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Compact holonomy G_2 manifolds need not be formal

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We construct a compact simply connected manifold with holonomy G_2 that is nonformal. This manifold is the resolution of a flat G_2 orbifold with $\mathbb{Z}_2$ isotropy, which can be resolved using both the generalized Kummer construction by D. D. Joyce and the method developed by Joyce and S. Karigiannis. A nonvanishing triple Massey product is obtained by arranging the singular locus in a particular configuration.

1 Introduction

The link between special holonomy and formality was discovered by P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan in [10]. They proved that compact complex manifolds satisfying the dd^c-lemma are formal. In particular, Kähler, Calabi–Yau, and hyper-Kähler compact manifolds are formal. Later, M. Amann and V. Kapovitch showed in [1] that compact quaternionic-Kähler manifolds with positive scalar curvature are also formal. For that, they used that their twistor space admits a Kähler metric, along with results from rational homotopy.

The question of formality for compact manifolds with exceptional holonomy has remained open since they were first constructed by D. D. Joyce in [13; 14; 15; 16]. In fact, these were conjectured to be formal in [1, Conjecture 2]. Various approaches have been explored to better understand the G_2 case. Analytical methods were employed by K. F. Chan, S. Karigiannis and C. C. Tsang [7] and M. Verbitsky [27], but these seem to be inconclusive on their own. More precisely, in [27] Verbitsky extended the Kähler identities to Riemannian manifolds equipped with a parallel form, and built a cohomology algebra with special properties using them. From this, he provided an alternative proof of formality for compact Kähler manifolds. This idea was later applied in [7] to torsion-free G_2 manifolds, where it is shown that these are almost formal — meaning that all triple Massey products vanish except possibly those involving three degree-2 cohomology classes. This condition is satisfied by all 7-manifolds with $b_1 = 0$.

Another strategy is contained in [9], where D. Crowley and J. Nordström studied the rational homotopy type of simply connected 7-manifolds, among others. In particular, they built the Bianchi–Massey tensor as a tool to detect nonformality of 7-manifolds, and they concluded that if there was a nonformal simply connected manifold with holonomy G_2, it would have $b_2 \geq 4$. Prior to [9], G. R. Cavalcanti proved a weaker version of this result in [6], and also obtained a compact nonformal 7-dimensional manifold that satisfies the known topological obstructions for holonomy G_2 manifolds.

The relative scarcity of holonomy G_2 manifolds makes it hard to determine what phenomena might influence whether they are formal. Formality of one of Joyce's examples was proved by Amann and

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Taimanov [2]. The examples of Nordström in [22] are formal because they have $b_2 \leq 1$. The author is unaware of any attempts to address the formality of the examples with $b_2 \geq 4$ provided by A. Kovalev [19], Kovalev and N. H. Lee [20], or A. Corti, M. Haskins, Nordström and T. Pacini [8]. We provide a negative answer to the question of formality for compact manifolds with holonomy G_2, proving the following result:

Theorem 1 *There exists a compact simply connected manifold with holonomy G_2 that is nonformal.*

The constructed manifold $(\widetilde{M}, \tilde{\varphi})$ is obtained by resolving an orbifold (X, φ) which itself is the quotient of a flat 7-manifold under the action of an involution with fixed points. The singular locus of X consists of 10 disjoint associative manifolds, $N_1, \dots, N_{10}$, each of which is 2:1 covered by a torus. In Theorem 3, we obtain $(\widetilde{M}, \tilde{\varphi})$ using the desingularization method of Joyce and Karigiannis [18], and in Remark 4, we argue that a resolution diffeomorphic to $\widetilde{M}$ is obtained from the generalized Kummer construction of Joyce [15]. As explained in Remark 4, the choice of the first was made due to the author's familiarity with the topological properties of resolutions of orbifolds fitting into the setup of [18].

It was known that the cohomology groups of $\widetilde{M}$ are determined by those of X and $N_1, \dots, N_{10}$ (see (17)), and the product structure depends on the embedding of the singular locus of X. More precisely, the square of the Thom class $\boldsymbol{x}_j$ of a connected component E_j in the exceptional divisor lies in $H^*(X)$, and it depends on the Thom class of the corresponding connected component N_j from which it arises; see (20). Here the orientation of N_j is determined by φ.

The novelty of this example is that we obtain a nonvanishing triple Massey product, even when the orbifold and the singular locus are formal, by positioning four of the connected components in a special configuration. Aside from the cohomology product formulas, the following facts ensure that the triple Massey product $\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{x}_7 + \boldsymbol{x}_3, \boldsymbol{x}_7 - \boldsymbol{x}_3 \rangle$ is both well-defined and nonvanishing:

(i) There are oriented cobordisms C and $\widetilde{C}$ with $\partial C = N_3 \sqcup -N_7$ and $\partial \widetilde{C} = N_1 \sqcup -N_2$. Here the minus sign indicates that the orientation is reversed.

(ii) The cobordism C intersects N_1 negatively and N_2 positively.

Alternatively, this can be expressed by saying that $N_1 \sqcup -N_2$ and $N_3 \sqcup -N_7$ are nullcobordant and their linking number is nonzero.

This contrasts with the example examined by Amann and Taimanov [2], where both the orbifold and the singular locus are formal, but the cobordisms between singular components can be chosen to avoid transverse intersections with other components. This also differs from the examples of compact manifolds with $b_1 = 1$ that are equipped with a closed G_2-structure and obtained by orbifold resolution techniques. The formal manifold in work of Fernández, Fino, Kovalev, and Muñoz [12] satisfies the same properties as the example analyzed in [2]. In work of Martín-Merchán [21], the orbifold is nonformal and the resolution carries a nonvanishing triple Massey product obtained by pulling back a nontrivial one from it. For topological reasons, the examples in [12; 21] do not admit metrics with holonomy contained in G_2.

This paper is organized as follows: In Section 2 we construct $\widetilde{M}$ and we prove that it is simply connected and it admits a metric with holonomy G_2. In Section 3 we obtain its cohomology algebra and show it is nonformal. The author has chosen to present most of the computations explicitly to ensure that the discussion is self-contained.

2 A simply connected compact manifold with holonomy G_2

Consider the flat 6-torus $T^6 = (\mathbb{C}/\Gamma)^3$, where $\Gamma = \mathbb{Z}\langle 1, i\rangle$, endowed with its standard flat SU(3)-structure $(\omega, \Theta) = \left(\frac{1}{2}i \sum_k dz_{k\bar{k}}, dz_{123}\right)$. The isometry

$$(1) \qquad f: T^6 \to T^6, \quad f[z_1, z_2, z_3] = [iz_1, iz_2, -z_3]$$

preserves the SU(3)-structure. The manifold

$$(2) \qquad M = \mathbb{R} \times T^6/(t, [z]) \sim (t+k, f^k[z]) \quad \text{for } k \in \mathbb{Z},$$

equipped with the flat metric has a parallel G_2-structure $\varphi = dt \wedge \omega + \mathrm{Re}(\Theta)$. Since $f^4 = \mathrm{Id}$, the 7-torus $\mathbb{T} = \mathbb{R}/4\mathbb{Z} \times T^6$ covers M. The deck transformation group of the covering is $\mathbb{Z}_4$, generated by $F[t, [z]] = [t+1, f[z]]$. We quotient M by the involutions

$$(3) \qquad \iota_1, \iota_2: M \to M, \quad \iota_1[t, [z]] = \left[t, \left[-z_1, -z_2, z_3 + \tfrac{1}{2}\right]\right], \quad \iota_2[t, [z]] = \left[t, \left[-z_1, -z_2, z_3 + \tfrac{1}{2}i\right]\right].$$

Neither of these involutions, nor their compositions, have fixed points. Since they preserve φ, the quotient space $M' = M/\langle \iota_1, \iota_2\rangle$ is a smooth manifold with a parallel G_2-structure and a flat metric. Observe that there is a fibration $T^6/\mathbb{Z}_2^2 \hookrightarrow M' \to S^1$. In addition, the involution

$$(4) \qquad \kappa: M \to M, \quad \kappa[t, [z]] = [1-t, [\bar{z}_1, \bar{z}_2, \bar{z}_3]],$$

is well-defined because $\kappa[t+1, f[z]] = [-t, [-i\bar{z}_1, -i\bar{z}_2, -\bar{z}_3]] = [1-t, [\bar{z}_1, \bar{z}_2, \bar{z}_3]] = \kappa[t, [z]]$. It determines a map $\kappa: M' \to M'$ as it commutes with both ι_1 and ι_2. Moreover, κ has fixed points and $\kappa^*(\varphi) = \varphi$.

The orbifold $X = M'/\kappa$ can be alternatively described as the quotient of the torus $\mathbb{T} = \mathbb{R}/4\mathbb{Z} \times T^6$ by the group $D_4 \times \mathbb{Z}_2^2$. Here D_4 is the dihedral group spanned by κ and F (note that $\kappa \circ F = F^{-1} \circ \kappa$), whereas $\mathbb{Z}_2^2$ is generated by ι_1 and ι_2.

Remark 2 The orbifold X is related to the examples provided by Joyce in [15]. More precisely, a change of variables transforms $X_1 = \mathbb{T}^7/\langle F, \kappa\rangle$ into Joyce's Example 9, and $X_2 = \mathbb{T}^7/\langle F, \kappa, \iota_1 \circ \iota_2\rangle$ into Example 10. In particular, there is a branched covering $X_2 \to X$, because ι_1 has fixed points when it acts on X_2 (a fixed point of ι_1 on X_2 is the projection of $p = \left[0, \left[0, 0, \tfrac{1}{4}\right]\right]$ as $\iota_1(\kappa(p)) = \left[1, \left[0, 0, \tfrac{3}{4}\right]\right] = p$).

2.1 Singular locus

The singular locus of X is the projection of the fixed locus of κ, $\kappa \circ \iota_1$, $\kappa \circ \iota_2$, and $\kappa \circ \iota_1 \circ \iota_2$ on M. We observe that if γ is one of these maps, and we denote it by $\gamma[t, [z]] = [1-t, \gamma'[z]]$, then a point $[t, [z]]$ with $t \in [0, 1)$ is fixed by γ if and only if $t = \frac{1}{2}$ and $[z] \in \mathrm{Fix}(\gamma')$, or $t = 0$ and $[z] \in \mathrm{Fix}(f^{-1} \circ \gamma')$. We now compute these and analyze their projections to X.

