Geometry & Topology

Volume 30 (2026)

The level structure in quantum K-theory and mock theta functions

Yongbin Ruan, Ming Zhang and Yaoxiong Wen

msp

Geometry & Topology **30**:3 (2026) 835–882
DOI: 10.2140/gt.2026.30.835
Published: April 20, 2026

The level structure in quantum K-theory and mock theta functions

YONGBIN RUAN, MING ZHANG AND YAOXIONG WEN

This is the first in a sequence of papers to develop the theory of levels in quantum K-theory and study its applications. Here, we give an adelic characterization of the range of the J-function in quantum K-theory with level structure. As an application, we prove a mirror theorem for permutation-equivariant quantum K-theory with level structures of toric varieties. In the study of the mirrors of some simple examples, we see the surprising appearance of Ramanujan's mock theta functions.

1 Introduction

1.1 Overview

More than a decade ago, quantum K-theory was introduced in [Givental 2000; Lee 2004] as the K-theoretic analog of quantum cohomology. Its recent revival stems partially from a physical interpretation of quantum K-theory as a 3D-quantum field theory in a 3-manifold of the form $S^1 \times \Sigma$ (see [Kapustin and Willett 2013; Jockers and Mayr 2020]). Because of this mysterious physical connection, the B-model counterpart of quantum K-theory is q-hypergeometric series, itself a classical subject. This connection was recently confirmed by Givental [2015e] as the mirror of the so-called *J-function* of the permutation-equivariant quantum K-theory.

Classically, K-theory is more closely related to representation theory, comparing to cohomology theory. It is natural to revisit quantum K-theory from the representation-theoretic point of view. In fact, a variant of quantum K-theory was already studied in [Okounkov 2017; Aganagic et al. 2018; Aganagic and Okounkov 2017; Pushkar et al. 2020; Koroteev et al. 2021; Okounkov and Smirnov 2022; Su et al. 2020] in relation to quantum groups. One of the predominant features of representation theory is the existence of an additional parameter, called the level. A natural question is whether it is possible to extend the current version of quantum K-theory to include this notion of level. In this paper, we answer the question affirmatively in the context of quasimap theory. This is the first in a sequence of papers to develop the theory of levels in quantum K-theory and study its applications.

Our motivating example is an old physical result [Witten 1995], which relates the quantum cohomology ring of the Grassmannian to the Verlinde algebra. Early explicit physical computations [Gepner 1991; Vafa 1992; Intriligator 1991] indicate that they are isomorphic as algebras, but have different pairings. Witten [1995] gave a conceptual explanation of the isomorphism, by proposing an equivalence between the quantum field theories which govern the Verlinde algebra and the quantum cohomology of the Grassmannian. His physical derivation of the equivalence naturally leads to a mathematical problem that

MSC2010: primary 14N35; secondary 14D20, 19E08, 33D15.

these two objects are conceptually isomorphic (without referring to the detailed computations). A great deal of work has been done in [Agnihotri 1995; Marian and Oprea 2007a; 2007b; 2010; Belkale 2008]. However, to the best of our knowledge, a complete conceptual proof of the equivalence is missing.

Assuming a basic knowledge of quantum K-theory, a key and yet more or less trivial observation is that the Verlinde algebra is a *K-theoretic invariant*. To be more precise, let X be a smooth projective variety. Suppose that $\overline{\mathcal{M}}_{g,k}(X,\beta)$ is the moduli space of stable maps to X. Quantum cohomology studies integrals of the form

$$\int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}} \alpha,$$

where $[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is the so-called *virtual fundamental cycle* and α is a "tautological" cohomology class. In quantum K-theory, we replace the virtual fundamental cycle by the *virtual structure sheaf* $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)}$. We also replace the integral by the holomorphic Euler characteristic

$$\chi(\overline{\mathcal{M}}_{g,k}(X,\beta), E\otimes\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)}),$$

where E is some natural K-theory class on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. For the Verlinde algebra, the relevant moduli space is the moduli space of semistable parabolic $U(n)$-bundles $\mathcal{M}_{U(n)}(\alpha_1,\dots,\alpha_k)$ on a fixed genus-g marked curve $(C,p_1,\dots,p_k)$ with parabolic structure at p_i indexed by $\alpha_i\in V_l(U(n))$. Here, l is a nonnegative integer and $V_l(U(n))$ denotes the level-l Verlinde algebra. A new ingredient is a certain determinant line bundle, denoted by det, over the moduli space $\mathcal{M}_{U(n)}(\alpha_1,\dots,\alpha_k)$. The level-$l$ Verlinde algebra calculates the holomorphic Euler characteristic

$$\langle\alpha_1,\dots,\alpha_k\rangle^l_{g,\mathrm{Verlinde}}=\chi(\mathcal{M}_{U(n)}(\alpha_1,\dots,\alpha_k),\det^l).$$

Based on the above description, the Verlinde algebra is clearly a K-theoretic object, and we should compare it with the quantum K-theory of the Grassmannian (with an appropriate notion of levels). Let $\mathfrak{Bun}_G$ be the moduli stack of principal bundles over curves. Let $\pi:\mathfrak{C}_{\mathfrak{Bun}_{g,k}}\to\mathfrak{Bun}_G$ be the universal curve and let $\mathfrak{P}\to\mathfrak{C}_{\mathfrak{Bun}_{g,k}}$ be the universal principal bundle. Given a finite-dimensional representation R of G, we consider the inverse determinant of cohomology

$$\det_R:=(\det R\pi_*(\mathfrak{P}\times_G R))^{-1}.$$

This is a line bundle over $\mathfrak{Bun}_G$. Suppose $X=Z/\!\!/G$ is a GIT quotient. Let $\mathcal{Q}^\epsilon$ be the moduli stack of ϵ-stable quasimaps to $X=Z/\!\!/G$ (see Section 2.2). Then there is a natural forgetful morphism $\mu:\mathcal{Q}^\epsilon\to\mathfrak{Bun}_G$. We define the level-$l$ determinant line bundle as

$$\mathcal{D}^{R,l}=\mu^*(\det_R)^l.$$

We will often refer to $\mathcal{D}^{R,l}$ as the *level structure*. In general, when X is a smooth complex projective variety but not a GIT quotient, one can still define determinant line bundles over the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$ as follows. Let $\mathcal{R}$ be a vector bundle over X. Let $\pi:\mathcal{C}\to\overline{\mathcal{M}}_{g,k}(X,\beta)$ be the universal curve and let $\mathrm{ev}: \mathcal{C} \to X$ be the universal evaluation map. We define the level-l determinant line bundle as

$$\mathcal{D}^l := (\det R\pi_*(\mathrm{ev}^* \mathcal{R}))^{-l}.$$

This definition agrees with the previous one when X is a GIT quotient (see Definition 2.4 and Remark 2.5).

With the above definition of the level-l determinant line bundle $\mathcal{D}^{R,l}$, we can define the level-l quantum K-invariants and quasimap invariants by twisting with $\mathcal{D}^{R,l}$ (see Section 3). The ordinary quantum K-theory corresponds to the case $l = 0$. One of the main purposes of this article is to verify that the level-l quantum K-theory satisfies all the axioms of the ordinary quantum K-theory (see Section 2.3).

One can use irreducible $\mathrm{GL}_n(\mathbb{C})$-representations to define two sets of invariants: Verlinde invariants from the theory of (semi)stable parabolic vector bundles, and level-l quantum K-theory invariants of the Grassmannian $\mathrm{Gr}(n, n+l)$. The *K-theoretic Verlinde/Grassmannian correspondence* describes a conjectural formula between these two sets of invariants (see [Ruan and Zhang 2023] for the precise formula).

1.2 Mirror theorem and mock theta functions

The proof of the Verlinde/Grassmannian correspondence will be discussed in different papers. The other main results of this paper are various mirror theorems, in the same style as the recent work of Givental [2017a; 2015a; 2015b; 2015c; 2015d; 2015e; 2015f; 2015g; 2017b; 2020; 2017c]. In the study of quantum K-theory with level structures, we see the surprising appearance of Ramanujan's mock theta functions in some of the simplest examples.

Let X be a smooth complex projective variety and let $\mathcal{R}$ be a vector bundle over X. When $X = Z /\!/ G$ is a GIT quotient, we assume $\mathcal{R}$ is of the form $(Z \times R) /\!/ G$, with R a finite-dimensional representation of G. By abuse of terminology, we refer to the vector bundle $\mathcal{R}$ as the "representation" R. Let Q be the Novikov variables. We fix a λ-algebra Λ which is equipped with Adams operations Ψ^i for $i = 1, 2, \dots$. Let $\{\phi_a\}$ be a basis of $K^0(X) \otimes \mathbb{Q}$ and let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing (5). Let q be a formal variable and let $\boldsymbol{t}(q)$ be a Laurent polynomial in q with coefficients in $K^0(X) \otimes \mathbb{Q}$. The *permutation-equivariant* K-theoretic J-function $\mathcal{J}^l_{S_\infty}(\boldsymbol{t}(q), Q)$ of level l and representation R is defined by

$$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t}(q), Q) := 1 - q + \boldsymbol{t}(q) + \sum_a \sum_{\beta \neq 0} Q^\beta \phi^a \left\langle \frac{\phi_a}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle^{R,l,S_k}_{0,k+1,\beta}.$$

Here $\langle \cdot \rangle^{R,l,S_k}_{0,k+1,\beta}$ denotes the permutation-equivariant quantum K-invariants of level l and L denote the cotangent line bundles. The J-function $\mathcal{J}^{R,l}_{S_\infty}$ can be viewed as elements in the loop space $\mathcal{K}$ defined by

$$\mathcal{K} := [K^0(X) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[\![Q]\!],$$

where $\mathbb{C}(q)$ denotes the field of complex rational functions in q. There is a natural Lagrangian polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where $\mathcal{K}_+$ consists of Laurent polynomials and $\mathcal{K}_-$ consists of reduced rational functions that are regular at $q=0$ and vanishing at $q=\infty$. We denote by $\mathcal{L}^{R,l}_{S_\infty}$ the range of $\mathcal{J}^{R,l}_{S_\infty}$, i.e.,

$$\mathcal{L}^{R,l}_{S_\infty} = \bigcup_{\boldsymbol{t}(q)\in\mathcal{K}_+} \mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t}(q), Q) \subset \mathcal{K}.$$

Due to the stacky structure of the moduli space of stable maps, the J-function $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t}(q), Q)$, as a function in q, has poles at roots of unity. The main technical tool is a generalization of Givental and Tonita's *adelic characterization* of points on the cone $\mathcal{L}^{R,l}_{S_\infty}$ with the presence of level structure, i.e., we describe the Laurent expansion of $\mathcal{J}^{R,l}_{S_\infty}$ at each primitive root of unity in terms of a certain twisted *fake* quantum K-theory. The precise statement is rather technical, and we present it in Theorem 4.5. When $l=0$, the determinant line bundle $\mathcal{D}^{R,l}$ is trivial, and we recover the ordinary quantum K-theory. The adelic characterization of the cone in the ordinary quantum K-theory was introduced in [Givental and Tonita 2014], and its generalization to the permutation-equivariant theory is given in [Givental 2015b]. The proofs of all these results are based on application of the virtual Kawasaki Riemann–Roch formula to moduli spaces of stable maps (see Section 4).

Let $\mathcal{L}_{S_\infty}$ denote the range of the permutation-equivariant big J-function in ordinary quantum K-theory (i.e., with trivial level structure). As an application of Theorem 4.5, we prove that certain "determinantal" modifications of points on $\mathcal{L}_{S_\infty}$ lie on the cone $\mathcal{L}^{R,l}_{S_\infty}$ of quantum K-theory of level l.

Theorem 1.1 *If*

$$I = \sum_{\beta\in \mathrm{Eff}(X)} I_\beta Q^\beta$$

lies on $\mathcal{L}_{S_\infty}$, then the point

$$I^{R,l} := \sum_{\beta\in \mathrm{Eff}(X)} I_\beta Q^\beta \prod_i (L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/2})^l$$

lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ of permutation-equivariant quantum K-theory of level l. Here $\mathrm{Eff}(X)$ denotes the semigroup of effective curve classes on X, L_i are the K-theoretic Chern roots of $\mathcal{R}$, and $\beta_i := \int_\beta c_1(L_i)$.

In Proposition 5.1, we give explicit formulas for level-l (torus-equivariant) I-functions of toric varieties. Moreover, we prove the following toric mirror theorem:

Theorem 1.2 *Assume that X is a smooth quasiprojective toric variety. The level-l torus-equivariant I-function $(1-q)I^{R,l}_X$ of X lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ in the permutation- and torus-equivariant quantum K-theory of level l of X.*

In the study of toric mirror theorems for quantum K-theory with level structure, a remarkable phenomenon is the appearance of Ramanujan's mock theta functions. We first establish some notation. Denote the standard representation of G by St and its dual by $\mathrm{St}^\vee$. When $G=\mathbb{C}^*$, any n-dimensional representation of G is determined by a *charge vector* $(a_1,\dots,a_n)$ with $a_i\in\mathbb{Z}$: a $\mathbb{C}^*$-action on $\mathbb{C}^n$ can be explicitly described by

$$\lambda\cdot(x_1,\dots,x_n)\to(\lambda^{a_1}x_1,\dots,\lambda^{a_n}x_n), \quad \text{where } \lambda\in\mathbb{C}^*.$$

In the following propositions, we consider GIT quotients $\mathbb{C}^n /\!\!/ \mathbb{C}^*$, and we refer to the $\mathbb{C}^*$-actions by their associated charge vectors.

Proposition 1.3 *Consider $X = \mathbb{C} /\!\!/ \mathbb{C}^* = [(\mathbb{C} \setminus 0)/\mathbb{C}^*]$, where the $\mathbb{C}^*$-action is the standard action by multiplication. The $\mathbb{C}^*$-equivariant K-ring $K_{\mathbb{C}^*}(X)$ is isomorphic to the representation ring* $\mathrm{Repr}(\mathbb{C}^*)$. *Let $\lambda \in K_{\mathbb{C}^*}(X)$ be the equivariant parameter corresponding to the standard representation. For the $\mathbb{C}^*$-representations* St *and* $\mathrm{St}^\vee$, *we have the explicit formulas of the equivariant small I-functions*

$$I_X^{\mathrm{St},l}(q,Q) = 1 + \sum_{n\geq 1} \frac{q^{n(n-1)l/2}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2)\cdots(1-\lambda^{-1}q^n)} Q^n,$$

$$I_X^{\mathrm{St}^\vee,l}(q,Q) = 1 + \sum_{n\geq 1} \frac{q^{n(n+1)l/2}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2)\cdots(1-\lambda^{-1}q^n)} Q^n.$$

By choosing certain specializations of the parameters, we obtain Ramanujan's mock theta functions of order 3,

$$I_X^{\mathrm{St},l=1}(q^2,Q)|_{\lambda=-1,Q=q} = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$I_X^{\mathrm{St},l=1}(q^2,Q)|_{\lambda=q,Q=q} = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})},$$

$$I_X^{\mathrm{St},l=1}(q^2,Q)|_{\lambda=-q,Q=1} = 1 + \sum_{n\geq 1} \frac{q^{n(n-1)}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})},$$

and Ramanujan's mock theta functions of order 5,

$$I_X^{\mathrm{St},l=2}(q,Q)|_{\lambda=-1,Q=q} = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)},$$

$$I_X^{\mathrm{St},l=2}(q^2,Q)|_{\lambda=q,Q=q^2} = 1 + \sum_{n\geq 1} \frac{q^{2n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})},$$

$$I_X^{\mathrm{St}^\vee,l=2}(q,Q)|_{\lambda=-1,Q=1} = 1 + \sum_{n\geq 1} \frac{q^{n(n+1)}}{(1+q)(1+q^2)\cdots(1+q^n)},$$

$$I_X^{\mathrm{St}^\vee,l=2}(q^2,Q)|_{\lambda=q,Q=1} = 1 + \sum_{n\geq 1} \frac{q^{2n^2+2n}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}.$$

Proposition 1.4 *Let a_1 and a_2 be two positive integers which are coprime. We consider the target $X_{a_1,a_2} = [(\mathbb{C}^2 \setminus 0)/\mathbb{C}^*]$ with charge vector (a_1,a_2) and a line bundle $p = [\{(\mathbb{C}^2 \setminus 0) \times \mathbb{C}\}/\mathbb{C}^*]$ with charge vector $(a_1,a_2,1)$. Let λ_1 and λ_2 be the equivariant parameters. For the $\mathbb{C}^*$-representations* St *and* $\mathrm{St}^\vee$, *we have the explicit formulas of the equivariant small* I*-functions*

$$I^{\mathrm{St},l}_{X_{a_1,a_2}}(q,Q)=1+\sum_{n\geq 1}\frac{p^{nl}q^{n(n-1)l/2}}{(1-p^{a_1}\lambda_1^{-1}q)\cdots(1-p^{a_1}\lambda_1^{-1}q^{a_1n})(1-p^{a_2}\lambda_2^{-1}q)\cdots(1-p^{a_2}\lambda_2^{-1}q^{a_2n})}Q^n,$$

$$I^{\mathrm{St}^\vee,l}_{X_{a_1,a_2}}(q,Q)=1+\sum_{n\geq 1}\frac{p^{nl}q^{n(n+1)l/2}}{(1-p^{a_1}\lambda_1^{-1}q)\cdots(1-p^{a_1}\lambda_1^{-1}q^{a_1n})(1-p^{a_2}\lambda_2^{-1}q)\cdots(1-p^{a_2}\lambda_2^{-1}q^{a_2n})}Q^n.$$

By choosing $(a_1,a_2)=(1,1)$ *and certain specializations of the parameters, we obtain the following four Ramanujan's mock theta functions of order* 3:

$$I^{\mathrm{St},l=2}_{X_{1,1}}(q,Q)|_{p=1,\lambda_1=\lambda_2=-1,Q=q}=1+\sum_{n\geq 1}\frac{q^{n^2}}{((1+q)(1+q^2)\cdots(1+q^n))^2},$$

$$I^{\mathrm{St},l=2}_{X_{1,1}}(q,Q)|_{p=1,\lambda_1=(1+\sqrt{3}i)/2,\lambda_2=(1-\sqrt{3}i)/2,Q=q}=1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})},$$

$$\frac{1}{(1-q)^2}I^{\mathrm{St}^\vee,l=2}_{X_{1,1}}(q^2,Q)|_{p=1,\lambda_1=\lambda_2=q^{-1},Q=1}=\sum_{n\geq 0}\frac{q^{2n^2+2n}}{((1-q)(1-q^3)\cdots(1-q^{2n+1}))^2},$$

$$\frac{1}{1+q+q^2}I^{\mathrm{St}^\vee,l=2}_{X_{1,1}}(q^2,Q)|_{p=1,\lambda_1=(-1+\sqrt{3}i)/2q^{-1},\lambda_2=(-1-\sqrt{3}i)/2q^{-1},Q=1}$$
$$=\sum_{n\geq 0}\frac{q^{2n^2+2n}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}.$$

Proposition 1.5 *Let* a *and* b *be two positive integers which are coprime. We consider the target* $X_{a,-b}=[\{(\mathbb{C}\setminus 0)\times\mathbb{C}\}/\mathbb{C}^*]$ *with charge vector* $(a,-b)$ *and a line bundle* $p=[\{(\mathbb{C}\setminus 0)\times\mathbb{C}\times\mathbb{C}\}/\mathbb{C}^*]$ *with charge vector* $(a,-b,1)$*. Let* λ *and* μ *be the equivariant parameters of the standard* $(\mathbb{C}^*)^2$*-action on* $X_{a,-b}$*. For the* $\mathbb{C}^*$*-representation* St*, we have the explicit formula for the equivariant small* I*-function*

$$I^{\mathrm{St},l}_{X_{a,-b}}(q)=$$
$$1+\sum_{n\geq 1}(-1)^{bn}\frac{p^{nl-b^2n}q^{n(n-1)l-bn(bn-1)/2}\mu^{-bn}(1-p^b\mu)(1-p^b\mu q)\cdots(1-p^b\mu q^{bn-1})}{(1-p^a\lambda^{-1}q)(1-p^a\lambda^{-1}q^2)\cdots(1-p^a\lambda^{-1}q^{an})}Q^n.$$

In particular, we have order 7 *mock theta functions*

$$I^{\mathrm{St},l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=1,\mu=q,Q=-q^2}=1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q^{n+1})\cdots(1-q^{2n})},$$

$$\frac{q}{1-q}I^{\mathrm{St},l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=q^{-1},\mu=q,Q=-q^4}=\sum_{n\geq 1}\frac{q^{n^2}}{(1-q^n)\cdots(1-q^{2n-1})},$$

$$\frac{1}{1-q}I^{\mathrm{St},l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=q^{-1},\mu=q,Q=-q^3}=\sum_{n\geq 1}\frac{q^{n^2-n}}{(1-q^n)\cdots(1-q^{2n-1})}.$$

Remark 1.6 The targets that we consider in Propositions 1.4 and 1.5 are, in general, orbifolds. The full I-functions have components corresponding to the twisted sectors of the (rigidified) inertia stacks of the orbifold targets. However, to match with mock theta functions, we only consider the components of I-functions corresponding to the untwisted sectors. See the discussion on the orbifold case in Remarks 4.14 and 4.15.

It is interesting that we can recover Ramanujan's mock theta functions using only very simple targets.

