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Springer theory for symplectic Galois groups

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A classical and beautiful story in geometric representation theory is the construction by Springer of an action of the Weyl group on the cohomology of the fibres of the Springer resolution of the nilpotent cone. We establish a natural extension of Springer's theory to arbitrary symplectic resolutions of conical symplectic singularities. We analyse features of the action in the case of affine quiver varieties, constructing Weyl group actions on the cohomology of ADE quiver varieties, and also consider "symplectically dual" examples arising from slices in the affine Grassmannian. Along the way, we document some basic features of the symplectic geometry of quiver varieties.

1 Introduction

A complex *symplectic variety* is a (quasiprojective) variety X with at worst rational, Gorenstein singularities and with a symplectic form ω, ie a closed, nondegenerate algebraic 2-form, on its smooth locus. A resolution of singularities $\pi : Y \to X$ is a *symplectic resolution* if ω extends to a symplectic form on all of Y.

A celebrated example is the cotangent bundle to a flag variety of a reductive group, which, as was first observed by a number of people including Springer and Steinberg, is a symplectic resolution of the nilpotent cone of the corresponding Lie algebra. In recent years, much effort has been focused on discovering to what extent one can generalise known representation-theoretic phenomena from Springer resolutions to all symplectic resolutions.

It is a beautiful observation of Markman [2010] in the projective case — shortly thereafter extended to the conical affine case by Namikawa [2010] — that if $\pi : Y \to X$ is a symplectic resolution, then the Poisson deformation base of Y is a Galois covering of the corresponding deformation base of X. The corresponding "symplectic Galois group", in the case of the cotangent bundle of the flag variety, is

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precisely the Weyl group, and in general it is the product of reflection groups attached to the geometry of slices to the resolution in codimension two.

In the present paper, we show that the cohomologies of the fibres of a symplectic resolution $\pi : Y \to X$ carry representations of its symplectic Galois group, generalising the seminal work of Springer [1976]. We also show that these representations can be constructed in a number of ways, mirroring the various constructions in the classical case: in particular, one can use both small resolutions and a version of the nearby cycles construction over a higher-dimensional base. Along the way, we establish some fundamental but not well-documented features of the symplectic geometry of Nakajima quiver varieties, and compute their symplectic Galois groups in the finite-type case.

A more precise description of our results is the following. Suppose that X is a conical symplectic variety (Definition 2.7) with symplectic resolution $Y \xrightarrow{\pi} X$. Let $\mathcal{Y} \to B_Y$ and $\mathcal{X} \to B_X$ be the corresponding versal Poisson deformations, with morphism $\mathcal{Y} \xrightarrow{\tilde{\pi}} \mathcal{X}$, as constructed in [Markman 2010; Namikawa 2010] (and reviewed in Section 2.2), with symplectic Galois group W. Fixing a coefficient field k, we define constructible complexes

$$\mathsf{HC} := \tilde{\pi}_! k_{\mathcal{Y}}[\dim(\mathcal{Y})] \quad \text{and} \quad \mathsf{Spr} := \pi_! k_Y[\dim(Y)],$$

the *symplectic Harish-Chandra sheaf* and *symplectic Springer sheaf*. Let $\mathrm{Perv}_{\mathrm{sympl}}(X)$ denote the category of perverse sheaves on X (with coefficients in k) which are smooth along the stratification by symplectic leaves. We also define, in Section 3.2, a nearby cycles sheaf $\mathcal{P}$ for the family $\mathcal{X} \to B_X$ (again, notation as in Section 2.2). This sheaf carries a natural action of the fundamental group of the complement of the discriminant locus, which we call the *braid group* of the symplectic resolution.

Theorem 1.1 (Theorem 3.2, Corollary 3.12, Theorem 3.21, Proposition 3.24) (1) *The complexes* HC *and* Spr *are perverse sheaves, which are semisimple if* $\mathrm{char}(k) = 0$. *Moreover* HC *is the intersection cohomology extension of the local system on* $\mathcal{X}^{\mathrm{reg}}$ *given by* $k[W]$, *the regular representation of the symplectic Galois group* W.

(2) *We have* $i_X^* \mathsf{HC} \cong \mathsf{Spr}$, *where* $i_X : X \hookrightarrow \mathcal{X}$ *is the inclusion.*

(3) *We have a natural algebra isomorphism*

$$k[W] \xrightarrow{a} \mathrm{End}(\mathsf{HC}).$$

(4) *We get an adjoint pair of functors*

$$(- \otimes_{k[W]} \mathsf{Spr}) : k[W]\text{-mod} \leftrightarrows \mathrm{Perv}_{\mathrm{sympl}}(X) : \mathrm{Hom}(\mathsf{Spr}, -).$$

(5) *The group* W *acts faithfully on the sheaf* Spr, *yielding, for each* $x \in X$, *an action of* W *on* $H^*(\pi^{-1}(x))$.

(6) *We have* $\mathcal{P} \cong \mathsf{Spr}$ *and the action of the braid group factors through the symplectic Galois group* W. *Moreover, this action coincides with the action of* W *induced by the intermediate extension functor.*

Assuming that k is coprime to $|W|$, the W action on Spr decomposes it into W-isotypic components: $\mathsf{Spr} \cong \bigoplus_{\rho\in\widehat{W}} \mathscr{E}_\rho \otimes \rho$, for certain perverse sheaves $\mathscr{E}_\rho$. In the classical case, the perverse sheaves $\mathscr{E}_\rho$ are irreducible, and hence one obtains[1] the "Springer correspondence" between irreducible W-representations and certain local systems on nilpotent orbits. In Section 3.3 we discuss some examples which show that in our more general setting the situation is more subtle: the sheaves $\mathscr{E}_\rho$ need not be irreducible and indeed may be zero or have nonisomorphic irreducible constituents. It would be interesting to better understand the structure of the sheaves $\mathscr{E}_\rho$.

Turning to Nakajima's quiver varieties, we review several features of their geometry, most likely known to experts, but not all of which appear yet to be documented in the literature. Fixing a quiver Q as in Section 4.1, let $Y(\boldsymbol{v},\boldsymbol{w})$, respectively $X(\boldsymbol{v},\boldsymbol{w})$, be the smooth symplectic quiver variety and affine quiver variety associated to the dimension vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Theorem 1.2 [Bellamy and Schedler 2021, Propositions 4.6 and 5.2] (1) *$X(\boldsymbol{v},\boldsymbol{w})$ is a conical affine symplectic variety.*

(2) *$Y(\boldsymbol{v},\boldsymbol{w})$ is a symplectic resolution of some conical affine symplectic variety.*

(3) *If Q is a Dynkin quiver of finite type, then for each $\boldsymbol{v}$ and $\boldsymbol{w}$ there exists $\boldsymbol{v}'$ so that $X(\boldsymbol{v}',\boldsymbol{w}) \cong X(\boldsymbol{v},\boldsymbol{w})$ and the natural projective morphism $\pi_{\boldsymbol{v}'} : Y(\boldsymbol{v}',\boldsymbol{w}) \to X(\boldsymbol{v}',\boldsymbol{w})$ is a symplectic resolution.*

We caution the reader that in general $Y(\boldsymbol{v},\boldsymbol{w})$ need *not* be a resolution of $X(\boldsymbol{v},\boldsymbol{w})$; cf Example 5.3. Bellamy and Schedler [2021], as well as establishing part (1) of the above theorem, also determine precisely when the varieties $X(\boldsymbol{v},\boldsymbol{w})$ have symplectic resolutions.

Furthermore, we calculate the symplectic Galois group of quiver varieties for finite-type Dynkin quivers. More precisely, associated to the finite-type Dynkin quiver Q and dimension vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is a parabolic subroot system (denoted by Φ_μ in Section 5) with associated Weyl group W_μ. We show:

Theorem 1.3 (Theorem 5.8) *If Q is a Dynkin quiver of finite type, the symplectic Galois group of the quiver variety $X(\boldsymbol{v},\boldsymbol{w})$ is W_μ.*

Combining Theorem 1.3 with our general theory, we obtain Weyl group actions on the cohomology of ADE quiver varieties. Weyl group actions have previously been obtained by Lusztig [2000], Nakajima [2003] and Maffei [2002].

Outside of finite type, as we show in Example 5.12, the naive generalisation of Theorem 1.3 fails. It is not clear to the authors what the correct general statement should be. It is plausible that one needs to take into account the existence of actions by larger algebras than the Kac–Moody algebras on topological invariants of quiver varieties. Bellamy and Schedler [2021, Section 7] establish some important results in the general setting, but they work in the unframed setting, so a direct comparison with our results is not immediate. See Remark 5.13 for more details.

We should note that much has already been written about categorical actions of braid groups on (derived categories of) quiver varieties and their quantisations. For example, if $\mathfrak{D}$ is the discriminant

[1] Since a simple perverse sheaf is the intermediate extension of a local system.

locus in $H^2(Y, \mathbb{C})$ (see Theorem 2.12) and $\mathscr{D}^c$ denotes its complement, then it is shown in [Braden et al. 2016] that there is an action of $\pi_1(\mathscr{D}^c/W)$ on $D^b(\mathbb{O}_g)$, the bounded derived category of the geometric category $\mathbb{O}$ defined in [Braden et al. 2016]. By taking Hochschild cohomology in an appropriate sense, this action should "decategorify" to an action of the braid group on the cohomology of the symplectic resolution, and compatibility with supports should ensure one also obtains actions on the cohomology of the fibres of the resolution also. These actions should then descend to the Weyl group actions constructed here. We hope to explore this relation in more detail in future work. However, even if one views our actions, at a lower categorical level, as a poor relation of categorical actions, our approach does work uniformly, independent of coefficient field. It is also quite concrete and topological; one important feature of Springer theory is the interplay amongst various approaches.

Convention 1.4 Throughout the paper, all varieties are over the field $\mathbb{C}$ of complex numbers, unless otherwise stated. We will also need the analytic topology, and in that case we will, by abuse of notation, identify a variety X with its analytification X^{an}. We write k to denote a coefficient field for sheaves, cohomology, etc.

2 Symplectic resolutions

2.1 Symplectic varieties and Poisson structures

Symplectic varieties were introduced by Beauville [2000].

A *symplectic variety* X is a normal variety which admits an (algebraic) symplectic form ω on its smooth locus X^{reg} such that, for some (or equivalently, every) resolution of singularities $\pi : Y \to X$, the pullback of ω extends to a regular (though not necessarily nondegenerate) 2-form on all of Y. In the case where the pullback of ω does extend to a symplectic form on all of Y, we say that π is a *symplectic resolution*. It is not necessarily the case that a symplectic variety admits a symplectic resolution (see the work of Fu [2003] on nilpotent orbit closures for examples).

We restrict our attention to affine symplectic varieties and their symplectic resolutions. These have been studied by a number of people. We will rely heavily on Kaledin's lovely paper [2006], in which he provides a basic structure theory for symplectic varieties which we now review. We begin by noting that, since X is normal, the Poisson bracket on functions on X^{reg} given by ω extends to a Poisson bracket on all of X. A subvariety of a Poisson variety X is said to be Poisson if it is the vanishing locus of a (radical) Poisson ideal sheaf.

Definition 2.1 If X is a Poisson variety, then, considering it as an analytic space, we may define the *symplectic leaves* of X to be the equivalence classes of the equivalence relation generated by Hamiltonian flows on X. Thus $x, y \in X$ are equivalent if there is a finite sequence of Hamiltonian flows

$$\mathscr{H}_t^i : [0, a_i] \to \mathrm{Symp}(X)$$

for $1 \le i \le n$ such that if $x_0 = x$ and $x_i = \mathscr{H}_{a_i}(x_{i-1})$ then $x_n = y$. Note that, in general, symplectic leaves need not be algebraic.

Definition 2.2 We say that a Poisson variety Z is *generically nondegenerate* if the bivector giving the Poisson bracket yields a generically nondegenerate alternating two-form. A Poisson variety is said to be *holonomic* if the restriction of the Poisson bracket to any Poisson subvariety is generically nondegenerate.

It is shown in [Kaledin 2006] that any symplectic variety is a holonomic Poisson variety. Note however that the converse is false (already in dimension 2, for example for simple elliptic singularities).

Recall that the *singularity stratification* of a reduced scheme X of finite type is the stratification whose closed subsets $X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_k$ have the property that $X_k^\circ := X_k \setminus X_{k+1}$ equals X_k^{reg}, the smooth locus of X_k. This stratification is canonical and thus is preserved by all automorphisms of X and vector fields on X. The strata of this stratification however need not be connected. We will normally refine it so as to obtain connected strata. By abuse of notation we refer to such a refinement as the singularity stratification. We note that these strata are still clearly preserved by all vector fields on X.

Theorem 2.3 [Kaledin 2006] *Let X be a complex symplectic variety. The locally closed strata of the singularity stratification of X are locally closed smooth symplectic subvarieties. The closures X_i of the X_i^0 are irreducible Poisson subvarieties and every Poisson subvariety of X is the closure of a stratum. Moreover, the normalisation of any Poisson subvariety $Y \subset X$ is also a symplectic variety.*

Remark 2.4 The singularity stratification is, in general, too coarse, and many desirably properties of a stratification will only hold once it is refined. One virtue of symplectic varieties, however, is that the singularity stratification already possesses many of these properties. The most basic of these is for the strata to satisfy the "axiom of closure", that is, the closure of a stratum of the partition is a union of strata. The following lemma, which is implicit in the proof of Proposition 3.1 of [Kaledin 2006], shows this is indeed the case for the singularity stratification of a symplectic variety.

Lemma 2.5 *The strata of the singularity stratification of a holonomic Poisson variety X (hence, in particular, a symplectic variety) are given by its symplectic leaves and thus they satisfy the axiom of closure.*

Proof We use induction on $\dim(X)$. The smooth locus of X^{reg} is a smooth holonomic variety, and hence, by Lemma 1.4 of [Kaledin 2006] is in fact smooth symplectic, and thus is a single symplectic leaf. The singular locus X^{sing} is a proper closed subscheme preserved by all automorphisms of X and all vector fields. In particular if $C^1, \dots, C^k$ are the irreducible components of X^{sing}, then each C^i is preserved by all Hamiltonian vector fields, and is therefore a Poisson subscheme that is necessarily holonomic. By induction it follows that the singularity stratification of each $\overline{C}_i$ is given by its symplectic leaves, and since by definition the symplectic leaves of $\overline{C}_i$ are precisely the symplectic leaves of X contained in $\overline{C}_i$, the stratifications of the components agree on their overlaps, which completes the inductive step. □

Remark 2.6 We thank the referee for simplifying our original proof of the previous lemma.

2.2 Deformation theory and the symplectic Galois group

While standard deformation theory for nonprojective or singular varieties is generally rather intractable, it turns out that one often obtains a more tractable theory of *Poisson deformations* for algebraic varieties equipped with a Poisson bracket. This deformation theory has been studied by Namikawa [2011] and Ginzburg and Kaledin [2004]; here we will largely follow Namikawa. Given a Poisson variety X there is a Poisson deformation functor PD_X from the category of local Artinian rings with residue field $\mathbb{C}$ to sets. This assigns to an Artinian ring R the set of pairs $(\mathcal{X}, \phi)$ consisting of a flat morphisms[2] $\mathcal{X} \to T = \mathrm{Spec}(R)$ equipped with a relative Poisson bracket $\{\,.\,,.\,\}$ and an isomorphism $\phi : X \to X \times_T \mathbb{C}$, taken up to the natural notion of isomorphism. We write $\mathbb{C}[\epsilon]$ for the ring of dual numbers over $\mathbb{C}$.

Definition 2.7 A *conical symplectic variety* is an affine symplectic variety X equipped with a $\mathbb{G}_m$-action with nonnegative weights on $\mathbb{C}[X]$ and $\mathbb{C}[X]^{\mathbb{G}_m} = \mathbb{C}$, and for which the symplectic form transforms with weight $\ell > 0$. When X has a symplectic resolution $\pi : Y \to X$, the $\mathbb{G}_m$-action lifts uniquely to Y.

Now suppose that X is an affine symplectic variety and $\pi : Y \to X$ is a symplectic resolution, we thus have Poisson deformation functors for both X and Y. If $p : \mathcal{Y} \to T$ is a deformation of Y, then it can be checked that $\mathrm{Spec}(p_*(\mathbb{O}_{\mathcal{Y}})) \to T$ is a deformation of X, so that one obtains a natural transformation $\pi_* : \mathrm{PD}_Y \to \mathrm{PD}_X$. One then has the following theorem of Namikawa:

Theorem 2.8 [Lehn et al. 2012, Theorem 2.1] *In the above setting both* PD_X *and* PD_Y *are unobstructed and are pro-representable. More precisely, one has a* $\mathbb{G}_m$*-equivariant commutative diagram*:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\theta_Y} & B_Y \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{\theta_X} & B_X \end{array} \tag{2-1}$$

where $B_Y = \mathrm{PD}_Y(\mathbb{C}[\epsilon]) \cong \mathbb{A}^d$ *and* $B_X = \mathrm{PD}_X(\mathbb{C}[\epsilon]) \cong \mathbb{A}^d$ *with* $d = \dim(H^2(Y, \mathbb{C}))$. *The deformations* $\mathcal{Y} \to B_Y$ *and* $\mathcal{X} \to B_X$ *are formally universal Poisson deformations of* Y *and* X *respectively at* 0. *The map* q *is a finite Galois with* $q(0) = 0$.

The action of $\mathbb{G}_m$ on the affine spaces $\mathbb{A}^d$ is of weight ℓ by construction. There is a natural period map $B_Y \to H^2(Y, \mathbb{C})$ for the deformation $\mathcal{Y}$ defined, for example, in step 2 of the proof of Theorem 1.1 in [Namikawa 2010]; see also [Kaledin and Verbitsky 2002] and [Ginzburg and Kaledin 2004]. Roughly, this map assigns to each fibre of the deformation the class of its symplectic form (for this to make sense one must show that the H^2 of the fibres can be canonically identified, which follows from the topological triviality of the deformation — see Proposition 2.9 below). The period map identifies B_Y with $H^2(Y, \mathbb{C})$. It follows from this that the $\mathbb{G}_m$-action in the above theorem has weight ℓ on $B_Y \cong H^2(Y, \mathbb{C})$. The symplectic Galois group W arises as the covering group for the map q at 0.

Note that the affinisation $\mathcal{Y}^{\mathrm{aff}}$ is given by $\mathcal{Y}^{\mathrm{aff}} = \mathcal{X} \times_{B_X} B_Y$. It is well known to experts that the family $\mathcal{Y} \to H^2(Y, \mathbb{C})$ is topologically trivial. For example, it is noted by Namikawa [2010] that the arguments of Slodowy [1980, Section 4.2] for slices in the nilpotent cone generalise, as we discuss in some more detail below. The resulting triviality of cohomology is alternatively a consequence of Theorem 1.13 of [Ginzburg and Kaledin 2004] or Theorem 10.2.2 of [Dobrovolska et al. 2016].

[2] In this section we must work with schemes and not just varieties.

Proposition 2.9 *The morphism $\mathcal{Y} \to H^2(Y,\mathbb{C})$ is trivial as a topological fibration.*

Proof We may choose a contracting linear $\mathbb{G}_m$-action on $\mathbb{A}^N$ for some positive integer N, and a $\mathbb{G}_m$-equivariant embedding of the affinisation $\mathcal{Y}^{\mathrm{aff}} = \mathcal{X} \times_{B_X} B_Y$ in $\mathbb{A}^N \times B_Y$ for some N, compatibly with projection to B_Y. Pull the function $\overline{\phi}(z_1,\dots,z_N) = \sum_i |z_i|^2$ on $\mathbb{A}^N$ back to a function $\phi : \mathcal{Y} \to \mathbb{R}$ that is proper on each fibre of $\mathcal{Y} \to B_Y$. This is an exhaustion function, in the sense that it is bounded below (by zero) and proper: for any $a > 0$ the preimage $\phi^{-1}([0,a])$ has compact fibres over the compact closed hyperdisk $\{(z_1,\dots,z_N) : \sum |z_i|^2 \le a\}$.

It follows that $\phi : \mathcal{Y} \to \mathbb{R}$ is a proper map and hence the subset $\mathcal{Y}_{\le r} := \phi^{-1}([0,r]) \subset \mathcal{Y}$ is thus compact for each $r > 0$. When r is sufficiently small, the boundary of $\mathcal{Y}_{\le r}$ is a smooth manifold. This follows from the transversality of the affinisation map to $\phi^{-1}(r)$. To see this, note that because the $\mathbb{G}_m$-action has positive weights, near 0 the boundary of the disk $\{z \in \mathbb{A}^N : \phi(z) \le r\}$ is transverse to $\mathbb{R}^\times$-orbits, and the affinisation map $\mathcal{Y} \to \mathcal{Y}^{\mathrm{aff}}$ is equivariant. This argument also shows that $\mathcal{Y}_{\le r}$ intersects $Y = \theta_Y^{-1}(0)$, where $0 \in B_Y$, transversely. It follows by continuity that the boundary of $\mathcal{Y}_{\le r}$ intersects the fibres $\mathcal{Y}_\lambda = \theta_Y^{-1}(\lambda)$, where $\lambda \in B_Y$, transversely for small λ. It follows that, near $0 \in B_Y$, the map $\mathcal{Y}_{\le r} \to B_Y$ is a proper submersion and hence by the Ehresmann fibration lemma (for manifolds with boundary) it is a locally trivial fibration over U some open ball around $0 \in B_Y$. Explicitly, we have shown that for sufficiently small $r > 0$ there is an open ball U about 0 such that θ_Y restricted to $\mathcal{Y}_{\le r} \cap \theta_Y^{-1}(U)$ is a locally trivial fibration.

