure μ_1 induced by the pushout along $X \times I$, i.e., $\mu_1(Y, Z) := Y \cup_{X \times I} Z$. Sliding gives a homotopy between μ_0 and μ_1: intuitively, the maps

$$\mu_t\colon s\mathcal{C}^h(X \times I) \times s\mathcal{C}^h(X \times I) \to s\mathcal{C}^h(X \times I), \quad (Y, Z) \mapsto Y_{X \times [1-t,1]} \cup_{X \times [0,t]} Z, \quad t \in [0,1],$$

---

[11] This means that Y is generated by the image of $X \times I \hookrightarrow Y$ and finitely many simplices.

constitute the homotopy. More precisely, consider the simplicial category $s\widetilde{\mathcal{C}}^h_\bullet(X)$ [Waldhausen et al. 2013, Definition 3.1.1] whose objects in simplicial degree q consist of commutative diagrams

$$\begin{tikzcd}
X \times \Delta^q_\bullet \arrow[rr, hook, "i", "\sim"'] \arrow[dr, "\mathrm{pr}"] & & Y \arrow[dl, "\pi"] \\
& \Delta^q_\bullet &
\end{tikzcd}$$

where i is an acyclic cofibration and π is a Serre fibration. The inclusion $s\mathcal{C}^h(X) \hookrightarrow s\widetilde{\mathcal{C}}^h_\bullet(X)$ as the 0-simplices is a homotopy equivalence by [loc. cit., Corollary 3.5.2], where a simplicial category is seen as a bisimplicial set by taking its nerve, and a bisimplicial set as an ordinary simplicial set by taking its totalisation. The monoidal structures μ_t make perfectly good sense in $s\widetilde{\mathcal{C}}^h_\bullet(X \times I)$, and one can indeed define a simplicial homotopy between μ_0 and μ_1 in this setting resembling the idea above.

But by Proposition 3.1.1 and Theorems 3.1.7 and 3.3.1 of [Waldhausen 1985] (see also [Waldhausen et al. 2013, p. 5]), for any simplicial set T, there is a zigzag of equivalences connecting $|s\mathcal{C}^h(T)|$ and $\Omega\mathrm{Wh}^{\mathrm{PL}}(T) := \Omega^{\infty+1}(\mathbf{Wh}^{\mathrm{PL}}(T))$ which is monoidal up to homotopy with respect to μ_1 in the domain and the loop structure on the looped Whitehead space. It hence follows that (†) is indeed a zigzag of H-maps. □

Claim 2 *For each $k \geq 1$, the diagram*

$$\begin{tikzcd}
\Omega^\infty(S^{2k\cdot(\sigma-1)} \wedge \boldsymbol{H}(M)) \arrow[d, "\wr"] & H(M \times I^{2k}) \arrow[l] \arrow[r] \arrow[d, "\Sigma^2"] & \Omega^\infty(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \arrow[d, "\wr"] \\
\Omega^\infty(S^{(2k+2)\cdot(\sigma-1)} \wedge \boldsymbol{H}(M)) & H(M \times I^{2k+2}) \arrow[l] \arrow[r] & \Omega^\infty(S^{(d+2k+2)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M))
\end{tikzcd} \tag{5-11}$$

is homotopy commutative, where the rows are the zigzags (5-10) *and the vertical external maps are induced by the homotopy C_2-equivariant equivalence $\mathbb{S}^0 \approx \mathbb{S}^{2\cdot(\sigma-1)}$.*

Proof The right-hand square is clearly commutative, for both right horizontal maps factor through $\mathcal{H}(M)$. For the commutativity of the left one, by Lemma 5.15 it suffices to argue that the square

$$\begin{tikzcd}
H(M \times I^{2k}) \arrow[r, "\mathrm{alex}"] \arrow[d, "\Sigma"] & \Omega^\infty \boldsymbol{H}(M \times I^{2k}) \arrow[d, dash, "e_1"', "\wr"] \\
H(M \times I^{2k+1}) \arrow[r, "\mathrm{alex}"] & \Omega^\infty \boldsymbol{H}(M \times I^{2k+1})
\end{tikzcd} \tag{5-12}$$

homotopy commutes (nonequivariantly).[12] Recall that the map e_1, nonequivariantly, is induced by the zigzags

$$\boldsymbol{H}(M \times I^{2k})_n \xrightarrow{s^\vee} \Omega \boldsymbol{H}(M \times I^{2k})_{n+1} \xleftarrow[\sim]{\mathrm{alex}} \boldsymbol{H}(M \times I^{2k+1})_n,$$

---

[12] Homotopy C_2-equivariant maps that are homotopic (as ordinary maps) induce the same morphism in the homotopy category of C_2-spaces.

where $s^\vee$ is the (adjoint to the) structure map of $\boldsymbol{H}(M \times I^{2k})$. Since (5-12) is a diagram of $\mathbb{E}_1$-groups by stacking in the first of the I^{2k}-coordinates, it suffices to provide a homotopy for the diagram

$$\begin{tikzcd}
H^s(M \times I^{2k}) \arrow[rr, "\text{(5-3)}"] \arrow[d, "\Sigma"] & & \boldsymbol{H}(M \times I^{2k})_0 \arrow[d, "s^\vee"] \\
H^s(M \times I^{2k+1}) \arrow[r, "\text{(5-3)}"] & \boldsymbol{H}(M \times I^{2k+1})_0 \arrow[r, "\text{alex}"] & \Omega \boldsymbol{H}(M \times I^{2k})_1
\end{tikzcd}$$

As in Remark 3.9, such homotopy is obtained by delooping (with respect to stacking in the second of the I^{2k}-coordinates) the diagram (3-5) of Proposition 3.8. □

Clearly Σ^2 is an H-map and also homotopy C_2-equivariant (with respect to ι_H) by Lemma 5.16(b). Therefore by Claim 1, all of the maps involved in (5-11) are H-maps and homotopy C_2-equivariant. Taking the homotopy colimit as $k \to \infty$, we obtain a homotopy C_2-equivariant zigzag

$$\Omega^\infty \boldsymbol{H}(M) \xleftarrow{\approx} \operatorname*{hocolim}_k H(M \times I^{2k}) \xrightarrow{\approx} \Omega^\infty(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \tag{5-13}$$

of H-maps. The connectivity of, say, the upper horizontal maps in (5-11) is $\phi(d+2k) \gtrsim \frac{1}{3}(d+2k)$ by Igusa's theorem and, as this lower bound increases linearly with k, the horizontal maps in (5-13) are indeed equivalences. The equivalence in the statement now follows by Corollary 5.4(ii) applied to (5-13), and because taking homotopy C_2-orbits commutes up to equivalence with $\Omega^\infty(-)$ if 2 is inverted (as in Corollary 5.4). □

Corollary 5.17 *If M^d is stably parallelisable and $P \subset M^d$ is a codimension-zero submanifold with* $p \le d-3$ (in the notation of Theorem A), *then there is an equivalence away from* 2,

$$\Omega^\infty(\mathbf{CE}(P,M)_{hC_2}) \simeq_{[\frac{1}{2}]} \Omega^\infty\big((S^{d\cdot(\sigma-1)-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M, M-P;\epsilon))_{hC_2}\big),$$

where $\mathbf{Wh}^{\mathrm{Diff}}(M, M-P;\epsilon)$ *stands for the homotopy cofibre of* $\mathbf{Wh}^{\mathrm{Diff}}(M-P;\epsilon) \to \mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)$.

Proof Note that $\overline{M-P}$ is stably parallelisable because M is. The zigzag (5-13) is functorial with respect to codimension-zero embeddings of stably parallelisable manifolds, and hence taking homotopy fibres in the map from (5-13) with M replaced by $\overline{M-P}$ to (5-13) itself, we obtain another homotopy C_2-equivariant zigzag of equivalences

$$\Omega^\infty\,\mathbf{CE}(P,M) \xleftarrow{\approx} \operatorname*{hocolim}_k C\mathrm{Emb}(P \times I^{2k}, M \times I^{2k}) \xrightarrow{\approx} \Omega^\infty(S^{d\cdot(\sigma-1)-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M, M-P)).$$

The same line of reasoning as before yields the desired result. □

5.4 The canonical involution in algebraic K-theory

We now define the canonical involution τ_ϵ on $A(X)$, for X based, and relate it to an involution in the model of A-theory via "spaces of matrices with values in the ring up to homotopy" $Q_+\Omega X$ [Waldhausen 1985, Section 2.2]. Throughout, let $G := GX$ denote the topological monoid of Moore loops on X, and write $\mathbb{S}[G]$ for the $\mathbb{E}_1$-ring spectrum $\mathbb{S} \wedge G_+$.

We work over the ∞-category $\mathrm{Mod}_{\mathbb{S}[G]}$ of right $\mathbb{S}[G]$-module spectra; we also write ${}_{\mathbb{S}[G]}\mathrm{Mod}$ for the ∞-category of left $\mathbb{S}[G]$-modules. Then for $m \geq 1$, if $\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)$ denotes the homotopy invertible components of the mapping space $\mathrm{Mod}_{\mathbb{S}[G]}\bigl(\bigoplus^m \mathbb{S}[G], \bigoplus^m \mathbb{S}[G]\bigr)$, Waldhausen [1985, Theorem 2.2.1] showed that for X connected, there is a natural equivalence

$$A(X) \simeq \mathbb{Z} \times \operatorname*{hocolim}_m B\,\mathrm{Aut}_G\biggl(\bigoplus^m \mathbb{S}[G]\biggr)^+. \tag{5-14}$$

In order to define τ_ϵ, we will introduce compatible anti-involutions on $\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)$ defined in terms of Spanier–Whitehead duality. As $\mathbb{S}[G]$ is not commutative, this duality really arises as an instance of a *duality in the symmetric closed bicategory* $\mathrm{Bimod}_{\mathbb{S}}$ of bimodule spectra, in the sense of May and Sigurdsson [2006, Section 16.4]. This duality coincides with the one considered by Vogell [1985, Section 1].

Remark 5.18 We can safely import the duality theory of May and Sigurdsson [2006]: even though it is developed only 2-categorically, and $\mathrm{Bimod}_{\mathbb{S}}$ (aka the Morita category of the sphere spectrum [Haugseng 2017]) is an $(\infty, 2)$-category, the arguments that rely on duality only involve the homotopy 2-category $\mathrm{Ho}_2(\mathrm{Bimod}_{\mathbb{S}})$, which is *symmetric closed* in the sense of [May and Sigurdsson 2006, Definitions 16.2.1 and 16.3.1].

First observe that a right $\mathbb{S}[G]$-module M can always be regarded as a left $\mathbb{S}[G]$-module by

$$\mathbb{S}[G] \otimes M \xrightarrow{\text{swap}} M \otimes \mathbb{S}[G] \xrightarrow{\mathrm{Id}_M \otimes \mathrm{inv}} M \otimes \mathbb{S}[G^{\mathrm{op}}] = M \otimes \mathbb{S}[G] \xrightarrow{\text{act}} M,$$

where “inv” stands for inversion in the monoid G — write M_ℓ for this left $\mathbb{S}[G]$-module. Here $\otimes = \otimes_{\mathbb{S}}$ stands for the usual smash product of spectra. Note also that

$${}_{\mathbb{S}[G]}\mathrm{Mod}(M_\ell, M_\ell) \simeq \mathrm{Mod}_{\mathbb{S}[G]}(M, M)$$

as $\mathbb{E}_1$-algebras. If $\nu\colon \mathbb{S} \to \mathbb{S}[G]_\ell$ denotes the unit, consider the map of spectra

$$\eta_1\colon \mathbb{S} \simeq \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S} \xrightarrow{1\otimes\nu} \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S}[G]_\ell$$

and the map of $(\mathbb{S}[G], \mathbb{S}[G])$-bimodules

$$I_1\colon \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \xrightarrow{\mathrm{inv}\otimes 1} \mathbb{S}[G] \otimes \mathbb{S}[G] \xrightarrow{\text{act}} \mathbb{S}[G].$$

Then (η_1, I_1) exhibits $(\mathbb{S}[G], \mathbb{S}[G]_\ell)$ as a dual pair in the sense of Definition 16.4.1 of [May and Sigurdsson 2006] by Example 16.4.3[13] of [loc. cit.]. More generally, the map of spectra

$$\eta_m\colon \mathbb{S} \xrightarrow{\bigoplus_{i,j} \delta_{ij}\eta_1} \bigoplus_{i,j=1}^m \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \mathbb{S}[G]_\ell \cong \bigoplus_{j=1}^m \mathbb{S}[G] \otimes_{\mathbb{S}[G]} \biggl(\bigoplus_{i=1}^m \mathbb{S}[G]\biggr)_\ell$$

---

[13] This example is really concerned with ordinary rings and modules, but the same argument applies to bimodule spectra. Moreover, it really shows that $\mathbb{S}[G]$, as a right $\mathbb{S}[G]$-module, is *left* dual to $\mathbb{S}[G]$ as a left $\mathbb{S}[G]$-module. We have identified the latter with $\mathbb{S}[G]_\ell$ via the isomorphism of left $\mathbb{S}[G]$-modules $\mathrm{inv}\colon \mathbb{S}[G]_\ell \cong \mathbb{S}[G]$.

together with the map of $(\mathbb{S}[G],\mathbb{S}[G])$-bimodules

$$I_m\colon \Big(\bigoplus_{i=1}^{m} \mathbb{S}[G]\Big)_{\ell} \otimes \bigoplus_{j=1}^{m} \mathbb{S}[G] \cong \bigoplus_{i,j=1}^{m} \mathbb{S}[G]_{\ell} \otimes \mathbb{S}[G] \xrightarrow{\bigoplus_{i,j}\delta_{ij} I_1} \mathbb{S}[G],$$

exhibit $\big(\bigoplus^m \mathbb{S}[G]\big)_\ell$ as a *right* dual to $\bigoplus^m \mathbb{S}[G]$. (This pretty much follows from $(\mathbb{S}[G],\mathbb{S}[G]_\ell)$ being a dual pair.) Therefore, by [loc. cit., Proposition 16.4.9], I_m induces an equivalence of left $\mathbb{S}[G]$-modules

$$\tilde{I}_m\colon \Big(\bigoplus^{m} \mathbb{S}[G]\Big)_{\ell} \simeq D_r\Big(\bigoplus^{m} \mathbb{S}[G]\Big) := \underline{\mathrm{Hom}}_{\mathbb{S}[G]}\Big(\bigoplus^{m} \mathbb{S}[G], \mathbb{S}[G]\Big),$$

where $\underline{\mathrm{Hom}}_{\mathbb{S}[G]}$ denotes the right $\mathbb{S}[G]$-linear mapping spectrum.

