



Geometry & Topology

Volume 30 (2026)

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We introduce the theory of unipotent morphisms of algebraic stacks and prove a surprising local to global principle for a class of vector bundles. Two sample applications of our methods are a unipotent analogue of Gabber’s theorem for torsion \mathbf{G}_m -gerbes, and that smooth Deligne–Mumford stacks with quasiprojective coarse spaces satisfy the resolution property in positive characteristic. Our main tool is a descent result for flags, which we prove using results of Schäppi.

1 Introduction

In the early 1990s, Gabber answered a question of Grothendieck for quasiprojective schemes admitting an ample line bundle: every cohomological Brauer class on such a scheme is represented by an Azumaya algebra; see Gabber [14] and de Jong [22]. This result has since been reinterpreted in terms of the *resolution property* for algebraic stacks (i.e., every quasicoherent sheaf is a quotient of a direct sum of vector bundles) and gerbes; see Edidin, Hassett, Kresch and Vistoli [12], Gross [17], Kresch and Vistoli [24], Mathur [27] and Totaro [45]. In this language, Gabber’s result can be restated as follows:

Theorem (Gabber) *Let X be a quasiprojective scheme that admits an ample line bundle. Let $\mathcal{X} \rightarrow X$ be a \mathbf{G}_m -gerbe. If the cohomology class $[\mathcal{X}] \in H^2(X, \mathbf{G}_m)$ of \mathcal{X} is torsion, then \mathcal{X} is a global quotient stack (see Definition 2.1) and has the resolution property.*

Using quasiprojective methods and Gabber’s theorem, Kresch and Vistoli show that smooth, separated and generically tame Deligne–Mumford stacks with quasiprojective coarse moduli spaces have the resolution property [24]. It has long been a challenge to produce nontrivial vector bundles without such hypotheses.

Here we introduce new methods to construct nontrivial vector bundles on schemes, algebraic spaces and algebraic stacks. Our key idea is to leverage the abundance of cohomology in unipotent settings. A sample application of our results is the following additive analogue of Gabber’s result (also see Theorem 6.3):

Theorem A *Let X be a quasiprojective algebraic stack with affine diagonal and the resolution property. If $\mathcal{G} \rightarrow X$ is a \mathbf{G}_a -gerbe, then \mathcal{G} has the resolution property.*

In particular, this implies that \mathbf{G}_a -gerbes over smooth separated schemes have the resolution property. Note that the analogous multiplicative statement is still unknown, even in dimension 3. We can also extend Kresch and Vistoli’s result to the generically wild setting:

Theorem B *Let \mathcal{X} be a smooth and separated Deligne–Mumford stack of finite type over a field k . Let $\pi: \mathcal{X} \rightarrow X$ be the associated coarse moduli space. If X is quasiprojective over $\text{Spec } k$, then \mathcal{X} has the resolution property.*

MSC2020: primary 14A20; secondary 14G17.

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We expect the key new technical input for the above results to be of general interest: it states that one may descend certain vector bundles along flat morphisms after locally taking a direct sum (see Theorem 4.8 for a more general statement):

Theorem C *Let $\mathcal{X} \rightarrow X$ be a morphism of quasicompact and quasiseparated algebraic stacks. Suppose that X has affine diagonal and satisfies the resolution property. If there is a faithfully flat morphism $X' \rightarrow X$, where X' is affine, and a vector bundle V' on $\mathcal{X}' = \mathcal{X} \times_X X'$ that admits a filtration*

$$0 = V'_0 \subseteq V'_1 \subseteq \cdots \subseteq V'_n = V'$$

*with $V'_i/V'_{i-1} \simeq \mathcal{O}_{\mathcal{X}'}$ for all i (i.e., V' is **trivially graded**), then there is a vector bundle V on \mathcal{X} and a split surjection $V_{\mathcal{X}'} \rightarrow V'$.*

In fact, our results apply in the broader setting of *unipotent morphisms*, a notion that we develop and characterize in Section 5. Roughly speaking, the theory is modeled on gerbes with unipotent stabilizers and algebraic stacks of the form $[Z/U_n]$, where Z is an algebraic space and U_n is the unipotent subgroup of GL_n consisting of unitriangular matrices. Compare with [12; 17; 45], where quotients of the form $[Z/\mathrm{GL}_n]$ and the resolution property are considered. In fact, we present a unipotent enrichment of their results (see Theorem 5.4) and, better still, we prove a striking descent statement. Indeed, Theorem C reveals the following local to global principle: *a locally unipotent morphism over a base with enough flags is globally unipotent* (see Theorem 5.5).

These notions are closely related to many foundational questions of independent interest. For example, over a field k a *unipotent group scheme* is a closed subgroup of $U_{n,k}$ (see Demazure and Grothendieck [10, Theorem XVII.3.5] and Remark 5.14 for several other characterizations). It is natural to ask: to what extent is this true over a general base?

Question 1.1 *Let $G \rightarrow S$ be a group scheme with unipotent geometric fibers. Is there an embedding $G \rightarrow U_{n,S}$ for some $n \geq 1$? Is this true locally on S ?*

As a consequence of our methods, if S is a regular separated scheme or admits an ample line bundle, then the existence of a flat-local embedding in U_n implies the existence of a Zariski-local embedding of G in U_n (see Corollary 5.13 for a more general statement). In Section 5.4, we refine Question 1.1 and explain some cases where the answer is positive (see Examples 5.22 and 5.23) and one case, in positive characteristic, where it is negative (Example 5.21(1)). One may also compare Question 1.1 with a question of Conrad on when smooth affine group schemes can be embedded into GL_n ; see Conrad [7].

Unipotent morphisms are closely related to the notion of a faithful moduli space, due to Alper (personal communication, 2017). This is a morphism of algebraic stacks $f: \mathcal{X} \rightarrow X$ such that

- (1) the natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism (i.e., f is Stein), and
- (2) if F is quasicoherent on \mathcal{X} and $f_*F = 0$, then $F = 0$.

Faithful moduli spaces should help with the analysis of moduli stacks with only unipotent stabilizers (compare with good or adequate moduli spaces of Alper [2; 3]) that appear in the unstable locus of GIT

quotients. In contrast to good moduli spaces, however, faithful moduli spaces are rarely compatible with base change. This makes them difficult to reduce to simple cases where we can apply deformation theory. Faithful moduli spaces are also stable under composition.

We show that faithful moduli spaces and locally unipotent morphisms have unipotent stabilizers (Propositions 5.15 and 7.10) and that faithful moduli spaces are generically gerbes for a unipotent group (Proposition 7.12). We also show that certain Stein unipotent morphisms are faithful moduli spaces (Proposition 7.1).

2 Vector bundles, global quotients, and the resolution property

We briefly recall a circle of ideas from [12; 17; 45]. We also make some minor refinements, which will be useful. To agree with the conventions of [17, Section 5], a *vector bundle* on X will mean a locally free \mathcal{O}_X -module of finite constant rank.

Let X be an algebraic stack and let n be a nonnegative integer. If V is a vector bundle on X of rank n , there is an associated *frame bundle* $\mathrm{Fr}(V) = \underline{\mathrm{Isom}}_X(\mathcal{O}_X^{\oplus n}, V)$. Then $\mathrm{Fr}(V) \rightarrow X$ is a GL_n -torsor and sits in the following 2-cartesian diagram:

$$\begin{array}{ccc} \mathrm{Fr}(V) & \longrightarrow & \mathrm{Spec} \mathbf{Z} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{BGL}_n \end{array}$$

It is well known that this association induces an equivalence of categories between vector bundles on X of rank n and GL_n -torsors.

Definition 2.1 Let X be an algebraic stack. We say that X is a *global quotient* if $X \cong [Z/\mathrm{GL}_n]$, where Z is a quasicompact and quasiseparated algebraic space with an action of GL_n for some nonnegative integer n . If V denotes the vector bundle of rank n associated to the GL_n -torsor $Z \rightarrow X$, we say that V is *faithful*. A morphism of algebraic stacks $f: X \rightarrow S$ is a *global quotient* if there is a vector bundle V of rank n on X with frame bundle that is quasicompact, quasiseparated and representable over S . In this case, we say that V is *f-faithful*.

Remark 2.2 A vector bundle V on X is easily seen to be f -faithful for $f: X \rightarrow S$ if and only if the relative stabilizer groups of f act faithfully on V at every geometric point of X . In particular, the relative stabilizer groups are affine [8, Corollary VI_B.1.4.2]. Moreover, if $X \xrightarrow{f} S \xrightarrow{g} T$ are quasicompact and quasiseparated morphisms of algebraic stacks and V is $(g \circ f)$ -faithful, then V is f -faithful. Also, X is a global quotient if and only if the morphism $X \rightarrow \mathrm{Spec} \mathbf{Z}$ is a global quotient.

Remark 2.3 It is natural to ask if algebraic stacks such as $X_1 = \coprod_{n \geq 1} \mathrm{BGL}_{n, \mathbf{Z}}$ or $X_2 = \mathrm{B}\mathbf{Z}$, where \mathbf{Z} denotes the constant group scheme associated to the integers, should be considered global quotients. There is certainly a locally free sheaf on X_1 with a faithful action, but it has nonconstant and unbounded rank; also, X_2 admits a vector bundle of rank 2 with a faithful action whose frame is quasicompact but

not quasiseparated. We exclude these as we are unable to shed much light on them and also to adhere to the conventions of [17, Section 5].

Let \mathcal{A} be an abelian category. Let Λ be a set of objects of \mathcal{A} . Recall that Λ *generates* \mathcal{A} (or Λ is *generating*) if the functor

$$H_\Lambda: \mathcal{A} \rightarrow \mathbf{Ab}: x \mapsto \prod_{a \in \Lambda} \text{Hom}_{\mathcal{A}}(a, x)$$

is faithful. If \mathcal{A} is closed under small coproducts, then Λ is generating if and only if every $x \in \mathcal{A}$ admits a presentation of the form $\bigoplus_{a \in \Lambda} (a^{\oplus I_a}) \twoheadrightarrow x$.

Remark 2.4 Let \mathcal{A} be an abelian category and Λ a set of objects of \mathcal{A} . Then H_Λ is *conservative* (i.e., if $f: x \rightarrow x'$ in \mathcal{A} and $H_\Lambda(f)$ is bijective, then f is an isomorphism) if and only if Λ generates \mathcal{A} . Note that while H_Λ conservative implies that it is *zero-reflecting* (i.e., $H_\Lambda(x) = 0$ implies that $x = 0$), the converse does not hold in general — it does when the $a \in \Lambda$ are all \mathcal{A} -projective. For a counterexample, consider the affine line with the doubled origin; then $H_{\{\mathcal{O}\}}$ is zero-reflecting but not conservative.

Example 2.5 If X is an algebraic stack, then the abelian category $\mathbf{QCoh}(X)$ always admits a set of generators [43, Tag 0781]. If X is quasicompact and quasiseparated, then a set of generators can always be found amongst the \mathcal{O}_X -modules of finite presentation [39]. Note this result is much simpler if X is noetherian [25, Proposition 15.4] or has quasifinite and separated diagonal [37].

The *resolution property*, which we refine and recall below, is about algebraic stacks that have a generating set of vector bundles.

Definition 2.6 Let X be an algebraic stack. Let \mathbf{V} be a set of vector bundles on X . We say that X has the *\mathbf{V} -resolution property* if it is quasicompact, quasiseparated and \mathbf{V} generates $\mathbf{QCoh}(X)$. Following [17, Definition 5.1], we say X has the *resolution property* if it has the \mathbf{V} -resolution property for some set \mathbf{V} .

Remark 2.7 Fix a noetherian algebraic stack X and let \mathbf{V} denote the set of isomorphism classes of vector bundles on X . Then X has the \mathbf{V} -resolution property if and only if every coherent sheaf on X is the quotient of a locally free sheaf of finite rank; see [17, Remark 5.2].

Just as in [17], we will be interested in a relative version of the resolution property, so we recall the following definition:

Definition 2.8 [17, Definition 2.7] Let $f: X \rightarrow S$ be a morphism of algebraic stacks. Let \mathbf{V} be a set of finitely presented quasicohherent sheaves on X . We say that \mathbf{V} is *f -generating* if f is quasicompact and quasiseparated and there is a set of (not necessarily finitely presented) quasicohherent \mathcal{O}_S -modules \mathbf{W} such that $\mathbf{V} \otimes_{\mathcal{O}_X} f^* \mathbf{W} = \{V \otimes_{\mathcal{O}_X} f^* W : V \in \mathbf{V}, W \in \mathbf{W}\}$ generates $\mathbf{QCoh}(X)$. We say that \mathbf{V} is *universally f -generating* if for every morphism of algebraic stacks $s: S' \rightarrow S$, the set $s'^* \mathbf{V} = \{s'^* V : V \in \mathbf{V}\}$, where $s': X' = X \times_S S' \rightarrow X$ and $f': X' \rightarrow S'$ denote the projections, is *f' -generating*.