To extend this to the entire family $\theta_Y : \mathcal{Y} \to B_Y$, one must again use the $\mathbb{G}_m$-action: Fix some r as above so that if $K = \mathcal{Y}_{\le r} \cap \theta_Y^{-1}(U)$ then $\theta_{Y|K}$ is a locally trivial fibration. Then using the action of $t \in [1,\infty) \in \mathbb{R}^\times$ we see that the boundary of tK intersects the fibres of the family $\mathcal{Y} \to B_Y$ transversely. Using a Hermitian metric compatible with the projection to B_Y one can then project the radial vector field given by the infinitesimal $\mathbb{R}^\times$-action to obtain a nonvanishing vector field along the fibres, which is complete by the properness of ϕ. Standard arguments in Morse theory (cf Theorem 3.1 of [Milnor 1963]) now show that the rescaled downward flow of this vector field defines a diffeomorphism $\mathcal{Y}_{\ge r} \cap \phi_Y^{-1}(U) \cong (\mathcal{Y}_r \cap \phi_Y^{-1}(U)) \times [r,\infty)$. It follows that $\phi_Y^{-1}(U) \to U$ is a topologically trivial fibration. (In fact with appropriate arguments one can make it C^∞ trivial.) One can then again use the $\mathbb{R}^\times$ action to expand from U to all of B_Y and conclude that θ_Y is a locally trivial (and hence trivial as B_Y is contractible) fibration. □

2.3 The universal conic deformation

The deformations $\mathcal{Y} \to B_Y$ and $\mathcal{X} \to B_X$, in addition to being formally Poisson universal at the origin 0, are also universal once one takes the $\mathbb{G}_m$-action into account:[3] Given a symplectic variety Y with a $\mathbb{G}_m$-action $\rho : \mathbb{G}_m \to \mathrm{Aut}(Y)$ which scales the symplectic form ω_Y with weight ℓ, we will need to consider flat $\mathbb{G}_m$-equivariant families over a linear base $\mathfrak{a}$, that is, families $f : \mathcal{Y}_{\mathfrak{a}} \to \mathfrak{a}$ of algebraic symplectic varieties over a vector space $\mathfrak{a}$ with central fibre Y, where $\mathbb{G}_m$ acts on the vector space $\mathfrak{a}$ by the ℓ-th power of the scaling action and the action on $\mathcal{Y}_{\mathfrak{a}}$ induces the action ρ on the central fibre Y, with moreover

[3] Everything here is presumably well known to the experts, but we failed to find a suitable reference, so for the convenience of the reader we outline the details here.

$t.\widetilde{\omega} = t^\ell \widetilde{\omega}$, where $\widetilde{\omega}$ is the relative symplectic form of the family $\mathscr{Y}_\mathfrak{a}$ over $\mathfrak{a}$. We shall call such a family a *weight-ℓ conic deformation*. The following result is a rephrasing of the work of Namikawa [2008]:

Proposition 2.10 *The family $\mathscr{Y} \to H^2(Y,\mathbb{C})$ is universal among weight-ℓ conic deformations, that is, any such family $\mathscr{Y}_\mathfrak{a} \to \mathfrak{a}$ is obtained by pullback from a linear map $\alpha : \mathfrak{a} \to H^2(Y,\mathbb{C})$. Moreover, the family $\mathscr{X} \to B_X = H^2(Y,\mathbb{C})/W$ is also universal among weight-ℓ conic deformations of X, that is, any such family $\mathscr{X}_\mathfrak{b} \to \mathfrak{b}$ is obtained by pullback along a $\mathbb{G}_m$-equivariant morphism $\beta : \mathfrak{b} \to B_X$.*

Proof We sketch a proof. As already noted above in Theorem 2.8, the deformation $\mathscr{Y} \to B_Y$ is formally Poisson-universal at the origin 0. The space B_Y is obtained from the pro-representing object by taking $\mathbb{G}_m$-finite vectors, and one can check that the period map p is compatible with the $\mathbb{G}_m$-actions (with weight ℓ on $H^2(Y,\mathbb{C})$).

Now if we have any weight-ℓ conic deformation $\mathscr{Y}_\mathfrak{a} \to \mathfrak{a}$, then by universality its formal completion at 0 is pulled back from the universal deformation $(\widehat{\mathscr{Y}}, \widehat{\omega})$ by a map $\widetilde{f}$, say. The $\mathbb{G}_m$-actions then allow us to spread this out to a map $f : \mathfrak{a} \to H^2(Y,\mathbb{C})$, which by homogeneity (everything in sight has weight ℓ for the $\mathbb{G}_m$-action) must be linear. The claim for $\mathscr{X} \to B_X$ is established in an entirely similar manner. □

2.4 The symplectic Galois group

It is shown in [Namikawa 2010] that q is a Galois covering.

Definition 2.11 We call the covering group W of q the *Galois group* of our symplectic resolution, or simply the *symplectic Galois group*.

Namikawa's work shows that W can be described quite explicitly in terms of the exceptional divisors of the resolution π: Let Σ denote the singular locus of X and Σ_0 denote the union of the strata of X of codimension 4 or higher, and set $U = X \backslash \Sigma_0$. Let $\mathscr{B}$ denote the set of connected components of $\Sigma \backslash \Sigma_0$, the singular locus of U. For each B in $\mathscr{B}$, a transverse slice S_B through a point $b \in B$ yields a two-dimensional symplectic singularity — that is, a rational surface singularity, whose minimal resolution is given by the restriction of π. Let W'_B denote the Weyl group of type ADE attached to this singularity. As the point $b \in B$ varies, the type of this singularity is locally constant, and thus one obtains an action of $\pi_1(B)$ on the components of the exceptional divisor of $\pi^{-1}(S_B)$. As these components can be identified with a set of simple roots for the simply laced root system attached to W_B, this monodromy action thus induces a diagram automorphism of W'_B and we set $W_B \subset W'_B$ to be its fixed-point subgroup. The main theorem of [Namikawa 2010] then shows that the symplectic Galois group is

$$W = \prod_{B \in \mathscr{B}} W_B,$$

and so $B_X = H^2(Y,\mathbb{C})/W$. Moreover, the space $H^2(Y,\mathbb{C})$ can be identified with the direct sum of the reflection representations of the W_B.

Note that in diagram (2-1) each square is commutative, but the right-hand square is *not* Cartesian, since for example the fibre product $\mathscr{X}' := \mathscr{X} \times_{B_X} B_Y$ is affine. Following Namikawa [2015], we let $\Pi : \mathscr{Y} \to \mathscr{X}'$

be the natural map and, for $t \in H^2(Y,\mathbb{C})$, let $\Pi_t : \mathcal{Y}_t \to \mathcal{X}'_t$ be the restriction of Π to the preimage $\mathcal{Y}_t = \psi_Y^{-1}(t)$.

We now recall a result of Namikawa. Let $\mathcal{D} \subset H^2(Y,\mathbb{C})$ be the discriminant locus: that is, $\mathcal{D}$ is defined to be the closure of the locus where Π_t is not an isomorphism. It can be shown that this is also the locus where $\mathcal{Y}_t$ is not affine, or equivalently, where $\mathcal{X}_t$ is singular; see [Namikawa 2011] for details. For a $\mathbb{Q}$-vector space H write $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} H$.

Theorem 2.12 [Namikawa 2015] *There are finitely many linear subspaces $\{H_i\}_{i\in I}$ of $H^2(Y,\mathbb{Q})$ of codimension 1 such that $\mathcal{D} = \bigcup_{i\in I}(H_i)_{\mathbb{C}}$. Moreover, the open chambers in $H^2(Y,\mathbb{R})$ complementary to the arrangement of real hyperplanes $\{(H_i)_{\mathbb{R}}\}_{i\in I}$ are given by $\{w(\mathrm{Amp}(\pi_k))\}$, where $w \in W$ and π_k are the projective crepant resolutions of X, and $\mathrm{Amp}(\pi_k)$ denotes the ample cone of π_k.*

Thus, in particular, the discriminant locus of the map Π is an algebraic hypersurface: more precisely, a finite union of hyperplanes.

2.5 (Semi)smallness and Poisson deformations

Definition 2.13 A morphism of varieties $f : Y \to X$ is said to be *semismall* if the fibre product $Z = Y \times_X Y$ has dimension at most $d = \dim(Y)$ (and hence exactly d). If moreover any irreducible component of Z which has dimension d maps to a dense subset of X then f is said to be *small*.

Kaledin shows that any symplectic resolution $Y \to X$ is semismall. We know show that suitable families of resolutions form small maps.

Definition 2.14 Let $f : \mathcal{X} \to B$ be a family of affine symplectic varieties such that for Zariski-generic $b \in B$ the variety $X_b = f^{-1}(b)$ is smooth and $\dim(X_b)$ is independent of $b \in B$. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be a family of fibrewise symplectic resolutions, that is, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\ \pi\ } & \mathcal{X} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & B & \end{array}$$

such that $\pi_b : Y_b \to X_b$ is a symplectic resolution, where $Y_b = (f \circ \pi)^{-1}(b)$. We will call f a *smoothing family*. Note that by assumption, for Zariski-generic $b \in B$, the resolution π_b is an isomorphism.

Proposition 2.15 *Suppose that $f : \mathcal{X} \to B$ is a smoothing family of d-dimensional affine symplectic varieties and $\pi : \mathcal{Y} \to \mathcal{X}$ a family of fibrewise symplectic resolution as above. Then π is a small morphism.*

Proof Let D denote the closure of the locus of $b \in B$ for which X_b is singular. By assumption, D is a proper subvariety of B. Let $\mathcal{X}_{\mathrm{sing}} = f^{-1}(D)$, a proper subvariety of $\mathcal{X}$. To show that π is small it suffice to show that $\dim(\mathcal{Y} \times_{\mathcal{X}_{\mathrm{sing}}} \mathcal{Y}) < \dim(\mathcal{Y})$.

To see this, fix $b \in B$. By Lemma 2.11 of [Kaledin 2006], since $\pi_b : Y_b \to X_b$ is a projective birational morphism from the smooth variety Y_b to the symplectic variety X_b, it is semismall. It follows that $\dim(Y_b \times_{X_b} Y_b) = \dim(Y_b) = d$. Considering the natural map $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to B$ we see that

$$\dim(\mathcal{Y} \times_{\mathcal{X}_{\mathrm{sing}}} \mathcal{Y}) \le d + \dim(D) < \dim(\mathcal{Y}). \qquad \square$$

Corollary 2.16 *Suppose that $\mathcal{Y}_{\mathfrak{a}} \to \mathfrak{a}$ is a conic deformation of a conic symplectic resolution $\pi : Y \to X$ with the generic fibre of $\mathcal{Y}_{\mathfrak{a}}$ affine. Then if $\mathcal{X}_{\mathfrak{a}}$ is the affinisation of $\mathcal{Y}_{\mathfrak{a}}$, the natural morphism $\mathcal{Y}_{\mathfrak{a}} \to \mathcal{X}_{\mathfrak{a}}$ is small.*

Proof By Proposition 2.10, the family $\mathcal{Y}_{\mathfrak{a}}$ is a pullback of the universal family by a linear map $\alpha : \mathfrak{a} \to H^2(Y, \mathbb{C})$. Since the generic fibre of $\tilde{\pi}$ is affine, the image of α is not contained in $\mathcal{D}$ the discriminant locus of the universal deformation, hence the locus in $\mathfrak{a}$ for which the fibres of the map $\mathcal{X}_{\mathfrak{a}} \to \mathfrak{a}$ are singular is a proper closed subset, hence we may apply the previous proposition. □

Remark 2.17 It might arguably be more natural to consider, in the context of a conic symplectic resolution $\pi : Y \to X$, the morphism $\mathcal{Y} \to \mathcal{X}$ between respective universal deformations (over the deformation base B_X rather than $H^2(Y, \mathbb{C})$). Since the map from $H^2(Y, \mathbb{C}) \to B_X$ is finite, however, it follows immediately from the above that this map is also small.

3 Basics of symplectic Springer theory

We lay out some general algebraic and topological features of symplectic Springer theory. We let k denote a coefficient field for constructible sheaves (and their cohomologies). Unless otherwise stated we will assume that $\mathrm{char}(\mathsf{k}) \nmid |W|$.

3.1 Harish-Chandra and Springer sheaves

We maintain the notation of diagram (2-1).

Definition 3.1 The constructible complex

$$\mathsf{HC} := \tilde{\pi}_! k_{\mathcal{Y}}[\dim(\mathcal{Y})]$$

is the *symplectic Harish-Chandra sheaf* on $\mathcal{X}$. The constructible complex

$$\mathsf{Spr} := \pi_! k_Y[\dim(Y)]$$

is the *symplectic Springer sheaf* on X. When we wish to emphasise the particular symplectic resolution relevant, we will write Spr_X, and similarly when we wish to emphasise which coefficient field is being used, we will write $\mathsf{Spr}(\mathsf{k})$.

We collect some basic properties below. Let $\mathrm{Perv}_{\mathcal{S}}(X)$ denote the category of perverse sheaves on X (with coefficients in k) which are smooth along the stratification by symplectic leaves.

Theorem 3.2 *Let W denote the symplectic Galois group.*

1. *The complexes HC and Spr are perverse sheaves, which are semisimple when k has characteristic zero. Moreover, HC is the intersection cohomology complex on $\mathcal{X}$ extending the local system on $\mathcal{X}^{\mathrm{reg}}$ given by the regular representation of W the symplectic Galois group.*
2. *We have $i_X^* \mathsf{HC} \cong \mathsf{Spr}$, where $i_X : X \hookrightarrow \mathcal{X}$.*

(3) *We have a natural algebra isomorphism*

$$k[W] \xrightarrow{a} \mathrm{End}(\mathsf{HC}).$$

(4) *We get an adjoint pair of functors*

$$(-\otimes_{k[W]} \mathsf{Spr}) : k[W]\text{-mod} \leftrightarrows \mathrm{Perv}_{\mathrm{sympl}}(X) : \mathrm{Hom}(\mathsf{Spr}, -).$$

Proof Statement (1) is immediate from the semismallness of $\pi : Y \to X$ and the smallness of the map $\widetilde{\pi}$ established in Proposition 2.15. Note that the semisimplicity statement about the local system will follow from the proof of (3) below. Statement (2) follows from proper base change.

For statement (3), let B_X^{reg} denote the open subset of the base B_X of the versal deformation over which $\mathscr{X}$ has smooth fibres. Write $\mathscr{X}^{\mathrm{reg}}$ for $\mathscr{X} \times_{B_X} B_X^{\mathrm{reg}}$ and let $j : \mathscr{X}^{\mathrm{reg}} \hookrightarrow \mathscr{X}$ denote the inclusion. Note that when restricted to the regular locus $\widetilde{\pi}^{-1}(\mathscr{X}^{\mathrm{reg}}) \to \mathscr{X}^{\mathrm{reg}}$ is an unramified Galois W-cover. Now, observe that smallness of $\widetilde{\pi}$ implies that $\mathsf{HC} = j_{!*}\widetilde{\pi}_* k_{\widetilde{\pi}^{-1}(\mathscr{X}^{\mathrm{reg}})}[-\dim(\mathscr{X})]$. To complete the proof of (3) we use Lemma 3.3 below. This shows that

$$\mathrm{End}(\mathsf{HC}) \cong \mathrm{End}(\widetilde{\pi}_* k_{\widetilde{\pi}^{-1}(\mathscr{X}^{\mathrm{reg}})}) \cong k[W].$$

This defines the isomorphism a.

Finally, the existence of the adjunction claimed in (4) is a version of the standard tensor-Hom adjunction, provided we prove that the sheaf Spr lies in $\mathrm{Perv}_{\mathscr{S}}(X)$, that is, it is smooth along the stratification of X by symplectic leaves. We will give a proof of this using the nearby cycles description of the Springer sheaf which we will give in Section 3.2, since the nearby cycles sheaf is constructible with respect to a Whitney stratification of the exceptional fibre that satisfies the a_f condition. □

Lemma 3.3 *For a dense open subset $U \xrightarrow{\iota} X$ and a local system $\mathscr{L}$ on U,*

$$\mathrm{End}_X(\iota_{!*}\mathscr{L}) = \mathrm{End}_U(\mathscr{L}).$$

Proof We have homomorphisms

$$\mathrm{End}_U(\mathscr{L}) \to \mathrm{End}_X(\iota_{!*}\mathscr{L}) \to \mathrm{End}_U(\mathscr{L}),$$

the first given by $(-)_{!*}$ and the second by restriction to U. The composite is the identity; hence it suffices to show that the latter map is injective. If $\phi \in \mathrm{End}_X(\iota_{!*}\mathscr{L})$ and $\phi|_U = 0$, it follows that $\phi(\iota_{!*}\mathscr{L})$ is supported on $X \setminus U$. But the intermediate extension $\iota_{!*}\mathscr{L}$ is the unique extension that has no sub- or quotient object supported on $X \setminus U$, so $\mathrm{im}(\phi) = 0$, as required. □

Remark 3.4 In the case of HC this is a direct consequence of the definitions — the pushforward is, by the smallness assumption, an intersection cohomology sheaf, and the local systems that arise will be semisimple provided $\mathrm{char}(\mathsf{k})$ is coprime to $|W|$.

On the other hand, for Spr, although its perversity is an immediate consequence of semismallness, the semisimplicity of Spr is not automatic if one only assumes that $\mathrm{char}(\mathsf{k}) \nmid |W|$. Indeed, [de Cataldo and Migliorini 2002] shows that the decomposition theorem for semismall maps can be related to the nondegeneracy of certain intersection forms, and the symplectic Galois group is not sufficient to determine these forms in general. We will give an explicit example of this later in the paper.

Definition 3.5 If P is an irreducible constituent of Spr, then, since Spr is constructible with respect to the stratification of X by its symplectic leaves, $\mathsf{P} \cong \mathrm{IC}(\mathscr{L})$ for some local system $\mathscr{L}$ on a stratum S, and the pair $(S, \mathscr{L})$ uniquely determine P (it is the "middle-extension" of $\mathscr{L}$ to X). We call such a local system a *Springer local system.*

Remark 3.6 From the point of view of $\mathrm{Perv}_{\mathscr{S}}(X)$, an immediate deficiency in the theory described above is that $\mathsf{k}[W]$-mod does not detect all of the category $\mathrm{Perv}_{\mathscr{S}}(X)$. This is of course already true in the classical setting of Springer theory when X is the nilpotent cone of a reductive group. In that setting, however, Lusztig [1984] shows that one can understand all of the simple objects in $\mathrm{Perv}_{\mathscr{S}}(X)$ by a Harish-Chandra-style classification in which each object is associated to an irreducible representation of a finite Coxeter group, which arises by studying the endomorphism algebra of complexes induced from what Lusztig calls *cuspidal local systems*. In this context, the simple perverse sheaves arising in the Springer correspondence are those induced from the trivial local system on a maximal torus. (This is a cuspidal local system only because a torus has no proper parabolic subgroups.) Indeed more turns out to be true — one in fact gets an orthogonal decomposition of corresponding derived category of $\mathrm{Perv}_{\mathscr{S}}(X)$, where each block of the decomposition can be labelled by a cuspidal local system on a Levi subalgebra. The objects of $\mathrm{Perv}_{\mathscr{S}}(X)$ one can understand using the Springer sheaf might thus be thought of as the "principal block" of $\mathrm{Perv}_{\mathscr{S}}(X)$.

Perhaps the easiest examples of cuspidal local systems arise from the fact that, when the group is SL_n, the equivariant fundamental group of the regular nilpotent orbit is cyclic of order n. The nontrivial representations of the cyclic group then correspond to local systems on the regular orbit which are cuspidal. In any case where the open symplectic leaf of our symplectic variety X has a nontrivial étale fundamental group, the nontrivial local systems on X^{reg} cannot be Springer local systems, ensuring $\mathrm{Perv}_{\mathscr{S}}(X)$ has more simple objects that the "principal block".

It is not immediately obvious how to even frame a generalisation to the setting of symplectic resolutions of Lusztig's generalisation of Springer's results. One reason for this is of course that it is difficult to define an appropriate analogue of a parabolic subgroup. At the very least it seems one must require X to possess additional torus symmetries which lift to the symplectic resolution.

Remark 3.7 As we will see in examples later in this section, the stronger version of assertion (3), which readers familiar with the classical theory might have expected to see, namely that the composition of the map a with the natural map $\mathrm{End}(\mathsf{HC}) \to \mathrm{End}(\mathsf{Spr})$ is an isomorphism, is not true in general. Neither, then, is it true that the induction functor from $k[W]$-modules to $\mathrm{Perv}_{\mathrm{sympl}}(X)$ is faithful. Indeed we will see later in this section examples which show that the map from $k[W]$ to $\mathrm{End}(\mathsf{Spr})$ need neither be injective nor surjective. The following lemmas will, however, imply a weaker statement, namely that W embeds into the automorphisms of Spr.

Next we note that the deformation $\tilde{\pi} : \mathscr{Y} \to \mathscr{X}$ gives us a more elementary way of constructing an action of W on the cohomology of Y. We first remark that the endomorphism algebra of the sheaf Spr has a natural geometric interpretation. Let $Z = X \times_Y X$ be the fibre product of X with itself over Y (the "symplectic

Steinberg variety”). Its Borel–Moore homology has a natural convolution structure preserving the middle-dimensional piece $H^{\mathrm{mid}}_{\mathrm{BM}}(Z)$; the latter has a natural basis given by the irreducible components of Z.

Lemma 3.8 $$\mathrm{End}(\mathsf{Spr}) \cong H^{\mathrm{mid}}_{\mathrm{BM}}(Z).$$

Proof The argument is standard: see, for example, Section 3.4 of [Chriss and Ginzburg 1997]. □

In terms of the identification $\mathrm{End}(\mathsf{Spr}) \cong H^{\mathrm{mid}}_{\mathrm{BM}}(Z)$, the map from the symplectic Galois group W to $H^{\mathrm{mid}}_{\mathrm{BM}}(Z)$ can be seen as a specialisation-of-cycles map, similar to the construction of Springer representations in [Chriss and Ginzburg 1997].

Since we are assuming that our symplectic resolution $\pi : Y \to X$ is conic, the fixed point locus $F = Y^{\mathbb{G}_m}$ is a smooth projective variety contained in $L = \pi^{-1}(0)$ (where as usual we write 0 for the cone point of X). Moreover, since this $\mathbb{G}_m$ action contracts Y to F, it follows by an argument of Slodowy [1980, Section IV] that L is a homotopy retract of Y, and so we have isomorphisms in cohomology $H^*(Y) \cong H^*(L)$. (Note however that in general, L is not smooth and $F \subsetneq L$.)

Definition 3.9 At a regular point $b \in B_Y$, that is, at a point outside the discriminant locus, the map $\tilde{\pi}$ is W-sheeted local isomorphism and we get an isomorphism between each fibre of θ_Y over the orbit $W \cdot b \subset B_Y$ and $\theta_X^{-1}(\pi(b))$. This induces an action of W on the cohomology of the fibres (for example induced by the action of W on the complex $(\theta_Y)_!(k_{\mathcal{Y}})$). But the deformation $\mathcal{Y}$ is topologically trivial (by Proposition 2.9), so the cohomologies of the fibres of θ_Y are canonically identified, and hence we get an action on the cohomology of Y. We will denote this action by $\rho : W \to \mathrm{GL}(H^*(Y))$. Note that the action on the fibres at regular points is precisely the one inducing the morphism a as in Theorem 3.2.