With (5-14) and Lemma 5.6 in mind, the involution τ_ϵ on $A(X;\epsilon)$ is then induced by the map of $\mathbb{E}_1$-algebras

$$\mathrm{Aut}_G\Big(\bigoplus^{m} \mathbb{S}[G]\Big) \xrightarrow{D_r} {}_G\,\mathrm{Aut}\Big(D_r\Big(\bigoplus^{m} \mathbb{S}[G]\Big)\Big)^{\mathrm{op}} \overset{\tilde{I}_\#}{\simeq} {}_G\,\mathrm{Aut}\Big(\Big(\bigoplus^{m} \mathbb{S}[G]\Big)_{\ell}\Big)^{\mathrm{op}} \simeq \mathrm{Aut}_G\Big(\bigoplus^{m} \mathbb{S}[G]\Big)^{\mathrm{op}}, \tag{5-15}$$

where $\tilde{I}_\#$ stands for conjugation with the equivalence $\tilde{I}_m$. It will be convenient to think of (5-15) in the following way: Let $\mathrm{GL}_m(Q_+G)$ denote the union of path components in $(Q_+G)^{m\times m}$ in the image of $\mathrm{Aut}_G\big(\bigoplus^m \mathbb{S}[G]\big)$ under the natural equivalence

$$u\colon \mathrm{Mod}_{\mathbb{S}[G]}\Big(\bigoplus^{m} \mathbb{S}[G], \bigoplus^{m} \mathbb{S}[G]\Big) \xrightarrow{\sim} \mathrm{Mod}_{\mathbb{S}[G]}\Big(\bigoplus^{m} \mathbb{S}[G], \prod^{m} \mathbb{S}[G]\Big) \simeq \mathrm{Sp}(\mathbb{S},\mathbb{S}[G])^{m\times m} \simeq (Q_+G)^{m\times m},$$

where $\mathrm{Sp}\simeq\mathrm{Mod}_{\mathbb{S}}$ stands for the ∞-category of spectra. So $u\colon \mathrm{Aut}_G\big(\bigoplus^m \mathbb{S}[G]\big)\simeq \mathrm{GL}_m(Q_+G)$ and, just as in standard linear algebra, under this equivalence the anti-involution (5-15) corresponds to the rule that sends a matrix A to its conjugate transpose $A^\dagger$ (conjugate with respect to inversion of G in $\mathbb{S}[G]$). More precisely:

Proposition 5.19 *Write* $\mathrm{End}_G\big(\bigoplus^m \mathbb{S}[G]\big) := \mathrm{Mod}_{\mathbb{S}[G]}\big(\bigoplus^m \mathbb{S}[G], \bigoplus^m \mathbb{S}[G]\big)$ *with the action of the cyclic group* C_m *by conjugation with the permutation automorphisms of* $\bigoplus^m \mathbb{S}[G]$*. Let* C_m *act similarly on* $(Q_+G)^{m\times m}$ *by conjugation. Then the following square is commutative in the homotopy category of* C_m*-spaces:*

$$\begin{tikzcd}
\mathrm{End}_G\big(\bigoplus^m \mathbb{S}[G]\big) \arrow[r, "\text{(5-15)}", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{End}_G\big(\bigoplus^m \mathbb{S}[G]\big) \arrow[d, "u", "\wr"'] \\
(Q_+G)^{m\times m} \arrow[r, "\dagger", "\sim"'] & (Q_+G)^{m\times m}
\end{tikzcd} \tag{5-16}$$

Remark 5.20 Passing to the homotopy invertible components in (5-16), we obtain a commutative square in the homotopy category of C_m-spaces:

$$\begin{tikzcd}
\mathrm{Aut}_G\big(\bigoplus^m \mathbb{S}[G]\big) \arrow[r, "\text{(5-15)}", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{Aut}_G\big(\bigoplus^m \mathbb{S}[G]\big) \arrow[d, "u", "\wr"'] \\
\mathrm{GL}_m(Q_+G) \arrow[r, "\dagger", "\sim"'] & \mathrm{GL}_m(Q_+G)
\end{tikzcd}$$

We have suppressed the $(-)^{\mathrm{op}}$ in the codomain of the map (5-15) in the previous squares to emphasise that such squares take place in the homotopy category of C_m-spaces, and not that of $\mathbb{E}_1$-spaces.

Proof of Proposition 5.19 First note that all the maps involved in (5-16) are indeed C_m-maps: the only one that is not obviously so is (5-15), but this follows from the observation that I_m is C_m-equivariant for the diagonal action on the domain and the trivial action on the target. Note also that the C_m-action on $(Q_+G)^{m\times m}$ restricts to a cofree C_m-action on each of the right C_m-cosets of the diagonal subspace, and hence $(Q_+G)^{m\times m} = \prod^m \operatorname{coInd}_e^{C_m} Q_+G$ as a C_m-space. Thus, in order to show that (5-16) commutes in the homotopy category of C_m-spaces, it suffices to prove that it commutes in the homotopy category of spaces after postcomposing it with the map

$$(Q_+G)^{m\times m} = \prod^m \operatorname{coInd}_e^{C_m} Q_+G \to \prod^m Q_+G$$

that records the first column of a matrix.

Given an endomorphism h of $\bigoplus^m \mathbb{S}[G]$, the $m\times m$ matrix $u(h) = (h_{ij}) \in \mathrm{GL}_m(Q_+G)$ has components

$$h_{ij} \colon \mathbb{S} \xrightarrow{\nu} \mathbb{S}[G] \xrightarrow{\mathrm{inc}_j} \bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{h} \bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{\mathrm{pr}_i} \mathbb{S}[G].$$

Slightly abusing the notation, we will write τ_ϵ to mean (5-15). Then we must only check that $\tau_\epsilon(h)_{ij}$ is homotopic to $\bar{h}_{ji}$, coherently in h (and for $j = 1$, though it is still true for all j of course). Observe now that $(-)_\ell \colon \mathrm{Mod}_{\mathbb{S}[G]} \to {}_{\mathbb{S}[G]}\mathrm{Mod}$ is a functor over Sp, and hence the last equivalence in (5-15) happens over the automorphism space of $\bigoplus^m \mathbb{S}[G]$ as a regular spectrum. Consequently, $\tau_\epsilon(h)_{ij}$ is by definition the top horizontal composition in the diagram of spectra

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\nu"] \arrow[dr, "\nu"'] & \mathbb{S}[G]_\ell \arrow[rr, "\mathrm{inc}_j"] \arrow[d, "\mathrm{inv}", "\wr"'] & & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[rr, "\tau_\epsilon(h)"] \arrow[dd, "\tilde{I}_m", "\wr"'] & & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[rr, "\mathrm{pr}_i"] \arrow[dd, "\tilde{I}_m", "\wr"'] & & \mathbb{S}[G]_\ell \\
& \mathbb{S}[G] \arrow[d, phantom, sloped, "\cong"] & (*_1) & & (*_2) & & (*_3) & \mathbb{S}[G] \arrow[u, "\mathrm{inv}", "\wr"'] \arrow[d, phantom, sloped, "\cong"] \\
& \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G]) \arrow[rr, "\mathrm{pr}_j^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\bigoplus^m \mathbb{S}[G], \mathbb{S}[G]) \arrow[rr, "h^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\bigoplus^m \mathbb{S}[G], \mathbb{S}[G]) \arrow[rr, "\mathrm{inc}_i^*"] & & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G])
\end{tikzcd}$$

On the other hand, one recognises the composite that goes through the bottom row to be $\bar{h}_{ji}$. Note that the left triangle is commutative, and that the square $(*_2)$ is too by definition of $\tau_\epsilon(h)$. Moreover, $(*_1)$ and $(*_3)$ do not depend on h; we must then argue that $(*_1)$ and $(*_3)$ are commutative up to homotopy.

For the commutativity of $(*_1)$, first observe that the left vertical composite equivalence of $(*_1)$ coincides up to homotopy with the equivalence of left $\mathbb{S}[G]$-modules $\tilde{I}_1 \colon \mathbb{S}[G]_\ell \simeq \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G])$. This is because, under the usual tensor–hom adjunction, both maps represent the same element in

$$\pi_0\big({}_{\mathbb{S}[G]}\mathrm{Mod}(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G]))\big) \cong \pi_0({}_{\mathbb{S}[G]}\mathrm{Mod}_{\mathbb{S}[G]}(\mathbb{S}[G]_\ell \otimes \mathbb{S}[G], \mathbb{S}[G]))$$

by definition of I_1. But now $(*_1)$, with $\tilde{I}_1$ in place of the left vertical composite, commutes up to homotopy, as both composites represent the same element in

$$\pi_0\Big({}_{\mathbb{S}[G]}\mathrm{Mod}\Big(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}\Big(\bigoplus^m \mathbb{S}[G], \mathbb{S}[G]\Big)\Big)\Big) \cong \pi_0\Big({}_{\mathbb{S}[G]}\mathrm{Mod}_{\mathbb{S}[G]}\Big(\mathbb{S}[G]_\ell \otimes \Big(\bigoplus^m \mathbb{S}[G]\Big), \mathbb{S}[G]\Big)\Big)$$

simply because the following diagram commutes by definition of I_1 and I_m:

$$\begin{tikzcd}
\mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "\mathrm{inc}_j \otimes 1"] \arrow[d, "1\otimes \mathrm{pr}_j"] & \bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[d, "I_m"] \\
\mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "I_1"] & \mathbb{S}[G]
\end{tikzcd}$$

Finally the commutativity of $(*_3)$ follows by similar reasoning, using that

$$\begin{tikzcd}
\bigoplus^m \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "\mathrm{pr}_i \otimes 1"] \arrow[d, "1\otimes \mathrm{inc}_i"] & \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[d, "I_1"] \\
\bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "I_m"] & \mathbb{S}[G]
\end{tikzcd}$$

is also commutative by definition. □

Warning 5.21 The canonical involution τ_ϵ induces an involution on the Whitehead group

$$\mathrm{Wh}(X) := \pi_1^s(\mathbf{Wh}^{\mathrm{Diff}}(X)) \cong \mathrm{GL}(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}/(\pm\pi_1(X)).$$

In his foundational paper, Milnor [1966] also defined an involution $\mathrm{Wh}(X) \ni \kappa \mapsto \bar{\kappa}$ induced by sending a matrix in $\mathrm{GL}(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}$ to its conjugate transpose (conjugate with respect to inversion in $\pi_1(X)$). This is an actual homomorphism because of the abelianisation present in the general linear group of $\mathbb{Z}[\pi_1(X)]$. It is worth being aware that these two involutions on $\mathrm{Wh}(X)$ are only the same after introducing a minus sign, i.e.,

$$\tau_\epsilon(\kappa) = -\bar{\kappa}. \tag{5-17}$$

This does *not* contradict the commutativity of (5-16). On the contrary, this extra minus sign is the result of having to deloop the anti-involution (5-15) in the sense of Lemma 5.6 in order to obtain τ_ϵ.

5.5 A-theory of a suspension

In this section we focus our attention on the homotopy type of $A(X;\epsilon)$ when X is the suspension ΣY of a connected based space Y. By a theorem of Carlsson, Cohen, Goodwillie, and Hsiang[14] [Carlsson et al. 1987, Theorem 3], in such cases there is an equivalence of infinite loop spaces

$$\theta\colon \prod_{m\geq 1} Q(Y^{\wedge m}_{hC_m}) \xrightarrow{\sim} \Omega \widetilde{A}(\Sigma Y), \tag{5-18}$$

---

[14] As pointed out in [Bökstedt et al. 1996, p. 543], the proof in [Carlsson et al. 1987] has a serious flaw around page 71. This issue was fixed in [Bökstedt et al. 1996, Corollary 4.15], and in particular the map θ of (5-18) constructed in [Carlsson et al. 1987, Section 1] is still an equivalence. We are indebted to Tom Goodwillie for his help in clearing up this matter and for carefully explaining to us another more general principle for which (5-18) holds — namely, that if F is a functor (from based spaces to based spaces, say) whose m-th derivative spectrum is of the form $X \mapsto X^{\wedge m}_{hC_m}$ for every $m \geq 1$, then its Taylor tower must split globally. This is indeed the case for the functor $F(-) := \Omega \circ A \circ \Sigma(-)$.

where $\widetilde{A}(-) := \mathrm{hofib}(A(-) \to A(*))$ and C_m acts on $Y^{\wedge m}$ by cyclic permutation of the factors. In this section, we argue that (5-18) can be upgraded to be C_2-equivariant up to homotopy.

Proposition 5.22 *Let Y be a connected based C_2-space. There is an equivalence of spectra*

$$\theta\colon \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}((ED_m)_+ \wedge_{C_m} Y^{\wedge m}) \xrightarrow{\sim} \widetilde{A}(\Sigma^\sigma Y;\epsilon)$$

that is C_2-equivariant up to homotopy, and whose underlying (nonequivariant) equivalence induces (5-18)*. Here $\Sigma^\sigma Y := S^\sigma \wedge Y$ and $D_m \subset \Sigma_m$ acts on $Y^{\wedge m}$ by*

$$g \cdot (y_1 \wedge \cdots \wedge y_m) := g \cdot y_{g(1)} \wedge \cdots \wedge g \cdot y_{g(m)} \quad \text{for } g \in D_m \text{ and } y_i \in Y,$$

where Y is now seen as a based D_m-space (on which C_m acts trivially). Finally $C_2 = D_m/C_m$ acts on $(ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ by its residual diagonal action.

Remark 5.23 The C_2-space $\Sigma^\sigma Y$ induces an involution on $A(\Sigma^\sigma Y)$ which commutes with the canonical involution τ_ϵ described in the previous section, by naturality of its construction. Therefore its composite gives the involution on $\widetilde{A}(\Sigma^\sigma Y;\epsilon)$ appearing in the statement of Proposition 5.22. Alternatively, we can allow X in the previous section to mean a C_2-space (e.g., $\Sigma^\sigma Y$), and agree that $\mathrm{inv}\colon G = GX \to G^{\mathrm{op}}$ there stands for inversion in the monoid G followed by the C_2-action on X.

In practice, we will apply Proposition 5.22 to the case when $Y = S^\sigma \wedge Z$ for some trivial C_2-space Z, as then $\Sigma^\sigma Y \simeq S^{2\sigma} \wedge Z \approx S^2 \wedge Z$ because of the homotopy C_2-equivariant equivalence $S^{2\sigma} \approx S^2$. In such a case, as $A(-;\epsilon)$ is a homotopy functor, Proposition 5.22 provides a simple description of the homotopy C_2-equivariant homotopy type of $\widetilde{A}(\Sigma^2 Z;\epsilon) \approx \widetilde{A}(\Sigma^{2\sigma} Z;\epsilon)$. This, together with Corollary 5.4, can then be used to analyse the homotopy type of $\widetilde{A}(\Sigma^2 Z;\epsilon)_{hC_2}$ away from 2.

We will need the following observation for the proof of Proposition 5.22:

Lemma 5.24 *Let X be a based connected C_m-space. The equivalence*

$$\iota\colon B((C_m \ltimes \Omega X)^{\mathrm{op}}) \simeq B(C_m \ltimes \Omega X)$$

of Lemma 5.6 coincides up to equivalence with the delooping of the inversion map

$$\mathrm{inv}\colon (C_m \ltimes \Omega X)^{\mathrm{op}} \to C_m \ltimes \Omega X, \quad (s^i, \gamma) \mapsto (s^{-i}, s^i \cdot \bar{\gamma}),$$

where $\bar{\gamma}$ stands for the loop γ with the reversed orientation.

Proof Given a topological monoid M equipped with a C_m-action, it is well-known (see, e.g., [Adem and Milgram 2004, Section II, Theorem 1.12]) that the classifying space of the semidirect product $C_m \ltimes M$ is equivalent to $EC_m \times_{C_m} BM$. On the simplicial level, this equivalence is given by

$$\begin{gathered}\beta\colon B_\bullet(C_m \ltimes M) \xrightarrow{\sim} E_\bullet C_m \times_{C_m} B_\bullet M,\\ ((s^{i_1}, m_1), \dots, (s^{i_q}, m_q)) \mapsto [(e, s^{i_1}, \dots, s^{i_q}), (s^{i_1} \cdot m_1, s^{i_1+i_2} \cdot m_2, \dots, s^{i_1+\cdots+i_q} \cdot m_q)].\end{gathered}$$

Now, for simplicity, we may assume that $\Omega(-)$ stands for the Moore loop space, so that $C_m \ltimes \Omega X$ is strictly associative (a *Moore loop* is a pair (γ, t) where $t \geq 0$ and $\gamma\colon [0,t] \to Y$ is a map with $\gamma(0) = \gamma(t) = *$; multiplication of Moore loops is given by concatenation of loops and addition of its lengths). Also recall that there is an identification $\Delta^q \cong \{\boldsymbol{v} = (v_1, \dots, v_q) : 0 \leq v_1 \leq \cdots \leq v_q \leq 1\}$; under this identification, the self-isomorphism $\Phi_q\colon \Delta^q \cong \Delta^q$ in the proof of Lemma 5.6 becomes the rule that sends a partition $\boldsymbol{v} = (0 \leq v_1 \leq \cdots \leq v_q \leq 1)$ to $1 - \boldsymbol{v} := (0 \leq 1 - v_q \leq \cdots \leq 1 - v_1 \leq 1)$. There is a C_m-equivariant map

$$\xi\colon B\Omega X \to X, \quad [(\gamma_1, t_1), \dots, (\gamma_q, t_q), \boldsymbol{v}] \mapsto \gamma_q \cdot \gamma_{q-1} \cdot \dots \cdot \gamma_1\Bigl(\sum_{i=1}^{q} t_i v_i\Bigr),$$

that is an equivalence if X is connected. This map satisfies the property that

$$\xi([(\bar{\gamma}_1, t_1), \dots, (\bar{\gamma}_q, t_q), \boldsymbol{v}]) = \xi([(\gamma_q, t_q), \dots, (\gamma_1, t_1), 1 - \boldsymbol{v}]).$$

With all of this in mind, one verifies that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
B((C_m \ltimes \Omega X)^{\mathrm{op}}) \arrow[r, "\iota", "\sim"'] \arrow[d, "B(\mathrm{inv})"] & B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[d, "EC_m \times_{C_m} \xi", "\wr"'] \\
B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[r, "EC_m \times_{C_m} \xi", "\sim"'] & EC_m \times_{C_m} X
\end{tikzcd}$$

□

Proof of Proposition 5.22 In the notation of the previous section, we let $X = \Sigma^\sigma Y$ now, so that $G = GX = \Omega^\sigma \Sigma^\sigma Y$ as a monoid with anti-involution (i.e., "inv" now means inversion in the monoid G followed by the C_2-action on $X = \Sigma^\sigma Y$). For each $m \geq 1$, let us write $\varrho\colon \mathrm{GL}_m(Q_+G) \simeq \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)$ for u^{-1} (meaning u as a wrong-way equivalence); it should be thought of as given by the rule that sends a matrix $h = (h_{ij}) \in \mathrm{Map}(\mathbb{S}, \mathbb{S}[G])^{m \times m} \cong \mathrm{Map}_G(\mathbb{S}[G], \mathbb{S}[G])^{m \times m}$ to

$$\varrho(h)\colon \bigoplus_{j=1}^{m} \mathbb{S}[G] \xrightarrow{\oplus_j (\oplus_i h_{ij})} \bigoplus_{i=1}^{m} \mathbb{S}[G].$$

The map θ of (5-18) is constructed in several steps in [Carlsson et al. 1987, Section 1], each of which we now upgrade to the homotopy C_2-equivariant setting. As these homotopy C_2-actions will get mixed up with strict C_m-actions, it will be more convenient and clear, at least throughout the first few steps, to avoid speaking about "homotopy equivariance" and rather regard a homotopy involution as what it is, i.e., a map whose square happens to be homotopic to the identity.

