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The proposals of Joyce (2018) and Doan and Walpuski (2019) on counting closed associative submanifolds of G_2-manifolds depend on various conjectural transitions. This article contributes to the study of transitions arising from the degenerations of associative submanifolds into conically singular (CS) associative submanifolds. First, we study the moduli space of CS associative submanifolds with isolated singularities modeled on associative cones in $\mathbb{R}^7$, establishing transversality results in both fixed and one-parameter families of coclosed G_2-structures. We prove that for a generic coclosed G_2-structure (or a generic path thereof) there are no CS associative submanifolds having singularities modeled on cones with stability index greater than 0 (or 1, respectively). We establish that associative cones whose links are null-torsion holomorphic curves in S^6 have stability index greater than 4, and all special Lagrangian cones in $\mathbb{C}^3$ have stability index greater than or equal to 1 with equality only for the Harvey–Lawson T^2-cone and a transverse pair of planes. Next, we study the desingularizations of CS associative submanifolds in a one-parameter family of coclosed G_2-structures. Consequently, we derive desingularization results relating the above transitions for CS associative submanifolds with a Harvey–Lawson T^2-cone singularity and for associative submanifolds with a transverse self-intersection.

1. Introduction: main results 1371
2. Preliminaries: G_2-manifolds and associative submanifolds 1383
3. Associative cones 1391
4. AC and CS associative submanifolds 1404
5. Deformations of CS associative submanifolds 1416
6. Desingularizations of CS associative submanifolds 1430
7. Desingularizations for special cases of singularities 1441
Appendix. Lemma for quadratic estimates 1445
Acknowledgements 1447
References 1447

1 Introduction: main results

Joyce [2018] and Doan and Walpuski [2019] have proposed constructing enumerative invariants of G_2-manifolds by counting closed associative submanifolds. It has been observed that the counting (possibly with sign) of associative submanifolds does not lead to an invariant due to various transitions that may occur along a generic path of G_2-structures. One type of such transition is the degeneration of associative submanifolds into conically singular associative submanifolds. In particular, it has been conjectured

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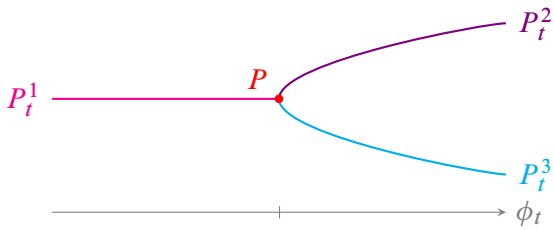
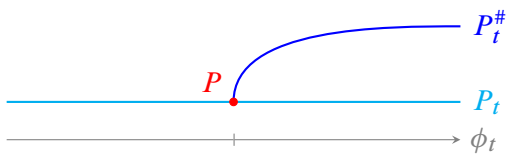
(see [Joyce 2018, Conjectures 4.4 and 5.3]) that at least the following transitions can occur along a generic path of G_2-structures ϕ_t on a 7-dimensional manifold Y.

(1) Three families of embedded closed associative submanifolds, P_t^1 with $-T < t < 0$ and P_t^2, P_t^3 with $0 < t < T$ in (Y, ϕ_t), converge in the sense of currents to an associative submanifold P with a Harvey–Lawson (HL) T^2-cone singularity at x in (Y, ϕ_0), as $t \to 0$. The P_t^i are diffeomorphic to the Dehn fillings of $P^o := P \setminus B(x)$ along simple closed curves $\mu_i \subset \partial P^o \cong T^2$ which satisfy $\mu_1.\mu_2 = \mu_2.\mu_3 = \mu_3.\mu_1 = -1$. Here is a depiction of three associatives arising from a singular associative:



(2) Two families of embedded closed associative submanifolds, P_t with $0 \neq t \in (-T, T)$ and $P_t^\#$ with $0 < t < T$ in (Y, ϕ_t), converge in the sense of currents to an associative submanifold P with a self-intersection in (Y, ϕ_0), as $t \to 0$. The $P_t^\#$ are diffeomorphic to the connected sums $P_t \# (S^1 \times S^2)$ if P_t are connected, and otherwise to $P_t^+ \# P_t^-$, where $P_t = P_t^+ \sqcup P_t^-$. Here is a depiction of the birth of an associative out of an associative with self intersection:



One of the contributions of this article is to confirm the above two transitions, provided there is an associative submanifold with a Harvey–Lawson T^2-cone singularity (Theorem 1.28) and an associative submanifold with a self-intersection (Theorem 1.35), respectively. More generally, we prove a desingularization theorem (Theorem 1.22) for conically singular associative submanifolds under a certain hypothesis, equation (1.23). This is done in a one-parameter family of coclosed G_2-structures[1] by gluing rescaled asymptotically conical associative submanifolds of $\mathbb{R}^7$. Consequently we derive the first transition (Theorem 1.28) and partially derive the second transition (Theorem 1.35).

Desingularization results of a similar nature have been obtained previously for special Lagrangians in Calabi–Yau manifolds and for coassociatives in G_2-manifolds [Joyce 2004b, 2004c; Lotay 2009, 2014],

[1] These structures are weaker than torsion-free G_2-structures, but torsion-free structures form a special subclass; see Definition 2.5. We work with coclosed rather than closed structures, as the deformation operator for associatives is self-adjoint only in the coclosed case — a property used extensively in this article, and also being used in [Joyce 2018] to define canonical orientations. While closed structures are more suited to compactness issues, [Joyce 2018, Section 2.5] addresses this by introducing the subclass of tamed structures. Since compactness is not relevant here, coclosed structures suffice.

but the associative case poses significantly greater difficulties. In particular, to account for the scaling freedom in the gluing construction, the G_2-structure must vary in a one-parameter family; otherwise, one would produce a one-parameter family of associatives in a fixed G_2-manifold, contradicting the expected dimension zero. This necessity, captured in the hypothesis (1.23), introduces further analytical complexity. Moreover, in contrast to the special Lagrangian and coassociative cases, associative deformations can be obstructed even in the smooth setting, making the problem substantially more delicate.

Another contribution of this article is the study of the deformation theory of conically singular (CS) associative submanifolds. This not only underpins the aforementioned desingularization results but also makes further progress toward the conjectural enumerative theories proposed by Joyce [2018] and Doan and Walpuski [2019] for G_2-manifolds. Specifically, one needs to understand all possible degenerations of closed associative submanifolds that may arise in a generic d-parameter family of coclosed G_2-structures for $d = 0, 1$. It is known in geometric measure theory (Theorem 2.37) that the associative submanifolds can only degenerate into an associative integral current with a singular set of Hausdorff dimension at most 1. However, the precise regularity of this singular set remains unknown. If all the tangent cones are Jacobi integrable multiplicity-1 associative cones with smooth link, then the associative integral current is conically singular with isolated singular points (Definition 4.13). This naturally leads to the following question.

Question 1.1 What are all the possible conically singular (CS) associative submanifolds that may appear in a generic d-parameter family of coclosed G_2-structures with $d = 0, 1$?

To answer this question we study the deformation theory of conically singular associative submanifolds. We explain (Theorem 1.12) that the index of the deformation operator can be expressed completely in terms of a certain nonnegative integer associated to the tangent cones, which we define to be the stability index (Definition 1.5). In particular, CS associative submanifolds having one singularity modeled on cones with stability index 0 and 1 can only appear in a generic zero- and one-parameter family of coclosed G_2-structures, respectively (Theorem 1.17). Therefore we ask the following question.

Question 1.2 What are all the associative cones in $\mathbb{R}^7$ with stability index equal to 0 and 1?

We establish that the Harvey–Lawson T^2-cone and a union of two transverse special Lagrangian planes have stability index 1 (Theorem 1.10). Also we prove with the help of a result obtained by [Haskins 2004a] that all other special Lagrangian cones in $\mathbb{C}^3$ have stability index strictly greater than 1. Furthermore, building on a result of Madnick [2022], we prove that all associative cones in $\mathbb{R}^7$ whose links are null-torsion holomorphic curves in S^6 (see [Bryant 1982]) have stability index strictly greater than 4. In particular, this applies to all associative cones whose link in S^6 is of genus 0 but not a totally geodesic sphere. We also establish that associative cones with link of genus 1 always have stability index greater than or equal to 1. Although these results do not fully resolve Question 1.2, they do rule out a large class of associative cones, narrowing down the possibilities and paving the way for future exploration.

We now provide a summary of the main results established in this article, encompassing the discussions above.

Associative cones

Since our aim is to study the deformation theory of conically singular associative submanifolds — where the associative cones play an important role — we begin in Section 3 with an investigation of associative cones in $\mathbb{R}^7$. These are the cones whose links in (S^6, J) are holomorphic curves; see Section 3. For this purpose, we express the moduli space of holomorphic curves in S^6 locally as the zero set of a nonlinear map (Definition 3.18), as described in the following theorem.

Theorem 1.3 *Let Σ be a closed holomorphic curve in (S^6, J). Then a neighborhood of Σ in the moduli space of holomorphic curves is homeomorphic to the zero-set of a smooth map (often called the obstruction map or Kuranishi map)*

$$\mathrm{ob}_\Sigma : \mathcal{I}_\Sigma \to \mathrm{coker}(\boldsymbol{D}_\Sigma + 2J),$$

where the operator $\boldsymbol{D}_\Sigma$ is defined in equation (3.15), and $\mathcal{I}_\Sigma$ is an open neighborhood of 0 in $\ker(\boldsymbol{D}_\Sigma + 2J)$. Moreover, the index of the deformation operator $\boldsymbol{D}_\Sigma + 2J$ is zero. This deformation operator is related to another standard deformation operator, the normal Cauchy–Riemann operator $\overline{\partial}^N_\nabla$ (Definition 3.7), by a J-antilinear isomorphism $\boldsymbol{\gamma}_\Sigma : \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N\Sigma) \to N\Sigma$ as follows:

$$\boldsymbol{\gamma}_\Sigma \circ \overline{\partial}^N_\nabla = \boldsymbol{D}_\Sigma + 2J.$$

We denote by $\mathcal{M}^{\mathrm{hol}}$ the moduli space of closed holomorphic curves in (S^6, J). Its subset $\mathcal{M}^{\mathrm{hol}}_\bullet$ consists of closed, connected holomorphic curves. This space naturally carries a structure of a real-analytic space, essentially by Theorem 1.3, but it is not necessarily a smooth manifold. In order to establish the results in Theorems 1.12 and 1.17 concerning the moduli space of conically singular associative submanifolds and their transversality properties, we equip $\mathcal{M}^{\mathrm{hol}}_\bullet$ with a *canonical minimal Whitney stratification* in the sense of [Gibson et al. 1976, Chapter I, Sections 1–2], a decomposition

$$\mathcal{M}^{\mathrm{hol}}_\bullet = \bigsqcup_{k \in I} \mathcal{Z}^{(k)}, \tag{1.4}$$

where

- $I \subset \mathbb{Z}_{\geqslant 0}$ is a countable index set corresponding to the dimensions of the strata,
- each stratum $\mathcal{Z}^{(k)}$ is a smooth manifold of dimension k and is preserved under the natural action of the group G_2,
- the strata are pairwise disjoint and collectively form a Whitney stratification,
- the stratification is minimal and canonical: it consists of the fewest strata necessary to satisfy the above properties and is uniquely determined up to local equivalence of stratified spaces.