Remark 2.9 Note that \mathbf{V} is universally f -generating if and only if it is so on an fpqc covering of the target; see [17, Proposition 2.8 (iii)]. Also, if \mathbf{V} is f -generating, then it remains so after quasiaffine base change

(combine [17, Proposition 2.8 (ii), (v) and (vi)]). In particular, if S has quasiaffine diagonal, then \mathbf{V} is f -generating if and only if it is universally f -generating; see [17, Corollary 2.10]. Ignoring set-theoretic issues, Example 2.5 implies that we can take $\mathbf{W} = \mathbf{QCoh}(S)$ in the definition of f -generating.

Definition 2.10 Let $f: X \rightarrow S$ be a morphism of algebraic stacks. If \mathbf{V} is a set of vector bundles on X , we say f has the \mathbf{V} -resolution property if \mathbf{V} is universally f -generating. Following [17, Definition 5.1], we say that f has the resolution property if f has the \mathbf{V} -resolution property for some \mathbf{V} . If such a \mathbf{V} arises as a set of submodules of the polynomials in a fixed vector bundle V and its dual that are split by restriction to the frame bundle of V , then we say that V is an f -tensor generator [17, Definition 6.1]. Such a V is f - R -faithful if it is also f -faithful (note that f -tensor-generators are automatically f -faithful if f has relatively affine stabilizers by [17, Theorem 6.4]).

Remark 2.11 An algebraic stack X has the \mathbf{V} -resolution property if and only if the morphism $X \rightarrow \mathrm{Spec} \mathbf{Z}$ has the \mathbf{V} -resolution property. In general, if Y has the \mathbf{V}' -resolution property and $f: X \rightarrow Y$ is a morphism of algebraic stacks with the \mathbf{V} -resolution property, then X has the $(\mathbf{V} \otimes_{\mathcal{O}_X} f^* \mathbf{V}')$ -resolution property [17, Proposition 2.8(v)].

Example 2.12 Let A be an abelian scheme of positive dimension over a field k . If $g: \mathrm{Spec} k \rightarrow BA$ denotes the standard covering, then $\mathcal{O}_{\mathrm{Spec} k}$ is g -generating but not universally g -generating. Also, g^* induces an equivalence $\mathbf{QCoh}(BA) \simeq \mathbf{QCoh}(\mathrm{Spec} k)$, so $f: BA \rightarrow \mathrm{Spec} k$ has the resolution property with \mathcal{O}_{BA} as a f -tensor-generator. However, BA admits no f -faithful vector bundles.

Example 2.13 If X admits an ample family L_1, \dots, L_r of line bundles [43, Tag 0FXR] (e.g., X is quasiprojective over a field or is noetherian, normal and \mathbf{Q} -factorial with affine diagonal), then it satisfies the \mathbf{V} -resolution property [17, Example 5.9(i)], where $\mathbf{V} = \{L_i^m : m \leq 0 \text{ for } i = 1, \dots, r\}$. In dimensions 2 and above, there are proper schemes with no nontrivial line bundles. However, 2-dimensional separated algebraic spaces always have enough vector bundles; see [42, Theorem 2.1; 16, Theorem 5.2; 27, Theorem 41]. Nothing is known for smooth separated algebraic spaces or normal separated schemes over a field in dimension ≥ 3 , except that they enjoy the resolution property after removing a closed subset of codimension ≥ 3 [29].

Example 2.14 For algebraic stacks the situation is more complicated. Kresch and Vistoli showed that a Deligne–Mumford stack which is smooth, generically tame and separated over a field satisfies the resolution property if its coarse moduli space is quasiprojective [24, Theorem 1.3]. The case of 2-dimensional normal tame algebraic stacks with finite diagonal over a field was settled in [27, Theorem 1]. There are also nontorsion (and hence, nonregular) \mathbf{G}_m -gerbes which do not have the resolution property [18, Remark 1.11b]. As far as the authors are aware, there is no known example of a separated or smooth algebraic stack with affine diagonal which does not have the resolution property.

Surprisingly, Totaro and Gross (see [45, Theorem 1.1; 17, Theorems 1.1, and 6.10]) were able to relate quotient stacks and the resolution property. We have the following refinement, whose proof is the same as Gross', but we just keep track of the generating set:

Theorem 2.15 *Let $f: X \rightarrow S$ be a morphism of quasicompact and quasiseparated algebraic stacks. Assume that f has affine stabilizers.*

- (1) *Let \mathbf{V} be a set of vector bundles on X that is closed under direct sums. If \mathbf{V} is universally f -generating, then there is a vector bundle $V \in \mathbf{V}$ of rank n on X whose frame bundle is quasiaffine over S .*
- (2) *Let V be a vector bundle on X . Then V has frame bundle quasiaffine over S if and only if V is an f -tensor generator.*

In particular, a quasicompact and quasiseparated algebraic stack with affine stabilizers at closed points has the resolution property if and only if it can be written as $[U/\mathrm{GL}_n]$, where U is a quasiaffine scheme.

Proof Claim (2) is [17, Theorem 6.4]. For (1), consider the natural inverse system of X -stacks formed by

$$F_J: \prod_{W \in J} \mathrm{Fr}(W) \rightarrow X,$$

where $J \subset \mathbf{V}$ is a finite subset. Since each frame bundle is affine over X , the transition maps in the inverse system are affine. Hence the inverse limit is an algebraic stack $p: F \rightarrow X$, which is affine over X [37, Theorem C(i)]. Since \mathbf{V} is generating for f and every vector bundle in \mathbf{V} becomes trivial when restricted to F , it follows that \mathcal{O}_F is a tensor generator for the morphism $f \circ p: F \rightarrow X \rightarrow S$. Thus the morphism $f \circ p$ is quasiaffine [17, Proposition 4.1]. By absolute approximation [37, Theorem C(i)], there is a finite subset $J \subset \mathbf{V}$ such that $F_J \rightarrow S$ is quasiaffine. By [35, Lemma 1.1], the frame bundle of $V = \bigoplus_{W \in J} W \in \mathbf{V}$ is quasiaffine. For the final claim, it suffices to show that an algebraic stack with the resolution property and affine stabilizers at closed points, has affine stabilizers everywhere. This is just [17, Lemma 5.15]. □

If a quasicompact and quasiseparated scheme has an ample line bundle, then it can be written in the form $[U/\mathrm{GL}_1]$, where U is a quasiaffine scheme. From this perspective, it is natural to view the resolution property as a higher-dimensional analogue of quasiprojectivity. Note that the converse does not hold (e.g., take $U = \mathbf{A}^2 - \{0\}$ with GL_1 acting with weights $(1, -1)$; then $[U/\mathrm{GL}_1]$ is the line with the doubled origin, which is not separated, so does not admit an ample line bundle—but it does admit an ample family of line bundles).

Definition 2.16 Let S be an algebraic stack. We say that a morphism $G \rightarrow S$ is a *group* if it is a group object in representable morphisms over S . If $G \rightarrow S$ is a group that is flat and of finite presentation, then we say that it is

- (1) *embeddable* if it admits a group monomorphism to $\mathrm{GL}(E)$ for some vector bundle E of rank n on S ,
- (2) *R-embeddable* if it admits a group monomorphism to $\mathrm{GL}(E)$ for some vector bundle E of rank n on S such that the quotient $\mathrm{GL}(E)/G$ is quasiaffine over S (note that this implies that $G \rightarrow \mathrm{GL}(E)$ is a closed immersion).

Example 2.17 An embedded group is not necessarily R-embedded, even over a field. Indeed, if $B_{2,C} \subset GL_{2,C}$ is a Borel subgroup, then $GL_{2,C}/B_{2,C} \simeq \mathbf{P}^1_C$.

Remark 2.18 If $G \rightarrow S$ is a group that is flat and of finite presentation, then $G \rightarrow S$ is embeddable if and only if the structure morphism of the classifying stack $BG \rightarrow S$ is a global quotient. Indeed, if $BG \rightarrow S$ is a global quotient, then there is a vector bundle E on BG whose total space is representable over S . This gives a representable S -morphism $BG \rightarrow BGL_{\text{rank } E}$; the induced map on inertia groups gives a monomorphism $G \hookrightarrow GL(E')$ for some vector bundle E' on S . The converse is similar. In particular, $G \rightarrow S$ is embeddable if and only if there is a vector bundle E on S with a faithful action of G . We call these *faithful* G -representations. If, in addition, $G \rightarrow S$ has affine fibers, then it is R-embeddable if and only if $BG \rightarrow S$ has the resolution property (Theorem 2.15). In particular, such a $G \rightarrow S$ is R-embeddable if and only if there is a vector bundle E on S with a faithful action such that the quotient $GL(E)/G \rightarrow S$ is a quasiaffine morphism. We will call these G -representations *R-faithful*.

We expect the following result to be well known to experts. Over a field, it is due to Rosenlicht [34, Theorem 3].

Proposition 2.19 *Let S be an algebraic stack.*

- (1) *The quotient morphism $GL_{n,S}/U_{n,S} \rightarrow S$ is quasiaffine.*

Let $G \subseteq U_{n,S}$ be a closed subgroup that is flat and of finite presentation over S .

- (2) *If S is a normal scheme and the quotient $U_{n,S}/G$ is representable by a scheme (e.g., if S is Dedekind or the spectrum of a field, see [8, Remark VI_B.9.3(b)]), then the quotient $U_{n,S}/G \rightarrow S$ is quasiaffine.*
- (3) *If $G \rightarrow S$ is finite, then the quotient $U_{n,S}/G \rightarrow S$ is affine.*

Proof Since the inclusion $U_{n,S} \subset GL_{n,S}$ is defined over $\text{Spec } \mathbf{Z}$, in (1) it suffices to assume $S = \text{Spec } \mathbf{Z}$. In particular, we may assume that S is Dedekind. We will now prove that both of the homogenous spaces $U_{n,S}/G$ and $GL_{n,S}/U_{n,S}$ are quasiaffine over S using [33, Theorem VII.2.1], thereby establishing both (1) and (2). Since $GL_{n,S}$ and $U_{n,S}$ are both smooth with connected fibers, it suffices to show that the abelian groups $\text{Pic}(U_{n,k(s)}/G_{k(s)})$ and $\text{Pic}(GL_{n,k(s)}/U_{n,k(s)})$ are torsion. To this end, first note that the cohomology of the complex

$$\text{Hom}(K, \mathbf{G}_m) \rightarrow \text{Pic}(H/K) \rightarrow \text{Pic}(H)$$

is torsion for any inclusion of finite-type group schemes $K \subset H$ over a field by [33, Theorem VII.1.5]. Moreover, by [9, Proposition XVII.2.4(ii)] we always have $\text{Hom}(K, \mathbf{G}_m) = 0$ for every unipotent group scheme K over a field. Further, the Picard groups of $U_{n,k(s)}$ and $GL_{n,k(s)}$ both vanish. Indeed, in the former case the underlying scheme is isomorphic to affine space, and in the latter case it is a principal open of affine space. Thus the two outer terms in the complex are zero in both of our cases and it follows that $\text{Pic}(U_{n,k(s)}/G_{k(s)})$ and $\text{Pic}(GL_{n,k(s)}/U_{n,k(s)})$ are torsion, as desired.

For (3), the morphism $U_{n,S} \rightarrow U_{n,S}/G$ is finite flat. Since $U_{n,S} \rightarrow S$ is affine, the same is true for $U_{n,S}/G$; e.g., [37, Theorem 8.1]. □

Proposition 2.20 *Let $f: X \rightarrow S$ be a quasicompact and quasiseparated morphism of algebraic stacks.*

- (1) *Let W and V be vector bundles on X . If the frame bundle F_W is representable (resp. quasiaffine) over S , then the frame bundle of $V \oplus W$ is also representable (resp. quasiaffine) over S .*
- (2) *If f is adequately affine (e.g., a gerbe banded by a reductive group scheme, a good/adequate moduli space morphism or a coarse moduli space morphism [3, Definition 4.1.1]), then every f -faithful vector bundle V on X has a frame bundle which is affine over S .*

Proof For the representability part of (1), the hypothesis implies that the relative stabilizers act faithfully on the fibers of W and therefore they must act faithfully on the fibers of $V \oplus W$, as desired. For the latter claim, see [35, Lemma 1.1]. For (2), $F_V \rightarrow S$ is representable and adequately affine, and so is affine [3, Theorem 4.3.1]. \square

3 Schäppi's theorem

Let A be a ring and let M be a flat A -module. Lazard's theorem [26] tells us that M is a filtered colimit of free A -modules of finite rank. Extending this result to the nonaffine situation is surprisingly subtle [13]. Nonetheless, Schäppi recently proved a remarkable Lazard-type theorem for a restricted but very useful class of flat modules that arise in algebraic geometry [40, Theorem 1.3.1].