Step 1 For each $m \geq 2$, consider the D_m-equivariant map of spaces

$$\tilde{\theta}_{m,1}\colon Y^{\times m} \to \mathrm{GL}_m(Q_+G), \quad (y_1, \dots, y_m) \mapsto \begin{pmatrix} 1 & y_1 - 1 & & & & \\ & 1 & y_2 - 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & & 1 & y_{m-1} - 1 \\ y_m - 1 & & & & & 1 \end{pmatrix},$$

where, if $D_m := \langle s, r \mid s^m = r^2 = rsrs = e \rangle$, the notation is as follows:

- A point $y \in Y$ is identified in G with the path $\eta(y) := (t \mapsto t \wedge y) \in G$, which is itself identified with a point in $\{1\} \times QG \subset QS^0 \times QG \simeq Q_+G$. Then $y - 1$ is the corresponding point in $\{0\} \times QG \subset Q_+G$. Here the n-th component of QS^0 has been fixed a basepoint $n \in QS^0$.

- The action of D_m on $(y_1, \dots, y_m) \in Y^{\times m}$ is given by

$$s \cdot (y_1, \dots, y_m) := (y_m, y_1, \dots, y_{m-1}), \quad r \cdot (y_1, \dots, y_m) := (y^*_{m-1}, y^*_{m-2}, \dots, y^*_1, y^*_m),$$

where $y \mapsto y^*$ denotes the C_2-action on Y.

- Let $S, R \in \mathrm{GL}_m(\mathbb{Z})$ be the permutation matrices that send the i-th unit vector e_i to e_{i+1} and e_{m+1-i} (with subindexes taken modulo m), respectively. Then D_m acts on $A \in \mathrm{GL}_m(Q_+G)$ by

$$s \cdot A := SAS^{-1}, \quad r \cdot A := RA^{\dagger}R.$$

In other words, r acts by transposition along the "$x{=}y$"-axis together with conjugation on G.

We also define $\tilde{\theta}_{1,1} \colon Y \to \mathrm{GL}_1(Q_+G) = (Q_+G)^{\times}$ by sending $y \in Y$ to $y \in \{1\} \times QG \subset Q_+G$. We note that $\tilde{\theta}_{m,1}(y_1, \dots, y_m)$ is homotopy invertible by choosing a path from each of the y_i to the basepoint $* \in Y$. Then for $m \geq 1$, define $\theta_{m,1}$ as the composite

$$\theta_{m,1} \colon Y^{\times m} \xrightarrow{\tilde{\theta}_{m,1}} \mathrm{GL}_m(Q_+G) \xrightarrow[\sim]{\varrho} \mathrm{Aut}_G\Bigl(\bigoplus^m \mathbb{S}[G]\Bigr).$$

By construction, $\theta_{m,1}$ is a C_m-map as $\tilde{\theta}_{m,1}$ and u are. Recall that $s \in C_m \subset D_m$ acts on $\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)$ by conjugation with $S \in \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)$, which we denote by $S_{\#}$.

Step 2 Recall that the free $\mathbb{E}_1$-algebra on a based connected space X is naturally equivalent to $\Omega\Sigma X$. Therefore we can extend $\theta_{m,1}$ to a C_m-equivariant $\mathbb{E}_1$-map

$$\theta_{m,2} \colon \Omega\Sigma(Y^{\times m}) \to \mathrm{Aut}_G\Bigl(\bigoplus^m \mathbb{S}[G]\Bigr).$$

For any based space X, let us write $\sigma \colon \Omega X \to (\Omega X)^{\mathrm{op}}$ for inversion in ΩX (i.e., reversing the loop direction). Given a C_m-space X, we will write $X^{\mathrm{op}_{C_m}}$ for X with the opposite C_m-action (i.e., that in which s acts by s^{-1}, which is a valid left action as C_m is abelian). If X additionally has an $\mathbb{E}_1$-structure, we will write $X^{\mathrm{op},\mathrm{op}_{C_m}}$ for X with both the opposite $\mathbb{E}_1$-structure and the opposite C_m-action. Then the square of $\mathbb{E}_1$-maps

$$\begin{tikzcd}
\Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,2}"] \arrow[d, "\sigma \circ r"] & \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr) \arrow[d, "R_{\#} \circ \tau_{\epsilon}"] \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op},\mathrm{op}_{C_m}} \arrow[r, "\theta_{m,2}^{\mathrm{op},\mathrm{op}_{C_m}}"] & \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)^{\mathrm{op},\mathrm{op}_{C_m}}
\end{tikzcd} \tag{5-19}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces with a C_m-action. Here $r\colon \Omega\Sigma(Y^{\times m}) \to \Omega\Sigma(Y^{\times m})^{\mathrm{op}C_m}$ is induced by the action of $r \in D_m$ on $Y^{\times m}$ together with the flip of the suspension coordinate, and τ_ϵ really stands for (5-15). To see this, consider the diagram

(5-20)
$$\begin{tikzcd}
Y^{\times m} \arrow[rr, "\tilde{\theta}_{m,1}"] \arrow[dr, "r"] \arrow[dd, "\sigma\circ r\circ\eta"'] & & \mathrm{GL}_m(Q_+G) \arrow[r, "\varrho", "\sim"'] \arrow[d, "r=R_\#\circ\dagger"] & \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right) \arrow[dd, "R_\#\circ\tau_\epsilon"] \\
& (Y^{\times m})^{\mathrm{op}C_m} \arrow[r, "\tilde{\theta}^{\mathrm{op}C_m}_{m,1}"] \arrow[dl, "\eta^{\mathrm{op}C_m}"] & \mathrm{GL}_m(Q_+G)^{\mathrm{op}C_m} \arrow[dr, "\varrho^{\mathrm{op}C_m}", "\sim"'] & \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op}C_m} \arrow[rrr, "\theta^{\mathrm{op}C_m}_{m,2}"] & & & \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right)^{\mathrm{op}C_m}
\end{tikzcd}$$

of C_m-spaces. By definition, $\theta_{m,2}$ is the $\mathbb{E}_1$-map induced from the top horizontal composite in (5-20). Thus, in order to show that (5-19) homotopy commutes as C_m-equivariant $\mathbb{E}_1$-maps, it suffices to show that the outer square of (5-20) commutes in the homotopy category of C_m-spaces. But each of its subsquares/triangles commute in this category; indeed the lower subsquare does so by definition of $\theta_{m,2}$ (after applying $(-)^{\mathrm{op}C_m}$), the left subtriangle and the upper subsquare too by an easy check, and the right subsquare by Proposition 5.19 and the observation that $R_\# \circ u = u^{\mathrm{op}C_m} \circ R_\#$.

Step 3 The C_m-equivariant $\mathbb{E}_1$-map $\theta_{m,2}$ gives rise to an $\mathbb{E}_1$-map

$$\theta_{m,3}\colon C_m \ltimes \Omega\Sigma(Y^{\times m}) \xrightarrow{C_m\ltimes\theta_{m,2}} C_m \ltimes \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right) \xrightarrow{\mu} \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right),$$

where $\mu(s^i, h) := S^i h$. Observe that μ is indeed an $\mathbb{E}_1$-map, as

$$\mu((s^i, h)\cdot(s^j, h')) = \mu(s^{i+j}, S^{-j} h S^j h') = S^i h S^j h' = \mu(s^i, h)\mu(s^j, h').$$

Now from the homotopy commutativity of (5-19), it immediately follows that the left subsquare in

(5-21)
$$\begin{tikzcd}
C_m\ltimes\Omega\Sigma(Y^{\times m}) \arrow[r, "C_m\ltimes\theta_{m,2}"] \arrow[d, "C_m\ltimes(\sigma\circ r)"] \arrow[rr, bend left, "\theta_{m,3}"] & C_m\ltimes\mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right) \arrow[r, "\mu"] \arrow[d, "C_m\ltimes(R_\#\circ\tau_\epsilon)"] & \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right) \arrow[d, "R_\#\circ\tau_\epsilon"] \\
C_m\ltimes\Omega\Sigma(Y^{\times m})^{\mathrm{op},\mathrm{op}C_m} \arrow[r, "C_m\ltimes\theta^{\mathrm{op},\mathrm{op}C_m}_{m,2}"] & C_m\ltimes\mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right)^{\mathrm{op},\mathrm{op}C_m} \arrow[r, "\mu^{\mathrm{op}}"] & \mathrm{Aut}_G\left(\bigoplus^m \mathbb{S}[G]\right)^{\mathrm{op}}
\end{tikzcd}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces. Here $\mu^{\mathrm{op}}(s^i, h) := hS^i$, and since $\tau_\epsilon(S) = S^\dagger = S^{-1}$ and $RS = S^{-1}R$, it easily follows that the right subsquare also commutes as $\mathbb{E}_1$-maps. So the outer square of (5-21) commutes in the homotopy category of $\mathbb{E}_1$-spaces.

But given an $\mathbb{E}_1$-space X equipped with a C_m-action, there is an isomorphism of $\mathbb{E}_1$-spaces

$$\alpha\colon C_m \ltimes X^{\mathrm{op},\mathrm{op}C_m} \xrightarrow{\cong} (C_m \ltimes X)^{\mathrm{op}}, \quad (s^i, x) \mapsto (s^i, s^{-i}\cdot x).$$

Under this identification, the lower horizontal composite of (5-21) becomes $\theta_{m,3}^{\mathrm{op}}$, and hence

$$\begin{tikzcd} C_m \ltimes \Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,3}"] \arrow[d, "\alpha\circ(C_m\ltimes(\sigma\circ r))"] & \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr) \arrow[d, "R_\#\circ\tau_\epsilon"] \\ (C_m \ltimes \Omega\Sigma(Y^{\times m}))^{\mathrm{op}} \arrow[r, "\theta_{m,3}^{\mathrm{op}}"] & \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr)^{\mathrm{op}} \end{tikzcd} \tag{5-22}$$

is commutative in the homotopy category of $\mathbb{E}_1$-spaces.

Step 4 We wish to deloop (5-22), viewing the vertical maps as anti-involutions of their respective domains, and appealing to Lemma 5.6 to do so. But by Lemma 5.24, the delooping of the anti-involution $\alpha\circ(C_m\ltimes(\sigma\circ r))$ is homotopic to the delooping of the involution $\mathrm{inv}\circ\alpha\circ(C_m\ltimes(\sigma\circ r))$, where inv stands for inversion in the $\mathbb{E}_1$-space $C_m\ltimes\Omega\Sigma(Y^{\times m})$. It is given explicitly by $\mathrm{inv}(s^i,\gamma):=(s^{-i},s^i\cdot\sigma(\gamma))$, and hence

$$\mathrm{inv}\circ\alpha\circ(C_m\ltimes(\sigma\circ r))\colon (s^i,\gamma)\mapsto(s^{-i},r\cdot\gamma).$$

We denote this map simply by $\mathrm{inv}\ltimes r$. From now on we treat r as the action map on the D_m-space $\Sigma^\sigma(Y^{\times m})$, where σ is seen as a D_m-representation on which C_m acts trivially. Putting this together, the delooped version of (5-22) yields a homotopy commutative square of spaces

$$\begin{tikzcd} B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] \arrow[d, "B(\mathrm{inv}\ltimes r)"] & B\,\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr) \arrow[d, "\overline{B}(R_\#\circ\tau_\epsilon)"] \\ B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] & B\,\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr) \end{tikzcd}$$

where the notation $\overline{B}(-)$ stands for delooping in the sense of Lemma 5.6.

To simplify the terms in this last diagram, first observe that as $Y^{\times m}$ is connected, we have

$$B(C_m\ltimes\Omega\Sigma^\sigma(Y^{\times m}))\simeq ED_m\times_{C_m}B\Omega\Sigma^\sigma(Y^{\times m})\simeq ED_m\times_{C_m}\Sigma^\sigma(Y^{\times m}).$$

The inversion on C_m coincides with the residual $C_2=D_m/C_m$-action on C_m by conjugation, which explains why we chose to write ED_m instead of EC_m. As for the right-hand side, note that $R_\#$ is an inner automorphism of $\mathrm{Aut}_G\bigl(\bigoplus^m\mathbb{S}[G]\bigr)$, and hence it induces a map homotopic to the identity on the classifying space level [Adem and Milgram 2004, Section II, Theorem 1.9]. But delooping is functorial, so $\overline{B}(R_\#\circ\tau_\epsilon)$ and $\overline{B}(\tau_\epsilon)=:\tau_\epsilon$ are homotopic involutions on $B\,\mathrm{Aut}_G\bigl(\bigoplus^m\mathbb{S}[G]\bigr)$. All together, we obtain a homotopy C_2-equivariant map

$$\theta_{m,4}\colon ED_m\times_{C_m}\Sigma^\sigma(Y^{\times m})\xrightarrow{B(\theta_{m,3})^+}B\,\mathrm{Aut}_G\biggl(\bigoplus^m\mathbb{S}[G]\biggr)^+\subset\mathbb{Z}\times B\,\mathrm{Aut}_G\biggl(\bigoplus^m\mathbb{S}[G]\biggr)^+\to A(\Sigma^\sigma Y;\epsilon),$$

where the last map is the passage to the colimit as $m\to\infty$; see (5-14).