Such a canonical minimal Whitney stratification exists for any real analytic space, as established in classical results; see, for example, [Hironaka 1973] and [Hardt 1975]. While such stratifications are not necessarily unique globally, any two are locally equivalent as stratified spaces. This guarantees that many of the quantities considered below are independent of the particular choice of the above stratification.

Using any canonical minimal Whitney stratification as in equation (1.4), we associate to each associative cone C in $\mathbb{R}^7$ an integer invariant s-ind(C), referred to as the stability index. It turns out that the negative of this stability index is essentially the virtual dimension of the moduli space of conically singular associative submanifolds modeled on C.

Definition 1.5 (stability index) Let C be an associative cone in $\mathbb{R}^7$. Denote the link by Σ, which is a closed J-holomorphic curve in S^6. Let d_λ be the dimension of homogeneous kernels from Definition 3.20. Let $\Sigma = \bigsqcup_{j=1}^{l} \Sigma_j$ be the decomposition into connected components. Let $\mathcal{Z}_j$, $j = 1, \dots, l$, be the stratum in the decomposition (1.4) containing Σ_j. The *stability index* of C is defined by

$$\text{s-ind}(C) := \frac{d_{-1}}{2} + \sum_{-1<\lambda\leqslant 1} d_\lambda - 7 - \sum_{j=1}^{l} \dim \mathcal{Z}_j.$$

Remark 1.6 Since canonical minimal Whitney stratifications are locally equivalent and $\dim \mathcal{Z}_j$ coincides with the dimension of the tangent space of $\mathcal{Z}_j$ at Σ_j, s-ind(C) is independent of the particular choice of such a stratification. We will see in Remark 3.32 that if C is not a 3-plane then s-ind$(C) \geqslant 0$.

In what follows, we introduce the notions of upper and lower stability indices, which provide effective upper and lower bounds on the stability index.

Definition 1.7 (upper and lower stability-indices) Let C be an associative cone with link components Σ_j as in Definition 1.5. Denote the cone of Σ_j by C_j. Let H_j be the maximal subgroup of G_2 that fixes Σ_j under the standard action of G_2 on S^6. The subgroup H_j is called the *symmetry group* of C_j. We define the *upper stability index* by

$$\text{s-ind}_+(C) := \frac{d_{-1}}{2} + \sum_{-1<\lambda\leqslant 1} d_\lambda - 7 - \sum_{j=1}^{l} (\dim G_2 - \dim H_j),$$

and the *lower stability index* by

$$\text{s-ind}_-(C) := \frac{d_{-1}}{2} + \sum_{-1<\lambda<1} d_\lambda - 7.$$

Remark 1.8 Since d_1 represents the sum of the dimensions of the space of infinitesimal deformations of Σ_j, which definitely contains all the actual deformations induced by the G_2-action, it is larger than the sum of $\dim G_2/H_j$. Furthermore, since each stratum is G_2-invariant, $\mathcal{Z}_j$ always contains $G_2 \cdot \Sigma_j$ and therefore $\dim \mathcal{Z}_j \geqslant \dim G_2/H_j$. Hence,

$$\text{s-ind}_+(C) \geqslant \text{s-ind}(C) \geqslant \text{s-ind}_-(C).$$

Finally, we introduce a natural class of cones that will be the primary focus of this article: the rigid cones.

Definition 1.9 (rigid associative cones) An associative cone C is said to be *rigid* if

$$\text{s-ind}_+(C) = \text{s-ind}_-(C),$$

or equivalently all the infinitesimal deformations of each component of the link are induced by the G_2-action, that is,

$$d_1 = \sum_{j=1}^{l} (\dim G_2 - \dim H_j).$$

Having introduced the necessary definitions, we now present the theorem proved in Section 3.

Theorem 1.10 *Let C be an associative cone in $\mathbb{R}^7$ with link $\Sigma \subset S^6$.*

(i) *If the genus of Σ is 1 then $\text{s-ind}_-(C) \geqslant 1$.*

(ii) *If Σ is a null-torsion holomorphic curve in S^6 (Example 3.4), then*

$$\text{s-ind}_-(C) > 4.$$

In particular, this holds for any holomorphic curve of genus 0 in S^6 which is not a totally geodesic sphere.

(iii) *If C is the Harvey–Lawson T^2-cone (Example 3.3) or a union of two special Lagrangian planes with transverse intersection at the origin (Example 3.2) then it is rigid and*

$$\text{s-ind}(C) = \text{s-ind}_{\pm}(C) = 1.$$

(iv) *If C is a special Lagrangian cone in $\mathbb{C}^3$ that is not a plane, then*

$$\text{s-ind}(C) \geqslant \text{s-ind}_-(C) \geqslant \frac{b^1(\Sigma)}{2} + b^0(\Sigma) - 1 \geqslant 1,$$

with equality if and only if C is one of the cones in part (iii).

Moduli space of conically singular associative submanifolds

Let (Y, ϕ) be a coclosed G_2-manifold (Definition 2.11). We denote by $\mathscr{P}$ the space of all coclosed G_2-structures on Y, and by $\mathscr{P}$ the space of all smooth paths $[0, 1] \to \mathscr{P}$. We consider conically singular (CS) associative submanifolds with isolated singularities at a finite number of points in (Y, ϕ) (Definition 4.13). These singularities are locally modeled on associative cones in $\mathbb{R}^7$. The moduli space of all CS associative submanifolds in (Y, ϕ) is denoted by $\mathcal{M}_{\text{cs}}^{\phi}$ (Definition 5.1). Given a path of coclosed G_2-structures $\boldsymbol{\phi} \in \mathscr{P}$, the one-parameter moduli space of all CS associative submanifolds is denoted by $\mathfrak{M}_{\text{cs}}^{\boldsymbol{\phi}}$ (Definition 5.30). More explicitly,

$$\mathfrak{M}_{\text{cs}}^{\boldsymbol{\phi}} = \{(t, P) : t \in [0, 1],\, P \in \mathcal{M}_{\text{cs}}^{\phi_t}\}.$$

Both these moduli spaces are equipped with the weighted C^∞-topology (Definition 5.12). We will now decompose the moduli spaces $\mathcal{M}_{\text{cs}}^{\phi}$ and $\mathfrak{M}_{\text{cs}}^{\boldsymbol{\phi}}$ as a countable union of moduli subspaces whose deformation theory will be studied.

Definition 1.11 Let $\mathcal{Z}=\prod_{i=1}^m \mathcal{Z}_i$, where $\mathcal{Z}_i=\prod_{j=1}^l \mathcal{Z}_i^j$, with $\mathcal{Z}_i^j$ one of the strata in the decomposition of equation (1.4). We denote by $\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ the set of all CS associative submanifolds with m conical singularities, whose asymptotic cones C_i have links $\Sigma_i=\bigsqcup_{j=1}^l \Sigma_i^j$ with $\Sigma_i^j\in\mathcal{Z}_i^j$. We also define

$$\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}:=\{(t,P):t\in[0,1],\,P\in\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi_t}\}.$$

We can express the moduli spaces $\mathcal{M}_{\mathrm{cs}}^{\phi}$ and $\boldsymbol{\mathcal{M}}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ as a countable union of moduli subspaces, namely

$$\mathcal{M}_{\mathrm{cs}}^{\phi}=\bigcup_{\mathcal{Z}}\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}\quad\text{and}\quad\boldsymbol{\mathcal{M}}_{\mathrm{cs}}^{\boldsymbol{\phi}}=\bigcup_{\mathcal{Z}}\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}.$$

Here the unions run over all possible $\mathcal{Z}=\prod_{i=1}^m\mathcal{Z}_i$, where $\mathcal{Z}_i=\prod_{j=1}^l\mathcal{Z}_i^j$ as above.

We prove the following theorem in Section 5.1 about local Kuranishi models for the above moduli spaces.

Theorem 1.12 *Let $\mathcal{Z}$, $\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ and $\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}$ be as in Definition 1.11. Let $P\in\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ be a conically singular associative submanifold in a coclosed G_2-manifold (Y,ϕ) with m singularities modeled on cones $C_i\in\mathcal{Z}_i$ for $i=1,\dots,m$ (Definition 4.13). Let $\boldsymbol{\phi}:[0,1]\to\mathscr{P}$ be a path of coclosed G_2 structures such that $\phi(t_0)=\phi$. Then there exist open neighborhoods $\widetilde{\mathcal{I}}_{P,\mathcal{Z}}$ and $\overline{\mathcal{I}}_{P,\mathcal{Z}}$ of 0 in $\ker\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ and $\ker\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$, respectively (where $\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ and $\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ are defined in Definitions 5.28 and 5.30, respectively, and μ is chosen as in Definition 5.12) such that*:

(i) *The moduli space $\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ near P is homeomorphic to $\mathrm{ob}_{P,\mathcal{Z}}^{-1}(0)$, the zero set of a smooth map*

$$\mathrm{ob}_{P,\mathcal{Z}}:\widetilde{\mathcal{I}}_{P,\mathcal{Z}}\to\operatorname{coker}\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}.$$

Moreover, $\operatorname{index}\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}=-\sum_{i=1}^m\text{s-ind}(C_i)$.

(ii) *The one-parameter moduli space $\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}$ near (t_0,P) is homeomorphic to $\mathrm{ob}_{t_0,P,\mathcal{Z}}^{-1}(0)$, the zero set of a smooth map*

$$\mathrm{ob}_{t_0,P,\mathcal{Z}}:\overline{\mathcal{I}}_{P,\mathcal{Z}}\to\operatorname{coker}\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}.$$

Moreover, $\operatorname{index}\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}=\operatorname{index}\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}+1=-\sum_{i=1}^m\text{s-ind}(C_i)+1$.

The moduli spaces $\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ and $\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}$ are not always smooth manifolds. The following definition will make these moduli spaces smooth in a generic situation.