One of the attractive features of quantum K-theory is the appearance of q-hypergeometric series as mirrors of K-theoretic J-functions. Recall the definition of the q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{for } n>0,$$

and $(a;q)_0 := 1$. A general q-hypergeometric series can be written as

$${}_r\phi_s = \sum_{n\geq 0} \frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(\beta_1;q)_n\cdots(\beta_s;q)_n}\frac{z^n}{(q;q)_n}[(-1)^n q^{n(n-1)/2}]^{1+s-r}.$$

For the quantum K-theory of level 0, i.e., Givental and Lee's quantum K-theory, we only see special q-hypergeometric series of the form

$$\sum_{n\geq 0} \frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(\beta_1;q)_n\cdots(\beta_s;q)_n}\frac{z^n}{(q;q)_n}.$$

The level structure naturally introduces the term

$$[(-1)^n q^{n(n-1)/2}]^{1+s-r}.$$

Proposition 1.7 *Consider the target* $X_{\mathbf{1},-\mathbf{1}} := \mathcal{O}(-1)^{\oplus r}_{\mathbb{P}^s} = [\{(\mathbb{C}^{s+1}\setminus 0)\times\mathbb{C}^r\}/\mathbb{C}^*]$ *with the charge vector* $(1,1,\dots,1,-1,-1,\dots,-1)$. *Let* $p = [\{(\mathbb{C}^{s+1}\setminus 0)\times\mathbb{C}^r\times\mathbb{C}\}/\mathbb{C}^*]$ *be a line bundle with charge vector* $(1,\dots,1,-1,\dots,-1,1)$. *Let* $\lambda_1,\dots,\lambda_{s+1},\mu_1,\dots,\mu_r$ *be the equivariant parameters of the standard* $(\mathbb{C}^*)^{s+r+1}$*-action on* $X_{\mathbf{1},-\mathbf{1}}$. *Then the equivariant small I-function has the explicit form*

$$I^{\mathrm{St},l=1+s}_{X_{\mathbf{1},-\mathbf{1}}}(q) = 1+\sum_{n\geq 1}(-1)^{nr}\prod_{i=1}^{r}(p\mu_i)^{-n}p^{(1+s)n}\frac{(p\mu_1,q)_n\cdots(p\mu_r;q)_n}{(p\lambda_1^{-1}q;q)_n\cdots(p\lambda_s^{-1}q;q)_n}\frac{Q^n}{(p\lambda_{s+1}^{-1}q;q)_n}(q^{n(n-1)/2})^{1+s-r}.$$

Hence we can recover the general q-hypergeometric series by setting $p=1$, $\lambda_i^{-1}q=\beta_i$ *for* $i=1,\dots,s$, $\lambda_{s+1}^{-1}=1$ *and* $\mu_j=\alpha_j$ *for* $j=1,\dots,r$, $Q=(-1)^{1+s}z\prod_{i=1}^r\mu_i$.

Recall that Gromov–Witten theory (of Calabi–Yau varieties) is related to quasimodular forms. Mock modular forms are another class of modular objects, which are different from the quasimodular forms. Yet, they share some common properties. The above mirror theorems suggest an exciting possibility: that the natural geometric home of mock modular forms is quantum K-theory with nontrivial level structures. We certainly would like to investigate this surprising connection further.

Plan of the paper

In Section 2, we introduce the notion of level in the K-theoretic quasimap theory and establish its main properties. In Section 3, we define the K-theoretic quasimap invariants with level structure and their permutation-equivariant version. In Section 4, we characterize the cone $\mathcal{L}^{R,l}_{S_\infty}$ of level l in terms of the cone $\mathcal{L}_{S_\infty}$ of level 0. In Section 5, we compute the K-theoretic toric I-functions of level l, and prove a toric mirror theorem for quantum K-theory with level structure.

2 Level structure

In this section, we assume, unless otherwise indicated, that X is given by a geometric invariant theory (GIT) quotient. As mentioned in the introduction, the level structure is defined by determinant line bundles. We first recall the definition of determinant line bundles and then define the notion of level in quasimap theory.

2.1 Determinant line bundles

In this subsection, we briefly review the construction of determinant line bundles.

Let $\mathcal{X}$ be a Deligne–Mumford stack. Let $\mathcal{E}$ be a locally free, finitely generated $\mathcal{O}_{\mathcal{X}}$ module. We define the determinant line bundle of $\mathcal{E}$ as

$$\det(\mathcal{E}) := \bigwedge\nolimits^{\operatorname{rank}(\mathcal{E})} \mathcal{E},$$

where $\bigwedge^i$ denotes the i-th wedge product. In general, let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on $\mathcal{X}$ which has a bounded locally free resolution, i.e., there exists a bounded complex of locally free, finitely generated $\mathcal{O}_{\mathcal{X}}$ modules $\mathcal{G}^\bullet$ and a quasi-isomorphism

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet.$$

We define the determinant line bundle associated to $\mathcal{F}^\bullet$ by

$$\det(\mathcal{F}^\bullet) := \bigotimes_n \det(\mathcal{G}^n)^{(-1)^n}.$$

We summarize some basic properties of this construction in the following proposition:

Proposition 2.1 *Let $\mathcal{F}$ be a complex of coherent sheaves which has a bounded locally free resolution. Then*:

(1) *The construction of* $\det(\mathcal{F}^\bullet)$ *does not depend on the locally free resolution.*

(2) *For every short exact sequence of complexes of sheaves which have bounded locally free resolutions*

$$0 \to \mathcal{F}^\bullet \xrightarrow{\alpha} \mathcal{G}^\bullet \xrightarrow{\beta} \mathcal{H}^\bullet \to 0,$$

we have a functorial isomorphism

$$i(\alpha,\beta) : \det(\mathcal{F}^\bullet) \otimes \det(\mathcal{H}^\bullet) \xrightarrow{\sim} \det(\mathcal{G}^\bullet).$$

(3) *The operator* det *commutes with base change. To be more precise*, *for every* (*representable*) *morphism of Deligne–Mumford stacks*

$$g : \mathcal{X} \to \mathcal{Y},$$

we have an isomorphism

$$\det(\mathrm{L}g^*) \xrightarrow{\sim} g^* \det.$$

In the case when $\mathcal{X}$ is a scheme, the above proposition is proved in [Knudsen and Mumford 1976]. Note that these properties are preserved under flat base change. Therefore they hold for stacks as well.

2.2 Level structure in quasimap theory

In this subsection, we first recall the quasimap theory for nonsingular GIT quotients introduced in [Ciocan-Fontanine et al. 2014]. Then we define the level structure in this setting and discuss its generalizations in orbifold quasimap theory.

Let $Z = \mathrm{Spec}(A)$ be a complex affine algebraic variety in $\mathbb{C}^n$ and let G be a reductive group acting on it. Let $\theta : G \to \mathbb{C}^*$ be a character determining a G-equivariant line bundle $L_\theta := Z \times \mathbb{C}$. Let $Z^{\mathrm{s}}(\theta)$ and $Z^{\mathrm{ss}}(\theta)$ be the stable and semistable loci, respectively. Throughout the paper, we assume $Z^{\mathrm{s}}(\theta) = Z^{\mathrm{ss}}(\theta)$ is nonsingular. Furthermore, we assume that G acts freely on $Z^{\mathrm{s}}(\theta)$. It follows that the GIT quotient $Z /\!\!/_\theta G$ is nonsingular and quasiprojective. For simplicity, we drop θ from the notation of the GIT quotient. The unstable locus is defined as $Z_{\mathrm{us}} := Z - Z^{\mathrm{s}}(\theta)$. Recall that we can identify the G-equivariant Picard group $\mathrm{Pic}^G(Z)$ with the Picard group $\mathrm{Pic}([Z/G])$ of the quotient stack $[Z/G]$ by sending an G-equivariant line bundle L to $[L/G]$. Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(Z), \mathbb{Z})$.

Definition 2.2 [Ciocan-Fontanine et al. 2014] A quasimap is a tuple $(C, p_1, \dots, p_k, P, s)$ where

- $(C, p_1, \dots, p_k)$ is a connected, at most nodal, k-pointed projective curve of genus g,
- P is a principal G-bundle on C,
- s is a section of the induced fiber bundle $P \times_G Z$ on C such that (P, s) is of class β, i.e., the homomorphism

$$\mathrm{Pic}^G(Z) \to \mathbb{Z}, \quad L \mapsto \deg_C(s^*(P \times_G L)),$$

is equal to β.

We require that there be only finitely many basepoints, i.e., points $p \in C$ such that $s(p) \in Z_{\mathrm{us}}$. An element $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(Z), \mathbb{Z})$ is called L_θ-effective if it can be represented as a finite sum of classes of quasimaps. We also refer to β as a curve class. Denote by E the semigroup of L_θ-effective (curve) classes.

A quasimap $(C, p_1, \dots, p_k, P, s)$ is called *prestable* if the basepoints are disjoint from the nodes and marked points on C. Given a rational number $\epsilon > 0$, a prestable quasimap is called ϵ-*stable* if it satisfies the following conditions:

(1) $\omega_{C,\log} \otimes \mathcal{L}_\theta^\epsilon$ is ample, where $\omega_{C,\log} := \omega_C\bigl(\sum_{i=1}^k p_i\bigr)$ is the twisted dualizing sheaf of C and
$$\mathcal{L}_\theta := u^*(P \times_G L_\theta) \cong P \times_G \mathbb{C}_\theta.$$

(2) $\epsilon l(x) \le 1$ for every point x in C, where
$$l(x) := \mathrm{length}_x(\mathrm{coker}(u^*\mathcal{J}) \to \mathcal{O}_C).$$

Here $\mathcal{J}$ is the ideal sheaf of the closed subscheme $P \times_G Z_{\mathrm{us}}$ of $P \times_G Z$.

Let $\mathcal{Q}^\epsilon_{g,k}(Z/\!\!/G,\beta) = \{(C, p_1,\dots,p_k, P, s)\}$ be the moduli stack of ϵ-stable quasimaps. Ciocan-Fontanine et al. [2014] show that this stack is a separated Deligne–Mumford stack of finite type and it is proper over the affine quotient $Z/_{\mathrm{aff}}G := \mathrm{Spec}(A^G)$. When Z has only local complete intersection singularities, the ϵ-stable quasimap space $\mathcal{Q}^\epsilon_{g,k}(Z/\!\!/G,\beta)$ admits a canonical perfect obstruction theory.

Remark 2.3 There are two extreme chambers for the stability parameter ϵ.

(1) ($\epsilon = \infty$)-stable quasimaps: One can check that when $(g,k) \neq (0,0)$ and $\epsilon > 1$, the quasimap space $\mathcal{Q}^\epsilon_{g,k}(Z/\!\!/G,\beta)$ is isomorphic to the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(Z/\!\!/G,\beta)$. When $(g,k) = (0,0)$, the same holds with $\epsilon > 2$. Therefore, when ϵ is sufficiently large, we denote the ϵ-stable quasimap space by
$$\mathcal{Q}^\infty_{g,k}(Z/\!\!/G,\beta) = \overline{\mathcal{M}}_{g,k}(Z/\!\!/G,\beta)$$
and refer to it as the ($\epsilon = \infty$)-theory.

(2) ($\epsilon = 0+$)-stable quasimaps: Fix $\beta \in E$. For each $\epsilon \in (0, 1/\beta(L_\theta)]$, the ϵ-stability is equivalent to the condition that the underlying curve C of a quasimap does not have rational tails and, on each rational bridge, the line bundle $\mathcal{L}_\theta$ has strictly positive degree. Since we need to consider different β at the same time, we reformulate the stability condition as
$$\omega_{C,\log} \otimes \mathcal{L}_\theta^\epsilon \text{ is ample for all } \epsilon \in \mathbb{Q}_{>0}.$$
Quasimaps which satisfy the above stability condition are referred to as ($\epsilon = 0+$)-stable quasimaps.

To define the level structure, we introduce some notation first. Let $\mathfrak{M}_{g,k}$ be the algebraic stack of prestable nodal curves and $\mathfrak{Bun}_G$ be the relative moduli stack
$$\mathfrak{Bun}_G \xrightarrow{\phi} \mathfrak{M}_{g,k}$$
of principal G-bundles on the fibers of the universal curve $\mathfrak{C}_{g,k} \to \mathfrak{M}_{g,k}$. The morphism ϕ is smooth. There is a forgetful morphism, which forgets the section s,
$$\mathcal{Q}^\epsilon_{g,k}(Z/\!\!/G,\beta) \xrightarrow{\mu} \mathfrak{Bun}_G.$$
Let $\tilde{\pi} : \mathfrak{C}_{\mathfrak{Bun}_{g,k}} \to \mathfrak{Bun}_{g,k}$ be the universal curve which is the pullback of $\mathfrak{C}_{g,k}$ along ϕ. Let $\widetilde{\mathfrak{P}} \to \mathfrak{C}_{\mathfrak{Bun}_{g,k}}$ be the universal principal G-bundle. We denote by $\pi : \mathcal{C}_{g,k} \to \mathcal{Q}^\epsilon_{g,k}(Z/\!\!/G,\beta)$ the universal curve on the quasimap space. Let $\mathcal{P} \to \mathcal{C}_{g,k}$ be the universal principal bundle given by the pullback of $\widetilde{\mathfrak{P}} \to \mathfrak{C}_{\mathfrak{Bun}_{g,k}}$.

Definition 2.4 Given a finite-dimensional representation R of G, we define the level-l determinant line bundle over $\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)$ as

$$\mathcal{D}^{R,l} := (\det R\pi_*(\mathcal{P}\times_G R))^{-l}. \tag{1}$$

Alternatively, one can define $\mathcal{D}^{R,l}$ to be the pullback of the determinant line bundle $(\det R\pi_*(\widetilde{\mathfrak{P}}\times_G R))^{-l}$ on $\mathfrak{Bun}_{g,k}$ via μ.

Remark 2.5 The definition mentioned in the introduction is the second one. It is conceptually better in the sense that it does not depend on the different moduli spaces over $\mathfrak{Bun}_{g,k}$. In our case, these moduli spaces are the ϵ-stable quasimap spaces $\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)$ for different ϵ. However, $\mathfrak{Bun}_{g,k}$ is an Artin stack, and it is technically more difficult to work with it. Formally, we will use the first definition as the working definition.

Remark 2.6 Note that in Definition 2.4, the bundle $\mathcal{P}\times_G R$ is the pullback of the vector bundle $[Z\times R/G]\to[Z/G]$ along the evaluation map to the quotient stack $[Z/G]$. Therefore, given a vector bundle $\mathcal{R}$ on X, we can use (1) to define a determinant line bundle over the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$, even when X is not a GIT quotient. To be more precise, let $\pi:\mathcal{C}\to\overline{\mathcal{M}}_{g,k}(X,\beta)$ be the universal curve and let $\mathrm{ev}:\mathcal{C}\to X$ be the universal evaluation map. We define the level-l determinant line bundle as

$$\mathcal{D}^{R,l} := (\det R\pi_*(\mathrm{ev}^*\mathcal{R}))^{-l}.$$

We will often abuse notation by referring to the vector bundle $\mathcal{R}$ as the "representation" R.

The above construction can be easily generalized to orbifold quasimap theory. To be more precise, suppose the target can be written as $[Z^{\mathrm{s}}/G]$. Now, we do not assume G acts freely on the stable locus $Z^{\mathrm{s}}(\theta)$. Therefore $[Z^{\mathrm{s}}/G]$ is in general a Deligne–Mumford stack. The quasimap theory for such orbifold GIT targets is established in [Cheong et al. 2015]. According to [ibid., Section 2.4.5], we still have universal curves and universal principal G-bundles over moduli spaces of ϵ-stable orbifold quasimaps. Therefore the level-l determinant line bundle can still be defined using (1).

2.3 Properties of the level structure in quasimap theory

In this subsection, we study the level-l determinant line bundle in the case $\beta=0$ and its pullbacks along some natural morphisms between moduli spaces of quasimaps. An important property of the level structure $\mathcal{D}^{R,l}$ is that it splits "correctly" among nodal strata (see Proposition 2.9). In the following discussion, we assume $X=Z/\!\!/G$ is a GIT quotient, so that moduli spaces of quasimaps are defined. When X is a smooth projective variety but not a GIT quotient, the same results hold for the determinant line bundles defined in Remark 2.6. In fact, the arguments used in the proofs are identical for both cases.

2.3.1 Mapping to a point Assume that $\beta=0$. Then any quasimap is a constant map and the morphism

$$\mathcal{Q}^{\epsilon}_{g,k}(X,0)\xrightarrow{\mathrm{stab}\times\mathrm{ev}}\overline{M}_{g,k}\times X$$

is an isomorphism. Here $\mathrm{stab}: \mathcal{Q}^{\epsilon}_{g,k}(X,0) \to \overline{M}_{g,k}$ denotes the stabilization morphism of source curves of quasimaps, and $\mathrm{ev}: \mathcal{Q}^{\epsilon}_{g,k}(X,0) \to X$ denotes the constant evaluation morphism. Let P be the principal G-bundle $Z^{\mathrm{s}} \to X = Z/\!\!/G$.

Lemma 2.7 *The universal bundle $\mathcal{P}$ over the universal curve $\mathcal{C} = \overline{C}_{g,k} \times X$ is equal to $\pi_2^*(P)$, where $\overline{C}_{g,k}$ is the universal curve over $\overline{M}_{g,k}$ and $\pi_2 : \overline{C}_{g,k} \times X \to X$ is the second projection.*

Proof In general, there is an evaluation map from the universal curve $\mathcal{C}$ to the quotient stack $[Z/G]$ and $\mathcal{P}$ is the pullback of P along this map. The lemma follows from the observation that the evaluation map is given by the second projection π_2 in this case. □

Corollary 2.8 *Let $\mathcal{R} := P \times_G R$ be the associated vector bundle on X and let $\pi : \overline{C}_{g,k} \to \overline{M}_{g,k}$ be the canonical morphism. We have*

$$\mathcal{D}^{R,l} = \big(\textstyle\bigwedge^{\mathrm{rk}(R)g}(R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}} \boxtimes \mathcal{R})\big)^l \otimes \big(\textstyle\bigwedge^{\mathrm{rk}(R)}\mathcal{R}\big)^{-l}.$$

Proof By Lemma 2.7, we have $\mathcal{P} \times_G R = \pi_2^*(\mathcal{R})$. Therefore the pushforward $R\pi_*(\pi_2^*(\mathcal{R}))$ is equal to $\mathcal{O}_{\overline{M}_{g,k}} \boxtimes \mathcal{R} - R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}} \boxtimes \mathcal{R}$ via the projection formula. Note that $\mathrm{rk}(R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}}) = g$. □

2.3.2 Cutting edges For $i = 1, 2$, we denote by $\mathrm{ev}_{k_i} : \mathcal{Q}^{\epsilon}_{g_i,k_i}(X,\beta_i) \to X$ the evaluation morphism at the last marking. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2) & \xrightarrow{\Phi} & \mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{k_1} \times \mathrm{ev}_{k_2}} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

where Δ is the diagonal embedding of X. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the universal curves over

$$\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2) \quad \text{and} \quad \mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2),$$

respectively. Let $\mathcal{P}$ and $\mathcal{P}'$ be the universal principal G-bundles over $\mathcal{C}$ and $\mathcal{C}'$, respectively. We can define level structures

$$\mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2)} = \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1)} \boxtimes \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2)} \quad \text{and} \quad \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2)}$$

using (1). The following proposition shows that the level structure splits "correctly" among nodal strata:

Proposition 2.9 *Let $x : \mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2) \to \mathcal{C}$ be the section corresponding to the node. Then*

$$\Phi^*\big(\mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1)} \boxtimes \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2)}\big) = \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2)} \otimes \det(x^*(\mathcal{P} \times_G R))^{-l}. \tag{2}$$

Proof Consider the commutative diagram

$$\begin{array}{ccc} \Phi^*\mathcal{C}' & \xrightarrow{p} & \mathcal{C} \\ & {\scriptstyle \pi'} \searrow & \downarrow {\scriptstyle \pi} \\ & & \mathcal{Q}^{\epsilon}_{g_1,k_1}(X,\beta_1) \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2}(X,\beta_2) \end{array}$$

Notice that $\mathcal{C}$ is obtained by gluing along two sections of marked points of $\Phi^*\mathcal{C}'$. For any locally free sheaf $\mathcal{F}$ on $\mathcal{C}$, we have a short exact sequence (the normalization exact sequence)

$$0 \to \mathcal{F} \to p_* p^* \mathcal{F} \to x_* x^* \mathcal{F} \to 0.$$

which induces the natural isomorphism

$$\det(R\pi_*(p_* p^* \mathcal{F}))^{-1} \cong \det(R\pi_*(\mathcal{F}))^{-1} \otimes \det(R\pi_*(x_* x^* \mathcal{F}))^{-1}. \tag{3}$$

Note that

$$\det(R\pi_*(p_* p^* \mathcal{F}))^{-1} = \det(R\pi'_*(p^* \mathcal{F}))^{-1} \quad \text{and} \quad \det(R\pi_*(x_* x^* \mathcal{F}))^{-1} = \det(x^* \mathcal{F})^{-1}.$$

Now take $\mathcal{F} = \mathcal{P} \times_G R$ and $\mathcal{F}' = \mathcal{P}' \times_G R$. Finally, the lemma follows from the isomorphism $p^*\mathcal{F} \cong \Phi^*\mathcal{F}'$, equation (3), and cohomology and base change. □

2.3.3 Contractions Fix g_1, g_2 and k_1, k_2 such that $g = g_1 + g_2$ and $k = k_1 + k_2$. We denote the basic gluing maps by

$$r : \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} \to \overline{M}_{g,k}, \quad q : \overline{M}_{g-1,k+2} \to \overline{M}_{g,k}.$$

Let k' be a nonnegative integer. Let $\underline{k}' = (k'_1, \dots, k'_{m+1})$ and $\underline{\beta} = (\beta_1, \dots, \beta_{m+1})$ be partitions of k' and β, respectively. For simplicity, we denote by $\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}$ the fiber product

$$\mathcal{Q}^\epsilon_{g_1,k_1+k'_1+1}(X,\beta_1) \times_X \underbrace{\mathcal{Q}^\epsilon_{0,2+k'_3}(X,\beta_3) \times_X \cdots \times_X \mathcal{Q}^\epsilon_{0,2+k'_{m+1}}(X,\beta_{m+1})}_{m-1 \text{ factors}} \times_X \mathcal{Q}^\epsilon_{g_2,k_2+k'_2+1}(X,\beta_2).$$

Let $\mathrm{st} : \mathcal{Q}^\epsilon_{g,k+k'}(X,\beta) \to \overline{M}_{g,k}$ be the morphism defined by forgetting the (quasi)map and the last k' markings, then stabilizing the source curve. Consider the commutative diagram

$$\begin{array}{ccc} \bigsqcup \mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}} & \xrightarrow{\sqcup r_{m,\underline{k}',\underline{\beta}}} & \mathcal{Q}^\epsilon_{g,k+k'}(X,\beta) \\ \downarrow & & \downarrow \mathrm{st} \\ \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} & \xrightarrow{r} & \overline{M}_{g,k} \end{array}$$

Here the disjoint union is over partitions $\underline{k}'$ of k' and partitions $\underline{\beta}$ of β. The above commutative diagram induces a morphism

$$\Psi_m : \bigsqcup \mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}} \to (\overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1}) \times_{\overline{M}_{g,k}} \mathcal{Q}^\epsilon_{g,k+k'}(X,\beta).$$

Using the same argument as in the proof of [Lee 2004, Proposition 11], one can show that the virtual structure sheaves satisfy

$$\sum_m (-1)^{m+1} \Psi_{m*} \sum_{\underline{k}',\underline{\beta}} \mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}} = r^! \mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta)}.$$

Let $\mathcal{C}_{g,k+k'}$ and $\mathcal{C}_{m,\underline{k}',\underline{\beta}}$ be the universal curves on $\mathcal{Q}^{\epsilon}_{g,k+k'}(X,\beta)$ and $\mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}}$, respectively. Let $\mathcal{P}_{g,k+k'}$ and $\mathcal{P}_{m,\underline{k}',\underline{\beta}}$ be the corresponding universal principal G-bundles. The level structures

$$\mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g,k+k'}(X,\beta)} \quad \text{and} \quad \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}}}$$

can be defined using (1). To prove that quantum K-theory with level structure satisfies the same axioms as Givental and Lee's quantum K-theory, we need the following proposition:

Proposition 2.10 $$\mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}}} = (r_{m,\underline{k}',\underline{\beta}})^* \mathcal{D}^{R,l}_{\mathcal{Q}^{\epsilon}_{g,k+k'}(X,\beta)}.$$

Proof The proposition follows from the following cartesian diagram and cohomology and base change:

$$\begin{array}{ccc} \mathcal{C}_{m,\underline{k}',\underline{\beta}} & \longrightarrow & \mathcal{C}_{g,k+k'} \\ \downarrow & & \downarrow \\ \mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}} & \xrightarrow{r_{m,\underline{k}',\underline{\beta}}} & \mathcal{Q}^{\epsilon}_{g,k+k'}(X,\beta) \end{array}$$

□

3 The K-theoretic quasimap theory with level structure

In this section, we first define the K-theoretic quasimap invariants with level structure and their permutation-equivariant version. For most of the discussion, we assume that the GIT quotient $Z /\!/ G$ is nonsingular and projective. The cases when the target $[Z^{\mathrm{s}}/G]$ is noncompact or an orbifold are discussed at the end of this section.

