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On the Donaldson–Scaduto conjecture

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Motivated by G_2-manifolds with coassociative fibrations in the adiabatic limit, Donaldson and Scaduto conjectured the existence of associative submanifolds homeomorphic to a three-holed 3-sphere with three asymptotically cylindrical ends in the G_2-manifold $X \times \mathbb{R}^3$, or equivalently similar special Lagrangians in the Calabi–Yau 3-fold $X \times \mathbb{C}$, where X is an A_2-type ALE hyperkähler 4-manifold. We prove this conjecture by solving a real Monge–Ampère equation with a singular right-hand side, which produces a potentially singular special Lagrangian. Then, we prove the smoothness and asymptotic properties for the special Lagrangian using inputs from geometric measure theory. The method produces many other asymptotically cylindrical $U(1)$-invariant special Lagrangians in $X \times \mathbb{C}$, where X arises from the Gibbons–Hawking construction.

1 Introduction

Simon Donaldson [2017] initiated a program to study G_2-manifolds through coassociative K3 fibrations $\pi : M^7 \to B^3$ over a 3-dimensional base B^3, in the adiabatic limit where the diameters of the K3 fibers shrink to zero. This program is expected to lead to large classes of new examples of compact torsion-free G_2-manifolds.

Subsequent work of Donaldson and Scaduto [2022] provided a conjectural limiting description of certain associative submanifolds in the adiabatic setting. Roughly speaking, the generic part of the associative submanifolds is fibered over one-dimensional gradient flowlines inside the base B^3. These flowlines are allowed to end on the discriminant locus of $\pi : M^7 \to B^3$, and generically, three flowlines can meet to form a triple junction point. Donaldson and Scaduto made several conjectures concerning the existence of the local model for the triple junction.

In the "global" version of the conjecture, let (X^4, I_1, I_2, I_3) be a hyperkähler K3 surface, and let $\alpha_1, \alpha_2, \alpha_3$ be (-2)-classes in $H_2(X^4; \mathbb{R})$, namely $\alpha_i^2 = -2$ with respect to the intersection product, such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Each α_i determines a complex structure J_i in the S^2-family of complex structures, so that for suitable choices of $v_i \in \mathbb{R}^3$, we have three cylindrical associative submanifolds $P_i := \Sigma_i \times (\mathbb{R}^+ v_i) \subset X^4 \times \mathbb{R}^3$ for $i \in \{1, 2, 3\}$.

Conjecture (Donaldson and Scaduto) *There is an associative submanifold homeomorphic to a three-holed 3-sphere $P \subset X^4 \times \mathbb{R}^3$ with three ends asymptotic to cylinders P_1, P_2 and P_3.*

In the "local" version of the conjecture, the K3 surface is replaced with an A_2-type ALE gravitational instanton $X^4_{A_2}$. In the Gibbons–Hawking ansatz, $X^4_{A_2}$ is defined as the completion of a $U(1)$-bundle over $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$. We make the genericity assumption that p_1, p_2, p_3 are three noncolinear points

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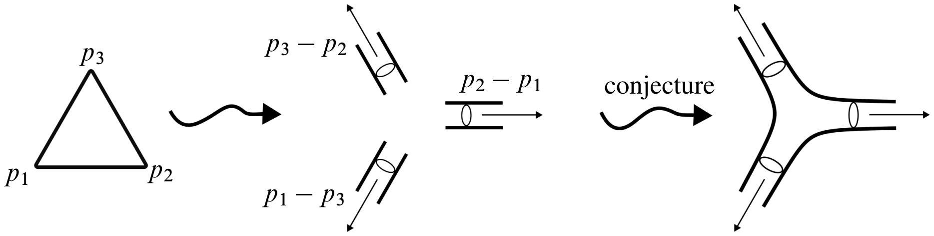

Figure 1: Donaldson-Scaduto conjecture.

in $\mathbb{R}^3$, which corresponds to the condition that the three (-2)-spheres Σ_i are holomorphic with respect to different complex structures. The relation to the global version is that when the K3 surface is the small desingularization (e.g., smoothing or resolution) of an orbifold with local A_2-singularity, then the local version captures the metric behavior near the desingularization region.

We prove the local version in this note.

Theorem 1 (Donaldson–Scaduto conjecture, local version) *There exists a $U(1)$-invariant associative submanifold $P \subset X^4_{A_2} \times \mathbb{R}^3$ homeomorphic to a three-holed 3-sphere, with three ends asymptotic to the half-cylinders $\Sigma_i \times (\mathbb{R}^+ v_i)$, where $i \in \{1,2,3\}$.*

In fact, since the vectors v_1, v_2, and v_3 lie in a plane, say $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, the associative submanifold P can be equivalently interpreted as a special Lagrangian submanifold in $X^4 \times \mathbb{R}^2$ with an appropriate Calabi–Yau structure. Our method readily generalizes to the A_{n-1}-type ALE or ALF gravitational instantons X where $n \geq 3$ and the monopole points $p_1, \dots, p_n$ in the Gibbons–Hawking ansatz are in the "convex position", namely they are the vertices of a convex polygon in a plane in $\mathbb{R}^3$, arranged in the counterclockwise orientation. We equip $X \times \mathbb{R}^2$ with a natural product Calabi–Yau 3-fold structure.

Theorem 2 (generalization) *There is an $(n-1)$-dimensional family of $U(1)$-invariant special Lagrangian submanifolds in the Calabi–Yau 3-fold $X \times \mathbb{R}^2$, each homeomorphic to an n-holed 3-sphere and with n asymptotically cylindrical ends, modeled on the product of $\Sigma_i \subset X$ and $\{(p_{i+1} - p_i) \cdot y = c_i\} \subset \mathbb{R}^2$, where the parameters $\{c_i\}_{i=1}^n$ satisfy one constraint, $\sum_{i=1}^n c_i = 0$.*

The $n-1$ parameters geometrically correspond to the translation of the n asymptotic cylinder ends, subject to one constraint coming from the vanishing of the integral of $\mathrm{Im}(\Omega)$. Specifically, two of these parameters account for global translations along the $\mathbb{R}^2$ direction, while the remaining $n-3$ parameters yield geometrically distinct special Lagrangians. Moreover, these special Lagrangians remain rigid after fixing the asymptotic conditions, as studied in [Habibi Esfahani 2022, Theorem 43].

Remark 3 In [Bera et al. 2025, Theorem 1], it is shown that any special Lagrangian submanifold homeomorphic to an n-holed 3-sphere with n asymptotically cylindrical ends, satisfying the conditions of Theorem 2, belongs to the $(n-1)$-dimensional family of $U(1)$-invariant special Lagrangians constructed in this paper.

We expect these special Lagrangians to be useful as building blocks in the gluing construction of new special Lagrangians in "local Calabi–Yau 3-folds" admitting a fibration of A_{n-1}-type spaces over a Riemann surface.

Plan of the paper We focus on Theorem 2. In Section 2, we introduce the geometric structures on the ambient spaces. In Section 3.1, we dimensionally reduce the $U(1)$-invariant special Lagrangian conditions to an equation for surfaces in the symplectic quotient. Under an additional graphical assumption, this leads to a 2-dimensional real Monge–Ampère equation for some potential φ over the convex polygon with vertices $p_1, \dots, p_n$. In Section 3.2, we solve the appropriate Dirichlet problem for φ, where the boundary data is given by affine linear functions on each edge of the polygon. The $U(1)$-bundle L° over the gradient graph of the solution over the open solid convex polygon is an open $U(1)$-invariant special Lagrangian in $X \times \mathbb{R}^2$.

In Section 4.1, we take the closure of L°, denoted by L. To prove L is a closed submanifold, we need two extra ingredients. First, the gradient of φ diverges to infinity near the edges of the convex polygon away from the vertices, and therefore, the edges give no contribution to the closure of the special Lagrangian; this uses some analysis on the real Monge–Ampère equation. Second, the vertex contributions introduce only smooth points to the special Lagrangian; this uses some geometric measure theory, as well as the classification of $U(1)$-invariant special Lagrangian cones in $\mathbb{C}^3$, following an earlier idea of Joyce [2005]. In Section 4.2, we prove an exponential decay estimate and show that L has the expected asymptotic behavior. In Section 4.3, we prove L is homeomorphic to an n-holed 3-sphere. This concludes the proof of Theorem 2.

2 Preliminaries: ambient spaces

We recall the hyperkähler structure on the $U(1)$-invariant gravitational instanton X, and describe the Calabi–Yau structure on $Z = X \times \mathbb{R}^2$ and the G_2-structure on $M = X \times \mathbb{R}^3$.

Hyperkähler structure Let X be a complete noncompact $U(1)$-invariant hyperkähler 4-manifold, given by the Gibbons–Hawking construction as follows. Let $n \geq 3$, and let $p_1, p_2 \dots, p_n$ be n distinct points in $\mathbb{R}^3$. We will assume that p_i are contained in the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, and in the "convex position", namely they are the vertices of a convex polygon, arranged in counterclockwise order. In the $n = 3$ case, up to coordinate rotation, this amounts to the genericity assumption that p_1, p_2, p_3 are noncolinear.