We begin with the level set $t=0$. From $f^{-1}[z]=[-iz_1,-iz_2,-z_3]$, we obtain

$$f^{-1}\circ\kappa'[z]=[-i\bar z_1,-i\bar z_2,-\bar z_3],\qquad f^{-1}\circ(\kappa\circ\iota_1)'[z]=\big[i\bar z_1,i\bar z_2,-\bar z_3+\tfrac12\big],$$
$$f^{-1}\circ(\kappa\circ\iota_2)'[z]=\big[i\bar z_1,i\bar z_2,-\bar z_3+\tfrac12 i\big],\quad f^{-1}\circ(\kappa\circ\iota_1\circ\iota_2)'[z]=\big[-i\bar z_1,-i\bar z_2,-\bar z_3+\tfrac12(1+i)\big].$$

Hence $\mathrm{Fix}(f^{-1}\circ(\kappa\circ\iota_2)')=\mathrm{Fix}(f^{-1}\circ(\kappa\circ\iota_1\circ\iota_2)')=\varnothing$ $\big($these act on y_3 as $y_3\mapsto y_3+\tfrac12\big)$, and

$$\mathrm{Fix}(f^{-1}\circ\kappa')=\big\{[x_1-ix_1,x_2-ix_2,\varepsilon_3+iy_3]\mid \varepsilon_3\in\big\{0,\tfrac12\big\},\ x_1,x_2,y_3\in\big[-\tfrac12,\tfrac12\big]\big\},\tag{5}$$
$$\mathrm{Fix}(f^{-1}\circ(\kappa\circ\iota_1)')=\big\{[x_1+ix_1,x_2+ix_2,\delta_3+iy_3]\mid \delta_3\in\big\{-\tfrac14,\tfrac14\big\},\ x_1,x_2,y_3\in\big[-\tfrac12,\tfrac12\big]\big\}.\tag{6}$$

We denote by $N_1^1,N_1^2\subset M$ the connected components of $\mathrm{Fix}(\kappa)$ at $t=0$ determined by $\varepsilon_3=0$ and $\varepsilon_3=\frac12$, respectively. Similarly, we denote by $N_2^1,N_2^2\subset M$ the components of $\mathrm{Fix}(\kappa\circ\iota_1)$ at $t=0$ determined by $\delta_3=-\frac14$ and $\delta_3=\frac14$. Each of these is diffeomorphic to a 3-torus. The map ι_1 interchanges N_j^1 with N_j^2, while ι_2 preserves each connected component. In addition, $N_j^k\to N_j^k/\iota_2$ is a 2:1 covering, and $H^1(N_j^k/\iota_2)=\langle[dy_3]\rangle$. The form dy_3 is harmonic because N_j^k is equipped with the flat metric. In fact, the map $N_1^1/\iota_2\to\mathbb{R}/\mathbb{Z}$, $[x_1-ix_1,x_2-ix_2,iy_3]\mapsto[2y_3]$ is a submersion with a T^2 fiber. A similar statement holds for the remaining connected components. Therefore the fixed locus of κ on M' at $t=0$ has two connected components, N_1' and N_2'. Both admit the nowhere-vanishing harmonic 1-form dy_3. We denote by N_1 and N_2 their projections to X.

To analyze the singularities at $t=\frac12$, we compute

$$\kappa'[z]=[\bar z_1,\bar z_2,\bar z_3],\qquad (\kappa\circ\iota_1)'[z]=\big[-\bar z_1,-\bar z_2,\bar z_3+\tfrac12\big],$$
$$(\kappa\circ\iota_2)'[z]=\big[-\bar z_1,-\bar z_2,\bar z_3+\tfrac12 i\big],\quad (\kappa\circ\iota_1\circ\iota_2)'[z]=\big[\bar z_1,\bar z_2,\bar z_3+\tfrac12(1+i)\big].$$

Hence $\mathrm{Fix}((\kappa\circ\iota_1)')=\mathrm{Fix}((\kappa\circ\iota_1\circ\iota_2)')=\varnothing$, and

$$\mathrm{Fix}(\kappa')=\big\{[x_1+i\varepsilon_1,x_2+i\varepsilon_2,x_3+i\varepsilon_3]\mid \varepsilon_1,\varepsilon_2,\varepsilon_3\in\big\{0,\tfrac12\big\},\ x_1,x_2,x_3\in\big[-\tfrac12,\tfrac12\big]\big\},\tag{7}$$
$$\mathrm{Fix}((\kappa\circ\iota_2)')=\{[\varepsilon_1+iy_1,\varepsilon_2+iy_2,x_3+i\delta_3]\mid \varepsilon_1,\varepsilon_2,\delta_3-\tfrac14\in\big\{0,\tfrac12\big\},\ y_1,y_2,x_3\in\big[-\tfrac12,\tfrac12\big]\}.\tag{8}$$

We denote by $N_3^1,N_3^2\subset M$ the connected components of $\mathrm{Fix}(\kappa)$ at $t=\frac12$ determined by $\varepsilon_1=\varepsilon_2=0$ and $\varepsilon_3=0$ or $\varepsilon_3=\frac12$, respectively. We use lexicographic order for the pairs $(\varepsilon_1,\varepsilon_2)$ and label the remaining connected components of $\mathrm{Fix}(\kappa)$ at $t=\frac12$ by N_4^1, N_4^2, N_5^1, N_5^2, N_6^1, and N_6^2, with the upper index 1 corresponding to $\varepsilon_3=0$. Similarly, we denote the connected components of $\mathrm{Fix}(\kappa\circ\iota_2)$ by N_j^k, where $j=7,\dots,10$ and $k=1,2$, and the upper index $k=1$ now indicates $\delta_3=-\frac14$. For instance,

$$N_7^1=\big\{\big[\tfrac12,\big[iy_1,iy_2,x_3-\tfrac14 i\big]\big]\mid y_1,y_2,x_3\in\big[-\tfrac12,\tfrac12\big]\big\}.$$

All these components are diffeomorphic to a 3-torus. We observe that ι_1 preserves each connected component, while ι_2 swaps N_j^1 with N_j^2. Also, $N_j^k\to N_j^k/\iota_1$ is a 2:1 covering, and $H^1(N_j^k/\iota_1)=\langle[dx_3]\rangle$. Again, dx_3 is indeed harmonic on N_j^k and there are submersions $N_j^k/\iota_1\to\mathbb{R}/\mathbb{Z}$ with a T^2 fiber. The fixed locus of κ on M' at $t=\frac12$ has eight connected components, $N_3',\dots,N_{10}'$. These admit the nowhere-vanishing harmonic 1-form dx_3. We denote by $N_3,\dots,N_{10}$ their projections to X.

The submanifolds N_j^k and N_j' are associative submanifolds of (M,φ) and (M',φ), respectively; see [18, Proposition 2.13]. *We always assume that N_j^k (resp. N_j', N_j) are oriented by $\varphi|_{N_j^k}=\mathrm{Re}(dz_{123})|_{N_j^k}$ (resp. $\varphi|_{N_j'}$, $\varphi|_{N_j}$).* Using the coordinates at $t=0$ and $t=\frac{1}{2}$ provided by (5)–(8), these forms are

(9) $$\varphi|_{N_1^k}=2\,dx_1\wedge dx_2\wedge dy_3 \qquad \text{for } k=1,2,$$

(10) $$\varphi|_{N_2^k}=-2\,dx_1\wedge dx_2\wedge dy_3 \qquad \text{for } k=1,2,$$

(11) $$\varphi|_{N_j^k}=dx_1\wedge dx_2\wedge dx_3 \qquad \text{for } j=3,4,5,6 \text{ and } k=1,2,$$

(12) $$\varphi|_{N_j^k}=-dy_1\wedge dy_2\wedge dx_3 \qquad \text{for } j=7,8,9,10 \text{ and } k=1,2.$$

Note that if $j=1$ (or $j=2$), the projections of N_j^k to the first and second factors of $(\mathbb{C}/\Gamma)^3$ are the loops $[x-ix]$ (or $[x+ix]$) for $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$. These have length $\sqrt{2}$. The projection to the third factor has, of course, length 1. Therefore the volume of these tori is 2, which is why the factor of 2 appears in (9) and (10).

2.2 Resolution

The resolution method developed in [18] allows us to desingularize the orbifold X and to obtain a metric with holonomy G_2 on the resolution.

Theorem 3 *The resolution $\widetilde{M}$ of X is a compact simply connected manifold that admits a metric with holonomy G_2.*

Proof Since each connected component of the singular locus of X admits a nowhere-vanishing harmonic 1-form, the resolution method [18, Theorem 1.1] ensures that there is a resolution $\rho\colon\widetilde{M}\to X$ and a torsion-free G_2-structure $\tilde{\varphi}$ on $\widetilde{M}$. It satisfies $\pi_1(\widetilde{M})=\pi_1(X)$. According to [17, Proposition 10.2.2], the holonomy of $(\widetilde{M},\tilde{\varphi})$ equals G_2 if and only if $\pi_1(\widetilde{M})$ is finite. To finish the proof, it suffices to show that X is simply connected.

We first compute $\pi_1(M,p_0)$ with $p_0=\left[\frac{1}{2},[0,0,0]\right]$. From the long exact sequence of the fibration $T^6\hookrightarrow M\to\mathbb{R}/\mathbb{Z}$, $[t,[z]]\mapsto[t]$, we obtain the short exact sequence

$$\{1\}=\pi_2(S^1)\to\pi_1(T^6)\to\pi_1(M)\to\pi_1(S^1)\to\{1\}.$$

Of course, the loop $\gamma_0(s)=\left[\frac{1}{2}+s,[0,0,0]\right]$ in M maps to the generator of $\pi_1(S^1)$, and provides a splitting of the short exact sequence. Therefore $\pi_1(M)=\pi_1(S^1)\ltimes\pi_1(T^6)$. We now analyze the product of $[\gamma_0]$ with elements in the image of $\pi_1(T^6)\hookrightarrow\pi_1(M)$. Given a loop $\gamma(s)=\left[\frac{1}{2},\hat{\gamma}(s)\right]$ on M with $\hat{\gamma}(0)=[0,0,0]$, the loop $\gamma_0^{-1}\cdot\gamma\cdot\gamma_0$ is homotopic to $s\mapsto\left[\frac{3}{2},\hat{\gamma}(s)\right]=\left[\frac{1}{2},f^{-1}\circ\hat{\gamma}(s)\right]$ on M. (As usual, we write $\gamma^{-1}(s)=\gamma(1-s)$.) The homotopy $H_r(s)$ is a sequence of three paths. First, move along the t-direction from $\left[\frac{3}{2},[0,0,0]\right]$ to $\left[\frac{3}{2}-r,[0,0,0]\right]$. Then travel on the level set $\frac{3}{2}-r$ via the map $s\mapsto\left[\frac{3}{2}-r,\hat{\gamma}(s)\right]$, and finally return to $\left[\frac{3}{2},[0,0,0]\right]$ by reversing the initial path.

We denote by $\gamma_k(s)=\left[\frac{1}{2},\hat{\gamma}_k(s)\right]$ for $k=1,\dots,6$ the canonical generators of $\pi_1\left(\left\{\frac{1}{2}\right\}\times T^6,p_0\right)$. Namely, $\hat{\gamma}_1(s)=[s,0,0],\dots,\hat{\gamma}_6(s)=[0,0,is]$. For $k=1,2$ we have $f^{-1}\circ\hat{\gamma}_{2k-1}=\hat{\gamma}_{2k}^{-1}$ and $f^{-1}\circ\hat{\gamma}_{2k}=\hat{\gamma}_{2k-1}$.