3.1 K-theoretic quasimap invariants with level structure

In this subsection, we first briefly recall Givental and Lee's quantum K-theory. Then we define K-theoretic quasimap invariants with level structure.

The quantum K-theory or K-theoretic Gromov–Witten theory was introduced in [Givental 2000; Lee 2004]. Let X be a smooth projective variety and let $\overline{\mathcal{M}}_{g,k}(X,\beta)$ be the moduli space of stable maps to X. The moduli space is known to be a proper Deligne–Mumford stack (see for example [Behrend and Manin 1996]). In particular, for any coherent sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g,k}(X,\beta)$, we can consider its K-theoretic pushforward to the point $\operatorname{Spec}\mathbb{C}$, i.e., we can take its Euler characteristic

$$\chi(\mathcal{E}) = \sum_i (-1)^i h^i(\mathcal{E}),$$

where $h^i(\mathcal{E}) := \dim_{\mathbb{C}} H^i(\overline{\mathcal{M}}_{g,k}(X,\beta),\mathcal{E})$.

From perfect obstruction theory, Lee [2004] constructs a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}} \in K_0(\overline{\mathcal{M}}_{g,k}(X,\beta))$, where $K_0(\overline{\mathcal{M}}_{g,k}(X,\beta))$ denotes the Grothendieck group of coherent sheaves on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. The virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ has the following properties:

(1) If the obstruction sheaf is trivial and hence $\overline{\mathcal{M}}_{g,k}(X,\beta)$ is smooth, then $\mathcal{O}^{\mathrm{vir}}$ is the structure sheaf of $\overline{\mathcal{M}}_{g,k}(X,\beta)$.

(2) If the obstruction sheaf Obs is locally free, then $\mathcal{O}^{\mathrm{vir}} = \sum_i (-1)^i \bigwedge^i \mathrm{Obs}^\vee$. Here $\bigwedge^i \mathrm{Obs}^\vee$ denotes the i-th wedge product of the dual of the obstruction bundle.

Since we assume X to be smooth, the Grothendieck group of locally free sheaves on X, denoted by $K^0(X)$, is isomorphic to the Grothendieck group of coherent sheaves $K_0(X)$. We denote both of them by $K(X)$. Suppose that E_i are K-theory elements in $K(X)$ and let L_i denote the i-th cotangent line bundles. The *K-theoretic Gromov–Witten invariants* are defined by

$$\langle E_1 L_1^{l_1}, \dots, E_k L_k^{l_k} \rangle_{g,k,\beta} = \chi\Big(\overline{\mathcal{M}}_{g,k}(X,\beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}}\Big),$$

where $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(X,\beta) \to X$ are the evaluation morphisms at the i-th marking. Let $E \subset H_2(X,\mathbb{Z})$ be the semigroup generated by effective curve classes on X. We define the *quantum K-potential of genus* 0 by

$$\mathcal{F}(t,Q) := \tfrac{1}{2}(t,t) + \sum_{k=0}^{\infty} \sum_{\beta \in E} \frac{Q^\beta}{k!} \langle t, \dots, t \rangle_{0,k,\beta},$$

where $t \in K(X)_{\mathbb{Q}} := K(X) \otimes \mathbb{Q}$ and $(t,t) := \chi(t \otimes t)$ is the Mukai pairing. Let $\phi_0 = \mathcal{O}_X, \phi_1, \phi_2, \dots$ be a basis of $K(X)_{\mathbb{Q}}$. One can define the "quantized" pairing on $K(X)_{\mathbb{Q}}$ by

$$((\phi_i, \phi_j)) := \mathcal{F}_{ij} = \partial_{t_i} \partial_{t_j} \mathcal{F}(t,Q).$$

Lee [2004] shows that quantum K-theory satisfies all the usual axioms of cohomological Gromov–Witten theory except the flat identity axiom.

In the following discussion, we assume X can be represented as a GIT quotient $Z /\!/ G$. As mentioned before, the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(Z /\!/ G, \beta)$ can be identified with $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$ for large ϵ. According to [Ciocan-Fontanine et al. 2014], for general ϵ, the ϵ-stable quasimap space $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$ is proper and it admits a two-term perfect obstruction theory, assuming Z has only lci singularities. Hence, by [Lee 2004], one can construct a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ on $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$.

Definition 3.1 The *K-theoretic quasimap invariants of level l* are defined by

$$\langle E_1 L_1^{l_1}, \dots, E_k L_k^{l_k} \rangle_{g,k,\beta}^{Z/\!/G,R,l,\epsilon} = \chi\Big(\mathcal{Q}^{\epsilon}_{g,k}(Z /\!/ G, \beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}} \otimes \mathcal{D}^{R,l}\Big) \in \mathbb{Z},$$

where $E_i \in K(Z /\!/ G) \otimes \mathbb{Q}$.

We shall usually suppress $Z /\!/ G$ from the notation if there is no confusion. Note that these invariants are all integers.

3.2 Quasimap graph space and $J^{R,l,\epsilon}$-function

In this subsection, we first recall the definition and properties of the ϵ-stable quasimap graph space. Then we define an important generating series $\mathcal{J}^{R,l,\epsilon}$ of K-theoretic quasimap invariants of level l.

Given a rational number $\epsilon > 0$, the *quasimap graph space*, denoted by $\mathcal{QG}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$, is introduced in [Ciocan-Fontanine et al. 2014]. It is the moduli space of the tuples

$$((C, x_1, \dots x_k), P, u, \varphi),$$

where $((C, x_1, \dots x_k), P, u)$ is a prestable quasimap satisfying $\epsilon l(x) < 1$ for every point x on C, and the new data φ is a degree-1 morphism from C to $\mathbb{P}^1$. The curve C has a unique rational component C_0 such that $\varphi|_{C_0} : C_0 \to \mathbb{P}^1$ is an isomorphism and the complement C/C_0 is contracted by φ. The ampleness condition imposed on the tuples is modified to

$$\omega_{\overline{C \setminus C_0}}\Big(\sum x_i + \sum y_j\Big) \otimes \mathcal{L}_{\theta}^{\epsilon} \quad \text{is ample,}$$

where x_i are marked points on $\overline{C \setminus C_0}$ and y_i are the nodes $\overline{C \setminus C_0} \cap C_0$. Ciocan-Fontanine et al. [2014] show that the quasimap graph space is also a separated Deligne–Mumford stack which is proper over the affine quotient. Moreover, when Z has only lci singularities, the canonical obstruction theory on the graph space is perfect. Similarly, we can define the level-l determinant line bundle $\mathcal{D}^{R,l}_{\mathcal{QG}}$ on $\mathcal{QG}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$ using the universal principal G-bundles over its universal curve.

There is a natural $\mathbb{C}^*$-action on the graph spaces. Let $[x_0, x_1]$ be homogeneous coordinates on $\mathbb{P}^1$, and set $0 := [1, 0]$ and $\infty := [0, 1]$. We consider the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$,

$$t \cdot [x_0, x_1] = [t x_0, x_1] \quad \text{for all } t \in \mathbb{C}^*. \tag{4}$$

This induces an action on the ϵ-stable quasimap graph space $\mathcal{QG}^{\epsilon}_{g,k}(Z /\!/ G, \beta)$ by rescaling the parametrized rational component. According to [Ciocan-Fontanine and Kim 2014, Section 4.1], the $\mathbb{C}^*$-fixed locus can be described as

$$(\mathcal{QG}^{\epsilon}_{g,k}(Z /\!/ G, \beta))^{\mathbb{C}^*} = \coprod F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2},$$

where the disjoint union is over all possible splittings

$$g = g_1 + g_2, \quad k = k_1 + k_2, \quad \beta = \beta_1 + \beta_2$$

with $g_i, k_i \ge 0$ and β_i effective. In the stable cases, an ϵ-stable parametrized quasimap

$$((C, x_1, \dots, x_k), P, u, \varphi) \in F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2}$$

is obtained by gluing two ϵ-stable quasimaps of types (g_1, k_1, β_1) and (g_2, k_2, β_2) to a constant map $\mathbb{P}^1 \to p \in Z /\!/ G$ at 0 and ∞, respectively. Therefore, the component $F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2}$ is isomorphic to the fiber product

$$\mathcal{Q}^{\epsilon}_{g_1,k_1+\bullet}(Z /\!/ G, \beta_1) \times_{Z /\!/ G} \mathcal{Q}^{\epsilon}_{g_2,k_2+\bullet}(Z /\!/ G, \beta_2)$$

over the evaluation maps at the special marked points $\bullet$. When one of the components at 0 or ∞ is unstable, we use the following conventions:

(1) For the unstable cases $(g_1,k_1,\beta_1)=(0,0,0)$ or $(0,1,0)$ (and likewise for (g_2,k_2,β_2)), we define

$$\mathcal{Q}^{\epsilon}_{0,0+\bullet}(Z/\!/G,0):=Z/\!/G,\quad \mathcal{Q}^{\epsilon}_{0,1+\bullet}(Z/\!/G,0):=Z/\!/G,\quad \mathrm{ev}_\bullet=\mathrm{Id}_{Z/\!/G}.$$

(2) For the unstable cases $(g_1,k_1,\beta_1)=(0,0,\beta_1)$ with $\beta_1\neq 0$ and $\epsilon\le 1/\beta_1(L_\theta)$, we denote by

$$\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_1)_0$$

the moduli space of quasimaps $(C=\mathbb{P}^1,P,u)$ such that $u(x)\in P\times_G Z^{\mathrm{s}}$ for $x\neq 0\in\mathbb{P}^1$ and $0\in\mathbb{P}^1$ is a basepoint of length $\beta_1(L_\theta)$. Similarly, we define $\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_2)_\infty$ to be the moduli space of quasimaps whose only basepoint is of length $\beta_2(L_\theta)$ and located at ∞. Using these definitions, we have

$$F^{g,k,\beta_1}_{0,0,\beta_2}\cong\mathcal{Q}^{\epsilon}_{g,k+\bullet}(Z/\!/G,\beta_1)\times_{Z/\!/G}\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_2)_\infty$$

for $k\ge 1$ and $\epsilon\le 1/\beta_2(L_\theta)$. Similarly,

$$F^{0,0,\beta_1}_{g,k,\beta_2}\cong\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_1)_0\times_{Z/\!/G}\mathcal{Q}^{\epsilon}_{g,k+\bullet}(Z/\!/G,\beta_2)$$

for $k\ge 1$ and $\epsilon\le 1/\beta_1(L_\theta)$. When $g=k=0$ and $\epsilon\le\min\{1/\beta_1(L_\theta),1/\beta_2(L_\theta)\}$, we have

$$F^{0,0,\beta_1}_{0,0,\beta_2}\cong\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_1)_0\times_{Z/\!/G}\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta_2)_\infty.$$

We denote by $\mathcal{R}$ the vector bundle $Z\times_G R\to[Z/G]$ and its restriction to $Z/\!/G$. We define the twisted pairing on $K(Z/\!/G)_{\mathbb{Q}}$ by

$$(u,v)^{R,l}:=\chi(u\otimes v\otimes(\det\mathcal{R})^{-l}),\quad \text{where } u,v\in K(Z/\!/G)_{\mathbb{Q}}. \tag{5}$$

Let $\{\phi_a\}$ be a basis of $K(Z/\!/G)_{\mathbb{Q}}$ and let $\{\phi^a\}$ be the dual basis with respect to the above twisted pairing $(\cdot,\cdot)^{R,l}$. Let $t=\sum_i t^i\phi_i\in K(Z/\!/G)_{\mathbb{Q}}$. We define the $J^{R,l,\epsilon}$*-function of level* l to be[1]

$$\mathcal{J}^{R,l,\epsilon}(t,Q)=1-q+t+\sum_a\sum_{(k,\beta)\neq(0,0),(1,0)}\frac{Q^\beta}{k!}\phi^a\Big\langle\frac{\phi_a}{1-qL},t,\dots,t\Big\rangle^{R,l,\epsilon}_{0,k+1,\beta}. \tag{6}$$

In the above summation, the quasimap moduli spaces are empty when $k=0$, $\beta\neq 0$, $\beta(L_\theta)\le 1/\epsilon$, and the unstable terms are defined by $\mathbb{C}^*$-localization on the graph space $\mathcal{QG}^{\epsilon}_{0,0}(Z/\!/G,\beta)$. To be more precise, we consider the fixed-point locus $F_{0,\beta}:=\mathcal{Q}_{0,0+\bullet}(Z/\!/G,\beta)_0$ of the $\mathbb{C}^*$-action. The unstable terms in (6) are defined to be

$$(1-q)\sum_a\sum_{\substack{\beta\neq 0\\ \beta(L_\theta)\le 1/\epsilon}}Q^\beta(\mathrm{ev}_\bullet)_*\bigg(\mathcal{O}^{\mathrm{vir}}_{F_{0,\beta}}\otimes\bigg(\frac{\mathrm{tr}_{\mathbb{C}^*}\mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*}\bigwedge^* N^\vee_{F_{0,\beta}}}\bigg)\bigg),$$

where $\mathrm{ev}_\bullet$ is the evaluation map at $\infty\in\mathbb{P}^1$, $N_{F_{0,\beta}}$ is the *virtual* normal bundle of the fixed locus $F_{0,\beta}$ in $\mathcal{QG}^{\epsilon}_{0,0}(Z/\!/G,\beta)$ and $\bigwedge^* N^\vee_{F_{0,\beta}}:=\sum_i(-1)^i\bigwedge^i N^\vee_{F_0}$ is the *K-theoretic Euler class* of $N_{F_{0,\beta}}$. Here,

[1] In Ciocan-Fontanine and Kim's convention, the J-function starts at 1. The definition given here agrees with Givental's convention in which the J-function starts at the dilaton shift $1-q$.

the trace of a $\mathbb{C}^*$-equivariant bundle V, when restricted to the fixed-point locus, is a virtual bundle defined by the eigenspace decomposition with respect to the $\mathbb{C}^*$-action, i.e., we have

$$\operatorname{tr}_{\mathbb{C}^*}(V) := \sum_i q^i V(i),$$

where $t \in \mathbb{C}^*$ acts on $V(i)$ as multiplication by t^i.

For $1 < \epsilon \le \infty$, i.e., the $(\epsilon = \infty)$-theory, (6) defines $J^{R,l,\infty}$-function (or $J^{R,l}$-function for short) in the quantum K-theory of level l. In this case, we use $\langle \cdot \rangle^{R,l,\infty}$ or simply $\langle \cdot \rangle^{R,l}$ to denote quantum K-invariants of level l. Following [Givental and Tonita 2014], we introduce the *symplectic loop space formalism*. Recall that E denotes the semigroup of L_θ-effective curve classes on $Z/\!\!/_\theta G$. The *Novikov ring* $\mathbb{C}[\![Q]\!]$ is defined as

$$\mathbb{C}[\![Q]\!] := \overline{\left\{ \sum_{\beta \in E} c_\beta Q^\beta \;\middle|\; c_\beta \in \mathbb{C} \right\}}.$$

Here the completion is taken with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ denotes the maximal ideal generated by nonzero elements of E. We define the *loop space* as

$$\mathcal{K} := [K(Z/\!\!/G) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[\![Q]\!],$$

where $\mathbb{C}(q)$ is the field of complex rational functions in q. By viewing the elements in $\mathbb{C}(q) \otimes \mathbb{C}[\![Q]\!]$ as the coefficients, we extend the twisted pairing $(\cdot,\cdot)^{R,l}$ to $\mathcal{K}$ via linearity. There is a natural symplectic form Ω on $\mathcal{K}$ defined by

$$\Omega(f,g) := [\operatorname{Res}_{q=0} + \operatorname{Res}_{q=\infty}](f(q), g(q^{-1}))^{R,l} \frac{dq}{q}, \quad \text{where } f, q \in \mathcal{K}. \tag{7}$$

With respect to Ω, there is a *Lagrangian polarization* $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where

$$\mathcal{K}_+ = [K(Z/\!\!/G) \otimes \mathbb{C}[q,q^{-1}]\!] \otimes \mathbb{C}[\![Q]\!] \quad \text{and} \quad \mathcal{K}_- = \{ f \in \mathcal{K} \mid f(0) \neq \infty,\ f(\infty) = 0 \}.$$

As before, let $\{\phi_a\}$ be a basis of $K(Z/\!\!/G)_{\mathbb{Q}}$ and let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $(\cdot,\cdot)^{R,l}$. Let $\boldsymbol{t}(q) = \sum_{i,j} t^i_j \phi_i q^j \in \mathcal{K}_+$ be an arbitrary Laurent polynomial. We define the *big $J^{R,l}$-function of level l* to be the function $\mathcal{J}^{R,l}(\boldsymbol{t}(q), Q) : \mathcal{K}_+ \to \mathcal{K}$ given by

$$\mathcal{J}^{R,l}(\boldsymbol{t}(q), Q) = 1 - q + \boldsymbol{t}(q) + \sum_a \sum_{(k,\beta) \neq (0,0),(1,0)} \frac{Q^\beta}{k!} \phi^a \left\langle \frac{\phi_a}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \right\rangle^{R,l,\infty}_{0,k+1,\beta}.$$

Define the *genus*-0 *K-theoretic descendant potential of level l* by

$$\mathcal{F}^{R,l}(\boldsymbol{t}, Q) := \sum_{k,\beta} \frac{Q^\beta}{k!} \langle \boldsymbol{t}(L), \dots, \boldsymbol{t}(L) \rangle^{R,l,\infty}_{0,k,\beta}. \tag{8}$$

We can identify the cotangent bundle $T^*\mathcal{K}_+$ with the symplectic loop space $\mathcal{K}$ via the Lagrangian polarization and the *dilaton shift* $f \to f + (1-q)$. Then $\mathcal{J}^{R,l}$ coincides with the differential of the descendant potential up to the dilaton shift, i.e., we have

$$\mathcal{J}^{R,l} = 1 - q + \boldsymbol{t}(q) + d_t \mathcal{F}^{R,l}(\boldsymbol{t}, Q).$$

In the case $l = 0$, the above fact is proved in [Givental and Tonita 2014, Section 2]. The same argument works for arbitrary l.

For ($\epsilon = 0+$)-stable quasimap theory, the definition (6) gives the $I^{R,l}$-function of level l of $Z /\!/ G$,

$$\mathcal{I}^{R,l}(t, Q) := \frac{\mathcal{J}^{R,l,0+}(t, Q)}{1-q} = 1 + \frac{t}{1-q} + \sum_a \sum_{\beta \neq 0} Q^\beta (\mathrm{ev}_\bullet)_* \biggl(\mathcal{O}^{\mathrm{vir}}_{F_{0,\beta}} \otimes \biggl(\frac{\mathrm{tr}_{\mathbb{C}^*} \mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*} \bigwedge^* N^\vee_{F_{0,\beta}}} \biggr) \biggr) + \sum_a \sum_{k \geq 1, (k,\beta) \neq (1,0)} \frac{Q^\beta}{k!} \phi^a \biggl\langle \frac{\phi_a}{(1-q)(1-qL)}, t, \dots, t \biggr\rangle^{R,l,\epsilon=0+}_{0,k+1,\beta},$$

where $t \in K(Z /\!/ G)_{\mathbb{Q}}$.

3.3 The permutation-equivariant quasimap K-theory with level structure

Givental [2017a; 2015a; 2015b; 2015c; 2015d; 2015e; 2015f; 2015g; 2017b; 2020; 2017c] introduced the *permutation-equivariant* quantum K-theory, which takes into account the S_n-action on the moduli spaces of stable maps by permuting the marked points. The definition can be easily generalized to incorporate the level structure.

Let Λ be a λ-algebra, i.e., an algebra over $\mathbb{Q}$ equipped with abstract Adams operations Ψ^k for $k = 1, 2, \dots$. Here $\Psi^k : \Lambda \to \Lambda$ are ring homomorphisms which satisfy $\Psi^r \Psi^s = \Psi^{rs}$ and $\Psi^1 = \mathrm{id}$. We often assume that Λ includes the Novikov variables, the algebra of symmetric polynomials in a given number of variables, and the torus-equivariant K-ring of a point. We also assume that Λ has a maximal ideal Λ_+ and is equipped with the Λ_+-adic topology. For example, we can choose

$$\Lambda = \mathbb{Q}[\![N_1, N_2, \dots]\!][\![Q]\!][\Lambda_0^\pm, \dots, \Lambda_N^\pm],$$

where N_i are the Newton polynomials (in infinitely or finitely many variables) and Q denotes the Novikov variable(s). The parameters Λ_i denote the torus-equivariant parameters. The Adams operations Ψ^r act on N_m and Q by $\Psi^r(N_m) = N_{rm}$ and $\Psi^r(Q^\beta) = Q^{r\beta}$, respectively. We assume their actions on the torus-equivariant parameters are trivial.