Step 5 The diagram

$$\begin{tikzcd}
\theta_{m,4}\colon ED_m/C_m \simeq BC_m \arrow[r, "Bj"] \arrow[d] & B\,\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}\bigr) \arrow[r] \arrow[d] & A(*;\epsilon) \arrow[d] \\
\theta_{m,4}\colon ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}) \arrow[r] & B\,\mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}[G]\bigr) \arrow[r] & A(\Sigma^\sigma Y;\epsilon)
\end{tikzcd}$$

commutes up to homotopy, where $j\colon C_m \to \mathrm{Aut}_G\bigl(\bigoplus^m \mathbb{S}\bigr)$ is the inclusion of the permutation automorphisms. As $A(-;\epsilon) = \Omega^\infty \boldsymbol{A}(-;\epsilon)$, we can adjoin the $\Omega^\infty(-)$ to get a similar homotopy commutative diagram of spectra. Then passing to vertical cofibres and noting that C_m acts trivially on the suspension coordinate of $\Sigma^\sigma(Y^{\times m})$, we get a homotopy C_2-equivariant map of spectra

$$\theta_{m,5}\colon \Sigma^{\infty+\sigma}((ED_m)_+ \wedge_{C_m} Y^{\times m}) \to \widetilde{\boldsymbol{A}}(\Sigma^\sigma Y;\epsilon).$$

Step 6 Now by [Carlsson et al. 1987, Lemma 1.4] (see also [Carlsson and Cohen 1987, Lemma 2.4]), the obvious projection $(ED_m)_+ \wedge_{C_m} Y^{\times m} \to (ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ has a stable section

$$\Sigma^\infty((ED_m)_+ \wedge_{C_m} Y^{\wedge m}) \to \Sigma^\infty((ED_m)_+ \wedge_{C_m} Y^{\times m})$$

that is D_m/C_m-equivariant. This observation gives rise to a homotopy C_2-equivariant map of spectra

$$\boldsymbol{\theta}_{m,6}\colon \Sigma^{\infty+\sigma}((ED_m)_+ \wedge_{C_m} Y^{\wedge m}) \to \widetilde{\boldsymbol{A}}(\Sigma^\sigma Y;\epsilon).$$

Finally set $\boldsymbol{\theta}$ to be

$$\boldsymbol{\theta}\colon \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}((ED_m)_+ \wedge_{C_m} Y^{\wedge m}) \xrightarrow{\bigvee_{m\geq 1}(\boldsymbol{\theta}_{m,6})} \bigvee_{m\geq 1} \widetilde{\boldsymbol{A}}(\Sigma^\sigma Y;\epsilon) \twoheadrightarrow \widetilde{\boldsymbol{A}}(\Sigma^\sigma Y;\epsilon).$$

This map is homotopy C_2-equivariant by construction, and nonequivariantly yields (5-18) after applying $\Omega^{\infty+\sigma}(-)$. This latter map is an equivalence of infinite loop spaces by [Carlsson et al. 1987, Theorem 1.6], and as both the domain and codomain of $\boldsymbol{\theta}$ are 1-connective, it follows that $\boldsymbol{\theta}$ is itself an equivalence of spectra. □

Corollary 5.25 *Let Y be a connected based C_2-space. Denote by $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(-)$ the homotopy fibre* $\mathrm{hofib}(\mathbf{Wh}^{\mathrm{Diff}}(-) \to \mathbf{Wh}^{\mathrm{Diff}}(*))$. *Then there is an equivalence of spectra*

$$\boldsymbol{\theta}\colon \bigvee_{m\geq 2} \Sigma^{\infty+\sigma}((ED_m)_+ \wedge_{C_m} Y^{\wedge m}) \xrightarrow{\simeq} \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$$

that is C_2-equivariant up to homotopy.

Proof It is clear from the construction that the map

$$\Sigma^\infty(\Sigma^\sigma Y) \simeq \Sigma^{\infty+\sigma}((ED_1)_+ \wedge_{C_1} Y^{\wedge 1}) \xrightarrow{\boldsymbol{\theta}_{1,6}} \widetilde{\boldsymbol{A}}(\Sigma^\sigma Y;\epsilon)$$

is the (reduced version of the) usual inclusion of the stable homotopy into A-theory. Thus its cofibre is $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$, and the claim follows immediately. □

Remark 5.26 As the reader may have noticed, the last two sections are a tiny bit technical, and one may wonder if there are alternative approaches to deal with them. One that may come to mind is to use trace methods to analyse $\widetilde{A}(\Sigma^\sigma Y;\epsilon)$, since it coincides (nonequivariantly) with the reduced TC of $\mathbb{S}[\Omega\Sigma Y]$ (as Y is connected). In fact, recent developments have been made towards a (genuine) C_2-equivariant version of topological cyclic homology for ring spectra with anti-involutions, commonly known as *real topological cyclic homology*; see [Høgenhaven 2016; Hesselholt and Madsen 2015; Dotto et al. 2024]. This approach has two caveats:

- A *real cyclotomic trace* map does not yet exist (at the time of writing). The construction of such a map was supposed to appear in [Hesselholt and Madsen 2015], but it never saw the light of day, in the end. This is, nevertheless, current work in progress by Harpaz, Nikolaus, and Shah [Harpaz et al. 2021, p. 24].
- Even though much is known about the p-complete homotopy type of the TC of spherical group rings (see [Bökstedt et al. 1996] or [Nikolaus and Scholze 2018, Section 4.3]), the analysis of its integral homotopy type does not seem to be present in the literature.

For these two reasons, we preferred to proceed as we have.

6 The homotopy type of spaces of long knots

This section is devoted to Theorem B, which describes the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $p \le d-3$ and $d \ge 5$, localised at odd primes and up to the concordance embedding stable range $\phi_{C\mathrm{Emb}}(d,p)$. After its proof, which will not take too much effort given the results in the preceding sections, we will draw some conclusions on the homotopy groups of spaces of long knots. For convenience let us recall the statement of Theorem B. Recall that ψ_m stands for the real m-dimensional permutation representation of the dihedral group D_m and σ for the sign representation, regarded as a D_m-representation by restricting along the determinant

$$D_m \hookrightarrow O(2) \xrightarrow{\det} \{\pm 1\} = C_2.$$

Theorem B *For $p \le d-3$ and $d \ge 5$, consider the virtual D_m-representations*

$$\rho_m := (d+1)(\sigma-1) + \psi_m \otimes (d-p-3+\sigma).$$

Then the homotopy fibre sequence (6-1), upon localising away from 2 and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, takes the form

$$\prod_{m\ge 2} \Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}) \to \mathrm{Emb}_\partial(D^p, D^d) \to \Omega^p\,\mathrm{hofib}(G(d-p)/O(d-p) \to G/O).$$

The resulting sequence is split if $p \ge 2$, and splits after being looped once if $p = 1$.

Recall from Remark 1.6(ii) what we mean by localising the spaces $\mathrm{Emb}_\partial(D^p, D^d)$ and $\widetilde{\mathrm{Emb}}_\partial(D^p, D^d)$.

6.1 Proof of Theorem B

Recall from (1-4) that $\mathrm{Emb}_\partial^{(\sim)}(P,M)$ denotes $\mathrm{hofib}_\iota(\mathrm{Emb}_\partial(P,M)\to\widetilde{\mathrm{Emb}}_\partial(P,M))$ when ι is clear from the context. We saw in Corollary 4.4 that the fibration sequence

$$\mathrm{Emb}_\partial^{(\sim)}(D^p,D^d)\to\mathrm{Emb}_\partial(D^p,D^d)\to\widetilde{\mathrm{Emb}}_\partial(D^p,D^d), \tag{6-1}$$

upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, is split for $2\le p\le d-3$, and splits for $p=1$ after looping once. So we need to describe the exterior terms of (6-1) after inverting 2.

For the block embeddings, the graphing map

$$\Gamma\colon\Omega^p\,\widetilde{\mathrm{Emb}}(*,D^{d-p})\xrightarrow{\sim}\widetilde{\mathrm{Emb}}_\partial(D^p,D^{d-p}\times D^p)\cong\widetilde{\mathrm{Emb}}_\partial(D^p,D^d) \tag{6-2}$$

of (4-1) is an equivalence by inspection. Then by [Goodwillie et al. 2001, Theorem 2.2.1] and the example right after it, when $d-p\ge 3$ and $d\ge 5$ (see Remark 6.1 below), it follows that

$$\begin{aligned}\widetilde{\mathrm{Emb}}_\partial(D^p,D^d)&\simeq\Omega^p\,\mathrm{hofib}(O/O(d-p)\to G/G(d-p))\\&\simeq\Omega^p\,\mathrm{hofib}(G(d-p)/O(d-p)\to G/O),\end{aligned} \tag{6-3}$$

yielding the base of (6-1).

Remark 6.1 The equivalence (6-3) is only valid if $d-p\ge 3$ and $d\ge 5$. As pointed out right after [Goodwillie et al. 2001, Theorem 2.2.1], the second condition is not that important. For instance in the case $p=1$ and $d=4$, it follows directly from (6-2) and (6-3) that

$$\Omega\,\widetilde{\mathrm{Emb}}_\partial(D^1,D^4)\simeq\widetilde{\mathrm{Emb}}_\partial(D^2,D^5)\simeq\Omega^2\,\mathrm{hofib}(G(3)/O(3)\to G/O).$$

The codimension condition $d-p\ge 3$, however, is essential.

For the fibre of (6-1), we know by Theorem A that for $N=\phi_{C\mathrm{Emb}}(d,p)-1$, there is an equivalence

$$\tau_{\le N}\,\mathrm{Emb}_\partial^{(\sim)}(D^p,D^d)\simeq\tau_{\le N}\Omega^\infty(\mathbf{CE}(D^p,D^d)_{hC_2}).$$

We now use Corollaries 5.17 and 5.25 to describe the right-hand side of the equivalence above.

Proposition 6.2 *For ρ_m as in Theorem B, there is an equivalence*

$$\Omega^\infty(\mathbf{CE}(D^p,D^d)_{hC_2})\simeq_{[\frac12]}\prod_{m\ge 2}\Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}).$$

Proof First observe that there is a homotopy C_2-equivariant equivalence $S^2\approx S^{2\sigma}$. So by Corollary 5.25, for each $n\ge 2$ there is a homotopy C_2-equivariant equivalence of spectra

$$\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^n;\epsilon)\approx\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma S^{n-2+\sigma};\epsilon)\approx\bigvee_{m\ge 2}\Sigma^{\infty+\sigma}(S^{\psi_m\otimes(n-2+\sigma)}_{hC_m}).$$

Using this for $n = d-p-1 \geq 2$, we obtain a chain of equivalences

$$\begin{aligned}\Omega^\infty(\mathbf{CE}(D^p, D^d)_{hC_2}) &\simeq_{[\frac{1}{2}]} \Omega^\infty\big((S^{d\cdot(\sigma-1)-1} \wedge \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}))_{hC_2}\big) \\ &\simeq_{[\frac{1}{2}]} \Omega^\infty\bigg(\bigg(\bigvee_{m\geq 2} \mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathbb{S}^{\psi_m\otimes(n-2+\sigma)}_{hC_m}\bigg)_{hC_2}\bigg) = \prod_{m\geq 2} \Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}).\end{aligned}$$

The first equivalence follows from Corollary 5.17, together with the observation that $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}) \simeq \Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(D^p, S^{d-p-1}; \epsilon)$ as both τ_ϵ and $\mathbf{Wh}^{\mathrm{Diff}}(-)$ are homotopy invariants of $(-)$. The second equivalence is a consequence of the previous argument and Corollary 5.4. □

All together, this concludes the proof of Theorem B. □

Remark 6.3 (topological version of Theorem B) The space $\mathrm{Emb}^{\mathrm{Top}}_{\partial}(D^p, D^d)$ of topological long knots is contractible (for all $p \leq d$) by the Alexander trick. We could still be interested in the homotopy type of the space $\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d)$ of thickened topological long knots with $p \leq d-3$, and one can get a description of it localised away from 2 and up to the concordance embedding stable range, similar to the one in Theorem B — let us explain how. As before, we have a homotopy fibre sequence

$$\mathrm{Emb}^{\mathrm{Top},(\sim)}_{\partial_0}(D^p \times D^{d-p}, D^d) \to \mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \to \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d)$$

which, upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d,p){-}1)$-th Postnikov sections, is split for $2 \leq p \leq d-3$, and splits for $p = 1$ after looping once. So we should describe the side terms.

For the block embeddings, consider the space[15]

$$B\widetilde{\mathrm{Top}}(q) := \operatorname{holim}\begin{pmatrix} & B\mathrm{Top} \\ & \downarrow \\ BG(q) \to & BG \end{pmatrix},$$

which is responsible for the classification of topological block normal bundles if $q \geq 3$; see [Rourke and Sanderson 1968a, Section 2; 1968b; Wall 1999, Section 11]. As $\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial}(D^p, D^d)$ is contractible by the Alexander trick, it then follows that

$$\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \simeq \mathrm{Map}_\partial(D^p, \widetilde{\mathrm{Top}}(d-p)) = \Omega^p\widetilde{\mathrm{Top}}(d-p).$$

As for the pseudoisotopy embeddings, the topological version of Theorem A (see Remark 1.4) tells us that for $N = \phi_{C\mathrm{Emb}(d,p)} - 1$, there is an equivalence

$$\tau_{\leq N}\, \mathrm{Emb}^{\mathrm{Top},(\sim)}_{\partial_0}(D^p \times D^{d-p}, D^d)) \simeq \tau_{\leq N}\Omega^\infty(\mathbf{CE}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d)_{hC_2}), \tag{6-4}$$

where $\mathbf{CE}^{\mathrm{Top}}(P, M)$ stands for the first orthogonal derivative of the topological analogue of $F(-)$; as in Corollary 5.17, this is a C_2-spectrum whose infinite loop space is equivalent to that of

$$\Sigma^{-2}\mathbf{Wh}^{\mathrm{Top}}(M, M-P; \epsilon).$$

---

[15] If $q \geq 3$, this definition of $\widetilde{\mathrm{Top}}(q)$ coincides with the one given in Remark 3.4.

Noting that there is fibre sequence of C_2-spectra $\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon) \to \mathbf{Wh}^{\mathrm{Top}}(M;\epsilon) \to \Sigma\mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon)\wedge M_+$, one verifies that the infinite loop space in the right-hand side of (6-4) fits in a fibre sequence away from 2,

$$\prod_{m\geq 2}\Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}) \to \Omega^\infty(\mathbf{CE}^{\mathrm{Top}}(D^p\times D^{d-p}, D^d)_{hC_2}) \to \Omega^\infty((S^{d\cdot\sigma-p-2}\wedge \mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon))_{hC_2}).$$

In particular, it is easy to check (e.g., rationally) that the left-hand side of (6-4) is not contractible (at least if d is sufficiently large). This was claimed in Remark 3.4.

6.2 On the homotopy groups of spaces of long knots

We can get plenty of information about the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$ from Theorem B. First observe that by Morlet's lemma of disjunction [Burghelea et al. 1975, Theorem 3.1] (and Proposition 3.3 to reduce to the codimension-zero case), the pseudoisotopy embedding space $\mathrm{Emb}^{(\sim)}_\partial(D^p, D^d)$ is at least $(2(d-p-2)-1)$-connected. So by (6-3), it follows that

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)) \quad \text{for } * < 2(d-p-2). \tag{6-5}$$

Remark 6.4 This should be compared to work of Budney [2008, Proposition 3.9]. The main result there is the computation of the first nontrivial homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d-p\geq 3$, which lies in degree $2d-3p-3$, together with a geometric interpretation of the generators. This is

$$\pi_{2d-3p-3}(\mathrm{Emb}_\partial(D^p, D^d)) \cong \begin{cases} \mathbb{Z} & \text{if } p=1 \text{ or } d-p \text{ is odd,} \\ \mathbb{Z}/2 & \text{if } p\geq 2 \text{ and } d-p \text{ is even.}\end{cases} \tag{6-6}$$

From our point of view, he shows that $\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)$ is exactly $(2d-2p-4)$-connected, which follows by work of Haefliger [1966, Section 3, (4.11) and Corollary 6.6], and computes the group $\pi_{2d-2p-3}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))$.

Regarding the Gromoll filtration on $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))$ for $d-p\geq 3$, two things are stated:

(1) The Gromoll degree of the elements of $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))$ is at least $2d-2p-4$.

(2) When $2d-3p-3=0$, the Gromoll degree of the elements of $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))$ is $p-1$.

Part (1) follows from the fact that $\mathrm{Emb}^{(\sim)}_\partial(D^{p-j}, D^{d-j})$ is $(2d-2p-5)$-connected and that the block graphing map (6-2) is an equivalence. Given (6-6), we can deduce (2) from our work *only* when $\pi_0(\mathrm{Emb}_\partial(D^p, D^d)) \cong \mathbb{Z}$ (using the simple observation that a homomorphism $A\to\mathbb{Z}$ from a finitely generated abelian group A is surjective if and only if $A\left[\frac{1}{2}\right] \to \mathbb{Z}\left[\frac{1}{2}\right]$ is so).

If we allow ourselves to localise away from 2, though, much more can be said about the Gromoll filtration on $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))\left[\frac{1}{2}\right]$. For example, by Corollary 4.6(i), it follows that:

(3) For $d, p\geq 0$ with $d-p\geq 3$, the elements of $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))\left[\frac{1}{2}\right]$ have Gromoll degree at least

$$j := \min\left(p-1, \left\lfloor \tfrac{1}{2}(2d-p-6)\right\rfloor\right).$$

See Corollary 4.6 for further consequences for the Gromoll filtration of $\pi_k(\mathrm{Emb}_\partial(D^p, D^d))\left[\frac{1}{2}\right]$ for $k\geq 0$.