In addition, if $k=5,6$ then $f^{-1}\circ\hat{\gamma}_k=\hat{\gamma}_k^{-1}$. This implies the relation

$$[\gamma_{2k}]^{-1}=[\gamma_0]^{-1}[\gamma_{2k-1}][\gamma_0],\qquad [\gamma_{2k-1}]=[\gamma_0]^{-1}[\gamma_{2k}][\gamma_0],\qquad \text{for } k=1,2, \tag{13}$$

$$[\gamma_5]^{-1}=[\gamma_0]^{-1}[\gamma_5][\gamma_0],\qquad [\gamma_6]^{-1}=[\gamma_0]^{-1}[\gamma_6][\gamma_0]. \tag{14}$$

Since $\langle\iota_1,\iota_2\rangle$ acts freely on M, the projection $q\colon M\to M'$ is a 4:1 regular cover with deck transformation group $\langle\iota_1,\iota_2\rangle=\mathbb{Z}_2^2$; see [5, Chapter 3, Proposition 7.2]. Indeed, there is a short exact sequence

$$\{1\}\to\pi_1(M)\to\pi_1(M')\to\langle\iota_1,\iota_2\rangle\to\{1\},$$

where the last map is defined by lifting a representative of an element in $\pi_1(M')$ to M with basepoint p_0 and then considering the map of the deck group that transforms p_0 to the endpoint of the lift. Consider the paths

$$\tilde{\gamma}_5,\tilde{\gamma}_6\colon[0,1]\to M,\quad \tilde{\gamma}_5(s)=\big[\tfrac{1}{2},\big[0,0,\tfrac{1}{2}s\big]\big],\quad \tilde{\gamma}_6(s)=\big[\tfrac{1}{2},\big[0,0,\tfrac{1}{2}is\big]\big].$$

Then $\gamma_5'=q\circ\tilde{\gamma}_5$ and $\gamma_6'=q\circ\tilde{\gamma}_6$ are loops on M'. In the exact sequence, their homotopy classes project onto ι_1 and ι_2, respectively. If $0\le k\le 4$ we write $\gamma_k'=q\circ\gamma_k$. In addition, define the paths

$$\tilde{\gamma}_0,\gamma_7\colon[0,1]\to M,\quad \tilde{\gamma}_0(s)=\big[\tfrac{1}{2}+s,\big[0,0,\tfrac{1}{2}\big]\big],\quad \gamma_7(s)=\big[\tfrac{3}{2},\big[0,0,\tfrac{1}{2}s\big]\big]=\big[\tfrac{1}{2},\big[0,0,-\tfrac{1}{2}s\big]\big].$$

There is a homotopy with fixed endpoints on M from $\gamma_0^{-1}\cdot\tilde{\gamma}_5\cdot\tilde{\gamma}_0$ to γ_7. The way it is constructed is the same as before. Since $\gamma_0'=q\circ\tilde{\gamma}_0$ and $q\circ\gamma_7(s)=q\big[\tfrac{1}{2},\big[0,0,\tfrac{1}{2}-\tfrac{1}{2}s\big]\big]=(\gamma_5')^{-1}(s)$ we obtain $[\gamma_0']^{-1}[\gamma_5'][\gamma_0']=[\gamma_5']^{-1}$ on M'; a similar argument shows $[\gamma_0']^{-1}[\gamma_6'][\gamma_0']=[\gamma_6']^{-1}$.

The involution $\kappa\colon M'\to M'$ has fixed points. Therefore the map $q'_*\colon\pi_1(M')\to\pi_1(X)$ induced by the projection $q'\colon M'\to X$ is surjective; see [4, Chapter 2, Corollary 6.3]. Observe that κ reverses the direction of γ_0', γ_2', γ_4', and γ_6'. That is, if γ is one of these loops, then $\kappa\circ\gamma(s)=\gamma^{-1}(s)$. This implies that $q'_*[\gamma]=1$ because $q'\circ\gamma=q'\circ\gamma''(s)$, where

$$\gamma''(s)=\begin{cases}\gamma(s) & \text{if } t\le\frac{1}{2},\\ \gamma^{-1}(s) & \text{if } t\ge\frac{1}{2},\end{cases}$$

and γ'' represents the unit element in $\pi_1(M')$. Hence $q'_*[\gamma_0']=q'_*[\gamma_2']=q'_*[\gamma_4']=q'_*[\gamma_6']=1$. From the relations (13), and $[\gamma_5']^{-1}=[\gamma_0']^{-1}[\gamma_5'][\gamma_0']$, we obtain $q'_*[\gamma_1']=q'_*[\gamma_3']=q'_*[\gamma_5']^2=1$. We finally show that, indeed, $q'_*[\gamma_5']=1$. We first consider the paths

$$\gamma_8,\gamma_9,\gamma_{10}\colon[0,1]\to M,\quad \gamma_8(s)=\big[\tfrac{1}{2}+\tfrac{1}{2}s,[0,0,0]\big],\ \gamma_9(s)=\big[1,\big[0,0,\tfrac{1}{2}s\big]\big],\ \gamma_{10}(s)=\big[\tfrac{1}{2}+\tfrac{1}{2}s,\big[0,0,\tfrac{1}{2}\big]\big].$$

There is a basepoint preserving homotopy on M from $\gamma_8\cdot\gamma_9\cdot\gamma_{10}^{-1}$ to $\tilde{\gamma}_5$. Since $q\circ\gamma_8=q\circ\gamma_{10}$, this implies that $(q\circ\gamma_8)\cdot(q\circ\gamma_9)\cdot(q\circ\gamma_8)^{-1}$ is homotopic to γ_5'. We now observe that

$$\kappa\circ(q\circ\gamma_9)(s)=q\big[0,\big[0,0,\tfrac{1}{2}s\big]\big]=q\big[1,\big[0,0,-\tfrac{1}{2}s\big]\big]=q\big[1,\big[0,0,\tfrac{1}{2}-\tfrac{1}{2}s\big]\big]=(q\circ\gamma_9)^{-1}(s).$$

The argument above ensures that the loop $q'\circ q\circ\gamma_9$ is trivial on X. Hence

$$q'\circ\gamma_5'\sim(q'\circ q\circ\gamma_8)\cdot(q'\circ q\circ\gamma_9)\cdot(q'\circ q\circ\gamma_8)^{-1}\sim(q'\circ q\circ\gamma_8)\cdot(q'\circ q\circ\gamma_8)^{-1}\sim 1.$$

Therefore $\pi_1(\widetilde{M})=\pi_1(X)=\{1\}$. ◻

We denote by $\rho\colon \widetilde{M} \to X$ the projection map, and $E_j = \rho^{-1}(N_j)$. The resolution described in [18] occurs on the fibers of the normal bundle $\nu(N_j)$ of $N_j \subset X$, which are of the form $\mathbb{C}^2/\mathbb{Z}_2$. These singular fibers are replaced by their algebraic resolution,

$$Y = \{(w,\ell) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid w \in \ell\}/(w,\ell) \sim (-w,\ell).$$

We describe this procedure in our setup. First, $N_j' = N_j^1/\iota$, where $\iota = \iota_2$ if $j = 1,2$ or $\iota = \iota_1$ if $3 \le j \le 10$. Indeed, there is a $2:1$ cover between the normal bundles $\nu(N_j^1) \to \nu(N_j')$ induced by the projection map $M \to M'$. The deck transformation group of the cover is generated by the involution ι_*. Both bundles have a complex structure I_j induced by the cross product with the vector field χ_j determined by the harmonic form; see [18, Remark 4.1]. That is, $\chi_j = \partial_{y_3}$ if $j = 1,2$ or $\chi_j = \partial_{x_3}$ if $3 \le j \le 10$, and $I_j(X)$ is determined by the expression

$$g(I_j(X), Y) = \varphi(\chi_j, X, Y).$$

The action of ι_* on $\nu(N_j^1)$ preserves I_j because ι_* fixes φ, χ_j and g, and the projection $\nu(N_j^1) \to \nu(N_j')$ is a complex homomorphism. In addition, the bundle $\nu(N_j^1)$ is trivial. Since $I_j(\partial_t) \in \langle \partial_{x_3}, \partial_{y_3}\rangle$, we pick a trivialization of the form $(\partial_t, I_j(\partial_t), X, I_j(X))$ where $X \in \langle \partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}\rangle \cap \nu(N_j^1)$ is a unit-length vector field. For instance, if $j = 1$ this is $\big(\partial_t, \partial_{x_3}, (1/\sqrt{2})(\partial_{x_1} + \partial_{y_1}), (-1/\sqrt{2})(\partial_{x_2} + \partial_{y_2})\big)$, and if $j = 3$ this is $(\partial_t, -\partial_{y_3}, \partial_{y_1}, -\partial_{y_2})$. The orientation of $\nu(N_j^1)$ induced from N_j and M is determined by the form $(\chi_j \lrcorner \varphi|_{\nu(N_j^1)})^2$, and the frame above is positively oriented. Hence the trivialization yields an isomorphism of oriented vector bundles $\nu(N_j^1) \cong N_j^1 \times \mathbb{C}^2$, where $\mathbb{C}^2$ has the standard orientation. For any j, under the isomorphism, the generator of the deck group of $\nu(N_j^1) \to \nu(N_j')$ is $\iota_*(x, w_1, w_2) = (\iota(x), w_1, -w_2)$.

The involution κ induces the $\mathbb{Z}_2$-action on the fibers of $\nu(N_j')$ given by $v_x \to \kappa_*(v_x) = -v_x$; see the proof of [21, Lemma 2.8]. This can be lifted to a map $\hat{\kappa}\colon N_j^1 \times \mathbb{C}^2 \to N_j^1 \times \mathbb{C}^2$ by means of the expression $\hat{\kappa}(x, w_1, w_2) = (x, -w_1, -w_2)$, which is induced by the involution on M fixing the points of that connected component. This is κ_* if $j = 1,3,4,5,6$, or $(\kappa \circ \iota_1)_*$ if $j = 2$, or $(\kappa \circ \iota_2)_*$ if $j = 7,8,9,10$. This action commutes with ι_*. Thus $\nu(N_j) = \nu(N_j')/\kappa_* \cong (N_j^1 \times \mathbb{C}^2)/\langle \iota_*, \hat{\kappa}\rangle \cong (N_j^1 \times \mathbb{C}^2/\mathbb{Z}_2)/\iota_*$, and there is a $2:1$ cover $N_j^1 \times (\mathbb{C}^2/\mathbb{Z}_2) \to (N_j^1 \times \mathbb{C}^2/\mathbb{Z}_2)/\iota_*$.

The resolution of $N_j^1 \times \mathbb{C}^2/\mathbb{Z}_2$ is $N_j^1 \times Y$. The deck transformation ι_* lifts to $N_j^1 \times Y$ by means of the expression $\iota_*(x, [(w_1, w_2), [W_1 : W_2]]) = (\iota(x), [(w_1, -w_2), [W_1 : -W_2]])$. The resolution $P(N_j)$ of $\nu(N_j)$ is the quotient $(N_j^1 \times Y)/\iota_*$. Its exceptional divisor is $Q(N_j) = (N_j^1 \times \mathbb{CP}^1)/\iota_* \subset (N_j^1 \times Y)/\iota_*$, which inherits the product orientation from $N_j^1 \times \mathbb{CP}^1$, where $\mathbb{CP}^1$ is oriented by the Fubini–Study form. It turns out that $Q(N_j)$ is isomorphic to the trivial bundle because it is homotopic to it via

$$E_t = N_j^1 \times \mathbb{CP}^1/(x, [W_1 : W_2]) \sim (\iota(x), [W_1 : e^{\pi i t} W_2]) \quad \text{for } t \in [0,1]. \tag{15}$$

To obtain $\widetilde{M}$, the authors of [18] replace a neighborhood of N_j in X with a neighborhood of $Q(N_j)$ in $P(N_j)$. Therefore $E_j = Q(N_j)$. The identification between these pieces is given by the composition of the natural projection $P(N_j) \to \nu(N_j)$, a rescaling of the fibers of $\nu(N_j)$, and a modification of the exponential map obtained by precomposing it with the flow of a vector field of $\nu(N_j)$. The latter coincides with the exponential map over the zero section, preserves the orientation, and is compatible with the involution; see [18, Definition 3.2, Lemma 3.5, and (6.1)].

Remark 4 The resolution method of Joyce [15, Theorem 2.2.3] also provides a resolution of the orbifold X and a metric with holonomy G_2 on the resolution. The orbifold X fits into this setup as $X = \mathbb{T}/(D_4 \times \mathbb{Z}_2^2)$, and a neighborhood of N_j is isometric to a neighborhood of the singular set of $(N_j^1 \times \mathbb{C}^2/\mathbb{Z}_2)/\iota_*$ where N_j^1 is a flat 3-torus and ι_* is an isometry of $N_j^1 \times \mathbb{C}^2/\mathbb{Z}_2$ acting freely on N_j^1. The resolution is obtained by replacing a neighborhood of N_j in X with a neighborhood of $Q(N_j)$ in $P(N_j)$; the identification is the composition of the natural projection and the exponential map. Hence the resolution is diffeomorphic to $\widetilde{M}$. We used the method in [18], even if it involves harder analytic tools that are not necessary here, to take advantage of the author's previous knowledge of the cohomology algebra of the resolution; see [21].