Schäppi approaches his result through comodules over flat Hopf algebroids. Since Schäppi's result is crucial for our article and ought to be better known amongst algebraic geometers, we have translated his category-theoretic proof into a direct proof for algebraic stacks. We have made some simple but algebro-geometrically natural generalizations to his hypotheses. Note that while nonaffine schemes tend to not have interesting projective objects, there are interesting algebraic stacks (e.g., those with affine tame [1] or good [2] moduli spaces) that do.

Theorem 3.1 (Schäppi) *Let $f: Y \rightarrow X$ be a flat morphism of quasicompact and quasiseparated algebraic stacks. Let \mathbf{V} be a set of isomorphism classes of vector bundles on X . Let M be a vector bundle on Y that is a projective object of $\mathbf{QCoh}(Y)$. If X has the \mathbf{V} -resolution property, then f_*M is a filtered colimit of finite direct sums of objects of \mathbf{V} and their duals. In particular, if Y is cohomologically affine (e.g., an affine scheme), then $f_*\mathcal{O}_Y$ is a filtered colimit of vector bundles.*

Proof We may replace \mathbf{V} by the set of all finite direct sums of objects of \mathbf{V} and their duals. Let G be a quasicohherent \mathcal{O}_X -module. Consider the category \mathbf{V}_G whose objects are pairs (H, η) , where $H \in \mathbf{V}$ and $\eta: H \rightarrow G$ is an \mathcal{O}_X -module homomorphism. A morphism $h: (H, \eta) \rightarrow (H', \eta')$ in \mathbf{V}_G is an \mathcal{O}_X -module homomorphism $h: H \rightarrow H'$ such that $\eta = \eta' \circ h$. Consider the functor $\mu_G: \mathbf{V}_G \rightarrow \mathbf{QCoh}(X)$ that sends (H, η) to H . It remains to establish the following two claims:

Claim 3.1.1 *Every pair of objects in \mathbf{V}_G has an upper bound and $\text{colim}(\mu_G) \simeq G$.*

Proof If (H_1, η_1) and $(H_2, \eta_2) \in \mathbf{V}_G$, then $(H_1 \oplus H_2, \eta_1 \oplus \eta_2) \in \mathbf{V}_G$. This proves the existence of upper bounds for every pair. We now prove that $\text{colim}(\mu_G) \simeq G$. To do this, consider a quasicohherent

\mathcal{O}_X -module N together with \mathcal{O}_X -module homomorphisms $v_{(H,\eta)}: H \rightarrow N$ for every $(H, \eta) \in \mathbf{V}_G$, such that if $h: (H, \eta) \rightarrow (H', \eta')$ is a morphism in \mathbf{V}_G , then $v_{(H,\eta)} = v_{(H',\eta')} \circ h$. It suffices to show that there is a uniquely induced morphism $v: G \rightarrow N$ such that $v_{(H,\eta)} = v \circ \eta$ for every $(H, \eta) \in \mathbf{V}_G$. Since X has the \mathbf{V} -resolution property, G admits a presentation

$$\bigoplus_{j \in J} P_j \xrightarrow{\bigoplus p_j} \bigoplus_{(H,\eta) \in \mathbf{V}_G} H \xrightarrow{\bigoplus_{(H,\eta) \in \mathbf{V}_G} \eta} G \rightarrow 0,$$

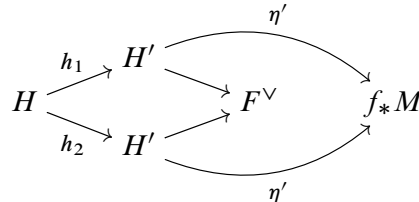
where P_j belongs to \mathbf{V} for all $j \in J$. Let $j \in J$ and note that the morphism $p_j: P_j \rightarrow \bigoplus_{(H,\eta) \in \mathbf{V}_G} H$ factors as

$$P_j \xrightarrow{\tilde{p}_j} \bigoplus_{(H,\eta) \in I_j} H \subseteq \bigoplus_{(H,\eta) \in \mathbf{V}_G} H,$$

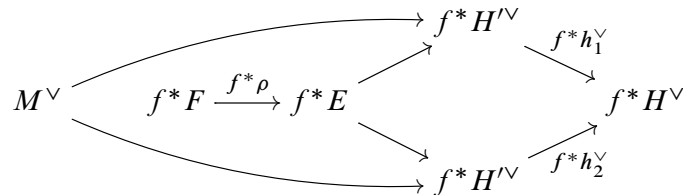
where $I_j \subseteq \mathbf{V}_G$ is finite. Let $\tilde{Q}_j = \bigoplus_{(H,\eta) \in I_j} H$ and let $\tilde{q}_j = \bigoplus_{(H,\eta) \in I_j} \eta: \tilde{Q}_j \rightarrow G$ be the resulting morphism. Then $\tilde{q}_j \circ \tilde{p}_j = 0$ and so $\tilde{p}_j: (P_j, 0) \rightarrow (\tilde{Q}_j, \tilde{q}_j)$ is a morphism in \mathbf{V}_G . Hence $v_{(\tilde{Q}_j, \tilde{q}_j)} \circ \tilde{p}_j = v_{(P_j, 0)} = v_{(P_j, 0)} \circ 0 = 0$. But this means $(\bigoplus_{(H,\eta) \in \mathbf{V}_G} v_{(H,\eta)}) \circ (\bigoplus_j p_j) = 0$. By the universal property of cokernels, there is a unique morphism $v: G \rightarrow N$ such that $v \circ \eta = v_{(H,\eta)}$ for all $(H, \eta) \in \mathbf{V}_G$. \square

Claim 3.1.2 *If M is a vector bundle on Y that is a projective object in $\mathbf{QCoh}(Y)$, then \mathbf{V}_{f_*M} is filtered.*

Proof Consider a pair of morphisms $h_1, h_2: (H, \eta) \rightarrow (H', \eta')$; we must show that these can be coequalized in \mathbf{V}_{f_*M} . Take the duals of h_1 and h_2 , which results in morphisms $h_1^\vee, h_2^\vee: H'^\vee \rightarrow H^\vee$. Let E be their equalizer in $\mathbf{QCoh}(X)$. If $(F, \rho) \in \mathbf{V}_E$, then taking duals of everything results in a commutative diagram:



Thus we just need to produce $(F, \rho) \in \mathbf{V}_E$ that admits a compatible morphism $F^\vee \rightarrow f_*M$. By taking adjoints in the above diagram, we see that it is sufficient to produce a compatible morphism $f^*F^\vee \rightarrow M$. But M is a vector bundle, so it is sufficient to produce a compatible morphism $M^\vee \rightarrow f^*F$. Dualizing their defining diagrams we obtain



But f is flat, so f^*E is also the equalizer of $f^*h_1^\vee$ and $f^*h_2^\vee$; hence there is a compatibly induced morphism $M^\vee \rightarrow f^*E$. By Claim 3.1.1, $E \simeq \text{colim}(\mu_E)$; therefore $f^*E \simeq f^* \text{colim}(\mu_E)$. Now

M^\vee is a vector bundle and Y is quasicompact and quasiseparated, so the functor $\text{Hom}_{\mathcal{O}_Y}(M^\vee, -) = \Gamma(Y, M \otimes_{\mathcal{O}_Y} -)$ preserves filtered colimits of quasicoherent sheaves [20, Lemma 1.2(iii)]. Since M is projective in $\mathbf{QCoh}(Y)$, M^\vee is projective¹ in $\mathbf{QCoh}(Y)$. Hence $\text{Hom}_{\mathcal{O}_Y}(M^\vee, -)$ is also exact and so commutes with *all* colimits of quasicoherent sheaves. Thus we obtain

$$\text{Hom}_{\mathcal{O}_Y}(M^\vee, f^* \text{colim}(\mu_E)) \simeq \text{Hom}_{\mathcal{O}_Y}(M^\vee, \text{colim}(f^* \mu_E)) \simeq \text{colim}_{(F, \rho) \in \mathbf{V}_E} \text{Hom}_{\mathcal{O}_Y}(M^\vee, f^* F).$$

In particular, $M^\vee \rightarrow f^* E$ factors through some morphism $f^* \rho: f^* F \rightarrow f^* E$, where $(F, \rho) \in \mathbf{V}_E$ [43, Tag 09WR]. The claim follows. \square

Example 3.2 In Theorem 3.1, it is necessary to include the duals of the generating set \mathbf{V} . Indeed, let $X = \mathbf{P}_{x,y}^1$ over a field k with coordinates x and y . Let $Y = D_+(x)$ and take $f: Y \rightarrow X$ to be the inclusion. Then $L = \mathcal{O}(1)$ is ample and X has the $\mathbf{V} = \{L^m\}_{m \leq 0}$ -resolution property (Example 2.13). But $f_* \mathcal{O}_Y \simeq \varinjlim_n L^n$ cannot be written as a filtered colimit of direct sums of objects from \mathbf{V} .

The converse to Theorem 3.1 is an earlier result of Hovey [21, Proposition 1.4.4]:

Proposition 3.3 *Let $f: Y \rightarrow X$ be a faithfully flat affine morphism of quasicompact and quasiseparated algebraic stacks, where Y is quasiaffine. Let \mathbf{V} be a set of vector bundles on X . If $f_* \mathcal{O}_Y$ is a filtered colimit of duals of finite direct sums of objects of \mathbf{V} , then X has the \mathbf{V} -resolution property.*

Proof It suffices to prove that \mathbf{V} generates $\mathbf{QCoh}(X)$. Write $f_* \mathcal{O}_Y = \text{colim}_{\lambda \in \Lambda} H_\lambda^\vee$, where $H_\lambda = \bigoplus_{V \in J_\lambda} V^{\oplus n_\lambda(V)}$ for $J_\lambda \subseteq \mathbf{V}$ a finite subset and $n_\lambda(V)$ finite. Let $M \in \mathbf{QCoh}(X)$; then there are natural isomorphisms

$$\begin{aligned} \Gamma(Y, f^* M) &= \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, f_* f^* M) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, f_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} M) && (f \text{ is affine}) \\ &\simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \text{colim}_\lambda H_\lambda^\vee \otimes_{\mathcal{O}_X} M) \simeq \text{colim}_\lambda \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, H_\lambda^\vee \otimes_{\mathcal{O}_X} M) \\ &\simeq \text{colim}_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, M) \end{aligned}$$

It follows that if $\phi: M \rightarrow N$ is a homomorphism of quasicoherent \mathcal{O}_X -modules, then there is a commutative diagram

$$\begin{array}{ccc} \Gamma(Y, f^* M) & \xrightarrow{\simeq} & \text{colim}_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, M) \\ \Gamma(Y, f^* \phi) \downarrow & & \downarrow \text{colim}_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, \phi) \\ \Gamma(Y, f^* N) & \xrightarrow{\simeq} & \text{colim}_\lambda \text{Hom}_{\mathcal{O}_X}(H_\lambda, N) \end{array}$$

In particular, if $\text{Hom}_{\mathcal{O}_X}(V, \phi): \text{Hom}_{\mathcal{O}_X}(V, M) \rightarrow \text{Hom}_{\mathcal{O}_X}(V, N)$ is the zero map for all $V \in \mathbf{V}$, then $\Gamma(Y, f^* \phi)$ is the zero map. Since Y is quasiaffine, $f^* \phi = 0$. But f is faithfully flat, so $\phi = 0$. That is, \mathbf{V} generates $\mathbf{QCoh}(X)$. \square

Let S be an integral quasicompact quasiseparated algebraic stack with generic point ξ . We say that a quasicoherent \mathcal{O}_S -module F is *torsion free* if $\ker(F \rightarrow (i_\xi)_* i_\xi^* F) = 0$, where $i_\xi: S_\xi \hookrightarrow S$ is the generic residual gerbe [36, Theorem B.2]. We record here the following folklore result:

¹This follows from three observations: a direct summand of a projective is projective, if M is projective then $M \otimes_{\mathcal{O}_Y} Q$ is projective for any vector bundle Q , and M^\vee is a direct summand of $M^\vee \otimes_{\mathcal{O}_Y} M \otimes_{\mathcal{O}_Y} M^\vee$.