Definition 1.13 Let $\mathcal{Z}$, $\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ and $\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}$ be as in Definition 1.11. We define $\mathscr{P}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}}$ to be the subset consisting of all $\phi\in\mathscr{P}$ with the property that for all CS associative $P\in\mathcal{M}_{\mathrm{cs},\mathcal{Z}}^{\phi}$ the linear operator

$$\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}\quad\text{is surjective.}$$

Similarly, we define $\boldsymbol{\mathscr{P}}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}}$ to be the subset consisting of all $\boldsymbol{\phi}\in\boldsymbol{\mathscr{P}}$ with the property that for all $(t_0,P)\in\boldsymbol{\mathcal{M}}_{\mathrm{cs},\mathcal{Z}}^{\boldsymbol{\phi}}$ the linear operator

$$\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}\quad\text{is surjective.}$$

The operators $\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ and $\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ are defined in Definitions 5.28 and 5.30, respectively.

In addition, we define

$$\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}} := \bigcap_{\mathcal{Z}} \mathscr{P}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}} \quad \text{and} \quad \boldsymbol{\mathscr{P}}_{\mathrm{cs}}^{\mathrm{reg}} := \bigcap_{\mathcal{Z}} \boldsymbol{\mathscr{P}}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}}.$$

Here the intersections run over all possible $\mathcal{Z} = \prod_{i=1}^{m} \mathcal{Z}_i$, where $\mathcal{Z}_i = \prod_{j=1}^{l} \mathcal{Z}_i^j$ as in Definition 1.11, and hence are countable intersections.

Remark 1.14 Since the definitions of the operators $\tilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ and $\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ in Definitions 5.28 and 5.30 depend only on the tangent space to the stratum $\mathcal{Z}_i^j$ at Σ_i^j, and not on the full stratum itself, the spaces $\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ and $\boldsymbol{\mathscr{P}}_{\mathrm{cs}}^{\mathrm{reg}}$ are independent of the choice of a canonical minimal Whitney stratification (1.4).

We prove the following theorem about generic transversality of the above moduli spaces in Section 5.2. It also tells us about what type of singularity model cones appear in a generic coclosed G_2-structure, as well as in a generic path of coclosed G_2-structures. Before stating the theorem, let us clarify the notion of generic.

Definition 1.15 Let X be a topological space and let $S \subset X$. The set S is said to be *meager* if it is contained in a countable union of closed subsets with empty interior. The complement of a meager set is called *comeager*.

Remark 1.16 In a completely metrizable space (such as $\mathscr{P}$ or $\boldsymbol{\mathscr{P}}$), the Baire category theorem asserts that meager sets have empty interior, and comeager sets are necessarily dense. Therefore, meager sets are often viewed as nongeneric, while comeager sets are seen as generic.

Theorem 1.17 *Let $\mathcal{Z} = \prod_{i=1}^{m} \mathcal{Z}_i$, where $\mathcal{Z}_i = \prod_{j=1}^{l} \mathcal{Z}_i^j$ as in Definition 1.11. Then the subsets $\mathscr{P}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}}$, $\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ in $\mathscr{P}$, and $\boldsymbol{\mathscr{P}}_{\mathrm{cs},\mathcal{Z}}^{\mathrm{reg}}$, $\boldsymbol{\mathscr{P}}_{\mathrm{cs}}^{\mathrm{reg}}$ in $\boldsymbol{\mathscr{P}}$ are comeager. In particular, the following holds. Let C_i for $i = 1, \ldots, m$ be associative cones in $\mathbb{R}^7$ having links $\Sigma_i = \bigsqcup_{j=1}^{l} \Sigma_i^j$ with $\Sigma_i^j \in \mathcal{Z}_i^j$.*

(i) *If $\sum_{i=1}^{m} \text{s-ind}(C_i) > 0$, then for any coclosed G_2-structure $\phi \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$, the moduli space $\mathcal{M}_{\mathrm{cs}}^{\phi}$ contains no conically singular associative submanifolds having singularities modeled on cones with links in a neighborhood of Σ_i in $\mathcal{Z}_i$.*

(ii) *If $\sum_{i=1}^{m} \text{s-ind}(C_i) > 1$, then for any path of coclosed G_2-structures $\boldsymbol{\phi} \in \boldsymbol{\mathscr{P}}_{\mathrm{cs}}^{\mathrm{reg}}$, the moduli space $\boldsymbol{\mathcal{M}}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ contains no conically singular associative submanifolds having singularities modeled on cones with links in a neighborhood of Σ_i in $\mathcal{Z}_i$.*

Remark 1.18 We conclude from Theorem 1.17 that if $\phi \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$, then the moduli space $\mathcal{M}_{\mathrm{cs}}^{\phi}$ essentially can contain only conically singular associative submanifolds having singularities modeled on cones C with $\text{s-ind}(C) = 0$. Similarly, if $\boldsymbol{\phi} \in \boldsymbol{\mathscr{P}}_{\mathrm{cs}}^{\mathrm{reg}}$, then the moduli space $\boldsymbol{\mathcal{M}}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ essentially can contain only conically singular associative submanifolds having singularities modeled on cones C with $\text{s-ind}(C) = 0$ or 1.

Desingularizations of conically singular associative submanifolds

Let P be a conically singular (CS) associative submanifold in a coclosed G_2-manifold (Y,ϕ), with an isolated singularity modeled on a cone C whose link is $\Sigma \subset S^6$ (Definition 4.13). To desingularize P, we glue rescaled asymptotically conical (AC) associative submanifolds to obtain approximate associative submanifolds, which we then aim to deform into genuine associative submanifolds, referred to as desingularizations. However, there is an obstruction: the deformation operator associated to this approximate one is not surjective due to the freedom of scaling. Due to the self-adjointness, this is equivalent to the fact that its kernel — identified with the matching kernel, consisting of bounded kernel elements on the CS and AC sides that agree to leading order — never vanishes. To compensate for this obstruction, we perform the deformation within a one-parameter family of coclosed G_2-structures under the hypothesis that both the matching kernel and its extension over the family, called the extended matching kernel, are one-dimensional and generated by scaling. Under this assumption, we obtain the desired desingularizations in this one-parameter family.

More precisely, let L be an AC associative submanifold in $\mathbb{R}^7$ with the same asymptotic cone C (Definition 4.2). The deformation operators $\boldsymbol{D}_P$ and $\boldsymbol{D}_L$ are defined in Definition 2.32, and the kernels $\ker \boldsymbol{D}_{P,\lambda}$ and $\ker \boldsymbol{D}_{L,\lambda}$ can be informally described as follows:

$$\ker \boldsymbol{D}_{P,\lambda} := \{u \in C^\infty(NP) : \boldsymbol{D}_P u = 0, u = O(r^\lambda) \text{ as } r \to 0\},$$
$$\ker \boldsymbol{D}_{L,\lambda} := \{u \in C^\infty(NL) : \boldsymbol{D}_L u = 0, u = O(r^\lambda) \text{ as } r \to \infty\}.$$

Definition 1.19 We define the *matching kernel* $\mathcal{K}^{\mathrm{m}}$ by

$$\mathcal{K}^{\mathrm{m}} := \{(u_L, u_P) \in \ker \boldsymbol{D}_{L,0} \oplus \ker \boldsymbol{D}_{P,0} : i_\infty u_L = i_0 u_P\},$$

where i_∞ and i_0 are the asymptotic limit maps $i_{L,0}$ and $i_{P,0}$, respectively, defined in Definition 5.25.

The dilation action of $\mathbb{R}^+$ on $\mathbb{R}^7$ induces a *canonical Fueter section* $\hat{s}_L \in C^\infty(NL)$, which satisfies

$$\boldsymbol{D}_L \hat{s}_L = 0. \tag{1.20}$$

Moreover, $\hat{s}_L$ vanishes at infinity, that is, $i_\infty \hat{s}_L = 0$. As a consequence, the matching kernel $\mathcal{K}^{\mathrm{m}}$ defined in Definition 1.19 always contains the element $(\hat{s}_L, 0)$. This observation will be useful in formulating the hypothesis of the desingularization theorem (Theorem 1.22).

Let $\boldsymbol{\phi} \in \mathscr{P}$ be a path of coclosed G_2-structures on Y and $t_0 \in (0,1)$ such that $\boldsymbol{\phi}(t_0) = \phi$. We set $\phi_t := \boldsymbol{\phi}(t)$. The G_2-structure ϕ_t induces the 4-form ψ_t (Definition 2.3).

Definition 1.21 We define the *extended matching kernel* by

$$\widetilde{\mathcal{K}}^{\mathrm{m}} := \{(u_L, u_P, t) \in \ker \boldsymbol{D}_{L,0} \oplus C^\infty_{P,0} \oplus \mathbb{R} : \boldsymbol{D}_P u_P + t\,\hat{f}_P = 0, i_\infty u_L = i_0 u_P\},$$

where $\hat{f}_P$ is the linearization at t_0 of the nonlinear map (relevant for associative deformations) along the path $\boldsymbol{\phi} \in \mathscr{P}$ with P fixed, defined in Definition 5.30. Here, $C^\infty_{P,0}$ means bounded (i.e., of order $O(r^0)$) smooth normal vector fields on P.

Note that the inclusion $\langle(\hat{s}_L,0)\rangle_{\mathbb{R}} \subset \mathcal{K}^{\mathrm{m}} \subset \widetilde{\mathcal{K}}^{\mathrm{m}}$ always holds. Equality in both of these inclusions implies that the relevant extended deformation operator for the family is surjective. This fact, together with other technical estimates, allows us to establish the following desingularization theorem, that is proved in Section 6.3.

Theorem 1.22 *Let P be a conically singular associative submanifold in a coclosed G_2-manifold (Y,ϕ) with singularity at a single point modeled on a cone C. Let L be an asymptotically conical associative submanifold in $\mathbb{R}^7$ with the same asymptotic cone C and rate $\nu<0$. Let $\boldsymbol{\phi}\in\mathscr{P}$ be a path of coclosed G_2-structures on Y and $t_0\in(0,1)$ be such that $\phi_{t_0}=\phi$. Assume that the matching kernel $\mathcal{K}^{\mathrm{m}}$ and the extended matching kernel $\widetilde{\mathcal{K}}^{\mathrm{m}}$ both are one-dimensional, that is,*

$$\widetilde{\mathcal{K}}^{\mathrm{m}} = \mathcal{K}^{\mathrm{m}} = \langle(\hat{s}_L,0)\rangle_{\mathbb{R}}, \tag{1.23}$$

where $\hat{s}_L$ is the canonical Fueter section from (1.20). Then there exist $\widetilde{\varepsilon}_0>0$, a continuous function $t:[0,\widetilde{\varepsilon}_0)\to[0,1]$ with $t(0)=t_0$, and smooth closed embedded associative submanifolds $\widetilde{P}_{\varepsilon,t(\varepsilon)}$ in $(Y,\phi_{t(\varepsilon)})$ for all $\varepsilon\in(0,\widetilde{\varepsilon}_0)$ such that $\widetilde{P}_{\varepsilon,t(\varepsilon)}\to P$ as $\varepsilon\to 0$ in the sense of integral currents.