For later purposes, we also consider $q_0 = [1:0] \in \mathbb{CP}^1$ and observe that

$$L_j := (N_j^1 \times \{q_0\})/\iota_* = N_j \times \{q_0\} \tag{16}$$

is a submanifold of E_j that is diffeomorphic to N_j. Its orientation is determined by that on N_j.

3 Nonformality of the constructed manifold

3.1 Cohomology groups

We introduce some notation. Fix small tubular neighborhoods $O_j' \subset O_j''$ of N_j' on M', and denote by $\pi_j \colon O_j'' \to N_j'$ the nearest-point projection. Define $O_j = q'(O_j')$ and $O_j^2 = q(O_j'')$, where $q' \colon M' \to X$ is the quotient projection. To ease notation, let $\pi_j \colon O_j^2 \to N_j$ denote the nearest-point projection. In addition, we let $\mathrm{pr}_j = \pi_j \circ \rho \colon \rho^{-1}(O_j^2) \to N_j$. From the discussion in Section 2.2 we deduce that it is a smooth map.

According to [18, Proposition 6.1], there are group isomorphisms

$$H^k(\widetilde{M}) \cong H^k(X) \oplus \bigoplus_{j=1}^{10} H^{k-2}(N_j) \otimes \langle \boldsymbol{x}_j \rangle. \tag{17}$$

Recall that Satake defines in [24] the de Rham algebra for orbifolds and proves that its cohomology algebra is isomorphic to the Čech cohomology with real coefficients. In [12, Remark 5], Fernández, Fino, Kovalev, and Muñoz deal with orbifolds obtained as the orbit space of a finite group action on a compact manifold. In this case, the de Rham algebra of the orbifold coincides with the subalgebra of invariant forms on the manifold. By averaging cohomology classes with respect to the group action, they observe that the cohomology algebra coincides with the subalgebra of invariant cohomology classes. Hence $H^*(X)$ is computed from the complex $(\Omega(M')^\kappa, d)$, and $H^*(X) = H^*(M')^\kappa$.

The inclusion $H^k(X) \subset H^k(\widetilde{M})$ is determined by ρ^* (see Lemma 5). Each $\boldsymbol{x}_j$ has degree 2, and is represented by the Thom class $[\tau_j]$ of the submanifold $E_j \subset \widetilde{M}$, oriented as in Section 2.2. Recall that the Thom class of E_j is the generator of the compactly supported cohomology group $H_c^2(\rho^{-1}(O_j))$ that integrates to 1 over each fiber of the normal bundle of E_j inside $\widetilde{M}$; see [3, Proposition 6.18]. In addition, a closed form in $H^*(\widetilde{M})$ that maps to $[\alpha] \otimes \boldsymbol{x}_j \in H^*(N_j) \otimes \boldsymbol{x}_j$ is $\mathrm{pr}_j^*(\alpha) \wedge \tau_j$.

Given $\alpha \in \Omega^k(X)$, the restriction $\rho^*(\alpha)|_{\widetilde{M}-\bigcup_j E_j}$ is a smooth form since $\rho\colon \widetilde{M}-\bigcup_j E_j \to X-\bigcup_j N_j$ is a diffeomorphism. The next lemma describes how to find a representative α_s of a cohomology class $[\alpha] \in H^*(X)$ whose pullback to $\widetilde{M}$ is smooth.

Lemma 5 *Let $W_j' \subset W_j'' \subset O_j''$ be tubular neighborhoods of N_j' on M' and define $W_j = q(W_j')$ and $W_j^2 = q(W_j'')$. For any $1 \le j \le 10$, if $\alpha \in \Omega^k(X)$ is closed on W_j^2, then there is $\alpha_j \in \Omega^k(X)$ such that:*

(i) *The forms α_j and α coincide on $X - W_j^2$. In addition, $\alpha_j - \alpha = d\beta_j$ where $\beta_j \in \Omega^{k-1}(X)$ is supported on W_j^2.*

(ii) *On W_j we have $\alpha_j = \pi_j^*(\alpha_j|_{N_j})$. In particular, $\rho^*(\alpha_j)$ is smooth on $\rho^{-1}(W_j)$ because $\rho^*(\alpha_j) = \mathrm{pr}_j^*(\alpha_j|_{N_j})$ there.*

Therefore, given $[\alpha] \in H^k(X)$, there is a representative $\alpha_s \in \Omega^k(X)$ such that $\rho^*(\alpha_s)$ is smooth and $\rho^*(\alpha_s)|_{\rho^{-1}(O_j)} = \mathrm{pr}_j^*(\alpha_s|_{N_j})$ for every $1 \le j \le 10$.

Proof The form $\beta = \pi_j^*(\alpha|_{N_j'}) \in \Omega^2(W_j'')$ is κ-invariant because $\pi_j \circ \kappa = \pi_j$. In addition, it is closed and satisfies $\beta|_{N_j'} = \alpha|_{N_j'}$. The Poincaré lemma allows us to find a form θ_j on W_j'' such that $d\theta_j = (\beta-\alpha)|_{W_j''}$. Indeed, since β and α are κ-invariant, we can assume that θ_j is. Let h_j be a κ-invariant bump function which equals 0 on $M' - W_j''$ and 1 on W_j'. Then $h_j\theta_j \in \Omega^{k-1}(X)$ is supported on W_j^2 and $\alpha_j = \alpha + d(h_j\theta_j)$ satisfies the required properties. □

Remark 6 In the proof of [18, Proposition 6.1], and following our notation, they use that the bundles $Q(N_j) \to N_j$ are trivial. Their justification is that the normal bundle $\nu(N_j)$ is trivial as a real bundle (see their Remark 2.14). There could be a gap between these statements, because the chosen complex structure might not be constant in the trivialization suggested by the authors. The space of complex lines could then differ from fiber to fiber. This is why we showed in (15) that $Q(N_j)$ is trivial.

Proposition 7 *The Betti numbers of $\widetilde{M}$ are $(b_1, b_2, b_3) = (0, 11, 16)$. The cohomology groups are*

$$\begin{aligned} H^2(\widetilde{M}) &= \langle [dz_{1\bar{2}} + dz_{\bar{1}2}] \rangle \oplus \langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_{10} \rangle, \\ H^3(\widetilde{M}) &= \langle [i\, dt \wedge dz_{1\bar{1}}], [i\, dt \wedge dz_{2\bar{2}}], [i\, dt \wedge dz_{3\bar{3}}], [i\, dt \wedge (dz_{1\bar{2}} - dz_{\bar{1}2})] \rangle \\ &\qquad \oplus \langle [dz_{123} + dz_{\bar{1}\bar{2}\bar{3}}], [dz_{12\bar{3}} + dz_{\bar{1}\bar{2}3}] \rangle \oplus \langle [dy_3] \otimes \boldsymbol{x}_1, [dy_3] \otimes \boldsymbol{x}_2, [dx_3] \otimes \boldsymbol{x}_3, \dots, [dx_3] \otimes \boldsymbol{x}_{10} \rangle. \end{aligned}$$

Proof We first compute the cohomology groups of M. We observe,

$$H^k(M) = H^k(\mathbb{T})^F = [dt] \wedge H^{k-1}(T^6)^f \oplus H^k(T^6)^f.$$

It is then clear that $H^1(M) = \langle [dt] \rangle$; in addition, $\kappa^*(dt) = -dt$. For the remaining degrees, we analyze the map $f^*\colon H^k(T^6, \mathbb{C}) \to H^k(T^6, \mathbb{C})$ and then take real parts to find $H^k(T^6)^f$. That is, since f is a holomorphic map, it preserves each of the subspaces $H^{p,q}(T^6)$, and given a (p,q)-form α with $p > q$, the map f^* fixes $[\alpha + \overline{\alpha}]$ if and only if it fixes $[\alpha]$. Then it suffices to study the action of f^* on $H^{p,q}(T^6)$ with $p \ge q$.

The action of f^* in terms of the basis $([dz_{13}], [dz_{23}], [dz_{12}])$ of $H^{2,0}(T^6, \mathbb{C})$ is $\mathrm{diag}(-i, -i, -1)$. For the basis of $H^{1,1}(T^6, \mathbb{C})$ given by $([dz_{1\bar{1}}], [dz_{2\bar{2}}], [dz_{3\bar{3}}], [dz_{1\bar{2}}], [dz_{\bar{1}2}], [dz_{1\bar{3}}], [dz_{2\bar{3}}], [dz_{\bar{1}3}], [dz_{\bar{2}3}])$, the action of f^* is $\mathrm{diag}(1,1,1,1,1,-i,-i,i,i)$. Hence

$$H^2(T^6)^f = \langle [i\, dz_{1\bar{1}}], [i\, dz_{2\bar{2}}], [i\, dz_{3\bar{3}}], [dz_{1\bar{2}} + dz_{\bar{1}2}], [i(dz_{1\bar{2}} - dz_{\bar{1}2})] \rangle.$$

All these classes are invariant under ι_1 and ι_2. However, κ acts as $\mathrm{diag}(-1,-1,-1,1,-1)$.

The map f^* acts trivially on $H^{3,0}(T^6)$. For $H^{2,1}(T^6,\mathbb{C})$, consider the basis $([dz_{12\bar{3}}], [dz_{13\bar{2}}], [dz_{23\bar{1}}])$. Then f^* acts as $\mathrm{diag}(1,-1,-1)$. Thus

$$H^3(T^6)^f = \langle [dz_{123} + dz_{\bar{1}\bar{2}\bar{3}}], [i(dz_{123} - dz_{\bar{1}\bar{2}\bar{3}})], [dz_{12\bar{3}} + dz_{\bar{1}\bar{2}3}], [i(dz_{12\bar{3}} - dz_{\bar{1}\bar{2}3})] \rangle.$$

These forms are fixed by ι_1 and ι_2. In addition, the action of κ is $\mathrm{diag}(1,-1,1,-1)$. Finally, one deduces the result from our previous calculations, the equality $H^k(X) = H^k(M)^{\langle \kappa, \iota_1, \iota_2 \rangle}$, and (17). □

Remark 8 As [17, Figure 12.3] shows, there is no holonomy G_2 manifold with $(b_2, b_3) = (11, 16)$ within the examples provided in Section 12 of that book. This pair does not appear in [23], where the Joyce–Karigiannis construction is used to resolve orbifolds of the form $(\mathrm{K3} \times T^3)/\mathbb{Z}_2^2$. The examples in [19, section 8] have $b_2 \le 9$, and those in [20, section 6] with $b_2 = 11$ have $b_3 \ge 66$. The author could not find a twisted connected sum in [8] with this pair of Betti numbers.

3.2 Algebra structure

We now focus on the product under the isomorphism provided by (17). This discussion is based on [21, Proposition 6.4] and uses the notation introduced in Section 2.2. Similar computations have been done in [12; 25].