Let u_1, u_2, u_3 denote the coordinates on $\mathbb{R}^3$. Let

$$\pi : X^\circ \to \mathbb{R}^3 \setminus \{p_1, p_2, \dots, p_n\}$$

be a principal $U(1)$-bundle, with Chern class 1 on small S^2 around each point p_i. Define a positive harmonic function $V : \mathbb{R}^3 \setminus \{p_1, p_2, \dots, p_n\} \to \mathbb{R}$ by

$$V(u) = A + \sum_{i=1}^{n} \frac{1}{2|u - p_i|}, \quad \text{where } A = \text{constant} \geq 0,$$

and let θ be a $U(1)$-connection on X° with curvature 2-form $-*dV$. The Gibbons–Hawking ansatz describes a hyperkähler structure on X° given by the symplectic forms

$$\omega_1 = \theta \wedge du_1 + V\,du_2 \wedge du_3, \quad \omega_2 = \theta \wedge du_2 + V\,du_3 \wedge du_1, \quad \omega_3 = \theta \wedge du_3 + V\,du_1 \wedge du_2,$$

and the metric $g = V^{-1}\theta^2 + V\sum_{i=1}^{3} du_i^2$.

The coordinates u_1, u_2, u_3 are the moment maps with respect to the symplectic forms $\omega_1, \omega_2, \omega_3$, respectively. The manifold X is obtained by adding a point above each p_i, and the hyperkähler structure extends smoothly to X, with the corresponding complex structures I_1, I_2, I_3. In fact, for each $(a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3$, we obtain a complex structure $\sum_{i=1}^{3} a_i I_i$ on X. For $A = 0$, the hyperkähler manifold X is an A_{n-1}-type ALE space, and for $A > 0$, it is an A_{n-1}-type ALF space.

Let $\Sigma_i = \pi^{-1}[p_i, p_{i+1}]$ be the preimage of $[p_i, p_{i+1}]$, the line segment from p_i to p_{i+1}, and for convenience write $p_{n+1} = p_1$. Each Σ_i is a 2-sphere, which is holomorphic with respect to the complex structure associated with the vector v_i,

$$v_i = \frac{(p_{i+1} - p_i)}{|p_{i+1} - p_i|} \in S^2 \cap (\mathbb{R}^2_{(u_1,u_2)} \times \{0\}) \subset \mathbb{R}^3_{(u_1,u_2,u_3)}.$$

Calabi–Yau structure Let $Z = X \times \mathbb{R}^2_{(y_1,y_2)}$. The 6-dimensional manifold Z can be equipped with the Calabi–Yau structure

$$g_Z = g_X + g_{\mathbb{R}^2}, \quad \omega = \omega_3 + dy_2 \wedge dy_1, \quad \Omega = (\omega_1 + i\omega_2) \wedge (dy_2 + i\,dy_1),$$

where y_1, y_2 denote the coordinates on $\mathbb{R}^2$. Note that with our convention,

$$\omega^3 = \frac{3\sqrt{-1}}{4}\Omega \wedge \overline{\Omega}.$$

We extend the $U(1)$-action on X to a $U(1)$-action on Z by $e^{it} \cdot (q, y) \to (e^{it} \cdot q, y)$ for any $q \in X$ and $y \in \mathbb{R}^2$.

Let $\tilde{v}_i = Rv_i$, where $R : \mathbb{R}^2_{(u_1,u_2)} \to \mathbb{R}^2_{(y_1,y_2)}$ is the linear transformation given by the 90-degree rotation,

$$R(\partial_{u_1}) = -\partial_{y_2}, \quad R(\partial_{u_2}) = \partial_{y_1}.$$

Let L_i be $\Sigma_i \times (\mathbb{R}^+ \cdot \tilde{v}_i) \subset X \times \mathbb{R}^2_{(y_1,y_2)}$ translated along some vector in the $\mathbb{R}^2_{(y_1,y_2)}$, so that L_i is contained inside

$$\Sigma_i \times \{y \in \mathbb{R}^2 \mid (p_{i+1} - p_i) \cdot y = c_i\} \subset X \times \mathbb{R}^2, \quad \text{for } i \in \{1, 2, \dots, n\}.$$

A direct computation shows that the L_i are $U(1)$-invariant special Lagrangians in Z,

$$\omega_{|L_i} \equiv 0 \quad \text{and} \quad \operatorname{Im}(\Omega)_{|L_i} \equiv 0 \quad \text{for } i \in \{1, 2, \dots, n\}.$$

Remark 4 The 90-degree rotation is an artifact of our choice of holomorphic volume form Ω. This choice will be useful when we later derive the real Monge–Ampère equation.

G_2 structure Let $M = X \times \mathbb{R}^3$. The 7-dimensional manifold M can be equipped with a torsion-free G_2-structure

$$\begin{aligned}\phi = dy_1 \wedge dy_2 \wedge dy_3 + dy_1 \wedge du_1 \wedge \theta - V dy_1 \wedge du_2 \wedge du_3 \\ + dy_2 \wedge du_2 \wedge \theta - V dy_2 \wedge du_3 \wedge du_1 \\ + dy_3 \wedge du_3 \wedge \theta - V dy_3 \wedge du_1 \wedge du_2,\end{aligned}$$

where y_1, y_2 and y_3 denote the coordinates on $\mathbb{R}^3$. The associated G_2-metric is given by $g_M = g_X + g_{\mathbb{R}^3}$.

Let $P_i = \Sigma_i \times (\mathbb{R}^+ \cdot v_i) \subset X \times \mathbb{R}^3$ for $i \in \{1, 2, 3\}$, where we regard v_i as a vector in the space $\mathbb{R}^3_{(y_1,y_2,y_3)} \cong \mathbb{R}^3_{(u_1,u_2,u_3)}$. The cylinders P_i are $U(1)$-invariant associatives in M, namely $\phi_{|P_i} = \mathrm{vol}_{P_i}$, where the volume form vol_{P_i} is defined with respect to the restriction of the Riemannian metric g_M to P_i.

Identifying M as $Z \times \mathbb{R}_{y_3}$, we have $\phi = -dy_3 \wedge \omega - \mathrm{Im}(\Omega)$. For any submanifold L in $X \times \mathbb{R}^2$, we define $\widetilde{L} = \{(q, y) \in X \times \mathbb{R}^2 \mid (q, R^{-1} y) \in L\}$. For any $a \in \mathbb{R}$, the submanifold $P = L \times \{a\}$ is an associative submanifold of M if and only if $\widetilde{L}$ is a special Lagrangian submanifold in $X \times \mathbb{R}^2$. The upshot is that Theorem 1 follows from Theorem 2.

3 Dimensional reduction and real Monge–Ampère equation

We look for the dimensional reduction on the desired special Lagrangian submanifolds $L \subset X \times \mathbb{R}^2$ with cylindrical ends L_i in the symplectic quotient, where the case $n = 3$ has been conjectured by Donaldson and Scaduto. Under some heuristic assumptions, this leads to a certain real Monge–Ampère equation with a specific Dirichlet boundary condition. The rest of this section will rigorously establish the existence and the properties of the solution.

3.1 The setup

The $U(1)$-invariance of the asymptotic ends L_i motivates us to search for L among $U(1)$-invariant submanifolds. The $U(1)$-moment map associated to the symplectic form ω on Z is u_3, which must be constant on the $U(1)$-invariant Lagrangian L, and the constant is fixed to be zero, since it is zero on the asymptotic cylinders. The special Lagrangian conditions $\omega_{|L} \equiv 0$ and $\mathrm{Im}(\Omega)_{|L} \equiv 0$ for the $U(1)$-invariant $L \subset X \times \mathbb{R}^2$ reduce to

$$V\, du_1 \wedge du_2 - dy_1 \wedge dy_2 = 0 \quad \text{and} \quad du_1 \wedge dy_1 + du_2 \wedge dy_2 = 0,$$

respectively. The dimensionally reduced Lagrangian is

$$L_{\mathrm{red}} := L/U(1) \subset Z_{\mathrm{red}} := u_3^{-1}(0)/U(1).$$

Topologically, Z_{red} can be identified with $\mathbb{R}^4$ with coordinates (u_1, u_2, y_1, y_2), but equipped with a degenerate reduced Kähler structure. Let

$$\pi_1 : Z_{\mathrm{red}} \to \mathbb{R}^2_{(u_1,u_2)} \quad \text{and} \quad \pi_2 : Z_{\mathrm{red}} \to \mathbb{R}^2_{(y_1,y_2)}$$

be the natural projection maps.

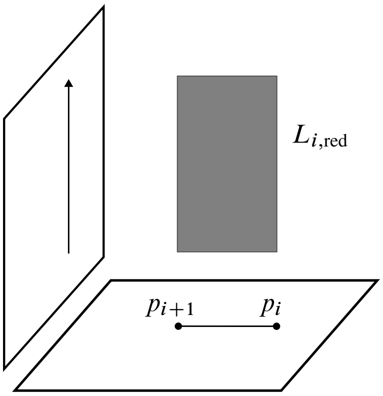

Figure 2: $L_{i,\mathrm{red}} \subset \mathbb{R}^4 = \mathbb{R}^2_{(u_1,u_2)} \times \mathbb{R}^2_{(y_1,y_2)}$.