As applications of Theorem 1.22, we proceed to desingularize CS associative submanifolds with a Harvey–Lawson T^2-cone singularity and associative submanifolds with a transverse unique self-intersection.

Desingularizations for CS associatives with a Harvey–Lawson T^2-cone singularity Let $\boldsymbol{\phi}\in\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ be a generic path of coclosed G_2-structures on Y (Definition 1.13). Consider a conically singular associative submanifold P in (Y,ϕ_{t_0}) for some $t_0\in(0,1)$, with a singularity modeled on the Harvey–Lawson T^2-cone at a single point (Example 3.3). Since this cone is a rigid special Lagrangian in $\mathbb{C}^3$ with a genus-one link, the genericity of the path implies that P is an isolated point in the moduli space $\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ (Lemma 7.3). Moreover, there exists a normal vector field $\hat{v}_P$ of order $O(r^{-1})$ which spans $\ker \boldsymbol{D}_{P,-1}$ and gives rise to the following nonzero quantity that will be useful in Theorem 1.28:

$$a := \langle \hat{v}_P, \widehat{f}_P\rangle_{L^2} \neq 0, \tag{1.24}$$

where $\widehat{f}_P$ is defined in Definition 5.30. The quantity a essentially represents the first-order obstruction to deform P as a CS associative along the path $\boldsymbol{\phi}$. This also implies that the kernels contributing to both the matching kernel and the extended matching kernel originate entirely from the AC side and consist of elements decaying at the rate $O(r^{-1})$ at infinity. Notably, the AC special Lagrangians employed in the desingularization process always possess a one-dimensional kernel of this decay type, which corresponds precisely to the canonical Fueter section arising from scaling. In summary, the genericity assumption on the path $\boldsymbol{\phi}$ is sufficient to ensure that the hypothesis (1.23) required in the desingularization Theorem 1.22 is satisfied.

There are three AC special Lagrangians in $\mathbb{C}^3$, $L^i := L_1^i$ for $i=1,2,3$ from Example 4.10, asymptotic to the Harvey–Lawson T^2-cone, which will be used to construct three one-parameter families of desingularizations. However, to realize the first transition discussed at the beginning of this article, we need to

impose additional transversality conditions. These restrict the choice of the path of G_2-structures $\boldsymbol{\phi}$ to a subset $\mathscr{P}^{\bullet}$ of $\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$, which we expect remains comeager — so the choice of $\boldsymbol{\phi}$ is still generic, though a proof is not included in this article. More precisely, we assume that $\hat{v}_P$ appearing in equation (1.24) is transverse, in leading order, to each L^i. The following definition formalizes this assumption.

Definition 1.25 $\mathscr{P}^{\bullet} \subset \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ consists of all $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ such that for all CS associative submanifolds P in $\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ with Harvey–Lawson T^2-cone singularity at a single point, the asymptotic limit of $\hat{v}_P$ appearing in equation (1.24) has the form

$$i_{P,-1}\hat{v}_P = b_1\xi_1 + b_2\xi_2, \quad \text{with } b_1 \neq 0,\ b_2 \neq 0,\ b_1 \neq b_2, \tag{1.26}$$

where ξ_1 and ξ_2 denote the leading order $O(r^{-1})$ terms in the expansions of the AC special Lagrangians L_1^1 and L_1^2, respectively, as given in equation (4.11) of Example 4.10. The quantities b_1, b_2 and $b_1 - b_2$ essentially represent the first-order obstructions to desingularizations of P by gluing L_1^1, L_1^2 and L_1^3 within the fixed G_2-structure ϕ_{t_0}.

Remark 1.27 The ratios b_1/a and b_2/a do not depend on the particular choice of $\hat{v}_P$.

We now state the desingularization theorem for conically singular associatives with Harvey–Lawson T^2-cone singularity at a single point, that is proved in Section 7.1.

Theorem 1.28 *Let $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ be a path of coclosed G_2-structures on Y and $t_0 \in (0,1)$. Let P be a conically singular associative submanifold of (Y, ϕ_{t_0}) in $\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ with Harvey–Lawson T^2-cone singularity at a single point x. There exist $\widetilde{\varepsilon}_0 > 0$, three continuous functions $t^i : [0, \widetilde{\varepsilon}_0) \to [0,1]$ with $t^i(0) = t_0$ for $i = 1,2,3$, and smooth closed embedded associative submanifolds $\widetilde{P}_{\varepsilon,t^i(\varepsilon)}$ in $(Y, \phi_{t^i(\varepsilon)})$ for all $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ such that $\widetilde{P}_{\varepsilon,t^i(\varepsilon)} \to P$ as $\varepsilon \to 0$ in the sense of integral currents. The $\widetilde{P}_{\varepsilon,t^i(\varepsilon)}$ are diffeomorphic to the Dehn filling of $P^o := P \setminus B_\varepsilon(x)$ along simple closed curves $\mu_i \subset \partial P^o \cong T^2$ that satisfy $\mu_1.\mu_2 = \mu_2.\mu_3 = \mu_3.\mu_1 = -1$. Furthermore, if $\boldsymbol{\phi} \in \mathscr{P}^{\bullet}$, then there is a constant $c \neq 0$ such that*

$$t^1(\varepsilon) = t_0 - \frac{cb_2}{a}\varepsilon^2 + o(\varepsilon^2), \quad t^2(\varepsilon) = t_0 + \frac{cb_1}{a}\varepsilon^2 + o(\varepsilon^2), \quad t^3(\varepsilon) = t_0 + \frac{c(b_2 - b_1)}{a}\varepsilon^2 + o(\varepsilon^2). \tag{1.29}$$

Remark 1.30 We would like to acknowledge that our derivation of the leading-order expression in equation (1.29) is influenced by [Joyce 2018, Remark 5.4(a)].

Desingularizations for associative submanifolds with transverse intersection Let $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ be a generic path of coclosed G_2-structures on Y. Consider an associative submanifold P in (Y, ϕ_{t_0}) for some $t_0 \in (0,1)$, exhibiting a unique transverse self-intersection. More precisely, P is a conically singular associative submanifold with a singularity at a single point modeled on the union of two transverse associative planes $\Pi_{\pm} \subset \mathbb{R}^7$ and there exists $B \in G_2$ such that $B\Pi_0 = \Pi_+$, $B\Pi_\theta = \Pi_-$, and

$$\mathbb{R}^7 = \langle \boldsymbol{n} \rangle_{\mathbb{R}} \oplus B\Pi_0 \oplus B\Pi_\theta, \tag{1.31}$$

which is an orientation-compatible splitting as described in Example 3.33. To resolve this intersection, we aim to glue in a Lawlor neck — an asymptotically conical (AC) special Lagrangian submanifold asymptotic to the planes $\Pi_{\pm}$ (Example 4.7).

Unlike the case of desingularizations for CS associative submanifolds with a Harvey–Lawson T^2-cone singularity, the genericity assumption $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ does not guarantee the absence of nontrivial elements in the extended matching kernel on the CS side. That is, a family of intersecting associative submanifolds may persist along the path $\boldsymbol{\phi}$, potentially increasing the dimension of the extended matching kernel and thereby violating the hypothesis (1.23) required for the desingularization result of Theorem 1.22. To address this issue, we restrict our choice of the path of G_2-structures $\boldsymbol{\phi}$ to a subset $\mathscr{P}^{\dagger}$ of $\mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ consisting of those paths for which associative submanifolds with a transverse, unique self-intersection are unobstructed as immersed associatives. This ensures that P can be deformed into a family of embedded, closed associatives that separate the two sheets of the self-intersection along the path. As a consequence, any nontrivial elements in the extended matching kernel on the CS side must have different components in the direction perpendicular to the union of the two tangent planes $\Pi_{\pm}$ at the intersection point. However, any elements in the extended matching kernel arising from the AC special Lagrangian (Lawlor neck) side always have vanishing perpendicular components. Therefore, for this restricted subset of paths, the hypothesis (1.23) is satisfied. We note that this subset is still expected to be comeager, so the choice of $\boldsymbol{\phi}$ remains generic, although a formal proof of this fact is not included in the present article. The following definition formalizes this restriction.

Definition 1.32 $\mathscr{P}^{\dagger} \subset \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ consists of all $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ such that along it, all associative submanifolds P with transverse unique self-intersection are (rigid) unobstructed as immersed associative submanifolds. In other words, $\ker \boldsymbol{D}_{P,0} = \{0\}$ and there exists $\widehat{v}_P \in C^{\infty}_{P,0}$ such that

$$\boldsymbol{D}_P \widehat{v}_P = \widehat{f}_P \quad \text{and} \quad a := (\widehat{v}_P^{+}(0) - \widehat{v}_P^{-}(0)) \cdot \boldsymbol{n} \neq 0, \tag{1.33}$$

where $\widehat{v}_P^{\pm}(0) \in \Pi_{\pm}^{\perp}$ are the asymptotic limits of $\widehat{v}_P^{\pm}$ (Lemma 4.31 and Example 3.33), and $\widehat{f}_P$ is from Definition 5.30.

The above implies that if $\boldsymbol{\phi} \in \mathscr{P}^{\dagger}$ and P is an associative submanifold in (Y, ϕ_{t_0}) for some $t_0 \in (0,1)$ with a transverse unique self-intersection then there exists a family of immersed closed associative submanifolds

$$\{P_t : |t - t_0| \ll 1,\ P_{t_0} = P,\ P_t \text{ is an embedded closed associative in } (Y, \phi_t) \text{ for all } t \neq t_0\}. \tag{1.34}$$

The quantity a essentially measures the speed at which the two local sheets of P_t move across each other near the crossing point as the parameter t passes through t_0.

We obtain the following theorem, whose proof is given in Section 7.2.