We first claim that the Thom form of the normal bundle of $N_j^1 \times \mathbb{CP}^1 \subset N_j^1 \times Y$ is the extension of

$$\tau_j = \frac{1}{\pi i} d(b(r^2)\partial \log(r^2)) = \frac{2}{\pi i} b'(r^2) r\, dr \wedge \partial \log(r^2) + \frac{b(r^2)}{\pi i} \bar{\partial}\partial \log r^2, \tag{18}$$

where r is the radial coordinate on $\mathbb{C}^2$, and $b(s)$ is an increasing function that equals -1 on $s \le \varepsilon^2$ and 0 on $s \ge 4\varepsilon^2$. Of course, we choose ε small enough that τ_j is supported in $\rho^{-1}(O_j)$. Observe that τ_j is smooth because the second term is proportional to the pullback of the Fubini–Study form on $\mathbb{CP}^1$.

To verify this claim, consider a unit-length vector $(u_1, u_2) \in \mathbb{C}^2$ and the holomorphic embedding to a fiber of the tautological line bundle $F = \{(w, [u_1 : u_2]) \mid w \in [u_1 : u_2]\}$, given by $e\colon \mathbb{C} \to F$, $e(\lambda) = (\lambda u_1, \lambda u_2, [u_1 : u_2])$. This induces an orientation-preserving embedding onto the fiber $F_p = \{p\} \times F/\mathbb{Z}_2$ of the bundle $N_j^1 \times Y \to N_j^1 \times \mathbb{CP}^1$ that we denote by $e_p\colon \mathbb{C}/\mathbb{Z}_2 \to F_p$.

We observe that $e^*(\partial \log(r^2)) = \partial(\log|\lambda|^2) = (d\lambda)/\lambda$, so that $e^*\tau_j = 1/(\pi i) b'(|\lambda|^2) d\bar{\lambda} \wedge d\lambda$. Since $e^*\tau_j$ vanishes when $|\lambda| \notin [\varepsilon, 2\varepsilon]$ we have

$$\begin{aligned}\int_{\mathbb{C}/\mathbb{Z}_2} e_p^*(\tau_j) = \frac{1}{2}\int_{\mathbb{C}} e^*(\tau_j) &= \frac{1}{2\pi i}\int_{|\lambda| \in [\varepsilon, 2\varepsilon]} d\left(\frac{b(|\lambda|^2) d\lambda}{\lambda}\right) \\ &= \frac{1}{2\pi i}\int_{|\lambda| = 2\varepsilon} \frac{b(|\lambda|^2) d\lambda}{\lambda} - \frac{1}{2\pi i}\int_{|\lambda| = \varepsilon} \frac{b(|\lambda|^2) d\lambda}{\lambda} = \frac{1}{2\pi i}\int_{|\lambda| = \varepsilon} \frac{d\lambda}{\lambda} = 1.\end{aligned}$$

This proves that τ_j integrates to 1 over the fibers of the normal bundle, and therefore it is a representative of the Thom class of that bundle. Note in addition that the Thom class of the tautological line bundle over $\mathbb{CP}^1$ is $\frac{1}{2}\tau_j$.

The form τ_j is ι_* invariant, and hence the Thom form of $Q(N_j) \subset P(N_j)$ is the pushforward of τ_j. We also denote it by τ_j. Since the second term in (18) is proportional to the pullback of the Fubini–Study form of $\mathbb{CP}^1$, the form τ_j^2 vanishes in a neighborhood of $Q(N_j)$. Hence it induces by pushforward a form $\rho_*(\tau_j^2)$ on X with compact support on O_j. Let V_j be a tubular neighborhood of N_j with $O_j \subset V_j$ and let $\mathrm{Th}[N_j] \in H_c^4(V_j)$ be the Thom class of $N_j \subset X$ oriented by $\varphi|_{N_j}$. Then

$$[\rho_*(\tau_j^2)] = -2\,\mathrm{Th}[N_j] \in H_c^4(V_j) \subset H^4(X), \tag{19}$$

because $\int_{\mathbb{C}^2/\mathbb{Z}_2} \rho_*(\tau_j^2) = \int_Y \tau_j^2 = \int_{\mathbb{CP}^1} \tau_j = 1/(\pi i) \int_{\mathbb{CP}^1} \partial\bar\partial \log(r^2) = -2$ (or alternatively, $\int_Y \tau_j^2 = [\mathbb{CP}^1][\mathbb{CP}^1] = -2$). In addition, when $j \neq k$ the supports of τ_j and τ_k are disjoint; hence $\tau_j \wedge \tau_k = 0$. Finally, letting $[\alpha] \in H^k(X)$, by Lemma 5 we can assume that it satisfies $\alpha = \pi_j^*(\alpha|_{N_j})$ on O_j; then $\rho^*(\alpha) \wedge \tau_j = \mathrm{pr}_j^*(\alpha|_{N_j}) \wedge \tau_j$. Under the isomorphism in (17), our discussion yields the following product rules:

$$\boldsymbol{x}_j^2 = -2\,\mathrm{Th}[N_j], \qquad \boldsymbol{x}_j \cdot \boldsymbol{x}_k = 0 \quad \text{if } j \neq k, \tag{20}$$

$$[\alpha] \cdot \boldsymbol{x}_j = [\alpha|_{N_j}] \otimes \boldsymbol{x}_j \quad \text{if } [\alpha] \in H^*(X). \tag{21}$$

The Thom class of N_j (viewed in $H^*(X)$) coincides with its Poincaré dual. Recall (see [3, p. 51]) that given a compact oriented n-dimensional manifold M, the Poincaré dual of a k-dimensional compact oriented submanifold $N \subset M$ is the unique cohomology class $\mathrm{PD}[N] \in H^{n-k}(M)$ such that for any representative $\upsilon \in \Omega^{n-k}(M)$,

$$\int_N \alpha = \int_M \alpha \wedge \upsilon \quad \text{for all } [\alpha] \in H^k(M). \tag{22}$$

This is a consequence of Poincaré duality, namely $(H^k(M))^* = H^{n-k}(M)$, which also holds for orbifolds; see [24, Theorem 3]. Hence a similar definition applies to orbifolds. In Lemma 9 we find $\mathrm{PD}[N_j] \in H^4(X)$ and in Lemma 10 we deduce from (19) that the self-intersection of $[E_j]$ is $-2[L_j]$, where L_j is defined by (16).

Lemma 9 *Write $\beta_1 = i\,dt \wedge (dz_{123} - dz_{\bar1\bar2\bar3})$, and $\beta_2 = i\,dt \wedge (dz_{12\bar3} - dz_{\bar1\bar23})$. Then*

$$\mathrm{PD}[N_1] = \mathrm{PD}[N_2] = [\beta_1 + \beta_2], \qquad \mathrm{PD}[N_3] = \mathrm{PD}[N_j] = \tfrac{1}{2}[\beta_1 - \beta_2] \quad \textit{for } 4 \le j \le 10.$$

Proof Following the proof of Proposition 7, we obtain that $[\beta_1], [\beta_2] \in H^4(X)$. Let $[\alpha] \in H^3(X)$; then we can choose $\alpha = dt \wedge \beta + \lambda_1(dz_{123} + dz_{\bar1\bar2\bar3}) + \lambda_2(dz_{12\bar3} + dz_{\bar1\bar23})$, where $f^*[\beta] = [\beta]$ and $\kappa^*[\beta] = -[\beta]$. Let $\mu_1, \mu_2 \in \mathbb{R}$. Since $M - \bigcup_{j,k} N_j^k \to X - \bigcup_j N_j$ is an $8:1$ cover,

$$\int_X \alpha \wedge (\mu_1\beta_1 + \mu_2\beta_2) = \frac{1}{8}\int_M -2i(\mu_1\lambda_1 - \mu_2\lambda_2)\,dt \wedge dz_{1\bar12\bar23\bar3} = 2(\mu_1\lambda_1 - \mu_2\lambda_2).$$

Using the coordinates described in (5), we compute $\alpha|_{N_1^k} = 4(\lambda_1 - \lambda_2)\,dx_{12} \wedge dy_3$. Similarly, from (6) we obtain $\alpha|_{N_2^k} = -4(\lambda_1 - \lambda_2)\,dx_{12} \wedge dy_3$. Taking into account the induced orientations (9) and (10) of N_j^k, we deduce $\int_{N_j} \alpha = \frac{1}{2}\int_{N_j^1} \alpha = 2(\lambda_1 - \lambda_2)$ for $j = 1, 2$. In addition, if $3 \le j \le 6$ then $\alpha|_{N_j^k} = 2(\lambda_1 + \lambda_2)\,dx_{123}$ and if $7 \le j \le 10$ then $\alpha|_{N_j^k} = -2(\lambda_1 + \lambda_2)\,dy_{12} \wedge dx_3$. Of course, we used the coordinates in (7) and (8). Equations (11) and (12) ensure $\int_{N_j} \alpha = \frac{1}{2}\int_{N_j^1} \alpha = (\lambda_1 + \lambda_2)$ for $j \ge 3$. The statement follows from these calculations. For instance, if $3 \le j \le 7$,

$$\int_{N_j} \alpha = \lambda_1 + \lambda_2 = 2\big(\tfrac{1}{2}\lambda_1 - (-\tfrac{1}{2}\lambda_2)\big) = \int_X \alpha \wedge \big(\tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_2\big),$$

and (22) implies $\mathrm{PD}[N_j] = \frac{1}{2}[\beta_1 - \beta_2]$. □

Lemma 10 *The following equalities hold on $H^4(\widetilde{M})$:*

$$[\tau_1]^2 = [\tau_2]^2 = -2\,\mathrm{PD}[L_1] = -2\,\mathrm{PD}[L_2], \qquad [\tau_3]^2 = [\tau_j]^2 = -2\,\mathrm{PD}[L_3] = -2\,\mathrm{PD}[L_j] \quad \text{for } 4 \le j \le 10.$$

Proof Fix $1 \le j \le 10$. We first prove that $[\tau_j]^2 = -2\,\mathrm{PD}[L_j]$. Let $[\alpha] \in H^3(\widetilde{M})$; according to (17) and Lemma 5 we can write $\alpha = \rho^*(\beta) + \sum_{k=1}^{10} \mathrm{pr}_k^*(\lambda_k) \wedge \tau_k$, where $[\lambda_k] \in H^1(N_k)$ and $[\beta] \in H^3(X)$ satisfies $\beta|_{O_k} = \pi_k^*(\beta|_{N_k})$ for every $1 \le k \le 10$.

On the one hand, $\alpha \wedge \tau_j^2 = \rho^*(\beta) \wedge \tau_j^2$ because $\tau_j \wedge \tau_k = 0$ if $j \ne k$ and $\tau_j^3 = 0$. For the last equality note that Y is 4-dimensional and that for every $[p,q] \in (N_j^1 \times Y)/\iota_*$ we have $\tau_j|_{[p,q]} \in \Lambda^2 T_q Y$ by definition (18). Equation (19) yields $\int_{\widetilde{M}} \alpha \wedge \tau_j^2 = \int_{\widetilde{M}} \rho^*(\beta) \wedge \tau_j^2 = \int_X \beta \wedge \rho_*(\tau_j^2) = -2\int_{N_j} \beta$.