Corollary 3.4 *Let T be an integral quasicompact algebraic stack with affine diagonal. If every torsion free quasicoherent \mathcal{O}_T -module of finite type is a vector bundle, then T has the resolution property.*

Proof Since T is quasicompact with affine diagonal, there is a smooth covering $\tau: \text{Spec } A \rightarrow T$ and τ is affine. We may write $\tau_* \mathcal{O}_{\text{Spec } A} = \varinjlim_{\lambda} N_{\lambda}$, where the N_{λ} are finite-type quasicoherent \mathcal{O}_T -submodules of $\tau_* \mathcal{O}_{\text{Spec } A}$ [38]. Then the N_{λ} are torsion free, and so are all vector bundles. By Proposition 3.3, the result follows. \square

Remark 3.5 Corollary 3.4 gives a simple generalization of [6, 1.4.5]. Indeed, if T is an integral quasicompact quasiseparated algebraic stack that admits a faithfully flat cover by a finite disjoint union of spectra of Dedekind or Prüfer domains (i.e., every finitely generated ideal is invertible), then every torsion free quasicoherent \mathcal{O}_T -module of finite type is a vector bundle [23, Theorem 1].

4 Flags

Our goal is to construct interesting vector bundles on algebraic stacks. The main problem—even for schemes—is that vector bundles are glued from local data. The key idea here is to add the additional structure of a flag to a vector bundle. This ends up being surprisingly useful.

Let Y be an algebraic stack. Let V_{\bullet} be a quasicoherent \mathcal{O}_Y -module V with a finite filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V.$$

We call n the *length* of the filtration. Recall that V_{\bullet} is a *flag* if the graded pieces $\text{gr}_i(V_{\bullet}) = V_i/V_{i-1}$ are vector bundles for all $i = 1, \dots, n$. Note that this condition implies that the V_i are also vector bundles. A flag is *complete* if the graded pieces are line bundles. We say that V_{\bullet} is *trivially graded* if $\text{gr}_i(V_{\bullet})$ is a trivial vector bundle for all i . Note that a vector bundle which admits the structure of a trivially graded flag is sometimes referred to as a unipotent vector bundle in the literature; see [31, Section 1; 30, Definition 4.5]. We now have the following key definition:

Definition 4.1 Let $f: X \rightarrow S$ be a morphism of algebraic stacks. Let V_{\bullet} be a flag on X . We say that V_{\bullet} is *f -graded* (or *S -graded*) if for each i there is a vector bundle E_i on S and an isomorphism $\phi_i: \text{gr}_i(V_{\bullet}) \simeq f^* E_i$. If the vector bundle V is f -(R-)faithful, then we say the same of the flag V_{\bullet} .

Note that trivially graded flags are graded by every morphism. Remarkably, we can establish that graded flags often descend (Theorem 4.8). We are not aware of any related result in the literature. We begin, however, with the universal example:

Example 4.2 Let S be an algebraic stack. Fix an ordered sequence of n line bundles $\mathcal{L} = (L_1, \dots, L_n)$ on S . Define a category fibered in groupoids $\text{Flag}_{\mathcal{L}} \rightarrow \mathbf{Sch}/S$ as follows: Its objects over $f: X \rightarrow S$ are pairs $(V_{\bullet}, \{\phi_i\}_{i=1}^n)$, where

- (1) V_{\bullet} is a flag of length n , and
- (2) $\phi_i: \text{gr}_i(V_{\bullet}) \simeq f^* L_i$ are isomorphisms for $i = 1, \dots, n$.

That is, an X -point of $\text{Flag}_{\mathcal{L}}$ is an S -graded flag on X whose graded pieces are isomorphic to the f^*L_i . A morphism in $\text{Flag}_{\mathcal{L}}$ is an isomorphism of vector bundles that is compatible with the filtrations and isomorphisms to the L_i . Note that there is a distinguished object in $\text{Flag}_{\mathcal{L}}$ defined over S :

$$\ell = \left(L_1 \subseteq \cdots \subseteq \bigoplus_{j=1}^{n-1} L_j \subseteq \bigoplus_{j=1}^n L_j, \{\text{id}_{L_i}\}_{i=1}^n \right).$$

Every other object in $\text{Flag}_{\mathcal{L}}$ becomes isomorphic to ℓ after passing to a smooth covering. In particular, $\text{Flag}_{\mathcal{L}}$ is a gerbe and the section defined by ℓ induces an equivalence $\text{Flag}_{\mathcal{L}} \simeq BU(\mathcal{L})$, where $U(\mathcal{L}) = \text{Aut}(\ell)$. Clearly $U(\mathcal{L}) \subseteq \text{GL}(\bigoplus_{j=1}^n L_j)$. Moreover, the following diagram 2-commutes:

$$\begin{array}{ccc} X & \xrightarrow{\text{Fr}(V)} & B\text{GL}_{n,S} \\ \downarrow (V_\bullet, \{\phi_i\}) & & \uparrow \wr \text{Fr}(\bigoplus_j L_j) \\ BU(\mathcal{L}) & \longrightarrow & B\text{GL}(\bigoplus_{j=1}^n L_j) \end{array}$$

If $\mathcal{L} = (\mathcal{O}_S, \dots, \mathcal{O}_S)$, then there is a natural identification

$$\text{Aut}(\ell) \simeq U_{n,S}.$$

Thus for any choice of \mathcal{L} , $\text{Aut}(\ell)$ is locally isomorphic to $U_{n,S}$. In particular, $U(\mathcal{L}) \rightarrow S$ is a flat group of finite presentation and it follows from Proposition 2.19 that $\text{GL}(\bigoplus_{j=1}^n L_j)/U(\mathcal{L}) \rightarrow S$ is quasiaffine, and $U(\mathcal{L}) \rightarrow S$ is R-embeddable. Hence $BU(\mathcal{L})$ is an algebraic stack, the morphism $BU(\mathcal{L}) \rightarrow S$ has the resolution property and the universal flag $F_\bullet(\mathcal{L})$ on $BU(\mathcal{L})$ is a $(BU(\mathcal{L}) \rightarrow S)$ -tensor generator. In particular, if S has the resolution property, then so does $BU(\mathcal{L})$.

If $B_{n,\mathbb{Z}}$ denotes the Borel subgroup of upper triangular matrices in $\text{GL}_{n,\mathbb{Z}}$ and $B_{n,\mathbb{Z}} \rightarrow \mathbf{G}_m^n$ is the induced quotient, then the following diagram is 2-cartesian:

$$(4.2.1) \quad \begin{array}{ccc} BU(\mathcal{L}) & \longrightarrow & BB_{n,\mathbb{Z}} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{L}} & B\mathbf{G}_m^n \end{array}$$

Let $f: X \rightarrow S$ be a morphism of algebraic stacks and let $\mathbf{Vect}(X)$ denote the set of isomorphism classes of vector bundles on X . We let

$$\mathbf{Flag}_\emptyset \subseteq \mathbf{Flag}_f \subseteq \mathbf{Flag} \subseteq \mathbf{Vect}(X)$$

denote the sets of isomorphism classes of vector bundles that admit trivially graded complete flags, complete f -graded flags, and complete flags, respectively.

Example 4.3 Standard arguments show that \mathbf{Flag} , \mathbf{Flag}_f and \mathbf{Flag}_\emptyset are stable under finite direct sums, the taking of duals, finite tensor products and extensions.

Example 4.4 Let X be an algebraic stack. If

- (1) X is affine,
- (2) X is quasiaffine,
- (3) X is quasiprojective over affine,
- (4) X admits an ample family of line bundles,
- (5) $\mathbf{QCoh}(X)$ is generated by a set of line bundles (e.g., $X = \mathbf{BG}_{m,\mathbf{Z}}^n$), or
- (6) $X = \mathbf{BU}_{n,k}$, where k is a field,

then X has the **Flag**-resolution property. These assertions are trivial. If X has the **Flag**-resolution property, then Schüppi's theorem (Theorem 3.1) implies that if $p: \mathrm{Spec} A \rightarrow X$ is flat, then $p_* \mathcal{O}_{\mathrm{Spec} A} \simeq \varinjlim_{\lambda} F_{\lambda}$, where the F_{λ} are complete flags on X . Note that $\mathbf{BGL}_{n,\mathbf{Z}}$ does not have the **Flag**-resolution property if $n > 1$.

The following result yields a useful sufficient condition for the existence of a flag structure on a vector bundle:

Lemma 4.5 *Let T be an integral quasicompact quasiseparated algebraic stack with generic point $\eta: \mathrm{Spec} k \rightarrow T$, where k is a field. Let $\gamma: G \rightarrow T$ be a flat group of finite presentation and let $f: \mathbf{BG} \rightarrow T$ be the induced morphism. Let V be a vector bundle on \mathbf{BG} .*

- (1) *If the adjunction $z: f^* f_* V \rightarrow V$ is an isomorphism after restriction along η , then it is an isomorphism.*
- (2) *If every torsion free quasicoherent \mathcal{O}_T -module of finite type is a vector bundle (Remark 3.5) and G_{η} is unipotent, then V admits an f -graded flag.*

Proof By [36, Theorem B.2], η factors as $\mathrm{Spec} k \xrightarrow{\eta'} T_{\eta} \xrightarrow{i_{\eta}} T$, where T_{η} is the residual gerbe and η' is faithfully flat. For (1), since z_{η} is an isomorphism, $z_{T_{\eta}}$ is an isomorphism. Let $p: T \rightarrow \mathbf{BG}$ be the usual section to f ; by descent, V is described by a group homomorphism $v: G \rightarrow \mathrm{GL}(p^*V)$. Note that z is an isomorphism if and only if $\ker(v) = G$ as z corresponds to the inclusion of G -invariants of V into V . Now $\ker(v) \subseteq G$ is a closed immersion as $\mathrm{GL}(p^*V) \rightarrow T$ is separated and contains $\gamma^{-1}(T_{\eta})$, which is dense since γ is open. Hence $\ker(v) = G$.

We prove (2) by induction on $\mathrm{rank}(V)$, the case $\mathrm{rank}(V) = 0$ being trivial; assume that $\mathrm{rank}(V) > 0$. If $z_{T_{\eta}}$ is an isomorphism, then (1) says that z_V is an isomorphism and so we would be done. Otherwise, if $z_{T_{\eta}}$ is not an isomorphism, then $(f^* f_* V)_{T_{\eta}}$ is nonzero as after restriction along η it is the inclusion of the invariants of a nontrivial representation of the unipotent group G_{η} [10, Theorem XVII.3.5]. In particular, there is a nontrivial quotient $V_{T_{\eta}} \rightarrow W$ on T_{η} . Now define \overline{W} to be the image of the composition $V \rightarrow (i_{T_{\eta}})_* V_{T_{\eta}} \rightarrow (i_{T_{\eta}})_* W$. Since the map $V \rightarrow (i_{T_{\eta}})_* W$ is nonzero, \overline{W} is a nonzero torsion free quasicoherent $\mathcal{O}_{\mathbf{BG}}$ -module of finite type, and so is a vector bundle. Also, $\overline{W}_{T_{\eta}} = W$ and so $\mathrm{rank}(\overline{W}) = \mathrm{rank}(W) < \mathrm{rank}(V)$. We now have an exact sequence of vector bundles on \mathbf{BG} :

$$0 \rightarrow K \rightarrow V \rightarrow \overline{W} \rightarrow 0.$$

The ranks of K and \overline{W} are less than that of V , and so by induction they admit f -graded flags; consequently so does V . □

Using Lemma 4.5, we can now prove the following:

Proposition 4.6 *Let \mathcal{L} be a sequence of n line bundles on an algebraic stack S .*

- (1) *The morphism $f: BU(\mathcal{L}) \rightarrow S$ has the **Flag** _{f} -resolution property.*
- (2) *If S is affine, then f has the **Flag** _{\mathcal{O}} -resolution property.*

Proof It suffices to show (1) universally by [17, Proposition 2.8(iv)]. Therefore we may suppose $f: BU(\mathcal{L}) \rightarrow BG_{m,\mathbf{Z}}^n$, where \mathcal{L} is the universal sequence of n line bundles. Since $BB_{n,\mathbf{Z}} \simeq BU(\mathcal{L})$ has the resolution property (Example 4.2), it suffices to show that every vector bundle V is a complete f -graded flag. Since $\text{Spec } \mathbf{Z} \rightarrow BG_{m,\mathbf{Z}}^n$ is a smooth covering, Lemma 4.5 and Remark 3.5 imply that V admits an f -graded flag. But every vector bundle on $BG_{m,\mathbf{Z}}^n$ is a direct sum of line bundles, so (1) follows from Example 4.3.