Lemma 3.10 *The restriction of ρ to $H^2(Y)$ agrees with the action of W on $B_Y = H^2(Y)$.*

Proof This follows from the construction of the period map, which is given by taking the homology class of the symplectic form of the fibres of ψ_Y. See for example Step 2 in the proof of Theorem 1.1 in [Namikawa 2010] for details. □

Lemma 3.11 *The action of W on $H^*(Y)$ via ρ coincides with the action of W on $\pi_!(\mathsf{k}_Y)_0 = H^*(L)$ via the isomorphism $H^*(L) \cong H^*(Y)$ described in Definition 3.9.*

Proof We have the diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\theta_Y} & B_Y & & \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} & & \\ \mathcal{X} & \xrightarrow{\theta_X} & B_X & \xrightarrow{=} & B_Y/W \end{array}$$

As already noted, the $\mathbb{G}_m$-action on X extends to the deformations $\mathcal{X}$ and $\mathcal{Y}$, and hence the sheaf $\tilde{\pi}_!(\mathsf{k}_{\mathcal{Y}})$ is equivariant with respect to the $\mathbb{G}_m$-action (since the constant sheaf $\mathsf{k}_{\mathcal{Y}}$ clearly is, and the morphism $\tilde{\pi}$ is equivariant). It follows that the natural morphism $H^*(\mathcal{X}, \tilde{\pi}_!(\mathsf{k}_{\mathcal{Y}})) \to \tilde{\pi}_!(\mathsf{k}_{\mathcal{Y}})_0$ is an isomorphism; see for example [Kaledin 2006, Lemma 5.6]. Now $H^*(\mathcal{Y}, \mathsf{k}) = H^*(\mathcal{X}, \tilde{\pi}_*(k_{\mathcal{Y}})) = H^*(\mathcal{X}, \tilde{\pi}_!(\mathsf{k}_{\mathcal{Y}})))$. But the map θ_Y is a topologically trivial fibration, so the restriction-to-fibres maps $H^*(\mathcal{Y}, k) \to H^*(\mathcal{Y}_t, k)$ are all isomorphisms. It follows readily that the composition

$$H^*(Y, k) = H^*(\mathcal{Y}_0, \mathrm{k}) \cong H^*(\mathcal{Y}, k) \to \tilde{\pi}_!(k)_0 \cong \pi_!(k)_0$$

is W-equivariant, as required. □

Corollary 3.12 *The symplectic Galois group W acts faithfully on the sheaf* Spr. *Moreover this action yields, for any $x \in X$, a natural action of W on the cohomology $H^*(\tilde{\pi}^{-1}(x))$.*

Proof The second assertion follows immediately from the action of W on Spr induced by the restriction of the action on HC, since the cohomology in question is just the stalk of Spr at $x \in X$. For the first part, note that it is enough to check that the induced action on some stalk cohomologies of Spr is faithful; but this follows for the stalk at $0 \in X$ by the preceding lemmas. Indeed, since $\pi^{-1}(0) = L$, these show that action of W on $H^2(\pi^{-1}(0))$ is just that of W on B_Y, which is faithful by construction. □

3.2 Stratifications and Thom's a_f condition for Poisson deformations

Let $X \subset U \subset \mathbb{C}^n$ be an analytic subset of U an open set in $\mathbb{C}^n$, and suppose $f : X \to \mathbb{C}$ is an analytic map, and $x \in f^{-1}(0)$. For $\delta \ll \varepsilon \ll 1$ the restriction of f to $f^{-1}(\dot{D}_\delta \backslash \{0\}) \cap B_\varepsilon(x)$ is a topological fibration. Here $\dot{D}_\delta$ is the open disc of radius δ centred at 0, and $B_\varepsilon(x)$ denotes the closed ball of radius ε around x (with respect to some Hermitian metric say). This was first proved by Lê Dũng Tráng [1977], generalising results of Milnor, who proved the result when X is a hypersurface in $\mathbb{C}^n$. The fibre of the resulting fibration is therefore known as the *Milnor fibre*. If we take $f : X \to Y$ where $\dim(Y)$ is greater than 1, the corresponding statement is false.

Example 3.13 Let $f : \mathbb{C}^3 \to \mathbb{C}^2$ be given by $f(x, y, z) = (x^2 - y^2 z, y)$. It can be checked that f is flat, however it is not a topological fibration in any ball around $0 \in \mathbb{C}^3$: indeed it is easy to see that some fibres are connected and some are not. (See the introduction to [Sabbah 1983] for a detailed discussion.)

However, one can impose a condition, going back to R Thom, on the map f which ensures that an analogue of Lê's result holds. In order to describe this we will need some basic material from stratification theory, as discussed for example in [Mather 2012]. First note that it follows from Theorem 2.3 that any symplectic variety is holonomic in the sense of Definition 2.2, and has finitely many symplectic leaves.

Lemma 3.14 *Let X be a symplectic variety. Then the canonical singularity stratification of X consists of the symplectic leaves of X and moreover it satisfies the Whitney conditions.*

Proof We have already seen in Theorem 2.3 that the (connected components of) the singularity stratification are the symplectic leaves of X, so it remains to check the Whitney conditions. To see this we use the basic fact which establishes the existence of Whitney stratifications: given any analytic variety X with a partition $X = \bigsqcup_{i \in I} X_i^0$, and strata X_i^0, X_j^0 satisfying $X_j^0 \subset X_i = \overline{X}_i^0$, the set of points $x \in X_j^0$ which do not satisfy the Whitney a and b condition is an analytic set of dimension strictly smaller than X_j^0. We call such points irregular.

But now if $x \in X_j^0 \subset X_i$ is a point of irregularity for a pair of strata X_i^0 and X_j^0, then its orbit under any Hamiltonian flow (which preserves both X_i^0 and X_j^0) will consist entirely of points of irregularity. It follows that the set of points of irregularity is a union of symplectic leaves in X. But then if the irregular locus is nonempty it must be all of X_j^0, contradicting the fact that it has smaller dimension than $\dim(X_j^0)$. □

Remark 3.15 The iterative procedure of taking (a component of) the smooth locus, removing it, and repeating until only smooth components remain yields, in the case of a holonomic variety X, the symplectic leaves of that variety, and the above shows that these form a Whitney stratification of X. The same procedure can of course be applied to an arbitrary singular variety, but in general it will only yield a partition of the variety into locally closed pieces, which need not satisfy the Whitney conditions.

Definition 3.16 Let M be a connected smooth complex analytic variety of dimension n and suppose that $f : M \to V$ is an analytic map, where $V \subset \mathbb{C}^r$ is a neighbourhood of $0 \in \mathbb{C}^r$. Let V^{reg} be the set of regular values of f and V^{sing} the set of nonregular values. We assume that U^{sing} is a proper analytic subvariety of V. Consider $M^0 = f^{-1}(U^{\mathrm{reg}})$, a smooth submanifold of M. Suppose that $E = f^{-1}(0)$ is equipped with a Whitney stratification $\mathcal{S}$. For $x \in M$, let $\mathrm{Fib}_x = f^{-1}(f(x))$ be the fibre of the map f which contains x. We will say that $(f, M, \mathcal{S})$ satisfies Thom's A_f condition if for each stratum $S \in \mathcal{S}$, whenever a sequence of points $(x_i)_{i=1}^{\infty}$ in M^0 converges to $y \in S$, is such that the limit

$$T = \lim_{i\to\infty} T_{x_i}\mathrm{Fib}_{x_i} \subset T_y M$$

exists, it contains the tangent space to S at y, that is, we have $T \supseteq T_y S$.

When (f, M, S) satisfies the A_f condition, then Grinberg [1998] (following an earlier observation of Lê Dũng Tráng [1977]) showed that it makes sense to talk about the nearby cycles sheaf of f, and the higher-dimensional base results in a richer monodromy action.

The construction is as follows: Let D be a neighbourhood of zero in $\mathbb{C}$, and let $\gamma : D \to U$ be an embedded analytic arc, such that $\gamma(0) = 0$ and $\gamma(t) \in M^0$ for $t \neq 0$. Form the fibre product $M_\gamma = M \times_\gamma D$, and let $f_\gamma : M_\gamma \to V$ be the obvious map. We may consider the nearby cycles sheaf $\psi_{f_\gamma}(\mathsf{k}_{M_\gamma}[d])$ (where d shifts appropriately to ensure perversity). Then we have the following:

Proposition 3.17 [Grinberg 1998, Proposition 2.4] *Let $(f, M, \mathcal{S})$ be as in Definition 3.16 and let $\gamma : D \to U$ be an embedded analytic arc as above.*

(i) *The sheaves $\psi_{f_\gamma}(\mathsf{k}_{M_\gamma[d]})$ for different γ are all isomorphic. We may therefore omit the subscript γ, and call the sheaf $\mathcal{P}_f$ the nearby cycles of f. It is a perverse sheaf on E, constructible with respect to $\mathcal{S}$.*

(ii) *The local fundamental group $\pi_1((U \cap B_\epsilon)$, where B_ϵ is a small ball around the origin, acts on $\mathcal{P}_f$ by monodromy. We denote this action by $\mu : \pi_1(U \cap B_\epsilon) \to \mathrm{Aut}(P_f)$.*

(iii) *The sheaf $\mathcal{P}_f$ is Verdier self-dual.*

Proof The monodromy action of $\pi_1(U \cap B_\epsilon)$ is the assertion of the proposition which is not standard, and we include Grinberg's proof for the convenience. If $x \in E$ is a point in the exceptional fibre lying in a stratum S, take a slice N at x transverse to $\mathcal{S}$ and take $0 < \epsilon \ll \delta \ll 1$ (chosen in decreasing order). Then for λ a regular value with $\|\lambda\| < \epsilon$ the *Milnor fibre* of f is $M_{f,N,x} = f^{-1}(\lambda) \cap N \cap B_{x,\delta}$, where $B_{x,\delta}$ is the ball of radius $\delta > 0$ centred at x. The stalk cohomology of P_{f_γ} at x is $H^*(M_{f,N,x})$ and these can be shown to vary as a local system using the argument in Section 1 of [Lê Dũng Tráng 1977]. □

Example 3.18 In the case of $T^*\mathcal{B}$, the resolution of the nilpotent cone, the symplectic deformation space is the Grothendieck resolution $\tilde{\mathfrak{g}}$ equipped with its canonical map to the universal Cartan $\mathfrak{h}$, which can be identified with $H^2(T^*\mathcal{B}, \mathbb{C})$ via Borel's description of $H^*(\mathcal{B})$. The symplectic Galois group W is the (usual) Weyl group of the algebraic group G, and the universal Poisson deformation of $\mathcal{N}$, the nilpotent cone, is $\mathfrak{g}$ the Lie algebra of the group G, equipped with the adjoint quotient map from $\mathfrak{g}$ to $\mathfrak{g} /\!/ G \cong \mathfrak{h}/W$.

Let $\mathcal{A}$ denote the union of the root hyperplanes in $\mathfrak{h}$. Then $U^{\text{sing}} = \mathcal{A}/W \subset \mathfrak{h}/W = B_X$ (and hence U^{reg} is its complement). The orbit stratification of $\mathcal{N}$ satisfies the A_f condition for the adjoint quotient map.

We will now show that this theory can be applied in our symplectic setting. We keep the notation of diagram (2-1) and of Theorem 2.12.

Definition 3.19 Let $B_X^{\text{reg}} = (H^2(Y, \mathbb{C}) \backslash \mathcal{D})/W$, and let $U = \theta_X^{-1}(B^{\text{reg}})$.

Lemma 3.20 *The symplectic leaves of X satisfy Thom's A_{θ_X} condition with respect to the regular locus U of θ_X.*

Proof That the set U is the regular locus of the morphism θ_X follows from [Namikawa 2015]; see Theorem 2.12.

Let $x \in S$ be a point in a symplectic leaf $S \subseteq X$, and suppose $(y_n)_{n \geq 1}$ is a sequence in U which converges to $x \in S$ with the tangent spaces $T_{y_i}\text{Fib}_{y_i}$ converging in a suitable Grassmannian to T. We must show that $T_x S \subseteq T$.

To show this, it suffices to produce, for $\eta \in T_x S$, a sequence of tangent vectors in $T_{y_i}\text{Fib}_{y_i}$ which converge to η. Since S is an integral Poisson subscheme, we may write the derivation given by η via the Poisson structure as $g \mapsto \{f_\eta, g\}(x)$ for some function f_η. Now picking any extension of this function to a function $\widetilde{f_\eta}$ on the variety $\mathcal{X}$ we see that if (y_n) is a sequence of points in U tending to x, since the fibre of $\tilde{\pi}$ at each y_n is smooth symplectic, the derivation $\eta_n = \{\widetilde{f_\eta}, \cdot\}(y_n)$ is tangent to Fib_{y_n} and clearly

$$\lim_{n \to \infty} \eta_n = \eta,$$

so that the tangent space $T_x S$ lies in the limit of the tangent spaces of the fibres of $\tilde{\pi}$ as required. □

Combining the previous lemma with Proposition 3.17 we obtain a perverse sheaf $\mathcal{P} = \mathcal{P}_{\theta_X}$ on X equipped with a monodromy action $\mu : \pi_1(B^{\text{reg}} \cap \mathbb{B}_\epsilon)$, where $\mathbb{B}_\epsilon$ is a small ball centred at $0 \in B_X$. By the conic nature of our setting, restricting to this ball does not change the fundamental group.

Theorem 3.21 *We have $\mathcal{P} \cong \pi_!(\mathsf{k}_Y)$, and the action of the braid group $\pi_1(U)$ factors through the symplectic Galois group W.*

Proof Let us first see that $\mathcal{P}$ is isomorphic to $\pi_!(\mathsf{k}_Y)$: pick a line $K \subseteq H^2(Y, \mathbb{C}))$ which intersects $\mathcal{D}$ only at $0 \in H^2(Y, \mathbb{C})$ (this is possible since $\mathcal{D}$ is a finite union of hyperplanes) and let $C = q(K)$ be the W-orbit of K, that is, its image under $q : H^2(Y, \mathbb{C}) \to H^2(Y, \mathbb{C})/W$ the quotient map. We may construct $\mathcal{P}$ by forming the (one-dimensional) nearby cycles for the restriction $\theta_C : \mathcal{X}_C \to C$ of the map θ_X to $\theta_X^{-1}(C)$. Let $\theta_K : \mathcal{Y}_K \to K$ denote the restriction of θ_Y to K, so that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_K & \xrightarrow{\tilde{\pi}_K} & \mathcal{X}_C \\ \theta_K \downarrow & & \downarrow \theta_C \\ K & \xrightarrow{q|_K} & C \end{array}$$

where $\tilde{\pi}_K$ denotes the restriction of $\tilde{\pi}$ to $\theta_Y^{-1}(K)$. Since the map $\tilde{\pi}_K$ is proper and the nearby cycles construction commutes with pushforward by $\tilde{\pi}$, we have

$$\psi_{\theta_C}(\tilde{\pi}_{K!}(\mathsf{k}_{\mathcal{Y}_K})) \cong \tilde{\pi}_{K!}(\psi_{\theta_K}(\mathsf{k}_{\mathcal{Y}_K})).$$

Now $\psi_{\theta_K}(\mathsf{k}_{\mathcal{Y}_K}) \cong \mathsf{k}_Y$, since, by Proposition 2.9, $\tilde{\pi} : \mathcal{Y} \to H^2(Y, \mathbb{C})$ is a topologically trivial fibre bundle. On the other hand, since $\tilde{\pi}$ is a local isomorphism over U^{reg} (see Definition 3.19) it follows that the nearby cycles sheaves $\psi_{\theta_C}(\tilde{\pi}_{K!}(\mathsf{k}_{\mathcal{Y}_K}))$ and $\psi_{\theta_C}(\mathsf{k}_{\mathcal{X}_C})$ are isomorphic, and hence $\mathcal{P} \cong \pi_!(\mathsf{k}_Y) = \mathsf{Spr}$. Finally, the topological triviality of the fibre bundle θ_Y shows that the action of the fundamental group on $\mathcal{P}$ must factor through W as required. □

Corollary 3.22 *The Springer sheaf* Spr *is constructible with respect to the stratification of X by its symplectic leaves. Moreover, the action of the monodromy action of the symplectic Galois group on the cohomology of the fibres of π respects the ring structure on cohomology.*

Proof This follows immediately from the isomorphism $\mathsf{Spr} \cong \mathcal{P}$ and the general fact that the nearby cycles sheaf is constructible with respect to a Whitney stratification satisfying the A_f-condition; see for example [Goresky and MacPherson 1988]. For the final sentence, the compatibility with the ring structure on cohomology comes from the fact that the nearby cycles construction shows that monodromy action actually acts on the homotopy type of the fibres of π. This is essentially contained in Slodowy's work, and explained in detail in the case of classical Springer theory in [Treumann 2009]. □

Remark 3.23 As the referee pointed out to us, it should be possible to deduce this using the Hamiltonian flows on a leaf.

Finally, we show that the monodromy action of the symplectic Galois group given by the nearby cycles construction is the same as that given by taking the intermediate extension.

Proposition 3.24 *The monodromy action of W on $\pi_!(\mathsf{k}_Y)$ induced by the isomorphism*

$$\mathcal{P} \cong \pi_!(\mathsf{k}_Y[\dim(Y)])$$

coincides with the action of W induced by the intermediate extension.

Proof By Lemmas 3.10 and 3.11 we see that the action of W on $H^*(\pi^{-1}(0)) = H^*(L)$ is faithful, where $L = \pi^{-1}(0)$. It thus suffices to compare the two actions on $H^*(L)$. But by Lemma 3.11 again, the intermediate extension action is given by the "naive action" of Definition 3.9 and it is easy to see that Grinberg's monodromy action also coincides with this action on $\pi^{-1}(0)$. □

3.3 Examples and a question

We continue with a discussion of some examples. For simplicity, we take $k = \mathbb{C}$.

Example 3.25 The simplest cases of symplectic singularities are the Kleinian singularities. Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, let $S = \mathbb{C}^2/\Gamma$ be the corresponding quotient singularity, and let $\pi : Y \to S$ be the minimal resolution. Now $\mathsf{Spr} = \pi_!(\mathbb{C}_Y[2])$ is a semisimple perverse sheaf. The only sheaves which can occur are the constant sheaf (which is perverse in this case because S is rationally smooth) and the skyscraper sheaf supported at the cone point. Using proper base change to compute the cohomology of the fibres, this shows that $\mathsf{Spr} \cong \mathbb{C}_S[2] \oplus \mathbb{C}_0^{\oplus r}$, where r is the number of components in the exceptional fibre. The action of the Weyl group is then via the trivial representation on $\mathbb{C}_S[2]$ and via the reflection representation on the summand supported on singular point. The action on $\mathbb{C}_0^{\oplus r}$ is just the action of W on $H^2(\widetilde{\mathbb{C}^2/\Gamma})$, and by Corollary 3.22 the W respects the intersection form, hence we see this is the reflection representation of W.

This example shows that in general one cannot expect that the map from $\mathbb{C}[W]$ to $\mathrm{End}(\mathsf{Spr})$ is injective (even though the map $W \to \mathrm{Aut}(\mathsf{Spr})$ is injective). Indeed as soon as W has more than two conjugacy classes (which is the case unless $W = Z_2$) the map from $\mathbb{C}[W]$ to $\mathrm{End}(\mathsf{Spr})$ will fail to be injective. (We thank the referee for suggesting we set out how stark the failure of injectivity is here.)

Example 3.26 Let $Y = T^*\mathbb{P}^n$ be the cotangent bundle of n-dimensional projective space. This can be described explicitly (viewing $\mathbb{P}^n$ as the space of lines in $\mathbb{C}^{n+1}$) as

$$Y = \{(\alpha, \ell) \in \mathfrak{gl}_{n+1} \times \mathbb{P}^n : \mathrm{im}(\alpha) \subseteq \ell \subseteq \ker(\alpha) \subseteq \mathbb{C}^{n+1}\}.$$

The natural map π given by the restriction of the projection to the first factor is the moment map for the natural GL_{n+1}-action on Y, and it realises Y as a symplectic resolution of its image $X = \overline{\mathbb{O}_{\min}}$ the closure of the minimal nonzero nilpotent orbit in $\mathfrak{gl}_{n+1}$. It is well-known and easy to see that $\dim(X) = 2n$, and moreover, if $\alpha \neq 0$ then $\ell = \mathrm{im}(\alpha)$, so that π is an isomorphism over $\mathbb{O}_{\min}$. Thus for $n > 1$ we see that X is smooth in codimension 2, so that its symplectic Galois group is trivial. On the other hand, for any coefficient field k, we have the Springer sheaf $\mathsf{Spr}_{\mathsf{k}} = \pi_!(\mathsf{k}_Y[2n])$ and, as noted in Lemma 3.8, its endomorphism algebra $\mathrm{End}(\mathsf{Spr})$ is isomorphic to the middle-dimensional Borel–Moore homology of $Z_Y = Y \times_X Y$, and it is easy to see that this has two components, C_1 and C_2 say, where C_1 is the fibre of the map $Z_Y \to Y$ over $0 \in \mathfrak{gl}_{n+1}$ and C_2 is the closure of the locus $\{((\alpha, \ell), (\alpha, \ell) \in Y \times Y : \alpha \neq 0\}$, so that $\mathrm{End}(\mathsf{Spr}_{\mathsf{k}}$ is 2-dimensional. However, a straightforward computation using the techniques of [Chriss and Ginzburg 1997] shows that, if we write $[C_i]$ for the class in Borel–Moore homology associated to the component C_i, then $[C_1] * [C_1] = (n+1)[C_2]$ (where $n + 1 = \chi(\mathbb{P}^n)$) and so $\mathsf{Spr}(\mathsf{k})$ is semisimple if and only if p and $n + 1$ are coprime.

This example also shows that the map $\mathbb{C}[W] \to \operatorname{End}(\mathsf{Spr})$ need not be surjective. (Note that the case $n = 1$ recovers the ordinary Springer sheaf for $\mathfrak{sl}_2$, where the Weyl group does of course control the semisimplicity of Spr.)

Example 3.27 Let $X = \operatorname{Sym}^n(\mathbb{A}^2)$ be the symmetric product of the affine plane $\mathbb{A}^2$, equipped with the usual scaling action of $\mathbb{G}_m$. It is well-known that a resolution of singularities is given by the Hilbert–Chow morphism $\pi : \operatorname{Hilb}^n(\mathbb{A}^2) \to \operatorname{Sym}^n(\mathbb{A}^2)$, and viewing $\mathbb{A}^2 = T^*\mathbb{A}^1$, it becomes a symplectic resolution.

The symmetric product X is naturally stratified into locally closed pieces X_ν given by partitions ν of n (the partition measures the multiplicities of points) and these are evidently integral Poisson subvarieties. Moreover, as in [Göttsche and Soergel 1993] for example, we have $\dim(X_\nu) = 2n - 2\ell(\nu)$, where $\ell(\nu)$ is the length of the partition ν (ie the number of parts in the partition). It follows that the only codimension-2 stratum is that labelled by $(2, 1^{n-2})$. In order to calculate the symplectic Galois group it is thus enough to calculate a slice through this stratum (and check monodromies). However, in our case since we know it must be a Kleinian singularity, and moreover the description of the fibres of the Hilbert–Chow morphism show that the resolution of the slice has exceptional fibre consisting of a single $\mathbb{P}^1$ (the punctual Hilbert scheme of two points) the slice must be a type A_1 singularity and hence $W = S_2$, the symmetric group on two letters.