We now return to the explicit computation of the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$. Recall that $\phi_{C\mathrm{Emb}}(d, p) \geq 2d - p - 5$ by work of Goodwillie, Krannich, and Kupers [Goodwillie et al. 2024], and so the space $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ has interesting homotopy in degrees from $2d-2p-4$ up to that range that we can understand by Theorem B. For any odd prime ℓ there are isomorphisms in degrees $* \leq \phi_{C\mathrm{Emb}}(d, p) - 1$,

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)} \oplus \bigoplus_{m\geq 2} \pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)},$$

and if $\phi = \phi_{C\mathrm{Emb}}(d, p)$, there is also an exact sequence of abelian groups

$$\bigoplus_{m\geq 2} \pi_\phi^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \to \pi_\phi(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \twoheadrightarrow \pi_{\phi+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)},$$

where $A_{(\ell)}$ denotes $A \otimes \mathbb{Z}_{(\ell)}$, for A an abelian group. It remains to understand the groups $\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$, which are easier to study when ℓ is coprime to m.

Proposition 6.5 *Let $m \geq 2$ and $d - p \geq 3$. For $\ell \nmid 2m$ a prime*:

- *If d is even and p is even, then*
$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \text{if } m = 3, 5, 7, \dots, \\ 0 & \text{otherwise.} \end{cases}$$
- *If d is odd and p is odd, then*
$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \text{if } m = 2, 4, 6, \dots, \\ 0 & \text{otherwise.} \end{cases}$$
- *If d is even and p is odd, then*
$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \text{if } m = 5, 9, 13, \dots, \\ 0 & \text{otherwise.} \end{cases}$$
- *If d is odd and p is even, then*
$$\pi_*^s(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)} & \text{if } m = 3, 7, 11, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.6 (rational homotopy of spaces of long knots) The rational homology and homotopy of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d - p \geq 3$ has been extensively studied in recent years (see, e.g., [Turchin 2010; Arone and Turchin 2014; 2015]) through the lens of embedding calculus and its relation to the little disks operads and their formality, finally culminating in the work of Fresse, Turchin, and Willwacher [Fresse et al. 2017]. There they compute the rational homotopy groups of $\overline{\mathrm{Emb}}_\partial(D^p, D^d) := \mathrm{hofib}_\iota(\mathrm{Emb}_\partial(D^p, D^d) \to \mathrm{Imm}_\partial(D^p, D^d))$ as the homology of the *hairy graph complex* (shifted appropriately). Observationally, our results correspond to the 0- and 1-loop order parts of this graph complex up to degree $\phi_{C\mathrm{Emb}}(d, p) \geq 2d - p - 5$, where higher loop orders are still not seen. More precisely, the 0-loop part corresponds to the rational homotopy of $G/G(d-p)$, the lowest summand (i.e., $m = 1$ when $d - p$ is even) of the 1-loop part appears as that of $O/O(d-p)$, and the higher summands of the 1-loop part come from the rational homotopy of the spectra $\mathbb{S}^{\rho_m}_{hD_m}$ for $m \geq 2$, which we just computed. It is worth noting that:

- The first nontrivial rational homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ coming from the 2-loop part of the hairy graph complex lies in degree $2d-p-4$ when both d and p are odd; see [Fresse et al. 2017, (3)]. Therefore the lower bound $\phi_{C\mathrm{Emb}}(d,p) \geq 2d-p-5$ on the concordance stable range of Goodwillie, Krannich, and Kupers is quite sharp.

- The 1-loop part of the hairy graph complex seems to be completely generated by the spectra $\mathbb{S}^{\rho_m}_{hD_m}$ for $m \geq 2$ in all degrees outside of the concordance embedding stable range. In other words, the computations in [Fresse et al. 2017] give evidence for the existence of a rational left splitting of the Weiss–Williams map

$$\Phi^{\mathrm{Emb}}\colon \mathrm{Emb}^{(\sim)}_\partial(D^p, D^d) \to \prod_{m\geq 2} \Omega^\infty(\mathbb{S}^{\rho_m}_{hD_m}).$$

To investigate this, one should first understand the attachment of the second orthogonal derivative $\Theta F^{(2)}$ of the orthogonal functor $F(U) := \mathrm{Emb}^b_\partial(D^p \times U, D^d \times U)$ to its Taylor tower (2-1). It would also be interesting to understand the integral picture.

Proof of Proposition 6.5 When ℓ is coprime to $2m$,

$$\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)} \cong H_0(D_m; \pi^s_*(\mathbb{S}^{\rho_m}))_{(\ell)}$$

by the homotopy fixed point spectral sequence, because the higher group homology of D_m is $2m$-torsion. Let t and r denote the generators of D_m with $t^m = e$ and $rtr = t^{-1}$ such that

$$\psi_m(t)\colon \mathbb{R}^m \ni (a_1,\dots,a_m) \mapsto (a_m, a_1, \dots, a_{m-1}), \qquad \psi_m(r)\colon \mathbb{R}^m \ni (a_1,\dots,a_m) \mapsto (a_m, a_{m-1}, \dots, a_1).$$

Then t and r act on $\pi^s_*(\mathbb{S}^{(d+1)(\sigma-1)+\psi_m\otimes(d-p-3+\sigma)})$ by $(-1)^{\epsilon_t}$ and $(-1)^{\epsilon_r}$, respectively, where

$$\begin{aligned}\epsilon_t &= (m-1)(d-p-2),\\ \epsilon_r &= \underbrace{d+1}_{(1)} + \underbrace{\tfrac{1}{2}m(m-1)(d-p-3)}_{(2)} + \underbrace{m+\tfrac{1}{2}m(m-1)}_{(3)} \equiv d+1+m+\tfrac{1}{2}m(m-1)(d-p-2) \mod 2.\end{aligned}$$

The terms (1), (2), and (3) are the contributions coming, respectively, from the summands $(d+1)(\sigma-1)$, $\psi_m \otimes (d-p-3)$, and $\psi_m \otimes \sigma$ of ρ_m. One then readily verifies that the groups $H_0(D_m; \pi^s_*(\mathbb{S}^{\rho_m}))$ are given by the formulae in the statement. □

A bit more interesting are the homotopy groups $\pi^s_*(\mathbb{S}^{\rho_m}_{hD_m})_{(\ell)}$ when ℓ is odd but divides m. We treat the case when $\ell = m = 3$, which hopefully serves as a sample computation for other cases.

Proposition 6.7 *The first few homotopy groups $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ of the spectrum $\mathbb{S}^{\rho_3}_{hD_3}$, localised at 3 and when $d-p=3$, are given in Table 1. Similarly coloured groups in this table correspond to the same case, depending on whether certain differentials in Figure 5 vanish or not. Entries containing “?” correspond to potentially more complicated answers that do not conveniently fit in the table.*

$*$	3	4	5	6	7	8	9	10	11	12	13
p even	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/9$	0	0	0	$\mathbb{Z}/9$	0	0	$\mathbb{Z}/3$
p odd	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0
$*$	14	15	16	17	18	19	20	21	22	23	24
p even	$\mathbb{Z}/27$	0	0	$\mathbb{Z}/3$ $\mathbb{Z}/3\oplus\mathbb{Z}/3$	$\mathbb{Z}/3^4$ $\mathbb{Z}/3^5$	0	?	?	$\mathbb{Z}/9$	$\mathbb{Z}/3$	0
p odd	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9\oplus\mathbb{Z}/3$

Table 1: $\pi_*^s(\mathbb{S}_{hD_3}^{\rho_3})\otimes\mathbb{Z}_{(3)}$ for $d-p=3$ and low values of $*\geq 3$.

Remark 6.8 Since this article was written, more extended computations of the groups $\pi_*^s(\mathbb{S}_{hD_3}^{\rho_3})\otimes\mathbb{Z}_{(3)}$ for $d-p=3$ have become available in the Master's thesis of Andrés Morán Lamas [2024]. For instance, he computes that in the "p even" case, the groups in degrees $*=20$ and 21 are both $\mathbb{Z}/3$. He also provides an extended version of Table 1 in [Lamas 2024, Tables 3.3 and 3.4], computing most of the groups up to degree $*\leq 39$. We are grateful to him for his enthusiasm in this particular computation.

To prove Proposition 6.7, we need to understand the cohomology of $\mathbb{S}_{hC_3}^{\rho_3}$ as a module over the Steenrod algebra $\mathcal{A}_3$, which we recall is generated by the Steenrod powers P^k and the Bockstein operation β.

Lemma 6.9 *The spectrum cohomology of* $\mathbb{S}_{hC_3}^{\rho_3}$ *is given by*

$$H^*(\mathbb{S}_{hC_3}^{\rho_3};\mathbb{F}_3)\cong\mathbb{F}_3\langle u\rangle\otimes_{\mathbb{F}_3}\mathbb{F}_3[\alpha,s]/(\alpha^2)\quad\text{for } |\alpha|=1,\ |s|=2,\ \text{and } |u|=3(d-p-2),$$

with

$$P^k(u\alpha^i s^j)=\Biggl(\sum_{r=0}^{k}\binom{d-p-2}{r}\binom{j}{k-r}\Biggr)u\alpha^i s^{j+2k},\qquad \beta(u\alpha^i s^j)=\begin{cases}0 & \text{if } i=0,\\ -us^{j+1} & \text{if } i=1.\end{cases}$$

Moreover $C_2=D_3/C_3$ *acts on* $H^*(\mathbb{S}_{hC_3}^{\rho_3};\mathbb{F}_3)$ *by* $u\alpha^i s^j\mapsto(-1)^{p+i+j}u\alpha^i s^j$.

Proof The key observation to carry out this calculation is that the C_3-representation $\psi_3|_{C_3}$ decomposes as $1+\theta$, where θ is the 2-dimensional representation pulled back from the standard complex $U(1)\cong\mathrm{SO}(2)$-representation on $\mathbb{C}\cong\mathbb{R}^2$. In particular, the associated vector bundle of the representation $\psi_m\otimes(d-p-3+\sigma)|_{C_m}$ is orientable; write $u\in H^*(\mathbb{S}_{hC_3}^{\rho_3};\mathbb{F}_3)$ for the corresponding Thom class. The $\mathbb{F}_3$-cohomology of BC_3 is $\mathbb{F}_3[\alpha,s]/(\alpha^2)$ with $|\alpha|=1$, $|s|=2$ and

$$\beta(\alpha)=s,\qquad P^k(\alpha^i s^j)=\binom{j}{k}\alpha^i s^{j+2k}.$$

So it remains to understand the action of $\mathcal{A}_3$ on u. Clearly $\beta(u)=0$, as if $\beta(u)=nu\alpha$ for some $n\in\mathbb{F}_3$, then $0=\beta^2(u)=n^2u\alpha^2-nus$ and hence $n=0$. For u_θ the Thom class of θ, $P^1(u_\theta)=u_\theta^3=u_\theta c_1(\theta)^2=u_\theta s^2$. It then easily follows that

$$P^k(u)=\binom{d-p-2}{k}us^{2k}.$$

The residual D_3/C_3-action on the $\mathbb{F}_3$-cohomology of BC_3 sends s to $-s$, and hence α to $-\alpha$. This action also switches the orientation of the vector bundle associated to θ, and hence that of $\psi_3=1+\theta$.

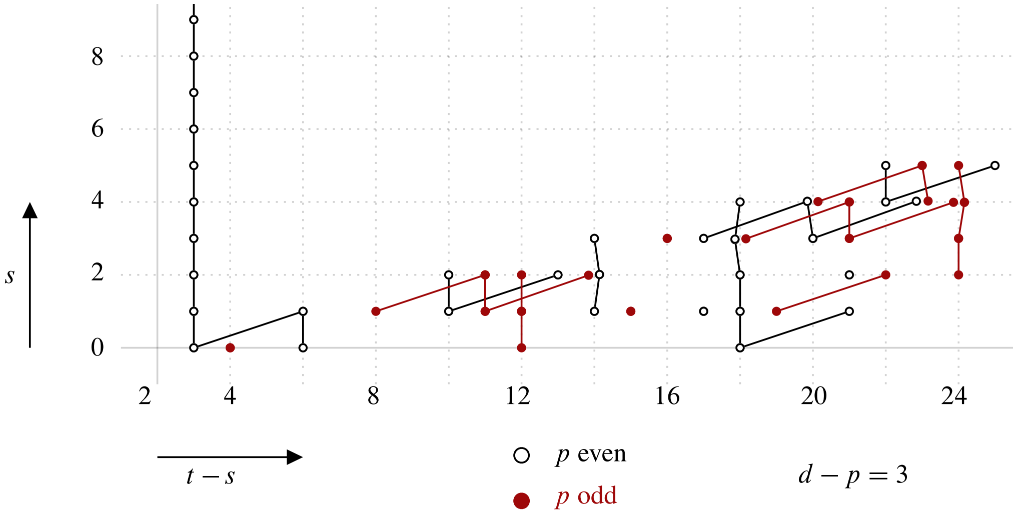

Figure 5: The E_2-page of the Adams spectral sequence at the prime 3 for $\mathbb{S}^{\rho_3}_{hD_3}$ when $d-p=3$. Here s denotes the degree in the Adams filtration and $t-s$ is the total degree.

So that of $\psi_3\otimes\sigma$ does not change, as ψ_3 is odd dimensional. All in all, this means that u is sent by the C_2-action to $(-1)^\epsilon u$ with $\epsilon=(d+1)+(d-p-3)\equiv p \bmod 2$, where the first term is the contribution of $\mathbb{S}^{(d+1)(\sigma-1)}$. □

Proof of Proposition 6.7 Consider the $\mathcal{A}_3$-submodules of $H^*(\mathbb{S}^{\rho_3}_{hC_3};\mathbb{F}_3)$ given by

$$J_0 := \langle u\alpha^i s^j : i+j\equiv 0 \bmod 2\rangle, \quad J_1 := \langle u\alpha^i s^j : i+j\equiv 1 \bmod 2\rangle.$$

Then $H^*(\mathbb{S}^{\rho_3}_{hC_3};\mathbb{F}_3)=J_0\oplus J_1$ as $\mathcal{A}_3$-modules, and J_p, where p is taken mod 2, is the $(+1)$-eigenspace of the residual $C_2=D_3/C_3$-action. Therefore $H^*(\mathbb{S}^{\rho_3}_{hD_3};\mathbb{F}_3)=J_p$ as an $\mathcal{A}_3$-module, and the Adams spectral sequence of J_p then converges to the stable homotopy of $\mathbb{S}^{\rho_3}_{hD_3}=(\mathbb{S}^{\rho_3}_{hC_3})_{hC_2}$. The software [Chua et al. 2022] computes the E_2-page of this spectral sequence, from which we determine some of the first homotopy groups of $\mathbb{S}^{\rho_3}_{hD_3}$. We illustrate the case $d-p=3$ in Figure 5.