For (2), it suffices to show that the vector bundle V underlying every f -graded complete flag V_\bullet is a direct summand of a vector bundle W which admits a trivially graded complete flag structure. We proceed by induction on n , the length of V_\bullet . If $n = 0$, then the result is trivial. If $n > 0$, then we may write V as an extension

$$0 \rightarrow V_{n-1} \rightarrow V \rightarrow f^*L \rightarrow 0$$

for some line bundle L on S . Since S is affine, there is a split surjection $\gamma: \mathcal{O}_S^{\oplus m} \twoheadrightarrow L$. Then $f^*\gamma: \mathcal{O}_{BU(\mathcal{L})}^{\oplus m} \twoheadrightarrow f^*L$ is a split surjection. Now pull the extension back along $f^*\gamma$, so we may replace f^*L with a trivial bundle of rank m . By induction, there is a split surjection $\delta: U \twoheadrightarrow V_{n-1}$, where U admits a trivially graded complete flag. Pushing forward along a section of δ , V is a direct summand of a vector bundle that is an extension of trivially graded complete flags. Now apply Example 4.3. □

The following concept will be useful. A *refinement* of a flag V_\bullet of length n is another flag $V'_0 \subseteq \dots \subseteq V'_m = V$ of length $m \geq n$ together with a strictly order-preserving function $\rho: \{0 < 1 < \dots < n\} \rightarrow \{0 < 1 < \dots < m\}$ such that $V_i = V'_{\rho(i)}$ for all i .

Example 4.7 Let Y be an algebraic stack. Let V_\bullet be a flag on Y . Suppose that $\text{gr}_i(V_\bullet)$ admits the structure of a complete flag for each $i \geq 0$; then V_\bullet admits a complete refinement. To see this, use Example 4.3.

Theorem 4.8 *Consider a 2-cartesian diagram of algebraic stacks:*

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{p} & S \end{array}$$

Assume that:

- (1) S is quasicompact with affine diagonal and has the resolution property.
- (2) f is quasicompact and quasiseparated.
- (3) p is faithfully flat and S' is affine.

If X' admits a q -graded flag V_\bullet of length n , then X admits a flag W_\bullet of length n and a degree-preserving split surjection $q^*W_\bullet \twoheadrightarrow V_\bullet$. Moreover:

- (a) If V_\bullet is f' -(R -)faithful, then W_\bullet is f -(R -)faithful.
- (b) If $p_*\mathcal{O}_{S'}$ is a filtered colimit of vector bundles F_λ on X (see Theorem 3.1) and $\text{gr}_i(V_\bullet) \simeq q^*E_i$, then there exists λ_i with $\text{gr}_i(W_\bullet) \simeq f^*F_{\lambda_i} \otimes_{\mathcal{O}_X} E_i$.
- (c) If V_\bullet is fq -graded, then W_\bullet is f -graded.
- (d) Assume that S has the **Flag**-resolution property.
 - (i) If V_\bullet is complete, then W_\bullet admits a complete refinement.
 - (ii) If V_\bullet is fq -graded, then W_\bullet admits a complete f -graded refinement.

Proof Part (a) follows from Proposition 2.20(1), the main claim and [17, Proposition 2.8(iii)]. Part (c) follows from (b). Part (d) follows from (b) and Examples 4.7 and 4.4. Part (b) will follow from the construction of W_\bullet , which we prove by induction on $n \geq 0$. The base case is trivial so we assume $n \geq 1$. Choose a vector bundle E on X such that $V_1 = q^*E$; then $(V/V_1)_\bullet$ is a q -graded flag of length $n - 1$. By induction, there is a flag W_\bullet of length $n - 1$ on X and a split surjection $q^*W_\bullet \rightarrow (V/V_1)_\bullet$ as in the statement of the theorem. We now pull back the defining short exact sequence

$$0 \rightarrow q^*E \rightarrow V \rightarrow V/q^*E \rightarrow 0$$

along the surjection $q^*W \twoheadrightarrow V/q^*E$. This results in a commutative diagram with exact rows, whose vertical arrows are easily checked to be split surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & q^*E & \longrightarrow & V' & \longrightarrow & q^*W \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & q^*E & \longrightarrow & V & \longrightarrow & V/q^*E \longrightarrow 0 \end{array}$$

Thus we may replace V by V' and assume that $(V/V_1)_\bullet = q^*W_\bullet$.

Since S' is affine and S has affine diagonal, it follows that p and q are affine and faithfully flat morphisms. It follows that we may push forward the top row in the above diagram to obtain an exact sequence of quasicoherent sheaves on X :

$$0 \rightarrow q_*q^*E \rightarrow q_*V' \rightarrow q_*q^*W \rightarrow 0.$$

Pulling this sequence back along the injective adjunction $W \hookrightarrow q_*q^*W$, we obtain a diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & q_*q^*E & \longrightarrow & W' & \longrightarrow & W \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & q_*q^*E & \longrightarrow & q_*V' & \longrightarrow & q_*q^*W \longrightarrow 0 \end{array}$$

Since E is a vector bundle and q is affine, the projection formula and flat base change produce natural isomorphisms

$$q_*q^*E \simeq q_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} E \simeq (f^*p_*\mathcal{O}_{S'}) \otimes_{\mathcal{O}_X} E.$$

Assume we are given that $p_*\mathcal{O}_{S'} \simeq \varinjlim_{\lambda} F_{\lambda}$, where the F_{λ} are vector bundles, or we produce such a description by Sch\"apfi's theorem (Theorem 3.1). Since the adjunction $p^*p_*\mathcal{O}_{S'} \rightarrow \mathcal{O}_{S'}$ is split surjective, taking λ sufficiently large, we can assume that the induced morphisms $p^*F_{\lambda} \rightarrow \mathcal{O}_{S'}$ are also split surjective. But we also have natural isomorphisms

$$\begin{aligned} \text{Ext}_{\mathcal{O}_X}^1(W, q_*q^*E) &\simeq H^1(X, W^{\vee} \otimes_{\mathcal{O}_X} q_*q^*E) \\ &\simeq H^1(X, \varinjlim_{\lambda} (f^*F_{\lambda} \otimes_{\mathcal{O}_X} W^{\vee} \otimes_{\mathcal{O}_X} E)) \\ &\simeq \varinjlim_{\lambda} H^1(X, f^*F_{\lambda} \otimes_{\mathcal{O}_X} W^{\vee} \otimes_{\mathcal{O}_X} E) && [20, \text{Lemma 1.2(iii)}] \\ &\simeq \varinjlim_{\lambda} \text{Ext}_{\mathcal{O}_X}^1(W, f^*F_{\lambda} \otimes_{\mathcal{O}_X} E). \end{aligned}$$

Thus the extension W' is the pushforward of an extension

$$0 \rightarrow f^*F_{\lambda} \otimes_{\mathcal{O}_X} E \rightarrow W'_{\lambda} \rightarrow W \rightarrow 0$$

along the morphism $f^*F_{\lambda} \otimes_{\mathcal{O}_X} E \rightarrow f^*(p_*\mathcal{O}_{S'}) \otimes_{\mathcal{O}_X} E \simeq q_*q^*E$. Pulling this exact sequence back along q , we obtain an exact sequence

$$0 \rightarrow q^*f^*F_{\lambda} \otimes_{\mathcal{O}_{X'}} q^*E \rightarrow q^*W'_{\lambda} \rightarrow q^*W \rightarrow 0.$$

But the pushforward of this extension along the split surjection $q^*f^*F_{\lambda} \otimes_{\mathcal{O}_{X'}} q^*E \rightarrow q^*E$ is clearly isomorphic to V' and the resulting morphism $q^*W'_{\lambda} \rightarrow V'$ is split surjective. \square

5 Unipotent morphisms and groups

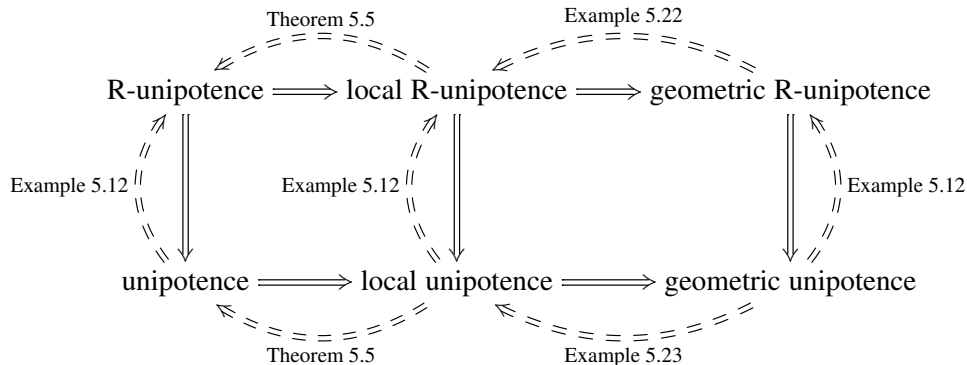
5.1 Unipotent morphisms

Motivated by the results of the previous section, we make the following definition:

Definition 5.1 Let $f: X \rightarrow S$ be a morphism of algebraic stacks. Then f is

- *(R-)unipotent* if X admits a complete f -graded f -(R-)faithful flag,
- *locally (R-)unipotent* if there is an fpqc covering $S' \rightarrow S$ [43, Tag 022B] such that $f \times_S S'$ is (R-)unipotent, and
- *geometrically (R-)unipotent* if for every algebraically closed field k and morphism $\text{Spec } k \rightarrow S$, the induced morphism $X \times_S \text{Spec } k \rightarrow \text{Spec } k$ is (R-)unipotent.

Locally R-unipotent and unipotent morphisms have affine and quasiaffine diagonals, respectively. We have the following sequence of implications:



The solid arrows follow from the definitions. The dashed arrows are partial converses. The dashed vertical implications are valid only for certain groups over normal bases. The dashed horizontal implications are valid only in special cases.

Quasicompact and quasiseparated representable morphisms are always unipotent. There are two prototypical examples of R-unipotent morphisms:

- (1) $BU(\mathcal{L}) \rightarrow S$ (Example 4.2), and
- (2) quasiaffine morphisms $X \rightarrow S$.

We will return to these examples often. We begin this section with the following two trivial lemmas:

Lemma 5.2 *Let $f: X \rightarrow S$ be an (R-)unipotent morphism of algebraic stacks. If $S' \rightarrow S$ is a morphism of algebraic stacks, then $f \times_S S'$ is (R-)unipotent. Moreover, the same holds for the local and geometric versions.*

Lemma 5.3 *Let $X \xrightarrow{h} Y \xrightarrow{g} S$ be morphisms of algebraic stacks.*

- (1) *If g is unipotent and h is quasicompact, quasiseparated and representable, then $g \circ h$ is unipotent.*
- (2) *If g is R-unipotent and h is quasiaffine, then $g \circ h$ is R-unipotent.*

Moreover, the analogous properties hold for the local and geometric versions.

In general, unipotent morphisms are not stable under composition (Example 5.6). We are optimistic that locally unipotent morphisms have better stability properties under composition, but this appears to be surprisingly subtle — even when S is the spectrum of a field, Y is quasiaffine and X is a gerbe over Y .

We now have the following characterization of (R-)unipotent morphisms, which provides a unipotent enrichment of the Totaro–Gross theorem (Theorem 2.15).

Theorem 5.4 *Let $f: X \rightarrow S$ be a morphism of algebraic stacks with S quasicompact. Then the following are equivalent:*

- (1) *The morphism f is R-unipotent.*

- (2) For some integer $n \geq 0$, there is an ordered sequence of n line bundles \mathcal{L} on S and a factorization of f as $X \xrightarrow{h} BU(\mathcal{L}) \rightarrow S$, where h is quasiaffine.
- (3) The morphism f has the **Flag** $_f$ -resolution property and the relative inertia stack $I_f \rightarrow X$ has affine fibers.

Moreover, f is unipotent if and only if there is a quasicompact, quasiseparated and representable morphism $X \xrightarrow{h} BU(\mathcal{L})$ as in (2).

Proof The statement regarding unipotent morphisms will follow from our arguments from the equivalence (1) \iff (2).

We have (2) \implies (1) by Lemma 5.3 and the R-unipotence of $BU(\mathcal{L}) \rightarrow S$. As for (1) \implies (2), by definition there is a complete f -graded flag V_\bullet of length n on X that is f -(R-)faithful. That is, there is an ordered sequence of line bundles $\mathcal{L} = (L_1, \dots, L_n)$ on S together with isomorphisms $\phi_i: \text{gr}_i(V_\bullet) \simeq f^* L_i$ for all i . By Example 4.2, we obtain a morphism $X \xrightarrow{h} BU(\mathcal{L})$. Postcomposing this with the quasiaffine morphism $BU(\mathcal{L}) \rightarrow BGL(\bigoplus_{i=1}^n L_i) \simeq BGL_{n,S}$ yields a quasiaffine morphism by Theorem 2.15(2), and the result follows from [43, Tag 054G].