The $U(1)$-reduction of the cylindrical special Lagrangians L_i results in half-strips $L_{i,\mathrm{red}}$ contained inside the cylinder

$$[p_i, p_{i+1}] \times \{(p_{i+1} - p_i) \cdot y = c_i\} \subset \mathbb{R}^2_{(u_1,u_2)} \times \mathbb{R}^2_{(y_1,y_2)},$$

where $[p_i, p_{i+1}]$ denotes the closed line segment in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ that connects the points p_i and p_{i+1}, and we will require the n parameters c_i to sum to zero; see the Appendix. Therefore,

$$\pi_1(L_{i,\mathrm{red}}) = [p_i, p_{i+1}] \subset \mathbb{R}^2_{(u_1,u_2)} \quad \text{and} \quad \pi_2(L_{i,\mathrm{red}}) = \text{a translation of } (\mathbb{R}^+ \cdot \tilde{v}_i) \subset \mathbb{R}^2_{(y_1,y_2)},$$

as shown in Figure 2. Therefore, $\pi_1\big(\bigcup_i L_{i,\mathrm{red}}\big)$ is the boundary of the convex polygon with vertices $p_1, \dots, p_n$, and $\pi_2\big(\bigcup_i L_{i,\mathrm{red}}\big)$ is the union of n rays, as shown in Figure 3 for the case $n = 3$.

Graphical case The asymptotic cylindrical requirement motivates us to look for the reduction L_{red} of the conjectural special Lagrangian L, as (the closure of) the graph of a map

$$F : U \subset \mathbb{R}^2_{(u_1,u_2)} \to \mathbb{R}^2_{(y_1,y_2)}, \quad \text{with } (y_1, y_2) = F(u_1, u_2),$$

where U is the interior of the convex polytope with vertices $p_1, \dots, p_n$.

The projection of L_{red} to $\mathbb{R}^2_{(y_1,y_2)}$ is expected to be a thickening of the union of the n rays, whose shape resembles an amoeba, as illustrated in Figure 4. We will study these shapes more in Section 3.4.

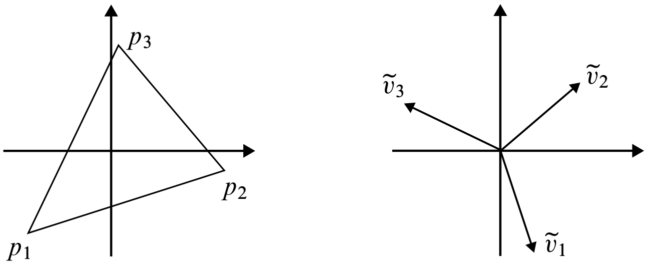

Figure 3: $\pi_1\big(\bigcup L_{i,\mathrm{red}}\big)$, left, and $\pi_2\big(\bigcup L_{i,\mathrm{red}}\big)$, right.

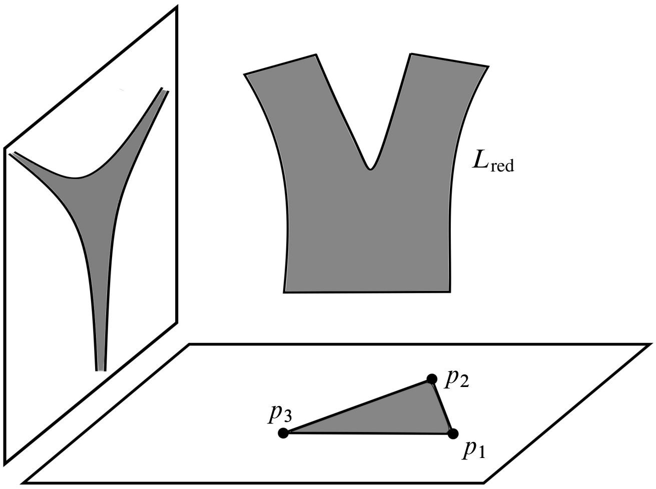

Figure 4: L_{red} in $Z_{\mathrm{red}}=\mathbb{R}^4$, in the case $n=3$.

The equation $du_1\wedge dy_1+du_2\wedge dy_2=0$ is equivalent to $\partial y_2/\partial u_1=\partial y_1/\partial u_2$. This implies that we can define a function $\varphi:U\to\mathbb{R}$ such that $F=\nabla\varphi$, namely

$$y_1=\frac{\partial\varphi}{\partial u_1}\quad\text{and}\quad y_2=\frac{\partial\varphi}{\partial u_2}.$$

The equation $V\,du_1\wedge du_2-dy_1\wedge dy_2=0$ reduces to the following real Monge–Ampère equation for φ:

$$\det D^2\varphi=V=A+\sum_{i=1}^{n}\frac{1}{2|u-p_i|}. \tag{1}$$

Note that V has singularities at the vertices of the convex polygon U.

Dirichlet boundary condition We now use the expected asymptote of the special Lagrangian L to heuristically motivate the Dirichlet boundary condition on the real Monge–Ampère equation. The rest of the paper will start from the Dirichlet problem and construct the conjectured special Lagrangian L. In Section 4.2, we will see that the Dirichlet boundary condition results in the correct asymptotic behavior.

Notice that the vector $\widetilde{v}_i\in\mathbb{R}^2$ is normal to the edge $[p_i,p_{i+1}]$. Since L_{red} is the graph of $\nabla\varphi$ and its asymptote at infinity is the union of L_i, we expect that when $u\in U$ approaches the open line segment (p_i,p_{i+1}) on the boundary of the convex polygon, the normal derivative tends to infinity, while the tangential derivative tends to a constant, $\nabla\varphi\cdot(p_{i+1}-p_i)\to c_i$. Since $\sum c_i=0$, we can write $c_i=b_{i+1}-b_i$, and notice that adding a global constant to b_i will be inconsequential. Thus the boundary value of φ on the edge is expected to be *affine linear*, namely

$$\varphi(p_i)=b_i\qquad\text{and}\qquad\varphi(tp_i+(1-t)p_{i+1})=tb_i+(1-t)b_{i+1}\quad\text{for all }t\in[0,1]. \tag{2}$$

Remark 5 The real Monge–Ampère equation is invariant under adding an affine linear function. Geometrically, adding a constant to φ has no effect on the special Lagrangian L, and adding a linear function amounts to translating L along some vector in $\mathbb{R}^2_{(y_1,y_2)}$. In particular, for the triangle case $n = 3$, we can reduce to the zero boundary data.

The real Monge–Ampère equation is also invariant under the Euclidean motion of the convex polygon U, with the corresponding change to V. Later, when we analyze the local behavior of φ near an open edge, we will sometimes reduce to the "standard position", where the open edge is contained in the u_2-axis, and U lies inside the right half-plane, to simplify notation.

3.2 Solving the Dirichlet problem

Theorem 6 (Dirichlet problem) *Let* $b_1, \dots, b_n \in \mathbb{R}$. *There is a unique continuous convex function* $\varphi : \overline{U} \to \mathbb{R}$, *with boundary data* (2), *which is smooth in* U *and solves the real Monge–Ampère equation* (1).

The remainder of this section is dedicated to the proof of this theorem. We use an approximation strategy to deal with the failure of strict convexity of the domain.

Proof Step 1 (approximate solutions): We take a smooth 1-parameter expanding family of strictly convex smooth domains $U_t \subset U$, where $t \in (0,1)$, converging to $U_1 = U$ as $t \to 1$, as in Figure 5. The piecewise linear boundary data (2) can be extended to some Lipschitz function $\overline{\phi}$ on $\overline{U}$. We consider the following Dirichlet problem for $\varphi_t : U_t \to \mathbb{R}$ for each $t \in (0,1)$:

$$\begin{cases} \det D^2\varphi_t = V & \text{on } U_t, \\ \varphi_t = \overline{\phi} & \text{on } \partial U_t. \end{cases} \tag{3}$$

Lemma 7 [Rauch and Taylor 1977] *Let* $\Omega \subset \mathbb{R}^2$ *be a strictly convex domain*, $g : \partial\Omega \to \mathbb{R}$ *a continuous function*, *and* μ *a nonnegative Borel measure on* Ω *with* $\mu(\Omega) < \infty$. *Then*, *there exists a unique convex function* $f \in C(\overline{\Omega})$ *such that*

$$\begin{cases} \det D^2 f = \mu & \text{on } \Omega, \\ f = g & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In our case, for every $t \in (0,1)$, we let $\Omega = U_t$ and define $\mu = V\,du_1\,du_2$. Notice here that V is strictly positive within U_t and uniformly bounded in L^1,

$$\int_{U_t} V\,du_1\,du_2 \le \int_U V\,du_1\,du_2 < +\infty \quad \text{for all } t \in (0,1).$$

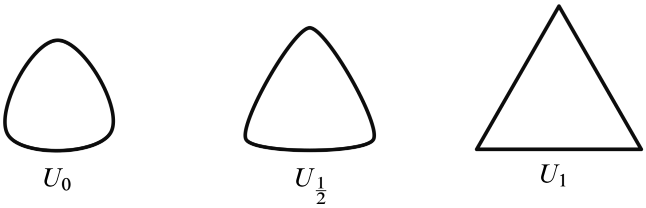

Figure 5: Approximating domains $U_t \to U$, in the case $n = 3$.

Therefore, by Lemma 7, there is a unique convex function $\varphi_t \in C(\overline{U}_t)$ satisfying (3).

Step 2 (uniform bounds and the limit): By the convexity of φ_t, and the Lipschitz property of $\bar{\phi}$, we obtain the uniform upper bound

$$\varphi_t(u) \le \bar{\phi}(u) + C\,\mathrm{dist}(u, \partial U_t) \quad \text{for all } u \in U_t.$$

We now derive a uniform lower bound.

Lemma 8 *There is a uniform constant, independent of t, such that*

$$\varphi_t(u) \ge \bar{\phi}(u) - C\,\mathrm{dist}(u, \partial U)^{1/2} \quad \textit{for all } u \in U_t.$$

Proof As in Remark 5, we put U into the standard position, namely that the boundary edge of interest $[p_i, p_{i+1}]$ lies on the u_2-axis, and U lies in the right half-plane, as in Figure 6, and without loss of generality, $\bar{\phi} = 0$ on this boundary edge. It suffices to show $\varphi_t(u) \ge -Cu_1^{1/2}$ for some constant C independent of t.