On the other hand, $\alpha|_{L_j} = \rho^*(\beta)|_{L_j}$ because $\tau_k|_{L_j} = 0$ for any $1 \le k \le 10$. Observe that if $j = k$ this follows from $\tau_j|_{[p,q_0]} \in \Lambda^2 T_{q_0} Y$ when $(p, q_0) \in L_j$. Hence $\int_{L_j} \alpha = \int_{L_j} \rho^*(\beta) = \int_{N_j \times \{q_0\}} \mathrm{pr}_j^*(\beta|_{N_j}) = \int_{N_j} \beta = -\frac{1}{2}\int_{\widetilde{M}} \alpha \wedge \tau_j^2$. This implies $[\tau_j]^2 = -2\,\mathrm{PD}[L_j]$; the result now follows from Lemma 9. □

3.3 A nonvanishing triple Massey product

We first recall the definition of a triple Massey product; see [11, Definition 2.89] or [26, Definition 6.1]. For cohomology classes $[\alpha_1], [\alpha_2], [\alpha_3] \in H^*$, we say that their triple Massey product $\langle[\alpha_1], [\alpha_2], [\alpha_3]\rangle$ is well-defined if

$$[\alpha_1][\alpha_2] = 0 \quad \text{and} \quad [\alpha_2][\alpha_3] = 0.$$

If these conditions are satisfied, we pick forms ξ_{12} and ξ_{23} with $d\xi_{12} = \alpha_1 \wedge \alpha_2$ and $d\xi_{23} = \alpha_2 \wedge \alpha_3$. Then

$$y = \xi_{12} \wedge \alpha_3 - (-1)^{\deg(\alpha_1)} \alpha_1 \wedge \xi_{23} \tag{23}$$

is a closed form. Let $\mathcal{I}$ be the ideal in H^* generated by $[\alpha_1]$ and $[\alpha_3]$, and denote the projection map by $\pi\colon H^* \to H^*/\mathcal{I}$. The triple Massey product is defined as $\langle[\alpha_1], [\alpha_2], [\alpha_3]\rangle = \pi[y]$. We say that it vanishes if $\pi[y] = 0$, that is, if $[y] \in \mathcal{I}$.

It is well-known that all triple Massey products vanish on a formal manifold; see, for instance, [11, Proposition 2.90]. The goal of this section is to prove that the triple Massey product on $H^*(\widetilde{M})$, $\langle[\tau_1 + \tau_2], [\tau_7 + \tau_3], [\tau_7 - \tau_3]\rangle$, does not vanish. As shown in the proof of Theorem 17 and Remarks 13, 15, and 18, this occurs because the oriented submanifolds $N_1' \sqcup -N_2'$ and $N_3' \sqcup -N_7'$ of M' are nullcobordant, and their linking number is nonzero. Here the minus sign indicates that the orientation of the submanifold is reversed. The linking number (see [3, pp. 229–234]) of two disjoint nullhomologous oriented 3-dimensional submanifolds A and B of M' is

(24) $$\int_{M'} \xi_A \wedge \tau_B = \int_{M'} \tau_A \wedge \xi_B, \quad d\xi_A = \tau_A, \quad d\xi_B = \tau_B, \quad [\tau_A] = \mathrm{Th}[A], \quad [\tau_B] = \mathrm{Th}[B].$$

If $A = \partial C_A$ and C_A intersects B transversally, the linking number is the oriented intersection number between C_A and B.

Note that $\langle [\tau_1 + \tau_2], [\tau_7 + \tau_3], [\tau_7 - \tau_3] \rangle$ is well-defined because $\tau_j \wedge \tau_k = 0$ when $j \neq k$ and $[\tau_7]^2 - [\tau_3]^2 = 0$ by Lemma 10. To find a primitive of $\tau_7^2 - \tau_3^2$, we first construct a form $\beta \in \Omega^3(M)$ with $d\beta = \upsilon_3^1 - \upsilon_7^1$, where υ_j^1 is a representative of the Thom class of N_j^1 for $j = 3, 7$. The construction uses a cobordism between N_3^1 and N_7^1.

Proposition 11 *There are small tubular neighborhoods V_3^1 and V_7^1 of N_3^1 and N_7^1 and forms $\beta \in \Omega^3(M)$ and $\upsilon_3^1, \upsilon_7^1 \in \Omega^4(M)$ such that:*

- (i) *The forms υ_3^1 and υ_7^1 are closed and supported on V_3^1 and V_7^1, respectively. The cohomology class $[\upsilon_j^1] \in H_c^4(V_j^1)$ is the Thom class of N_j^1.*
- (ii) *The form β vanishes in a neighborhood of the level set $t = \frac{1}{2}$ and $d\beta = \upsilon_3^1 - \upsilon_7^1$. In particular, it is closed on $M - (V_3^1 \cup V_7^1)$.*
- (iii) *We have $[\beta|_{N_1^k}] = -\frac{1}{2}[\varphi|_{N_1^k}]$, $[\beta|_{N_2^k}] = \frac{1}{2}[\varphi|_{N_2^k}]$.*

Proof For convenience, we use $t \in \left[\frac{1}{2}, \frac{3}{2}\right]$ instead of $t \in [0,1]$. We first find a cobordism on M between N_3^1 and N_7^1. We view points in N_3^1 at the level set $t = \frac{1}{2}$ and we consider the coordinates in (7) and the orientation in (11). Instead, we view N_7^1 inside the level set $t = \frac{3}{2}$. That is, since $\left[\frac{1}{2}, \left[iy_1, iy_2, x_3 - \frac{1}{4}i\right]\right] = \left[\frac{3}{2}, \left[-y_1, -y_2, -x_3 + \frac{1}{4}i\right]\right]$, we describe

(25) $$N_7^1 = \left\{\left[\tfrac{3}{2}, \left[x_1, x_2, x_3 + \tfrac{1}{4}i\right]\right] \mid x_1, x_2, x_3 \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\}.$$

Its orientation is given by $\varphi|_{N_7^1} = dx_{123}$. Let $\mathsf{f}\colon \left[\frac{1}{2}, \frac{3}{2}\right] \to \left[0, \frac{1}{4}\right]$ be a nondecreasing function that equals 0 if $t \le \frac{9}{8}$, and $\frac{1}{4}$ if $t \ge \frac{5}{4}$. Define the embedding $\Psi\colon \left[\frac{1}{2}, \frac{3}{2}\right] \times N_3^1 \to M$ by

(26) $$\Psi\left(t, \left[\tfrac{1}{2}, [x_1, x_2, x_3]\right]\right) = [t, [x_1, x_2, x_3 + i\,\mathsf{f}(t)]].$$

Then $C = \mathrm{Im}(\Psi)$ is a manifold with boundary, $N_3^1 \cup N_7^1$. We trivialize and orient the bundle TC using the frame $(-\partial_t - \mathsf{f}'(t)\partial_{y_3}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. The normal bundle $\nu(C)$ is then oriented and trivialized by $(\partial_{y_1}, \partial_{y_2}, \partial_{y_3} - \mathsf{f}'(t)\partial_t)$. The basis $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ orients N_j^1, and the outward-pointing vector is $V_o = -\partial_t$ on N_3^1 and $V_o = \partial_t$ on N_7^1. Since a basis (e_1, e_2, e_3) of ∂C is positive if (V_o, e_1, e_2, e_3) is positive on C, we have that $\partial C = N_3^1 \sqcup -N_7^1$ as oriented manifolds.

We now find β using the cobordism; the Thom forms υ_3^1 and υ_7^1 will later be constructed from $d\beta$. We observe that if $j = 3, 7$, the bundle $\nu(N_j^1)$ is oriented by $(\partial_t, \partial_{y_1}, \partial_{y_2}, \partial_{y_3})$. Hence

$$\nu(N_j^1) \cong \langle \partial_t|_{N_j^1} \rangle \oplus \nu(C)|_{N_j^1}$$

as oriented bundles. The Thom form of $\nu(N_j^1)$ is then the product of the Thom forms of the bundles $\langle \partial_t|_{N_j^1}\rangle$ and $\nu(C)|_{N_j^1}$; see [3, Proposition 6.19]. This is the idea that makes our strategy possible.

Let $\zeta\colon \left[-\frac{1}{2},\frac{1}{2}\right]^3 \to \mathbb{R}$ be a positive radial function whose integral is 1 and whose support is contained in $B_{2\delta}(0)-B_\delta(0)$, for a small δ. Consider a periodic extension of ζ to $\mathbb{R}^3/\mathbb{Z}^3$. Set $h\colon \left[\frac{1}{2},\frac{3}{2}\right]\to\mathbb{R}$ a bump function that equals 0 on $\left[\frac{1}{2},\frac{1}{2}+\varepsilon\right]\cup\left[\frac{3}{2}-\varepsilon,\frac{3}{2}\right]$, and 1 on $\left[\frac{1}{2}+2\varepsilon,\frac{3}{2}-2\varepsilon\right]$. Then

$$\beta|_{[t,[x_1+iy_1,x_2+iy_2,x_3+iy_3]]} = h(t)\zeta(y_1,y_2,y_3-\mathsf{f}(t))\,dy_1\wedge dy_2\wedge(dy_3-\mathsf{f}'(t)\,dt) \quad \text{for } t\in\left[\tfrac{1}{2},\tfrac{3}{2}\right] \tag{27}$$

induces a well-defined form on M that vanishes around $t=\frac{1}{2}$. Indeed, around a small neighborhood of the interior of C, the expression $\zeta(y_1,y_2,y_3-\mathsf{f}(t))\,dy_1\wedge dy_2\wedge(dy_3-\mathsf{f}'(t)\,dt)$ defines a closed form whose pullback to $\nu(\mathrm{Int}(C))$ represents the Thom class of that oriented bundle. The support of the form

$$d\beta = h'(t)\zeta(y_1,y_2,y_3-\mathsf{f}(t))\,dt\wedge dy_1\wedge dy_2\wedge dy_3 \quad \text{for } t\in\left[\tfrac{1}{2},\tfrac{3}{2}\right]$$

is contained on a neighborhood of $N_3^1\cup N_7^1$, and we can write $d\beta=\upsilon_3^1-\upsilon_7^1$, where the support of υ_j^1 is contained in the following neighborhood V_j^1 of N_j^1:

$$V_3^1 = \left\{[t,[x_1+iy_1,x_2+iy_2,x_3+iy_3]] \mid \left|t-\tfrac{1}{2}\right|<2\varepsilon,\ \|(y_1,y_2,y_3)\|<2\delta\right\},$$
$$V_7^1 = \left\{[t,[x_1+iy_1,x_2+iy_2,x_3+iy_3]] \mid \left|t-\tfrac{3}{2}\right|<2\varepsilon,\ \left\|(y_1,y_2,y_3-\tfrac{1}{4})\right\|<2\delta\right\}.$$

Moreover, if $[t,[z]]\in V_3^1$ satisfies $\frac{1}{2}-2\varepsilon<t<\frac{1}{2}$ and $\|(y_1,y_2,y_3)\|<2\delta$, from $\mathsf{f}(t+1)=\frac{1}{4}$ we obtain $d\beta|_{[t,[z]]}=d\beta|_{[t+1,[iz_1,iz_2,-z_3]]}=h'(t+1)\zeta\left(x_1,x_2,-y_3-\frac{1}{4}\right)dt\wedge dx_1\wedge dx_2\wedge d(-y_3)$. This vanishes because $\left|-y_3-\frac{1}{4}\right|\ge\frac{1}{4}-2\delta$ and ζ is supported around 0. Since h is constant on $\left[\frac{1}{2},\frac{1}{2}+\varepsilon\right]$, we also have $d\beta=0$ at points $[t,[z]]\in V_3^1$ with $\frac{1}{2}\le t<\frac{1}{2}+\varepsilon$. In the same way, $d\beta=0$ at points $[t,[z]]\in V_7^1$ with $\frac{3}{2}-\varepsilon<t<\frac{3}{2}+2\varepsilon$.