Similar to the "ordinary" quasimap K-theory with level structure, we define the loop space by

$$\mathcal{K} := [K(Z /\!/ G) \otimes \Lambda] \otimes \mathbb{C}(q).$$

As before, this is equipped with a symplectic form defined by (7), and it has a Lagrangian polarization

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

where $\mathcal{K}_+$ is the subspace of Laurent polynomials in q and $\mathcal{K}_-$ is the subspace of reduced rational functions which are regular at $q = 0$ and vanish at $q = \infty$.

Consider the natural S_k-action on the quasimap moduli space $\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)$ by permuting the k marked points. Notice that the virtual structure sheaf $\mathcal{O}_{\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)}$ and the determinant line bundle $\mathcal{D}^{R,l}$ are invariant under this action. Therefore, we have the S_k-module

$$[\boldsymbol{t}(L),\dots,\boldsymbol{t}(L)]_{g,k,\beta} := \sum_m (-1)^m H^m\biggl(\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta), \mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)} \otimes \mathcal{D}^{R,l} \otimes \bigotimes_{i=1}^{k} \boldsymbol{t}(L_i)\biggr)$$

for $\boldsymbol{t}(q) \in \mathcal{K}_+$.

Definition 3.2 The correlators of the permutation-equivariant quasimap K-theory of level l are defined by

$$\langle \boldsymbol{t}(L),\dots,\boldsymbol{t}(L)\rangle^{R,l,\epsilon,S_k}_{g,k,\beta} := \pi_*\biggl(\mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)} \otimes \mathcal{D}^{R,l} \otimes \bigotimes_{i=1}^{k} \boldsymbol{t}(L_i)\biggr),$$

where π_* is the K-theoretic pushforward along the projection

$$\pi : [\mathcal{Q}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)/S_k] \to [\mathrm{pt}].$$

Remark 3.3 When the λ-algebra Λ is chosen to be $\mathbb{Q}[\![Q]\!]$, we refer to the invariants as the *symmetrized* invariants. The pushforward map in Definition 3.2 carries information only about the dimensions of the S_k-invariant parts of sheaf cohomology $[\boldsymbol{t}(L),\dots,\boldsymbol{t}(L)]_{g,k,\beta}$. We refer to [Givental 2017a, Example 4] for more details.

For the permutation-equivariant quasimap K-theory, we also consider the $J^{R,l,\epsilon}$-function, and define the cone $\mathcal{L}_{S_\infty}$ to be the range of the $J^{R,l,\infty}$-function.

Definition 3.4 The permutation-equivariant K-theoretic $J^{R,l,\epsilon}$-function of $Z/\!\!/G$ of level l is defined by

$$\mathcal{J}^{R,l,\epsilon}_{S_\infty}(\boldsymbol{t}(q),Q) := 1-q+\boldsymbol{t}(q)+\sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} Q^\beta \biggl\langle \frac{\phi_a}{1-qL}, \boldsymbol{t}(L),\dots,\boldsymbol{t}(L)\biggr\rangle^{R,l,\epsilon,S_k}_{0,k+1,\beta} \phi^a, \tag{9}$$

where the unstable terms in the summation are the same as those in (6).

Note that, in the above definition of permutation-equivariant $J^{R,l,\epsilon}$-function, we do not need to divide each term by $k!$.

Definition 3.5 We define the Givental cone $\mathcal{L}^{R,l}_{S_\infty}$ as the range of $\mathcal{J}^{R,l,\infty}_{S_\infty}$, i.e.,

$$\mathcal{L}^{R,l}_{S_\infty} := \bigcup_{\boldsymbol{t}(q)\in\mathcal{K}_+} \mathcal{J}^{R,l,\infty}_{S_\infty}(\boldsymbol{t}(q),Q) \subset \mathcal{K}.$$

Remark 3.6 In the ordinary — i.e., permutation-*nonequivariant* — quantum K-theory, the range of the J-function is a cone which coincides with the differential of the descendant potential (up to the dilaton shift). Therefore the range of the ordinary K-theoretic J-function is a Lagrangian cone in the loop space $\mathcal{K}$. However, in the permutation-equivariant theory, it is explained in [Givental 2015f] that the cone $\mathcal{L}^{R,l}_{S_\infty}$ is not Lagrangian.

3.4 The level structure in equivariant quasimap theory and orbifold quasimap theory

When $Z/\!\!/G$ is not proper, one can still define the *equivariant* quasimap invariants if $Z/\!\!/G$ has an additional torus action such that the fixed-point loci in the quasimap moduli spaces are proper. Ciocan-Fontanine et al. [2014, Section 6.3] explain how to define cohomological quasimap invariants via virtual localization. Similarly, one can define equivariant K-theoretic quasimap invariants (with level structure) for noncompact GIT targets using the K-theoretic virtual localization formula (see [Qu 2018, Section 3.2]). With this understood, we define the $J^{R,l,\epsilon}$-function of equivariant K-theoretic quasimap invariants of level l using (6). Its permutation-equivariant generalization is straightforward.

If we do not assume G acts freely on the stable locus Z^{s}, then the target $X := [Z^{\mathrm{s}}/G]$ is naturally an orbifold. For such orbifold GIT targets, a quasimap is a tuple $((C, x_1, \dots, x_k), [u])$, where $(C, x_1, \dots, x_k)$ is a k-pointed, genus-g twisted curve (see [Abramovich et al. 2008, Section 4]) and $[u]$ is a representable morphism from $(C, x_1, \dots, x_k)$ to X. We refer the reader to [Cheong et al. 2015, Section 2.3] for the details of the ϵ-stability imposed on those tuples. We denote by $\mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ the moduli stack of ϵ-stable quasimaps to the orbifold X. Cheong et al. [2015, Theorem 2.7] show that this moduli stack is Deligne–Mumford and proper over the affine quotient. Furthermore, if Z only has lci singularities, then $\mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ has a canonical perfect obstruction theory. Let $\mathcal{C} \to \mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ be the universal curve. The universal principal G-bundle $\mathcal{P} \to \mathcal{C}$ is defined as the pullback of the principal G-bundle $Z \to [Z/G]$ via the universal morphism $[u] : \mathcal{C} \to [Z/G]$. In the orbifold setting, we can still define the level-l determinant line bundle $\mathcal{D}^{R,l}$ using (1).

According to [Cheong et al. 2015, Section 2.5.1], there are natural evaluation morphisms

$$\mathrm{ev}_i : \mathcal{Q}^{\epsilon}_{g,k}(X, \beta) \to \bar{I}_{\mu} X, \qquad ((C, x_1, \dots, x_k), [u]) \mapsto [u]_{|x_i} \quad \text{for } i = 1, \dots, k.$$

Here $\bar{I}_{\mu} X$ denotes the rigidified cyclotomic inertia stack of X which parametrizes representable maps from gerbes banded by finite cyclic groups to X. Let L_i be the universal cotangent line bundle whose fiber at $((C, x_1 \dots, x_k), [u])$ is the cotangent space of the coarse curve $\underline{C}$ of C at the i-th marked point $\underline{x_i}$. For nonnegative integers l_i and classes $E_i \in K^0(\bar{I}_{\mu} X) \otimes \mathbb{Q}$, we define the K-theoretic quasimap invariants of level l as

$$\langle E_1 L_1^{l_1}, \dots, E_k L_k^{l_k} \rangle^{X,R,l,\epsilon}_{g,k,\beta} = \chi\biggl(\mathcal{Q}^{\epsilon}_{g,k}(X, \beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}} \otimes \mathcal{D}^{R,l} \biggr).$$

When $\epsilon = \infty$ and $l = 0$, this definition recovers the K-theoretic Gromov–Witten invariants of X defined in [Tonita and Tseng 2013].

In the orbifold setting, one can still define the quasimap graph space $\mathcal{QG}^{\epsilon}_{g,k}(X, \beta)$ (see [Cheong et al. 2015, Section 2.5.3]). The definition of the determinant line bundle $\mathcal{D}^{R,l}$ over the graph space is straightforward. We choose a basis $\{\phi_a\}$ of $K^0(\bar{I}_{\mu} X) \otimes \mathbb{Q}$. Let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $(\cdot\,,\cdot)^{R,l}$ on $K^0(\bar{I}_{\mu} X) \otimes \mathbb{Q}$ given by

$$(u, v)^{R,l} := \chi(\bar{I}_{\mu} X, u \otimes \bar{\iota}^* v \otimes \det{}^{-l}(\bar{I}_{\mu} \mathcal{R})).$$

Here $\bar{\iota}$ is the involution induced by $(x,g)\mapsto(x,g^{-1})$ and $\bar{I}_\mu\mathcal{R}$ is a vector bundle over $\bar{I}_\mu X$ such that the fiber over (x,H), with $H\subset \operatorname{Aut} x$, is the H-fixed subspace of $\mathcal{R}_x$. With all the notation understood, it is straightforward to adapt the definition of cohomological orbifold quasimap $\mathcal{J}^\epsilon$-function [Cheong et al. 2015, Definition 3.1] to the (permutation-equivariant) K-theoretic setting.

4 Adelic characterization in quantum K-theory with level structure

In this section, we focus on quantum K-theory, i.e., ($\epsilon=\infty$)-quasimap theory. We first recall the (virtual) Lefschetz–Kawasaki Riemann–Roch formula. This is the main tool in analyzing the poles of the $J^{R,l}$-function. We give an adelic characterization of points on the cone $\mathcal{L}^{R,l}_{S_\infty}$ in Theorem 4.5. As an application of the adelic characterization, we prove that certain "determinantal" modifications of points on the cone $\mathcal{L}_{S_\infty}$ of level 0 lie on the cone $\mathcal{L}^{R,l}_{S_\infty}$ of level l. This result will be used in the proof of the toric mirror theorem in Section 5. We assume in this section that X is a smooth projective variety which is not necessarily a GIT quotient. The determinant line bundle $\mathcal{D}^{R,l}$ is defined as in Remark 2.6.

4.1 Virtual Lefschetz–Kawasaki Riemann–Roch formula

To understand the poles of the generating series of the permutation-equivariant quantum K-invariants, we recall the *Lefschetz–Kawasaki Riemann–Roch formula* in [Givental 2017b].

Let h be a finite-order automorphism of a holomorphic orbibundle E over a compact smooth orbifold $\mathcal{M}$. The (super)trace of h on the sheaf cohomology $H^*(\mathcal{M},E)$ can be computed as an integral over the h-fixed-point locus $I\mathcal{M}^h$ in the inertia orbifold $I\mathcal{M}$:

$$\operatorname{tr}_h H^*(\mathcal{M},E)=\chi^{\text{fake}}\Bigl(I\mathcal{M}^h,\frac{\operatorname{tr}_{\tilde h} E}{\operatorname{tr}_{\tilde h}\bigwedge^* N^\vee_{I\mathcal{M}^h}}\Bigr):=\int_{[I\mathcal{M}^h]}\operatorname{td}(T_{I\mathcal{M}^h})\operatorname{ch}\Bigl(\frac{\operatorname{tr}_{\tilde h} E}{\operatorname{tr}_{\tilde h}\bigwedge^* N^\vee_{I\mathcal{M}^h}}\Bigr). \tag{10}$$

We explain the ingredients of this formula as follows. By definition, we can choose an atlas of local charts $U\to U/G(x)$ of $\mathcal{M}$. The local description of the inertia orbifold $I\mathcal{M}$ near $x\in\mathcal{M}$ is given by $\bigl[\coprod_{g\in G(x)} U^g/G(x)\bigr]$, where $U^g\subset U$ denotes the fixed-point locus of g. The automorphism h can be lifted to an automorphism $\tilde h$ of the chart U^g. More precisely, the orbibundle lifts to an $G(x)$-equivariant bundle over U^g. Then the transformation h can be lifted to an automorphism $\tilde h$ of the chart U^g and that of the $G(x)$-equivariant bundle in $|G(x)|$ many ways. We denote by $(U^g)^{\tilde h}$ the fixed-point locus of $\tilde h$ in U^g. Then the local description of the orbifold $I\mathcal{M}^h$ is given by $\bigl[\coprod_g (U^g)^{\tilde h}/G(x)\bigr]$. We refer to the connected components of $I\mathcal{M}^h$ as *Kawasaki strata*. Near a point $(x,[g])\in I\mathcal{M}^h$, the tangent and normal orbifold bundles $T_{I\mathcal{M}^h}$ and $N_{I\mathcal{M}^h}$ are identified with the tangent bundle and normal bundle to $(U^g)^{\tilde h}$ in U, respectively. In the denominator of the right side of (10), $\bigwedge^* N^\vee_{I\mathcal{M}^h}:=\sum_{i\ge 0}(-1)^i\bigwedge^i N^\vee_{I\mathcal{M}^h}$ is the K-theoretic Euler class of the normal bundle $N_{I\mathcal{M}^h}$. The trace bundle $\operatorname{tr}_{\tilde h} F$ is the virtual orbifold bundle

$$\operatorname{tr}_{\tilde h} F:=\sum_\lambda \lambda F_\lambda,$$

where F_λ are the eigenbundles of $\tilde{h}$ corresponding to the eigenvalues λ. Finally, td and ch denote the *Todd class* and *Chern character*.

By choosing h to be the identity map, we obtain *Kawasaki's Riemann–Roch formula* [1979] from (10):

$$\chi(\mathcal{M}, E) = \chi^{\text{fake}}\left(I\mathcal{M}, \frac{\operatorname{tr}_g E}{\operatorname{tr}_g \bigwedge^* N_{I\mathcal{M}}^\vee}\right). \tag{11}$$

When $\mathcal{M}$ is no longer smooth, Tonita [2014b] proved a *virtual* Kawasaki formula: under the assumption that $\mathcal{M}$ has a perfect obstruction theory and admits an embedding into a smooth orbifold which has the resolution property, Kawasaki's formula still holds true if we replace the structure sheaves and the tangent and normal bundles in the formula by their virtual counterparts. According to [Abramovich et al. 2007], the moduli stacks of stable maps to smooth projective varieties satisfy the assumptions of Tonita's theorem. In the next subsection, we apply the virtual Kawasaki Riemann–Roch (KRR) formula to $\overline{\mathcal{M}}_{g,k}(X,\beta)/S_k$ to study the poles of the J-function.

4.2 Adelic characterization

In this subsection, we first recall the adelic characterization [Givental 2015b] of the cone $\mathcal{L}_{S_\infty}$ in the level-0 permutation-equivariant quantum K-theory. Then we generalize it to describe points on the cone $\mathcal{L}_{S_\infty}^{R,l}$ of level l.

4.2.1 The level-0 case In the level-0 case, i.e., Givental and Lee's quantum K-theory, the permutation-equivariant invariants are defined as

$$\langle \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\rangle_{g,k,\beta}^{S_k} := \chi\left(\overline{\mathcal{M}}_{g,k}(X,\beta)/S_k, \mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)} \otimes \bigotimes_{i=1}^{k} \boldsymbol{t}(L_i)\right),$$

where $\boldsymbol{t}(q)$ is a Laurent polynomial in q with coefficients in $K^0(X)\otimes\mathbb{Q}$. This is a special case of Definition 3.2 with ϵ sufficiently large and $l=0$. The J-function is defined as

$$\mathcal{J}_{S_\infty}(\boldsymbol{t}(q), Q) := 1 - q + \boldsymbol{t}(q) + \sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} Q^\beta \left\langle \frac{\phi_a}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\right\rangle_{0,k+1,\beta}^{S_k} \phi^a.$$

Here $\{\phi_a\}$ and $\{\phi^a\}$ are bases of $K^0(X)\otimes\mathbb{Q}$ dual with respect to the Mukai pairing

$$(\phi_a, \phi_b) := \chi(\phi_a \otimes \phi_b).$$

Recall the definition of the loop space $\mathcal{K}$ from Section 3.2,

$$\mathcal{K} := [K^0(X)\otimes\mathbb{C}(q)]\otimes\mathbb{C}[\![Q]\!].$$

With respect to the symplectic form (7) with $l=0$, there is a Lagrangian polarization $\mathcal{K}=\mathcal{K}_+\oplus\mathcal{K}_-$, where $\mathcal{K}_+$ consists of Laurent polynomials in q and $\mathcal{K}_-$ consists of reduced rational functions in q. The Givental cone $\mathcal{L}_{S_\infty}\subset\mathcal{K}$ is defined as the image of $\mathcal{J}_{S_\infty}:\mathcal{K}_+\to\mathcal{K}$.

To study the poles of the series $\mathcal{J}_{S_\infty}(q)$, we apply the virtual KRR formula to $\overline{\mathcal{M}}_{0,k+1}(X,\beta)/S_k$, where the symmetric group acts on the last k markings. By the virtual KRR formula, each term in the J-function can be written as a summation of fake Euler characteristics over the Kawasaki strata. Note that the Kawasaki strata parametrize stable maps with prescribed automorphisms, i.e., equivalence classes of pairs (C, f, h), where (C, f) is a stable map to X and h is an automorphism of the map. Here, h is allowed to permute the last k markings, but it has to preserve the first marking (with the insertion $1/(1-qL)$). Denote by η the eigenvalue of h on the *cotangent* line to the curve at the first marking.

There are two types of Kawasaki strata. Over the Kawasaki strata with $\eta = 1$, the input $\mathrm{tr}_h(1/(1-qL))$ in the fake Euler characteristics becomes $1/(1-q\overline{L})$, where $1-\overline{L}$ is nilpotent. From the finite expansion

$$\frac{1}{1-q\overline{L}} = \sum_{i\geq 0} \frac{q^i(\overline{L}-1)^i}{(1-q)^{i+1}},$$

we see that the contributions to the J-function from the Kawasaki strata with $\eta = 1$ have poles at $q = 1$.

Over the Kawasaki strata where $\eta \neq 1$ is a primitive m-th root of unity, the insertion $\mathrm{tr}_h(1/(1-qL))$ in the fake Euler characteristics becomes $1/(1-q\eta\overline{L})$, where $1-\overline{L}$ is nilpotent. By considering the finite expansion

$$\frac{1}{1-q\eta\overline{L}} = \sum_{i\geq 0} \frac{(q\eta)^i(\overline{L}-1)^i}{(1-q\eta)^{i+1}},$$

we see that they contribute terms with possible poles at the root of unity η^{-1} to the J-function. We refer the reader to [Givental 2015b] for a nice diagram cataloging all the strata.

For each primitive m-th root of unity η, we denote by $\mathcal{J}_{S_\infty}(\boldsymbol{t})_{(\eta)}$ the Laurent expansion of the J-function in $1-q\eta$ and regard it as an element in the loop space of power Q-series with vector Laurent series in $1-q\eta$ as coefficients:

$$\mathcal{K}^\eta := K^0(X)\Big[\frac{1}{1-q\eta}, 1-q\eta\Big]\!\Big] \otimes \mathbb{C}[\![Q]\!].$$

The contributions from the untwisted sector of $\overline{\mathcal{M}}_{0,k}(X,\beta)/S_k$ in the virtual KRR formula are called the *fake* K-theoretic Gromov–Witten (GW) invariants. More precisely, they are defined by

$$\langle \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\rangle^{\mathrm{fake}}_{0,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ch}(\boldsymbol{t}(L_i))\,\mathrm{Td}(T^{\mathrm{vir}}),$$

where $[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space and T^{vir} is the virtual tangent bundle of $\overline{\mathcal{M}}_{0,k}(X,\beta)$. We define the J-function in the fake quantum K-theory, $\mathcal{J}_{\mathrm{fake}} : \mathcal{K}^1_+ \to \mathcal{K}^1$, by

$$\mathcal{J}_{\mathrm{fake}}(\boldsymbol{t}(q), Q) := 1-q+\boldsymbol{t}(q) + \sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!}\phi^a \Big\langle \frac{\phi_a}{1-qL}, \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\Big\rangle^{\mathrm{fake}}_{0,k+1,\beta}.$$

Here the input $\boldsymbol{t}(q)$ belongs to

$$\mathcal{K}^1_+ := K^0(X)[\![1-q]\!] \otimes \mathbb{C}[\![Q]\!].$$

Denote by $\mathcal{L}_{\text{fake}} \subset \mathcal{K}^1$ the range of the series $\mathcal{J}_{\text{fake}}$. The negative space $\mathcal{K}^1_-$ of the polarization is spanned by $\{\phi^a q^k/(1-q)^{k+1} \mid a = 1, \dots, \dim K^0(X)_{\mathbb{Q}}, k = 0, 1, \dots\}$. The input $\boldsymbol{t}(q)$ of $\mathcal{J}_{\text{fake}}$ can be obtained from the projection of $\mathcal{J}_{\text{fake}}$ to $\mathcal{K}^1_+$ along $\mathcal{K}^1_-$.

Givental [2015b] gives the following adelic characterization of the values of $\mathcal{J}_{S_\infty}(\boldsymbol{t})$:

Theorem 4.1 [Givental 2015b] *The values of $\mathcal{J}_{S_\infty}(\boldsymbol{t})$ are characterized by the following requirements*:

(1) *$\mathcal{J}_{S_\infty}(\boldsymbol{t})$ has possible poles only at $0, \infty$ and roots of unity.*

(2) *$\mathcal{J}_{S_\infty}(\boldsymbol{t})_{(1)}$ lies on $\mathcal{L}_{\text{fake}}$.*

(3) *For every primitive root of unity η of order $m \neq 1$,*

$$\mathcal{J}_{S_\infty}(\boldsymbol{t})_{(\eta)}\left(\frac{q^{1/m}}{\eta}\right) \in \sqrt{\frac{\lambda_{-1}(T_X)}{\lambda_{-1}(\Psi^m T_X)}}\exp\sum_{i\geq 1}\left(\frac{\Psi^i T_X^\vee}{i(1-\eta^{-i}q^{i/m})} - \frac{\Psi^{im}T_X^\vee}{i(1-q^{im})}\right)\mathcal{T}_m(\mathcal{J}_{S_\infty}(\boldsymbol{t})_{(1)}),$$

where Ψ^m is the m-th Adams operation on $K^0(X)_{\mathbb{Q}}$, which acts on line bundles as $L \mapsto L^m$; $\lambda_{-1}(E) := \sum_i (-1)^i \bigwedge^i E^\vee$ is the K-theoretic Euler class of a vector bundle E; and $\mathcal{T}_m(\boldsymbol{f})$ is the space described in Definition 4.2 below.