By the Lipschitz property of $\bar{\phi}$, we know $|\bar{\phi}(u)| \le C_1 u_1$ on U, for some constant C_1. The boundary data of $\varphi_t + C_1 u_1$ is nonnegative, and it solves the same real Monge–Ampère equation (1).

By the Alexandrov estimate,

$$\max(0, -\varphi_t(u))^2 \le C_2\,\mathrm{diam}(U_t) \cdot \mathrm{dist}(u, \partial U_t) \cdot \int_{U_t} V\,du_1\,du_2 \le C_3 u_1 \quad \text{for all } u \in \overline{U}_t,$$

as required. □

Combining the upper and lower bounds, we obtain a uniform bound on the C^0-norm of φ_t. By the convexity of φ_t, this implies a uniform Lipschitz bound on any fixed compact subset of U, as $t \to 1$. By Arzelà–Ascoli, we can extract a subsequence of φ_t which C^0-converges to some continuous function φ on any compact subset of U, which is a viscosity solution to the real Monge–Ampère equation. By passing the upper and lower bounds to the limit as $t \to 1$, we obtain

$$\bar{\phi}(u) - C\,\mathrm{dist}(u, \partial U)^{1/2} \le \varphi(u) \le \bar{\phi}(u) + C\,\mathrm{dist}(u, \partial U). \tag{5}$$

Thus φ extends continuously to $\overline{U}$ and achieves the boundary data (2), which agrees with $\bar{\phi}$ on ∂U. The uniqueness of the solution is a standard consequence of the maximum principle.

Step 3 (interior smoothness): Notice V is smooth and strictly positive in U. As a standard fact about real Monge–Ampère equation in dimension two, the solution φ to the real Monge–Ampère equation is smooth in the interior domain U. This fact is the consequence of two standard results: Caffarelli [1990] proved that the singular set must propagate to the boundary along some line segment, while the partial regularity of Mooney [2015] showed that the singular set has codimension one Hausdorff measure zero. □

3.3 Gradient divergence near the edges

In this section, we study the behavior of φ near an open edge of the convex polygon U. The following theorem is essential in proving the smoothness of the special Lagrangian L.

Proposition 9 *Let u_* be any point on a boundary edge (p_i, p_{i+1}) in $\partial U \setminus \{p_1, \dots, p_n\}$. Then, as $u \in U$ tends to u_*, the normal and tangential gradient components satisfy*

$$\nabla^{\text{normal}}\varphi(u) \to +\infty \quad \textit{and} \quad \nabla\varphi(u) \cdot (p_{i+1} - p_i) \to c_i.$$

Proof The tangential component converges to a constant by the convexity of φ, the affine linearity of the boundary data, and the boundary continuity estimate (5) by considering the convex function restricted to line segments parallel to the boundary edge.

We now focus on the normal gradient component and place the convex polygon into the standard position by Remark 5, so the open boundary edge (p_i, p_{i+1}) containing u_* lies on the u_2-axis, the domain is contained in the right half-plane, and the boundary data is zero on this edge. The normal gradient component is just $-\partial_{u_1}\varphi$. Using convexity, $\partial_{u_1}\varphi$ is bounded from the above near u_*.

We suppose for contradiction that $\partial_{u_1}\varphi(u)$ stays bounded for some sequence $u \to u_*$, and therefore, the gradient at u stays bounded. Then there exists some subgradient for φ at u_*, which must be of the form $(-\Lambda, 0)$ for some $\Lambda \in \mathbb{R}$, because the tangential component is zero. Let

$$\Lambda_0 = \inf\{\Lambda \mid (-\Lambda, 0) \text{ is a subgradient of } \varphi \text{ at some point } v \in (p_i, p_{i+1})\},$$

so in particular $\varphi(u) \geq -\Lambda_0 u_1$ for any $u \in \overline{U}$. Thus $(-\Lambda_0, 0)$ is a subgradient at every point on the open edge, and $\partial_{u_1}\varphi \geq -\Lambda_0$ on U.

We fix a small constant $h > 0$ such that u_* has distance at least $2h$ to the vertices. Let $R(u_*, h, \varepsilon)$ be the rectangle with length h and width $\varepsilon \ll 1$, as shown in Figure 6. By the Monge–Ampère equation, and the strict positivity of V,

$$C^{-1}h\varepsilon \leq \int_{R(u_*,h,\varepsilon)} V\, du_1\, du_2 = \int_{\nabla\varphi(R(u_*,h,\varepsilon))} dy_1\, dy_2. \tag{6}$$

By the convexity of φ and the bound $-\Lambda_0 u_1 \leq \varphi \leq C u_1$, we deduce

$$|\partial_{u_2}\varphi(u)| \leq C(\Lambda_0, h) u_1 \quad \text{for all } u \in R(u_*, h, \varepsilon).$$

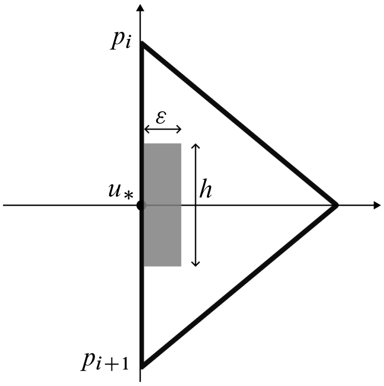

Figure 6: Standard position in the case $n = 3$. $R(u_*, h, \varepsilon)$ is the shaded region.

Thus by considering the gradient image,

$$\left|\int_{\nabla\varphi(R(u_*,h,\varepsilon))} dy_1\,dy_2\right| \le C(\Lambda_0,h)\,\varepsilon \sup_{R(u_*,h,\varepsilon)} (\partial_{u_1}\varphi(u)+\Lambda_0).$$

Contrasting with (6), for any small $\varepsilon > 0$,

$$\sup_{R(u_*,h,\varepsilon)} (\partial_{u_1}\varphi(u)+\Lambda_0) \ge C(\Lambda_0,h)^{-1}.$$

In the limit $\varepsilon \to 0$, we can extract some subgradient at some boundary point, which contradicts the minimality of Λ_0. □

3.4 Solutions near vertices

In this section, we examine the behavior of the gradient of φ near the vertices of the convex polygon, which will be important in studying the smoothness of L. The ideal picture to have in mind, which we justify in this section, is shown in Figure 7 for the case $n=3$. The Map $F=\nabla\varphi$ takes the bounded gray solid convex polygon to the unbounded gray area. Denote the subgradient sets at the vertices by

$$C_{p_i} = \{y \in \mathbb{R}^2_{(y_1,y_2)} \mid \varphi(u)-\varphi(p_i) \ge \langle y, u-p_i\rangle \text{ for all } u \in \overline{U}\}.$$

Lemma 10 *The sets C_{p_i} are disjoint convex closed subsets of $\mathbb{R}^2$ contained in the wedge region*

$$C_{p_i} \subset W_{p_i} = \{y \in \mathbb{R}^2 \mid y\cdot(p_{i+1}-p_i) \le b_{i+1}-b_i\} \cap \{y \in \mathbb{R}^2 \mid y\cdot(p_{i-1}-p_i) \le b_{i-1}-b_i\}.$$

Remark 11 The wedge region is a translated copy of the wedge

$$W'_{p_i} = \{y \in \mathbb{R}^2 \mid y\cdot(p_{i+1}-p_i) \le 0\} \cap \{y \in \mathbb{R}^2 \mid y\cdot(p_{i-1}-p_i) \le 0\},$$

in which the directions of its two extremal rays are specified by the vectors $\widetilde{v}_{i-1}$ and $\widetilde{v}_i$. For different i, the wedges W'_{p_i} can only intersect along boundary rays, and the intersections between different W_{p_i} have areas bounded by some constant depending only on $\{b_k\}_{k=1}^n$.

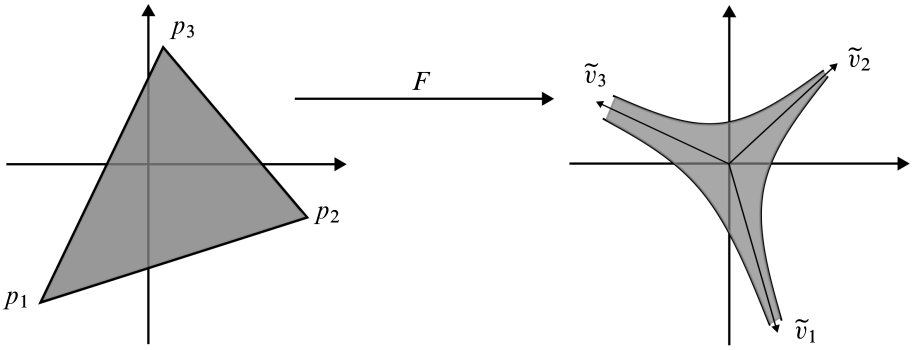

Figure 7: Mapping $F=\nabla\varphi : U \subset \mathbb{R}^2_{(u_1,u_2)} \to \mathbb{R}^2_{(y_1,y_2)}$.