On the other hand, it is shown in [Göttsche and Soergel 1993] that the Hilbert–Chow morphism is semismall with every stratum being relevant, and moreover

$$\pi_!(\mathbb{C}_Y[\dim(Y)]) = \bigoplus_\lambda \operatorname{IC}(X_\lambda),$$

where λ runs over the partitions of n. Thus the Springer sheaf Spr in this case is a direct sum of the simple perverse sheaves on X associated to the constant local systems on each stratum by the intermediate extension functor. Thus if $p(n)$ denotes the number of partitions of n, we see that for $n \geq 2$ there are thus $p(n)$ simple constituents in the Springer sheaf. On the other hand, the symplectic Galois group is S_2 for all n. It follows that $\operatorname{End}(\mathsf{Spr}_X) \cong \mathbb{C}^{p(n)}$, which is larger than $\mathbb{C}[S_2]$ provided $n \geq 2$, so that we obtain another example where the homomorphism from the group algebra to $\operatorname{End}(\mathsf{Spr}_X)$ fails to be surjective — indeed as $n \to \infty$ one can ensure that the failure is in some sense arbitrarily bad!

The last example also has another interesting consequence: Suppose for simplicity that we work over an algebraically closed field k for which the Springer sheaf is semisimple. Then it can be decomposed into a direct sum

$$\mathsf{Spr} \cong \bigoplus_{i \in I} \operatorname{IC}(S_i, \mathcal{L}_i) \otimes V_i, \tag{3-1}$$

where $i \in I$ runs over a finite set of local systems supported on symplectic leaves of X, and each V_i is a finite-dimensional vector space (which can be identified as usual with $\operatorname{Hom}(\operatorname{IC}(S_i, \mathcal{L}_i), \mathsf{Spr})$. The V_i thus become representations of W. In the classical theory, this yields a map from the irreducible representations of W to the set of local systems supported on nilpotent orbits, because each irreducible W-representation occurs uniquely as some V_i.

Notice that Example 3.27 shows that in our general setting, an irreducible representation of W the symplectic Galois group may be associated to more than one local system: Indeed when $n > 2$ there are $p(n) \geq n$ many local systems (the constant local systems on the various strata) but only two irreducible representations of S_2, the symplectic Galois group. Thus at least one of its two irreducible representations must be associated to more than one local system.

The following question was suggested to us by Gwyn Bellamy.

Question 3.28 When are the W-representations V_i in the isotypic decomposition (3-1) of Spr irreducible?

From the study of resolutions of slices in the affine Grassmannian in Section 7, we will obtain a large supply of examples where the irreducibility should be false in some sense "generically". We are very grateful to the referee for pointing out the following example, which also shows that a satisfactory answer to Bellamy's question will need to identify some natural conditions which can ensure the irreducibility of the representations V_i.

Example 3.29 Similarly to Example 3.27, consider $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$, where $\Gamma < \mathrm{SL}_2(\mathbb{C})$ is a nontrivial finite subgroup and $n \geq 2$. The symplectic Galois group is $W_\Gamma \times S_2$ (there are precisely two codimension-2 strata once $n \geq 2$, with slices equal to the slices we saw in Examples 3.25 and 3.27, respectively; for more details see Example 5.12.

If c denotes the number of conjugacy classes in W, the middle-dimensional homology of $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$ has dimension

$$\dim\bigl(H^{\mathrm{mid}}(\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}))\bigr) = \binom{n+c-2}{c-2}.$$

Moreover, by the bounds on the stalk cohomologies of intersection cohomology sheaves, this cohomology is precisely V_0, where 0 denotes the stratum of the cone point of $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ (indeed we saw in Example 3.25, the case of the minimal resolution of a Kleinian singularity, that the middle cohomology of the smooth surface was attached to the skyscraper sheaf at the singularity). Clearly, provided $|W| > 2$ so that $c > 2$, this grows with n, and so must eventually become a reducible representation of W.

4 Symplectic geometry of quiver varieties

We investigate symplectic features of Nakajima's quiver varieties. While much of this section is likely well understood by experts, we document the details where we could not find suitable references.

4.1 Basic set-up

We briefly recall the setting, using [Nakajima 1998] as our principal reference. Let $Q = (I, \Omega)$ be a quiver with vertex set I and edge set Ω, such that the edges $h \in \Omega$ are directed with starting vertex $s(h)$ and terminating vertex $t(h)$. The *adjacency matrix* of a quiver Q is the matrix $A = A_Q$ with entries $a_{ij} = |\{h \in \Omega : s(h) = i, t(h) = j\}|$. Let $\overline{Q}$ denote the doubled quiver, ie the quiver with vertex set I but edge set $H = \Omega \sqcup \overline{\Omega}$, where $\overline{\Omega}$ is in bijection with Ω via a map $h \mapsto \overline{h}$ so that $s(\overline{h}) = t(h)$, and $t(\overline{h}) = s(h)$.

For convenience we treat $\overline{(\cdot)}$ as an involution on H by setting $\overline{(\overline{h})} = h$. Note that $A_{\overline{Q}} = A_Q + A_Q^{\intercal}$ is symmetric.

For a doubled quiver $\overline{Q}$ we define the associated (symmetric) *Cartan matrix* $C = C_{\overline{Q}} = (c_{ij})$ to be $2I - A_{\overline{Q}}$. Let $R = \mathbb{Z}[I] = \{f : I \to \mathbb{Z}\}$ and $P = \operatorname{Hom}(R, \mathbb{Z})$ be its dual. For $i \in I$ we write $\breve{\alpha}_i$ for the indicator function of $i \in I$, and Λ_i for the corresponding element of the basis of P dual to $\{\breve{\alpha}_i : i \in I\}$. Now R carries a symmetric bilinear form $\langle\ ,\ \rangle$ given by $\langle \breve{\alpha}_i, \breve{\alpha}_j\rangle = c_{ij}$, and using this form one defines "reflections" $s_i : R \to R$, where $s_i(x) = x - \langle x, \breve{\alpha}_i\rangle\breve{\alpha}_i$ for all $x \in R$. The group $W(\overline{Q})$ is then defined to be the subgroup of $\operatorname{Aut}(R)$ generated by the $\{s_i : i \in I\}$. Note that $W(\overline{Q})$ acts on P in the usual way via $w(\lambda)(x) = \lambda(w^{-1}(x))$. The bilinear form $\langle\ ,\ \rangle$ yields a map $\theta_{\boldsymbol{R}} : R \to P$ given by $\theta_{\boldsymbol{R}}(x)(y) = \langle x, y\rangle$, and we will write α_i for $\theta_{\boldsymbol{R}}(\breve{\alpha}_i)$. We will also write $\Lambda_{\boldsymbol{w}} = \sum_{i\in I} w_i\Lambda_i$ and $\alpha_{\boldsymbol{v}} = \sum_{i\in I} v_i\alpha_i$. If $\nu = \sum_{i\in I} \nu_i\alpha_i \in R$ we write $\operatorname{supp}_{\boldsymbol{R}}(\nu) = \{i \in I : \nu_i \neq 0\}$, and similarly for $\mu = \sum_{i\in I} \mu_i\Lambda_i \in P$ we write $\operatorname{supp}_{\boldsymbol{P}}(\mu) = \{i \in I : \lambda_i \neq 0\}$.

A quiver variety is defined by a pair $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^I_{\geq 0}$. Given $\boldsymbol{v}, \boldsymbol{w}$, choose I-graded vector spaces V and W with $\dim(V_i) = \boldsymbol{v}_i$, $\dim(W_i) = \boldsymbol{w}_i$ for each $i \in I$, and let

$$\boldsymbol{M}(\boldsymbol{v}, \boldsymbol{w}) = \bigoplus_{h\in H} \operatorname{Hom}(V_{s(h)}, V_{t(h)}) \oplus \bigoplus_{i\in I} \operatorname{Hom}(W_i, V_i) \oplus \bigoplus_{i\in I} \operatorname{Hom}(V_i, , W_i).$$

We will write an element of this space as a triple (x, p, q). Picking a function $\varepsilon : H \to \{\pm 1\}$ such that $\varepsilon(h) + \varepsilon(\overline{h}) = 0$, the formula

$$\omega\big((x, p, q), (x', p', q')\big) = \sum_{h\in H} \operatorname{tr}(\varepsilon(h)x_h x'_{\overline{h}}) + \sum_{i\in I} \operatorname{tr}(p'_i q_i + p_i q'_i)$$

defines a symplectic form on $M(\boldsymbol{v}, \boldsymbol{w})$ for which the action of the group $G_{\boldsymbol{v}} = \prod_{i\in I} \mathrm{GL}(V_i)$ is Hamiltonian, with moment map

$$\mu(x, p, q) = \bigg(\sum_{h:s(h)=i} \varepsilon(h)x_{\overline{h}}x_h + pq\bigg)_{i\in I} \in \mathfrak{g}_{\boldsymbol{v}}^* = \bigoplus_{i\in I} \mathfrak{gl}(V_i)$$

(where we identify $\mathfrak{gl}_V$ with its dual using the trace pairings). We may then consider two kinds of quotients of $M(\boldsymbol{v}, \boldsymbol{w})$ by the action of $G_{\boldsymbol{v}}$. The first is the affine quotient, which we will denote by $X(\boldsymbol{v}, \boldsymbol{w})$. That is,

$$X(\boldsymbol{v}, \boldsymbol{w}) = \operatorname{Spec}(\mathbb{C}[\mu^{-1}(0)]^{G_{\boldsymbol{v}}}).$$

Since it is known since Hilbert that the invariant functions separate closed orbits in this setting, the points of $X(\boldsymbol{v}, \boldsymbol{w})$ thus correspond to the closed $G_{\boldsymbol{v}}$-orbits in $\mu^{-1}(0)$. Notice that the Poisson bracket associated to the symplectic form ω is clearly preserved by the action of $G_{\boldsymbol{v}}$, so that it induces a Poisson structure on $\mathbb{C}[\mu^{-1}(0)]^{G_{\boldsymbol{v}}}$, and hence $X(\boldsymbol{v}, \boldsymbol{w})$ is a Poisson variety.

The second quotient is the GIT quotient determined by the linearisation of the trivial line bundle given by the determinant character χ (more generally, one could take any sufficiently general character of $G_{\boldsymbol{v}}$):

$$Y(\boldsymbol{v}, \boldsymbol{w}) = \operatorname{Proj}\bigg(\bigoplus_{n\geq 0} \mathbb{C}[\mu^{-1}(0)]^{\chi^n}\bigg).$$

In [Nakajima 1998] it is shown that $Y(\boldsymbol{v}, \boldsymbol{w})$ coincides with the quotient of $\mu^{-1}(0)^s$ by the action of $G_{\boldsymbol{v}}$, where $\mu^{-1}(0)^s$ is the set of stable points of $\mu^{-1}(0)$, and $G_{\boldsymbol{v}}$ acts freely there, so the quotient is again an algebraic symplectic manifold. The description of $Y(\boldsymbol{v}, \boldsymbol{w})$ immediately shows that there is a projective morphism $\pi : Y(\boldsymbol{v}, \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$. It is easy to check also that the variety $Y(\boldsymbol{v}, \boldsymbol{w})$, if nonempty, has dimension $2\boldsymbol{v}^t\boldsymbol{w} - \boldsymbol{v}^t C\boldsymbol{v}$, where $C = C_{\overline{Q}}$ is the Cartan matrix of the doubled quiver.

Remark 4.1 Our notation, as is common in the subject, labels quiver varieties by the dimension vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ despite the definition using a choice of vector spaces of those dimensions. Since different choices of vector spaces yield isomorphic varieties however, the imprecision is harmless.

4.2 Affine quiver varieties are symplectic

Let (Q, I, H) be a quiver and $\boldsymbol{v}, \boldsymbol{w}$ be dimension vectors. Let $Y = Y(\boldsymbol{v}, \boldsymbol{w})$, respectively $X = X(\boldsymbol{v}, \boldsymbol{w})$, be the smooth symplectic quiver variety and affine quiver variety associated to the dimension vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ and let $\pi : Y \to X$ denote the natural projective morphism.

We first investigate the affine symplectic variety X. Note that since the morphism $\pi : Y \to X$ is not in general birational, the following is not obvious.

Theorem 4.2 [Bellamy and Schedler 2021, Theorem 1.2] *An affine quiver variety X is a symplectic variety.*

As mentioned in the introduction, Bellamy and Schedler [2021] also give explicit criteria for when an affine quiver variety admits a symplectic resolution.

It follows from the above theorem that every affine quiver variety is holonomic. In fact, one can give an explicit description of the symplectic leaves. Recall from [Nakajima 1998] that X is stratified by stabiliser-type (in fact it is an example of a Luna stratification): a point $z \in X$ corresponds to a closed orbit in $\mu^{-1}(0)$. Let $(x, p, q) \in \mu^{-1}(0)$ be a point lying in this orbit and let H denote its stabiliser in $G_{\boldsymbol{v}}$. Since the orbit of (x, p, q) is closed the subgroup H is a reductive subgroup of $G_{\boldsymbol{v}}$. If (H) denotes the conjugacy class of the subgroup H in $G_{\boldsymbol{v}}$ we say that (H) is the stabiliser type of $z \in X$. Conversely, given a conjugacy class (H) of reductive subgroups of $G_{\boldsymbol{v}}$ we let $X_{(H)}$ be the subset of all points of X of stabiliser type (H). This yields a finite[4] stratification $X = \bigsqcup_{(H)} X_{(H)}$.

Remark 4.3 This stratification is a special case of a stratification introduced by Luna [1973]. In the context of Hamiltonian reductions in the differentiable setting it was first studied by Sjaamar and Lerman [1991], who establish in that context its description in terms of symplectic leaves.

Lemma 4.4 *The variety X is a holonomic variety in the sense of Kaledin. Its symplectic leaves are precisely the strata of the orbit stratification in the sense of* [Nakajima 1998].

Proof As already noted, the variety X carries a natural Poisson variety structure. Martino [2008] showed that the strata of the orbit stratification of X are precisely the symplectic leaves of X, and hence X has only finitely many symplectic leaves. It follows from the discussion in Section 2.1 that X is holonomic. □

[4] There are finitely many conjugacy classes of such reductive subgroups, as can be seen by root combinatorics.

Remark 4.5 As noted in [Bellamy and Schedler 2021], the argument of [Martino 2008] incorrectly invokes a result of Schwarz to show that the orbit-type strata are connected. However, this gap is readily filled using the connectedness of quiver varieties as is explained in [Bellamy and Schedler 2021, Corollary 3.25].

4.3 Smooth quiver varieties are symplectic resolutions

While it is not, in general, true that an affine quiver variety has a symplectic resolution, it is nonetheless the case that a smooth symplectic quiver variety $Y(\boldsymbol{v}, \boldsymbol{w})$ *is* always a symplectic resolution — although not necessarily of the associated affine quiver variety $X(\boldsymbol{v}, \boldsymbol{w})$.

Proposition 4.6 *The smooth quiver variety $Y(\boldsymbol{v}, \boldsymbol{w})$ is a symplectic resolution.*

Although this seems to be known to experts, it does not seem to be documented anywhere. We thank the referee for suggesting the current organisation of our proof, and in particular for pointing out the virtue of first establishing the following lemma, which appears to be folklore, but we could not find a reference.

Lemma 4.7 *Suppose that Y is a smooth symplectic quasiprojective variety. Then Y is a symplectic resolution if and only if it is birational to its affinisation.*

Proof If $\pi : Y \to X$ exhibits Y as a symplectic resolution of an affine symplectic variety X, it follows from Grauert–Reimenschneider that $\pi_*(\mathbb{O}_Y) = \mathbb{O}_X$; hence Y is the affinisation of X, as required.

For the converse, suppose that the morphism $a : Y \to Y^{\mathrm{aff}}$ is birational. Let $V \subseteq Y^{\mathrm{aff}}$ denote the locus over which a is not an isomorphism. Since the affinisation of a smooth variety is automatically normal, we may apply Zariski's main theorem to deduce that the complement of V in N is codimension 2 in Y^{aff}. Now the Poisson bivector associated to the symplectic form of Y induces, through a the affinisation map, a Poisson structure on V. Since Y^{aff} is normal, and the complement of V is codimension 2 in Y^{aff} it follows that the Poisson structure extends to all of Y^{aff}, and hence a is a symplectic resolution. □

With the lemma in hand, Proposition 4.6 will follow if we can show that $Y = Y(\boldsymbol{v}, \boldsymbol{w})$ is birational to its affinisation. In general, this need *not* be $X(\boldsymbol{v}, \boldsymbol{w})$, but this can be rectified using reflection functors, as we now briefly review:

We have defined quiver varieties, following [Nakajima 1998], using the notion of stability provided by determinant character, but any choice of character θ of $G_{\boldsymbol{v}}$ yields varieties $Y_\theta(\boldsymbol{v}, \boldsymbol{w})$ and $X_\theta(\boldsymbol{v}, \boldsymbol{w})$. Now the existence of polystable representations for θ will in general mean that $Y_\theta(\boldsymbol{v}, \boldsymbol{w})$ need not be smooth, but for all suitably generic θ will possess no such representations and thus for all such generic θ the variety $Y_\theta(\boldsymbol{v}, \boldsymbol{w})$ is smooth. Note that we may identify the abelian group of characters of $G_{\boldsymbol{v}}$ with $R = \mathbb{Z}[I]$. Explicitly, if θ is a character of $G_{\boldsymbol{v}}$ and $g = (g_i)_{i \in I} \in G_{\boldsymbol{v}}$ then $\theta(g) = \prod_{i \in I} \det(g_i)^{n_i}$ and $\theta \mapsto \sum_{i \in I} n_i \check{\alpha}_i$.

Reflection functors for quiver varieties have been studied by Lusztig [2000], Nakajima [2003] and Maffei [2002]. In the algebrogeometric (as opposed to hyper-Kähler) setting these provide, for each vertex $i \in I$ an isomorphism of varieties $s_{i,w} : Y_\theta(\boldsymbol{v}, \boldsymbol{w}) \to Y_{s_i(\theta)}(s_i * \boldsymbol{v}, \boldsymbol{w})$. Here $s_i(\theta)$ is defined by viewing θ as an element of R, and $s_i * \boldsymbol{v}$ is defined so that, if $\alpha_{\boldsymbol{v}} = \sum_{i \in I} v_i \alpha_i$ and $\Lambda_{\boldsymbol{w}} = \sum_{i \in I} w_i \Lambda_i$, then $\Lambda_{\boldsymbol{w}} - \alpha_{s_i * \boldsymbol{v}} = s_i(\Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}})$.

Proposition 4.8 *Let $Y = Y(\boldsymbol{v}, \boldsymbol{w})$. The affinisation map $a : Y \to Y^{\mathrm{aff}}$ is a birational morphism.*

Proof Using the isomorphisms induced by reflection functors, it is shown in Proposition 46 of [Maffei 2002] that any quiver variety $Y_\theta(\boldsymbol{v}, \boldsymbol{w})$ is isomorphic to a quiver variety $Y_{\theta'}(\boldsymbol{v}', \boldsymbol{w})$ with the property that $(\Lambda_w - \alpha_v)(\check{\alpha}_i) \geq 0$ is, in the terminology of that paper, *dominant*. Moreover, the proof of Lemma 54 of that paper shows that, in this case, $\mu^{-1}(0)^{\theta'}$ is open dense in $\mu^{-1}(0)$, hence $Y_{\theta'}(\boldsymbol{v}', \boldsymbol{w})$. Since we work with generic θ (the action of $W(\bar{Q})$ preserves the locus of generic stability conditions) the action of $G_{\boldsymbol{v}'}$ on $\mu^{-1}(0)^{\theta'}$ is free, and hence it follows that $\pi_{\boldsymbol{v}'} : Y_{\boldsymbol{v}'}(\boldsymbol{v}', \boldsymbol{w})$ is birational. Since $\pi_{\boldsymbol{v}'}$ is also projective, and thus its image is closed, it follows that $\pi_{\boldsymbol{v}'}$ is a surjective birational morphism. Since $X(\boldsymbol{v}', \boldsymbol{w})$ is affine, if is clear that the affinisation map a factors through $\pi_{\boldsymbol{v}'}$, and since $\pi_{\boldsymbol{v}'}$ is birational, it follows a fortiori that a is birational. □

Remark 4.9 The preceding proof shows slightly more than is asserted in the statement: the smooth symplectic variety $Y(\boldsymbol{v}, \boldsymbol{w})$ is a symplectic resolution of the an affine quiver variety. The authors thank the referee for alerting them to the results of [Maffei 2002], which simplified considerably our argument in the proof of the preceding proposition.

Finally, we note that a smooth quiver variety is always a *conical* symplectic resolution. Indeed, if we let $\mathbb{G}_m$ act on the vector space $M(\boldsymbol{v}, \boldsymbol{w})$ with weight 1, we obtain $\mathbb{G}_m$-actions on $X(\boldsymbol{v}, \boldsymbol{w})$ and $Y(\boldsymbol{v}, \boldsymbol{w})$ for which π is equivariant. The action has weight 2 on the symplectic form on $Y(\boldsymbol{v}, \boldsymbol{w})$. The following is then clear:

Lemma 4.10 *The induced $\mathbb{G}_m$-action on $Y(\boldsymbol{v}, \boldsymbol{w})$ gives $Y(\boldsymbol{v}, \boldsymbol{w})$ the structure of a conical symplectic resolution.*

Remark 4.11 The $\mathbb{G}_m$-action need not be unique: in general, the automorphism group of a quiver variety may contain a higher-rank torus, and many of its one-parameter subgroups may yield conical structures on the quiver variety. Such group actions (or *flavour symmetries* of supersymmetric gauge theories) play a central role in the beautiful story of symplectic duality discovered by Braden, Licata, Proudfoot and Webster [Braden et al. 2016].

It should also be noted that Nakajima [1998] uses a different $\mathbb{G}_m$-action, which depends on a choice of orientation Ω of the quiver. For quivers with no oriented loops it is a conical action. See [Nakajima 1998, Proposition 3.22] for more details.

4.4 Universal Hamiltonian reduction and the universal Poisson deformation

The moment map for the action of $G_{\boldsymbol{v}}$ on $M(\boldsymbol{v}, \boldsymbol{w})$ takes values in $\mathfrak{g}_{\boldsymbol{v}}^*$. We have

$$\mathfrak{z}_{\boldsymbol{v}} = (\mathfrak{g}_{\boldsymbol{v}} / [\mathfrak{g}_{\boldsymbol{v}}, \mathfrak{g}_{\boldsymbol{v}}])^* \subseteq \mathfrak{g}_{\boldsymbol{v}}^*.$$

We may thus consider the flat family of reductions $\mu^{-1}(\mathfrak{z}_{\boldsymbol{v}}) /\!\!/_{\chi} G_{\boldsymbol{v}}$ over $\mathfrak{z}_{\boldsymbol{v}}$, and it follows from the universal property that this deformation has an associated period map $\kappa : \mathfrak{z}_{\boldsymbol{v}} \to H^2(Y(\boldsymbol{v}, \boldsymbol{w}))$. It is known that the map κ is actually given by the Chern character map: for details see [Losev 2012].