For (2) \implies (3), Proposition 4.6 and [17, Proposition 2.8(v)] imply that f has the **Flag** $_f$ -resolution property. Finally, for (3) \implies (2), the Gross–Totaro theorem (Theorem 2.15) produces a $V_\bullet \in \mathbf{Flag}_f$ which is f -R-faithful. □

A remarkable consequence of Theorem 4.8 is that local (R-)unipotence implies (R-)unipotence on bases with the **Flag**-resolution property.

Theorem 5.5 *Let $f: X \rightarrow S$ be a locally unipotent (resp. locally R-unipotent) morphism of algebraic stacks. Let S be quasicompact with affine diagonal.*

- (1) *If S has the resolution property, then X is a global quotient (resp. has the resolution property).*
- (2) *If S has the **Flag**-resolution property, then f is unipotent (resp. R-unipotent).*

Proof By assumption, there is a faithfully flat cover $p: S' \rightarrow S$ such that $f': X \times_S S' \rightarrow S'$ admits a complete f' -graded f' -faithful (resp. f' -R-faithful) flag V_\bullet . Passing to a smooth cover of S' , we may assume that S' is affine and that the flag is trivially graded, and so V_\bullet is pf' -graded. Now apply Theorem 4.8 to

$$\begin{array}{ccc}
 X \times_S S' & \xrightarrow{q} & X \\
 f' \downarrow & & \downarrow f \\
 S' & \xrightarrow{p} & S
 \end{array}$$

Then we obtain a flag W_\bullet on X and a split surjection $q^* W_\bullet \rightarrow V_\bullet$. Part (c) implies that W_\bullet is f -graded. Part (a) says that W_\bullet is f -faithful (resp. f -R-faithful), so X is a global quotient (resp. has the resolution property). Part (d) says that if S has the **Flag**-resolution property, then W_\bullet admits a complete refinement, which is also f -graded. Hence f admits a complete f -graded f -(R-)faithful flag. That is, f is unipotent (resp. R-unipotent). □

Example 5.6 Let S be a proper normal surface over \mathbf{C} with no nontrivial line bundles [41]. Let $Y \subseteq S$ be the inclusion of the regular locus, which is an open subscheme whose complement $S \setminus Y$ is a finite set of closed points. Then Y admits a nontrivial very ample line bundle L . Let $\mathcal{U} = BU(\mathcal{O}, L^\vee) \cong BL$. Then

- (1) $\pi: \mathcal{U} \rightarrow Y$ and $j: Y \rightarrow S$ are \mathbf{R} -unipotent,
- (2) the composition $p = j \circ \pi$ is locally \mathbf{R} -unipotent but not unipotent, and
- (3) S has the resolution property but not the **Flag**-resolution property.

Indeed, (1) is clear from Theorem 5.4. Also, (3) follows from (2) and Theorem 5.5.

For (2), to see that p is locally \mathbf{R} -unipotent, it suffices to restrict to affine opens $\text{Spec } A \subseteq S$; then $Y_A = Y \cap \text{Spec } A$ is quasiaffine and so $L_{Y_A}^\vee$ is globally generated. Hence there is a quasiaffine morphism $\mathcal{U} \rightarrow \mathbf{BG}_{a, Y_A}^n$ for some n ; after composition with the open immersion $\mathbf{BG}_{a, Y_A}^n \subseteq \mathbf{BG}_{a, \text{Spec } A}^n$, we see that p is locally \mathbf{R} -unipotent by Lemma 5.3(2).

If $p = j \circ \pi$ is unipotent, since S has no nontrivial line bundles, \mathcal{U} must admit a faithful trivially graded flag. Hence it suffices to show that every trivially graded flag V_\bullet on \mathcal{U} is of the form π^*W_\bullet for some trivially graded flag W_\bullet on Y . A simple calculation shows that $\mathbf{R}^1\pi_*\mathcal{O}_{\mathcal{U}} \simeq L^\vee$. Note that Y is not quasiaffine because $H^0(Y, \mathcal{O}_Y) = \mathbf{C}$ [43, Tag 01P9], and therefore L is nontrivial [43, Tag 01QE]. In turn, this implies $H^0(Y, L^\vee) = 0$ because by composing with a nonzero section of L , a nonzero section L^\vee would yield a trivialization of L^\vee . Therefore, since L^\vee has no nonzero global sections, if W_\bullet is a trivially graded flag W_\bullet on Y then $\text{Hom}_{\mathcal{O}_Y}(W, L^\vee) = 0$. Now let V_\bullet be a trivially graded flag on \mathcal{U} . By induction on the length of V_\bullet , we may assume that V_\bullet is an extension

$$e: 0 \rightarrow \pi^*\mathcal{O}_Y \rightarrow V_\bullet \rightarrow \pi^*W_\bullet \rightarrow 0,$$

where W_\bullet is a trivially graded flag on Y . There is also an exact sequence

$$0 \rightarrow \text{Ext}_{\mathcal{O}_Y}^1(W, \mathcal{O}_Y) \rightarrow \text{Ext}_{\mathcal{O}_{\mathcal{U}}}^1(\pi^*W, \pi^*\mathcal{O}_Y) \rightarrow \text{Hom}_{\mathcal{O}_Y}(W, \mathbf{R}^1\pi_*\mathcal{O}_{\mathcal{U}}).$$

Since $\text{Hom}_{\mathcal{O}_Y}(W, L^\vee) = 0$, the first map is bijective and so our extension e is pulled back from Y .

Two useful application of Theorem 5.5 are the following:

Corollary 5.7 *Let $f: X \rightarrow S$ be a locally (R-)unipotent morphism of algebraic stacks. If S is quasicompact, then there is a smooth surjection $S' \rightarrow S$ such that $f \times_S S'$ is (R-)unipotent.*

Proof Passing to a smooth cover of S , we may assume that S is an affine scheme and so has the **Flag**-resolution property. The result follows from Theorem 5.5. □

Corollary 5.8 *Let $f: X \rightarrow S$ be a morphism of algebraic stacks.*

- (1) *If f is geometrically (R-)unipotent, then $f \times_S \text{Spec } k: X \times_S \text{Spec } k \rightarrow \text{Spec } k$ is (R-)unipotent for every field k and morphism $\text{Spec } k \rightarrow S$.*
- (2) *Let $S' \rightarrow S$ be surjective. If $f': X \times_S S' \rightarrow S'$ is geometrically (R-)unipotent, then f is geometrically (R-)unipotent.*

Proof In (2), we may assume that $S = \text{Spec } k$ and $S' = \text{Spec } k'$, where $k \subseteq k'$ is a field extension and k' is algebraically closed. It suffices to prove that if f' is (R-)unipotent, then f is (R-)unipotent, which also implies (1). But $S' \rightarrow S$ is an fpqc covering, so f is locally (R-)unipotent. Since S is affine, it has the **Flag**-resolution property. The result now follows from Theorem 5.5. \square

5.2 Unipotent groups

A powerful source of unipotent morphisms will be unipotent groups and gerbes. Let k be a field. Let G be an algebraic group over k . There are several characterizations of unipotence over k [9, Theorem XVII.3.5]. We take the following: G is *unipotent* if there is an embedding into the upper triangular unipotent matrices $G \hookrightarrow U_{n,k} \subset \text{GL}_{n,k}$ for some n . Families of unipotent groups over general bases are more subtle, however, so we make the following definition:

Definition 5.9 Let S be an algebraic stack. A group $G \rightarrow S$ that is flat and of finite presentation is said to be *(R-)unipotent, locally (R-)unipotent or geometrically (R-)unipotent* if the corresponding gerbe $BG \rightarrow S$ is so.

Remark 5.10 There is a potential for confusion with this terminology: unipotence of the group $G \rightarrow S$ is not the same thing as unipotence of the morphism $G \rightarrow S$. For instance, every quasicompact, quasiseparated and representable morphism of algebraic stacks is unipotent, but an affine group scheme $G \rightarrow S$ need not be unipotent in the sense of Definition 5.9.

Remark 5.11 A group $G \rightarrow S$ that is flat and of finite presentation is unipotent (resp. R-unipotent) if and only if there exists an ordered sequence of line bundles \mathcal{L} on S , an object V_\bullet of $\text{Flag}_{\mathcal{L}}(S)$ and a monomorphism (resp. with quasiaffine quotient)

$$G \hookrightarrow \text{Aut}_{\text{Flag}_{\mathcal{L}}}(V_\bullet) \subset \text{GL}(V).$$

Thus (R-)unipotent groups are (R-)embeddable. Note that the above automorphism group is an inner form of $U(\mathcal{L})$. In particular, if S is a scheme, then G can be embedded in a Zariski form of $U_{n,S}$. Hence, if $S = \text{Spec } k$ is the spectrum of a field, then $G \rightarrow \text{Spec } k$ is unipotent (in the sense of Definition 5.9) if and only if there exists an embedding $G \subseteq U_{n,k}$ for some n , so both possible definitions of a unipotent group scheme agree over a field.

Example 5.12 By Proposition 2.19, the group $U_{n,S} \rightarrow S$ is R-unipotent, as is any flat and finitely presented closed subgroup $H \subset U_{n,S}$ when either S is the spectrum of a field or a Dedekind domain, or $H \rightarrow S$ is finite. In particular, if $G \rightarrow S$ is (locally) unipotent and either $G \rightarrow S$ is finite or S is the spectrum of a field, then it is (locally) R-unipotent.

Now translating Theorem 5.5 into groups we obtain the following:

Corollary 5.13 *Let S be a quasicompact algebraic stack with affine diagonal. Let $G \rightarrow S$ be a locally (R-)unipotent group.*

- (1) If S has the resolution property, then $G \rightarrow S$ is (R-)embeddable.
 (2) If S has the **Flag**-resolution property, then $G \rightarrow S$ is (R-)unipotent.

Note the stark contrast to the case of tori. Tori are always locally embeddable, but may not be embeddable even over a projective curve [9, X.1.6 and XI.4.6].

Remark 5.14 By Example 5.12 and Corollary 5.13, all the definitions in Definition 5.9 agree when S is the spectrum of a field. Thus one may speculate what the best definition of unipotence is over a general base. We argue that local R-unipotence is the best behaved as evidenced by Theorem 5.5, Examples 5.12, 5.16, 5.17 and 5.22, and Proposition 7.1.

We have the following proposition, which justifies our terminology:

Proposition 5.15 *Let $f: X \rightarrow S$ be a geometrically unipotent morphism of algebraic stacks. Then the relative geometric stabilizers of f are all unipotent groups.*

Proof We may assume that $S = \text{Spec } l$, where l is an algebraically closed field. Let $\bar{x}: \text{Spec } k \rightarrow X$ be a point of X , where k is an algebraically closed field. Let $G_{\bar{x}} = \text{Aut}_X(\bar{x})$ be its automorphism group. Then we have a representable morphism $\tilde{x}: BG_{\bar{x}} \rightarrow X$. By Lemma 5.3(1), $BG_{\bar{x}}$ is unipotent. Now apply the last sentence of Remark 5.11. \square

Example 5.16 Let S be an algebraic stack. Let E be a locally free sheaf on S of rank n . Then the vector group $\mathbf{V}(E) \rightarrow S$ is locally R-unipotent. We may work locally on S , so we may assume that $E \simeq \mathcal{O}_S^{\oplus n}$. In this case, $\mathbf{V}(E) \simeq \mathbf{G}_{a,S}^n$. Now $\mathbf{G}_{a,\mathbf{Z}}^n \subseteq U_{n+1,\mathbf{Z}} \subseteq \text{GL}_{n+1,\mathbf{Z}}$. Since \mathbf{Z} is a Dedekind domain, it follows from Example 5.12 that $\mathbf{G}_{a,\mathbf{Z}}^n \rightarrow \text{Spec } \mathbf{Z}$ is R-unipotent. Hence $\mathbf{V}(E) \rightarrow S$ is locally R-unipotent. If S is quasicompact with affine diagonal and the **Flag**-resolution property, then $\mathbf{V}(E) \rightarrow S$ is even R-unipotent (Corollary 5.13).

Example 5.17 Let $p > 0$ be a prime. Let S be an algebraic \mathbf{F}_p -stack. Let $G \rightarrow S$ be a finite étale group scheme of degree p^d for some $d \geq 0$. Then $G \rightarrow S$ is locally R-unipotent. Indeed, smooth locally on S we may assume that $G \rightarrow S$ is a constant group scheme. We may thus assume that $S = \text{Spec } \bar{\mathbf{F}}_p$ and G is a constant p -group. Standard facts from group theory show that G is unipotent, and the claim follows.