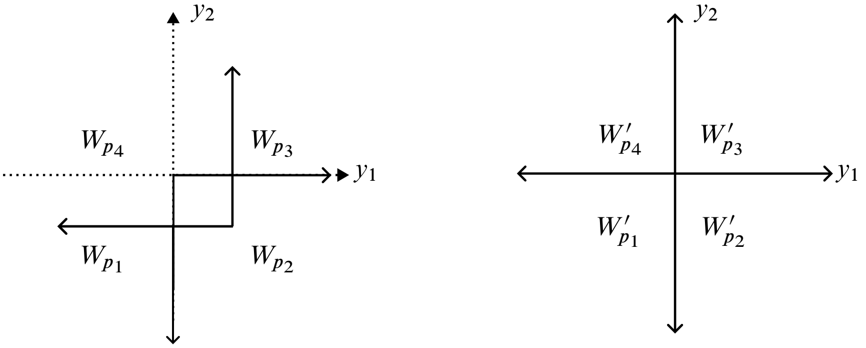

Figure 8: Adjacent wedges W_{p_i} and $W_{p_{i+1}}$ intersect along a common ray, but nonadjacent wedges may intersect nontrivially inside a compact set.

Proof of Lemma 10 Suppose C_{p_i} and C_{p_j} have a common vector. Then by convexity, this vector is a subgradient for all the points on the line segment $[p_i, p_j]$. If the open segment (p_i, p_j) is in the interior domain U, this would contradict the strict convexity of φ (which follows from the smoothness of the real Monge–Ampère solution), and if (p_i, p_j) is an edge in the boundary, this would imply the existence of subgradient at every point of (p_i, p_j), contradicting Proposition 9. This proves the disjointness of the C_{p_i}. □

Lemma 12 (image of the gradient) $\nabla\varphi(U) = \mathbb{R}^2 \setminus \Bigl(\bigcup_i C_{p_i}\Bigr)$.

Proof First we claim that $\mathbb{R}^2 = \bigcup_i C_{p_i} \cup \nabla\varphi(U)$. Given any $y \in \mathbb{R}^2$, we consider the graph of the affine linear function $a + \langle y, u\rangle$ on $\overline{U}$, where $a \in \mathbb{R}$ increases from negative infinity. There must be some a for which the graph first touches the graph of the convex function φ. This shows that y is a subgradient at some point on $\overline{U}$. But the divergence of the gradient on the edge (Proposition 9) shows that there is no subgradient at any point on the open edges, so y is a subgradient either at an interior point or at one of the vertices.

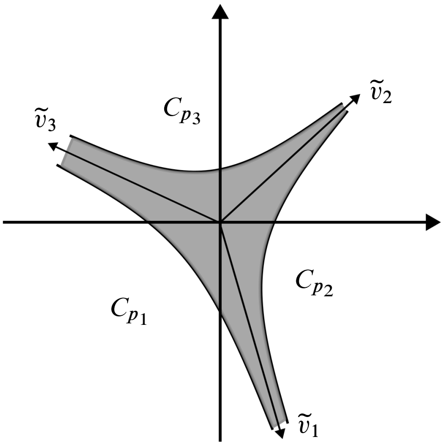

Figure 9: Subgradients C_{p_1}, C_{p_2} and C_{p_3}, in the case $n = 3$.

By the strict convexity of φ in U, we see

$$\nabla\varphi(U) \cap C_{p_i} = \varnothing.$$

Thus $\nabla\varphi(U) = \mathbb{R}^2 \setminus \bigl(\bigcup_i C_{p_i}\bigr)$. □

Lemma 13 *For any sequence of points $u \in U$ converging to p_i, after passing to a subsequence, either $\nabla\varphi(u) \to \infty$, or $\nabla\varphi(u)$ converges to a point in ∂C_{p_i}.*

Proof Suppose $|\nabla\varphi(u)|$ stays bounded. After passing to a subsequence, $|\nabla\varphi(u)|$ converges to some y. Then y is a subgradient at p_i, so lies in C_{p_i}. However, the limit y has to lie in the closure of $\nabla\varphi(U)$, which is disjoint from the interior of C_{p_i}. Thus the gradient lies on ∂C_{p_i}. □

Lemma 14 gives some asymptotic decay bound on the gradient image $\nabla\varphi(U)$.

Lemma 14 *Suppose $y \in \nabla\varphi(U)$ and $|y| \geq 1$. Then, for one of the boundary rays $\mathfrak{R}$ of some wedge region W_{p_i}, we have $\operatorname{dist}(y, \mathfrak{R}) \leq C/|y|$, where the distance is measured in the Euclidean $\mathbb{R}^2$, and the constant C is independent of y.*

In particular, Lemma 14 shows that a scenario similar to the one shown in Figure 10, where $\nabla\varphi(U)$ contains an infinite wedge, cannot happen.

Proof Without loss of generality, we assume that y is large compared to $\max_i |b_i - b_{i+1}|$. The rays $\mathbb{R}_{\geq 0}\tilde{v}_i$ partition $\mathbb{R}^2$ into wedge-shaped regions W'_{p_i}. We choose the direction $\tilde{v}_i$ which minimizes the angle with the direction of y. Notice $\tilde{v}_i$ is parallel to the boundary ray between W_{p_i} and $W_{p_{i+1}}$, and by the choice of $\tilde{v}_i$ and the largeness of y, we see that y must lie in either W_{p_i} or $W_{p_{i+1}}$, and without loss of generality we focus on $y \in W_{p_i}$.

Let q_i be the vertex point of W_{p_i}, and let $\mathfrak{R} = \mathbb{R}_{\geq 0}\tilde{v}_i + q_i$, and let y' be the intersection point of the ray $\mathfrak{R}$ with the ray $y + \mathbb{R}_{\leq 0}\tilde{v}_{i-1}$. If $y = y'$, then the distance to the ray is zero, and we are done. So without loss of generality $y \neq y'$, and we consider triangle $T(y)$ with vertices at y, y', q_i.

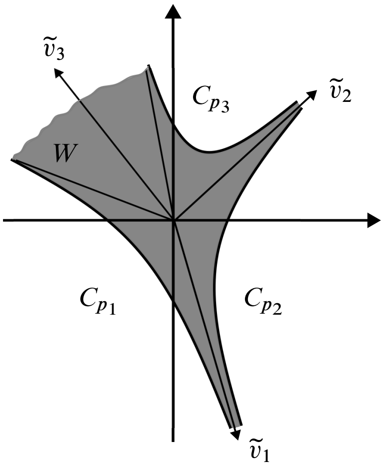

Figure 10: $\nabla\varphi(U)$ cannot contain an infinite wedge W.

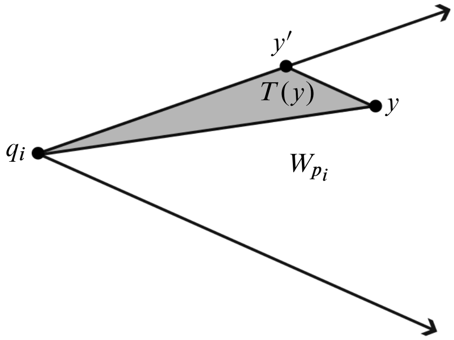

Figure 11: Triangle $T(y)$.

By construction the triangle $T(y)$ is contained in W_{p_i}. By elementary geometry, the area of this triangle is comparable to $|y|\operatorname{dist}(y,\mathcal{R})$ when $|y|$ is much larger than $\max_i|b_i-b_{i+1}|$.

Using the partition $\mathbb{R}^2=\nabla\phi(U)\cup\bigcup_j C_{p_j}$,

$$\begin{aligned}\operatorname{Area}(W_{p_i}\setminus C_{p_i}) &\le \int_{\nabla\varphi(U)} dy_1\,dy_2+\operatorname{Area}\Bigl(W_{p_i}\cap\bigcup_{j\neq i}C_{p_j}\Bigr)\\ &\le \int_{\nabla\varphi(U)} dy_1\,dy_2+\sum_{j\neq i}\operatorname{Area}(W_{p_i}\cap W_{p_j})\\ &\le \int_U V\,du_1\,du_2+C\le C.\end{aligned}\tag{7}$$

Here the second line uses $C_{p_j}\subset W_{p_j}$, and the third line uses the Monge–Ampère equation and the fact that nonadjacent wedge regions W_{p_j} only intersect within a compact set depending only on $\{b_k\}$ and the direction of the rays. Since $T(y)\subset W_{p_i}$, in particular, $\operatorname{Area}(T(y)\setminus C_{p_i})\le C$.

We claim that $C_{p_i}\cap T(y)$ is empty. Suppose $y''\in C_{p_i}\cap T(y)\subset W_{p_i}$. Then by the convexity of C_{p_i}, and the fact that $y\notin C_{p_i}$ because of $y\in\nabla\varphi(U)\subset\mathbb{R}^2\setminus\bigcup C_{p_i}$, we deduce that the ray starting from y in the direction $y-y''$ lies in the complement of C_{p_i}. Since the direction of $y-y''$ lies in the wedge region W'_{p_i}, by the convexity of C_{p_i} the set $W_{p_i}\setminus C_{p_i}$ contains some infinite wedge region with vertex at y. This is impossible by the finiteness of $\operatorname{Area}(W_{p_i}\setminus C_{p_i})$.

Combining the above,

$$|y|\operatorname{dist}(y,\mathcal{R})\le C\operatorname{Area}(T(y))\le C,$$

hence $\operatorname{dist}(y,\mathcal{R})\le C/|y|$ when $|y|$ is large. □

4 Regularity, asymptotics and topology

This section aims to prove Theorem 2 by producing the desired special Lagrangian from the solution of the real Monge–Ampère equation, and establishing its smoothness, asymptotic properties and topology.