Theorem 1.35 *Let $\boldsymbol{\phi} \in \mathscr{P}^{\dagger}$ be a path of coclosed G_2-structures on Y and take $t_0 \in (0,1)$. Let P be an associative submanifold of (Y, ϕ_{t_0}) with a transverse unique self-intersection at x. There exist $\widetilde{\varepsilon}_0 > 0$, a continuous function $t : [0, \widetilde{\varepsilon}_0) \to [0,1]$ with $t(0) = t_0$, and smooth closed embedded associative submanifolds $\widetilde{P}_{\varepsilon, t(\varepsilon)}$ in $(Y, \phi_{t(\varepsilon)})$ for all $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ such that $\widetilde{P}_{\varepsilon,t(\varepsilon)} \to P$ as $\varepsilon \to 0$ in the sense of integral currents. The $\widetilde{P}_{\varepsilon,t(\varepsilon)}$ are diffeomorphic to the connected sums $P_t \# (S^1 \times S^2)$ if P_t is connected, and otherwise to $P_t^{+} \# P_t^{-}$, where $P_t = P_t^{+} \amalg P_t^{-}$. Here P_t is from equation (1.34) with $t \neq t_0$.*

Remark 1.36 The condition $\boldsymbol{\phi} \in \mathscr{P}^{\dagger}$ suffices to establish the desingularization result Theorem 1.35 following Theorem 1.22, but realizing the second transition discussed at the beginning of this article requires additional transversality conditions. These impose further restrictions on the path of G_2-structures $\boldsymbol{\phi}$, confining it to a subset of $\mathscr{P}^{\dagger}$, which we denote by $\mathscr{P}^{\ddagger}$. We expect this subset to remain comeager — so the choice of $\boldsymbol{\phi}$ remains generic — though we do not provide a proof here. More precisely, define $\mathscr{P}^{\ddagger}$ to consist of those $\boldsymbol{\phi} \in \mathscr{P}^{\dagger}$ such that for every associative submanifold P (along the path) with a transverse, unique self-intersection, there exists $\widehat{u}_P \in \ker \boldsymbol{D}_{P,-2}$ satisfying

$$i_{P,-2}\widehat{u}_P = (B\xi^+, B\xi^-) \quad \text{and} \quad b := (\widehat{u}_P - B\xi^+)(0)\cdot \boldsymbol{n} - (\widehat{u}_P - B\xi^-)(0)\cdot \boldsymbol{n} \neq 0, \tag{1.37}$$

where $\xi^{\pm}$ denote the leading-order $O(r^{-2})$ terms over $\Pi_{\pm}$ for the Lawlor neck defined in equation (4.8) of Example 4.7, and $B \in G_2$ is as above. The quantity b essentially represents the first-order obstruction to deforming P into an embedded, closed associative submanifold within the fixed G_2-structure ϕ_{t_0}. Note also that choosing $-\boldsymbol{n}$ instead of $\boldsymbol{n}$ swaps Π_+ and Π_-, ensuring that the definitions of a and b remain well-defined. As we will discuss in Remark 7.9, following [Joyce 2018, Remark 4.5(a)], if $\boldsymbol{\phi} \in \mathscr{P}^{\ddagger}$, one would expect the existence of a constant $c \neq 0$ such that

$$t(\varepsilon) = t_0 - \frac{cb}{a}\varepsilon^3 + o(\varepsilon^3). \tag{1.38}$$

Establishing this expansion would confirm the second transition described at the beginning of the article. However, despite our efforts, we have not succeeded in proving equation (1.38), as explained in more detail in Remark 7.9. We would like to highlight that although Nordström [2013] previously established Theorem 1.35, he also did not derive (1.38). While we refine Nordström's argument by constructing an improved approximate desingularization that yields a smaller pre-gluing error, our analysis still lacks the precise estimates required to rigorously justify the expansion in equation (1.38). We hope that future refinements of the techniques developed in this article will lead to a complete proof. For this reason, as mentioned at the outset, our results concerning the second transition should be regarded as partial.

2 Preliminaries: G_2-manifolds and associative submanifolds

In this section we review definitions and basic facts about G_2-manifolds, more generally almost G_2-manifolds and associative submanifolds, which are important to understand this article. To delve further into these topics, we refer the reader to [Salamon and Walpuski 2017; Joyce 2007; Karigiannis and Lotay 2020; Harvey 1990] and other relevant sources mentioned throughout the discourse.

2.1 G_2-manifolds

This subsection reviews definitions and basic facts about G_2-manifolds.

Definition 2.1 The group G_2 is the automorphism group of the normed division algebra of octonions $\mathbb{O}$, that is,

$$G_2 := \mathrm{Aut}(\mathbb{O}) \subset \mathrm{SO}(7).$$

This is a simple, compact, connected, simply connected Lie group of dimension 14. Furthermore, there exists a fibration $\mathrm{SU}(3) \hookrightarrow G_2 \to S^6$. Writing $\mathbb{O} = \mathrm{Re}\,\mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \cong \mathbb{R} \oplus \mathbb{R}^7$, we define the *cross-product* $\times : \Lambda^2\mathbb{R}^7 \to \mathbb{R}^7$ by

$$(u, v) \mapsto u \times v := \mathrm{Im}(uv).$$

The 3-*form* $\phi_e \in \Lambda^3(\mathbb{R}^7)^*$ is defined by

$$\phi_e(u, v, w) := g_e(u \times v, w),$$

where $g_e : S^2(\mathbb{R}^7) \to \mathbb{R}$ is $g_e(u, v) = -\mathrm{Re}(uv)$, the standard Euclidean metric on $\mathbb{R}^7$. These are related by the important identity

$$\iota_u\phi_e \wedge \iota_v\phi_e \wedge \phi_e = 6g_e(u, v)\,\mathrm{vol}_{g_e}. \tag{2.2}$$

More explicitly, $\{e^1, \dots, e^7\}$ is an oriented orthonormal frame on $\mathbb{R}^7$ such that

$$\phi_e = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356},$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$. The Lie group G_2 can also be expressed as

$$G_2 := \{A \in \mathrm{GL}(\mathbb{R}^7) : A^*\phi_e = \phi_e\}.$$

There is also a 4-*form* $\psi_e := *_{g_e}\phi_e \in \Lambda^4(\mathbb{R}^7)^*$, which also can be defined by

$$\psi_e(u, v, w, z) := g_e([u, v, w], z),$$

where $[\cdot,\cdot,\cdot] : \Lambda^3(\mathbb{R}^7) \to \mathbb{R}^7$ is the *associator*, defined as

$$[u, v, w] := (u \times v) \times w + \langle v, w\rangle u - \langle u, w\rangle v.$$

Definition 2.3 A G_2*-structure* on a 7-dimensional manifold Y is a principal G_2-bundle over Y which is a reduction of the frame bundle $\mathrm{GL}(Y)$.

An *almost* G_2*-manifold* is a 7-dimensional manifold Y equipped with a G_2-structure or, equivalently, equipped with a *definite* 3-form $\phi \in \Omega^3(Y)$, that is, the bilinear form $G_\phi : S^2TY \to \Lambda^7(T^*Y)$ defined by

$$G_\phi(u, v) := \iota_u\phi \wedge \iota_v\phi \wedge \phi$$

is definite.

A G_2-structure ϕ on Y defines uniquely a Riemannian metric g_ϕ and a volume form vol_{g_ϕ} on Y satisfying the identity (2.2). Moreover, it defines

- a *cross product* $\times : \Lambda^2(TY) \to TY$,
- an *associator* $[\cdot,\cdot,\cdot] : \Lambda^3(TY) \to TY$,
- a 4-*form* $\psi := *_{g_\phi}\phi \in \Omega^4(Y)$.

Remark 2.4 A 7-dimensional manifold is an almost G_2-manifold if and only if it is spin; see [Gray 1969, Theorems 3.1–3.2]. A G_2-structure is also equivalent to a choice of a *definite* 4-form $\psi \in \Omega^4(Y)$ and an orientation on Y; see [Hitchin 2001, Section 8.4]. Here ψ being definite means that the bilinear form $G_\psi : S^2T^*Y \to \Lambda^7(T^*Y) \otimes \Lambda^7(T^*Y)$ defined by

$$G_\psi(u, v) := \iota_u\psi \wedge \iota_v\psi \wedge \psi$$

is definite. In this definition, the 4-form ψ is considered as a section of $\Lambda^3(TY) \otimes \Lambda^7(T^*Y) \cong \Lambda^4(T^*Y)$, and the contraction $\iota_u\psi$ is a section of $\Lambda^2(TY) \otimes \Lambda^7(T^*Y)$.

Definition 2.5 A *G_2-manifold* is a 7-dimensional manifold Y equipped with a torsion-free G_2-structure, that is, equipped with a definite 3-form $\phi \in \Omega^3(Y)$ such that $\nabla_{g_\phi}\phi = 0$ or, equivalently,

$$d\phi = 0 \quad \text{and} \quad d\psi = 0.$$

The equivalence in the above definition was established by [Fernández and Gray 1982, Theorem 5.2].

Example 2.6 $(\mathbb{R}^7, \phi_e)$ is a G_2-manifold.

Example 2.7 Let $(X, \omega_I, \omega_J, \omega_K)$ be a hyperkähler 4-manifold. The manifolds $\mathbb{R}^3 \times X$ and $T^3 \times X$ are G_2-manifolds with G_2-structure

$$\phi := dt^1 \wedge dt^2 \wedge dt^3 - dt^1 \wedge \omega_I - dt^2 \wedge \omega_J - dt^3 \wedge \omega_K,$$

where (t^1, t^2, t^3) are the coordinates of $\mathbb{R}^3$.

Example 2.8 Let (Z, ω, Ω) be a Calabi–Yau 3-fold, where ω is a Kähler form and Ω is a holomorphic volume form on Z satisfying

$$\frac{\omega^3}{3!} = -\Big(\frac{i}{2}\Big)^3 \Omega \wedge \bar{\Omega}.$$

The product with the unit circle, $Y := S^1 \times Z$, is a G_2-manifold with the G_2-structure

$$\phi := dt \wedge \omega + \operatorname{Re}\Omega, \quad \psi = \tfrac{1}{2}\omega \wedge \omega + dt \wedge \operatorname{Im}\Omega,$$

where t denotes the coordinate on S^1. In this case, the holonomy group satisfies $\operatorname{Hol}(Y, g_\phi) \subset \mathrm{SU}(3)$.

Remark 2.9 Any G_2-manifold (Y, ϕ) admits a nowhere vanishing parallel spinor and therefore the metric g_ϕ is Ricci-flat; see [Lawson and Michelsohn 1989, page 321] and [Hitchin 1974, Theorem 1.2]. A compact G_2-manifold Y has holonomy exactly equal to G_2 if and only if $\pi_1(Y)$ is finite [Joyce 2007, Proposition 11.2.1].