We now check that υ_3^1 represents the Thom class of N_3^1. First, υ_3^1 is closed because $d\beta$ is and the support of υ_3^1 is disjoint from that of υ_7^1. In addition, let $p\in N_3^1$ and let F_p be the fiber of V_3^1 at p, oriented by $(\partial_t,\partial_{y_1},\partial_{y_2},\partial_{y_3})$. Since υ_3^1 is supported on the level sets $\frac{1}{2}+\varepsilon<t<\frac{1}{2}+2\varepsilon$ where $\mathsf{f}=0$,

$$\int_{F_p}\upsilon_3^1 = \int_{t=1/2+\varepsilon}^{1/2+2\varepsilon} h'(t)\,dt \int_{\|(y_1,y_2,y_3)\|<2\delta}\zeta(y_1,y_2,y_3)\,dy_1\wedge dy_2\wedge dy_3 = h\left(\tfrac{1}{2}+2\varepsilon\right)-h\left(\tfrac{1}{2}+\varepsilon\right)=1.$$

Hence $[\upsilon_3^1]\in H_c^4(V_3^1)$ is the Thom class of N_3^1. A similar computation shows that υ_7^1 represents the Thom class of N_7^1.

Of course, β is closed on $M-(V_3^1\cup V_7^1)$. In particular, if $j,k=1,2$, then $\beta|_{N_j^k}$ determines an element in $H^3(N_j^k)$. To prove the third claim, it is sufficient to show $\int_{N_j^k}\beta|_{N_j^k}=\sigma_j$ where $\sigma_1=-1$ and $\sigma_2=1$ because $\int_{N_j^k}\varphi=2$. For convenience, we rewrite the formulas for N_j^k at the level set $t=1$:

$$N_1^k = \left\{[1,[x_1+ix_1,x_2+ix_2,-\varepsilon_k-iy_3]] \mid x_1,x_2,y_3\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right\},\quad \varepsilon_1=0,\quad \varepsilon_2=\tfrac{1}{2}, \tag{28}$$
$$N_2^k = \left\{[1,[-x_1+ix_1,-x_2+ix_2,-\delta_k-iy_3]] \mid x_1,x_2,y_3\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right\},\quad \delta_1=-\tfrac{1}{4},\quad \delta_2=\tfrac{1}{4}. \tag{29}$$

These are oriented by $\varphi|_{N_1^k} = 2\,dx_1 \wedge dx_2 \wedge dy_3$ and $\varphi|_{N_2^k} = -2\,dx_1 \wedge dx_2 \wedge dy_3$. Since $h(1) = 1$ and $\mathsf{f}(1) = 0$, we have

$$\int_{N_1^k} \beta = \int_{\|(x_1,x_2,-y_3)\|<2\delta} \zeta(x_1,x_2,-y_3)\,dx_1 \wedge dx_2 \wedge d(-y_3) = -1,$$
$$\int_{N_2^k} \beta = \int_{\|(x_1,x_2,-y_3)\|<2\delta} \zeta(x_1,x_2,-y_3)\,dx_1 \wedge dx_2 \wedge d(-y_3) = 1.$$

Here we used that $(x_1,x_2,y_3) \to (x_1,x_2,-y_3)$ reverses the orientation, and the expressions for $\varphi|_{N_j^k}$. □

Remark 12 Following the notation of the proof, we provide a geometric justification of the equality $\int_{N_j^k} \beta|_{N_j^k} = \sigma_j$ for $j,k = 1,2$. Observe that $N_j^k \cap C = \{p_j^k\}$, where $p_1^k = [1,[0,0,-\varepsilon_k]]$ and $p_2^k = [1,[0,0,-\delta_k]]$. Pick the basis $b_0 = (-\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ of $TC|_{N_j^k}$, and consider the positively oriented basis of N_j^k viewed at the level set $t = 1$ given by $b_j = (-\sigma_j\,\partial_{x_1} + \partial_{y_1}, -\sigma_j\,\partial_{x_2} + \partial_{y_2}, \sigma_j\,\partial_{y_3})$ — note that $\varphi(-\sigma_j\,\partial_{x_1} + \partial_{y_1}, -\sigma_j\,\partial_{x_2} + \partial_{y_2}, \sigma_j\,\partial_{y_3}) = 2$. Then (b_0,b_1) is a negative basis for $T_{p_1^k}M$ and (b_0,b_2) is a positive basis for $T_{p_2^k}M$ because

$$\begin{aligned}(-\partial_t) \wedge \partial_{x_1} \wedge \partial_{x_2} \wedge \partial_{x_3} \wedge (-\sigma_j\,\partial_{x_1} + \partial_{y_1}) \wedge (-\sigma_j\,\partial_{x_2} + \partial_{y_2}) \wedge (\sigma_j\,\partial_{y_3}) \\ = -\sigma_j\,\partial_t \wedge \partial_{x_1} \wedge \partial_{x_2} \wedge \partial_{x_3} \wedge \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \\ = \sigma_j\,\partial_t \wedge \partial_{x_1} \wedge \partial_{y_1} \wedge \partial_{x_2} \wedge \partial_{y_2} \wedge \partial_{x_3} \wedge \partial_{y_3}.\end{aligned}$$

Around C and on the level sets where $h = 1$, namely $\frac{1}{2} + 2\varepsilon < t < \frac{3}{2} - 2\varepsilon$, the form

$$\beta = \zeta(y_1, y_2, y_3 - \mathsf{f}(t))\,dy_1 \wedge dy_2 \wedge (dy_3 - \mathsf{f}'(t)\,dt)$$

is a representative of the Thom class of $\nu(C)$. Therefore $[\beta|_{N_j^k}]$ is the Thom class of p_j^k in N_j^k, oriented negatively if $j = 1$ or positively if $j = 2$. Hence the integral of β on N_1^k (resp. N_2^k) equals -1 (resp. 1).

Remark 13 The submanifold $N_1^1 \sqcup -N_2^1$ is nullcobordant. Similar to (26) we set

$$\widetilde{\Psi}\colon [0,1] \times N_1^1 \to M, \quad \widetilde{\Psi}(t,[0,[x_1 - ix_1, x_2 - ix_2, iy_3]]) = [t,[x_1 - ix_1, x_2 - ix_2, \tilde{\mathsf{f}}(t) + iy_3]],$$

where $\tilde{\mathsf{f}}\colon [0,1] \to \left[0, \frac{1}{4}\right]$ is a nondecreasing bump function with $\tilde{\mathsf{f}}|_{[0,3/4]} = 0$ and $\tilde{\mathsf{f}}|_{[7/8,1]} = \frac{1}{4}$. Then $\widetilde{C} = \operatorname{Im}(\widetilde{\Psi})$ oriented by $(-\partial_t - \tilde{\mathsf{f}}'(t)\partial_{x_3}, \partial_{x_1} - \partial_{y_1}, \partial_{x_2} - \partial_{y_2}, \partial_{y_3})$ satisfies $\partial\widetilde{C} = N_1^1 \sqcup -N_2^1$. The linking number on M of the submanifolds $N_3^1 \sqcup -N_7^1$ and $N_1^1 \sqcup -N_2^1$ is then well-defined and coincides with the oriented intersection number of C and $N_1^1 \sqcup -N_2^1$. As a consequence of Remark 12, this equals -2. In fact, set $\widetilde{C}_t = \widetilde{\Psi}(\{t\} \times N_1^1)$ oriented by $(\partial_{x_1} - \partial_{y_1}, \partial_{x_2} - \partial_{y_2}, \partial_{y_3})$. Then the oriented intersection number of C and $\widetilde{C}_t$ is -1 if $t \in \left[0, \frac{1}{2}\right)$, undefined if $t = \frac{1}{2}$, and 1 if $t \in \left(\frac{1}{2}, 1\right]$.

The projection to M' of the cobordism C obtained in the proof of Proposition 11 determines a cobordism between N_3' and N_7' that intersects N_1' negatively and N_2' positively. The average of the form α provides a primitive for the difference of the Thom forms of N_3' and N_7'. We make this precise in the following result:

Lemma 14 *There are small tubular neighborhoods V_3' and V_7' of N_3' and N_7' on M' and κ-invariant forms $\alpha \in \Omega^3(M')$ and $\upsilon_3, \upsilon_7 \in \Omega^4(M')$ such that:*

(i) *The forms* υ_3 *and* υ_7 *are closed and supported on* V_3' *and* V_7'*, respectively. The cohomology class* $[\upsilon_j] \in H^4_c(V_j')$ *is the Thom class of* N_j'*.*

(ii) *The pushforwards of* $2\upsilon_3$ *and* $2\upsilon_7$ *to* X *represent the Thom classes of* N_3 *and* N_7 *on* X*, respectively.*

(iii) *The form* α *vanishes in a neighborhood of the level set* $t = \frac{1}{2}$ *and* $d\alpha = \upsilon_3 - \upsilon_7$*. In particular,* α *is closed on* $M' - (V_3' \cup V_7')$*.*

(iv) *There are tubular neighborhoods* V_1' *and* V_2' *of* N_1' *and* N_2' *such that* $\alpha|_{V_1'} = -\pi_1^*(\varphi|_{N_1'})$*, and* $\alpha|_{V_2'} = \pi_2^*(\varphi|_{N_2'})$*, where* $\pi_j\colon V_j' \to N_j'$ *denotes the nearest-point projection for* $j = 1, 2$*.*

Proof Let $K = \langle \iota_1, \iota_2, \kappa \rangle = \mathbb{Z}_2^3$. For $j = 3, 7$ we write $V_j^2 = \iota_2(V_j^1) \subset M$, and set $K_3^1 = \{\mathrm{Id}, \iota_1, \kappa, \kappa \circ \iota_1\}$ and $K_7^1 = \{\mathrm{Id}, \iota_1, \kappa \circ \iota_2, \kappa \circ \iota_1 \circ \iota_2\}$. We first observe that every element of K_j^1 maps N_j^1 onto itself, and since the elements in K_j^1 are isometries, they preserve V_j^1 (and its fiber bundle structure). In addition, the elements of $K_j^2 = K - K_j^1$ swap N_j^1 with N_j^2, and therefore V_j^1 with V_j^2. Of course, if $\gamma \in K$ maps N_j^1 to N_j^k with $k = 1$ or $k = 2$, then the restrictions $\gamma\colon N_j^1 \to N_j^k$ and $\gamma_*\colon \nu(N_j^1) \to \nu(N_j^2)$ preserve the fixed orientations because the submanifolds N_j^k are oriented by $\varphi|_{N_j^k}$ and all the elements in K fix φ and preserve the orientation of M. Hence, if $j = 3, 7$, the form $\upsilon_j' = \frac{1}{4}\sum_{\gamma \in K_j^1} \gamma^*\upsilon_j^1$ is another representative of the Thom class of N_j^1 in M, and $\upsilon_j'' = \frac{1}{4}\sum_{\gamma \in K_j^2} \gamma^*\upsilon_j^1$ represents the Thom class of N_j^2.

The pushforward υ_j of $\upsilon_j' + \upsilon_j'' = \frac{1}{4}\sum_{\gamma \in K} \gamma^*\upsilon_j^1$ to M' is a κ-invariant form that represents the Thom class of $N_j' \subset M'$. It is supported in the projection of $V_j^1 \cup V_j^2$ to M'; we denote it by V_j'. We observe that (the pushforward of) $2\upsilon_j$ induces a representative of the Thom class of N_j on X. The reason is that the fiber of the normal bundle at a point $p \in N_j'$ is $F_p \cong \mathbb{C}^2$, and on X this is $F_p/\mathbb{Z}_2$; hence, $\int_{F_p/\mathbb{Z}_2} 2\upsilon_j = \frac{1}{2}\int_{F_p} 2\upsilon_j = 1$. Consider the K-invariant form $\alpha' = \frac{1}{4}\sum_{\gamma \in K} \gamma^*\beta$. On M we have

$$d\alpha' = \frac{1}{4}\sum_{\gamma \in K} \gamma^*\upsilon_3^1 - \frac{1}{4}\sum_{\gamma \in K} \gamma^*\upsilon_7^1 = (\upsilon_3' + \upsilon_3'') - (\upsilon_7' + \upsilon_7'').$$

Hence the pushforward of α' to M' (which we keep denoting by α') satisfies $d\alpha' = \upsilon_3 - \upsilon_7$. Since β vanishes in a neighborhood of the level set $t = \frac{1}{2}$ and the elements of K preserve that level set, α' vanishes in a neighborhood of $t = \frac{1}{2}$.