Theorem 4.12 [McGerty and Nevins 2018, Theorem 1.1] *For any quiver variety the map*

$$H^*_{G_{\boldsymbol{v}}}(\star) \cong H^*_{G_{\boldsymbol{v}}}(E_V) \to H^*_{G_{\boldsymbol{v}}}(\mu^{-1}(0)^{\mathrm{ss}}) \to H^*(Y(\boldsymbol{v}, \boldsymbol{w}))$$

is surjective.

Corollary 4.13 *The period map κ is surjective.*

Thus, up to identifying the kernel of κ, we obtain a realisation of the universal graded Poisson deformation.

5 Quiver varieties of type ADE

In this section, we provide refined information on the quiver varieties associated to finite-type simply laced diagrams. Thus, from now on, we will assume that the underlying graph of our quiver Q is a simply laced Dynkin diagram associated to a Lie algebra $\mathfrak{g} = \mathfrak{g}_Q$. As we noted before, the results in this subsection are presumably well known to experts. We use the notation introduced in Section 4.1.

Let $X(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{reg}}$ be the locus of points in $X(\boldsymbol{v}, \boldsymbol{w})$ corresponding to closed $G_{\boldsymbol{v}}$-orbits in $\mu^{-1}(0)$ with trivial stabiliser. (This is an open but possibly empty subset of $X(\boldsymbol{v}, \boldsymbol{w})$.) Note that, if we fix $\boldsymbol{w}$, then if $\boldsymbol{v}'$ has $v'_i \le v_i$ we may choose V'_i as a direct summand of V_i, and hence extending by zero yields a natural map $X(\boldsymbol{v}', \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$ which is injective.

Proposition 5.1 *For an affine quiver variety X attached to a simply laced finite-type Dynkin diagram, the orbit strata (and hence the symplectic leaves) are given by*

$$X(\boldsymbol{v}, \boldsymbol{w}) = \bigsqcup_{\boldsymbol{v}'} X(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{reg}},$$

where the $\boldsymbol{v}'$ that occur as nonempty strata are those for which both $v'_i \le v_i$ for all $i \in I$ and such that $\Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}'} \in P^+$.

Proof This is almost contained in [Nakajima 1998]. Indeed it is shown in Remark 3.28 of that paper that for quiver of type ADE, the symplectic leaves are all of the form $X(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{reg}}$, and moreover Corollary 10.8 therein shows that $X(\boldsymbol{v}, \boldsymbol{w})^{\mathrm{reg}}$ is nonempty if and only if $\Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}} \in P^+$, and $\Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}}$ is a weight of the irreducible representation $L(\Lambda_{\boldsymbol{w}})$ of highest weight $\Lambda_{\boldsymbol{w}}$. But it is known that the weights $\Pi(\Lambda_{\boldsymbol{w}})$ of the irreducible representation $L(\Lambda_{\boldsymbol{w}})$ are, in the sense of [Humphreys 1972, Section 13.4], a saturated set with highest weight $\Lambda_{\boldsymbol{w}}$, and so, by Lemma B of [loc. cit.] a weight μ lies in $P^+ \cap \Pi(\Lambda_{\boldsymbol{w}})$ if and only if $\Lambda_{\boldsymbol{w}} - \mu \in Q^+$. Applying this to $\mu = \Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}}$ gives the result. □

Finally we note that in the finite-type case, $X(\boldsymbol{v}, \boldsymbol{w})$ always has a symplectic resolution.

Proposition 5.2 *Let $X(\boldsymbol{v}, \boldsymbol{w})$ be a finite-type affine quiver variety. Then there is a $\boldsymbol{v}' \in \mathbb{Z}^I$ with $\alpha_{\boldsymbol{v}'} \leq \alpha_{\boldsymbol{v}}$ such that $X(\boldsymbol{v}', \boldsymbol{w}) \cong X(\boldsymbol{v}, \boldsymbol{w})$ and $\pi_{\boldsymbol{v}'} : Y(\boldsymbol{v}', \boldsymbol{w}) \to X(\boldsymbol{v}', \boldsymbol{w})$ is a symplectic resolution.*

Proof By Proposition 5.1 there is a $\boldsymbol{v}'$ with $X(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{reg}}$ dense and open in $X(\boldsymbol{v}, \boldsymbol{w})$. It follows that the natural injective morphism $i : X(\boldsymbol{v}', \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$ is surjective, and hence bijective. Since by the main result of [Crawley-Boevey 2003] the variety $X(\boldsymbol{v}, \boldsymbol{w})$ is normal, it follows that i is an isomorphism.

By Lemma 4.7, we must show that $\pi_{\boldsymbol{v}'}$ is the affinisation morphism of $Y(\boldsymbol{v}', \boldsymbol{w})$. Now [Nakajima 1998, Corollary 10.11] shows that the projective morphism $\pi_{\boldsymbol{v}}$ is a semismall resolution of its image, and moreover all strata contained in the image are relevant. Since $\dim(X(\boldsymbol{v}', \boldsymbol{w})) = \dim(Y(\boldsymbol{v}', \boldsymbol{w}))$, it follows that $\pi_{\boldsymbol{v}'}$ is dominant, and so the image is all of $X(\boldsymbol{v}', \boldsymbol{w})$. Since $X(\boldsymbol{v}', \boldsymbol{w})$ is affine, the morphism $\pi_{\boldsymbol{v}'}$ factors through the affinisation morphism $a : Y(\boldsymbol{v}', \boldsymbol{w}) \to Y(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{aff}}$, and the induced map $\pi'_{\boldsymbol{v}'} : Y(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{aff}} \to X(\boldsymbol{v}', \boldsymbol{w})$ is birational. Moreover, since $X(\boldsymbol{v}', \boldsymbol{w})$ is normal, the fibres of $\pi'_{\boldsymbol{v}'}$ are connected. Since they must also be proper and $Y(\boldsymbol{v}', \boldsymbol{w})^{\mathrm{aff}}$ is affine, they must be points. It follows that $\pi'_{\boldsymbol{v}'}$ is a bijective proper morphism, so again by the normality of $X(\boldsymbol{v}', \boldsymbol{w})$, it is an isomorphism, hence $\pi_{\boldsymbol{v}'}$ is the affinisation morphism for $Y(\boldsymbol{v}', \boldsymbol{w})$, as required. □

The above highlights one elementary criterion for a projective morphism to be a resolution of singularities, namely that it must at least be surjective! Despite its trivial nature, this criterion is essentially the only way the natural morphisms $\pi : Y(\boldsymbol{v}, \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$ can fail to be a resolution, and examples where this happens are not difficult to find even in low dimensions:

Example 5.3 Take Q to be the quiver with one node, and let $v = w = 2$. Then $Y(2, 2)$ has dimension 0 (and so is a point) while $X(2, 2)$ is the rational singularity $\mathbb{C}^2/\{\pm 1\}$, and thus π is not surjective. An explicit calculation shows that the zero fibre $\mu^{-1}(0)$ of the moment map has three irreducible components, two of which are picked out by stability conditions. These components each contain a free $G_{\boldsymbol{v}}$-orbit however, so that the invariant functions are constant on each, while on the third component there are no free orbits, and there are nonconstant $G_{\boldsymbol{v}}$-invariant functions on it.

Remark 5.4 In fact, the argument of the previous proposition can be modified to show that provided the map $\pi : Y(\boldsymbol{v}, \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$ is surjective, it will be a symplectic resolution. Moreover, since $X(\boldsymbol{v}, \boldsymbol{w})$ is irreducible and π is proper, it suffices to check that $\dim(Y(\boldsymbol{v}, \boldsymbol{w})) = \dim(X(\boldsymbol{v}, \boldsymbol{w}))$. We thank the referee for this clarifying remark.

If, on the other hand, π fails to be surjective, then a second difficulty arises: It is not clear (at least to the authors) that, in general, the Poisson subvarieties of $X(\boldsymbol{v}, \boldsymbol{w})$ (one of which must be the image of π) must be normal. Indeed this is false for arbitrary symplectic resolutions, as it is well-known that nilpotent orbit closures outside of type A can fail to be normal.

5.1 Weyl groups and symplectic Galois groups of quiver varieties of type ADE

We now compute the symplectic Galois groups for type ADE quiver varieties. As before, we let $\mathfrak{g}_Q$ denote the semisimple Lie algebra attached to the simply laced Dynkin diagram underlying the quiver Q. Nakajima [1998] has shown how to construct the irreducible highest-weight representations of $\mathfrak{g}_Q$ in the cohomology of quiver varieties, where if $\Lambda_{\boldsymbol{w}}$ is the highest weight, the variety $Y(\boldsymbol{v},\boldsymbol{w})$ corresponds to the weight space of weight $\Lambda_{\boldsymbol{w}}-\alpha_{\boldsymbol{v}}$. It turns out that if $W=W_Q$ is the Weyl group attached to the Lie algebra $\mathfrak{g}_Q$, then the symplectic Galois group of the symplectic resolution $Y(\boldsymbol{v},\boldsymbol{w})$ is simply the parabolic subgroup of the W_Q which stabilises this weight. The corresponding assertion can fail for simply laced quivers Q of affine type; see Example 5.12 for more details.

Let us introduce some notation: recall that the roots of $\mathfrak{g}$ are denoted by Φ. If $\lambda\in P$, then we write

$$\Phi_\lambda=\{\alpha\in\Phi:(\lambda,\alpha)=0\}$$

for the parabolic subroot system associated to λ, and Φ_λ^+ for the positive roots in Φ_λ. Let us denote by $\Phi_\lambda^{\max}$ the roots in Φ_λ which are maximal in Φ_λ with respect to the partial order on R^+.

Lemma 5.5 *Let $X(\boldsymbol{v},\boldsymbol{w})$ be a quiver variety such that $X(\boldsymbol{v},\boldsymbol{w})^{\mathrm{reg}}\neq\varnothing$. Then if $\mu=\Lambda_{\boldsymbol{w}}-\alpha_{\boldsymbol{v}}$, the codimension two strata of $X(\boldsymbol{v},\boldsymbol{w})$ are indexed by the roots in Φ_μ^{max}.*

Proof Recall that $X(\boldsymbol{v}',\boldsymbol{w})^{\mathrm{reg}}\neq\varnothing$ if and only if $\Lambda_{\boldsymbol{w}}-\alpha_{\boldsymbol{v}'}\in P^+$, and this is a stratum of $X(\boldsymbol{v},\boldsymbol{w})$ if $\alpha_{\boldsymbol{v}'}\leq\alpha_{\boldsymbol{v}}$. Let $\alpha_{\boldsymbol{v}}=\alpha_{\boldsymbol{v}'}+\beta$, and let $\mu=\Lambda_{\boldsymbol{w}}-\alpha_{\boldsymbol{v}}$. Then if $\Lambda_{\boldsymbol{w}}-\alpha_{\boldsymbol{v}'}\in P^+$, the dimension formula for quiver varieties shows that $X(\boldsymbol{v}',\boldsymbol{w})^{\mathrm{reg}}$ is a codimension two stratum precisely when

$$2(\mu,\beta)+(\beta,\beta)=2.$$

Since $\mu\in P^+$ and $\beta\in R^+$, we must have $(\mu,\beta)=0$ and $(\beta,\beta)=2$, that is, $\beta\in\Phi_\mu$, the parabolic subsystem of Φ corresponding to μ. Thus the codimension two strata are labelled by roots $\beta\in\Phi_\mu^+$ for which $\mu+\beta\in P^+$.

We now claim that this set is exactly $\Phi_\mu^{\max}$. Indeed if β is such that $\mu+\beta\in P^+$, then for any $\alpha\in\Phi_\mu^+$ we have $0\leq(\mu+\beta,\alpha)=(\beta,\alpha)$, and hence (as our root system is simply laced, so root strings have length at most 2) it follows that $\beta\in\Phi_\mu^{\max}$.

Conversely, if $\beta\in\Phi_\mu^{\max}$, and $\alpha_i\in\Phi$ is a simple root, consider $(\mu+\beta,\alpha_i)$. If $\alpha_i\in\Phi_\mu$ then $(\beta,\alpha_i)\geq0$ since β is maximal (otherwise $s_{\alpha_i}(\beta)=\beta+\alpha_i\in\Phi_\mu$) and so $(\mu+\beta,\alpha_i)\geq0$. If $\alpha_i\notin\Phi_\mu$, then since $\mu\in P^+$ we must have $(\mu,\alpha_i)\geq1$, and since our root system is simply laced, $(\alpha_i,\beta)\geq-1$, and hence $(\mu+\beta,\alpha_i)\geq0$ in this case also. □

Lemma 5.6 *Suppose that $X(\boldsymbol{v},\boldsymbol{w})$ is as above, and $\beta\in\Phi_\mu^{max}$. Then the slice through the stratum associated to β by Lemma 5.5 is isomorphic to the Kleinian singularity indexed by the subroot system $\Phi(\beta)=\{\alpha\in\Phi:\alpha\leq\beta\}$, an indecomposable subsystem of Φ.*

Proof Nakajima [2001] constructs an (analytic) slice through a stratum $X(\boldsymbol{v}',\boldsymbol{w})^{\mathrm{reg}}$ which is isomorphic to $X(\boldsymbol{v}-\boldsymbol{v}',\boldsymbol{w}-C\boldsymbol{v}')$. In our case it follows that the slices are isomorphic to $X(\boldsymbol{v}_0,\boldsymbol{w}_0)$ where $\alpha_{\boldsymbol{v}_0}=\beta$ and $\Lambda_{w_0}=\mu+\beta$). It is known that the support $\operatorname{supp}_{\boldsymbol{R}}(\beta)$ of a root β (see Section 4.1) is always a connected subdiagram of the Dynkin diagram. The condition that $(\beta,\mu)=0$ shows that $X(\boldsymbol{v}_0,\boldsymbol{w}_0)$ is isomorphic to the quiver variety $X(\boldsymbol{v}_0',C'(\boldsymbol{v}_0'))$ for the quiver associated to the subdiagram Q' given by $\operatorname{supp}_{\boldsymbol{R}}(\beta)$. (Here C' is the Cartan matrix attached to Q' and $\boldsymbol{v}_0'$ is just $\boldsymbol{v}_0$ viewed as a dimension vector for Q'.) It is well known that this quiver variety is the Kleinian singularity associated to the Weyl group of type Q', or equivalently, the root system $\Phi(\beta)$. □

Remark 5.7 The arguments above can also be used to show that the two-dimensional quiver varieties associated to an ADE quiver Q are precisely the Kleinian singularities attached to Dynkin diagrams which occur as subdiagrams of Q.

Theorem 5.8 *Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ are as above. Let W_μ be the Weyl group of the subroot system Φ_μ. The symplectic Galois group associated to the quiver variety $X(\boldsymbol{v},\boldsymbol{w})$ is W_μ.*

Proof We have seen that the codimension-two strata are labelled by the set $\Phi_\mu^{\max}$. Each $\beta \in \Phi_\mu^{\max}$ is associated to an indecomposable subroot system $\Phi(\beta)$ of Φ_μ. Lemma 5.6 shows that the slice through the associated stratum is the Kleinian singularity associated to the subroot system, which has symplectic Galois group $W(\Phi(\beta))$, the Weyl group of the root system $\Phi(\beta)$. Now Namikawa's description of the symplectic Galois groups shows that the group attached to $X(\boldsymbol{v},\boldsymbol{w})$ is

$$W = \prod_{\beta\in\Phi_\mu^{\max}} W(\Phi(\beta))^{\sigma_\beta},$$

where σ_β is a diagram automorphism of the Dynkin diagram induced by monodromy coming from the stratum. Thus it remains to show that this monodromy must be trivial. We show this in the following proposition. □

Proposition 5.9 *Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be such that $X(\boldsymbol{v},\boldsymbol{w})^{\mathrm{reg}} \neq \varnothing$. The monodromy automorphism σ_β for every codimension-two stratum in $X(\boldsymbol{v},\boldsymbol{w})$ is trivial.*

Proof Let $\eta = \Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}}$, and suppose that $X(\boldsymbol{v}_1,\boldsymbol{w})^{\mathrm{reg}}$ is a codimension-two stratum of $X(\boldsymbol{v},\boldsymbol{w})$. Then by Lemma 5.5, there is some $\beta \in \Phi_\eta^{\max}$ such that $\alpha_{\boldsymbol{v}} = \alpha_{\boldsymbol{v}_1} + \beta$.

Let $Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}) = Y(\boldsymbol{v},\boldsymbol{w}) \times_{X(\boldsymbol{v},\boldsymbol{w})} Y(\boldsymbol{v},\boldsymbol{w})$, and let $\tilde{\pi} : Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}) \to X(\boldsymbol{v},\boldsymbol{w})$ be the canonical map. By [Nakajima 1998, Theorem 7.2], $Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w})$ is a Lagrangian subvariety of $Y(\boldsymbol{v},\boldsymbol{w}) \times Y(\boldsymbol{v},\boldsymbol{w})$.

Let $H_{\mathrm{top}}(Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}))$ denote the top-dimensional Borel–Moore homology of $Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w})$ and write $[C]$ for the fundamental class of a component C of $Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w})$. Let

$$H_{\mathrm{top}}(Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{v}_1} = \mathrm{span}_{\Bbbk}\{[C] : C \text{ a component of } Z(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{w}) \cap \overline{\tilde{\pi}^{-1}(X(\boldsymbol{v}_1,\boldsymbol{w})^{\mathrm{reg}})}\}.$$

For the irreducible highest-weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}_Q$ associated to our quiver, let $L(\lambda)_\mu$ denote its μ-weight space. It is shown in [Nakajima 1998, Theorem 10.15] that there is a natural isomorphism

$$H_{\mathrm{top}}(Z(\boldsymbol{v}_1,\boldsymbol{v}_1,\boldsymbol{w}))_{\boldsymbol{v}} \cong L(\eta+\beta)_\eta^* \otimes L(\eta+\beta)_\eta.$$

But now the number s of components in $H_{\mathrm{top}}(Z(\boldsymbol{v},\boldsymbol{v},\boldsymbol{w}))_{\boldsymbol{v}_1}$ is just the number of orbits of the monodromy action on the components of $\tilde{\pi}^{-1}(x)$ for $x \in X(\boldsymbol{v}_1,\Lambda_{\boldsymbol{w}})^{\mathrm{reg}}$. Now clearly there are r^2 such components, where r is the number of components of $\pi_{\boldsymbol{v}}^{-1}(x)$. Since Theorem 10.2 of [Nakajima 1998] shows that the dimension of $L(\eta+\alpha)_\eta$ is exactly r, it follows that $s = r^2$, and hence the monodromy action must be trivial, as required. □

Example 5.10 It will be convenient in this example to write $X(\alpha_{\boldsymbol{v}}, \Lambda_{\boldsymbol{w}})$ rather than $X(\boldsymbol{v}, \boldsymbol{w})$. Suppose that Q is the A_3 quiver and $\Lambda_{\boldsymbol{w}} = \Lambda_1 + \Lambda_2 + \Lambda_3$, $\alpha_{\boldsymbol{v}} = \alpha_1 + \alpha_2 + \alpha_3$. Then $X(\alpha_{\boldsymbol{v}}, \Lambda_{\boldsymbol{w}})$ is 4-dimensional, and has four strata, labelled by 0, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$ and $\alpha_{\boldsymbol{v}} = \alpha_1 + \alpha_2 + \alpha_3$. Here the stratum corresponding to 0 is a point, and each of $X(\alpha_1 + \alpha_2, \Lambda_{\boldsymbol{w}})^{\mathrm{reg}}$ and $X(\alpha_2 + \alpha_3, \Lambda_{\boldsymbol{w}})^{\mathrm{reg}}$ is two-dimensional. The slice through the stratum $X(\alpha_1 + \alpha_2, \Lambda_{\boldsymbol{w}})$ is isomorphic to $X(\alpha_3, 2\Lambda_3)$, while that through $X(\alpha_2 + \alpha_3, \Lambda_{\boldsymbol{w}})$ is isomorphic to $X(\alpha_1, 2\Lambda_1)$. These are both Kleinian singularities of type A_1, and so the symplectic Galois group in this case is $S_2 \times S_2$, which is the parabolic subgroup of S_4 stabilizing the weight $\Lambda_{\boldsymbol{w}} - \alpha_{\boldsymbol{v}} = \Lambda_2$.

Our calculation of the symplectic Galois group for finite type quiver varieties, along with the general theory of Section 3, have the following immediate consequence:

Corollary 5.11 *Let $\pi : Y(\boldsymbol{v}, \boldsymbol{w}) \to X(\boldsymbol{v}, \boldsymbol{w})$ be as above, and let $\lambda = \Lambda_w - \alpha_v$. If $x \in X(\boldsymbol{v}, \boldsymbol{w})$, then the cohomologies $H^*(\pi^{-1}(x))$ carry an action of the group W_λ.*

Example 5.12 We give an example of a symplectic Galois group for a quiver variety of affine type. Let Q_Γ be the affine quiver associated by the McKay correspondence to the finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. Then if we take $\alpha_{\boldsymbol{v}} = n\delta$, where δ is a generator for the imaginary roots, and $\Lambda_{\boldsymbol{w}} = \Lambda_0$, where 0 denotes the extending node of the affine quiver, then it is known that $X(\boldsymbol{v}, \boldsymbol{w})$ is isomorphic to $\mathbb{C}^{2n}/\Gamma_n$, where $\Gamma_n = \Gamma \wr S_n$. Now for the quotient of a symplectic vector space by a symplectic reflection group such as Γ_n it is known that the stratification by stabiliser type coincides with that of the symplectic leaves. It follows that there are two codimension two strata: one labelled by the group Γ and the other by a subgroup of order two generated by a simple reflection in S_n (of course if $n = 1$ then this stratum does not exist). It can be checked that the symplectic Galois group is therefore either $W(\Gamma)$, the finite Weyl group associated to Γ if $n = 1$, or $S_2 \times W(\Gamma)$ for $n > 1$.

Let W_{aff} denote the affine Weyl group, that is, the Weyl group associated to the affine Lie algebra $\mathfrak{g}_Q$ whose Dynkin diagram underlies the affine quiver Q. Note that W_{aff} fixes the imaginary root δ, and thus $\Lambda_0 - \delta$ and $\Lambda_0 - n\delta$ for $n > 1$ have the same stabiliser in W_{aff}, thus the naive generalisation of Theorem 5.8 fails to hold.