Example 2.10 Bryant and Salamon [Bryant 1987; Bryant and Salamon 1989] first constructed local and complete manifolds with holonomy equal to G_2, respectively. Joyce [1996b] first constructed compact manifolds with holonomy equal to G_2 by smoothing flat T^7/Γ, where Γ is a finite group of isometries of T^7. This was generalized later by Joyce and Karigiannis [2021]. Kovalev [2003] introduced the twisted connected sum construction of G_2-manifolds, which glues a suitable matching pair of asymptotically

cylindrical G_2-manifolds. This construction was later improved by Kovalev and Lee [2011] and Corti, Haskins, Nordström and Pacini [Corti et al. 2015] to produce hundreds of thousands of examples of compact manifolds with holonomy equal to G_2.

The moduli space of torsion free G_2-structures over Y is a smooth manifold of dimension $b^3(Y)$ by [Joyce 1996a, Part I, Theorem C], therefore it is not enough to achieve transversality for various enumerative theories. To address this, one must consider an infinite-dimensional space of G_2-structures as in [Donaldson and Segal 2011, Section 3.2] and [Joyce 2018, Section 2.5], namely the following.

Definition 2.11 A G_2-structure ϕ is called a *coclosed G_2-structure* if $d\psi = 0$, where $\psi := *_{g_\phi}\phi$. An almost G_2-manifold (Y, ϕ) is called a *coclosed G_2-manifold* if ϕ is a coclosed G_2-structure. A G_2-structure ϕ is called *tamed* by a closed 3-form $\tau \in \Omega^3(Y)$ if for all $x \in Y$ and $u, v, w \in T_xY$ with $[u, v, w] = 0$ and $\phi(u, v, w) > 0$, we have $\tau(u, v, w) > 0$.

Remark 2.12 The definition of the above tamed G_2-structures can be ignored for this article; they are used only in Theorem 2.37 to bound the volume of associative submanifolds for compactness. This restricted class of G_2-structures has been employed to construct well-defined enumerative theories in [Joyce 2018] and [Doan and Walpuski 2019]. However, for the purposes of this article, such a restriction is not required.

Example 2.13 Nearly parallel G_2-manifolds, that is, coclosed G_2-manifolds (Y, ϕ) satisfying $d\phi = \lambda\psi$ for some constant $\lambda \in \mathbb{R}$, are examples of coclosed G_2-manifolds.

2.2 Associative submanifolds

In any almost G_2-manifold, we can consider a special class of 3-dimensional calibrated submanifolds, called associative submanifolds, which are the main objects of study in this article. These were first invented by [Harvey and Lawson 1982].

Definition 2.14 Let (Y, ϕ) be an almost G_2-manifold. A 3-dimensional oriented submanifold P of Y is called an *associative submanifold* if it is semicalibrated by the 3-form ϕ, that is, $\phi_{|P}$ is the volume form vol_{P,g_ϕ} on P or, equivalently, the associator $[u, v, w] = 0$ for all $x \in P$ and $u, v, w \in T_xP$.

Remark 2.15 If ϕ is a calibration (i.e., $d\phi = 0$) and P is compact, then it is a minimal submanifold and volume-minimizing in its homology class [Harvey and Lawson 1982, Theorem 4.2]. The equivalence in the above definition follows from the identity: $|u \wedge v \wedge w|^2 = \phi(u, v, w)^2 + |[u, v, w]|^2$; see [Harvey and Lawson 1982, Section IV, Theorem 1.6].

Example 2.16 In Example 2.7, $\mathbb{R}^3$ and T^3 are associative submanifolds.

Example 2.17 Let Z be a Calabi–Yau 3-fold and $S^1 \times Z$ be the G_2-manifold as in Example 2.8. For any holomorphic curve $\Sigma \subset Z$ and special Lagrangian $L \subset Z$ (i.e., L is calibrated by $\mathrm{Re}\,\Omega$), the 3-dimensional submanifolds $S^1 \times \Sigma$ and $\{e^{i\theta}\} \times L$, with $e^{i\theta} \in S^1$, are associative submanifolds of $S^1 \times Z$.

Example 2.18 Joyce [1996a, Section 4.2] has produced examples of closed associative submanifolds which are the fixed point loci of G_2-involutions in his generalized Kummer constructions. Recently, new examples of associative submanifolds in these G_2-manifolds have been constructed by Dwivedi, Platt and Walpuski [Dwivedi et al. 2023].

Example 2.19 Examples of closed associative submanifolds in the twisted connected sum (TCS) G_2-manifolds were first constructed by Corti, Haskins, Nordström and Pacini [Corti et al. 2015, Sections 5 and 7.2.2] from closed holomorphic curves and closed special Lagrangians in asymptotically cylindrical (ACyl) Calabi–Yau submanifolds. The author in [Bera 2022] has constructed more examples of closed associative submanifolds in the twisted connected sum G_2-manifolds. These are obtained from ACyl holomorphic curves and ACyl special Lagrangians in ACyl Calabi–Yau submanifolds using a gluing construction. It is also expected that this gluing construction can be used to produce infinitely many closed associative submanifolds in a certain TCS G_2-manifold studied by Braun, Del Zotto, Halverson, Larfors, Morrison and Schäfer-Nameki [Braun et al. 2018].

Remark 2.20 Examples of associative submanifolds of nearly parallel G_2-manifolds have been constructed by Lotay [2012] in S^7, Kawai [2015] in the squashed S^7 and Ball and Madnick [2023] in the Berger space.

2.3 Normal bundles and canonical isomorphisms

This subsection sets up our conventions for the normal bundle of a submanifold, the tubular neighborhood map, and various canonical isomorphisms, which will be used extensively throughout the article to describe the deformation theories of associative submanifolds and holomorphic curves.

Definition 2.21 (normal bundle) Let Y be a manifold and M be a submanifold of it. The *normal bundle* $\pi : NM \to M$ is characterized by the exact sequence

$$0 \to TM \to TY_{|M} \to NM \to 0. \tag{2.22}$$

Definition 2.23 (tubular neighborhood map) A *tubular neighborhood map* of M is a diffeomorphism between an open neighborhood V_M of the zero-section of the normal bundle NM of M that is convex in each fiber, and an open neighborhood U_M (tubular neighborhood) of M in Y,

$$\Upsilon_M : V_M \to U_M,$$

that takes the zero section 0 to M, and is such that the composition

$$NM \to 0^*TNM \xrightarrow{d\Upsilon_M} TY_{|M} \to NM$$

is the identity.

Definition 2.24 (canonical extension of normal vector fields) The tangent bundle TNM fits into the exact sequence

$$0 \to \pi^*NM \xrightarrow{i} TNM \xrightarrow{d\pi} \pi^*TM \to 0.$$

This induces an *canonical extension map* extending normal vector fields on M to vector fields on NM:

$$\widetilde{\bullet}: C^\infty(NM) \hookrightarrow \mathrm{Vect}(NM), \quad u \mapsto \widetilde{u} := i(\pi^* u).$$

Here $\pi^* u \in C^\infty(\pi^* NM)$ is the pullback section.

Notation 2.25 There are instances in this article where it is more appropriate to use the notation $\widetilde{u}$, but for simplicity, we will abuse notation and denote it by u.

Remark 2.26 The fact that $[\widetilde{u}, \widetilde{v}] = 0$ for all $u, v \in C^\infty(NM)$ will be useful later.

Definition 2.27 (canonical isomorphisms) For any section $u \in C^\infty(NM)$, we define the *graph of* u by

$$\Gamma_u := \{(x, u(x)) \in NM : x \in M\}.$$

This is a submanifold of NM, and the bundle $\pi^* NM_{|\Gamma_u}$ fits into the split exact sequence

$$0 \to T\Gamma_u \xrightarrow{\quad} TNM_{|\Gamma_u} \xrightarrow{\mathbf{1}-d(u\circ\pi)} \pi^* NM_{|\Gamma_u} \to 0. \tag{2.28}$$

($d(u\circ\pi)$: $T\Gamma_u \to TNM_{|\Gamma_u}$)

In particular, this induces a canonical isomorphism $N\Gamma_u \cong \pi^* NM_{|\Gamma_u}$. Moreover, the composition $T\Gamma_u \to TNM_{|\Gamma_u} \xrightarrow{d\pi} \pi^* TM_{|\Gamma_u}$ is an isomorphism. Let $\Upsilon_M : V_M \to U_M$ be a tubular neighborhood map of M. We define $C^\infty(V_M) := \{u \in C^\infty(NM) : \Gamma_u \subset V_M\}$. Let $u \in C^\infty(V_M)$. Denote by M_u the submanifold $\Upsilon_M(\Gamma_u)$ of Y. Then there is a *canonical bundle isomorphism*

$$\begin{array}{ccc} NM & \xrightarrow{\Theta_u^M} & NM_u \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Upsilon_M \circ u} & M_u \end{array} \tag{2.29}$$

induced by the commutative diagram of bundle isomorphisms

$$\begin{array}{ccc} NM & \xrightarrow{\Theta_u^M} & NM_u \\ {\scriptstyle \pi^*}\uparrow \downarrow{\scriptstyle u^*} & & {\scriptstyle d\Upsilon_M}\uparrow \\ \pi^* NM_{|\Gamma_u} & \xleftarrow[\mathbf{1}-d(u\circ\pi)]{} & N\Gamma_u \end{array}$$

Definition 2.30 (normal connection) The choice of a Riemannian metric g on Y induces a splitting of the exact sequence (2.22), that is,

$$TY_{|M} = TM \perp NM.$$

Denote by $\cdot^{\parallel}$ and $\cdot^{\perp}$ the projections onto the first and second summands, respectively. The Levi-Civita connection ∇ on $TY_{|M}$ decomposes as

$$\nabla = \begin{bmatrix} \nabla^{\parallel} & -\mathrm{II}^* \\ \mathrm{II} & \nabla^{\perp} \end{bmatrix}.$$

Here $\mathrm{II} \in \mathrm{Hom}(S^2TM, NM)$ is the *second fundamental form* of M, $\nabla^{\|}$ is the Levi-Civita connection on M, and $\nabla^{\perp}$ is the *normal connection* on NM.

2.4 Moduli space of closed associative submanifolds

This subsection reviews established results on the moduli space of closed associative submanifolds and motivates the study of conically singular associative submanifolds, which is the focus of this article.