We recall the following definition from [Tonita 2018]:

Definition 4.2 Let $\boldsymbol{f}$ be a point on $\mathcal{L}_{\text{fake}}$ and let $T(\boldsymbol{f})$ be the tangent space to $\mathcal{L}_{\text{fake}}$ at $\boldsymbol{f}$, considered as the image of a map $S(q,Q) : \mathcal{K}^1_+ \to \mathcal{K}$. We extend the Adams operations from $K^0(X)_{\mathbb{Q}}$ to $\mathcal{K}^1$ by $\Psi^m(q) = q^m$ and $\Psi^m(Q) = Q^m$. Let $\Psi^{1/m}$ be the inverse of Ψ^m, acting as $q \mapsto q^{1/m}$ and $Q \mapsto Q^{1/m}$. We define the space $\mathcal{T}_m(\boldsymbol{f})$ to be the image of the conjugate of $S(q,Q)$,

$$\Psi^m \circ S(q,Q) \circ \Psi^{1/m} : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Remark 4.3 As explained in [Tonita 2018, Remark 5.6], the explicit operator in condition (3) of Theorem 4.1 can be written as a composition $\square_\eta \square_m^{-1}$. The definitions of the operators $\square_\eta$ and $\square_m$ are given in Proposition 4.10.

4.2.2 The general case of level l Let R be a vector bundle over X. According to Remark 2.6, the level structure $\mathcal{D}^{R,l}$ is defined as

$$\mathcal{D}^{R,l} = (\det \mathcal{R}_{k,\beta})^{-l},$$

where $\mathcal{R}_{k,\beta} := R\pi_*(\text{ev}^* \mathcal{R})$ is the *index bundle*, $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,k}(X,\beta)$ is the universal curve, and $\text{ev} : \mathcal{C} \to X$ is the universal evaluation morphism. To state the adelic characterization theorem of the cone $\mathcal{L}^{R,l}_{S_\infty}$ in the permutation-equivariant quantum K-theory with level structure, we introduce the fake invariants of level l,

$$\langle \boldsymbol{t}(L), \dots, \boldsymbol{t}(L)\rangle^{\text{fake},R,l}_{0,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\text{vir}}} \prod_{i=1}^k \text{ch}(\boldsymbol{t}(L_i))\,\text{Td}(T^{\text{vir}})\,\text{ch}(\mathcal{D}^{R,l}),$$

and the fake J-function of level l,

$$\mathcal{J}^{R,l}_{\text{fake}}(\boldsymbol{t}(q),Q) := 1-q+\boldsymbol{t}(q)+\sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!}\phi^a\left\langle \frac{\phi_a}{1-qL},\boldsymbol{t}(L),\dots,\boldsymbol{t}(L)\right\rangle^{\text{fake},R,l}_{0,k+1,\beta}.$$

Here $\{\phi^a\}$ is the dual basis of $\{\phi_a\}$ with respect to the twisted pairing

$$(u,v)^{R,l} = \chi(u\otimes v\otimes(\det\mathcal{R})^{-l}).$$

Denote by $\mathcal{L}^{R,l}_{\text{fake}}\subset\mathcal{K}^1$ the range of the series $\mathcal{J}^{R,l}_{\text{fake}}$. Based on the relationship between gravitational descendants and ancestors of fake quantum K-theory [Givental 2004], one can show that $\mathcal{L}^{R,l}_{\text{fake}}$ is an overruled Lagrangian cone. We refer the reader to [Givental and Tonita 2014, Section 3] for more details.

Convention 4.4 We will consider various twisted theories, in which the pairings are usually different. In particular, the dual bases $\{\phi^a\}$ which appear in the definitions of various J-functions may not be the same. To relate J-functions in different theories, we need to regard them as elements of the same loop space. This is achieved by rescaling the elements in loop spaces. For example, there is a rescaling map

$$(\mathcal{K}^1,(\,,)^{R,l})\to(\mathcal{K}^1,(\,,)),\quad E\mapsto E\otimes(\det\mathcal{R})^{-l/2},$$

which identifies the loop space in fake quantum K-theory of level l with that in fake quantum K-theory of level 0.

One of our main results is the following adelic characterization of values of the big J-function in quantum K-theory of level l, generalizing Theorem 4.1:

Theorem 4.5 *The values of $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})$ are characterized by the following requirements*:

(1) *$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})$ has possible poles only at 0, ∞ and roots of unity.*
(2) *$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)}$ lies on $\mathcal{L}^{R,l}_{\text{fake}}$.*
(3) *For every primitive root of unity η of order $m\neq 1$,*

$$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(\eta)}\left(\frac{q^{1/m}}{\eta}\right)\in\sqrt{\frac{\lambda_{-1}(T_X)}{\lambda_{-1}(\Psi^m T_X)}}\exp\sum_{i\geq1}\left(\frac{\Psi^i T_X^\vee}{i(1-\eta^{-i}q^{i/m})}-\frac{\Psi^{im}T_X^\vee}{i(1-q^{im})}\right)\mathcal{T}^{R,l}_m(\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)}),$$

where the space $\mathcal{T}^{R,l}_m(\boldsymbol{f})$ is defined as in Definition 4.2 but starting with a point $\boldsymbol{f}\in\mathcal{L}^{R,l}_{\text{fake}}$.

4.2.3 The proof of Theorem 4.5 We follow the proofs of [Givental and Tonita 2014; Tonita 2018]. The first requirement in Theorem 4.5 is obviously satisfied. For the second condition, we apply the virtual KRR formula to the J-function. Let $\widetilde{\boldsymbol{T}}(q)$ be the sum of $\boldsymbol{t}(q)$ and all contributions in $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})$ which are regular at $q=1$. According to Proposition 2.9, the level structure $\mathcal{D}^{R,l}$ splits "correctly" over nodal strata. With this understood, the following proposition follows from an argument identical to the one given in [Tonita 2018, Proposition 5.2]:

Proposition 4.6 *We have*

$$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t}(q))_{(1)} = \mathcal{J}^{R,l}_{\text{fake}}(\widetilde{\boldsymbol{T}}(q))$$

as elements in $\mathcal{K}^1$. In particular, this shows that $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)}$ lies on the cone $\mathcal{L}^{R,l}_{\text{fake}}$.

Before we move on to prove the third condition in Theorem 4.5, we characterize the cone $\mathcal{L}^{R,l}_{\text{fake}}$ of fake quantum K-theory of level l in terms of the cone $\mathcal{L}_{\text{fake}}$ of level 0. This characterization will be needed later in the proof of Theorem 4.5.

Note that the fake quantum K-theory is a version of twisted cohomological Gromov–Witten theory. The machinery of twisted cohomological Gromov–Witten invariants was introduced in [Coates and Givental 2007], and generalized in various directions in [Tseng 2010; Tonita 2014a]. Let $\mathcal{H}$ be the loop space of the cohomological GW theory of X,

$$\mathcal{H} := H^{\text{even}}(X,\mathbb{C})[z^{-1},z]][[Q]].$$

This is equipped with a natural symplectic form Ω_H on $\mathcal{H}$ given by

$$\Omega_H(\boldsymbol{f},\boldsymbol{g}) = \text{Res}_{z=0}(\boldsymbol{f}(-z),\boldsymbol{g}(z))\,dz,$$

where the pairing $(\,,)$ is the Poincaré pairing. With respect to Ω_H, there is a Lagrangian polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ := H^{\text{even}}(X,\mathbb{C})[[z]][[Q]], \quad \mathcal{H}_- := \frac{1}{z}H^{\text{even}}(X,\mathbb{C})[z^{-1}][[Q]].$$

Inside $\mathcal{H}$, one can define an overruled Lagrangian cone $\mathcal{L}_H$ by the image of the cohomological big J-function. Since we will not use the explicit description of $\mathcal{L}_H$ in this paper, we refer the reader to [Coates and Givental 2007] for the basic definitions in the cohomological GW theory.

Convention 4.7 Throughout this subsection, we identify $\mathcal{K}^1$ with $\mathcal{H}$ via the Chern character

$$\text{qch}: \mathcal{K}^1 \to \mathcal{H}, \qquad E \mapsto \text{ch}(E), \quad q \mapsto e^z.$$

Hence, K-theoretic insertions (e.g., ϕ_a and L) in the correlators of twisted cohomological theories should be understood as their Chern characters (e.g., $\text{ch}\,\phi_a$ and $\text{ch}\,L$).

By definition, the fake quantum K-invariants are obtained from the cohomological invariants by inserting the Todd class $\text{Td}(T^{\text{vir}})$ of the virtual tangent bundle T^{vir}. According to [Coates 2003], the virtual tangent bundle can be written as

$$T^{\text{vir}} = \pi_*(\text{ev}^*\,T_X - 1) - \pi_*(L^\vee_{k+1} - 1) - (\pi_* i_* \mathcal{O}_{\mathcal{Z}})^\vee \tag{12}$$

in $K^0(\overline{\mathcal{M}}_{0,k}(X,\beta))$. Here, $i:\mathcal{Z}\to\mathcal{C}$ is the embedding of the nodal locus. The three parts correspond respectively to

(i) deformations of maps to X of a fixed source curve,

(ii) deformations of complex structure and configuration of markings, and

(iii) smoothing the nodes.

Coates and Givental [2006] prove that the cone $\mathcal{L}_{\text{fake}}$ (of level 0) is given explicitly in terms of $\mathcal{L}_H$ as

$$\operatorname{qch}(\mathcal{L}_{\text{fake}}) = \Delta \mathcal{L}_H,$$

where the *loop group transformation* Δ is the Euler–Maclaurin asymptotics of the infinite product

$$\Delta \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x - rz}{1 - e^{-x+rz}}.$$

Here, Δ acts on $\mathcal{H}$ by pointwise multiplication, and it is determined only by the Todd class of the first summand in the expression of T^{vir}. The second and third summands in (12) are respectively responsible for the changes of the dilaton shifts and the polarizations between $\mathcal{H}$ and $\mathcal{K}^1$. We refer to [Coates 2003] for the details.

The fake quantum K-invariants of level l are obtained from those of level 0 by inserting one more class $\operatorname{ch}(\mathcal{D}^{R,l})$. Its effect on the Lagrangian cone is described in the following proposition:

Proposition 4.8 *Under the identification $z = \log q$, we have the identity*

$$\mathcal{L}_{\text{fake}}^{R,l} = \exp\left(-l\left(\frac{\operatorname{ch}_2 \mathcal{R}}{z}\right)\right)\mathcal{L}_{\text{fake}}$$

in the loop space $(\mathcal{K}^1, (\,,\,))$.

Proof Recall that the level structure $\mathcal{D}^{R,l}$ is defined as a certain power of the determinant of the index bundle $\mathcal{R}_{k,\beta} = R\pi_*(\operatorname{ev}^* \mathcal{R})$. Note that

$$\operatorname{ch}(\mathcal{D}^{R,l}) = \exp(-l \cdot \operatorname{ch}_1(\mathcal{R}_{k,\beta})). \tag{13}$$

According to [Coates and Givental 2007], the cone of a theory twisted by a general multiplicative characteristic class of the form

$$\exp\left(\sum_{i\geq 0} s_i \operatorname{ch}_i(\mathcal{R}_{k,\beta})\right)$$

is obtained from the cone of the untwisted theory by applying the operator

$$\exp\left(\sum_{m,i\geq 0} s_{2m-1+i} \frac{B_{2m}}{(2m)!} \operatorname{ch}_i(\mathcal{R}) \cdot z^{2m-1}\right).$$

Here we set $s_{-1} = 0$ and the Bernoulli numbers B_{2m} are defined by

$$\frac{t}{1-e^{-t}} = \frac{t}{2} + \sum_{m\geq 0} \frac{B_{2m}}{(2m)!} t^{2m},$$

and the operator acts on $\mathcal{H}$ by pointwise multiplication. For the twisting class (13), we have $s_1 = -l$ and $s_i = 0$ if $i \neq 1$. By applying the above result, we obtain the corresponding loop group transformation,

$$\exp\left(-l\left(\frac{\operatorname{ch}_2(\mathcal{R})}{z} + \frac{\operatorname{ch}_0(\mathcal{R}) \cdot z}{12}\right)\right).$$

Note that the cone $\mathcal{L}_{\text{fake}}$, being overruled, is invariant under multiplication by functions of z. Therefore, we can ignore the second summand in the exponent of the above operator. □

Now let us prove the third condition in Theorem 4.5. Let $\eta \neq 1$ be a primitive root of unity of order m. The Kawasaki strata in $\overline{\mathcal{M}}_{0,k+1}(X,\beta)/S_k$ which contribute terms with poles at $q=\eta^{-1}$ to the J-function are called the *stem spaces* in [Givental and Tonita 2014]. We give a brief description of stem spaces here, and we refer the reader to [ibid., Section 8] for more details. Let (C', f, h) be a point in these strata. A node is *balanced* if the eigenvalues of h on the two branches are inverse. Let $C_+ \subset C'$ denote the unique maximal subcurve, containing the first marked point, that consists of a chain of balanced $\mathbb{P}^1$ such that the m-th power h^m acts as the identity. There are only two smooth points on C_+ which are fixed by h: the first marking on the first component, and one more on the last component. The second point is called the *butt* in [loc. cit.]. The butt can be a regular point, a marking or a node in C'. The automorphism h acts on the cotangent space at the butt by η^{-1}. The other marked points and unbalanced nodes on C_+ are cyclically permuted by h. We denote by C the quotient of C_+ by the $\mathbb{Z}_m$-symmetry generated by h. The quotient curve C together with the induced quotient stable map is called a *stem* in [loc. cit.]. Note that a stem curve can carry: unramified marked points, coming from symmetric configurations of m-tuples of markings on the cover; or nodes, coming from m-tuples of symmetric nodes on the cover, where further components of C', cyclically permuted by h, are attached.

One of the key observations in [loc. cit.] is that the data (C_+, C, f) also represents a stable map to the orbifold $X/\mathbb{Z}_m = X \times B\mathbb{Z}_m$ in the sense of [Chen and Ruan 2002; Abramovich et al. 2002]. Therefore, the contributions with poles at $q=\eta^{-1}$ in the KRR formula for the J-function can be expressed as cohomological integrals over the moduli space of stable maps to $X \times B\mathbb{Z}_m$, twisted by the Todd classes of the traces of the virtual tangent and normal bundles of the Kawasaki strata, and the Chern class of the trace of the level structure $\mathcal{D}^{R,l}$. To be more precise, we introduce some notation first. Let $\overline{\mathcal{M}}^{X,\beta}_{0,k+2}(\eta)$ denote the stem space. It parametrizes stems of degree β, which are quotient maps by the $\mathbb{Z}_m$-symmetry generated by g. Here g acts by η and η^{-1} on the cotangent lines at the first and last markings of the covering curve, respectively. The only markings on the covering curve fixed by h are the first and last markings. Note that the stem space is a Kawasaki stratum in $\overline{\mathcal{M}}_{0,mk+2}(X, m\beta)$. According to [Givental and Tonita 2014, Proposition 5], the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+2}(\eta)$ is isomorphic to the moduli space $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+2}(g,1,\dots,1,g^{-1})$ of stable maps to the orbifold $X/\mathbb{Z}_m$. Here the sequence $(g,1,\dots,1,g^{-1})$ indicates the sectors where the evaluation maps land. We also consider the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ parametrizing stems whose butts are regular points. Similarly, we have an isomorphism between $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ and $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+1}(g,1,\dots,1)$.

For simplicity, we denote the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ by $\overline{\mathcal{M}}$. Modeling on the contributions in the virtual KRR formula applied to the stack $\overline{\mathcal{M}}_{g,k+1}(X,\beta)/S_k$, we define the correlators in the stem theory of level l by

$$\left\langle \frac{\phi}{1-qL^{1/m}}, \boldsymbol{t}(L),\dots,\boldsymbol{t}(L) \right\rangle^{\text{stem},R,l}_{0,k+1,\beta} := \int_{[\overline{\mathcal{M}}]^{\text{vir}}} \operatorname{td}(T_{\overline{\mathcal{M}}}) \operatorname{ch}\left(\frac{\operatorname{ev}_1^*\phi \cdot \prod_{i=2}^k \operatorname{ev}_i^* \boldsymbol{t}(L_i) \cdot \operatorname{tr}_g \mathcal{D}^{R,l}}{(1-qL_1^{1/m}) \cdot \operatorname{tr}_g \left(\bigwedge^* N^\vee_{\overline{\mathcal{M}}} \right)} \right).$$

Here $[\overline{\mathcal{M}}]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+1}(g,1,\dots,1)$ of stable maps to $X/\mathbb{Z}_m$, and $T_{\overline{\mathcal{M}}}$ and $N_{\overline{\mathcal{M}}}$ are, respectively, the virtual tangent and normal bundles to $\overline{\mathcal{M}}$, considered as a Kawasaki stratum in $\overline{\mathcal{M}}_{0,mk+1}(X,m\beta)$. The line bundle L_1 is formed by the cotangent spaces of stem curves at the first markings, while $L_1^{1/m}$ corresponds to the cotangent line bundle of the covering curves (see [ibid., Section 7] for the explanation). From the definition, we see that the stem theory of level l is a type of twisted cohomological GW theory of $X\times B\mathbb{Z}_m$.

Before we investigate the stem theory further, let us recall some basic facts about the GW theory of the orbifold $X\times B\mathbb{Z}_m$. In this case, the Lagrangian cone of the cohomological GW theory of $X\times B\mathbb{Z}_m$ is the product of m copies of the Lagrangian cone of the GW theory of X. It lies inside the product of m copies of the Fock space $\mathcal{H}$. We refer to each copy of the Lagrangian cone as a *sector*. These sectors are labeled by elements of $\mathbb{Z}_m=\{1,g,\dots,g^{m-1}\}$.

The following proposition relates the Laurent expansion of $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})$ at $q=\eta^{-1}$ to generating series in stem theory:

Proposition 4.9 *Let $\delta\boldsymbol{t}(q)$ be the contributions in $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})$ which are regular at $q=\eta^{-1}$, i.e.,*

$$\delta\boldsymbol{t}(q)=1-q+\boldsymbol{t}(q)+\tilde{\boldsymbol{t}}(q),$$

where $\tilde{\boldsymbol{t}}(q)$ is the sum of all the contributions from Kawasaki strata $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\xi)$ with $\xi\neq\eta$. Then

$$\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(\eta)}=\delta\boldsymbol{t}(q)+\sum_a\sum_{(k,\beta)\neq(0,0)}\frac{Q^{m\beta}}{k!}\phi^a\Big\langle\frac{\phi_a}{1-q\eta L^{1/m}},\boldsymbol{T}(L),\dots,\boldsymbol{T}(L),\delta\boldsymbol{t}\Big(\frac{L^{1/m}}{\eta}\Big)\Big\rangle^{\mathrm{stem},R,l}_{0,k+2,\beta},$$

where:

(1) *The evaluation morphisms at the marked points land in the twisted sector of $B\mathbb{Z}_m$ labeled by the sequence $(g,1,\dots,1,g^{-1})$.*

(2) *$\boldsymbol{T}(L)=\Psi^m\widetilde{\boldsymbol{T}}(L)$, where Ψ^m acts on cotangent line bundles $L\mapsto L^m$, elements of $K^0(X)_{\mathbb{Q}}$, and Novikov variables $Q^\beta\mapsto Q^{m\beta}$.*

(3) *$\widetilde{\boldsymbol{T}}(q)$ is the input point of $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)}$, i.e., it is determined by*

$$1-q+\widetilde{\boldsymbol{T}}(q)=(\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)})_+,$$

where $(\cdot)_+$ denotes the projection along $\mathcal{K}^1_-$ to $\mathcal{K}^1_+$.

Proof Proposition 2.9 shows that the determinant line bundle $\mathcal{D}^{R,l}$ factorizes "nicely" over nodal strata. With this in mind, the argument of [ibid., Proposition 2] applies here with one slight change: when determining $\widetilde{\boldsymbol{T}}(q)$, we do not impose the same condition $\boldsymbol{t}(q)=0$ as in [loc. cit.]. This is because in the permutation-equivariant theory, we are allowed to permute marked points. □

Proposition 4.9 shows that the Laurent expansion $\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(\eta)}(\eta^{-1}q^{1/m})$ of the J-function around $q=\eta^{-1}$ can be identified with a tangent vector to the cone of stem theory of level l,

$$\delta\mathcal{J}^{\mathrm{st},R,l}(\delta\boldsymbol{t},\boldsymbol{T}') := \delta\boldsymbol{t}(q^{1/m}) + \sum_a \sum_{(k,\beta)\neq(0,0)} \frac{Q^\beta}{k!}\phi^a\Big\langle \frac{\phi_a}{1-q^{1/m}L^{1/m}}, \boldsymbol{T}'(L),\dots,\boldsymbol{T}'(L),\delta\boldsymbol{t}(L^{1/m})\Big\rangle^{\mathrm{stem},R,l}_{0,k+2,\beta},$$

after replacing Q^β with $Q^{m\beta}$ (but not in $\delta\boldsymbol{t}$). Note that $\boldsymbol{T}'$ belongs to the sector labeled by 1 and the tangent vector $\delta\mathcal{J}^{\mathrm{st},R,l}(\delta\boldsymbol{t},\boldsymbol{T})$ belongs to the sector labeled by g^{-1}.

Now let us study the stem theory of level l using the formalism of twisted cohomological GW theory of $X\times B\mathbb{Z}_m$. It follows from the definition that the stem theory of level l is obtained from the untwisted cohomological Gromov–Witten theory of $X\times B\mathbb{Z}_m$ by twisting the classes

$$\mathrm{td}(T_{\overline{\mathcal{M}}})/\mathrm{ch}\big(\mathrm{tr}_g\big(\textstyle\bigwedge^* N^\vee_{\overline{\mathcal{M}}}\big)\big) \tag{14}$$

and

$$\mathrm{ch}(\mathrm{tr}_g\,\mathcal{D}^{R,l}). \tag{15}$$

The trace in the first twisting class (14) is computed in [ibid., Section 8], and the effects of this twisting class on the Lagrangian cone and the genus zero potential of the untwisted theory are also studied in [loc. cit.]. We summarize them in the following proposition:

Proposition 4.10 [Givental and Tonita 2014] *The effects of the twisting class* (14) *on the Lagrangian cone and the genus zero potential of the untwisted theory are described as follows*:

(i) *The sectors labeled by* 1 *and* g^{-1} *are rotated by the operators* $\square_m$ *and* $\square_\eta$, *respectively. These two operators are the Euler–Maclaurin asymptotics of the infinite products*

$$\square_m \sim \prod_i\Bigg(\sqrt{\frac{x_i}{1-e^{-mx_i}}}\prod_{r=1}^\infty \frac{x_i-rz}{1-e^{-mx_i+rmz}}\Bigg),$$
$$\square_\eta \sim \prod_i\Bigg(\sqrt{\frac{x_i}{1-e^{-x_i}}}\prod_{r=1}^\infty \frac{x_i-rz}{1-\eta^{-r}e^{-x_i+rz/m}}\Bigg),$$

where x_i *are the Chern roots of the tangent bundle* T_X.