Apart from standard arguments exploiting the Leibniz rule and multiplicative structures of the differentials in Figure 5, we can appeal to some more refined tricks that allow us to solve the first few possible nonzero differentials in the case when p is even. Denote by $\hat\theta$ the D_3-representation pulled back from the standard $O(2)$-representation on $\mathbb{R}^2$. Observe that $\hat\theta|_{C_3}\equiv\theta$ in the notation of Lemma 6.9. Write $\underline{S}(\hat\theta\otimes\sigma|_{C_3})$ for the unit sphere bundle of the associated vector bundle $ED_3\times_{C_3}(\hat\theta\otimes\sigma)$. As it is an S^1-bundle over BC_3, $\underline{S}(\hat\theta\otimes\sigma|_{C_3})$ must itself be a $K(\pi,1)$, and in fact it must be homotopy equivalent to S^1 because $q\colon \underline{S}(\hat\theta\otimes\sigma|_{C_3})\to BC_3$ does not admit a section (its Euler class is $s\in H^2(BC_3;\mathbb{F}_3)=\mathbb{F}[\alpha,s]/(\alpha^2)$). Moreover the homology class represented by q is the dual of α, and hence the residual $C_2=D_3/C_3$-action on $\underline{S}(\hat\theta\otimes\sigma|_{C_3})\simeq S^1$ must have degree -1. So there is an equivalence of unbased spaces $\underline{S}(\hat\theta\otimes\sigma|_{C_3})\simeq S^\sigma$ which is C_2-equivariant up to homotopy. We thus get a cofibration

$$S^\sigma_+\simeq \underline{S}(\hat\theta\otimes\sigma|_{C_3})_+ \xrightarrow{q} (BC_3)_+ \twoheadrightarrow \mathrm{Th}(\hat\theta\otimes\sigma|_{C_3})\simeq S^{-\sigma}\wedge\mathrm{Th}(\psi_3\otimes\sigma|_{C_3})$$

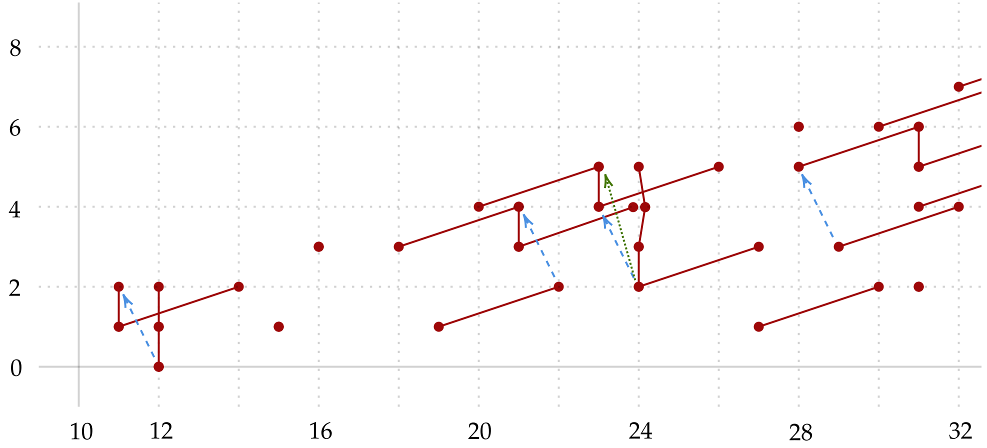

Figure 6: The E_2-page of Adams spectral sequence at the prime 3 for $\mathbb{S}^{\rho_3}_{hD_3}$ when $d-p=3$ and p is odd. Some of the d_2 (dashed blue) and d_3 (dotted green) differentials that will be analysed are depicted.

which is C_2-equivariant up to homotopy; see Notation 5.2(i). By equipping both S^σ and BC_3 with distinguished basepoints which are fixed under the respective involutions and which match under q, we can get rid of the added basepoints and obtain a homotopy cofibre sequence of C_2-spectra

$$\mathbb{S}^{(d+1)(\sigma-1)+2\sigma} \xrightarrow{q} S^{(d+1)(\sigma-1)+\sigma} \wedge \Sigma^\infty BC_3 \twoheadrightarrow \mathbb{S}^{\rho_3}_{hC_3}.$$

Then, upon inverting 2 and taking homotopy C_2-orbits in the sequence above, we obtain equivalences of spectra

$$\mathbb{S}^{\rho_3}_{hD_3} \simeq_{[\frac{1}{2}]} \begin{cases} (\Sigma^{\infty+1}BC_3)_{hC_2} \simeq_{[\frac{1}{2}]} \Sigma^{\infty+1}BD_3 & \text{if } d \text{ is even (so } p \text{ odd),} \\ \operatorname{hocofib}(q_{hC_2}\colon \mathbb{S}^2 \to (S^\sigma \wedge \Sigma^\infty BC_3)_{hC_2}) & \text{if } d \text{ is odd (so } p \text{ even).} \end{cases} \tag{6-7}$$

Note that the second equivalence in the "d even" case really only holds after inverting 2: by definition, there is an equivalence $(\Sigma^\infty_+ BC_3)_{hC_2} \simeq \Sigma^\infty_+ BD_3$ — we need to get rid of the "+"s. Now $\Sigma^\infty_+ BC_3 \simeq \mathbb{S} \oplus \Sigma^\infty BC_3$ is a C_2-equivariant equivalence, and $\mathbb{S}_{hC_2} = \Sigma^\infty_+ BC_2 \simeq \mathbb{S} \oplus \Sigma^\infty BC_2$; the first summand $\mathbb{S}$ cancels with that of $\Sigma^\infty_+ BD_3 \simeq \mathbb{S} \oplus \Sigma^\infty BD_3$, so a copy of $\Sigma^\infty BC_2$ remains, which is contractible only upon inverting 2.

Now by the Kahn–Priddy theorem [1978] at the prime 3, the transfer-like map $\Sigma^{\infty+1}BD_3 \to \tau_{>1}\mathbb{S}^1$ is split surjective on homotopy groups localised at 3, and hence by (6-7) the group $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3})_{(3)}$ split surjects onto $\pi^s_{*-1} \otimes \mathbb{Z}_{(3)}$ for $* > 1$ when p is odd. We will use this fact together with knowledge of π^s_* to determine the differentials of the red spectral sequence in Figure 5. For convenience, let us reillustrate a different portion of it in Figure 6.

The first possible nonzero differential goes from $(12,0)$ to $(11,2)$. Depending on its (non)vanishing, either

(a) $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \begin{cases} \mathbb{Z}/9 & \text{if } * = 11, \\ \mathbb{Z}/27 & \text{if } * = 12, \end{cases}$ or (b) $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \begin{cases} \mathbb{Z}/3 & \text{if } * = 11, \\ \mathbb{Z}/9 & \text{if } * = 12. \end{cases}$

Since $\pi^s_{10} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$, we must rule out possibility (a) as $\mathbb{Z}/9$ does not split surject onto $\mathbb{Z}/3$. In other words, the d_2 differential in Figure 5 from $(12, 0)$ to $(11, 2)$ is nonzero and (b) holds.

The d_2 differential from $(22, 2)$ to $(21, 4)$ must be nonzero, and hence so will that from $(19, 1)$ to $(18, 3)$ by the multiplicative structure. Indeed if such differential was zero, it would follow that $\pi^s_{21}(\mathbb{S}^{\rho_3}_{hD_3})_{(3)} \cong \mathbb{Z}/9$, which does not split surject onto $\pi^s_{20} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$.

The d_2 differential from $(24, 2)$ to $(23, 4)$ must also be nonzero (and hence that from $(24, 3)$ to $(23, 5)$) by a similar reason; indeed if it were trivial, then $\pi^s_{23}(\mathbb{S}^{\rho_3}_{hC_3})_{(3)}$ would be isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/81$ or $\mathbb{Z}/3 \oplus \mathbb{Z}/27$ (depending on whether the d_3 differential from $(24, 2)$ to $(24, 5)$ vanishes or not), neither of which split surject onto $\pi^s_{23} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9$.

The arguments we just made above, together with standard ones exploiting the Leibniz rule and the multiplicative structure of the differentials in the spectral sequence in Figure 5, establish the homotopy groups appearing in Table 1. □

One can keep using the Kahn–Priddy theorem and go quite far up, determining all possible nonzero differentials when p is odd; we leave it as a fun exercise to the eager reader. One would also hope that (6-7) could be used in the case when p is even, but this approach has somehow been inconclusive for us (at least for the first nonzero differentials that cannot be ruled out by elementary means).

Appendix A The first orthogonal derivative

Throughout, let $F \colon \mathcal{J}_0 \to \mathrm{Top}_*$ be an orthogonal functor. In this section, we present an explicit model for the structure maps (2-2) of the $O(1)$-spectrum $\Theta F^{(1)}$, and compare our convention for its $O(1)$-action with that of [Weiss 1995, Proposition 3.1].

A.1 An explicit model of the (pre)spectrum $\Theta F^{(1)}$

For V, we now describe an explicit model for the structure map (2-2)

$$s_V \colon S^1 \wedge F^{(1)}(V) \to F^{(1)}(V \oplus \mathbb{R}),$$

where $F^{(1)}(V) := \mathrm{hofib}(F(V) \to F(V \oplus \underline{\mathbb{R}}))$; here $\underline{\mathbb{R}}$ simply stands for a copy of $\mathbb{R}$ that we underline to distinguish it from the one appearing in the codomain of s_V. Our model for the homotopy fibre of a map $X \to Y$ of pointed spaces is the standard one, i.e., the subspace $\{(x, \gamma) \in X \times Y^{[0,1]} : \gamma(0) = x,\ \gamma(1) = *\}$.

The evident commutative diagram in $\mathcal{J}_0$

$$\begin{tikzcd}
V \arrow[r, "v \mapsto (v,0)"] \arrow[d, "v \mapsto (v,0)"'] & V \oplus \mathbb{R} \arrow[d, "(v,t) \mapsto (v,t,0)"] \\
V \oplus \underline{\mathbb{R}} \arrow[r, "(v,\underline{t}) \mapsto (v,0,\underline{t})"'] & V \oplus \mathbb{R} \oplus \underline{\mathbb{R}}
\end{tikzcd}$$

induces a commutative diagram of based spaces

$$\begin{tikzcd}
F^{(1)}(V) \arrow[r, dashed] \arrow[d, "i"] \arrow[dr, "\alpha"] & F^{(1)}(V\oplus\mathbb{R}) \arrow[d] \\
F(V) \arrow[r, "h"] \arrow[d, "\beta"] & F(V\oplus\mathbb{R}) \arrow[d, "\beta'"] \\
F(V\oplus\underline{\mathbb{R}}) \arrow[r, "h'"] & F(V\oplus\mathbb{R}\oplus\underline{\mathbb{R}})
\end{tikzcd}$$

Our task is to define a loop of dashed arrows (based at the constant map) — we will try to do so by providing a loop of nullhomotopies $\beta'\alpha\sim *$ (we will fail, but just slightly).

Note that, since βi is nullhomotopic via $\widetilde{H}^{(t)}(x,\gamma)=\gamma(t)$, we get a nullhomotopy $H_0:=h'\widetilde{H}\colon \beta'\alpha\sim *$. For each $\theta\in\left[-\frac{1}{2}\pi,+\frac{1}{2}\pi\right]$, let θ_* denote the automorphism of $F(V\oplus\mathbb{R}\oplus\underline{\mathbb{R}})$ induced by the rotation of the plane $0\oplus\mathbb{R}\oplus\underline{\mathbb{R}}$ with angle θ. Then note that by functoriality of F, we have that $\theta_*\beta'h=\beta'h$. Thus the maps $H_\theta:=\theta_*H_0\colon F^{(1)}(V)\times I\to F(V\oplus\mathbb{R}\oplus\underline{\mathbb{R}})$ provide a path of nullhomotopies $\beta'\alpha\sim *$.

As foreshadowed, the nullhomotopies $H_{-\pi/2}:=\left(-\frac{1}{2}\pi\right)_*h'\widetilde{H}$ and $H_{\pi/2}:=\left(\frac{1}{2}\pi\right)_*h'\widetilde{H}$ are distinct (as one can check). The crucial point then is that both are of the form $\beta'G$ for some nullhomotopy $G\colon\alpha\sim *$; indeed, if $\phi\colon F(V\oplus\underline{\mathbb{R}})\cong F(V\oplus\mathbb{R})$ denotes the map induced by identifying $\underline{\mathbb{R}}$ with $\mathbb{R}$, then

$$\left(-\tfrac{1}{2}\pi\right)_*h'=\beta'(-1_{\mathbb{R}})_*\phi,\quad \left(\frac{1}{2}\pi\right)_*h'=\beta'\phi,$$

where $(-1_{\mathbb{R}})_*$ is the automorphism of $F(V\oplus\mathbb{R})$ induced by $(v,t)\mapsto(v,-t)$. By noting that $\phi\beta=h$ and that $(-1_{\mathbb{R}})_*h=h$, one easily verifies that $G_{-\pi/2}:=(-1_{\mathbb{R}})\phi\widetilde{H}$ and $G_{\pi/2}:=\phi\widetilde{H}$ indeed provide the desired nullhomotopies $\alpha\sim *$.

Finally the nullhomotopies $G_{\pm\pi/2}$ give rise to canonical nullhomotopies of the maps $\sigma_{\pm\pi/2}\colon F^{(1)}(V)\to F^{(1)}(V\oplus\mathbb{R})$ induced by $H_{\pm\pi/2}$. All together, we obtain a loop of maps $F^{(1)}(V)\to F^{(1)}(V\oplus\mathbb{R})$ that is adjoint to the structure map σ_V of (2-2).

A.2 The $O(1)$-action

We have defined the first derivative spectrum $\Theta F^{(1)}$ of F to be the (naïve) $O(1)$-spectrum with n-th $O(1)$-space $\Theta F^{(1)}_n=F^{(1)}(\mathbb{R}^n)$ and with structure maps as defined just above. Now

$$F^{(1)}(V)=\operatorname{hofib}(F(V)\to F(V\oplus\underline{\mathbb{R}}))$$

is an $O(1)$-space by declaring $-1\in O(1)$ to act on V and $V\oplus\underline{\mathbb{R}}$ by -1 on *all* coordinates. A straightforward verification shows that, under this convention, the map

$$s_V\colon S^1\wedge F^{(1)}(V)\to F^{(1)}(V\oplus\mathbb{R})$$

is indeed $O(1)$-equivariant, where $O(1)$ acts trivially on S^1. The key point in this verification is that if R is a 2×2 matrix (e.g., a rotation of the plane), then $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)R$ and $R\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ agree on the subspace $\mathbb{R}\oplus 0\subset\mathbb{R}\oplus\underline{\mathbb{R}}$ even though they may not be equal.

In [Weiss 1995, Proposition 3.1], $O(1)$ instead acts on $F^{(1)}(V) = \operatorname{hofib}(F(V) \to F(V \oplus \mathbb{R}))$ by declaring the action of $-1 \in O(1)$ on V to be trivial and by -1 on the $\mathbb{R}$-summand of $V \oplus \mathbb{R}$. If we write $\underline{F}^{(1)}(V)$ for this $O(1)$-space, then the maps

$$s_V\colon S^\sigma \wedge \underline{F}^{(1)}(V) \to \underline{F}^{(1)}(V \oplus \mathbb{R}) \quad \text{and} \quad \underline{F}^{(1)}(V) \to F(V) \tag{A-1}$$

are $O(1)$-equivariant, where σ stands for the (1-dimensional) sign $O(1)$-representation and S^σ for its associated representation sphere. The corresponding (sequential) spectrum, call it $\underline{\Theta}F^{(1)}$, is *not* a naïve $O(1)$-spectrum in the usual sense anymore, as $O(1)$ acts nontrivially on the suspension coordinates. To solve this issue, Weiss [1995, p. 17] introduces the $O(1)$-spectrum $\Theta^\# F^{(1)}$ with n-th $O(1)$-space

$$\Theta^\# F^{(1)}_n := \Omega^{\infty\cdot\sigma}(S^n \wedge \underline{\Theta}F^{(1)}) = \operatorname*{hocolim}_k \Omega^{k\cdot\sigma}(S^n \wedge \underline{F}^{(1)}(\mathbb{R}^k)), \tag{A-2}$$

where $\Omega^{k\cdot\sigma}(-) := \operatorname{Map}_*(S^{k\cdot\sigma}, -)$ and $O(1)$ acts by conjugation on this mapping space.