Remark 5.13 If one is willing to work with all quivers and arbitrary dimension vectors, the distinction between "framed" and "unframed" quivers disappears thanks to the "Crawley-Boevey trick". In essence this exchanges framing vertices for "framing edges" all connected to a single framing vertex, denoted by ∞, equipped with a 1-dimensional vector space. The "diagonal" copy of $\mathbb{G}_m$ in the unframed quiver can be identified with the gauge group at the vertex ∞. Bellamy and Schedler [2021, Section 7] calculate the codimension-two strata in an arbitrary unframed affine quiver variety in terms of the root system combinatorics of the associate Kac–Moody Lie algebra $\mathfrak{g}_{Q'}$ and establish the isomorphism type of a transverse slice through each of these strata. They thus give the analogues of our Lemmas 5.5 and 5.6 in the unframed setting. It is however not clear, at least to the authors, how to interpret the results of [Bellamy and Schedler 2021] in terms of the Kac–Moody algebra $\mathfrak{g}_Q$ (rather than $\mathfrak{g}_{Q'}$).

Remark 5.14 The proof that the monodromy action on the components of the fibres of $\pi_{\boldsymbol{v}}$ should work in a somewhat more general context than that of finite-type quivers: Nakajima's results [1998, Section 10] produce an isomorphism from a modified form of the enveloping algebra to the "regular" part of the convolution algebra

$$\mathfrak{Z} = \bigoplus_{\boldsymbol{v}^1,\boldsymbol{v}^2} H_{\mathrm{top}}(Z(\boldsymbol{v}^1,\boldsymbol{v}^2,\boldsymbol{w})^{\mathrm{reg}}),$$

where the superscript "reg" means that one restricts to the open subset which is the complement of the closure of points in $Z(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{w})$ whose image under $\widetilde{\pi}$ does not lie in any locus $X(\boldsymbol{v}',\boldsymbol{w})^{\mathrm{reg}}$. In the finite-type case the regular set exhausts $Z(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{w})$, but in general it is a proper subset. The proof of Proposition 5.9 immediately extends to show that the monodromy action is trivial on the components of the fibres of codimension-two strata which are regular.

Remark 5.15 Many authors, including Nakajima [2003], Lusztig [2000] and Maffei [2002], have studied actions of the Weyl group on the cohomology of quiver varieties. It would be interesting to compare the actions constructed here with these actions. A novelty of our approach is that the action of the group W_λ is constructed without the use of a presentation of W_λ: previous approaches define the action of simple reflections and then must explicitly check the braid relations to deduce the existence of a Weyl group action. On the other hand, these approaches give a more explicit description of the action of the simple reflections.

A first step in such a comparison might be to exhibit an explicit homomorphism from the Weyl group to the symplectic Galois group. As the previous example shows, we should not expect this to be surjective in general, but it seems reasonable to expect that there should be a natural embedding. Evidence for this is provided, for example, by the generic isomorphisms constructed in [Maffei 2002]. We hope to address the existence of such an embedding in future work.

6 Bionic symplectic varieties and Springer theory

In this section we consider a special class of conic symplectic resolutions, which we call *bionic*. These are symplectic resolutions $\pi : Y \to X$ which, in addition to having a conical action $\lambda : \mathbb{G}_m \to \mathrm{Aut}(Y)$ acting with positive weight on the symplectic form, have another action $\mathbb{G}_m$-action $\rho : \mathbb{G}_m \to \mathrm{Aut}(Y)$, commuting with the λ-action, which is required to be Hamiltonian (and thus preserves the symplectic form on Y). This class of resolutions has been considered by a number of authors; see for example [Losev 2017] and [Braden et al. 2016].

Lemma 6.1 *The actions λ and ρ naturally extend to an action on the deformation $\pi : \mathcal{Y} \to \mathcal{X}$. Moreover, the Hamiltonian action acts trivially on the base of the deformation of Y, thus it induces an action on each deformation Y_t for $t \in H^2(Y,\mathbb{C})$.*

Proof The fact that the action λ extends is proved in [Namikawa 2011]. The arguments in [Namikawa 2008, Lemma 20] show that the Hamiltonian action naturally extends to a Hamiltonian action on the formal deformation. That argument moreover shows that the action preserves the base of the deformation. Since ρ commutes with λ, the action on the formal deformation extends to the algebraicisation, and moreover again fixes the base of the deformation, and hence induces a Hamiltonian action on each deformation. □

Though we will focus on the case where Y^ρ is finite, it is checked in [Losev 2017] that the connected components of the fixed point locus are always themselves conical symplectic resolutions.

Proposition 6.2 [Losev 2017, Proposition 5.1] *Each connected component of the fixed point locus Y^ρ for the ρ-action on Y is a conical symplectic resolution.*

Lemma 6.3 *Let $\rho_c : U_1 \to \mathrm{Aut}(Y)$ be the restriction of the $\mathbb{G}_m$ action to the circle U_1. The $\mathscr{C}^\infty$-trivial fibration $\mathscr{Y} \to H^2(Y,\mathbb{C})$ can be made ρ_c-equivariantly topologically trivial. In particular, the family $\mathscr{Y}^\rho \to H^2(Y,\mathbb{C})$ is topologically locally trivial.*

Proof That the trivialisation can be done U_1-equivariantly follows immediately from Thom's second isotopy theorem (see [Mather 2012] for example) that the $\mathscr{C}^\infty$-trivial fibration $\mathscr{Y} \to H^2(Y,\mathbb{C})$ can be made ρ_c-equivariantly topologically trivial. The local triviality of the fixed-point locus follows immediately. □

Lemma 6.4 *Suppose that ρ has finitely many fixed points on Y. Then $\mathscr{Y}^\rho$ is a trivial étale covering of $B_Y = H^2(Y,\mathbb{C})$.*

Proof The map $q_Y : \mathscr{Y} \to B_Y$ is smooth, hence it is surjective on tangent spaces. The action of ρ is trivial on B_Y, and if $x \in Y^\rho$ then dq_Y must induce an isomorphism on the ρ-fixed subspace of $T_x\mathscr{Y}$ (it must be surjective, and the kernel of dq_Y is the subspace T_xY, so since x is an isolated fixed point of the ρ-action on Y, it must also be injective). The dimension of the ρ-fixed subspace in the fibres of the restriction of $T\mathscr{Y}$ to $\mathscr{Y}^\rho$ is locally constant. Moreover, since λ commutes with ρ, it will contract $\mathscr{Y}^\rho$ to a subvariety of Y^ρ, hence the rank calculation for fixed points on Y imply that $dq_{Y|\mathscr{Y}^\rho}$ is an isomorphism at every point $y \in \mathscr{Y}^\rho$. Thus q_Y restricts to an étale map on $\mathscr{Y}^\rho$.

Next we take an equivariant compactification $\overline{\mathscr{Y}}$ of $\mathscr{Y}$, using for example [Brion 2017, Theorem 1.2]. Applying the same argument used above we see that the fixed locus on the compactification is proper and étale, and hence finite. By taking an equivariant compactification, one sees by the same argument that the fixed locus on the compactified locus is proper and étale, hence finite. If the restriction to the original locus was not finite, it can only be because the fibre multiplicities are nonconstant. But by Lemma 6.4, we know $\mathscr{Y}^\rho \to H^2(Y,\mathbb{C})$ is a locally trivial family, hence the fibre multiplicities are locally constant, and so, as $H^2(Y,\mathbb{C})$ is connected, constant. It follows that the fibre multiplicities are constant, and it follows that q_Y is finite, as required. Since $H^2(Y,\mathbb{C})$ is simply connected, it follows immediately that it is a trivial covering. □

Example 6.5 Let $Y = T^*\mathscr{B}$ be the flag variety for a reductive group G and $X = \mathscr{N}$ the nilpotent cone. It is well known that the moment map $\mu : Y \to X$ is a symplectic resolution. If T is a maximal torus of G and $\rho : \mathbb{G}_m \to T$ is a one-parameter subgroup, then the induced action of T on Y is Hamiltonian, so ρ gives a Hamiltonian $\mathbb{G}_m$-action on Y. Let $L = G^\rho$ be the subgroup of G fixed by the action of ρ. It is a Levi subgroup of G. The fixed-point locus of the ρ-action is a disjoint union of varieties isomorphic to $T^*\mathscr{B}_L$, where $\mathscr{B}_L$ is the flag variety of L. The connected components are in bijection with W/W_L, where W is the Weyl group of G and W_L is the Weyl group of L. In this case the restriction of the moment map is in fact a symplectic resolution.

We now restrict again to the case where ρ has finitely many fixed points. We will also, for simplicity, assume that all sheaves have $\mathbb{Q}$-coefficients. In this case we have a permutation action of W on the fixed point loci Y_t^ρ due to the trivialisation given by Lemma 6.4. Let us denote by P the corresponding permutation representation of W. We claim that the representation of the symplectic Galois group on $H^*(\pi^{-1}(0))$ is isomorphic to P. To show this, we will use equivariant cohomology, and hence we need to lift all of our constructions to the setting of the equivariant derived category. This is completely standard; see [Bernstein and Lunts 1994] for example. As we will only need to work with the group U_1, which is connected, the forgetful functor from equivariant perverse sheaves to perverse sheaves is fully faithful and hence equivariance, roughly speaking, is a property and not extra structure. The constant sheaf on any U_1-variety is equivariant, and where functors are compatible with the U_1 action, they lift to the equivariant setting, hence Spr lifts to a U_1-equivariant perverse sheaf. Since $H^*_{U_1}(\star) = H^*(B\mathbb{G}_m)$, in the equivariant setting all cohomologies are modules over $H^*(B\mathbb{G}_m)$, which is a polynomial ring in one variable.

We will show that $H^*(\pi^{-1}(0))$ is isomorphic to the permutation representation P using localisation and the nearby cycles description of Spr. The equivariant derived category has a six functor formalism allowing us to lift the nearby cycles functor. In the equivariant cohomology of the fixed points, the family $\mathcal{Y}^\rho \to H^2(Y, \mathbb{C})$ is trivial, with the generic action of W being given by the permutation action.

Fix a Hermitian metric on $\mathcal{X}$ which is invariant under the action of U_1 the maximal compact subgroup of ρ, the Hamiltonian $\mathbb{G}_m$-action (since X is affine this can be done globally), and a W-invariant metric on $H^2(Y, \mathbb{C})$, so that it descends to give a metric on $B_X = H^2(Y, \mathbb{C})/W$. The *Milnor fibre* of the map $\theta_X : \mathcal{X} \to B_X$ at a point $x \in X$ is $F_{x,\epsilon} = B(x, \epsilon) \cap \theta_X^{-1}(v)$, where $\delta = d(0, v) \ll \epsilon \ll 1$. We will write $\mathbb{B}_\epsilon$ for the ball of radius ϵ centred at $0_X \in \mathcal{X}$, and $B_X(\delta)$ for the ball of radius δ centred at $0 \in B_X$, and $B_X(\delta)^{\mathrm{reg}} = B_X(\delta) \cap B_X^{\mathrm{reg}}$ for the regular values of θ_X in $B_X(\delta)$ (cf Definition 3.19).

Lemma 6.6 *The stalk at $0 \in X$ of the U_1-equivariant lift of the Springer sheaf Spr_{U_1} is $H^*_{U_1}(F_{\epsilon,v})$, where $F_{\epsilon,v} = \mathbb{B}_\epsilon \cap \theta_X^{-1}(v)$ for $0 < \|v\| < \delta \ll \epsilon \ll 1$.*

Proof Note that since we have chosen our Hermitian metric to be U_1-invariant, the spaces $F_{x,\epsilon}$ are preserved by U_1, and so it makes sense to speak their U_1-equivariant cohomology. By Theorem 3.21, the Springer sheaf is given by a nearby cycles construction. Now it follows from the construction of the nearby cycles functor that for sufficiently small ϵ and δ, the spaces F_ϵ are all homeomorphic, and the stalk of the nearby-cycles sheaf $\mathcal{P}_{\theta_X,0}$ is given by the cohomology of F_ϵ, and hence working in the equivariant setting, this is $H^*_{U_1}(F_\epsilon)$. □

Theorem 6.7 *Suppose that the Hamiltonian $\mathbb{G}_m$ acts with finitely many fixed points. Then the symplectic Galois group W permutes the fixed points in a generic deformation $\mathcal{X}_t$ and the cohomology $H^*(X_t^\rho)$ has a filtration whose associated graded is isomorphic as a W-representation to $H^*(Y)$.*

Proof If the Hamiltonian $\mathbb{G}_m$-action ρ has finitely many fixed points on Y, then we have seen that the fixed-point locus $\mathcal{Y}^\rho$ is a trivial finite cover of $H^2(Y, \mathbb{C})$. Moreover, since the action of ρ is compatible with the deformation it follows that the symplectic Galois group action commutes with the action of ρ

and so acts on $\mathcal{Y}^\rho$, hence the triviality of the covering $\mathcal{Y}^\rho \to H^2(Y,\mathbb{C})$ yields a permutation action of W on the sheets of the covering.

Now the action of the symplectic Galois group on the stalks of the Springer sheaf is, via the nearby cycles description, given by the monodromy action arising from the fact that the construction of ψ_{θ_X} is locally constant as one varies the analytic arc used in the construction of ψ_{θ_X}; see Proposition 3.17. To prove the theorem we use localisation theory for the U_1-action, following the strategy of [Treumann 2009]. Let $F_\epsilon = \tilde{\pi}^{-1}(\mathbb{B}_\epsilon \cap \theta_X^{-1}(B_X(\delta)^{\mathrm{reg}}))$ and F_ϵ^ρ the fixed points of ρ in F_ϵ. Then if ψ is the restriction of $\theta_X \circ \tilde{\pi}$ to F_ϵ and $\iota : F_\epsilon^\rho \to F_\epsilon$ the inclusion map, and $\phi = \psi \circ \iota$, we have diagrams

$$\begin{array}{ccc} F_\epsilon^\rho & \xrightarrow{\ \iota\ } & F_\epsilon \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & B_X^{\mathrm{reg}}(\delta) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (F_\epsilon^\rho)_{hU_1} & \xrightarrow{\ (\iota)_{hU_1}\ } & (F_\epsilon)_{hU_1} \\ & {\scriptstyle\tilde{\phi}}\searrow \quad \swarrow{\scriptstyle\tilde{\psi}} & \\ & BU_1 \times B_X^{\mathrm{reg}}(\delta) & \end{array}$$

where if M is any topological space with a U_1-action we write M_{hU_1} for the Borel construction $EU_1 \times_{U_1} M$, and the top row of the diagram on the right is obtained from that on the left by applying the Borel construction to the inclusion map ι. The maps with the maps $\tilde{\phi}$ and $\tilde{\psi}$ induced by ϕ and ψ since the U_1-action preserves the fibres of those maps. In more detail, we have a natural map

$$\tilde{\psi} : (F_\epsilon)_{hU_1} = EU_1 \times_{U_1} F_\epsilon \to BU_1 \times B_X(\delta)^{\mathrm{reg}}, \quad \tilde{\psi}([e,x]) = (U_1.e, \psi(x)),$$

with fibre

$$\tilde{\psi}^{-1}(p,v) = F_{\epsilon,v} = \psi^{-1}(v) = \mathbb{B}_\epsilon \cap \theta_X^{-1}(v) \quad \text{for } v \in B_X^{\mathrm{reg}}(\delta),\ p \in BU_1.$$

The map $\tilde{\phi}$ is defined in the same way.

Now let $\pi_2 : BU_1 \times B_X(\delta)^{\mathrm{reg}} \to B_X(\delta)^{\mathrm{reg}}$ be the projection to the second factor and let

$$\mathcal{F}_\rho = \bigoplus_{i\in\mathbb{Z}} R^i(\pi_2 \circ \tilde{\phi})_*(\mathbb{Q}) \quad \text{and} \quad \mathcal{F} = \bigoplus_{i\in\mathbb{Z}} R^i(\pi_2 \circ \tilde{\psi})_*(\mathbb{Q}).$$

Then $\mathcal{F}_\rho$ and $\mathcal{F}$ are $\mathbb{Z}$-graded locally constant sheaves with fibres at $v \in B_X(\delta)^{\mathrm{reg}}$ isomorphic to the U_1-equivariant cohomology of the fibres $\phi^{-1}(v)$ and $\psi^{-1}(v)$, respectively. This shows that $\tilde{\psi}^{-1}(v) = (\psi^{-1}(v))_{hU_1}$ so that we may view p as a fibration relative to the B_X^{reg}, ie as a fibration in the category of locally trivial fibre bundles over B_X^{reg}: As a result, the associated Leray spectral sequence for the fibration p can be treated as a spectral sequence of locally constant sheaves over B_X^{reg}, that is, as a spectral sequence of $\pi_1(B_X^{\mathrm{reg}})$-representations.

Fix a ring isomorphism of the rational cohomology $H^*(BU_1,\mathbb{Q})$ with $Q[t]$ (so that t has degree 2). By Lemma 6.4, the map $\mathcal{Y}^\rho \to H^2(Y,\mathbb{C})$ is a trivial covering, ie it is isomorphic to the product $Y^\rho \times H^2(Y,\mathbb{C})$. Moreover, U_1 acts trivially on this space, hence the Borel construction for F_ϵ^ρ is just $BU_1 \times Y^\rho \times B_X^{\mathrm{reg}}(\delta)$. It follows that the graded local system $\mathcal{F}_\rho$ is just the constant local system with fibre $H^*(BU_1) \otimes H^0(Y^\rho) = \mathrm{Ind}_1^W(\mathbb{Q}[t] \otimes H^0(Y^\rho))$, where Ind_1^W denotes induction from the trivial subgroup of W.

First consider the map $\widetilde{\psi}$. If $\pi_2 : BU_1 \times B_X(\delta)^{\mathrm{reg}} \to B_X(\delta)^{\mathrm{reg}}$ denotes the projection to the second factor, we therefore have a commutative diagram

$$\begin{array}{ccc} (F_\epsilon)_{hU_1} & \xrightarrow{\quad p \quad} & BU_1 \times B_X^{\mathrm{reg}}(\delta) \\ & {\scriptstyle\widetilde{\psi}}\searrow \quad \swarrow{\scriptstyle\pi_2} & \\ & B_X^{\mathrm{reg}}(\delta) & \end{array}$$

The stalks of the sheaves on its E_2-page are $E_2^{ij} = H^i(BU_1, \mathbb{Q}) \otimes H^j(F_\epsilon, \mathbb{Q})$, with the sequence converging to $H^*_{U_1}(F_\epsilon)$. However, it is known that by [Kaledin 2006, Theorem 2.12] the cohomology of the fibres of a symplectic resolution $\pi : Y \to X$ are nonzero only in even degrees, hence $H^i(BU_1) \otimes H^j(F_\epsilon)$ is zero unless $i = 2k$ and $j = 2l$, in which case it is $t^k \otimes H^{2l}(F_\epsilon)$. Thus the differential on the E^2 page vanishes for degree reasons and so the spectral sequence collapses at the E_2-page. In particular, it follows that $H^*_{U_1}(F_\epsilon)$ is a free $\mathbb{Q}[t]$-module, and that, if we set $F_k(H^*_{U_1}(F_\epsilon)) = t^k H^*_{U_1}(F_\epsilon)$, then $\mathrm{gr}^F(H^*_{U_1}(F_\epsilon)) \cong \mathbb{Q}[t] \otimes H^*(F_\epsilon)$ as graded rings.

Now consider $\iota^* : H^*_{U_1}(F_\epsilon) \to H^*_{U_1}(F_\epsilon^\rho)$. By the localisation theorem in equivariant cohomology (see for example [Quillen 1971]) this map is injective, and becomes an isomorphism after localising. In view of the degeneration of the Leray spectral sequence for $H^*_{U_1}(F_\epsilon)$, it suffices to invert $t \in \mathbb{Q}[t]$. We may therefore identify $H^*_{U_1}(F_\epsilon)$ with its image in $\mathbb{Q}[t] \otimes H^0(Y^\rho)$. As $\mathbb{Q}[t]$ is a Euclidean domain, there is a basis $B = \{e_1, \dots, e_n\}$ of $\mathbb{Q}[t] \otimes H^0(Y^\rho)$ and $f_1, \dots, f_n \in \mathbb{Q}[t]$ for which $\{f_1 e_1, \dots, f_n e_n\}$ is a basis of $H^*_{U_1}(F_\epsilon)$. Moreover, since $\mathbb{Q}[t] \otimes H^0(Y^\rho)$ is a graded $\mathbb{Q}[t]$-module, and $H^*_{U_1}(F_\epsilon)$ a graded submodule, we may assume that the basis $\{e_1, \dots, e_n\}$ and the f_i are homogeneous elements. It follows that B is a basis of $H^0(Y^\rho) = H^0_{U_1}(Y^\rho)$ and that we may assume $f_i = t^{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. But now if we define $G_k(H^0(Y^\rho) = \mathrm{span}_{\mathbb{Q}}\{e_i : n_i \geq k\}$, we have

$$\mathrm{gr}^G(H^0(Y^\rho)) \cong H^*_{U_1}(F_\epsilon)/t.H^*_{U_1}(F_\epsilon) \cong H^*(F_\epsilon) \cong H^*(Y). \qquad \square$$

Remark 6.8 The action of W on the fixed-point locus of generic fibres of $\mathcal{X}$ can also be described as follows: fix $s_0 \in B_X^{\mathrm{reg}}$ and pick $t_0 \in H^2(Y, \mathbb{C})$ with $q(t_0) = s_0$. Now $\mathcal{Y}_{t_0} \cong \mathcal{X}_{s_0}$ via the restriction $\widetilde{\pi}_{t_0}$ of $\widetilde{\pi}$, so that $\mathcal{X}^\rho_{s_0}$ is identified with $\mathcal{Y}^\rho_{t_0}$. But W acts on $\mathcal{Y}$ compatibly with the ρ-action (by the universality of the deformation $\mathcal{Y}$ for example), where $w \in W$ satisfies $w(\mathcal{Y}_t) = \mathcal{Y}_{w(t)}$, where $\widetilde{\pi}_{w(t_0)} : \mathcal{Y}_{w(t_0)} \to \mathcal{X}_q(w(t_0))$ is an isomorphism and $q(w(t_0)) = q(t_0) = s_0$. It follows that we obtain an action of W on $\mathcal{X}_{q(t)}$ by setting $w(x_0) = \widetilde{\pi}_{w(t)} \circ w \circ \widetilde{\pi}_t^{-1}(x_0)$ for all $w \in W$ and $x_0 \in \mathcal{X}^\rho_{s_0}$.

Since $\mathcal{Y}^\rho \to H^2(Y, \mathbb{C})$ is a trivial covering, we may identify Y^ρ with the fixed point locus $\mathcal{Y}^\rho_t$ for any t, thus we may also view this as an action of W on Y^ρ.

7 Slices in the affine Grassmannian

The idea that slices in the affine Grassmannian should have a kind of Springer theory associated to them goes back to work of Ngô [1999, Section 2], and was studied in the context of knot invariants by Kamnitzer [2011]. A detailed study of the Poisson geometry of these slices was initiated in [Kamnitzer et al. 2014], and we will review briefly some results of that paper below. As should be clear to the reader, the main contribution of this section is to compare the Beilinson–Drinfel'd deformation with the universal conic deformation and obtain some basic insight into the symplectic Springer theory for these symplectic varieties which, following the insights of Braverman, Finkelberg and Nakajima [Braverman et al. 2019] should perhaps be thought of as the "symplectic duals" of the finite-type quiver varieties studied in Section 5.