4.1 Smoothness of the special Lagrangian

Let $q : u_3^{-1}(0) \to Z_{\mathrm{red}}$ be the quotient map, and $L^\circ = q^{-1}(\mathrm{Graph}(\nabla\varphi_{|U}))$. By the smoothness of the real Monge–Ampère solution φ, clearly L° is a smooth special Lagrangian submanifold in Z, diffeomorphic to $\mathbb{D}^2 \times S^1$; however, it is not a closed subset of Z. Let L be the current of integration defined by L°, so that L contains also points in the closure of L°. The goal of this section is to prove the following theorem.

Theorem 15 *L is represented by a smooth submanifold of Z without boundary.*

The proof of Theorem 15 follows from Lemmas 16 and 20.

Lemma 16 *L is a closed integral current, $\partial L = 0$. Moreover, any point in the support of L either lies on L° or lies above some vertex p_i.*

Proof We first show that L has locally finite mass inside any given ball in Z. Since the gradient $\nabla\varphi$ diverges to infinity near the open edges (p_i, p_{i+1}), we only need to show that mass cannot accumulate near the vertices p_i. Now by the special Lagrangian condition, for any $U(1)$-invariant compact set $K \subset Z$,

$$\mathrm{Mass}(L \cap K) = \int_{L^\circ \cap K} \mathrm{Re}(\Omega) = 2\pi \int_{(L^\circ \cap K)/U(1)} (du_1 \wedge dy_2 - du_2 \wedge dy_1).$$

Since φ is a smooth strictly convex function, both integrands are positive, and the maps $(u_1, u_2) \mapsto (u_1, y_2)$ and $(u_1, u_2) \mapsto (y_1, u_2)$ are injective, so $\int_{(L^\circ \cap K)/U(1)} du_1 \wedge dy_2$ is simply the Lebesgue measure of the (u_1, y_2) projection of $(L^\circ \cap K)/U(1)$, and likewise with the second integrand. By the boundedness of the range of $(u_1, y_2), (u_2, y_1)$ on $L^\circ \cap K$, we deduce that the integral is finite.

The support of L is contained in the closure of L°. Since the gradient $\nabla\varphi$ diverges to infinity, there cannot be any limiting point in the support of L whose projection to $\mathbb{R}^2_{(u_1,u_2)}$ lies above the open edges (p_i, p_{i+1}). Thus, the only points added in the closure lie above the vertices p_i, and by Lemma 13, they lie above $\{p_i\} \times \partial C_{p_i}$.

Let ψ be any smooth and compactly supported test 2-form on Z, and for small $r > 0$, let χ_r be a cutoff function supported in the union of $B_{X^4}(p_i, r) \times \mathbb{R}^2 \subset Z$, with value one on the union of $B_{X^4}(p_i, r/2) \times \mathbb{R}^2$, and whose g_X-gradient is bounded by Cr^{-1}. Since $(1 - \chi_r)\psi$ is supported away from the $U(1)$-fixed locus, integration by parts gives

$$\int_L \chi_r d\psi = -\int_L d\chi_r \wedge \psi = -\int_{L^\circ} d\chi_r \wedge \psi,$$

hence

$$\left| \int_L \chi_r d\psi \right| \le C(\psi) r^{-1} \sum_{i=1}^{n} \mathrm{Mass}(L^\circ \cap \mathrm{supp}(\psi) \cap B_{X^4}(p_i, r) \times \mathbb{R}^2).$$

Now y_1 and y_2 are bounded within the support of ψ, and from the Gibbons–Hawking ansatz we know $|u - p_i| \le Cr^2$ on $B_{X^4}(p_i, r)$. The same argument as in the local finiteness of measure now gives a bound $|\int_L \chi_r d\psi| \le C(\psi)r$. Taking the limit $r \to 0$, we deduce that $\int_L d\psi = 0$ for any test 2-form, which means $\partial L = 0$. $\square$

Now, we prove the smoothness. L is a special Lagrangian integral current and, in particular, a minimal integral current. Therefore, there exists a tangent cone at each point x on L. The proof of smoothness is based on the following implication of Allard's regularity theorem.

Proposition 17 *A point $x \in \operatorname{supp}(L)$ is a smooth point if and only if every tangent cone $N \subset \mathbb{C}^3$ at x is a 3-plane with multiplicity one.*

Let $q(x) \in \{p_i\} \times \partial C_{p_i}$. Any tangent cone $N \subset \mathbb{C}^3$ at x is a $U(1)$-invariant tangent cone in $\mathbb{C}^3$. To prove every tangent cone of L is a 3-plane with multiplicity one, we employ Joyce's classification [2005] of $U(1)$-invariant special Lagrangian cones in $\mathbb{C}^3$.

Proposition 18 [Joyce 2005; Haskins 2004] *Let N be a special Lagrangian cone without boundary in $\mathbb{C}^3$ invariant under the $U(1)$-action given by*

$$e^{i\theta} : (z_1, z_2, z_3) \to (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \quad \text{for } e^{i\theta} \in U(1),$$

where $N \setminus \{0\}$ is connected. Then there exists $A \in [-1, 1]$ and functions $w : \mathbb{R} \to (-1, 1)$, and $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ satisfying the system of differential equations

$$\left(\frac{dw}{dt}\right)^2 = 4((1-w)^2(1+2w) - A^2), \qquad \frac{d\alpha}{dt} = \frac{A}{1-w},$$
$$\frac{d\beta}{dt} = \frac{-2A}{1+2w}, \qquad (1-w)(1+2w)^{\frac{1}{2}} \cos(2\alpha+\beta) = A,$$

such that, away from points $(z_1, z_2, z_3) \in \mathbb{C}^3$ with $z_j = 0$ for some j, we may locally write N in the form $\Phi(r, s, t)$ with $r > 0$ and $s, t \in \mathbb{R}$, where

$$\Phi : (r, s, t) \mapsto (re^{i(\alpha(t)+s)}\sqrt{1-w(t)}, e^{i(\alpha(t)-s)}\sqrt{1-w(t)}, re^{i\beta(t)}\sqrt{1+2w(t)}),$$

and exactly one of the following holds:

- $A = 1$. *Then, N is the $U(1)^2$-invariant special Lagrangian $\mathbb{T}^2$-cone*
$$\{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}) \mid r > 0,\ \theta_1, \theta_2, \theta_3 \in \mathbb{R},\ \theta_1 + \theta_2 + \theta_3 = 0\}.$$
- $A = -1$. *Then, N is the $U(1)^2$-invariant special Lagrangian $\mathbb{T}^2$-cone*
$$\{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}) \mid r > 0,\ \theta_1, \theta_2, \theta_3 \in \mathbb{R},\ \theta_1 + \theta_2 + \theta_3 = \pi\}.$$
- $A = 0$. *Then, for some $\phi \in (-\pi, \pi]$, either $N = \Pi_\phi^+$, or $N = \Pi_\phi^-$, or N is the singular union $\Pi_\phi^+ \cup \Pi_\phi^-$, where $\Pi_\phi^\pm$ are the special Lagrangian 3-planes*
$$\Pi_\phi^+ = \{(z, ie^{-i\phi}\bar{z}, re^{i\phi}) \mid z \in \mathbb{C}, r \in \mathbb{R}\} \quad \text{and} \quad \Pi_\phi^- = \{(z, -ie^{-i\phi}\bar{z}, re^{i\phi}) \mid z \in \mathbb{C}, r \in \mathbb{R}\}.$$

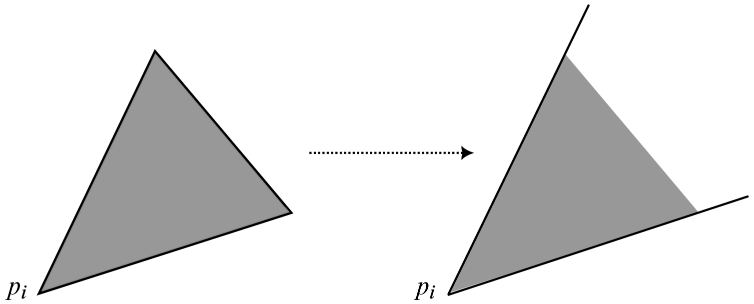

Figure 12: The π_1-projection of a tangent cone at p_i.

- $0 < |A| < 1$. *Then, the function $w(t)$ may be written in terms of the Jacobi elliptic functions. It is nonconstant and periodic in t with period T depending only on A, and $2\alpha + \beta$ is also nonconstant and periodic in t with period T.*

We proceed by ruling out every possibility on Joyce's list except 3-planes. We do this using the following lemma.

Lemma 19 *Let $N \subset \mathbb{C}^3$ be a special Lagrangian tangent cone of L at x, where $q(x) \in p_i \times \partial C_{p_i}$. Let $U = \pi_1(N/U(1)) \subset \mathbb{R}^2_{(u_1,u_2)}$. The set U is a subset of the infinite wedge with vertex p_i and two rays along the direction $\overrightarrow{p_i p_{i+1}}$ and $\overrightarrow{p_i p_{i-1}}$.*

Proof The image of L in $\mathbb{R}^2_{(u_1,u_2)}$ is a convex polygon. Let W be the infinite wedge with vertex at p_i, and two boundary rays $\overrightarrow{p_i p_{i+1}}$ and $\overrightarrow{p_i p_{i-1}}$. In particular its opening angle is less than π. The π_1-projections of all the special Lagrangians obtained by rescaling L around the base point x, are all contained in W, so by passing to the limit, the same holds for the projection of the tangent cone N to $\mathbb{R}^2_{(u_1,u_2)}$. □

Lemma 20 *Let $N \subset \mathbb{C}^3$ be a special Lagrangian tangent cone at x in L, where $q(x) \in \{p_i\} \times \partial C_{p_i}$. Then, N is a 3-plane with multiplicity 1.*

Proof We apply Joyce's classification to the connected components of the tangent cone, and rule out every other possibility of the list of Proposition 18.