Example 5.18 If X denotes the affine line with doubled origin, then it is an R-unipotent scheme. Indeed if $x, y \in X$ denote the two origins, then the trivially graded flag $\mathcal{O}_X = I_{\{x,y\}} \subset I_{\{x\}} \oplus I_{\{y\}}$ is a tensor generator, where I_C is the ideal sheaf associated to a reduced closed subscheme $C \subset X$.

5.3 Representable R-unipotent morphisms

We now briefly discuss representable (locally) R-unipotent morphisms $f: X \rightarrow S$. Characterizing such morphisms seems subtle. For instance, consider for simplicity the case when $S = \text{Spec } k$ is the spectrum of a field. It follows from Theorem 5.4 that representable R-unipotent morphisms $f: X \rightarrow \text{Spec } k$ are exactly quotients of the form T/U_n , where T is a quasiaffine scheme over k acted on freely by U_n . In particular, a quasiaffine morphism over k is R-unipotent and hence geometrically R-unipotent. The converse is not true, however.

Example 5.19 (1) Let k be a field and denote by X the complement of a single point in the exceptional locus of \mathbf{A}_k^2 blown up at the origin. We claim that the morphism $f: X \rightarrow \mathbf{A}^2$ is geometrically R-unipotent but is not quasiaffine. Indeed, the geometric fibers are all affine and hence the morphism is geometrically R-unipotent. On the other hand, if f was quasiaffine then the natural map $g: X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ would be an open immersion [43, Tag 01SM] and this cannot be true since $\Gamma(X, \mathcal{O}_X) = k[x, y]$.

(2) In [5, Example 3.16], Asok and Doran describe a free action of $U_1 = \mathbf{G}_a$ on \mathbf{A}^5 such that the quotient $\mathbf{A}^5/\mathbf{G}_a$ is not even a scheme. More generally, given a smooth affine scheme T over k equipped with a free and proper action of U_n , [5, Corollary 3.18] yields an effective criterion for determining when the quotient T/U_n will be affine or quasiaffine.

We have the following result, whose proof we defer until Section 7:

Proposition 5.20 *Let $f: X \rightarrow S$ be a representable and geometrically R-unipotent morphism of algebraic stacks. If f is proper, then f is finite.*

It follows from Proposition 5.20 that representable geometrically R-unipotent morphisms over a field do not contain any positive-dimensional proper subvarieties.

5.4 Geometrically unipotent vs locally unipotent groups

We conclude this section with some thoughts on the following natural refinement of Question 1.1: which geometrically unipotent groups are locally unipotent? Note that locally unipotent groups are *always* quasiaffine; in particular, they are separated and schematic. In positive characteristic, it is easy to produce geometrically unipotent group algebraic spaces that are not separated. There are also separated geometrically unipotent group algebraic spaces that are not schemes. In particular, these give examples of geometrically unipotent groups that are not locally unipotent.

Example 5.21 Let k be a field of characteristic 2. Let $S = \mathbf{A}_k^1$. Let $H \subseteq (\mathbf{Z}/2\mathbf{Z})_S = G$ be the étale subgroup obtained by deleting the nontrivial point over the origin.

(1) Let $Q = G/H \rightarrow S$ be the line with the doubled origin, viewed as a group scheme. Then $Q \rightarrow S$ is geometrically unipotent, but nonseparated. In particular, it is not locally unipotent.

(2) We have $H \subseteq G \subseteq \mathbf{G}_{a,S}$. Let $Q' = \mathbf{G}_{a,S}/H \rightarrow S$. Then $Q' \rightarrow S$ is smooth, geometrically unipotent, with connected fibers, nonseparated and not a scheme [8, VI_B.5.5]. This shows that the closed subgroup condition in Proposition 2.19(2) is necessary.

(3) A more sophisticated example was constructed in [33, X.14]: take $T = \mathbf{A}_k^2$; then there is a *closed* subgroup $N \subseteq \mathbf{G}_{a,T}^2$ with $N \rightarrow T$ étale. Taking $Q'' = \mathbf{G}_{a,T}^2/N$, we produce a smooth separated geometrically unipotent group of finite presentation with connected fibers. There it is shown that Q'' is not representable by a scheme. This shows that the condition that the quotient is representable by a scheme in Proposition 2.19(2) is necessary.

In characteristic 0, there are strong results in the existing literature, however:

Example 5.22 Let S be an algebraic stack of equicharacteristic 0. Let $G \rightarrow S$ be geometrically unipotent and flat-locally schematic. Then $G \rightarrow S$ is affine. Indeed, this is local on S , so we may assume that S is affine. By standard limit methods, we may assume that S is of finite type over $\text{Spec } \mathbf{Z}$ and so excellent; in particular, S has noetherian normalization. If S is either normal or $G \rightarrow S$ is embeddable, then there is an isomorphism of groups

$$\exp: \mathbf{V}(\text{Lie}(G)) \rightarrow G,$$

where we endow the left-hand side with the group structure coming from the Baker–Campbell–Hausdorff formula. In particular, when $G \rightarrow S$ is commutative it is a vector group, and so is locally \mathbf{R} -unipotent. The statement involving the exponential map is established in [44, Section 1.3; 33, XV.3]. That $G \rightarrow S$ is affine follows from the normal case and Chevalley’s theorem [37, Theorem 8.1].

Example 5.23 Let $G \rightarrow S$ be an affine and geometrically unipotent group.

- (1) If $S = \text{Spec } A$, where A is a Prüfer domain, then $G \rightarrow S$ is \mathbf{R} -unipotent.
- (2) If S is equicharacteristic $p > 0$ and $G \rightarrow S$ is a finite locally free commutative group scheme with dual of height 1, then $G \rightarrow S$ is locally \mathbf{R} -unipotent.

For (1), by Lemma 4.5 every vector bundle on BG admits the structure of a complete S -graded flag. In combination with Corollary 3.4, this implies that G is \mathbf{R} -unipotent. For (2), the exact sequence of [4, Proposition 1.1] implies there is an affine morphism $BG \rightarrow B\mathbf{V}(\omega)$ for some vector group $\mathbf{V}(\omega) \rightarrow S$. Since $\mathbf{V}(\omega) \rightarrow S$ is locally \mathbf{R} -unipotent (Example 5.16), (2) follows.

6 Applications

6.1 Unipotent gerbes and the resolution property

Theorem 5.5 immediately yields a proof of Theorem A.

Proof of Theorem A A \mathbf{G}_a -gerbe is locally \mathbf{R} -unipotent (Example 5.12). It follows from Theorem 5.5 that \mathcal{G} has the resolution property. \square

Remark 6.1 Assume that X lives over a scheme S and $G \rightarrow S$ is locally \mathbf{R} -unipotent: for example, $G = \mathbf{G}_a^n$, $G = U_{n,S}$, or S is the spectrum of a field and $G \subseteq U_{n,S}$ (see Example 5.12). Then the argument above for Theorem A also holds for G -gerbes $\mathcal{G} \rightarrow X$.

We next obtain the following corollary of Theorem 5.5, which is key for our proof of Theorem B:

Corollary 6.2 Let S be a quasicompact algebraic stack with affine diagonal over \mathbf{F}_p . Let $\pi: \mathcal{G} \rightarrow S$ be a gerbe that is relatively Deligne–Mumford, separated and with p -power order inertia.

- (1) If S has the resolution property, then \mathcal{G} has the resolution property.
- (2) If S has the **Flag**-resolution property, then π is \mathbf{R} -unipotent.

Proof Since π is relatively Deligne–Mumford and separated, there is a smooth covering $S' \rightarrow S$ such that $\mathcal{G} \times_S S' \simeq BG'$, where $G' \rightarrow S'$ is finite étale. The p -power order inertia assumption on π implies that $G' \rightarrow S'$ has degree p^d for some $d \geq 0$. Example 5.17 shows that $G' \rightarrow S'$ is locally R-unipotent. Now apply Theorem 5.5. □

6.2 Deligne–Mumford gerbes

Let X be an algebraic stack. Then there is an injective group homomorphism

$$\mathrm{Br}(X) \rightarrow \mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbf{G}_m)_{\mathrm{tors}}$$

from the geometric Brauer group of X to the cohomological Brauer group of X . However, it is still not known when the map is surjective (see, for instance, [28] for an introduction to this problem).

We are now in a position to prove one of our key results:

Theorem 6.3 *Let S be a connected quasicompact algebraic stack with affine diagonal and the resolution property over a field k . Let $\pi: \mathcal{G} \rightarrow S$ be a gerbe that is relatively Deligne–Mumford and separated. If $\mathrm{Br}(T) = \mathrm{Br}'(T)$ for all finite étale morphisms $T \rightarrow S$ (e.g., if S is a quasicompact scheme that admits an ample line bundle), then \mathcal{G} has the resolution property.*

Proof The gerbe $\mathcal{G} \rightarrow S$ is locally banded by a finite constant group scheme \underline{G}_S . It follows that the sheaf of isomorphisms (in the category of Bands over S)

$$I = \underline{\mathrm{Isom}}(\underline{\mathrm{Band}}(\mathcal{G}), \underline{G}_S)$$

is a torsor under the outer automorphism group $\underline{\mathrm{Out}}(\underline{G}_S) \rightarrow S$. This map is finite étale, so we may replace S with I as the resolution property descends under such maps [17, Proposition 5.3(vii)]. Hence we can assume that $\mathcal{G} \rightarrow S$ is banded by \underline{G}_S , which is constant. Now there is a factorization

$$\mathcal{Z} \rightarrow \mathcal{G} \rightarrow S,$$

where $\mathcal{Z} \rightarrow S$ is a gerbe banded by the center of \underline{G}_S , which is a finite abelian constant group scheme and $\mathcal{Z} \rightarrow \mathcal{G}$ is finite étale and surjective. Indeed, define \mathcal{Z} to be the stack of banded equivalences

$$\mathcal{Z} = \underline{\mathrm{HOM}}_{\mathrm{id}}(B\underline{G}_S, \mathcal{G}) \rightarrow S.$$

By [15, IV.2.3.2(iii)], \mathcal{Z} is banded by $Z(\underline{G}_S) = Z(G)_S$ and the natural evaluation map $\mathrm{ev}: \mathcal{Z} \rightarrow \mathcal{G}$ induces a map on bands equal to the inclusion $Z(G)_S \subset \underline{G}_S$. Again, by [17, Proposition 5.3(vii)], we may now replace \mathcal{G} by \mathcal{Z} . Thus \mathcal{G} is banded by the constant group scheme associated to the finite abelian group

$$\mathbf{Z}/p_1^{n_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p_l^{n_l}\mathbf{Z},$$

where the p_i are primes. Hence $\mathcal{G} \cong \mathcal{R}_1 \times_S \cdots \times_S \mathcal{R}_l$, where \mathcal{R}_i is a $\mathbf{Z}/p_i^{n_i}\mathbf{Z}$ -gerbe for all i . For each p_i that is prime to the characteristic of k , consider the finite and separable field extension $k \subseteq k'$ obtained by adding in all the $p_i^{n_i}$ -th roots of unity. As before, it suffices to prove the result after base changing to k' .

In particular, for those p_i we now have $\mathbf{Z}/p_i^{n_i}\mathbf{Z} \cong \mu_{p_i^{n_i}}$. Since $\mathrm{Br}(S) = \mathrm{Br}'(S)$, the resulting \mathcal{R}_i have the resolution property. For the remaining p_i that are the characteristic of k , these \mathcal{R}_i have the resolution property by Corollary 6.2. An easy calculation now shows that \mathcal{G} has the resolution property. \square

6.3 Deligne–Mumford stacks in positive characteristic

Proof of Theorem B We may assume that \mathcal{X} is connected. By [32, Proposition 2.1], the morphism $\mathcal{X} \rightarrow X$ can be rigidified. Hence it factors into a gerbe morphism and a coarse moduli space morphism:

$$\mathcal{X} \rightarrow \mathcal{X}^{\mathrm{rig}} \rightarrow X.$$

Here $\mathcal{X}^{\mathrm{rig}}$ has generically trivial stabilizers. Therefore $\mathcal{X}^{\mathrm{rig}}$ is a global quotient [12, Theorem 2.18]. By [24, Theorem 2.1] (see also [11, Corollary 4.5]), there is a finite flat cover $Z \rightarrow \mathcal{X}^{\mathrm{rig}}$, where Z is a quasiprojective scheme. Base changing along this cover, we may replace $\mathcal{X}^{\mathrm{rig}}$ by Z [17, Proposition 5.3(vii)]. Then $\mathcal{X} \rightarrow X$ is a separated Deligne–Mumford gerbe, where X is a quasiprojective scheme over $\mathrm{Spec} k$. The result now follows from Theorem 6.3 and Gabber’s theorem [22]. \square

7 Faithful moduli spaces

If V is a nonzero representation of the unipotent group $U_{n,k}$ over a field k , then the set of fixed points $V^{U_{n,k}}$ is nonzero [9, XVII, Proposition 3.2]. If X is a quasiaffine scheme and F is a nonzero quasicoherent sheaf on X , then $\Gamma(X, F)$ is nonzero. The following result shows that this property generalizes to any locally R -unipotent morphism:

Proposition 7.1 *Let $f: X \rightarrow S$ be a locally R -unipotent morphism of algebraic stacks. If F is quasicoherent on X and $f_*F = 0$, then $F = 0$.*

Proof By flat base change we may assume that f is R -unipotent and S is affine. Let N be a quasicoherent \mathcal{O}_S -module; then $\mathrm{Hom}_{\mathcal{O}_X}(f^*N, F) = \mathrm{Hom}_{\mathcal{O}_S}(N, f_*F) = 0$. It follows that if V belongs to \mathbf{Flag}_f , then $\mathrm{Hom}_{\mathcal{O}_X}(V, F) = 0$. As f is R -unipotent, it has the \mathbf{Flag}_f -resolution property (Theorem 5.4). Hence $F = 0$. \square

The following definition is due to Alper (personal communication, 2017).