(ii) *The dilaton shift changes from* $-z$ *to* $1-q^m$.

(iii) *There are changes of polarizations of symplectic loop spaces. More precisely*, *in the sector labeled by* 1, *the negative space of the polarization is spanned by*

$$\phi^a\Psi^m\frac{q^k}{(1-q)^{k+1}},$$

whereas in the sector labeled by g^{-1}, it is spanned by

$$\phi^a \frac{q^{k/m}}{(1-q^{1/m})^{k+1}}.$$

We compute the second twisting class (15), and describe its effect on the Lagrangian cone. Let p be the universal family of stem curves. By abuse of notation, we still use ev to denote the universal evaluation morphism from the universal family of quotient curves to $X/\mathbb{Z}_m$. Let $\mathbb{C}_{\eta^i}$ be the topologically trivial line bundle on $X/\mathbb{Z}_m$ on which g acts as multiplication by η^i. According to a simple argument in [Tonita 2018], the trace of the index bundle $R\pi_*(\mathrm{ev}^*\mathcal{R})$ can be expressed as

$$\mathrm{tr}_g(R\pi_*(\mathrm{ev}^*\mathcal{R})) = \sum_{i=0}^{m-1} \eta^i Rp_*(\mathrm{ev}^*\mathcal{R}\otimes\mathbb{C}_{\eta^i}).$$

For simplicity, we denote $Rp_*(\mathrm{ev}^*\mathcal{R}\otimes\mathbb{C}_{\eta^i})$ by $\overline{\mathcal{R}}_i$. Then

$$\begin{aligned}\mathrm{ch}\,\mathrm{tr}_g\,\mathcal{D}^{R,l} &= \mathrm{ch}\Big(\det\sum_{i=0}^{m-1}\eta^i\overline{\mathcal{R}}_i\Big)^{-l} \\ &= \mathrm{ch}\Big(\prod_{i=0}^{m-1}(\eta^i)^{\mathrm{ch}_0\overline{\mathcal{R}}_i}\prod_{i=0}^{m-1}\det\overline{\mathcal{R}}_i\Big)^{-l} \\ &= \prod_{i=0}^{m-1}\exp\big(-l(i\log(\eta)\,\mathrm{ch}_0\overline{\mathcal{R}}_i + \mathrm{ch}_1\overline{\mathcal{R}}_i)\big).\end{aligned} \tag{16}$$

Here, to take the logarithm of η, we may assume that η is taken in the principal branch of the logarithm.

Proposition 4.11 *Twisting by the class* (15) *rotates the sector labeled by the identity of the Lagrangian cone of $X\times B\mathbb{Z}_m$ by*

$$D_m := \exp\Big(-ml\Big(\frac{\mathrm{ch}_2\mathcal{R}}{z}\Big)\Big).$$

The sector labeled by g^{-1} is rotated by the same operator D_m.

Proof The proof is based on the orbifold quantum Riemann–Roch theorem developed in [Tseng 2010]. Let E be an orbifold vector bundle over $X\times B\mathbb{Z}_m$. Consider a general twisting class

$$\exp\Big(\sum_{j\ge 0} s_j\,\mathrm{ch}_j\,p_*(\mathrm{ev}^*E)\Big).$$

According to [Tseng 2010, Theorem 1], this corresponds to rotation by the operator

$$\exp\Big(\sum_{j\ge 0} s_j\Big(\sum_{n\ge 0}\frac{(A_n)_{j+1-n}z^{n-1}}{n!} + \frac{\mathrm{ch}_j\,E^{(0)}}{2}\Big)\Big).$$

Here A_n is an operator which acts on all sectors. The restriction $(A_n)|_{X,g^i}$ of A_n to the sector labeled by g^i is defined by

$$(A_n)|_{(X,g^i)} = \sum_{r=0}^{m-1} B_n\Big(\frac{r}{m}\Big)\operatorname{ch} E_i^{(r)},$$

where $E_i^{(r)}$ (respectively $E^{(0)}$) is the subbundle of the restriction of E to (X, g^i) on which g^i acts with eigenvalue $e^{2\pi\sqrt{-1}r/m}$ (respectively 1). The notation $(A_n)_j$ denotes the degree-j component of the operator A_n. The Bernoulli polynomials are defined by

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t-1}.$$

In our case, the twisting class is given by (16). Let $\widetilde{D}_{\eta,i}$ denote the symplectic transformations corresponding to the i-th factor of the twisting class (16), restricted to (X, g^{-1}). For each $i \in \{0,\dots,m-1\}$, the operator A_n in the definition of $\widetilde{D}_{\eta,i}$ is given by

$$(A_n)|_{(X,g^{-1})} = B_n\Big(\frac{i}{k}\Big)\operatorname{ch}\mathcal{R}.$$

By the orbifold quantum Riemann–Roch theorem, the operator $\widetilde{D}_{\eta,i}$ equals

$$\exp\bigg(-l\log(\eta)\Big(\frac{\operatorname{ch}_1\mathcal{R}}{z}+B_1\Big(\frac{i}{m}\Big)\operatorname{ch}_0\mathcal{R}\Big)-l\Big(\frac{\operatorname{ch}_2\mathcal{R}}{z}+B_1\Big(\frac{i}{m}\Big)\operatorname{ch}_1\mathcal{R}+\frac{B_2(i/m)\operatorname{ch}_0\mathcal{R}}{2}z\Big)-l\frac{\operatorname{ch}_1\mathcal{R}}{2}\bigg).$$

Let $\widetilde{D}_\eta = \prod_{i=0}^{m-1}\widetilde{D}_{\eta,i}$. To simplify the expression of $\widetilde{D}_\eta$, we use the fact that $n\log(\eta)=0$ if n is an integer divisible by m. Keeping this in mind, we obtain

$$\widetilde{D}_\eta = \exp\bigg(-l\Big(\frac{m\operatorname{ch}_2\mathcal{R}}{z}+\frac{\operatorname{ch}_0\mathcal{R}}{12m}z+\frac{\operatorname{ch}_0\mathcal{R}}{6}\log(\eta)\Big)\bigg).$$

Note that the factor $\exp\big(-l(z\operatorname{ch}_0\mathcal{R}/(12m)+\log(\eta)\operatorname{ch}_0\mathcal{R}/6)\big)$ in $\widetilde{D}_\eta$ is a scalar z-series and thus it preserves the overruled Lagrangian cone. We can drop it and obtain the operator D_m.

For the sector labeled by the identity, we denote by $\widetilde{D}_i$ the restriction of the operator corresponding to the i-th factor of the twisting class (16). It is easy to check that

$$(A_n)|_{(X,1)} = B_n(0)\operatorname{ch}\mathcal{R},$$

and the operator $\widetilde{D}_i$ equals

$$\exp\bigg(-il\log(\eta)\Big(\frac{\operatorname{ch}_1\mathcal{R}}{z}\Big)-ml\Big(\frac{\operatorname{ch}_2\mathcal{R}}{z}+\frac{\operatorname{ch}_0\mathcal{R}}{12}z\Big)\bigg).$$

Let $\widetilde{D}=\prod_{i=0}^{m-1}\widetilde{D}_i$. Again by using the fact that $n\log(\eta)=0$ if $m\,|\,n$, we can simplify the operator $\widetilde{D}$ to

$$\exp\bigg(-ml\Big(\frac{\operatorname{ch}_2\mathcal{R}}{z}+\frac{\operatorname{ch}_0\mathcal{R}}{12}z\Big)\bigg).$$

We can drop the second term in the exponent because it is a constant z-series. □

Recall the Lagrangian cone of $X \times B\mathbb{Z}_m$ is m copies of the Lagrangian cone of X. The tangent vector $\delta\mathcal{J}^{\mathrm{st},R,l}(\delta\boldsymbol{t},\boldsymbol{T})$ has application point in the sector labeled by 1 but is tangent in the direction labeled by g^{-1}. Using the above discussion, we locate $\mathrm{qch}\,\delta\mathcal{J}^{\mathrm{st},R,l}(\delta\boldsymbol{t},\boldsymbol{T})$ as follows:

Proposition 4.12 *$\mathrm{qch}\,\delta\mathcal{J}^{\mathrm{st},R,l}(\delta\boldsymbol{t},\boldsymbol{T})$ lies in the tangent space $\square_\eta\,\square_m^{-1}(\mathcal{T}_{\mathcal{I}^{tw}}\,\square_m\,D_m\,\mathcal{L}_H)$ to the cone of the stem theory of level l at a certain point $\mathcal{I}^{tw}$. The input $\boldsymbol{T}$ satisfies*

$$\mathrm{qch}(1-q^m+\boldsymbol{T}(q)) = [\mathcal{I}^{tw}]_+,$$

where $[\,\cdot\,]_+$ denotes the projection along the negative space of the polarization of the sector labeled by 1.

Proof The argument of [Tonita 2018, Proposition 5.8] applies here. We briefly explain the relation between the application point $\mathcal{I}^{tw}$ and the input $\boldsymbol{T}$. Note that the application point $\mathcal{I}^{tw}$ lies on the Lagrangian of the stem theory of level l in the sector labeled by 1. According to Proposition 4.10(ii), the new dilaton shift is $1-q^m$. This explains the equality in the proposition. □

To prove the third condition in Theorem 4.5, we need to identify $\mathcal{T}_{\mathcal{I}^{tw}}\,\square_m\,D_m\mathcal{L}_H$ with $\mathcal{T}_m(\mathcal{J}_{S_\infty}(\boldsymbol{t})_{(1)})$. We first show that:

Proposition 4.13 *We have*

$$\mathrm{qch}^{-1}(\square_m\,D_m\,\mathcal{L}_H) = \widetilde{\Psi}^m\mathcal{L}^{R,l}_{\mathrm{fake}}.$$

Here, the Adams operation $\widetilde{\Psi}^m$ acts on K-theory classes of X and q by $\Psi^m(q)=q^m$, but not on the Novikov variables Q.

Proof It is proved in [Givental and Tonita 2014, Proposition 9] that $\mathrm{qch}^{-1}(\square_m\,\mathcal{L}_H) = \Psi^m\mathcal{L}_{\mathrm{fake}}$. In that proof, one needs to extend the action of the Adams operator Ψ^m on cohomology classes via the Chern isomorphism:

$$\mathrm{ch}(\Psi^m(\mathrm{ch}^{-1}a)) = m^{\deg(a)/2}a.$$

By Proposition 4.8, we have $\mathcal{L}^{R,l}_{\mathrm{fake}} = D_1\mathcal{L}_{\mathrm{fake}}$. We conclude the proof by noticing that $\Psi^m(D_1)=D_m$. □

Let $\widetilde{\boldsymbol{T}}(q)$ be the input point of $\mathcal{J}^{R,l}_{\mathrm{fake}}(\widetilde{\boldsymbol{T}}(q))$ determined by

$$1-q+\widetilde{\boldsymbol{T}}(q) = (\mathcal{J}^{R,l}_{S_\infty}(\boldsymbol{t})_{(1)})_+,$$

where $(\,\cdot\,)_+$ denotes the projection along $\mathcal{K}^1_-$ to $\mathcal{K}^1_+$. Let $\boldsymbol{T}'(q)$ be the input point of $\mathcal{I}^{tw}$ such that

$$\widetilde{\Psi}^m\big(\mathcal{J}^{R,l}_{\mathrm{fake}}(\widetilde{\boldsymbol{T}}(q))\big) = \mathcal{I}^{tw}(\boldsymbol{T}'(q)). \tag{17}$$

We claim that $\widetilde{\Psi}^m(\widetilde{\boldsymbol{T}}(q)) = \boldsymbol{T}'(q)$. This equality holds because, according to Propositions 4.10 and 4.13, the operation $\widetilde{\Psi}^m:\mathcal{K}^1\to\mathcal{K}^1$ identifies the cone $\mathcal{L}^{R,l}_{\mathrm{fake}}$ with the cone $\square_m\,D_m\,\mathcal{L}_H$, the polarization of the fake quantum K-theory with the polarization in the sector labeled by 1 of the stem theory, and the old dilaton shift $1-q$ with the new one $1-q^m$. Therefore $\widetilde{\Psi}^m$ must also map the input point $\widetilde{\boldsymbol{T}}(q)$ of the fake J-function to the input point $\boldsymbol{T}'(q)$ of $\mathcal{I}^{tw}$. Recall from Proposition 4.9 that $\Psi^m(\widetilde{\boldsymbol{T}}(q)) = \boldsymbol{T}(q)$.

Then it follows from the definitions of Ψ^m and $\widetilde{\Psi}^m$ that $\boldsymbol{T}'(q)$ is obtained from $\boldsymbol{T}(q)$ by replacing Q^β with $Q^{\beta/m}$.

By differentiating the relation (17), we get

$$\widetilde{\Psi}^m\biggl(\boldsymbol{f}(q)+\sum\frac{Q^\beta}{k!}\phi^a\Bigl\langle\frac{\phi_a}{1-qL},\widetilde{\boldsymbol{T}}(L),\dots,\widetilde{\boldsymbol{T}}(L),\boldsymbol{f}(L)\Bigr\rangle_{0,k+2,\beta}^{\mathrm{fake},R,l}\biggr)\\ =\widetilde{\Psi}^m\boldsymbol{f}(q)+\sum\frac{Q^\beta}{k!}\widetilde{\Psi}^m\phi^a\Bigl\langle\frac{\widetilde{\Psi}^m\phi_a}{1-q^mL^m},\boldsymbol{T}'(L),\dots,\boldsymbol{T}'(L),\widetilde{\Psi}^m\boldsymbol{f}(L)\Bigr\rangle_{0,k+2,\beta}^{\mathrm{stem},R,l}.$$

The right-hand side is a tangent vector in $\mathcal{T}_{\mathcal{I}^{tw}\,\square_m\,D_m\mathcal{L}_H}$ along the direction of $\delta\boldsymbol{t}':=\widetilde{\Psi}^m\boldsymbol{f}(q)$. The left-hand side becomes $\Psi^m\circ S(q,Q)\circ\Psi^{1/m}(\delta\boldsymbol{t}')$ after we replace Q^β with $Q^{m\beta}$ (including such a change in $\widetilde{\boldsymbol{T}}$ but excluding it in $\boldsymbol{f}(q)$). This concludes the proof of Theorem 4.5.

Remark 4.14 When the target X is an orbifold, the adelic characterization of points on the cone $\mathcal{L}$ of the ordinary — i.e., permutation-*nonequivariant* — quantum K-theory is developed in [Tonita and Tseng 2013]. In this case, the Lagrangian cone $\mathcal{L}$ has different sectors, and each sector corresponds to a connected component of the rigidified inertia stack $\overline{I}_\mu X$. Let $f: C\to X$ be an orbifold stable map. Here C is an orbifold curve with possible orbifold structures at the marked points and nodes. Let $\underline{f}:\underline{C}\to\underline{X}$ be the map between coarse moduli spaces. There is a short exact sequence

$$1\to K\to\mathrm{Aut}(f)\to\mathrm{Aut}(\underline{f})\to 1.$$

The kernel K consists of automorphisms of $C\to X$ that fix $\underline{C}\to\underline{X}$. These automorphisms are referred to as “ghost automorphisms” in [Abramovich et al. 2003], and they arise from the stacky nodes of the source curve.

To analyze poles of K-theoretic J-functions in the orbifold setting, we still apply the virtual KRR formula to the moduli space of orbifold stable maps. In this case, there are extra contributions from twisted sectors corresponding to ghost automorphisms. The key observation in [Tonita and Tseng 2013] is that once we add the appropriate contributions from ghost automorphisms in the definition of fake K-theoretic GW invariants, the formalism of adelic characterizations carries over to the orbifold setting. We refer the reader to [ibid., Definition 3.1] for the precise definition of fake K-theoretic invariants and [ibid., Theorem 4.1] for the adelic characterization in the orbifold and permutation-nonequivariant setting. We only mention that if we restrict to the untwisted sector of the cone $\mathcal{L}$, the main theorem in [loc. cit.] specializes to Theorem 4.1 above.

The generalization of [ibid., Theorem 4.1] to the permutation-equivariant setting is straightforward: we only need to change the application point of the tangent space in [ibid., Definition 4.3] from $\mathcal{J}_1(0)$ to the Laurent expansion $(\mathcal{J}_{S_\infty}(\boldsymbol{t}))_1$ of the J-function at $q=1$. Since the determinant line bundle splits “correctly” among nodal strata, we can also generalize [ibid., Theorem 4.1] to permutation-equivariant quantum K-theory with level structure. In this paper, we focus on recovering examples of mock theta functions. For this purpose, we only need to consider the untwisted sector of Lagrangian cones of orbifold targets. Once we make this restriction, the statement of the adelic characterization is the same as in Theorem 4.5.

4.3 Determinantal modification

In this subsection, we use the adelic characterization to prove Theorem 1.1, which gives us a way to obtain points on $\mathcal{L}^{R,l}_{S_\infty}$ by making certain "determinantal" modifications to points on the level-0 cone $\mathcal{L}_{S_\infty}$.

Let us restate Theorem 1.1:

Theorem 1.1 *If*

$$I=\sum_{\beta\in\mathrm{Eff}(X)} I_\beta Q^\beta$$

lies on $\mathcal{L}_{S_\infty}$, *then the point*

$$I^{R,l}:=\sum_{\beta\in\mathrm{Eff}(X)} I_\beta Q^\beta \prod_i (L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/2})^l$$

lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ *of permutation-equivariant quantum* K*-theory of level* l*. Here* $\mathrm{Eff}(X)$ *denotes the semigroup of effective curve classes on* X*,* L_i *are the* K*-theoretic Chern roots of* $\mathcal{R}$*, and* $\beta_i:=\int_\beta c_1(L_i)$*.*

Proof Suppose $I=\sum_{\beta\in\mathrm{Eff}(X)} I_\beta Q^\beta$ is a point on $\mathcal{L}_{S_\infty}$. Let $I^{R,l}$ be its "determinantal" modification $\sum_\beta I_\beta Q^\beta \prod_i (L_i^{-\beta_i} q^{\beta_i(\beta_i-1)/2})^l$. According to Convention 4.4, we compare different cones in the same loop spare $\mathcal{K}^1$. In particular, $\mathcal{L}^{R,l}_{\text{fake}}$ and the tangent space in Theorem 4.5 are viewed as subspaces of $\mathcal{K}^1$. Therefore, to show $I^{R,l}$ lies on $\mathcal{L}^{R,l}_\infty$, we need to work with the series after the rescaling

$$\widetilde{I}^{R,l}:=(\det\mathcal{R})^{-l/2} I^{R,l}.$$

Note that the square root of the determinant line bundle may not exist. Instead, we treat it as a formal variable that rescales the elements in the loop space. This rescaling modifies the pairing in the loop space, which, nonetheless, remains well defined.

We denote by $I_{(\eta)}$ and $\widetilde{I}^{R,l}_{(\eta)}$ the Laurent expansions of I and $\widetilde{I}^{R,l}$ in $1-q\eta$, respectively. Let $Q_1,\dots,Q_n$ be the Novikov variables. Let p_i be the degree-2 cohomology classes corresponding to Q_i, and let $P_i=e^{-p_i}\in K^0(X)$. We denote by L_i the K-theoretic Chern roots of $\mathcal{R}$. In other words, $\mathrm{ch}\, L_i=e^{l_i}$, where l_i are cohomological Chern roots of $\mathcal{R}$. We write l_i as a linear function $f_i(p_1,\dots,p_n)$ in terms of the basis $p_1,\dots,p_n$.

It is clear that $\widetilde{I}^{R,l}$ satisfies the first condition in Theorem 4.5. Now, we check the second condition. It follows from the lemma in the proof of [Coates and Givental 2007, Theorem 2] that the operator

$$\Phi:=\prod_{i=1}^n \exp\biggl(l\biggl(\frac{(f_i(p_j-zQ_j\partial_{Q_j}))^2}{2z}+\frac{f_i(p_j-zQ_j\partial_{Q_j})}{2}\biggr)\biggr)$$

preserves $\mathcal{L}_{\mathrm{fake}}$. Define $d_j = -\langle c_1(P_j), \beta\rangle$ to be the components of degree β. By a simple computation, one can show that

$$\begin{aligned}\Phi(Q^\beta) &= \prod_{i=1}^n \exp\biggl(l\biggl(\frac{(f_i(p_j - zd_j))^2}{2z} - \frac{f_i(p_j - zd_j)}{2}\biggr)\biggr)Q^\beta \\ &= \exp\biggl(l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{2z} - \frac{\mathrm{ch}_1\,\mathcal{R}}{2}\biggr)\biggr)Q^\beta \\ &\qquad \cdot \prod_{i=1}^n \exp\biggl(l\biggl(\frac{(f_i(p_j - zd_j))^2}{2z} - \frac{f_i(p_j - zd_j)}{2}\biggr)\biggr)\exp\biggl(-l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{2z} - \frac{\mathrm{ch}_1\,\mathcal{R}}{2}\biggr)\biggr) \\ &= \exp\biggl(l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{z} - \frac{\mathrm{ch}_1\,\mathcal{R}}{2}\biggr)\biggr)Q^\beta \prod_i (L_i^{-\beta_i} q^{\beta_i(\beta_i+1)/2})^l.\end{aligned}$$

It follows that

$$\Phi(I_{(1)}) = \exp\biggl(l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\biggr)\biggr)\tilde{I}^{R,l}_{(1)}. \tag{18}$$

Since the left-hand side lies on $\mathcal{L}_{\mathrm{fake}}$, we conclude that the Laurent expansion of $\tilde{I}^{R,l}$ at $q=1$ lies on the cone $\mathcal{L}^{R,l}_{\mathrm{fake}} = \exp(-l(\mathrm{ch}_2\,\mathcal{R}/z))\mathcal{L}_{\mathrm{fake}}$.

Now we check the third condition. Suppose the tangent space to $\mathcal{L}_{\mathrm{fake}}$ at $I_{(1)}$ is given as the image of a map

$$S(q, Q) : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Then, by (18), the tangent space to $\mathcal{L}^{R,l}_{\mathrm{fake}}$ at $\tilde{I}^{R,l}_{(1)}$ is given as the image of a map

$$S'(q, Q) = \exp\biggl(-l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\biggr)\biggr)\Phi \circ S(q, Q) : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Here we use the fact that the Novikov variables are contained in the λ-algebra and hence they preserve tangent spaces.