7.1 Background on slices

In this section we review some facts about slices in the affine Grassmannian. An excellent introduction to the Geometric Satake correspondence and the affine Grassmannian is [Zhu 2017], and more details on the geometry of slices are given in [Kamnitzer et al. 2014].

Let G be a reductive algebraic group over $\mathbb{C}$. We write $G[\![t^{\pm 1}]\!]$ or $G(\!(t^{\pm 1})\!)$ for the $\mathbb{C}[\![t^{\pm 1}]\!]$ or $\mathbb{C}(\!(t^{\pm 1})\!)$ points of G, and similarly we write $G[t]$ for the $\mathbb{C}[t]$-points of G. The *thick affine Grassmannian* is the scheme $\mathbf{Gr} = G(\!(t^{-1})\!)/G[t]$, which contains the thin affine Grassmannian Gr via the maps

$$G(\!(t)\!)/G[\![t]\!] \cong G[t, t^{-1}]/G[t] \hookrightarrow G(\!(t^{-1})\!)/G[t].$$

A coweight λ for G may be viewed as an element of $G[t, t^{-1}]$ and hence yields a point in the thin affine Grassmannian, which we denote by t^{λ}. Set, for λ and μ dominant coweights of G,

$$\mathrm{Gr}^{\lambda} = G[t].t^{\lambda} \quad \text{and} \quad \mathbf{Gr}_{\mu} = G_1[\![t^{-1}]\!].t^{w_0\mu},$$

where $G_1[\![t^{-1}]\!]$ is the kernel of the natural homomorphism $G[\![t^{-1}]\!] \to G$ given by evaluation at $t^{-1} = 0$, and as usual w_0 denotes the longest element of the Weyl group of G.

For λ and μ as above with $\mu \le \lambda$, let $\mathrm{Gr}^{\bar{\lambda}}_{\mu} = \overline{\mathrm{Gr}^{\lambda}} \cap \mathbf{Gr}_{\mu}$. This is an affine slice to the orbit Gr^{μ} through the point $t^{w_0\mu}$, since $\mathbf{Gr}_{\mu}$ is transverse to every Gr^{ν} and intersects Gr^{μ} precisely at the point $t^{w_0\mu}$. It is known to have dimension $2\langle \rho, \lambda - \mu\rangle$ (where ρ as usual denotes the half-sum of positive roots). There is a natural action of $\mathbb{G}_m$ on Gr by "loop rotation". This action preserves both $G[t]$- and $G(\!(t^{-1})\!)$-orbits, and hence preserves each $\mathrm{Gr}^{\lambda}_{\mu}$.

Lemma 7.1 *Let λ and μ be dominant coweights of G such that $\mu \le \lambda$. Then the variety $\mathrm{Gr}^{\bar{\lambda}}_{\mu}$ is a conical symplectic variety in the sense of Definition 2.7. Moreover, its symplectic leaves are $\mathrm{Gr}^{\lambda}_{\mu} = \mathrm{Gr}^{\lambda} \cap \mathbf{Gr}_{\mu}$.*

Sketch of proof The details of the proof of these statements are contained in Theorems 2.5 and 2.7 of [Kamnitzer et al. 2014]. Here we outline the key ideas. The Poisson structure is constructed from the Manin triple $(\mathfrak{g}(\!(t^{-1})\!), \mathfrak{g}[t], t^{-1}\mathfrak{g}[\![t^{-1}]\!])$, where the pairing between $\mathfrak{g}[t]$ and $t^{-1}\mathfrak{g}[\![t^{-1}]\!]$ being given by the residue of the Killing form (note that $\mathrm{Lie}(G_1[\![t^{-1}]\!]) = t^{-1}\mathfrak{g}[\![t^{-1}]\!]$). The loop rotation action preserves $\mathrm{Gr}^{\bar{\lambda}}_{\mu}$ and contracts it to the unique fixed point $t^{w_0\mu}$, and its compatibility with the Manin triple described above ensures it makes $\mathrm{Gr}^{\bar{\lambda}}_{\mu}$ into a conic symplectic variety as claimed (acting with weight -1 on the Poisson bracket). The claim about the symplectic leaves follows from a calculation of the rank of the Poisson structure which is due to Lu and Yakimov [2008]. □

Remark 7.2 If X is an arbitrary conical symplectic variety, it may not have a symplectic resolution. It is known, however, using the minimal model programme, that X will always have a $\mathbb{Q}$-factorial terminalisation, that is, a crepant birational morphism $\pi : Y \to X$ such that Y is $\mathbb{Q}$-factorial and possesses only terminal singularities. The former condition means that if D is a Weil divisor then there is some integer N such that $N.D$ is Cartier, while the latter condition means that any smooth resolution of Y will have positive discrepancy (so that Y is as smooth as a crepant (partial) resolution can get). Namikawa [2010] shows that, if $\pi : Y \to X$ is such a morphism, then Y is sufficiently smooth that it possesses a universal conical Poisson deformation $\mathcal{Y} \to H^2(Y, \mathbb{C})$ to which the conclusions of Theorem 2.8 apply. But then if X has a smooth crepant resolution, Y' say, Theorem 2.8 for the universal deformation of Y' shows that the generic Poisson deformations of X will be smooth, but then since $\widetilde{\Pi} : \mathcal{Y} \to \mathcal{X} \times_{B_X} H^2(Y, \mathbb{C})$ induces an isomorphism $\mathcal{Y}_t \cong \mathcal{X}_{q(t)}$ for generic $t \in H^2(Y, \mathbb{C})$, we see that the generic deformation of Y must also be smooth. But Theorem 5.5 of [Namikawa 2011] shows that $\mathcal{Y} \to H^2(Y, \mathbb{C})$ is a locally trivial deformation (as a flat family of varieties without Poisson structure), so that $Y = \mathcal{Y}_0$ is smooth if $\mathcal{Y}_t$ is smooth. Thus remarkably, the existence of a single smooth crepant resolution implies that *any* $\mathbb{Q}$-factorial terminalisation of X will in fact be smooth!

The previous remark implies that, given a conical symplectic variety X, if one can produce an explicit $\mathbb{Q}$-factorial terminalisation $\pi : Y \to X$, then X has a symplectic resolution if and only if Y is in fact smooth, in which case of course, π is such a resolution! Perhaps surprisingly, [Kamnitzer et al. 2014] provide such a Y when $X = \mathrm{Gr}^{\overline{\lambda}}_{\mu}$. To describe it we must first recall the convolution product for $G[t]$-invariant subsets of $\mathbf{Gr}$. Let $p : G(\!(t^{-1})\!) \to \mathbf{Gr}$ be the natural map. Then if $A, B \subseteq \mathbf{Gr}$ are $G[t]$-invariant, we let $A \mathbin{\tilde{\times}} B = p^{-1}(A) \times_{G[t]} B$.

Now consider $\mathrm{Gr}^{\overline{\lambda}}_{\mu}$: given λ, assuming (as we shall) that G is of adjoint type, we may pick a sequence $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of fundamental coweights such that $\lambda = \lambda_1 + \dots + \lambda_n$. Then we have open and closed convolutions

$$\begin{array}{c} \mathrm{Gr}^{\vec{\lambda}} = \mathrm{Gr}^{\lambda_1} \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} \mathrm{Gr}^{\lambda_n} \\ \Big\downarrow m \\ \overline{\mathrm{Gr}^{\lambda}} \end{array} \qquad \text{and} \qquad \begin{array}{c} \overline{\mathrm{Gr}^{\vec{\lambda}}} = \overline{\mathrm{Gr}^{\lambda_1}} \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} \overline{\mathrm{Gr}^{\lambda_n}} \\ \Big\downarrow \bar{m} \\ \overline{\mathrm{Gr}^{\lambda}} \end{array}$$

where m and $\bar{m}$ denote the convolution maps. If we set $\mathrm{Gr}^{\overline{\vec{\lambda}}}_{\mu} = \bar{m}^{-1}(\mathrm{Gr}_{\mu})$ then by [Kamnitzer et al. 2014, Theorem 2.7], $\mathrm{Gr}^{\overline{\vec{\lambda}}}_{\mu}$ is a $\mathbb{Q}$-factorial terminalisation of $\mathrm{Gr}^{\overline{\lambda}}_{\mu}$, and therefore, by Remark 7.2, a symplectic resolution of $\mathrm{Gr}^{\overline{\lambda}}_{\mu}$ whenever such a resolution exists. Provided the fundamental coweights λ_i are minuscule as weights for the dual group $G^\vee$, the orbits Gr^{λ_i} are actually closed, and hence the convolution product $\overline{\mathrm{Gr}^{\vec{\lambda}}}$ is smooth. A refinement of this observation shows the following.

Theorem 7.3 [Kamnitzer et al. 2014, Theorem 2.9]. *The variety $\mathrm{Gr}^{\overline{\lambda}}_{\mu}$ has a symplectic resolution precisely when there do not exist coweights $\nu_1, \dots, \nu_k$ such that $\nu_1 + \dots + \nu_n = \mu$ and for all k, ν_k is a weight of $V(\lambda_k)$ (the irreducible representation of highest weight λ_k for the dual group $G^\vee$) and for some k, ν_k is not an extremal weight of $V(\lambda_k)$.*

Remark 7.4 We shall assume from now on that our variety $\overline{\mathrm{Gr}_\mu^\lambda}$ possesses a symplectic resolution. In this case, one can check (see Theorem 2.9 in [Kamnitzer et al. 2014]) that the preimage of $\overline{\mathrm{Gr}_\mu^\lambda}$ lies in the open convolution space $\mathrm{Gr}^{\lambda_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}^{\lambda_n}$, hence we will write $\tilde{\mathrm{Gr}}_\mu^{\bar{\lambda}}$ for the resolution.

Remark 7.5 The theorem shows that we must take λ to be a sum of minuscule coweights, if we wish to ensure $\overline{\mathrm{Gr}_\mu^\lambda}$ always has a symplectic resolution. When G is of type A, every fundamental weight is minuscule, hence this is automatic, but in general we see that the theorem gives a large supply of symplectic varieties which do not possess a symplectic resolution.

Namikawa's recipe for computing the symplectic Galois group can be carried out in these cases thanks to the work of [Malkin et al. 2005, Theorem 5.2]:

Lemma 7.6 *The codimension-two strata of the symplectic variety* $\overline{\mathrm{Gr}_\mu^\lambda}$ *are those* Gr_ν^λ *with* ν *dominant and* $\nu = \lambda - \check{\alpha}_i \geq \mu$ *for some simple coroot* $\check{\alpha}_i$. *Moreover, a transverse slice through this stratum is isomorphic to a Kleinian singularity of the form* A_{p_i-1}, *where* $p_i = \langle \check{\alpha}_i, \lambda \rangle$.

Proof This follows immediately from the dimension formula $\dim(\overline{\mathrm{Gr}_\mu^\lambda}) = 2\langle \rho, \lambda - \mu \rangle$ and Theorem 5.2 of [Malkin et al. 2005]. □

Once one has computed the codimensional 2 strata, in order to calculate the symplectic Galois group one must compute the monodromy action of the fundamental group of the strata on the exceptional fibres of minimal resolutions of the codimension two strata. However, it is asserted in [Kamnitzer et al. 2014] that these strata are simply connected, and hence the monodromy is trivial. For simplicity, in the remainder of this section we will fix $\mu = 0$.

Let $\{\check{\omega}_i\}_{i=1}^r$ be the set of fundamental coweights for G.

Proposition 7.7 *Let* $Y = \overline{\mathrm{Gr}_0^\lambda}$, *and assume that the conditions of Theorem 7.3 hold for* $(\lambda, \mu) = (\lambda, 0)$ (*and hence* Y *has a symplectic resolution*). *If* $\lambda = \sum_{i=1}^r p_i \check{\omega}_i$ *then its symplectic Galois group is isomorphic to* $\prod_{p_i > 1} S_{p_i}$.

Proof By Lemma 7.6, the slices through codimension two strata are Kleinian singularities of type A_{p_i-1}, and thus the associated reflection group is the symmetric group on p_i letters. To be more precise, given λ as in the statement of the proposition, we obtain a codimension-two stratum corresponding to $\nu = \lambda - \check{\alpha}_k$ provided ν is dominant. But if $C = (c_{ij})$ denotes the Cartan matrix of dual group $G^\vee$, then

$$\nu(\alpha_j) = p_j - c_{kj},$$

which is at least p_j if $j \neq k$, while if $j = k$ then $\nu(\alpha_k) = p_k - 2$ and hence $p_i \geq 2$. It follows that for each k, $1 \leq k \leq r$ we have a codimension 2 stratum in $\overline{\mathrm{Gr}_0^\lambda}$ provided $p_k > 1$, and in that case a slice through that stratum is a Kleinian singularity of type A_{p_k-1}. It follows that the symplectic Galois group is $\prod_{p_i > 1} S_{p_i}$ as required. □

Remark 7.8 If we take $\mu \neq 0$ then the symplectic Galois group is in general smaller: one needs in addition, for each fundamental weight λ_i, that we have $\mu \leq \lambda - \alpha_i$.

Next we recall the geometric Satake correspondence, discovered by Lusztig, whose full categorical meaning was elucidated in the beautiful paper [Mirković and Vilonen 2007]. This establishes an equivalence of Tannakian categories between the representations of the Langlands dual group $G^\vee$ and the category of $G[\![t]\!]$-equivariant perverse sheaves on Gr. Under this equivalence, the convolution product of sheaves (recalled above for subsets of Gr) corresponds to tensor product of representations. As such, we have the following:

Proposition 7.9 *If $m : \mathrm{Gr}_0^{\bar{\vec{\lambda}}} \to \mathrm{Gr}_0^{\bar{\lambda}}$ is a symplectic resolution, then the associated Springer sheaf Spr_λ corresponds to the tensor product $V(\lambda_1) \otimes \ldots \otimes V(\lambda_n)$ of representations of $G^\vee$. If i denotes the inclusion of our slice into the affine Grassmannian, it follows that*

$$\mathsf{Spr}_{\vec{\lambda}} \cong \bigoplus_{\nu} i^*(\mathrm{IC}(\mathbb{O}_\nu)) \otimes M_{\vec{\lambda},\nu},$$

where $M_{\vec{\lambda},\nu}$ is the multiplicity space, a vector space of dimension equal to the multiplicity of the irreducible V_ν in the tensor product representation $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$.

Proof The irreducible representations $V(\lambda_i)$ correspond under the Geometric Satake Correspondence to the simple perverse sheaves IC_{λ_i}, the intersection cohomology extension of the constant local system $\mathsf{k}_{\mathrm{Gr}^\lambda}[(2\rho,\lambda)]$. The tensor product of representations corresponds to the pushforward via $m_!$ of the twisted product

$$\mathrm{IC}_{\lambda_1} \,\tilde{\boxtimes} \cdots \tilde{\boxtimes}\, \mathrm{IC}_{\lambda_n} \tag{7-1}$$

of the sheaves IC_{λ_i}. On the other hand, we have

$$\mathsf{Spr}_{\vec{\lambda}} = m_!(\mathsf{k}_{\mathrm{Gr}_0^{\vec{\lambda}}}[(2\rho,\lambda)]).$$

However, it is known that the smooth locus of $\mathrm{Gr}^{\bar{\mu}} = \mathrm{Gr}^\mu$ for all μ and hence $\mathrm{Gr}^{\vec{\lambda}}_{\mu_0}$ is the smooth locus of $\mathrm{Gr}_0^{\bar{\vec{\lambda}}}$. But by our hypotheses $\mathrm{Gr}_0^{\bar{\vec{\lambda}}}$ is smooth, and hence it is exactly equal to the smooth locus of $\mathrm{Gr}^{\vec{\lambda}}_\mu$. It follows that on this locus the twisted product (7-1) is just the constant sheaf (with appropriate shift). The claim about the multiplicity spaces thus follows from the Satake equivalence. □

Remark 7.10 These tensor product multiplicities are computable, via a number of combinatorial tools, eg crystal bases, Littelmann paths, etc. In order for $\mathcal{M}_{\vec{\lambda},\nu}$ to be an irreducible representation of the symplectic Galois group W for all the irreducible constituents of $\mathsf{Spr}_{\vec{\lambda}}$ one would need the group algebra $\mathsf{k}[W]$ to be the full commutator algebra of the action of $\mathfrak{g}$ in $\mathrm{End}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})$. While this is true, famously, for the n-th tensor power of the vector representation of $\mathfrak{sl}_n$, where it is known as Schur–Weyl duality, in general one should expect $\mathsf{k}W$ to be too small to capture all of the commutant algebra. We will give an explicit example of this in Corollary 8.4.

7.2 Comparing the universal deformation and the BD deformation

The Beilinson–Drinfel'd deformation family arises by considering the affine Grassmannian spread over $\mathbb{P}^1$. We equip $\mathbb{P}^1$ with a $\mathbb{G}_m$-action, fixing points $\{0,\infty\} \subseteq \mathbb{P}^1$ and pick a local coordinate $t \in \Gamma(\mathbb{P}^1, \mathcal{K}_{\mathbb{P}^1})$

at 0 having a pole at $\infty \in \mathbb{P}^1$. We will require our $\mathbb{G}_m$-action to have weight 1 on our coordinate t. The basic observation is the following: let E^0 denote the trivial G-bundle on $\mathbb{P}^1$ and consider the set

$$\{(E,\phi) : E \text{ is a principal } G\text{-bundle on } \mathbb{P}^1 \text{ and } \phi : E_{|\mathbb{P}^1\backslash\{0\}} \to E^0_{|\mathbb{P}^1\backslash\{0\}} \text{ is an isomorphism}\},$$

where E^0 denotes the trivial G-bundle. Using our local coordinate t at 0 one can readily obtain a bijection between this set and the affine Grassmannian Gr. Since any principal G-bundle on $\mathbb{P}^1\backslash\{0\}$ is necessarily trivial, they can all be obtained by a modification of the trivial bundle at 0, and hence we see that Gr uniformises the moduli space of G-bundles on $\mathbb{P}^1$. Indeed observing that a trivialisation on $\mathbb{P}^1\backslash\{0\}$ is just a G-valued function on $\mathbb{P}^1\backslash\{0\}$, that is, and element of $G[t^{-1}]$, we see[5] that

$$\mathrm{Bun}_G(\mathbb{P}^1) \cong G[t^{-1}]\backslash\mathrm{Gr}.$$

Definition 7.11 If (E,ϕ) is a pair consisting of a principal G-bundle on $\mathbb{P}^1$ together with a trivialisation $\phi : E_{|\mathbb{P}^1\backslash\{a\}} \to E^0_{|\mathbb{P}^1\backslash\{a\}}$ of E on $\mathbb{P}^1\backslash\{a\}$ (thus, as before, we let E^0 denote the trivial bundle). The bundle E therefore corresponds to point on the affine Grassmannian $G(\!(s)\!)/G[\![s]\!]$, where $s = t - a$, and we say E has *Hecke type* λ and write $\mathrm{Inv}(\phi) = \lambda$ if this point lies in Gr^λ, and say E has Hecke type $\leq \lambda$ if $\mathrm{Inv}(\phi) = \nu$ with $\mathrm{Gr}^\nu \subseteq \overline{\mathrm{Gr}^\lambda}$.

The action of $G_1[\![t^{-1}]\!]$ yield changes of trivialisation which preserve ∞, and as a result one can describe the slices Gr_μ in terms of the isomorphism type of the bundle E and the parabolic reduction that it yields at ∞. We will restrict our attention, however, to the case where $\mu = 0$, which corresponds in the moduli interpretation simply to the triviality of the bundle E. The importance of the moduli description is that it allows us to deform the affine Grassmannian over $\mathbb{A}^n$: we write $\mathrm{Gr}_{\mathbb{A}^n}$ for the family over $\mathbb{A}^n$ whose fibre at $\underline{a} = (a_1,\dots,a_n)$ is just

$$\mathrm{Gr}_{\underline{a}} = \{(E,\phi) : E \text{ is a principal } G\text{-bundle on } \mathbb{P}^1 \text{ and } \phi : E_{|\mathbb{P}^1\backslash\{a_1,\dots,a_n\}} \to E^0_{|\mathbb{P}^1\backslash\{a_1,\dots,a_n\}} \text{ is an isomorphism}\}.$$

Now fix a coweight λ with a decomposition $\lambda = \lambda_1 + \cdots + \lambda_n$ as a sum of minuscule coweights. We think of the coweights λ_i as giving a "colouring" the divisor $\sum_{i=1}^n a_i$, that is, we set $D = \sum_{i=1}^n \lambda_i.a_i$ and view it as a finite-supported coweight-valued function on $\mathbb{A}^1$: if $x \in \mathbb{A}^1$ then we set $D(x) = \sum_{i:a_i=x} \lambda_i$, and define

$$\mathrm{Gr}^\lambda_{0,\underline{a}} = \{(E,\phi) \in \mathrm{Gr}_{0,\underline{a}} : (E,\phi) \text{ has Hecke type } D(x) \text{ for all } x \in \mathbb{P}^1\},$$
$$\mathrm{Gr}^{\overline{\lambda}}_{0,\underline{a}} = \{(E,\phi) \in \mathrm{Gr}_{0,\vec{a}} : (E,\phi) \text{ has Hecke type } \leq D(x) \text{ for all } x \in \mathbb{P}^1\}.$$

Letting $\underline{a} = (a_1,\dots,a_n)$ vary over $\mathbb{A}^n$ we thus obtain a family of symplectic varieties $\mathrm{Gr}^\lambda_{0,\mathbb{A}^n}$ (from a Poisson structure defined by a global version of the Manin triple used above) with central fibre $\mathrm{Gr}^{\overline{\lambda}}_0$.

[5] Though clearly we have only argued at the level of $\mathbb{C}$-points!

To deform our resolution $\mathrm{Gr}^{\vec\lambda}_{\mu}$ of $\mathrm{Gr}^{\bar\lambda}_{0}$ we use the twisted products of Gr^{λ_i} (which, since we are assuming each λ_i is minuscule, are smooth projective varieties). In more detail, we set

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\vec a} & = & \{(E^1, E^2, \dots, E^n, \phi_1, \dots, \phi_n) \mid \phi_i : E^i_{|\mathbb{P}^1\setminus\{a_i\}} \xrightarrow{\simeq} E^{i-1}_{|\mathbb{P}^1\setminus\{a_i\}}, \mathrm{Inv}(\phi_i) \le \lambda_i\} \\ \Big\downarrow m_\lambda & & \Big\downarrow \\ \mathrm{Gr}^{\bar\lambda}_{0,\underline{a}} & & (E^n, \phi_1 \circ \cdots \circ \phi_n) \end{array}$$

Here, by our convention that E^0 denotes the trivial bundle, ϕ_1 gives a trivialisation of E_1 over $\mathbb{P}^1\setminus\{a_1\}$. Note that the composition $\phi_1 \circ \ldots \circ \phi_n$ is defined on $\mathbb{P}^1\setminus\{a_1, \dots, a_n\}$. Now again letting $\vec a$ vary we thus obtain a family $\widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\mathbb{A}^n}$ which has $\mathrm{Gr}^{\vec\lambda}_0$ as its central fibre. Let n_i be the multiplicity with which the fundamental weight $\breve\omega_i$ occurs in the sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$, and let $\Sigma_{\vec\lambda} = \prod_{i=1}^{\ell} S_{n_i}$ (where ℓ is the rank of G).