Definition 7.2 Let $f: X \rightarrow S$ be a quasicompact and quasiseparated morphism of algebraic stacks. We say that f is a *faithful moduli space* if

- (1) the natural map $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$ is an isomorphism (i.e., f is Stein), and
- (2) if $F \in \mathbf{QCoh}(X)$ and $f_*F = 0$, then $F = 0$ (i.e., f_* is zero-reflecting).

Remark 7.3 Faithful moduli spaces are compatible with quasiaffine flat base change. Indeed, if $p: S' \rightarrow S$ is flat and quasiaffine and $f': X' = X \times_S S' \rightarrow S'$ is the induced morphism, then flat base change says that (1) is preserved. Let $p': X' \rightarrow X$ be the induced morphism; then p_* and p'_* are zero-reflecting on quasicoherent sheaves because they are quasiaffine. Hence $p_*f'_* \simeq f_*p'_*$ is zero-reflecting on

quasicoherent sheaves, and so f'_* is zero-reflecting on quasicoherent sheaves, which proves (2). In particular, if $f: X \rightarrow S$ is a faithful moduli space and S has quasiaffine diagonal, then f remains a faithful moduli space after arbitrary flat base change on S .

Example 7.4 A Stein and locally R-unipotent morphism is a faithful moduli space (Proposition 7.1). Also if S is a normal scheme, then any open immersion $U \subseteq S$ with complement of codimension ≥ 2 is Stein and R-unipotent. In particular, there is not necessarily an induced bijection on closed points, so some may find the term “moduli space” misleading.

Remark 7.5 A faithful moduli space f need not imply that the functor f_* is faithful. Indeed, f_* being faithful forces f to be quasiaffine (because the structure sheaf would be f -generating). Moreover, f being Stein implies that the quasiaffine morphism f is an open immersion. Thus the second example in Example 7.4 is essentially the only example of a faithful moduli space for which f_* is faithful.

Remark 7.6 The converse to Proposition 7.1 does not hold: if X denotes affine n -space with a doubled origin and $f: X \rightarrow A^n$ is the natural map, f is a faithful moduli space but is not locally R-unipotent unless $n = 1$ (see Example 5.18). Indeed, it cannot be locally R-unipotent when $n \geq 2$, because Theorem 5.5 would imply that X has affine diagonal. On the other hand, f is Stein. To see that f_* is zero-reflecting, use Lemma 7.7.

Lemma 7.7 *Let $f: X \rightarrow Y$ be a morphism of algebraic stacks that is representable and quasifinite. Then $f_*: \mathbf{QCoh}(X) \rightarrow \mathbf{QCoh}(Y)$ is zero-reflecting.*

Proof By passing to a smooth cover of Y , we may assume that Y is an affine scheme and X is an algebraic space. Let F be a quasicoherent \mathcal{O}_X -module with $f_*F = 0$; then the same is true for any of its finite-type subsheaves, so we may assume F is also of finite-type. If $F \neq 0$, then we may replace X with $V(\text{Ann}_{\mathcal{O}_X}(F))$ and then Y with $V(\ker(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X))$. Then $\text{Ann}_{\mathcal{O}_X}(F) = 0$ and f has schematically dense image. Since f is quasifinite, there is a nonempty open subset $V \subseteq Y$ such that $f^{-1}(V) \rightarrow V$ is finite. Then $F|_{f^{-1}(V)} = 0$ and so $F = 0$, which is a contradiction. \square

Example 7.8 Let k be a field. Let G be an algebraic group scheme over k . Then $BG \rightarrow \text{Spec } k$ is a faithful moduli space if and only if G is unipotent. One implication follows from Example 7.4. For the other, if V is a finite-dimensional representation of G over k , then $V^G \neq 0$. By induction, V is a trivially graded flag on BG . Taking V to be a faithful representation of G , we see that G admits a faithful flag. In particular, it is unipotent.

Example 7.9 Let $Y \xrightarrow{g} X \xrightarrow{f} S$ be quasicompact and quasiseparated morphisms of algebraic stacks.

- (1) If f and g are faithful moduli spaces, then $f \circ g$ is a faithful moduli space.
- (2) If f_* is zero-reflecting, then $X \rightarrow \underline{\text{Spec}}_S(f_*\mathcal{O}_X)$ is a faithful moduli space.

The following result is a version of Proposition 5.15 for faithful moduli spaces:

Proposition 7.10 *Let $f: X \rightarrow S$ be a faithful moduli space with affine stabilizers. Then the relative geometric stabilizers of f are unipotent groups.*

Proof We may assume that S is an affine scheme; then X is quasicompact and quasiseparated. Let $\bar{x}: \text{Spec } k \rightarrow X$ be a geometric point of X , where k is an algebraically closed field. Let $G = \text{Aut}(\bar{x})$, which is a group scheme of finite type over $\text{Spec } k$. There is an induced quasiaffine morphism $BG \rightarrow X$ [36, Theorem B.2] and so $BG \rightarrow \text{Spec } k$ is a faithful moduli space (Example 7.9(2)). By Example 7.8, the result follows. \square

The following lemma turns out to be very useful:

Lemma 7.11 *Let $f: X \rightarrow S$ be a quasicompact, quasiseparated and Stein morphism of algebraic stacks. If $I \subseteq \mathcal{O}_X$ is a quasicoherent sheaf of ideals, then $V(I) \subseteq f^{-1}(V(f_*I))$.*

Proof Indeed, the quasicoherent sheaf defining the closed substack $f^{-1}(V(f_*I)) \subseteq X$ is the image of $f^*f_*I \rightarrow \mathcal{O}_X$, which factors through I . \square

Proposition 7.12 *Let $f: X \rightarrow S$ be a faithful moduli space with affine stabilizers. If S is reduced and quasiseparated with affine stabilizers, then there is a nonempty open $U \subseteq S$ with $f^{-1}(U) \rightarrow U$ a locally R -unipotent gerbe.*

Proof We may assume that S is quasicompact; then [19, Proposition 2.6(i)] implies that there is a nonempty open quasicompact $U \subseteq S$ with affine diagonal. Let $\text{Spec } A \rightarrow U$ be a smooth cover; then the composition $\text{Spec } A \rightarrow U \rightarrow S$ is quasiaffine and smooth. It follows from Remark 7.3 that we may now assume that $S = \text{Spec } A$, where A is reduced.

Now there is a nonempty open immersion $\mathcal{G} \subseteq X_{\text{red}}$ such that \mathcal{G} is a gerbe with the resolution property with affine coarse space $g: \mathcal{G} \rightarrow T$ (combine [43, Tags 06RC and 06NH] with [19, Proposition 2.6(i)]). Let $I \subseteq \mathcal{O}_X$ be a quasicoherent ideal sheaf defining the reduced closed complement Z of $\mathcal{G} \subseteq X$. If $Z \neq \emptyset$, then $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Z, \mathcal{O}_Z)$ is nonzero because it is a ring homomorphism and $\Gamma(Z, \mathcal{O}_Z) \neq 0$. In particular, $\Gamma(X, I) \subsetneq A$. Since $I \neq 0$, it follows that we may choose $a \in \Gamma(X, I) \setminus \{0\}$ that is not a unit. Since A is reduced, a is not nilpotent and so $\text{Spec } A_a \neq \emptyset$. But Lemma 7.11 implies that $Z \cap f^{-1}(\text{Spec } A_a) = \emptyset$, and so we may replace A by A_a and assume that $Z = \emptyset$. Hence $\mathcal{G} = X_{\text{red}} \subseteq X$ is a surjective closed immersion with defining ideal J . By Lemma 7.11 again, we may further shrink S so that $J = 0$ and $\mathcal{G} = X$ is a gerbe. But the composition $\mathcal{O}_S \rightarrow t_*\mathcal{O}_T \rightarrow f_*\mathcal{O}_X$ is an isomorphism, as is $\mathcal{O}_T \rightarrow g_*\mathcal{O}_X$; hence, $T \simeq S$ and $X \rightarrow S$ is of finite presentation. By standard limit methods and Theorem 5.4, we may replace S by the spectrum of the localization at a minimal prime of A , which is a field [43, Tag 00EU]. By Example 7.8 and Remark 5.14 the result follows. \square

Corollary 7.13 *Let k be a field. Let $X \rightarrow \text{Spec } k$ be a faithful moduli space. Then $X \rightarrow \text{Spec } k$ is an R -unipotent gerbe. In particular, if X is an algebraic space, then $X \rightarrow \text{Spec } k$ is an isomorphism.*

We conclude with the proof of our characterization of proper and representable geometrically R -unipotent morphisms:

Proof of Proposition 5.20 We may assume that S is affine. By Zariski's main theorem, it suffices to prove that f is quasifinite. Hence we may further reduce to the situation where S is the spectrum of an

algebraically closed field k and f is \mathbb{R} -unipotent. Let $X \rightarrow S' \rightarrow S$ be the Stein factorization of f . Then $S' \rightarrow S$ is finite and the morphism $X \rightarrow S'$ is again \mathbb{R} -unipotent, and so we may further reduce to the situation where X is geometrically connected and f is Stein. It follows from Example 7.4 that $X \rightarrow \text{Spec } k$ is a faithful moduli space. By Corollary 7.13, $X \rightarrow \text{Spec } k$ is an isomorphism, which gives the claim. \square

Acknowledgements

We would like to thank Jarod Alper, Brian Conrad, Andrea Di Lorenzo, Aise Johan de Jong, Andrew Kresch, Nikolas Kuhn, Max Lieblich, Martin Olsson, Stefan Schröer, Tuomas Tajakka, and Angelo Vistoli for helpful comments. We are especially thankful to David Rydh for carefully reading an earlier draft and providing many invaluable suggestions. We would also like to thank the referees for a number of helpful suggestions. Bragg was supported by the National Science Foundation under NSF postdoctoral research fellowship DMS-1902875. Hall was partially supported by the Australian Research Council DP210103397 and FT210100405. Mathur conducted part of this research in the framework of the research training group GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology, which is funded by the Deutsche Forschungsgemeinschaft. Mathur was also supported by the Swedish Research Council under grant 2016-06596 while in residence at Institut Mittag-Leffler in Djursholm, Sweden during the Fall of 2021. Mathur was also supported by FONDECYT regular grant 1230402.

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Received: February 16, 2022

Revised: November 25, 2024

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

GT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

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GEOMETRY & TOPOLOGY

Volume 30 Issue 3 (pages 835–1201) 2026

The level structure in quantum K -theory and mock theta functions	835
YONGBIN RUAN, MING ZHANG and YAOXIONG WEN	
Springer theory for symplectic Galois groups	883
KEVIN MCGERTY and THOMAS NEVINS	
Unipotent morphisms	929
DANIEL BRAGG, JACK HALL and SIDDHARTH MATHUR	
On the Donaldson–Scaduto conjecture	959
SAMAN HABIBI ESFAHANI and YANG LI	
Milnor’s invariants for knots and links in closed orientable 3-manifolds	983
RYAN STEES	
Chromatic defect, Wood’s theorem and higher real K -theories	1051
CHRISTIAN CARRICK	
Compact holonomy G_2 manifolds need not be formal	1109
LUCÍA MARTÍN-MERCHÁN	
$K3$ surfaces associated with varieties of generalized Kummer type	1129
SALVATORE FLOCCARI	
Ulrich sheaves, the arithmetic writhe and algebraic isotopies of space curves	1155
DANIELE AGOSTINI and MARIO KUMMER	