Step 1 ($U(1)^2$-invariant $\mathbb{T}^2$-cone): The cases $A = 1$ and $A = -1$ are similar, so we focus on $A = 1$. In this scenario,

$$\pi_1(N/U(1)) = \{r^2 e^{-i\theta_3} \mid r > 0,\ \theta_3 \in \mathbb{R}\},$$

and $r^2 e^{-i\theta_3}$ can take any value in $\mathbb{C} = \mathbb{R}^2_{(u_1,u_2)}$; consequently, $\pi_1(N/U(1)) = \mathbb{R}^2$. In particular, $\pi_1(N)$ is not subset of a wedge with angle less than 180°, which contradicts Lemma 19.

Step 2 (union of two 3-planes): Suppose $\mathrm{supp}(N)$ contains $\Pi^{\phi}_{+} \cup \Pi^{\phi}_{-}$. We have

$$u_1(z, i e^{-i\varphi}\overline{z}) = \mathrm{Re}(z_1 z_2) = |z|^2 \sin(\varphi) \quad \text{and} \quad u_2(z, i e^{-i\varphi}\overline{z}) = \mathrm{Im}(z_1 z_2) = |z|^2 \cos(\varphi).$$

Hence

$$\pi_1(\Pi^{\phi}_{+}) = \{r^2(\sin(\varphi), \cos(\varphi)) \mid r \in \mathbb{R}\} \quad \text{and} \quad \pi_1(\Pi^{\phi}_{-}) = \{-r^2(\sin(\varphi), \cos(\varphi)) \mid r \in \mathbb{R}\}.$$

Therefore, $\pi_1(N/U(1)) = \{R(\sin(\varphi), \cos(\varphi)) \mid R \in \mathbb{R}\}$ forms a line. In particular, it is not subset of a wedge with angle less than 180°, contradicting Lemma 19.

Step 3 (multiplicity and graphicality): The special Lagrangian L projects with degree one to the (y_1, u_2)-plane. As in Theorem 5.6. in [Joyce 2005], for any given cutoff function $\chi \ge 0$ on $\mathbb{R}^2$,

$$\int_L \chi(y_1, u_2)(\theta \, du_1 - V \, du_2 \, du_3) \wedge dy_1 = 2\pi \int_S \chi(y_1, u_2) \, dy_1 \, du_2,$$

where S is the projection of $L/U(1)$ on the (y_1, u_2)-plane. By passing to a tangent cone N of L at x, we get

$$\int_N \chi(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2))\mathrm{Re}(dz_1 \wedge dz_2) \wedge \mathrm{Re}(dz_3) \le 2\pi \int_{\mathbb{R}^2} \chi(y_1, u_2) \, dy_1 \, du_2.$$

In a small open neighborhood around a generic point $(y_1, u_2) \in \mathbb{R}^2$, the tangent cone N is nonsingular and divides into k components, therefore for χ supported near this point we get

$$\int_N \chi(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2))\mathrm{Re}(dz_1 \wedge dz_2) \wedge \mathrm{Re}(dz_3) = 2\pi k \int_{\mathbb{R}^2} \chi(y_1, u_2) \, dy_1 \, du_2.$$

Therefore, $k = 0$ or 1. Thus the preimage of $(y_1, u_2) \in \mathbb{R}^2$ contains at most one point counting multiplicity. The same conclusion holds for the projections to (y_2, u_1), and indeed any choice of direction of y and the corresponding direction of u specified by the partial Legendre transform. This forces that there is at most one connected component in N, and it has multiplicity one.

Furthermore, following Proposition 5.5 in [Joyce 2005], in the Jacobi elliptic cone case $0 < |A| < 1$, the (y_1, u_2) projection has degree greater than one; hence this case is ruled out. The only remaining possibility is that N is a flat 3-plane with multiplicity one. □

4.2 Asymptotics of the special Lagrangians

In this section, we prove that L has the expected asymptotic behavior.

Theorem 21 *Asymptotically near infinity, L is an exponentially small C^k graph over the model special Lagrangian cylinders $\bigcup_{i=1}^{n} L_i$.*

We divide the proof into a few steps.

Step 1 (Cauchy–Riemann type equation): The region of L close to spatial infinity must project to a small neighborhood of one of the edges. Moreover, Lemma 14 shows that the projection to the (y_1, y_2) plane is close to some ray $\mathfrak{R}$, which we can without loss of generality take to be inside $\{(p_{i+1} - p_i) \cdot y = c_i\}$. Up to rotating and translating the coordinates, we reduce the problem to the standard position, so the edge lies in $\{u_1 = 0\}$, and the ray is simply $\{y_2 = 0, y_1 < 0\}$, and the region lies in $\{y_1 < 0, |y_1| \gg 1\} \cap \{u_2 \in [a, b]\}$, where $[a, b]$ describes the boundary edge.

Via the partial Legendre transform, we see the reduced Lagrangian L_{red} is graphical over the (u_2, y_1) variables, except at the vertices. The special Lagrangian condition can be rewritten as the Cauchy–Riemann type equation

$$\frac{\partial y_2}{\partial y_1} = -\frac{\partial u_1}{\partial u_2}, \quad \frac{\partial y_2}{\partial u_2} = V\frac{\partial u_1}{\partial y_1}.$$

Then Lemma 14 provides the preliminary decay estimate $|y_2| \le C/|y_1|$. Moreover, since φ is a convex function with $C^{1/2}$ boundary modulus of continuity, we obtain

$$|y_1|u_1 = -y_1u_1 = -\partial_{u_1}\varphi(u_1,u_2)u_1 \le \varphi(0,u_2) - \varphi(u_1,u_2) \le Cu_1^{1/2},$$

so we have the preliminary estimate $u_1 \le C|y_1|^{-2}$. Thus L_{red} is a C^0-small graph over the (u_2, y_1) variables, with decay rate estimate $O(|y_1|^{-1})$.

Now the Cauchy–Riemann type equation is quasilinear elliptic, so away from the singular locus $V = \infty$ corresponding to $u_2 \in \{a, b\}$, we can bootstrap the smallness of the C^0-norm to smallness of the C^k-norm. More geometrically, the asymptotic model is the half-cylinder $\Sigma_i \times \mathbb{R}^+$. For any given $\epsilon > 0$, we can find some $R \gg 1$ such that on the region $\{y_1 < -R, u_2 \in [a+\epsilon, b-\epsilon]\}$ away from the vertex, our special Lagrangian is a C^k-small perturbation of the model with C^1-norm bounded by ϵ.

Step 2 (quantitative smoothness): We need to prove quantitative smoothness estimate for L_{red} near the vertex region, and in our coordinates this means u_2 close to the endpoints a, b of the edges. This is based on Allard's regularity theorem. Notice the ambient manifold has bounded geometry in our region of interest.

Proposition 22 (Allard's regularity) *There exists a universal constant $\epsilon_0 \ll 1$, and some small fixed r_0 depending on the ambient manifold such that the following holds. Let X be an n-dimensional multiplicity one stationary integral varifold inside the coordinate ball $B(p,r)$ with $r \le r_0$. Assume that p lies on the support of X, and the volume $\mathcal{H}^n(X \cap B(p,r)) \le (\omega_n + \epsilon_0)r^n$. Then $X \cap B(p, r/2)$ is a $C^{1,\alpha}$ graph over the tangent plane through p, with the $C^{1,\alpha}$-norm bounded by $\frac{1}{100}$.*

Without loss of generality p lies in $a \le u_2 \le a+\epsilon$, where $\epsilon \ll r_0$ will be fixed later. We compute the volume on a small geodesic ball of radius $r \le r_0$,

$$\mathrm{Mass}(L_{\mathrm{red}} \cap B(p,r)) = \int_{L_{\mathrm{red}} \cap B(p,r)} \mathrm{Re}(\Omega) = 2\pi \int_{L_{\mathrm{red}} \cap B(p,r)} (du_1 \wedge dy_2 - du_2 \wedge dy_1).$$

On the integration region away from $a \le u_2 \le a+\epsilon$, by the smallness of the C^1-norms of y_2 and u_1, we see that the integral contribution is bounded by $\omega_3 r^3(1+O(\epsilon))$. On the other hand, the contribution from the region $\{a \le u_2 \le a+\epsilon\} \cap B(p,r)$ can be estimated by the same idea as in Lemma 16: the integral of $dy_1 \wedge du_2$ is the Lebesgue area of the (y_1, u_2) projection, which gives a contribution bounded by $O(\epsilon r)$. The integral of $du_1 \wedge dy_2$ is the Lebesgue area of the projection to the (u_1, y_2) plane. Since u_1, y_2 are both $O(|y|^{-1})$, this contribution is bounded by $O(|y|^{-2})$. In total,

$$\mathrm{Mass}(L_{\mathrm{red}} \cap B(p,r)) \le \omega_3 r^3(1+C\epsilon) + C\epsilon r + C|y|^{-2}.$$

Now for fixed r, we can choose $\epsilon \ll 1$ and $R \gg 1$ such that for $y_1 < -R$, all the remainder terms can be dominated by $\epsilon_0 r^3$, so that

$$\mathrm{Mass}(L_{\mathrm{red}} \cap B(p,r)) \le \omega_3(1+\epsilon_0)r^3.$$

Thus we can apply Allard regularity to deduce the quantitative smoothness of L close to infinity. Combining with the C^0-decay, it follows that L is a C^k-small graph over the model half-cylinder. The C^k-norms of u_1, y_2 are both bounded by $O(|y|^{-1})$. In particular,

$$\int_{\{y_1<-1,u_2\in[a,b]\}\cap L} |\nabla y_2|^2 < +\infty.$$

Step 3 (exponential decay): It remains to improve the C^k decay to exponential decay.