In order to relate the $O(1)$-spectra $\Theta F^{(1)}$ and $\Theta^\# F^{(1)}$, we observe that F can be naturally upgraded to a functor $\underline{F}\colon \mathcal{J}_0^{O(1)} \to \mathrm{Top}_*^{O(1)}$ enriched over Top_*, where $\mathcal{J}_0^{O(1)} := \operatorname{Fun}(O(1), \mathcal{J}_0)$ is regarded as the pointed topological category of inner product finite-dimensional $O(1)$-representations. We likewise define for $V \in \mathcal{J}_0^{O(1)}$ the $O(1)$-space $\underline{F}^{(1)}(V) := \operatorname{hofib}(\underline{F}(V) \to \underline{F}(V \oplus \sigma))$. Now tensoring such an $O(1)$-representation with the sign representation σ gives a self-isomorphism of $\mathcal{J}_0^{O(1)}$ denoted by $-\cdot\sigma$. One could stabilise $\underline{F}^{(1)}(-)$ with respect to $\mathbb{R}$ as in (A-1), or with the sign representation σ, giving rise to maps

$$s_{a,b}\colon S^{a\cdot\sigma+b} \wedge \underline{F}^{(1)}(V) \to \underline{F}^{(1)}(V \oplus \mathbb{R}^{a,b}) \quad \text{for } a,b \geq 0 \text{ and } V \in \mathcal{J}_0^{O(1)},$$

where $S^{a\cdot\sigma+b} := S^{a\cdot\sigma} \wedge S^b$ and $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b\cdot\sigma$. We then obtain a zigzag of maps of $O(1)$-spectra

$$\begin{tikzcd}[column sep=tiny] \Theta^\# F^{(1)} := \operatorname{hocolim}_{a\geq 0}\mathbb{S}^{-a\cdot\sigma}\wedge\underline{F}^{(1)}(\mathbb{R}^a) \arrow[dr, hook, "b=0", "\sim"'] & & \operatorname{hocolim}_{b\geq 0}\mathbb{S}^{-b}\wedge\underline{F}^{(1)}(b\cdot\sigma) =: \Theta F^{(1)} \arrow[dl, hook', "a=0"', "\sim"] \\ & \operatorname{hocolim}_{a,b\geq 0}\mathbb{S}^{-a\cdot\sigma-b}\wedge\underline{F}^{(1)}(\mathbb{R}^{a,b}) & \end{tikzcd} \tag{A-3}$$

where the maps in the colimit of the middle spectrum are induced by $s_{1,0}$ and $s_{0,1}$. Nonequivariantly, both of the maps in the zigzag are equivalences by Fubini's theorem. This establishes the desired natural $O(1)$-equivariant equivalence[16] $\Theta F^{(1)} \simeq \Theta^\# F^{(1)}$.

Convention A.1 In the body of the paper, $F^{(1)}(V)$ stands for

$$\underline{F}^{(1)}(V\cdot\sigma) := \operatorname{hofib}(\underline{F}(V\cdot\sigma) \to \underline{F}((V \oplus \mathbb{R})\cdot\sigma))$$

in the notation of this section, unless we explicitly say otherwise. This way (2-2) is $O(1)$-equivariant.

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[16] In the sense of Borel; see Notation 5.2(i).

Appendix B Bounded geometry

Throughout, M^d denotes a smooth compact d-manifold (possibly with boundary).

B.1 Models for bounded diffeomorphisms and embeddings

Let N be a (possibly noncompact) smooth manifold and fix some smooth embedding $\iota\colon M \hookrightarrow N$. For $V \in \mathcal{J}_0$, we will write $\mathrm{Emb}_\partial(M \times V, N \times V)$ for the space of smooth embeddings of M into N that agree with $\iota \times \mathrm{Id}_V$ on some neighbourhood of the boundary $\partial M \times V$, endowed with the Whitney weak C^∞-topology. Following Definition 2.4, the *space of bounded embeddings* of $M \times V$ into $N \times V$ relative to $\partial M \times V$ is the subspace of $[0, +\infty) \times \mathrm{Emb}_\partial(M \times V, N \times V)$ given by

$$\mathrm{Emb}^b_\partial(M \times V, N \times V) := \{(t, \varphi) \in [0, +\infty) \times \mathrm{Emb}_\partial(M \times V, N \times V) : \varphi \text{ is } t\text{-bounded}\}.$$

Define similarly its simplicial version $\mathrm{Emb}^b_\partial(M \times V, N \times V)_\bullet$ as in Definition 2.8. In this section we prove:

Proposition B.1 *There is a zigzag of weak equivalences of semisimplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M \times V)_\bullet \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M \times V)).$$

Similarly, there is a zigzag of weak equivalences of semisimplicial sets

$$\mathrm{Emb}^b_\partial(M \times V, N \times V)_\bullet \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} \mathrm{Sing}_\bullet(\mathrm{Emb}^b_\partial(M \times V, N \times V)).$$

We will only deal with the first part, as the proof for the embedding case is completely analogous. Let us first introduce some notation. Given a topological space X, let $\mathrm{Sing}^{\mathrm{col}}_\bullet(X)$ be the subsimplicial set of $\mathrm{Sing}_\bullet(X)$ consisting of those singular simplices that satisfy the ϵ-collaring condition of Section 2.2 for some $0 < \epsilon < \frac{1}{2}$. Denote by $\mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ the subsimplicial group of $\mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}_\partial(M \times V))$ consisting of those j-simplices which are adjoint to a bounded map $\Delta^j \times M \times V \to M \times V$. Then there is a zigzag of maps of simplicial group-like monoids

$$\text{(B-1)}\quad \mathrm{Diff}^b_\partial(M \times V)_\bullet \xhookrightarrow{①} \mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V)) \xleftarrow{②} \mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}^b_\partial(M \times V)) \xhookrightarrow{③} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M \times V)),$$

where the map ② forgets the explicit bounding constant of a simplex. We will show that all the maps in (B-1) are weak equivalences. We start with ③.

Lemma *The inclusion* $i\colon \mathrm{Sing}^{\mathrm{col}}_\bullet(X) \hookrightarrow \mathrm{Sing}_\bullet(X)$ *is a weak equivalence for every topological space* X.

Proof We show that the relative homotopy groups $\pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$ vanish for all $j \geq 0$. Indeed, a homotopy class $x \in \pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$ corresponds, by the Yoneda lemma, to a singular j-simplex $g\colon \Delta^j \to X$ which satisfies the ϵ-collaring condition for all faces $\sigma \subset \partial\Delta^j$ and some $\epsilon > 0$. Now fix some identification $\Delta^j \cong \Delta^j \cup_{\partial\Delta^j} (\partial\Delta^j \times [0, \epsilon])$, and consider the singular j-simplex

$$\bar{g} = g \cup (g|_{\partial_{\Delta^j}} \circ \mathrm{proj}_{\partial\Delta^j})\colon \Delta^j \cong \Delta^j \cup_{\partial\Delta^j} (\partial\Delta^j \times [0, \epsilon]) \to X.$$

By construction $\bar{g}$ now satisfies the δ-collaring conditions for all faces $\sigma \subset \Delta^j$ and some $0 < \delta \le \epsilon$, so the corresponding relative homotopy class $\bar{x}$ is trivial. But clearly g and $\bar{g}$ are homotopic relative to the boundary by shrinking the added collar, and hence $x = \bar{x} = 0$ in $\pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$, as claimed. □

Remark B.2 The inclusion $i\colon \mathrm{Sing}^{\mathrm{col}}_\bullet(X) \hookrightarrow \mathrm{Sing}_\bullet(X)$ is in fact a simplicial homotopy equivalence; a homotopy inverse is constructed by induction on the skeleta of Δ^j. We will not need this though.

Lemma B.3 *The map* ②: $\mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}^b_\partial(M \times V)) \to \mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ *of* (B-1) *is a weak equivalence.*

Proof We again show that the relative homotopy groups π_j(②) vanish for all $j \ge 0$. Such a homotopy x class can be represented, for some $\epsilon > 0$, by an ϵ-collared singular j-simplex $g\colon \Delta^j \to \mathrm{Diff}_\partial(M \times V)$, adjoint to a map $g^\vee\colon \Delta^j \times M \times V \to M \times V$ bounded by some $K \ge 0$, together with a continuous ϵ-collared map $r\colon \partial\Delta^j \to [0,\infty)$ such that $g(s)$ is $r(s)$-bounded for all $s \in \partial\Delta^j$. To show that x is trivial, we need to extend r to a continuous δ-collared map $R\colon \Delta^j \to [0,\infty)$, for some $0 < \delta \le \epsilon$, such that $g(s)$ is $R(s)$-bounded for all $s \in \Delta^j$. Fix some identification $\Delta^j \cong (\partial\Delta^j \times [0,\epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$; then $R|_{\partial\Delta^j \times [0,\epsilon/2]} \equiv r \circ \mathrm{proj}_{\partial\Delta^j}$ whilst $R|_{\partial\Delta^j \times [\epsilon/2,\epsilon]}$ is a linear interpolation along $\left[\frac{1}{2}\epsilon, \epsilon\right]$ between r and the constant map $c_K\colon \partial\Delta^j \times \{\epsilon\} \to [0,\infty)$ with value $K \ge 0$. Finally set R to be constant of value K in the inner $\Delta^j \subset (\partial\Delta^j \times [0,\epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$. Then R is as required, and hence the relative homotopy class $x \in \pi_j$(②) is trivial. □

Remark B.4 For CAT $=$ Top, the map ① of (B-1) is an equality and thus, at this point, Proposition B.1 is established in the topological case.

Proof of Proposition B.1 It remains to show that ① is a weak equivalence, i.e., that the relative homotopy groups π_k(①) vanish for all $k \ge 0$. This is clear for $k = 0$ by definition. Such a homotopy class in π_k(①) is represented by a bounded homeomorphism

$$\bar{g} = (\mathrm{proj}_{\Delta^k}, g)\colon \Delta^k \times M \times V \to \Delta^k \times M \times V$$

which is collared in the simplex direction and such that $g|_{\partial\Delta^k \times M \times V}$ is smooth. Therefore g is smooth on (a neighbourhood of) $\partial\Delta^k \times M \times V$. We need to smooth g outside of such a neighbourhood in the Δ^k-direction while preserving boundedness. For $r \in \Delta^k$, we will write $g_r \in \mathrm{Diff}_\partial(M \times V)$ for $g|_{\{r\} \times M \times V}$.

Standard smoothing techniques [Munkres 1966, Section 4] — see also [Kupers 2019, Proposition 6.4.2] or [Lurie 2009, Proposition 1] — can be used to prove the following: given nested compact subsets $L \subset K \subset \Delta^k \times M \times V$ with $L \subset \mathrm{int}\, K$ and any arbitrarily small $\epsilon > 0$, there exists a homotopy $H\colon I \times \Delta^k \times M \times V \to M \times V$ from g to some map $g'\colon \Delta^k \times M \times V \to M \times V$ satisfying that:

(i) H remains fixed on $\Delta^k \times M \times V - \mathrm{int}\, K$. In particular g and g' agree there.

(ii) g' is smooth on L. Moreover if g was already smooth on some (open neighbourhood of a) closed subset $\partial\Delta^k \times M \times V \subset F \subset \Delta^k \times M \times V$, the homotopy H remains fixed on F.

(iii) For each $t \in I$ and $r \in \Delta^k$, the map $H^t_r = H|_{\{t\} \times \{r\} \times M \times V}\colon M \times V \to M \times V$ is smooth.

(iv) H^t remains arbitrarily close to g for all $t \in I$. Consequently, if g is bounded by some $C \geq 0$, then for every $(r,t) \in I \times \Delta^k$ the map $H^t_r \colon M \times V \to M \times V$ is bounded by $C + \epsilon$, and is a diffeomorphism (as diffeomorphisms of compact manifolds are open in the space of smooth self-maps).

With this in mind, we construct a homotopy in $\mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ from the k-simplex $\bar{g} = (\mathrm{proj}_{\Delta^k}, g)$ to some $\bar{h} \in \mathrm{Diff}^b_\partial(M \times V)_k$ (relative to $\partial\Delta^k \times M \times V$) as follows: Without loss of generality assume $V = \mathbb{R}^n$. Also, for $v \in \mathbb{R}^n$ and $\delta > 0$, let $C_\delta(v) \subset \mathbb{R}^n$ denote the cube of side length 2δ and centred at v (i.e., $C_\delta(v) := v + [-\delta, \delta]^n$). Fix an $\epsilon > 0$ (e.g., $\epsilon = 1$). Then for each $v \in 3\mathbb{Z}^n \subset \mathbb{R}^n$, choose a homotopy as above starting from g with $(K, L) = (C_1(v), C_{2/3}(v))$, and perform all of these at the same time[17] to obtain some $g' \colon \Delta^k \times M \times V \to M \times V$. Now apply the same process to g' on $(K, L) = (C_1(v), C_{2/3}(v))$ for each $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$ with $v_1 \equiv 1 \bmod 3$ and $v_i \equiv 0 \bmod 3$ for $2 \leq i \leq n$, keeping in mind that, by (ii) above, the homotopies keep fixed the parts that have been smoothed in the previous step. Continue this process in a similar fashion. After 3^n steps, we will obtain a smooth $(C + 3^n \cdot \epsilon)$-bounded map $h \colon \Delta^k \times M \times V \to M \times V$ such that $\bar{h} := (\mathrm{proj}_{\Delta^k}, h)$ represents the required k-simplex of $\mathrm{Diff}^b_\partial(M \times V)_\bullet$. This means that the relative homotopy class $[\bar{g}] \in \pi_k(①)$ is trivial, as was to be shown. □

B.2 A moduli space model for classifying spaces of bounded diffeomorphism groups

Fix an embedding $\iota \colon M \hookrightarrow \mathbb{R}^m \subset \mathbb{R}^\infty$. Recall that the classifying space $B\mathrm{Diff}_\partial(M)$ of the diffeomorphism group of M admits a model as the moduli space of all d-manifolds $N^d \subset \mathbb{R}^\infty$ with $\partial N = \partial M$ which are diffeomorphic to M relative to the boundary. In this section we give an analogous description of the classifying space $B\mathrm{Diff}^b_\partial(M \times V)$ for any real finite-dimensional inner product vector space $V \in \mathcal{J}_0$.

Proposition B.5 *Set* $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V) := \mathrm{colim}_n \, \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^n \times V)$, *and let* $\mathrm{Diff}^b_\partial(M \times V)$ *act on it by precomposition. Then there is an equivalence*

$$B\mathrm{Diff}^b_\partial(M \times V) \simeq \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)/\mathrm{Diff}^b_\partial(M \times V).$$

In other words, $B\mathrm{Diff}^b_\partial(M \times V)$ is (equivalent to) the moduli space of all submanifolds in $\mathbb{R}^\infty \times V$ with boundary $\partial M \times V$ which are diffeomorphic to $M \times V$ boundedly in V and relative to $\partial M \times V$.

Proof By Proposition B.1, $B\mathrm{Diff}^b_\partial(M \times V) \simeq B|\mathrm{Diff}^b_\partial(M \times V)_\bullet| \simeq |B\mathrm{Diff}^b_\partial(M \times V)_\bullet|$ and

$$\frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)}{\mathrm{Diff}^b_\partial(M \times V)} \simeq \frac{|\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet|}{|\mathrm{Diff}^b_\partial(M \times V)_\bullet|} \simeq \left| \frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet}{\mathrm{Diff}^b_\partial(M \times V)_\bullet} \right|.$$

As the simplicial action of $\mathrm{Diff}^b_\partial(M \times V)_\bullet$ on $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is visibly free, we only need to show that $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is weakly contractible by [Goerss and Jardine 1999, Corollary 2.6]. To that end, let

$$\varphi = (\mathrm{proj}_{\Delta^k}, \varphi_n, \varphi_V) \colon \Delta^k \times M \times V \hookrightarrow \Delta^k \times \mathbb{R}^n \times V \quad \text{for } n \geq m$$

---

[17] This can be done as, by (i), the supports of such homotopies are disjoint by construction.

represent some homotopy class in $\pi_k(\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet)$ for some $k \geq 0$. We will show that $[\varphi] = [\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V]$ by constructing a simplicial map $H\colon \Delta^1_\bullet \to \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ such that, under the Yoneda isomorphism, $\partial_0 H = \varphi$ and $\partial_1 H = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$. The map H will be given by (a modification of) the usual straight-line homotopy between φ and $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$.

Let us fix some notation. Pick some open collar $c\colon [0,1) \times \partial M \hookrightarrow M$ of the boundary of M. We can arrange the embedding $\iota\colon M \hookrightarrow \mathbb{R}^m$ to be such that

(i) $\iota \equiv (\mathrm{Id}_{[0,1)]} \times i) \circ c^{-1}|_{c([0,1)\times\partial M)}$ for some embedding $i\colon \partial M \hookrightarrow \mathbb{R}^{m-1}$, and

(ii) $\iota(M \setminus c((0,1] \times \partial M)) \subset [1, +\infty) \times \mathbb{R}^{m-1}$.