Definition 2.31 Let (Y, ϕ) be an almost G_2-manifold. Let $\mathcal{S}_3$ be the set of all 3-dimensional oriented, closed smooth submanifolds of Y. We define the *C^k-topology* on the set $\mathcal{S}_3$ by specifying a basis, which is a collection of all the sets of the form $\{\Upsilon_P(\Gamma_u) : u \in \mathcal{V}_P^k\}$, where $P \in \mathcal{S}_3$, Υ_P is a tubular neighborhood map of P, and $\mathcal{V}_P^k$ is an open set in $C^\infty(V_P)$ whose topology is induced by the C^k-norm on $C^\infty(NP)$. The *C^∞-topology* on the set $\mathcal{S}_3$ is the inverse limit topology of C^k-topologies on it, that is, a set is open in the C^∞-topology if it is open for every C^k-topology.

The *moduli space* $\mathcal{M}^\phi$ of closed associative submanifolds in (Y, ϕ) is the subset of all submanifolds in $\mathcal{S}_3$ which are associatives. The C^∞-topology on $\mathcal{M}^\phi$ is the subspace topology of $\mathcal{S}_3$.

Let $\mathscr{P}$ be the set of all coclosed G_2-structures on Y. Denote the *universal moduli space* of closed associative submanifolds by

$$\mathcal{M} := \{(\phi, P) \in \mathscr{P} \times \mathcal{S}_3 : P \in \mathcal{M}^\phi\}.$$

Equip $\mathscr{P}$ with the C^∞-topology. The topology of $\mathcal{M} \subset \mathscr{P} \times \mathcal{S}_3$ is then given by the subspace topology of the product C^∞-topologies.

Let $\boldsymbol{\mathscr{P}}$ be the space of paths $\boldsymbol{\phi} : [0, 1] \to \mathscr{P}$ which are smooth as sections over $[0, 1] \times Y$. Set $\phi_t := \boldsymbol{\phi}(t)$. Define the *one-parameter moduli space* of closed associative submanifolds by the fiber product

$$\boldsymbol{\mathcal{M}^\phi} := [0, 1] \times_{\mathscr{P}} \mathcal{M} \cong \{(t, P) \in [0, 1] \times \mathcal{S}_3 : P \in \mathcal{M}^{\phi_t}\}.$$

The topology on $\boldsymbol{\mathcal{M}^\phi}$ is the fiber product topology, which is same as the subspace topology of the product topology of $[0, 1] \times \mathcal{S}_3$.

Definition 2.32 Let M be an associative submanifold (compact or noncompact) of an almost G_2-manifold (Y, ϕ). The operator $\boldsymbol{D}_M : C^\infty(NM) \to C^\infty(NM)$ is defined by

$$\langle \boldsymbol{D}_M v, w \rangle_{L^2} := \int_M \left\langle \sum_{i=1}^3 e_i \times \nabla^{\perp}_{M, e_i} v, w \right\rangle + \int_M \iota_w \nabla_v \psi$$

for all $v \in C^\infty(NM)$ and $w \in C^\infty_c(NM)$. Here NM is the normal bundle of M and $\nabla^{\perp}_M$ is the normal connection, and $\{e_1, e_2, e_3\}$ is an oriented local orthonormal frame for TM with respect to the metric g_ϕ.

If (Y, ϕ) is a G_2-manifold then $\nabla\psi = 0$ and $\boldsymbol{D}_M$ is a Dirac operator, called the *Fueter operator*. A closed associative submanifold M of a coclosed G_2-manifold is called *rigid*, or equivalently, *unobstructed*, if $\ker \boldsymbol{D}_M = \{0\}$.

Remark 2.33 The operator $\boldsymbol{D}_M$ is formally self-adjoint if $d\psi = 0$ (Proposition 5.20). This operator is the linearization of a nonlinear map $\mathfrak{F}_M^\phi$ which controls the deformation theory of associative submanifolds (Proposition 5.18). Let $\Upsilon_M : V_M \to U_M \subset Y$ be a tubular neighborhood map. The map $\mathfrak{F}_M^\phi : C^\infty(V_M) \to C^\infty(NM)$ is defined by

$$\langle \mathfrak{F}_M^\phi u, w\rangle_{L^2} = \int_{\Gamma_u} \iota_w \Upsilon_M^* \psi \quad \text{for } u \in C^\infty(V_M) \text{ and } w \in C_c^\infty(NM). \tag{2.34}$$

The notation w in the integrand is the extension vector field of $w \in C^\infty(NM)$ in the tubular neighborhood as in Notation 2.25. The associative submanifolds can also be thought of as critical points of a functional on the space of submanifolds; see [Donaldson and Thomas 1998, Section 8] and [Doan and Walpuski 2019, Section 2.3]. The differential of this functional is a 1-form which is locally of the form $\mathfrak{F}_M^\phi$. This led Doan and Walpuski [2019] to propose constructing Floer homology groups whose chain complex is generated by associative submanifolds.

The following theorem summarizes the deformation theory of closed associative submanifolds, as established in the literature.

Theorem 2.35 [McLean 1998, Theorem 5.2; Joyce 2018, Theorem 2.12; Doan and Walpuski 2019, Theorem 2.20, Proposition 2.23, Section 2.7] *Let (Y, ϕ) be a coclosed G_2-manifold.*

(i) *Let P be an associative submanifold of (Y, ϕ). Then there exists an open neighborhood $\mathcal{I}_P$ of 0 in* $\ker \boldsymbol{D}_P$ *such that the moduli space $\mathcal{M}^\phi$ near P is homeomorphic to* $\mathrm{ob}_P^{-1}(0)$, *the zero set of a smooth map (obstruction map/Kuranishi map)*

$$\mathrm{ob}_P : \mathcal{I}_P \to \operatorname{coker} \boldsymbol{D}_P.$$

Moreover, there is a comeager subset $\mathscr{P}^{\mathrm{reg}} \subset \mathscr{P}$ such that for all $\phi \in \mathscr{P}^{\mathrm{reg}}$, the moduli space $\mathcal{M}^\phi$ is a 0-dimensional manifold and consists only of unobstructed (rigid) closed associative submanifolds.

(ii) *There is a comeager subset $\mathscr{P}^{\mathrm{reg}} \subset \mathscr{P}$ such that for all $\boldsymbol{\phi} \in \mathscr{P}^{\mathrm{reg}}$, the moduli space $\boldsymbol{\mathcal{M}^\phi}$ is a 1-dimensional manifold, and there is a discrete subset $I_o \subset [0,1]$ having the property that*

- *for each $t \in [0,1]\backslash I_o$, the moduli space $\mathcal{M}^{\phi_t}$ consists only of unobstructed (rigid) closed associative submanifolds,*
- *for each $\hat{t} \in I_o$ the moduli space $\mathcal{M}^{\phi_{\hat{t}}}$ consists only of closed associative submanifolds $\hat{P}$ having* $\dim\ker \boldsymbol{D}_{\hat{P}} \leqslant 1$. *If $P_o \in \mathcal{M}^{\phi_{\hat{t}}}$ and* $\dim\ker \boldsymbol{D}_{P_o} = 1$, *then there exist nonzero constants a, b such that* $\mathrm{ob}_{(\hat{t}, P_o)}$ *can be written as*

$$\mathrm{ob}_{(\hat{t}, P_o)}(t, x) = at + bx^2 + \text{higher-order terms}.$$

Although Theorem 2.35 implies that for generic ϕ, the moduli space $\mathcal{M}^\phi$ of closed associative submanifolds is discrete, it does not guarantee finiteness, which is essential for defining counting invariants. This leads to the following natural — and still open — question.

Question 2.36 [Joyce 2018, Conjecture 2.16] Let (Y,ϕ) be a compact almost G_2-manifold and $\tau\in\Omega^3(Y)$ be a closed 3-form. Let $\mathscr{P}_\tau$ be the space of all coclosed G_2-structures that are tamed by τ. Does there exist a comeager subset $\mathscr{P}_\tau^\blacklozenge\subset\mathscr{P}_\tau$ such that for all $\phi\in\mathscr{P}_\tau^\blacklozenge$, the moduli space $\mathcal{M}^\phi$ is compact?

If (Y,ϕ) is a compact almost G_2-manifold tamed by $\tau\in\Omega^3(Y)$, then there is a constant $c>0$ such that for every closed associative submanifold P in (Y,ϕ) we have (see [Donaldson and Segal 2011, Section 3.2] and [Joyce 2018, Section 2.5])

$$\operatorname{vol}(P,g_\phi)\leqslant c[\tau]\cdot[P].$$

Therefore, we may use the following theorem of geometric measure theory to get a Federer–Fleming compactification of $\mathcal{M}^\phi$. For a discussion on the proof of the following theorem in the special Lagrangian context we refer the reader to [Joyce 2004a, Section 6] and [Doan and Walpuski 2021, Section 4].

Theorem 2.37 [Simon 1983, Sections 6 and 32; Spolaor 2019; Adams and Simon 1988, Theorem 1; Joyce 2004a, Theorem 6.8] *Let P_n be a sequence of closed associative submanifolds in a sequence of compact almost G_2-manifolds (Y,ϕ_n) that are tamed by a fixed $\tau\in\Omega^3(Y)$. Assume ϕ_n converges to ϕ in the C^∞-topology. Then after passing to a subsequence, P_n converges (in the sense of currents) to a closed integral current P_∞ which is calibrated by ϕ. Moreover*:

(i) *The Hausdorff dimension of the singular set of P_∞ is at most* 1.

(ii) *If all the tangent cones of P_∞ are Jacobi integrable*[2] *multiplicity-one associative cones in $\mathbb{R}^7$ with smooth links, then P_∞ is a conically singular associative submanifold of (Y,ϕ), in the sense of Definition 4.13.*

The difficulty in addressing Question 2.36 arises primarily from our limited understanding of the singular set in the generic setting. A natural starting point is to analyze the simplest degeneration scenario, as described in Theorem 2.37(ii), where the limits are conically singular associative submanifolds. This article contributes specifically to advancing the understanding of this case.

3 Associative cones

This section focuses on associative cones in $\mathbb{R}^7$ and their links, which are holomorphic curves in S^6, and establishes Theorem 1.3. We also examine the Fueter operator on these cones and compute or bound their stability index, leading to the proof of Theorem 1.10.