Recall from Definition 4.2 that the space $\mathcal{T}_m(I_{(1)})$ is defined as the image of a map

$$\Psi^m \circ S(q, Q) \circ \Psi^{1/m} : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Then $\mathcal{T}^{R,l}_m(\tilde{I}^{R,l}_{(1)})$ is given as the image of

$$\begin{aligned}\Psi^m \circ S'(q, Q) \circ \Psi^{1/m} &= \Psi^m \circ \exp\biggl(-l\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\biggr)\biggr)\Phi \circ S(q, Q) \circ \Psi^{1/m} \\ &= \exp\biggl(-ml\biggl(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\biggr)\biggr)\Psi^m \circ \Phi \circ S(q, Q) \circ \Psi^{1/m} \\ &= D_m \Phi^m(\Psi^m \circ S(q, Q) \circ \Psi^{1/m}),\end{aligned}$$

where $\Phi^m := \Psi^m(\Phi)$ is given by

$$\Psi^m(\Phi) = \prod_{i=1}^{n} \exp\biggl(l\biggl(\frac{(f_i(mp_j - zQ_j\partial_{Q_j}))^2}{2mz} + \frac{f_i(mp_j - zQ_j\partial_{Q_j})}{2}\biggr)\biggr).$$

Here we use the fact that the Adams operation Ψ^m acts on the degree-2 classes z and p_j as multiplication by m, and its action on the differential operator $zQ_j\partial Q_j$ is trivial.[2] This shows that $\mathcal{T}_m^{R,l}(\tilde{I}_{(1)}^{R,l}) = D_m\Phi^m(\mathcal{T}_m(I_{(1)}))$.

By assumption, we have

$$I_{(\eta)}\biggl(\frac{q^{1/m}}{\eta}\biggr) \in \square_\eta\, \square_m^{-1}\, \mathcal{T}_m(I_{(1)}).$$

Then

$$\begin{aligned}(19)\qquad \tilde{I}_{(\eta)}^{R,l}\biggl(\frac{q^{1/m}}{\eta}\biggr) &= \sum_\beta \exp\bigl(-\tfrac{1}{2}\operatorname{ch}_1 \mathcal{R}\bigr)(I_\beta)_{(\eta)}\biggl(\frac{q^{1/m}}{\eta}\biggr) Q^\beta \prod_i (L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/(2m)}\eta^{-(\beta_i+1)\beta_i/2})^l \\ &= \exp\Bigl(-l\operatorname{ch}_2\mathcal{R}\cdot\Bigl(\frac{z}{m} - \log\eta\Bigr)^{-1}\Bigr)(\Phi(I_{(\eta)}))\biggl(\frac{q^{1/m}}{\eta}\biggr) \\ &= \exp\biggl(-ml\biggl(\frac{\operatorname{ch}_2\mathcal{R}}{z}\biggr)\biggr)(\Phi(I_{(\eta)}))\biggl(\frac{q^{1/m}}{\eta}\biggr).\end{aligned}$$

Here we use the fact that $m\cdot\log(\eta) = 0$.

By an elementary computation, one can show that, for any series $\boldsymbol{f}$ in q and Q, we have

$$(20)\qquad (\Phi(\boldsymbol{f}))\biggl(\frac{q^{1/m}}{\eta}\biggr) = \Phi^m\mathcal{D}_\eta \boldsymbol{f}\biggl(\frac{q^{1/m}}{\eta}\biggr),$$

where the operator $\mathcal{D}_\eta$ is defined by

$$\mathcal{D}_\eta = \prod_{i=1}^{n}\exp\biggl(l\biggl(-\log(\eta)f_i(Q_j\partial Q_j)^2 + \frac{f_i(\log(\eta)Q_j\partial Q_j)}{2} - \frac{m-1}{m}\frac{f_i(mp_j - zQ_j\partial_{Q_j})}{2}\biggr)\biggr).$$

Here the substitution $q \mapsto q^{1/m}/\eta$ corresponds to the change $z \mapsto z/m - \log(\eta)$ in the expression of Φ.

It follows from (20) that (19) equals

$$(21)\qquad \biggl(D_m\Phi^m\mathcal{D}_\eta\biggl(I_\eta\biggl(\frac{q^{1/m}}{\eta}\biggr)\biggr)\biggr) \in D_m\Phi^m\mathcal{D}_\eta\,\square_\eta\,\square_m^{-1}\,\mathcal{T}_m(I_{(1)}).$$

Since all the operators above have constant coefficients (i.e., independent of Q), they commute. We claim that $\mathcal{D}_\eta$ preserves $\mathcal{T}_m(I_{(1)})$. This is because, by definition, we have

$$\mathcal{D}_\eta\mathcal{T}_m(I_{(1)}) = \mathcal{D}_\eta\Psi^m \circ S(q, Q)\circ \Psi^{1/m}\mathcal{K}_+^1.$$

Let

$$\mathcal{D}_{\eta,1} = \prod_{i=1}^{n}\exp\bigl(l\bigl(-\log(\eta)f_i(Q_j\partial Q_j)^2 + \tfrac{1}{2}f_i(\log(\eta)Q_j\partial Q_j)\bigr)\bigr)$$

[2]This is because $\Psi^m(zQ_j\partial Q_j) = mzQ_j^m\partial Q_j^m = zQ_j\partial Q_j$.

and

$$\mathcal{D}_{\eta,2}=\prod_{i=1}^{n}\exp\biggl(l\biggl(-\frac{(m-1)f_i(mp_j-zQ_j\partial_{Q_j})}{2m}\biggr)\biggr)$$

be the factors of $\mathcal{D}_\eta$. Then it is easy to check that the first factor $\mathcal{D}_{\eta,1}$ commutes with $\Psi^m\circ S(q,Q)\circ\Psi^{1/m}$, and hence preserves $\mathcal{T}_m(I_{(1)})$. The second factor $\mathcal{D}_{\eta,2}$ satisfies the commutation relation

$$\mathcal{D}_{\eta,2}\Psi^m=\Psi^m\prod_{i=1}^{n}\exp\biggl(l\biggl(-\frac{(m-1)f_i(p_j-zQ_j\partial_{Q_j})}{2m}\biggr)\biggr).$$

According to [Givental and Tonita 2014, Corollary 1], the second operator on the right-hand side preserves the tangent space $S(q,Q)\circ\Psi^{1/m}\mathcal{K}_+^1$. Therefore, we have shown that the space $\mathcal{T}_m(I_{(1)})$ is $\mathcal{D}_\eta$-invariant.

We can further simplify the space on the right-hand side of (21) as

$$\begin{aligned}D_m\Phi^m\mathcal{D}_\eta\,\Box_\eta\,\Box_m^{-1}\,\mathcal{T}_m(I_{(1)})&=D_m\Phi^m\,\Box_\eta\,\Box_m^{-1}\,\mathcal{D}_\eta\mathcal{T}_m(I_{(1)})\\&=D_m\Phi^m\,\Box_\eta\,\Box_m^{-1}\,\mathcal{T}_m(I_{(1)})\\&=D_m\Phi^m\,\Box_\eta\,\Box_m^{-1}(\Phi^m)^{-1}D_m^{-1}(\mathcal{T}_m^{R,l}(\tilde{I}_{(1)}^{R,l}))\\&=\Box_\eta\,\Box_m^{-1}(\mathcal{T}_m^{R,l}(\tilde{I}_{(1)}^{R,l})).\end{aligned}$$

□

Remark 4.15 Suppose the target is an orbifold. As explained in Remark 4.14, the adelic characterization of points on the untwisted sector of the Lagrangian cone of X is the same as the one given in Theorem 4.5. Using the same proof as above, we can show that, if I is a point on the untwisted sector of $\mathcal{L}_{S_\infty}$, then the determinantal modification $I^{R,l}$ lies on the untwisted sector of $\mathcal{L}_{S_\infty}^{R,l}$.

5 Toric mirror theorem and mock theta functions

In this section, we first recall a mirror theorem proved in [Givental 2015d] for ordinary quantum K-theory of toric varieties. This corresponds to Theorem 1.2 in the level 0 case. By combining Givental's result with Theorem 1.1, proved in the previous section, we obtain a toric mirror theorem for level-l quantum K-theory. In some simple cases, we recover Ramanujan's mock theta functions from toric I-functions with nontrivial level structure.

5.1 K-theoretic I-function and mock theta function

In this subsection, we first explicitly compute the (torus-equivariant) small I-functions with level structures for toric varieties, using quasimap graph spaces. Then we use torus localization to prove a toric mirror theorem (Theorem 1.2), following [Givental 2015d]. In the study of quantum K-theory with nontrivial level structures, a remarkable phenomenon is the appearance of Ramanujan's mock theta functions.

Let $M\cong\mathbb{Z}^n$ be a n-dimensional lattice and let N be its dual lattice. To every complete nonsingular fan $\Sigma\subset N_{\mathbb{R}}$, we can associate a n-dimensional smooth projective variety X_Σ. We denote by $\Sigma(1)$ the set of 1-dimensional cones in Σ. Let $m = |\Sigma(1)|$. Each $\rho \in \Sigma(1)$ determines a Weil divisor D_ρ on X_Σ and the Picard group of X_Σ is determined by the short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X_\Sigma) \to 0. \tag{22}$$

Here the inclusion is defined by $m \mapsto \sum_\rho \langle m, \rho\rangle D_\rho$. Now let us describe the quotient construction of X_Σ. Since $\mathrm{Pic}(X_\Sigma)$ is torsion free, we choose an integral basis $\{L_1, \dots, L_s\}$ of it, where $s = m - n$. Then the surjective map in (22) is given by the transpose of an integral $s \times m$ matrix $Q = (Q_{a\rho_b})_{a=1,\dots,s;b=1,\dots,m}$ which is called the charge matrix of X_Σ. Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ to the exact sequence (22), we get an exact sequence.

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to N \otimes \mathbb{C}^* \to 1,$$

where $G := \mathrm{Hom}(\mathrm{Pic}(X_\Sigma), \mathbb{C}^*) \cong (\mathbb{C}^*)^s$. The first map in the above short exact sequence defines the G-action on $\mathbb{C}^{\Sigma(1)}$

$$\boldsymbol{t} \cdot (z_{\rho_1}, \dots, z_{\rho_m}) = \left(\prod_{a=1}^{s} t_a^{Q_{a\rho_1}} z_{\rho_1}, \dots, \prod_{a=1}^{s} t_a^{Q_{a\rho_m}} z_{\rho_m} \right), \tag{23}$$

where $\boldsymbol{t} = (t_1, \dots, t_s) \in (\mathbb{C}^*)^s$. By choosing an appropriate linearization of the trivial line bundle on $\mathbb{C}^{\Sigma(1)}$ (see, e.g., [Dolgachev 2003, Chapter 12]), the semistable and stable loci are equal. We denote this linearized trivial line bundle by L_Σ and the stable loci by $U(\Sigma)$. Let z_ρ be the coordinates in $\mathbb{C}^{\Sigma(1)}$. We define a subvariety

$$Z(\Sigma) = \left\{ (z_\rho) \in \mathbb{C}^{\Sigma(1)} \;\middle|\; \prod_{\rho \not\subset \sigma} z_\rho = 0, \sigma \in \Sigma \right\}.$$

Then

$$U(\Sigma) = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma).$$

The toric variety X_Σ is the geometric quotient $U(\Sigma)/G$. Let P be the principal G-bundle $\mathbb{C}^{\Sigma(1)} \to [\mathbb{C}^{\Sigma(1)}/G]$. Let $\pi_i : G \to \mathbb{C}^*$ be the projection to the i-th component and let R_j be the characters given by $\boldsymbol{t} = (t_1, \dots, t_s) \mapsto \prod_{a=1}^{s} t_a^{Q_{a\rho_j}}$ for $1 \le j \le m$. Then the line bundles L_i and $\mathcal{O}(-D_{\rho_j})$ are the restrictions of the associated line bundles of P with the characters π_i and R_j, respectively, to X_Σ.

Note that X_Σ admits a $T^m := (\mathbb{C}^*)^{\Sigma(1)}$-action, i.e., T^m acts by diagonal matrix on $\mathbb{C}^{\Sigma(1)}$. We denote by P_i and U_ρ the T^m-equivariant line bundles corresponding to L_i and $\mathcal{O}(D_\rho)$, respectively. In the T^m-equivariant K-group $K^0_{T^m}(X_\Sigma) \otimes \mathbb{Q}$, we have the multiplicative relation

$$U_\rho = \prod_{i=1}^{s} P_i^{\otimes Q_{i\rho}} \Lambda_\rho^{-1},$$

where Λ_ρ are the generators of $\mathrm{Repr}(T^m)$ corresponding to the projection to the component labeled by ρ. See [Givental 2015d] for more details.

Now let us compute the (T^m-equivariant) small I-function of X_Σ with level structures, using the quasimap graph space. Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{T^m}(\mathbb{C}^{\Sigma(1)}), \mathbb{Z})$ be an L_Σ-effective class. According to

[Ciocan-Fontanine and Kim 2010, Lemma 3.1.8], a point in the quasimap graph space $QG^{\epsilon=0+}_{0,k}(X_\Sigma,\beta)$ is specified by the data

$$\big((C,p_1,\dots,p_k),\{\mathcal{P}_i \mid i=1,\dots,s\},\{u_\rho\}_{\rho\in\Sigma(1)},\varphi\big),$$

where

- $(C,p_1,\dots,p_s)$ is a connected, at most nodal, curve of genus 0 and p_i are distinct nonsingular points of C,
- $\mathcal{P}_i$ are line bundles on C of degree $f_i := \beta(L_i)$,
- $u_\rho\in\Gamma(C,\mathcal{L}_\rho)$, where $\mathcal{L}_\rho$ is defined by

$$\mathcal{L}_\rho := \bigotimes_{i=1}^{s}\mathcal{P}_i^{\otimes Q_{i\rho}},$$

- $\varphi : C\to\mathbb{P}^1$ is a regular map such that $\varphi_*[C]=[\mathbb{P}^1]$.

The stability conditions are discussed in Section 3.2. When $(g,k)=(0,0)$, we have $C\cong\mathbb{P}^1$ and $\mathcal{P}_i\cong\mathcal{O}_{\mathbb{P}^1}(f_i)$. The line bundles $\mathcal{L}_\rho$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}\big(\sum_{i=1}^s f_iQ_{i\rho}\big)=\mathcal{O}_{\mathbb{P}^1}(\beta_\rho)$, where $\beta_\rho:=\beta(\mathcal{O}(D_\rho))$. Therefore, a point on $QG^{\epsilon=0+}_{0,0}(X_\Sigma,\beta)$ is specified by sections $\{u_\rho\in\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(\beta_\rho))\mid\rho\in\Sigma(1)\}$. We choose coordinates $[x_0,x_1]$ on $\mathbb{P}^1$ and consider the standard action $\mathbb{C}^*$-action defined by (4). Let F_0 be the distinguished fixed-point locus parametrizing quasimaps whose degrees are concentrated only at 0. According to [Ciocan-Fontanine and Kim 2010, Section 7.2], we have the identification

$$F_0\cong\bigcap_{\{\rho\mid\beta_\rho<0\}}D_\rho\subset X_\Sigma,\quad (z_\rho x_0^{\beta_\rho})\mapsto(z_\rho), \tag{24}$$

where (z_ρ) are the coordinates on X_Σ.

Let R be a character of $G=(\mathbb{C}^*)^s$ defined by $\boldsymbol{t}\cdot z=\prod_{i=1}^s t_i^{r_i}z$ where $r_i\in\mathbb{Z}$. Recall that the small I-function of X_Σ of level l and representation R is defined by

$$I^{R,l}(q)=1+\sum_a\sum_{\beta\neq 0}Q^\beta\chi\bigg(F_0,\mathrm{ev}^*(\phi_a)\otimes\bigg(\frac{\mathrm{tr}_{\mathbb{C}^*}\mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*}\bigwedge^*(N^{\mathrm{vir}}_{F_0/QG})^\vee}\bigg)\bigg)\phi^a.$$

It is not difficult to check that, under the identification (24), we can identify the virtual normal bundle $N^{\mathrm{vir}}_{F_0/QG}$ in $K^0(F_0)$ with

$$N^{\mathrm{vir}}_{F_0/QG}=\sum_{\{\rho\mid\beta_\rho>0\}}\sum_{i=1}^{\beta_\rho}\mathcal{O}(D_\rho)|_{F_0}\otimes\mathbb{C}_{-i}-\sum_{\{\rho\mid\beta_\rho<0\}}\sum_{i=1}^{-\beta_\rho-1}\mathcal{O}(D_\rho)|_{F_0}\otimes\mathbb{C}_i, \tag{25}$$

where $\mathbb{C}_a$ denotes the representation of $\mathbb{C}^*$ on $\mathbb{C}$ with weight $a\in\mathbb{Z}$.[3] Let $\mathcal{P}$ be the universal principal G-bundle on $F_0\times\mathbb{P}^1\subset X_\Sigma\times\mathbb{P}^1$. Then the associated line bundle $\mathcal{P}\times_G R$ can be identified with

[3]Here, the $\mathbb{C}^*$ is the torus acting on the graph space.

$\bigotimes_{i=1}^{s} L_i^{r_i} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta_R)$, where $\beta_R := \sum_{i=1}^{s} r_i f_i$. We denote the line bundle $\bigotimes_{i=1}^{s} L_i^{r_i}$ by $\mathcal{R}$. Let $\pi : F_0 \times \mathbb{P}^1 \to F_0$ be the projection. When $\beta_R \geq 0$, we have

$$\mathcal{D}^{R,l} = \det{}^{-l} R\pi_*(\mathcal{R} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta_R)) = \det{}^{-l}\big(\mathcal{R} \otimes R^0\pi_*(\mathcal{O}_{\mathbb{P}^1}(\beta_R))\big) = \mathcal{R}^{-l(\beta_R+1)} \otimes \mathbb{C}_{l\beta_R(\beta_R+1)/2}. \tag{26}$$

When $\beta_R < 0$, a similar calculation derives the same formula $\mathcal{D}^{R,l} = \mathcal{R}^{-l(\beta_R+1)} \otimes \mathbb{C}_{l\beta_R(\beta_R+1)/2}$.

We give the explicit formulas of the (torus-equivariant) small I-functions of toric varieties in the following proposition:

Proposition 5.1 *The small I-function of a toric variety X_Σ of level l and character R is given by*

$$I^{R,l}(q) = 1 + \sum_{\beta \in \mathrm{Eff}(X)} Q^\beta \mathcal{R}^{-l\beta_R} q^{l\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \frac{\prod_{j=-\infty}^{0}(1-\mathcal{O}(-D_\rho)q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-\mathcal{O}(-D_\rho)q^j)},$$

and its equivariant version is given by

$$I^{R,l,\mathrm{eq}}(q) = 1 + \sum_{\beta \in \mathrm{Eff}(X)} Q^\beta \widetilde{\mathcal{R}}^{-l\beta_R} q^{l\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \frac{\prod_{j=-\infty}^{0}(1-U_\rho q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-U_\rho q^j)}.$$

Here $\mathcal{R} := \bigotimes_{i=1}^{s} {L_i}^{r_i}$ is the line bundle associated to the character R, and $\widetilde{\mathcal{R}} \otimes \bigotimes_{i=1}^{s} {P_i}^{r_i}$ and U_ρ are the equivariant line bundles corresponding to $\mathcal{R}$ and $\mathcal{O}(-D_\rho)$, respectively.

Proof Note that the evaluation map $\mathrm{ev}_\bullet$ makes F_0 as a closed subvariety of X_Σ, whose normal bundle is given by $\bigoplus_{\rho:\beta_\rho<0} \mathcal{O}(D_\rho)$.Then, the proposition follows easily from (25) and (26). Note that one factor $\mathcal{R}^{-l}$ in (26) disappears due to the change of pairings (see Convention 4.4). □

By extending Givental's localization argument [2015d] to the setting with level structure, we prove a toric mirror theorem. Now let us restate Theorem 1.2.

Theorem 1.2 *Assume that X_Σ is a smooth quasiprojective toric variety. Let $I^{R,l,\mathrm{eq}}(q)$ be the level-l torus-equivariant small I-function given in Proposition 5.1. Then the series $(1-q)I^{R,l,\mathrm{eq}}(q)$ lies on the cone $\mathcal{L}_{S_\infty}^{R,l,\mathrm{eq}}$ in the symmetrized torus-equivariant quantum $\mathfrak{v}$-theory of level l of X_Σ.*

Proof For simplicity, we denote by I the I-function $I^{R,l,\mathrm{eq}}$ of X_Σ. Let $\{\phi_\alpha\}_{\alpha \in X_\Sigma^{T^m}}$ be the fixed-point basis of $K_{T^m}^0(X_\Sigma)$ and let $\{\phi^\alpha\}$ be the dual basis with respect to the pairing (5). For each fixed point α, we denote by $J(\alpha) \subset \Sigma(1)$ the cardinality-s subset such that α equals the intersection $\bigcap_{\rho \notin J(\alpha)} D_\rho$. Write $I = \sum_\alpha I^{(\alpha)}\phi_\alpha$. We denote by $U_\rho(\alpha)$ and $\mathcal{R}(\alpha)$ the restrictions of U_ρ and $\mathcal{R}$ to the fixed point α, respectively. For $\rho \in J(\alpha)$, we have $U_\rho(\alpha) = 1$. Hence $I^{(\alpha)}$ can be explicitly written as

$$I^{(\alpha)}(q) = 1 + \sum_{\beta \in \mathrm{Eff}'(X_\Sigma)} Q^\beta \frac{\mathcal{R}(\alpha)^{-l\beta_R} q^{l\beta_R(\beta_R+1)/2}}{\prod_{\rho \in J(\alpha)} \prod_{j=1}^{\beta_\rho}(1-q^j)} \prod_{\rho \notin J(\alpha)} \frac{\prod_{j=-\infty}^{0}(1-U_\rho(\alpha)q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-U_\rho(\alpha)q^j)}.$$

Here $\mathrm{Eff}'(X_\Sigma)$ denotes the semigroup of effective curve classes β such that $\beta_\rho \geq 0$. The terms with $\beta_\rho < 0$ disappear because there is a factor $(1-q^0)$ in the numerators.

We first observe that, for the point target, the cone $\mathcal{L}^{\mathrm{pt},R,l}_{S_\infty}$ of level l coincides with the cone $\mathcal{L}^{\mathrm{pt}}_\infty$ of level 0. This is because, in the case of the point target, the determinant line bundle $\mathcal{D}^{R,l}$ is always topologically trivial (with possible equivariant weights). Combining this observation with the fact that the level structure $\mathcal{D}^{R,l}$ splits "nicely" among nodal strata, we can extend the argument in [Givental 2015a] to prove that a point $f=\sum_\alpha f^{(\alpha)}\phi_\alpha$ lies on $\mathcal{L}^{R,l,\mathrm{eq}}_{S_\infty}$ if and only if the following are satisfied:

(1) When expanded as meromorphic functions with poles only at roots of unity, $f^{(\alpha)}$ lie on the cone $\mathcal{L}^{\mathrm{pt}}_{S_\infty}$ in the permutation-equivariant quantum K-theory of the point target space.