We first notice that y_2 defines a harmonic function on L. This is because y_2 has zero Hessian on the ambient manifold, and L is a minimal surface.

Lemma 23 *For any sufficiently large $R \ge R_0$,*

$$\int_{y_1\le -R} |\nabla y_2|^2 \le Ce^{-\gamma R}, \tag{8}$$

for a constant $\gamma > 0$ independent of R.

Proof We already know that the end of L is a small $C^{k,\alpha}$-graph over the cylindrical model. We define

$$F(R) = \int_{\{y_1\le -R\}\cap L} |\nabla y_2|^2.$$

Since y_2 is harmonic, for any $c \in \mathbb{R}$, we can apply the divergence theorem to deduce

$$F(R) = \int_{L\cap\{y_1=-R\}} (y_2-c)\nabla y_2\cdot\vec{n},$$

whence by Cauchy–Schwarz and the Poincaré inequality,

$$F(R) \le \left(\int_{L\cap\{y_1=-R\}} |\nabla y_2|^2\right)^{1/2}\left(\int_{L\cap\{y_1=-R\}} |y_2-c|^2\right)^{1/2} \le C\int_{L\cap\{y_1=-R\}} |\nabla y_2|^2.$$

Thus $-CF' \ge F$, which implies the exponential decay. □

Since L is already C^k-regular, we can bootstrap this to C^k exponential decay for y_2. Using the elliptic system, it is then easy to see that L is asymptotically an exponentially small graph over the model cylindrical special Lagrangian.

4.3 Topology of the constructed special Lagrangians

We conclude by proving the last component of Theorem 2, thereby confirming the Donaldson–Scaduto conjecture, Theorem 1.

Theorem 24 *L is homeomorphic to an n-holed 3-sphere.*

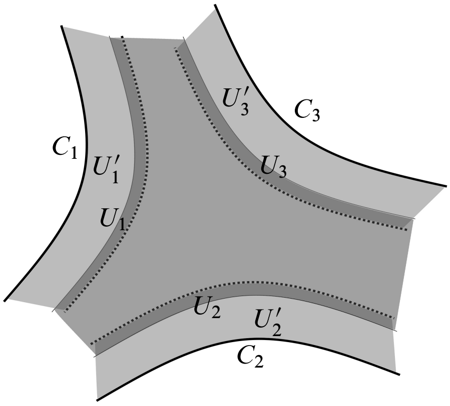

Figure 13: Boundary components C_i and sets U_0 and U_i, in the case $n = 3$.

Proof Let L' be the 3-manifold obtained by truncating the ends of L at a sufficiently large distance R, denoted by L_R, and sealing them by adding 3-balls, resulting in a closed 3-manifold. We show $L' \cong S^3$.

First argument (employing the Poincaré conjecture): We prove that L' is simply connected.

Note that L is fibered over L_{red}, where L_{red} is homeomorphic to $\overline{\mathbb{D}} \setminus \{a_1, \dots, a_n\}$ for n distinct boundary points $a_1, \dots, a_n$. The fiber over any interior point $z \in L^{\circ}_{\mathrm{red}}$ is a copy of $U(1)$, and the fibers collapse to a point when $z \in \partial L_{\mathrm{red}}$.

Let $C_1, \dots, C_n$ be the boundary components of L_{red}. Recalling the projection map $q : u_3^{-1}(0) \to Z_{\mathrm{red}}$, let $V_i := q^{-1}(U_i) \subset L$ be the preimage of an open neighborhood U_i of the boundary component C_i in L_{red} such that $U_i \cap U_j = \varnothing$ when $i \neq j$. Each V_i is homeomorphic to $\mathbb{D} \times \mathbb{R}$. Let U_i' be another open neighborhood of the boundary component C_i in L_{red} slightly smaller than U_i. Let V_0 be the open set in L defined as the preimage of $U_0 := L_{\mathrm{red}} \setminus \bigcup_{i=1}^{n} \overline{U}_i'$. The set V_0 is homeomorphic to $\mathbb{D} \times U(1)$. The configuration of open sets in L_{red} is shown in Figure 13.

Let $x_0 \in V_0 \cap V_1$ be the base point. We have $\pi_1(V_0, x_0) \cong \mathbb{Z}$, with a generator presented by a curve encircling the $U(1)$-fiber based at x_0. Furthermore, $\pi_1(V_1, x_0) = \{0\}$ and $\pi_1(V_0 \cap V_1, x_0) \cong \mathbb{Z}$, and the inclusion map $V_0 \cap V_1 \to V_0$ takes the generator of $\pi_1(V_0 \cap V_1, x_0)$ to the generator of $\pi_1(V_0, x_0)$. Therefore, by Van Kampen's theorem, $\pi_1(V_0 \cup V_1, x_0) = \{0\}$. Applying Van Kampen's theorem again repeatedly and adding V_i inductively yields $\pi_1(V_0 \cup V_1 \cup \dots \cup V_n, x_0) = \{0\}$, namely $\pi_1(L) = 0$. Consequently, $\pi_1(L') = 0$, and therefore, by the Poincaré conjecture, L' is a 3-sphere.

Second argument (without employing the Poincaré conjecture): We can extend q to obtain $q : L' \to L'_{\mathrm{red}}$, where $L'_{\mathrm{red}} \cong \overline{\mathbb{D}}^2$ is the truncated version of L_{red} capped off with n half-discs, where the preimage of any interior point under q is a copy of S^1, and the preimage of any boundary point is a single point. In other words, L'_{red} is an S^1-bundle over the interior of $\mathbb{D}^2$, which collapses to a point above each boundary point.

Let C be an embedded circle in L'_{red} which divides it into two regions: the interior D_1 and the exterior D_2. Let $\Sigma = q^{-1}(C)$, and $\Sigma_i = q^{-1}(D_i)$ for $i \in \{1, 2\}$. The Heegaard surface $\Sigma = \mathbb{T}^2$, and handlebodies $\Sigma_1 \cong \overline{\mathbb{D}}^2 \times S^1$ and $\Sigma_2 \cong S^1 \times \overline{\mathbb{D}}^2$, leading to a genus 1 Heegaard decomposition of L', where the gluing map of this decomposition maps the meridian of Σ_1 to the longitude of Σ_2 and the longitude of Σ_1 to the meridian of Σ_2. This description characterizes the genus 1 Heegaard decomposition of S^3. □

Appendix: Parameter count

Our main construction depends on the parameters $\{b_k\}_{k=1}^n$. Changing these parameters by the same additive constant amounts to adding a constant to the Dirichlet solution φ, which does not affect the special Lagrangian. In total, our construction depends on $n-1$ real parameters. Furthermore, as shown in [Habibi Esfahani 2022], each of these special Lagrangians with fixed asymptotics are rigid.

We now provide an alternative perspective on why the deformations of the n asymptotically cylindrical ends are subject to one additional constraint. We suppose that the n asymptotic half cylinders are contained in $L_i = \Sigma_i \times l_i$, where we recall that $\Sigma_i = \pi^{-1}[p_i, p_{i+1}]$ and $l_i \subset \mathbb{R}^2$ is defined by

$$l_i = \{y \in \mathbb{R}^2 \mid y \cdot (p_{i+1} - p_i) = c_i\} \quad \text{for some } c_i \in \mathbb{R}.$$

Lemma 25 *Let L be an asymptotically cylindrical special Lagrangian in $X \times \mathbb{R}^2$ with asymptotes ends L_i for $i = 1, \dots, n$. Then, we have $\sum_{i=1}^n c_i = 0$.*

Proof We can find a primitive for $\mathrm{Im}(\Omega)$,

$$\mathrm{Im}(\Omega) = \omega_1 \wedge dy_1 + \omega_2 \wedge dy_2 = d\lambda, \quad \text{with } \lambda = y_1\omega_1 + y_2\omega_2.$$

We apply Stokes' theorem to L truncated at a very large distance R, which is a manifold L_R with boundary diffeomorphic to $\bigcup_{i=1}^n \Sigma_i$. Since L is a special Lagrangian,

$$0 = \int_{L_R} \mathrm{Im}(\Omega) = \sum_{i=1}^n \int_{\Sigma_i} \lambda.$$

By definition, L is an exponentially small graph over L_i in the asymptotic regime. Consequently, we can evaluate the boundary integrals on L_i with an error of $O(e^{-cR})$, which disappears in the limit when $R \to +\infty$. We have

$$\left(\int_{\Sigma_i} \omega_1, \int_{\Sigma_i} \omega_2\right) = 2\pi(p_{i+1} - p_i) \in \mathbb{R}^2,$$

hence

$$\sum_{i=1}^n \int_{\Sigma_i} \lambda = \sum_{i=1}^n \big(y \cdot (p_{i+1} - p_i) + O(e^{-cR})\big) = \sum_{i=1}^n \big(c_i + O(e^{-cR})\big),$$

and letting $R \to +\infty$, we deduce $\sum_1^n c_i = 0$. □

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