Because we have coloured our divisor, our affine deformation no longer lives over $\mathrm{Sym}^n \mathbb{A}^1$. Instead it lives over $\mathbb{A}^n/\Sigma_{\vec\lambda}$. We thus obtain a diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\mathbb{A}^n} & \longrightarrow & \mathbb{A}^n \\ {\scriptstyle \pi_{\mathrm{BD}}}\Big\downarrow & & \Big\downarrow {\scriptstyle q_{\mathrm{BD}}} \\ \mathrm{Gr}^{\lambda}_{0,\mathbb{A}^n} & \longrightarrow & \mathbb{A}^n/\Sigma_{\vec\lambda} \end{array}$$

Note that since we are assuming the λ_i are all minuscule, the families $\mathrm{Gr}^{\lambda}_{0,\mathbb{A}^n}$ and $\mathrm{Gr}^{\bar\lambda}_{0,\mathbb{A}^n}$ coincide. Recall that we gave our choice of coordinate t weight 1 for our $\mathbb{G}_m$-action on $\mathbb{P}^1$. This action on $\mathbb{P}^1$ induces an action of $\mathbb{G}_m$ on the diagram above, and thus we have a deformation of a conical symplectic resolution over a linear base $\mathbb{A}^n$ (for the smooth deformation $\widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\mathbb{A}^n}$).

Now as in Proposition 2.10, Namikawa has shown that his deformation spaces are universal for conic Poisson deformations; see Proposition 2.10 for a careful statement. By universality, there is a unique linear map $\alpha : \mathbb{A}^n \to H^2(\mathrm{Gr}^{\vec\lambda}_0)$ such that the families $\widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\mathbb{A}^n} \to \mathbb{A}^n$ and $\mathrm{Gr}^{\lambda}_{0,\mathbb{A}^n} \to \mathbb{A}^n/\Sigma_{\vec\lambda}$ are obtained by pulling back the universal families.

Lemma 7.12 *The map $\alpha : \mathbb{A}^n \to H^2(\mathrm{Gr}^{\vec\lambda}_0, \mathbb{C})$ is an isomorphism, and thus the symplectic Galois group action on $H^2(\mathrm{Gr}^{\vec\lambda}_0$ is that of $\Sigma_{\vec\lambda}$.*

Proof First note that our $\mathbb{G}_m$-action gives $\mathbb{A}^n$ the structure of a vector space, with 0-vector the fixed point of the action. Now the twisted products forming $\mathrm{Gr}^{\vec\lambda}_0$ are in fact isomorphic to a product, and hence we have $\mathrm{Pic}(\mathrm{Gr}^{\vec\lambda}_0) = \prod_{i=1}^n \mathrm{Pic}(\mathrm{Gr}^{\lambda_i}_0) = \mathbb{Z}^n$, so that $\dim(H^2(\mathrm{Gr}^{\vec\lambda}_0)) = n$ (as the cohomology of a symplectic resolution is spanned by algebraic cycles). Since α is a linear map between two vector spaces of the same dimension, it follows that it is an isomorphism provided it is injective. But if $K = \ker(\alpha) \neq \{0\}$, then $(\widetilde{\mathrm{Gr}}^{\vec\lambda}_{0,\mathbb{A}^n})_{|K} = K \times Y$, and hence the $\mathbb{G}_m$ action must have weight 0 on K. But the $\mathbb{G}_m$ action has weight 1 on all of $\mathbb{A}^n$, hence α must be injective and hence an isomorphism as required. The final sentence of the statement of the lemma is immediate. □

Corollary 7.13 *Every symplectic resolution of* $\mathrm{Gr}_0^{\bar\lambda}$ *is of the form* $\mathrm{Gr}_0^{\vec\lambda}$ *for some decomposition of* λ.

Proof This follows from Namikawa's results: any conic symplectic resolution $\pi : Y \to X$ allows one to build its associated universal deformation, and the universality implies that its different movable cones will exhaust all symplectic resolutions of X. □

To identify $\Sigma_{\vec\lambda}$ with the symplectic Galois group we will use the following lemma, which was first observed in [Kamnitzer 2011, Proposition 2.8]. We outline a proof for the reader's convenience.

Lemma 7.14 *The monodromy action of* Σ_λ *on* Spr *corresponds, under the geometric Satake correspondence, to the action by permutation of tensor factors.*

Proof This is essentially a consequence of the fusion construction of the commutativity constraint in [Mirković and Vilonen 2007]. In particular, it is shown in Section 5 of that paper that the convolution product can be interpreted via the fusion product, and in the proof of Lemma 6.1 of Section 6 that the pushforward to $\mathbb{A}^1 \times \mathbb{A}^1$ of the fusion product sheaf has constant cohomologies. These facts imply the required compatibility. □

It follows that we can reduce the calculation the action of the symplectic Galois group on the Springer sheaf $\mathcal{S}_{\vec\lambda}$ to the question of calculating the action of $\prod_{i\in I} S_{n_i}$ on the multiplicity spaces $M_{\vec\lambda,\nu}$ (in the notation of the preceding subsection). We use this in the next section to show that the multiplicity spaces in the Springer sheaf need not always be irreducible.

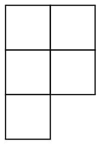
8 Tensor products of representations

The irreducible representations of $\mathfrak{sl}_{n+1}$, as for any semisimple Lie algebra, are indexed by highest weights. In the case of $\mathfrak{sl}_{n+1}$ however, these can naturally be identified with partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, where $k \le n$. Indeed, the root lattice of $\mathfrak{sl}_{n+1}$ is $Q = \{\alpha \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} \alpha_i = 0\}$, and hence the weight lattice P is naturally identified with $\mathbb{Z}^{n+1}/\mathbb{Z}.\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}^{n+1}$. The set of dominant weights P^+ consists of the elements $\lambda \in P$ whose representatives $\tilde\lambda = (\tilde\lambda_1, \dots, \tilde\lambda_{n+1}) \in \mathbb{Z}^{n+1}$ satisfy $\tilde\lambda_i \ge \tilde\lambda_{i+1}$ for $1 \le i \le n$. Clearly given $\lambda \in P$ it has a unique representative $\tilde\lambda$ with $\tilde\lambda_{n+1} = 0$. Thus any dominant weight λ of $\mathfrak{sl}_{n+1}$ has a partition with at most n parts associated to it, and we will write this partition as $(\lambda_1, \dots, \lambda_n)$. By abuse of terminology, we will refer to the dominant weight and its associated partition interchangeably. Write $V(\lambda)$ for the irreducible highest-weight representation with highest weight λ, though if we specify λ as a partition, eg $\lambda = (2, 1^2)$, we will write $V(2, 1^2)$ rather than $V((2, 1^2))$.

Definition 8.1 A partition of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of nonnegative integers (so $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge 0$) with $\sum_{i=1}^{l} \lambda_i = n$. The *length* of λ is defined to be $\ell(\lambda) = \max\{k : \lambda_k > 0\}$. A *Young diagram of shape* λ is a left-justified arrangement of boxes with λ_i boxes in row i, where the rows are arranged in descending order. By convention we take $\lambda_k = 0$ if $k > l = \ell(\lambda)$.

Now fix n a positive integer. A *tableau* T is a Young diagram of length at most n, where each box is assigned an integer between 1 and n. It is *standard* if the rows and columns are strictly increasing, and *semistandard* if its columns are strictly increasing but its rows are only weakly increasing. The *weight* $w(T)$ of a semistandard (skew)-tableau T is the sequence $(\mu_1, \mu_2, \dots, \mu_n)$, where μ_i is the number of boxes in the tableau with entry i. We write $\mathcal{S}(\lambda)$ for the set of semistandard tableaux of shape λ (ie whose underlying Young diagram is λ). Given a (skew)-tableau T we will write $T_{\geq j}$ for the subtableau of T formed by the entries in columns $j, j+1, \dots$.

Thus, for example, the Young diagram below is the diagram associated to the partition $(2,2,1)$ of 5, and the tableau is semistandard with shape $(4,3,2,1)$ and weight $(3,3,3,1)$:



1	1	1	2
2	2	3	
3	3		
4			

Theorem 8.2 (Littlewood–Richardson rule [Stembridge 2002]) *If $\mathfrak{g} = \mathfrak{sl}_n$ and λ, μ, ν are dominant weights for $\mathfrak{g}$, then the simple representation $V(\nu)$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$ with multiplicity $C^{\nu}_{\lambda,\mu}$, where $C^{\nu}_{\lambda,\mu}$ is the number of tableaux T of shape μ which satisfy*

(i) $\lambda + w(T_{\geq j}) \in P^+$ *for each* $j \geq 1$, *and*

(ii) $\lambda + w(T) = \nu$.

The fundamental representations of $\mathfrak{sl}_{n+1}$ are the exterior powers of V, the vector representation. They are the representations associated to the partitions (1^k) for $k = 1, \dots, n$. They are minuscule representations (that is, their weights form a single Weyl-group orbit).

Corollary 8.3 *Let $\omega_k = (1^k)$.*

(1) *We have $c^{\lambda}_{\mu,\omega_k} \in \{0,1\}$.*

(2) *If $n \geq 9$ and $\lambda = \omega_3 = (1^3)$, then*

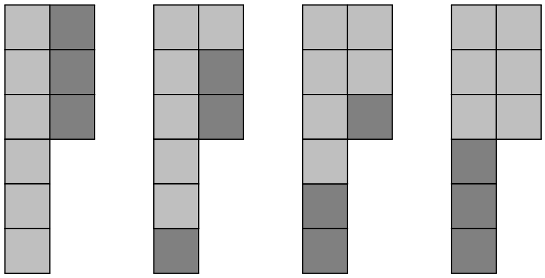
$$V(1^3) \otimes V(1^3) = V(1^6) \oplus V(2,1^5) \oplus V(2^2,1^4) \oplus V(2^3),$$

and $V(1^3)^{\otimes 3}$ contains $V(2^3,1^3)$ with multiplicity 4.

Proof For part (1), note that since $\omega_k = (1^k)$ has only 1 column, Stembridge's version of the Littlewood–Richardson rule shows that the multiplicity of $V(\lambda)$ in $V(\mu) \otimes V(\omega_k)$ is just the number of semistandard tableaux of shape 1^k and weight $\nu - \lambda$. But since (1^k) has only one column, the weight of the semistandard tableau uniquely determines the tableau.

For part (2), to calculate $V(1^3) \otimes V(1^3)$ we must find all (semi)standard tableaux T of shape (1^3) such that $(1^3) + w(T)$ is a partition. Now if $(1^3) + w(T) = (a_1, a_2, \dots, a_9)$, each $a_i \in \{0,1,2\}$. Let $m = \max\{a_i : 0 \leq i \leq 9\}$. If $m = 2$, then we must have $w_1 = 1$ and $w_2 \in \{2,4\}$ and $w_3 \in \{3,4\}$ or $\{5\}$, respectively, while if $m = 1$ then $(w_1, w_2, w_3) = (4,5,6)$.

Finally, it is easy to see that if we tensor with $V(1^3)^{\otimes 2}$ with $V(1^3)$, and let $\nu = (2^3, 1^3)$, then for each of the 4 summands $V(\mu)$ of $V(1^3)^{\otimes 2}$ the difference $\nu - \mu$ is a partition:



Hence $V(2^3, 1^3)$ occurs in $V(1^3)^{\otimes 3}$ with multiplicity 4 as required. □

Recall from Definition 3.5 that we say a local system $\mathscr{E}$ on a symplectic leaf is a Springer local system if $\mathrm{IC}(\mathscr{E})$ occurs as a constituent of the Springer sheaf.

Corollary 8.4 *Let* $\lambda = 3\varpi_3$ *and* $\mu = 0$, *and let* $\vec{\lambda} = (\varpi_3, \varpi_3, \varpi_3)$. *We may decompose the Springer sheaf* $\mathscr{S}$ *for* $\pi : \mathrm{Gr}_0^{\vec{\lambda}} \to \mathrm{Gr}_0^{\bar{\lambda}}$ *as*

$$\mathscr{S} \cong \bigoplus_{\nu \leq \lambda} \mathrm{IC}(\mathrm{Gr}_0^{\nu}) \otimes \mathscr{M}_\nu,$$

where each $\mathscr{M}_\nu$ *carries a natural action of* $W = S_3$ *the symplectic Galois group. When* $\nu = (2^3, 1^3)$, *this representation is reducible. Thus the symplectic Springer correspondence does not necessarily yield a function from Springer local systems to irreducible representations of the symplectic Galois group.*

Proof By Proposition 7.9, the dimension of the multiplicity space M_ν is the multiplicity of $V(2^3, 1^3)$ in $V(1^3)^{\otimes 3}$, and Corollary 8.3 shows that this multiplicity is 4. It follows that M_ν cannot be an irreducible representation of S_3. □

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References

[Beauville 2000] **A Beauville**, *Symplectic singularities*, Invent. Math. 139:3 (2000) 541–549 MR

[Bellamy and Schedler 2021] **G Bellamy**, **T Schedler**, *Symplectic resolutions of quiver varieties*, Selecta Math. 27:3 (2021) art. id. 36 MR

[Bernstein and Lunts 1994] **J Bernstein**, **V Lunts**, *Equivariant sheaves and functors*, Lecture Notes in Mathematics 1578, Springer (1994) MR

[Braden et al. 2016] **T Braden**, **A Licata**, **N Proudfoot**, **B Webster**, *Quantizations of conical symplectic resolutions, II: Category $\mathcal{O}$ and symplectic duality*, Astérisque 384, Soc. Math. France, Paris (2016) 75–179 MR

[Braverman et al. 2019] **A Braverman**, **M Finkelberg**, **H Nakajima**, *Coulomb branches of* 3*d* $\mathcal{N} = 4$ *quiver gauge theories and slices in the affine Grassmannian*, Adv. Theor. Math. Phys. 23:1 (2019) 75–166 MR

[Brion 2017] **M Brion**, *Algebraic group actions on normal varieties*, Trans. Moscow Math. Soc. 78 (2017) 85–107 MR

[de Cataldo and Migliorini 2002] **M A A de Cataldo**, **L Migliorini**, *The hard Lefschetz theorem and the topology of semismall maps*, Ann. Sci. École Norm. Sup. 35:5 (2002) 759–772 MR

[Chriss and Ginzburg 1997] **N Chriss**, **V Ginzburg**, *Representation theory and complex geometry*, Birkhäuser, Boston (1997) MR

[Crawley-Boevey 2003] **W Crawley-Boevey**, *Normality of Marsden–Weinstein reductions for representations of quivers*, Math. Ann. 325:1 (2003) 55–79 MR

[Dobrovolska et al. 2016] **G Dobrovolska**, **V Ginzburg**, **R Travkin**, *Moduli spaces, indecomposable objects and potentials over a finite field*, preprint (2016) arXiv 1612.01733

[Fu 2003] **B Fu**, *Symplectic resolutions for nilpotent orbits*, Invent. Math. 151:1 (2003) 167–186 MR

[Ginzburg and Kaledin 2004] **V Ginzburg**, **D Kaledin**, *Poisson deformations of symplectic quotient singularities*, Adv. Math. 186:1 (2004) 1–57 MR

[Goresky and MacPherson 1988] **M Goresky**, **R MacPherson**, *Stratified Morse theory*, Ergebnisse der Math. (3) 14, Springer (1988) MR

[Göttsche and Soergel 1993] **L Göttsche**, **W Soergel**, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. 296:2 (1993) 235–245 MR

[Grinberg 1998] **M Grinberg**, *A generalization of Springer theory using nearby cycles*, Represent. Theory 2 (1998) 410–431 MR

[Humphreys 1972] **J E Humphreys**, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics 9, Springer (1972) MR

[Kaledin 2006] **D Kaledin**, *Symplectic singularities from the Poisson point of view*, J. Reine Angew. Math. 600 (2006) 135–156 MR

[Kaledin and Verbitsky 2002] **D Kaledin**, **M Verbitsky**, *Period map for non-compact holomorphically symplectic manifolds*, Geom. Funct. Anal. 12:6 (2002) 1265–1295 MR

[Kamnitzer 2011] **J Kamnitzer**, *The Beilinson–Drinfeld Grassmannian and symplectic knot homology*, from "Grassmannians, moduli spaces and vector bundles" (D A Ellwood, E Previato, editors), Clay Math. Proc. 14, Amer. Math. Soc., Providence, RI (2011) 81–94 MR

[Kamnitzer et al. 2014] **J Kamnitzer**, **B Webster**, **A Weekes**, **O Yacobi**, *Yangians and quantizations of slices in the affine Grassmannian*, Algebra Number Theory 8:4 (2014) 857–893 MR

[Lê Dũng Tráng 1977] **Lê Dũng Tráng**, *Some remarks on relative monodromy*, from "Real and complex singularities" (Oslo, 1976) (P Holm, editor), Sijthoff & Noordhoff, Alphen aan den Rijn (1977) 397–403 MR

[Lehn et al. 2012] **M Lehn**, **Y Namikawa**, **C Sorger**, *Slodowy slices and universal Poisson deformations*, Compos. Math. 148:1 (2012) 121–144 MR

[Losev 2012] **I Losev**, *Isomorphisms of quantizations via quantization of resolutions*, Adv. Math. 231:3-4 (2012) 1216–1270 MR

[Losev 2017] **I Losev**, *On categories $\mathcal{O}$ for quantized symplectic resolutions*, Compos. Math. 153:12 (2017) 2445–2481 MR

[Lu and Yakimov 2008] **J-H Lu**, **M Yakimov**, *Group orbits and regular partitions of Poisson manifolds*, Comm. Math. Phys. 283:3 (2008) 729–748 MR

[Luna 1973] **D Luna**, *Slices étales*, from "Sur les groupes algébriques", Mém. Soc. Math. France 33, Soc. Math. France, Paris (1973) 81–105 MR

[Lusztig 1984] **G Lusztig**, *Intersection cohomology complexes on a reductive group*, Invent. Math. 75:2 (1984) 205–272 MR

[Lusztig 2000] **G Lusztig**, *Quiver varieties and Weyl group actions*, Ann. Inst. Fourier (Grenoble) 50:2 (2000) 461–489 MR

[Maffei 2002] **A Maffei**, *A remark on quiver varieties and Weyl groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 1:3 (2002) 649–686 MR

[Malkin et al. 2005] **A Malkin**, **V Ostrik**, **M Vybornov**, *The minimal degeneration singularities in the affine Grassmannians*, Duke Math. J. 126:2 (2005) 233–249 MR

[Markman 2010] **E Markman**, *Modular Galois covers associated to symplectic resolutions of singularities*, J. Reine Angew. Math. 644 (2010) 189–220 MR

[Martino 2008] **M Martino**, *Stratifications of Marsden–Weinstein reductions for representations of quivers and deformations of symplectic quotient singularities*, Math. Z. 258:1 (2008) 1–28 MR

[Mather 2012] **J Mather**, *Notes on topological stability*, Bull. Amer. Math. Soc. 49:4 (2012) 475–506 MR

[McGerty and Nevins 2018] **K McGerty**, **T Nevins**, *Kirwan surjectivity for quiver varieties*, Invent. Math. 212:1 (2018) 161–187 MR

[Milnor 1963] **J Milnor**, *Morse theory*, Annals of Mathematics Studies 51, Princeton Univ. Press (1963) MR

[Mirković and Vilonen 2007] **I Mirković**, **K Vilonen**, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. 166:1 (2007) 95–143 MR

[Nakajima 1998] **H Nakajima**, *Quiver varieties and Kac–Moody algebras*, Duke Math. J. 91:3 (1998) 515–560 MR

[Nakajima 2001] **H Nakajima**, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. 14:1 (2001) 145–238 MR

[Nakajima 2003] **H Nakajima**, *Reflection functors for quiver varieties and Weyl group actions*, Math. Ann. 327:4 (2003) 671–721 MR

[Namikawa 2008] **Y Namikawa**, *Flops and Poisson deformations of symplectic varieties*, Publ. Res. Inst. Math. Sci. 44:2 (2008) 259–314 MR

[Namikawa 2010] **Y Namikawa**, *Poisson deformations of affine symplectic varieties, II*, Kyoto J. Math. 50:4 (2010) 727–752 MR

[Namikawa 2011] **Y Namikawa**, *Poisson deformations of affine symplectic varieties*, Duke Math. J. 156:1 (2011) 51–85 MR

[Namikawa 2015] **Y Namikawa**, *Poisson deformations and birational geometry*, J. Math. Sci. Univ. Tokyo 22:1 (2015) 339–359 MR

[Ngô 1999] **B C Ngô**, *Faisceaux pervers, homomorphisme de changement de base et lemme fondamental de Jacquet et Ye*, Ann. Sci. École Norm. Sup. 32:5 (1999) 619–679 MR

[Quillen 1971] **D Quillen**, *The spectrum of an equivariant cohomology ring, I*, Ann. of Math. 94 (1971) 549–572 MR

[Sabbah 1983] **C Sabbah**, *Morphismes analytiques stratifiés sans éclatement et cycles évanescents*, from "Analysis and topology on singular spaces, II, III" (Luminy, 1981), Astérisque 101-102, Soc. Math. France, Paris (1983) 286–319 MR

[Sjamaar and Lerman 1991] **R Sjamaar**, **E Lerman**, *Stratified symplectic spaces and reduction*, Ann. of Math. 134:2 (1991) 375–422 MR

[Slodowy 1980] **P Slodowy**, *Four lectures on simple groups and singularities*, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht 11, Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht (1980) MR

[Springer 1976] **T A Springer**, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. 36 (1976) 173–207 MR

[Stembridge 2002] **J R Stembridge**, *A concise proof of the Littlewood–Richardson rule*, Electron. J. Combin. 9:1 (2002) Note 5, 4 MR

[Treumann 2009] **D Treumann**, *A topological approach to induction theorems in Springer theory*, Represent. Theory 13 (2009) 8–18 MR

[Zhu 2017] **X Zhu**, *An introduction to affine Grassmannians and the geometric Satake equivalence*, from "Geometry of moduli spaces and representation theory" (R Bezrukavnikov, A Braverman, Z Yun, editors), IAS/Park City Math. Ser. 24, Amer. Math. Soc., Providence, RI (2017) 59–154 MR

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