(2) Away from $q=0,\infty$ and roots of unity, $f^{(\alpha)}$ may have at most simple poles at $q=U_\rho(\alpha)^{-1/m}$ with $\rho\notin J(\alpha)$ and $m=1,2,\dots$ for generic values of $\Lambda_1,\dots,\Lambda_m$. The residues satisfy the recursion relations

$$\mathrm{Res}_{q=U_\rho(\alpha)^{-1/m}} f^{(\alpha)}(q)\frac{dq}{q}=-\frac{\phi^\alpha Q^{md_{\alpha\rho}}}{m\cdot C_{\alpha\rho}(m)}f^{(\rho)}(U_\rho(\alpha)^{-1/m}).$$

Here $C_{\alpha\rho}(m)=\lambda_{-1}(T_p\overline{\mathcal{M}}_{0,2}(X_\Sigma,md_{\alpha\rho}))\cdot(\mathcal{R}(\alpha)^{-md^R_{\alpha\rho}}q^{md^R_{\alpha\rho}(md^R_{\alpha\rho}+1)/2})^l$, where:

(a) T_p denotes the virtual tangent space to the moduli space at the point p represented by the m-multiple cover of the one-dimensional orbit connecting α and ρ. The explicit formula of the equivariant weights of the K-theoretic Euler class $\lambda_{-1}(T_p)$ is given in [Givental 2015d].

(b) $d_{\alpha\rho}$ denotes the degree of the one-dimensional orbit connecting α and ρ.

(c) $d^R_{\alpha\rho}:=\langle d_{\alpha\rho},c_1(\mathcal{R})\rangle$.

We want to show that $(1-q)I$ satisfies (1) and (2). It is proved in the main theorem[4] of [Givental 2015d] that the series

$$\widetilde{I}^{(\alpha)}=1+\sum_{\beta\in\mathrm{Eff}'(X_\Sigma)}\frac{Q^\beta}{\prod_{\rho\in J(\alpha)}\prod_{j=1}^{\beta_\rho}(1-q^j)}\prod_{\rho\notin J(\alpha)}\frac{\prod_{j=-\infty}^{0}(1-U_\rho(\alpha)q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-U_\rho(\alpha)q^j)}$$

represents a value of $\mathcal{J}^{\mathrm{pt}}_{S_\infty}(\boldsymbol{t}(q),Q)/(1-q)$, i.e., $(1-q)\widetilde{I}^{(\alpha)}$ lies on the cone $\mathcal{L}^{\mathrm{pt}}_{S_\infty}$. Note that $I^{(\alpha)}$ is obtained from $\widetilde{I}^{(\alpha)}$ by a "determinantal" modification. Therefore, it follows from Theorem 1.1 that $(1-q)I^{(\alpha)}$ lies on $\mathcal{L}^{\mathrm{pt},R,l}_{S_\infty}=\mathcal{L}^{\mathrm{pt}}_{S_\infty}$.

To prove that $(1-q)I^{(\alpha)}$ satisfies the second condition, we rewrite $\mathcal{R}(\alpha)^{-l\beta_R}q^{l\beta_R(\beta_R+1)/2}$ as

$$\frac{\prod_{j=-\infty}^{\beta_R}(\mathcal{R}(\alpha)^{-1}q^j)^l}{\prod_{j=-\infty}^{0}(\mathcal{R}(\alpha)^{-1}q^j)^l}. \tag{27}$$

Note that, for all j, we have $\mathcal{R}(\alpha)=\mathcal{R}(\beta)\lambda^{-d^R_{\alpha\rho}}$, where $\lambda=U_\rho(\alpha)$. Hence, at $q=\lambda^{-1/m}$,

$$\mathcal{R}(\alpha)^{-1}q^j=\mathcal{R}(\beta)^{-1}q^{j-md^R_{\alpha\rho}}.$$

[4]Givental [2015d] defines the I-function to sum over all $\beta\in\mathbb{Z}^m$. However, the same argument works if we restrict the summation to curve classes in the semigroup $\mathrm{Eff}(X_\Sigma)$.

The formula (27) is equivalent to

$$\mathcal{R}(\alpha)^{-mld^R_{\alpha\rho}} q^{mld^R_{\alpha\rho}(md^R_{\alpha\rho}+1)/2} \frac{\prod_{j=-\infty}^{\beta_R-md^R_{\alpha\rho}} (\mathcal{R}(\beta)^{-1}q^j)^l}{\prod_{j=-\infty}^{0} (\mathcal{R}(\beta)^{-1}q^j)^l}$$

at $q=\lambda^{-1/m}$. Combining the above equivalence with the result in [Givental 2015d, page 10], we obtain

$$\operatorname{Res}_{q=U_\rho(\alpha)^{-1/m}}(1-q)I^{(\alpha)}(q) = -\frac{\phi^\alpha Q^{md_{\alpha\rho}}}{m\cdot C_{\alpha\rho}(m)}(1-U_\rho(\alpha)^{-1/m})I^{(\rho)}(U_\rho(\alpha)^{-1/m}),$$

which is equivalent to the residue formula in condition (2).

Since I is defined over the λ-algebra $\mathbb{Z}[\Lambda_1^{\pm},\dots,\Lambda_m^{\pm}][[Q]]$, it takes values in the symmetrized theory (see Remark 3.3). □

When X_Σ is projective, we may pass to the nonequivariant limit in Theorem 1.2 to obtain:

Corollary 5.2 *When X_Σ is a smooth projective toric variety, the level-l small I-function $I^{R,l}$ given in Proposition 5.1 lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ in the symmetrized quantum K-theory of level l.*

We denote by St and $\mathrm{St}^\vee$ the standard representation and its dual representation of $\mathbb{C}^*$. As corollaries of Proposition 5.1, we give proofs for Propositions 1.3–1.5 and 1.7.

Proof of Proposition 1.3 Let the target be $X=(\mathbb{C}\setminus 0)/\mathbb{C}^*$, where the action is the standard action. Then the proposition follows directly from Proposition 5.1. □

Proof of Proposition 1.4 Let the target be $X_{a_1,a_2}=(\mathbb{C}^2\setminus\{(0,0)\})/\mathbb{C}^*$ with charge vector (a_1,a_2). As mentioned in Remarks 1.6, 4.14 and 4.15, we only consider the untwisted component of the orbifold I-function, and its formula is given by Proposition 5.1. □

Proof of Proposition 1.5 For positive integers a, b, we consider $X_{a,-b}=\{(\mathbb{C}-0)\times\mathbb{C}\}/\mathbb{C}^*$ with charge vector $(a,-b)$. Let λ, μ be the generators of $\mathrm{Repr}((\mathbb{C}^*)^2)$ corresponding to the first and second projections of $(\mathbb{C}^*)^2$ onto its factors. From Proposition 5.1, the untwisted component of the orbifold I-function is given by

$$\begin{aligned} &I^{\mathrm{St},l}_{X_{a,-b}}(q)\\ &=1+\sum_{n\ge 1}\frac{p^{nl}q^{n(n-1)l/2}(1-p^{-b}\mu^{-1})(1-p^{-b}\mu^{-1}q^{-1})\cdots(1-p^{-b}\mu^{-1}q^{1-bn})}{(1-p^a\lambda^{-1}q)(1-p^a\lambda^{-1}q^2)\cdots(1-p^a\lambda^{-1}q^{an})}Q^n\\ &=1+\sum_{n\ge 1}(-1)^{bn}\frac{p^{nl-b^2n}q^{(n(n-1)l-bn(bn-1))/2}\mu^{-bn}(1-p^b\mu)(1-p^b\mu q^1)\cdots(1-p^b\mu q^{bn-1})}{(1-p^a\lambda^{-1}q)(1-p^a\lambda^{-1}q^2)\cdots(1-p^a\lambda^{-1}q^{an})}Q^n. \end{aligned}$$ □

Proof of Proposition 1.7 We consider the target $O(-1)^{\oplus r}_{\mathbb{P}^{s-1}} = X_{\mathbf{1},-\mathbf{1}} = \{(\mathbb{C}^s - 0) \times \mathbb{C}^r\}/\mathbb{C}^*$ with the charge vector $(1, 1, \dots, 1, -1, -1, \dots, -1)$. It follows from Proposition 5.1 that

$$\begin{aligned} I^{\mathrm{St}, l=1+s}_{X_{\mathbf{1},-\mathbf{1}}}(q) &= 1 + \sum_{n \geq 1} Q^n p^{nl} q^{n(n-1)l/2} \frac{(p^{-1}\mu_1^{-1}, q)_n \cdots (p^{-1}\mu_r^{-1}; q)_n}{(p\lambda_1^{-1} q; q)_n \cdots (p\lambda_s^{-1} q; q)_n} \\ &= 1 + \sum_{n \geq 1} (-1)^{nr} \prod_{i=1}^r (p\mu_i)^{-n} p^{(1+s)n} \frac{(p\mu_1, q)_n \cdots (p\mu_r; q)_n}{(p\lambda_1^{-1} q; q)_n \cdots (p\lambda_s^{-1} q; q)_n} Q^n (q^{n(n-1)/2})^{1+s-r}. \quad \square \end{aligned}$$

Remark 5.3 Traditionally, there are two approaches to prove genus-zero mirror theorems: an older approach of Givental and Tonita using adelic descriptions of cones, and a more recent wall-crossing approach of Ciocan-Fontanine and Kim using quasimap theory. In the first version of our paper, we used the wall-crossing approach. Recently, we discovered a gap in the proof. Roughly speaking, the K-theoretic version of the polynomiality property [Ciocan-Fontanine and Kim 2014, Lemma 7.6.1] is not strong enough to determine the coefficients of the K-theoretic S-operators and J-functions recursively. In the current version, we switch back to Givental and Tonita's older technique. It is still an interesting question if we can improve the wall-crossing approach to prove the mirror theorem in the K-theory setting.

Acknowledgements

Ruan would like to thank Aaron Bertram, Prakash Belkale and Jie Zhou for interesting discussions. Zhang would like to thank Felix Janda, Feng Qu, Valentin Tonita, Rachel Webb and Fenglong You for helpful discussions. We also thank the referee for valuable comments and suggestions on a previous draft.

Ruan is partially supported by the NSF grant DMS 1405245 and the NSF FRG grant DMS 1159265.

References

[Abramovich et al. 2002] **D Abramovich**, **T Graber**, **A Vistoli**, *Algebraic orbifold quantum products*, from "Orbifolds in mathematics and physics" (Madison, WI, 2001) (A Adem, J Morava, Y Ruan, editors), Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 1–24 MR

[Abramovich et al. 2003] **D Abramovich**, **A Corti**, **A Vistoli**, *Twisted bundles and admissible covers*, Comm. Algebra 31:8 (2003) 3547–3618 MR

[Abramovich et al. 2007] **D Abramovich**, **T Graber**, **M Olsson**, **H-H Tseng**, *On the global quotient structure of the space of twisted stable maps to a quotient stack*, J. Algebraic Geom. 16:4 (2007) 731–751 MR

[Abramovich et al. 2008] **D Abramovich**, **T Graber**, **A Vistoli**, *Gromov–Witten theory of Deligne–Mumford stacks*, Amer. J. Math. 130:5 (2008) 1337–1398 MR

[Aganagic and Okounkov 2017] **M Aganagic**, **A Okounkov**, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J. 17:4 (2017) 565–600 MR

[Aganagic et al. 2018] **M Aganagic**, **E Frenkel**, **A Okounkov**, *Quantum q-Langlands correspondence*, Trans. Moscow Math. Soc. 79 (2018) 1–83 MR

[Agnihotri 1995] **S Agnihotri**, *Quantum cohomology and the Verlinde algebra*, PhD thesis, University of Oxford (1995)

[Behrend and Manin 1996] **K Behrend**, **Y Manin**, *Stacks of stable maps and Gromov–Witten invariants*, Duke Math. J. 85:1 (1996) 1–60 MR

[Belkale 2008] **P Belkale**, *Quantum generalization of the Horn conjecture*, J. Amer. Math. Soc. 21:2 (2008) 365–408 MR

[Chen and Ruan 2002] **W Chen**, **Y Ruan**, *Orbifold Gromov–Witten theory*, from "Orbifolds in mathematics and physics" (Madison, WI, 2001) (A Adem, J Morava, Y Ruan, editors), Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 25–85 MR

[Cheong et al. 2015] **D Cheong**, **I Ciocan-Fontanine**, **B Kim**, *Orbifold quasimap theory*, Math. Ann. 363:3-4 (2015) 777–816 MR

[Ciocan-Fontanine and Kim 2010] **I Ciocan-Fontanine**, **B Kim**, *Moduli stacks of stable toric quasimaps*, Adv. Math. 225:6 (2010) 3022–3051 MR

[Ciocan-Fontanine and Kim 2014] **I Ciocan-Fontanine**, **B Kim**, *Wall-crossing in genus zero quasimap theory and mirror maps*, Algebr. Geom. 1:4 (2014) 400–448 MR

[Ciocan-Fontanine et al. 2014] **I Ciocan-Fontanine**, **B Kim**, **D Maulik**, *Stable quasimaps to GIT quotients*, J. Geom. Phys. 75 (2014) 17–47 MR

[Coates 2003] **T H Coates**, *Riemann–Roch theorems in Gromov–Witten theory*, PhD thesis, University of California, Berkeley (2003) MR Available at https://www.proquest.com/docview/305341155

[Coates and Givental 2006] **T Coates**, **A Givental**, *Quantum cobordisms and formal group laws*, from "The unity of mathematics" (P Etingof, V Retakh, I M Singer, editors), Progr. Math. 244, Birkhäuser, Boston, MA (2006) 155–171 MR

[Coates and Givental 2007] **T Coates**, **A Givental**, *Quantum Riemann–Roch, Lefschetz and Serre*, Ann. of Math. 165:1 (2007) 15–53 MR

[Dolgachev 2003] **I Dolgachev**, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series 296, Cambridge Univ. Press (2003) MR

[Gepner 1991] **D Gepner**, *Fusion rings and geometry*, Comm. Math. Phys. 141:2 (1991) 381–411 MR

[Givental 2000] **A Givental**, *On the WDVV equation in quantum K-theory*, Michigan Math. J. 48 (2000) 295–304 MR

[Givental 2004] **A B Givental**, *Symplectic geometry of Frobenius structures*, from "Frobenius manifolds" (C Hertling, M Marcolli, editors), Aspects Math. E36, Friedr. Vieweg, Wiesbaden (2004) 91–112 MR

[Givental 2015a] **A Givental**, *Permutation-equivariant quantum K-theory, II: Fixed point localization*, preprint (2015) arXiv 1508.04374

[Givental 2015b] **A Givental**, *Permutation-equivariant quantum K-theory, III: Lefschetz' formula on $\overline{\mathcal{M}}_{0,n}/S_n$ and adelic characterization*, preprint (2015) arXiv 1508.06697

[Givental 2015c] **A Givental**, *Permutation-equivariant quantum K-theory, IV: D_q-modules*, preprint (2015) arXiv 1509.00830

[Givental 2015d] **A Givental**, *Permutation-equivariant quantum K-theory, V: Toric q-hypergeometric functions*, preprint (2015) arXiv 1509.03903

[Givental 2015e] **A Givental**, *Permutation-equivariant quantum K-theory, VI: Mirrors*, preprint (2015) arXiv 1509.07852

[Givental 2015f] **A Givental**, *Permutation-equivariant quantum K-theory, VII: General theory*, preprint (2015) arXiv 1510.03076

[Givental 2015g] **A Givental**, *Permutation-equivariant quantum K-theory, VIII: Explicit reconstruction*, preprint (2015) arXiv 1510.06116

[Givental 2017a] **A Givental**, *Permutation-equivariant quantum K-theory, I: Definitions: Elementary K-theory of $\overline{\mathcal{M}}_{0,n}/S_n$*, Mosc. Math. J. 17:4 (2017) 691–698 MR

[Givental 2017b] **A Givental**, *Permutation-equivariant quantum K-theory, IX: Quantum Hirzebruch–Riemann–Roch in all genera*, preprint (2017) arXiv 1709.03180

[Givental 2017c] **A Givental**, *Permutation-equivariant quantum K-theory, XI: Quantum Adams–Riemann–Roch*, preprint (2017) arXiv 1711.04201

[Givental 2020] **A Givental**, *Permutation-equivariant quantum K-theory, X: Quantum Hirzebruch–Riemann–Roch in genus* 0, Symmetry Integrability Geom. Methods Appl. 16 (2020) art. id. 031 MR

[Givental and Tonita 2014] **A Givental**, **V Tonita**, *The Hirzebruch–Riemann–Roch theorem in true genus*-0 *quantum K-theory*, from "Symplectic, Poisson, and noncommutative geometry" (T Eguchi, Y Eliashberg, Y Maeda, editors), Math. Sci. Res. Inst. Publ. 62, Cambridge Univ. Press (2014) 43–91 MR

[Intriligator 1991] **K Intriligator**, *Fusion residues*, Modern Phys. Lett. A 6:38 (1991) 3543–3556 MR

[Jockers and Mayr 2020] **H Jockers**, **P Mayr**, *A* 3*d gauge theory/quantum K-theory correspondence*, Adv. Theor. Math. Phys. 24:2 (2020) 327–457 MR

[Kapustin and Willett 2013] **A Kapustin**, **B Willett**, *Wilson loops in* 3*-dimensional $N = 6$ supersymmetric Chern–Simons theory and their string theory duals*, preprint (2013) arXiv 1302.2164

[Kawasaki 1979] **T Kawasaki**, *The Riemann–Roch theorem for complex V-manifolds*, Osaka Math. J. 16:1 (1979) 151–159 MR

[Knudsen and Mumford 1976] **F F Knudsen**, **D Mumford**, *The projectivity of the moduli space of stable curves: I: Preliminaries on "det" and "Div"*, Math. Scand. 39:1 (1976) 19–55 MR

[Koroteev et al. 2021] **P Koroteev**, **P P Pushkar**, **A V Smirnov**, **A M Zeitlin**, *Quantum K-theory of quiver varieties and many-body systems*, Selecta Math. 27:5 (2021) art. id. 87 MR

[Lee 2004] **Y-P Lee**, *Quantum K-theory, I: Foundations*, Duke Math. J. 121:3 (2004) 389–424 MR

[Marian and Oprea 2007a] **A Marian**, **D Oprea**, *Counts of maps to Grassmannians and intersections on the moduli space of bundles*, J. Differential Geom. 76:1 (2007) 155–175 MR

[Marian and Oprea 2007b] **A Marian**, **D Oprea**, *The level-rank duality for non-abelian theta functions*, Invent. Math. 168:2 (2007) 225–247 MR

[Marian and Oprea 2010] **A Marian**, **D Oprea**, *GL Verlinde numbers and the Grassmann TQFT*, Port. Math. 67:2 (2010) 181–210 MR

[Okounkov 2017] **A Okounkov**, *Lectures on K-theoretic computations in enumerative geometry*, from "Geometry of moduli spaces and representation theory" (R Bezrukavnikov, A Braverman, Z Yun, editors), IAS/Park City Math. Ser. 24, Amer. Math. Soc., Providence, RI (2017) 251–380 MR

[Okounkov and Smirnov 2022] **A Okounkov**, **A Smirnov**, *Quantum difference equation for Nakajima varieties*, Invent. Math. 229:3 (2022) 1203–1299 MR

[Pushkar et al. 2020] **P P Pushkar**, **A V Smirnov**, **A M Zeitlin**, *Baxter Q-operator from quantum K-theory*, Adv. Math. 360 (2020) art. id. 106919 MR

[Qu 2018] **F Qu**, *Virtual pullbacks in K-theory*, Ann. Inst. Fourier (Grenoble) 68:4 (2018) 1609–1641 MR

[Ruan and Zhang 2023] **Y Ruan**, **M Zhang**, *Verlinde/Grassmannian correspondence and rank* 2 *δ-wall-crossing*, Peking Math. J. 6:1 (2023) 217–306 MR

[Su et al. 2020] **C Su**, **G Zhao**, **C Zhong**, *On the K-theory stable bases of the Springer resolution*, Ann. Sci. Éc. Norm. Supér. 53:3 (2020) 663–711 MR

[Tonita 2014a] **V Tonita**, *Twisted orbifold Gromov–Witten invariants*, Nagoya Math. J. 213 (2014) 141–187 MR

[Tonita 2014b] **V Tonita**, *A virtual Kawasaki–Riemann–Roch formula*, Pacific J. Math. 268:1 (2014) 249–255 MR

[Tonita 2018] **V Tonita**, *Twisted K-theoretic Gromov–Witten invariants*, Math. Ann. 372:1-2 (2018) 489–526 MR

[Tonita and Tseng 2013] **V Tonita**, **H-H Tseng**, *Quantum orbifold Hirzebruch–Riemann–Roch theorem in genus zero*, preprint (2013) arXiv 1307.0262

[Tseng 2010] **H-H Tseng**, *Orbifold quantum Riemann–Roch, Lefschetz and Serre*, Geom. Topol. 14:1 (2010) 1–81 MR

[Vafa 1992] **C Vafa**, *Topological mirrors and quantum rings*, from "Essays on mirror manifolds" (S-T Yau, editor), Int. Press, Hong Kong (1992) 96–119 MR

[Witten 1995] **E Witten**, *The Verlinde algebra and the cohomology of the Grassmannian*, from "Geometry, topology, & physics" (S-T Yau, editor), Conf. Proc. Lecture Notes Geom. Topology IV, Int. Press, Cambridge, MA (1995) 357–422 MR

YONGBIN RUAN ruanyb@zju.edu.cn
Department of Mathematics, University of Michigan, Ann Arbor, MI, United States
Current address: *Institute of Advanced Study for Mathematics, Zhejiang University, Hangzhou, China*

MING ZHANG zhangmsq@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI, United States

YAOXIONG WEN y.x.wen.math@gmail.com
School of Mathematics, Korea Institute for Advanced Study, Seoul, South Korea

Proposed: András I Stipsicz Received: May 17, 2018
Seconded: Mark Gross, Arend Bayer Revised: February 11, 2025

GEOMETRY & TOPOLOGY
msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2026 is US $905/year for the electronic version, and $1275/year (+$80, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

GT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 30 Issue 3 (pages 835–1201) 2026