Similarly to Lemma 5, we now modify α' around the level set $t = 0$ to obtain α satisfying all these conditions. We again set $\sigma_1 = -1$ and $\sigma_2 = 1$. For $j = 1, 2$, if $\gamma \in K$ then γ maps N_j^1 onto itself, or it swaps N_j^1 and N_j^2. Since γ preserves the orientations in both cases, $\int_{N_j^k} \gamma^*\beta = \int_{\gamma(N_j^k)} \beta = \sigma_j = \frac{1}{2}\sigma_j \int_{N_j^k} \varphi$. Hence $[(\gamma^*\beta)|_{N_j^k}] = \frac{1}{2}\sigma_j[\varphi|_{N_j^k}]$ and therefore $[\alpha'|_{N_j^k}] = \sigma_j[\varphi|_{N_j^k}]$. Thus, $[\alpha'|_{N_j'}] = \sigma_j[\varphi|_{N_j'}]$ on M'.

By the Poincaré lemma, given a tubular neighborhood W_j' of N_j', there is $\theta_j \in \Omega^2(W_j')$ such that $d\theta_j = \sigma_j\pi_j^*(\varphi|_{N_j'}) - \alpha'|_{W_j'}$. Indeed, we can assume that θ_j is κ-invariant as both $\pi_j^*(\varphi|_{N_j'})$ and α' are. Consider V_1' and V_2' smaller tubular neighborhoods with $\overline{V}_j' \subset W_j'$, and κ-invariant bump functions h_j that equal 1 on V_j' and 0 on $M' - W_j'$. The form

$$\alpha = \alpha' + d(h_1\theta_1) + d(h_2\theta_2).$$

satisfies all the stated conditions. □

Remark 15 Similar to Remark 13, the linking number of $N_3' \sqcup -N_7'$ and $N_1' \sqcup -N_2'$ on M' is -2.

The equality (19), and Lemma 14 enable us to find a primitive of $\tau_7^2 - \tau_3^2$ in Lemma 16. To state it, we introduce some notation. For $j = 1, 2, 3, 7$, we define V_j as the projection of $V_j' \subset M'$ onto X and we let $U_j = \rho^{-1}(V_j)$. Recall that at the beginning of Section 3.1 we fixed tubular neighborhoods $O_j \subset O_j^2$ of $N_j \subset X$ and we assumed that τ_j is supported on $\rho^{-1}(O_j)$. Observe that it is not restrictive to assume that $O_j^2 \subset V_j$; we do so.

Lemma 16 *There is a form $\tilde\alpha \in \Omega^3(\widetilde M)$ such that $d(\tilde\alpha) = \tau_7^2 - \tau_3^2$. In particular, it is closed on $U_1 \cup U_2$. In addition,*

$$\tilde\alpha \wedge \tau_1 = -4\,\mathrm{pr}_1^*(\varphi|_{N_1}) \wedge \tau_1, \quad \tilde\alpha \wedge \tau_2 = 4\,\mathrm{pr}_2^*(\varphi|_{N_2}) \wedge \tau_2,$$

where $\mathrm{pr}_j = \pi_j \circ \rho\colon \rho^{-1}(V_j') \to N_j$.

Proof Let $j = 3, 7$. We first observe that $\rho^*(\upsilon_j)$ is smooth because it vanishes on a small neighborhood of N_j as α vanishes on a small neighborhood of $t = \frac{1}{2}$ and $d\alpha = \upsilon_3 - \upsilon_7$. Since τ_j^2 vanishes on a neighborhood of E_j, the pushforward $\rho_*(\tau_j^2)$ represents an element in $H_c^4(V_j)$. This group is generated by the Thom class of N_j, which is represented by $2\upsilon_j$ according to Lemma 14. In fact, (19) shows $[\rho_*(\tau_j^2)] = -2[2\upsilon_j] \in H_c^4(V_j)$. Hence there are forms $\eta_3, \eta_7 \in \Omega(X)$ with compact support on V_j such that $d\eta_j = \rho_*(\tau_j^2) + 4\upsilon_j$.

Lemma 5 allows us to modify η_j around N_j and find a form η_j' supported on V_j whose pullback $\rho^*(\eta_j')$ is smooth on $\widetilde M$ and $d\eta_j' = \rho_*(\tau_j^2) + 4\upsilon_j$. This is possible because on a small tubular neighborhood W_j^2 of N_j both $\rho_*(\tau_j^2)$ and υ_j vanish, so $d\eta_j|_{W_j^2} = (\rho_*(\tau_j^2) + 4\upsilon_j)|_{W_j^2} = 0$. We now argue that $\rho^*(\alpha)$ is smooth. Since $\alpha = 0$ on a neighborhood of the level set $t = \frac{1}{2}$, it is smooth on $\widetilde M - (E_1 \cup E_2)$. In addition, if $j = 1, 2$, then $\alpha|_{V_j'} = \sigma_j \pi_j^*(\varphi|_{N_j'})$ on M', where $\sigma_1 = -1$ and $\sigma_2 = 1$. The same identity holds for their pushforwards to X, so that $\rho^*(\alpha)|_{U_j} = \rho^*(\alpha|_{V_j}) = \sigma_j\,\mathrm{pr}_j^*(\varphi|_{N_j})$. Hence it is smooth.

Finally, we define $\tilde\alpha = \rho^*(\eta_7' + 4\alpha - \eta_3')$. Then $d\tilde\alpha = \tau_7^2 + 4\rho^*\upsilon_7 + 4\rho^*(\upsilon_3 - \upsilon_7) - (\tau_3^2 + 4\rho^*(\upsilon_3)) = \tau_7^2 - \tau_3^2$. Given $j = 1, 2$, the support of τ_j is contained in $\rho^{-1}(O_j) \subset U_j$ and both $\rho^*(\eta_3')$ and $\rho^*(\eta_7')$ vanish there. Hence $\tilde\alpha \wedge \tau_j = 4\rho^*(\alpha) \wedge \tau_j = \sigma_j 4\,\mathrm{pr}_j^*(\varphi|_{N_j}) \wedge \tau_j$. □

We finish by showing that $\langle [\tau_1 + \tau_2], [\tau_7 + \tau_3], [\tau_7 - \tau_3] \rangle$ does not vanish.

Theorem 17 *The triple Massey product $\langle [\tau_1 + \tau_2], [\tau_7 + \tau_3], [\tau_7 - \tau_3] \rangle$ is not trivial. Therefore $\widetilde M$ is nonformal.*

Proof Note that $(\tau_1 + \tau_2) \wedge (\tau_7 + \tau_3) = 0$ and $(\tau_7 + \tau_3) \wedge (\tau_7 - \tau_3) = \tau_7^2 - \tau_3^2 = d\tilde\alpha$. The cohomology class determined by (23) is

$$[y] = -[\tilde\alpha \wedge (\tau_1 + \tau_2)],$$

and the triple Massey product vanishes if and only if $[y] \in \mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $[\tau_1 + \tau_2]$ and $[\tau_7 - \tau_3]$. Observe that for any closed 5-form x, if $\int_{\widetilde M} x \wedge (\tau_1 - \tau_2) \neq 0$ then $[x] \notin \mathcal{I}$ because if $[x] = [c_1(\tau_1 + \tau_2) + c_2(\tau_7 - \tau_3)] \in \mathcal{I}$ with $[c_1], [c_2] \in H^3(\widetilde M)$, then $[x \wedge (\tau_1 - \tau_2)] = [c_1]([\tau_1^2] - [\tau_2^2]) = 0$ by Lemma 10. So we focus on proving $\int_{\widetilde M} y \wedge (\tau_1 - \tau_2) \neq 0$.

By Lemma 16, $[y] = 4[\mathrm{pr}_1^*(\varphi|_{N_1}) \wedge \tau_1] - 4[\mathrm{pr}_2^*(\varphi|_{N_2}) \wedge \tau_2]$. Therefore

$$[y \wedge (\tau_1 - \tau_2)] = 4[\mathrm{pr}_1^*(\varphi|_{N_1}) \wedge \tau_1^2] + 4[\mathrm{pr}_2^*(\varphi|_{N_2}) \wedge \tau_2^2] = 4[\rho^*(\beta)] \wedge ([\tau_1^2] + [\tau_2^2]),$$

where $\beta \in \Omega^3(X)$ is a closed form such that $[\beta] = [\varphi]$ and $\rho^*(\beta)|_{\rho^{-1}(O_j)} = \mathrm{pr}_j^*(\varphi|_{N_j})$ for every $1 \le j \le 10$, as in Lemma 5. From Lemma 10 we deduce

$$\int_{\widetilde{M}} y \wedge (\tau_1 - \tau_2) = -16 \int_{L_1} \rho^*(\beta) = -16 \int_{L_1} \mathrm{pr}_1^*(\varphi|_{N_1}) = -16.$$

So $[y] \notin \mathcal{I}$. □

Remark 18 Following the notation introduced in the proof of Theorem 17, we claim that the value of $\int_{\widetilde{M}} y \wedge (\tau_1 - \tau_2)$ is related to the linking number of $N_3' \sqcup -N_7'$ and $N_1' \sqcup -N_2'$ on M'.

In the proof of Lemma 16 we obtained $\tilde{\alpha} = \rho^*(\eta_7' + 4\alpha - \eta_3')$, where α is defined in Lemma 14 and η_j' is supported around N_j' for $j = 3, 7$. Since $y \wedge (\tau_1 - \tau_2) = -\tilde{\alpha} \wedge (\tau_1^2 - \tau_2^2)$ we obtain, $\int_{\widetilde{M}} y \wedge (\tau_1 - \tau_2) = -4 \int_X \alpha \wedge (\rho_*(\tau_1^2) - \rho_*(\tau_2^2)) = -2 \int_{M'} \alpha \wedge (\rho_*(\tau_1^2) - \rho_*(\tau_2^2))$.

Equation (19) implies that the pullback of $\rho_*(\tau_j^2)$ to M' (which we denote by the same name) equals $-4\,\mathrm{Th}[N_j'] \in H_c^4(V_j')$ because the fiber of the normal bundle of N_j' is $\mathbb{C}^2$ and that of N_j is $\mathbb{C}^2/\mathbb{Z}_2$. A representative of the Thom class of $N_1' \sqcup -N_2'$ is then the pullback of $-\frac{1}{4}(\rho_*(\tau_1^2) - \rho_*(\tau_2^2))$ to M'. Lemma 14 ensures that $d\alpha = \upsilon_3 - \upsilon_7$, where $[\upsilon_j] = \mathrm{Th}[N_j']$ for $j = 3, 7$. Equation (24) shows that the linking number of $N_3' \sqcup -N_7'$ and $N_1' \sqcup -N_2'$ on M' is

$$-\frac{1}{4} \int_{M'} \alpha \wedge (\rho_*(\tau_1^2) - \rho_*(\tau_2^2)) = \frac{1}{8} \int_{\widetilde{M}} y \wedge (\tau_1 - \tau_2).$$

Note that in the proof of Theorem 17 we obtained $\int_{\widetilde{M}} y \wedge (\tau_1 - \tau_2) = -16$, so the linking number is $-\frac{16}{8} = -2$. This is consistent with Remark 15.

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