From now on we will suppress ι and c from the notation, i.e., we canonically identify M (resp. $[0,1)\times\partial M$) with its image under ι (resp. c). Choose some increasing smooth function $\alpha\colon [0,1] \to [0,1]$ for which there exists some $0 < \delta$ with $\alpha|_{[0,\delta]} \equiv 0$, $\alpha|_{[1-\delta,1]} \equiv 1$, and $0 < \alpha(t) < 1$ for $\delta < t < 1-\delta$ (this δ is required for the collaring condition right before Definition 2.8). Now by the collaring condition,

$$\text{there exists some } 0 < \epsilon < 1 \text{ such that } \varphi \equiv \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V \text{ on } \Delta^k \times [0,\epsilon) \times \partial M \times V. \tag{B-2}$$

Finally, fix some smooth function $\rho\colon M \to [0,1]$ such that

$$\rho|_{[0,\epsilon/2]\times\partial M} \equiv 0 \quad \text{and} \quad \rho|_{M\setminus[0,\epsilon)\times\partial M} \equiv 1.$$

Then for $t \in [0,1]$, consider the map

$$H_t\colon \Delta^k \times M \times V \to \Delta^k \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{|V|} \times V \subset \Delta^k \times \mathbb{R}^\infty \times V,$$

$$(r,x,v) \mapsto \begin{pmatrix} r \\ \alpha(t)\cdot x + (1-\alpha(t))\cdot\varphi_n(r,x,v) \\ \rho(x)\alpha(t)(1-\alpha(t))\cdot x \\ \rho(x)\alpha(t)(1-\alpha(t))\cdot v \\ \alpha(t)\cdot v + (1-\alpha(t))\cdot\varphi_V(t,x,v) \end{pmatrix}. \tag{B-3}$$

Here $x \in M \subset \mathbb{R}^m \subset \mathbb{R}^n$ and $\mathbb{R}^{|V|}$ is a Euclidean space of the same dimension as V, treated as a copy of V.

Claim *Let $C \geq 0$ be the bound of φ on the V-coordinate. Then the map H_t is a C-bounded embedding for $t \in [0,1]$. Moreover, H_t agrees with $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ on $\Delta^k \times \left[0, \tfrac{1}{2}\epsilon\right] \times \partial M \times V$.*

Proof of Claim Indeed H_t is bounded by $C \geq 0$ (in the V-coordinate) as

$$\|(\alpha(t)\cdot v + (1-\alpha(t))\cdot\varphi_V(t,x,v)) - v\|_V = (1-\alpha(t))\cdot\|\varphi_V(t,x,v) - v\|_V \leq (1-\alpha(t))\cdot C \leq C.$$

To see that H_t is an embedding, suppose that $H_t(r,x,v) = H_t(r',x',v')$. Clearly then $r = r'$ by the first coordinate in (B-3). Note that $H_t = \varphi$ if $t \leq \delta$ and $H_t = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ if $t \geq 1-\delta$. As both are embeddings, we may assume that $\delta < t < 1-\delta$, so that $\alpha(t)(1-\alpha(t)) \neq 0$. To show that $x = x'$ we consider three cases:

- If $x, x' \in [0,\epsilon] \times \partial M$, then by (B-2) the equation on the second coordinate of (B-3) yields $x = x'$.

- If $x, x' \notin [0,\epsilon]\times\partial M$, then $\rho(x) = \rho(x') = 1$ and thus the third coordinate equation yields $x = x'$.
- If $x \in [0,\epsilon]\times\partial M$ but $x' \notin [0,\epsilon]\times\partial M$, then the third coordinate equation becomes $\rho(x)\cdot x = x' \in \mathbb{R}^n$. On the first coordinate of $[0,+\infty)\times\mathbb{R}^{n-1}\subset\mathbb{R}^n$, this implies, by items (i) and (ii) above, that $\rho(x) > 1$, which is a contradiction.

In all cases $x = x'$. Then the equation on the fourth coordinate of (B-3) implies that $v = v'$, as required.

Finally, the last part of the claim again follows from (B-2) and the nature of ρ. □

The family of C-bounded embeddings $\{H_t\}_{t\in[0,1]}$ gives rise to the required simplicial map H. □

Appendix C The h-cobordism stabilisation map

The (lower) stabilisation map $\Sigma\colon H(M)\to H(M\times I)$ of [Vogell 1985, p. 298] is depicted in Figure 7 below.

Recall that the h-cobordism space $H(M\times I)$ is an $\mathbb{E}_1$-space under stacking in the I-direction, denoted by $+_I$. In this section we argue that Σ anticommutes with the h-cobordism involution ι_H in the following sense:

Lemma C.1 (Lemma 5.16(a)) *The map $\iota_H\Sigma +_I \Sigma\iota_H\colon H(M)\to H(M\times I)$ is nullhomotopic*, i.e., *it is homotopic to the constant map at the trivial partition $*\in H(M\times I)$.*

Proof We describe the nullhomotopy in the topological setting; the smooth case is very similar, but one has to be slightly careful with issues regarding corners (which can be overcome by working with the collared version $H_{\mathrm{col}}(M\times I)$ of the h-cobordism space). It will be convenient to work with yet another (upgraded)

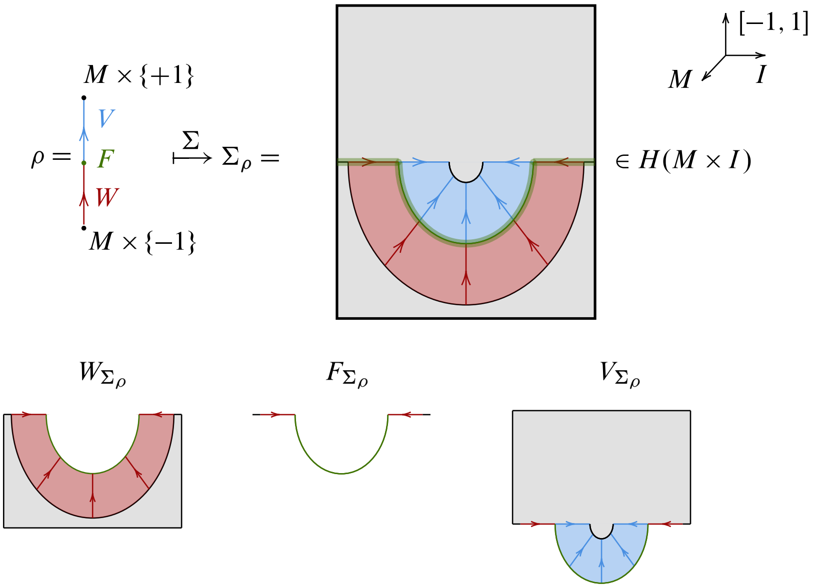

Figure 7: The h-cobordism (lower) stabilisation map $\Sigma_\ell = \Sigma\colon H(M)\to H(M\times I)$, sending $\rho = (W,F,V)$ to $\Sigma\rho := (W_{\Sigma\rho}, F_{\Sigma\rho}, V_{\Sigma\rho})$. A grey shaded region of shape S represents a manifold of the form $M\times S$.

version of the h-cobordism space: let $\overline{H}_{\mathrm{col}}(M)_\bullet$ denote the simplicial set in which a q-simplex consists of a pair (ρ,ϕ) with $\rho := (W,F,V) \in H_{\mathrm{col}}(M)_q$ and a diffeomorphism $\phi\colon V \cup_{M\times\Delta^q} W \cong F\times[-1,1]\times\Delta^q$ over Δ^q which fixes pointwise (a neighbourhood of) $\partial(F\times[-1,1])\times\Delta^q$. There is a Kan fibration

$$\mathrm{Diff}_\partial(M\times[-1,1])_\bullet \xrightarrow{j} \overline{H}_{\mathrm{col}}(M)_\bullet \xrightarrow{p} H_{\mathrm{col}}(M)_\bullet,$$

where $p(\rho,\phi) := \rho$. The inclusion j admits a (left) section up to homotopy

$$s\colon \overline{H}_{\mathrm{col}}(M)_\bullet \to \mathrm{Diff}_\partial(M\times[-1,1])_\bullet$$

given roughly by applying ϕ^{-1} on the collar of F in $M\times[-1,1] = W\cup_F V$ and then canonically identifying $W\cup_F V\cup_M W\cup_F V$ with $M\times[-1,1]$. This yields an equivalence

$$(p,s)\colon \overline{H}_{\mathrm{col}}(M)_\bullet \xrightarrow{\sim} H_{\mathrm{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M\times[-1,1])_\bullet.$$

But now the diagram

$$\begin{tikzcd}
\overline{H}_{\mathrm{col}}(M)_\bullet \arrow[d, "\wr"] \arrow[dr, "f_1"] & \\
H_{\mathrm{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M\times[-1,1])_\bullet \arrow[r, "f_2"] \arrow[d, shift right, "\mathrm{pr}_1"'] & H(M\times I)_\bullet \\
H_{\mathrm{col}}(M)_\bullet \arrow[u, shift right, "i"'] \arrow[ur, "f_3"] \arrow[d, "u", "\wr"'] & \\
H(M)_\bullet \arrow[uur, "f_4=\iota_H\Sigma+_I\Sigma\iota_H"'] &
\end{tikzcd}$$

commutes up to homotopy, where u is the map that forgets the collaring data, and all the horizontal maps (strictly) factor through the bottom horizontal map $f_4 := \iota_H\Sigma +_I \Sigma\iota_H$. Therefore, in order to show that f_4 is nullhomotopic, it suffices to show that f_1 is so; indeed this would imply that f_2 is nullhomotopic. But $f_2 = f_3\circ\mathrm{pr}_1$, so $f_3 \simeq f_3\circ\mathrm{pr}_1\circ i \simeq *$ too. This in turn would imply that f_4 is nullhomotopic, as we aim to prove.

We therefore need to describe a nullhomotopy of f_1. We will just describe a path (or rather, a 1-simplex) between $f_4(\rho,\phi) = (\iota_H\Sigma +_I \Sigma\iota_H)(\rho)$, for a fixed 0-simplex $(\rho = (W,F,V),\phi) \in \overline{H}_{\mathrm{col}}(M)_0$, and the trivial partition $* \in H(M\times I)_0$ — an exactly analogous argument yields an actual simplicial nullhomotopy. This path is depicted in Figures 8–11. The green shaded regions in each picture represent the F-part of a partition, i.e., the intersection of the two h-cobordisms making up the partition, which is a $(d+1)$-manifold embedded in $M\times I\times[-1,1]$. The partition $\rho = (W,F,V) \in H(M)_0$ is as depicted in Figure 7.

Firstly, the path between the partition $P_0 = \iota_H\Sigma(\rho) +_I \Sigma\iota_H(\rho)$ and P_1 is obtained from rescaling (and slightly shifting inwards). But as $W\cup_F V$ is canonically (in the sense that it does not depend on any other choice than that of ρ) identified with $M\times[-1,1]$ as part of the data of ρ, the outer bent regions of the form $(W\cup_F V)\times I$ added to P_1 in order to obtain P_2 are canonically identified with $M\times[-1,1]\times I$, and therefore $P_1 = P_2$ in $H(M\times I)$; see Figure 8.

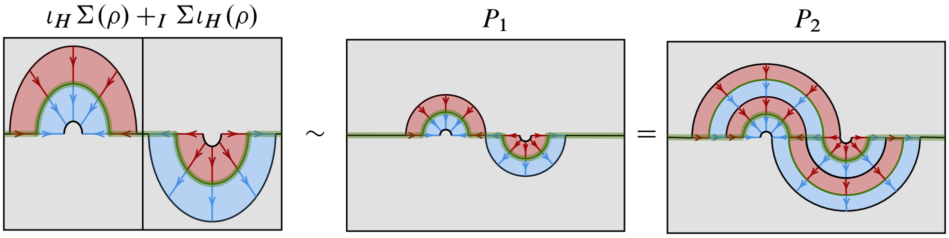

Figure 8: The path in $H(M \times I)$ between $\iota_H\Sigma(\rho) + \Sigma\iota_H(\rho)$ and P_2.

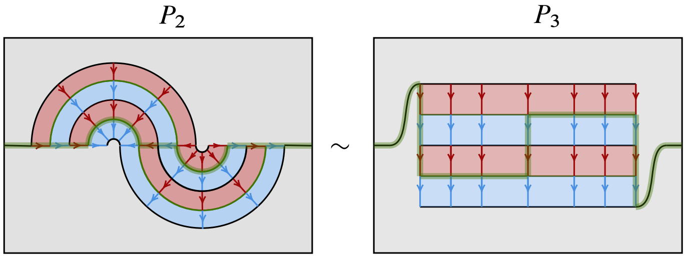

Figure 9: The path in $H(M \times I)$ between P_2 and P_3.

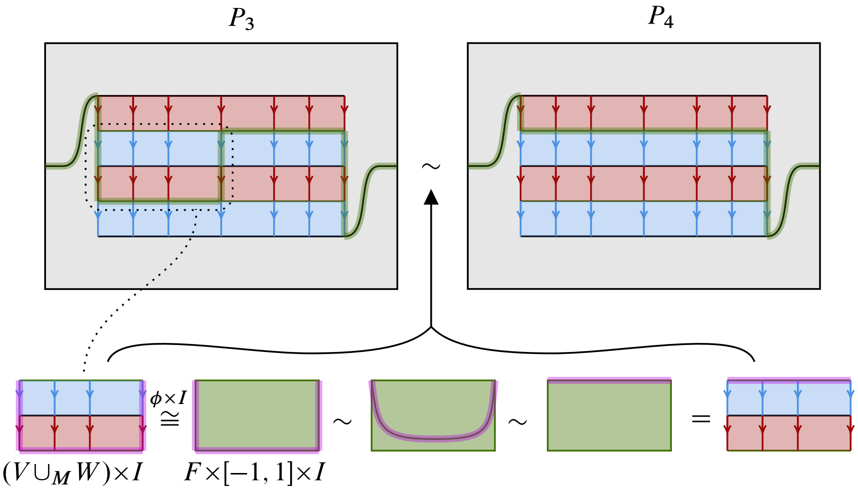

Figure 10: The path in $H(M \times I)$ between P_3 and P_4. The green shaded region in the lower part of the figure represents $F \times [-1,1] \times I$. The purple shaded region there represents the F-part of the partition (which used to be green, but is purple momentarily).

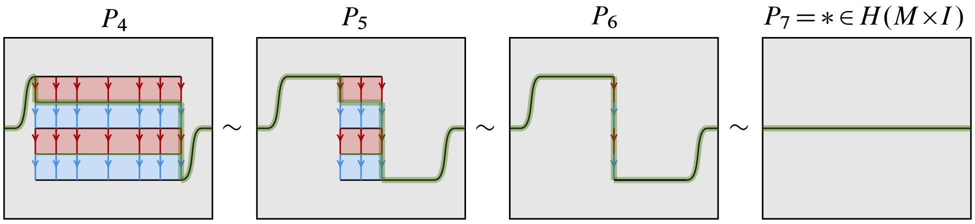

Figure 11: The path in $H(M \times I)$ between P_4 and P_7.

Unbending and straightening the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I \equiv M \times [-1,1] \times I$ in P_2, we get the path to the partition P_3 of Figure 9 (this step is not strictly necessary, but convenient for depiction).

We now use the diffeomorphism $\phi\colon V \cup_M \cup W \cong F \times [-1,1]$ (rel $\partial(F \times [-1,1])$) to carry out the path depicted in the lower part of Figure 10 locally in the circled subrectangle of P_3. This yields the path of Figure 10 between P_3 and P_4.

Retracting the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I$ in P_4 to $(W \cup_F V \cup_M W \cup_F V) \times \{\frac{1}{2}\}$, its midpoint, yields the path of Figure 11 between P_4 and P_6 (passing through P_5). But now as $W \cup_F V \cup_M W \cup_F V \equiv M \times [-1,3]$, the F-part of the partition P_6 is of the form $M \times \gamma$, for the 1-dimensional submanifold $\gamma \subset [-1,1] \times I$ depicted in P_6. By straightening γ to the submanifold $\{0\} \times I \subset [-1,1] \times I$, we get the path of Figure 11 between P_6 and the trivial partition $P_7 = * \in H(M \times I)$.

This process depends continuously on $(\rho, \phi) \in \overline{H}_{\mathrm{col}}(M)$ — i.e., can be set up as a homotopy of simplicial maps $\overline{H}_{\mathrm{col}}(M)_\bullet \to H(M \times I)_\bullet$ — and therefore gives the required nullhomotopy. □

Remark C.2 This result should be compared with [Hatcher 1978, Appendix I, Lemma] (or [Burghelea and Lashof 1982, Corollary A7]), which show that the concordance stabilisation map $\Sigma\colon C(M) \to C(M \times I)$ anticommutes (in the sense of Lemma 5.16) with the concordance involution ι_C of Warning 5.11.

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