3.1 Definition and examples of associative cones

Definition 3.1 Let Σ be a 2-dimensional closed submanifold of $S^6\subset\mathbb{R}^7$. Define the inclusion map $\iota:(0,\infty)\times\Sigma\to\mathbb{R}^7$ by $\iota(r,\sigma)=r\sigma$. A *cone* C with link Σ is the image of ι in $\mathbb{R}^7$. The Euclidean

[2] A cone in $\mathbb{R}^7$ with smooth link $\Sigma\subset S^6$ is called Jacobi integrable if, for every $\nu_\Sigma\in\ker(\boldsymbol{D}_\Sigma+2J)\subset C^\infty(N\Sigma)$, $\{\exp(t\nu_\Sigma):|t|\ll1\}$ is a one-parameter family of holomorphic curves in (S^6,J). Here, "exp" is the exponential map with respect to the round metric on S^6, and $\boldsymbol{D}_\Sigma+2J$ is the deformation operator controlling the deformation theory of holomorphic curves in (S^6,J) (Proposition 5.18).

metric on $\mathbb{R}^7$ induces a metric g_Σ on Σ and a cone metric g_C on C, that is, $g_C = dr^2 + r^2 g_\Sigma$. Furthermore, it induces a metric g_{NC} and a connection $\nabla_C^\perp$ on the normal bundle NC of the cone C. Let $\pi : \mathbb{R}^7 \setminus \{0\} \to S^6$ be the projection. Then $NC = \pi^*(N\Sigma)$, pullback of the normal bundle $N\Sigma$ of Σ in S^6, $g_{NC} = r^2 \pi^* g_{N\Sigma}$ and $\nabla_C^\perp = \pi^* \nabla_\Sigma^\perp$.

The *standard almost complex structure* on S^6, $J : T_x S^6 \to T_x S^6$ is defined by the standard cross product $\times$ on $\mathbb{R}^7$ as

$$J(v) = \partial_r \times v,$$

where $x \in S^6$ and $v \in T_x S^6 \subset \mathbb{R}^7$.

If the cone C is an associative submanifold then we call it an *associative cone*. This is equivalent to saying that the link Σ is a holomorphic curve in the almost complex manifold (S^6, J).

Any special Lagrangian cone in $\mathbb{C}^3$ is an associative cone in $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ and its link is special Legendrian in S^5. Examples 3.2 and 3.3 describe two examples of special Lagrangian cones in $\mathbb{C}^3$ that are important for the desingularization theorems (Theorems 1.35 and 1.28) discussed in this article. For more examples of special Lagrangian cones, see [Haskins 2004b; Haskins and Kapouleas 2007; Joyce 2001, 2002].

Example 3.2 (transverse pair of SL planes [Joyce 2003, page 328]) Let $C_\times$ be the union of a pair of special Lagrangian (SL) planes in $\mathbb{C}^3$ with transverse intersection at the origin. Then there exist a $B \in \mathrm{SU}(3)$ and unique $\theta_1, \theta_2, \theta_3 \in (0, \pi)$ satisfying $\theta_1 \leqslant \theta_2 \leqslant \theta_3$ and $\theta_1 + \theta_2 + \theta_3 = \pi$ such that $C_\times = B\Pi_0 \cup B\Pi_\theta$, where

$$\Pi_0 = \mathbb{R}^3 \quad \text{and} \quad \Pi_\theta := \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot \mathbb{R}^3.$$

We define $\Pi_+ := B\Pi_0$ and $\Pi_- := B\Pi_\theta$. Note that $\Pi_\pm$ are uniquely determined.

Example 3.3 (Harvey–Lawson T^2-cone [1982, Theorem 3.1]) The Harvey–Lawson T^2-cone is given by

$$\begin{aligned} C_{\mathrm{HL}} &:= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3|, z_1 z_2 z_3 \in (0, \infty)\} \\ &= \{r(e^{i\theta_1}, e^{i\theta_2}, e^{-i(\theta_1+\theta_2)}) \in \mathbb{C}^3 : r \in (0, \infty), \theta_1, \theta_2 \in [0, 2\pi)\}, \end{aligned}$$

which is a special Lagrangian cone in $\mathbb{C}^3$ whose link $\Sigma_{\mathrm{HL}} = C_{\mathrm{HL}} \cap S^5$ is isometric to the flat Clifford torus T^2.

Example 3.4 (null-torsion holomorphic curves [Bryant 1982, Section 4]) We follow here the exposition about null-torsion holomorphic curves given in [Madnick 2022]. Let Σ be a closed holomorphic curve in (S^6, J). Then the characteristic $\mathrm{SU}(3)$ connection $\widetilde{\nabla}$ (see equation (3.8)) induces holomorphic structures on $TS^6_{|\Sigma}$, $T\Sigma$ and $N\Sigma$. The second fundamental form of Σ is the obstruction $\mathrm{II} \in \mathrm{Ext}^1(N\Sigma, T\Sigma) \cong H^0(\Sigma, K_\Sigma^2 \otimes N\Sigma)$ (by Serre duality) to the holomorphic splitting of the exact sequence

$$0 \to T\Sigma \to TS^6_{|\Sigma} \to N\Sigma \to 0.$$

Moreover, $\mathrm{II} \neq 0$ if and only if Σ is not a totally geodesic S^2. In this case, denote the effective divisor of the zero set of II by Z. We define a holomorphic line bundle L_B by the exact sequence

$$0 \to L_N := K_\Sigma^{-2} \otimes \mathcal{O}(Z) \hookrightarrow N\Sigma \to L_B \to 0. \tag{3.5}$$

The torsion of Σ is the obstruction $\mathrm{III} \in \mathrm{Ext}^1(L_B, L_N) \cong H^0(\Sigma, K_\Sigma^3 \otimes \mathcal{O}(-Z) \otimes L_B)$ (by Serre duality) to the holomorphic splitting of the exact sequence in (3.5).

If Σ is *null-torsion* (i.e., $\mathrm{II} \neq 0, \mathrm{III} = 0$), then there is a holomorphic isomorphism

$$L_B \cong K_\Sigma^3 \otimes \mathcal{O}(-Z),$$

and we have

$$\mathrm{Area}(\Sigma) = 4\pi b \geqslant 24\pi,$$

where $b = -c_1(L_B) = 3\chi(\Sigma) + [Z]$. Moreover, no null-torsion holomorphic curves in S^6 are contained in a totally geodesic S^5. If Σ is of genus zero and not a totally geodesic S^2, then it must be a null-torsion holomorphic curve. Bryant [1982, Theorem 4.10] and later Rowland [1999] proved that closed Riemann surfaces of any genus can be conformally embedded as a null-torsion J-holomorphic curve in S^6.

We would like to point out that there are more examples of associative cones which are not special Lagrangians, discussed in [Lotay 2011, Section 7; 2007].

3.2 Moduli space of holomorphic curves in S^6

Definition 3.6 Let $\mathcal{S}$ be the set of all 2-dimensional oriented, closed smooth submanifolds of S^6. Equip $\mathcal{S}$ with the C^∞-topology in the same way as in Definition 2.31.

The *moduli space* $\mathcal{M}^{\mathrm{hol}}$ of embedded holomorphic curves in (S^6, J) is the subset of all submanifolds Σ in $\mathcal{S}$ that are J-holomorphic. The topology on $\mathcal{M}^{\mathrm{hol}}$ is the subspace topology of the above.

Let Σ be a holomorphic curve in S^6, that is, $\Sigma \in \mathcal{M}^{\mathrm{hol}}$. We denote the complex structure on Σ by j, which is just the restriction of J. Let $\Upsilon_\Sigma : V_\Sigma \to U_\Sigma$ be a tubular neighborhood map of Σ. For $u \in C^\infty(V_\Sigma)$ denote the submanifold $\Upsilon_\Sigma(\Gamma_u)$ by Σ_u. Note that Σ_u is J-holomorphic if and only if u satisfies the following *nonlinear Cauchy–Riemann equation*:

$$0 = \overline{\partial}_J u := \tfrac{1}{2}(du + \Upsilon_\Sigma^* J(u) \circ du \circ j) \quad \in C^\infty(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^* T V_\Sigma)).$$

The linearization of the nonlinear map $\overline{\partial}_J : C^\infty(V_\Sigma) \to C^\infty(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^* T V_\Sigma))$ at the zero section is described in [McDuff and Salamon 2012, Proposition 3.1.1]. This is the linear map

$$\mathfrak{d}_{\Sigma,J} : C^\infty(N\Sigma) \to C^\infty(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, TV_{\Sigma|\Sigma}))$$

defined by

$$\mathfrak{d}_{\Sigma,J}\xi := \tfrac{1}{2}(\nabla_\Sigma \xi + J \circ (\nabla_\Sigma \xi) \circ j + \nabla_\xi J \circ j) \quad \text{for } \xi \in C^\infty(N\Sigma).$$

The tangential component of $\mathfrak{d}_{\Sigma,J}$ can be discarded for the deformation theory. The following normal component actually controls the deformation theory.

Definition 3.7 For $\Sigma \in \mathcal{M}^{\text{hol}}$, the *normal Cauchy–Riemann operator*

$$\overline{\partial}^N_\nabla : C^\infty(N\Sigma) \to C^\infty(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, N\Sigma)) \cong \Omega^{0,1}(\Sigma, N\Sigma)$$

is defined by

$$\overline{\partial}^N_\nabla \xi := \tfrac{1}{2}\big(\nabla^\perp_\Sigma \xi + J \circ (\nabla^\perp_\Sigma \xi) \circ j + \nabla^\perp_\xi J \circ j\big).$$

The moduli space of holomorphic curves is usually studied using the nonlinear Cauchy–Riemann map [McDuff and Salamon 2012, Theorem 3.1.5] but here we study it in a different way.

On $\mathbb{R}^7$, the Euclidean metric is $g_e = dr^2 + r^2 g$ and the standard G_2-structure ϕ_e and $\psi_e = *_{g_e}\phi_e$ can be written as

$$\phi_e = r^2 dr \wedge \omega + r^3 \operatorname{Re}\Omega \quad \text{and} \quad \psi_e = \tfrac{1}{2} r^4 \omega^2 - r^3 dr \wedge \operatorname{Im}\Omega,$$

where $\omega(X, Y) := g(JX, Y)$, and Ω is a nowhere vanishing complex $(3,0)$-form. Together they give an $\mathrm{SU}(3)$-structure on S^6. Also, we have

$$\text{vol}_{\mathbb{R}^7} = r^6 dr \wedge \text{vol}_{S^6} \quad \text{and} \quad \text{vol}_{S^6} = \tfrac{1}{6}\omega^3 = \tfrac{1}{8} i\,\Omega \wedge \overline{\Omega} = \tfrac{1}{4} \operatorname{Re}\Omega \wedge \operatorname{Im}\Omega.$$

In particular, $*_g \operatorname{Re}\Omega = \operatorname{Im}\Omega$ and $*_g \operatorname{Im}\Omega = -\operatorname{Re}\Omega$. Moreover, $d\phi_e = 0$ and $d\psi_e = 0$ on $\mathbb{R}^7$ are equivalent to the following equations on S^6, respectively:

$$d\omega = 3 \operatorname{Re}\Omega \quad \text{and} \quad d \operatorname{Im}\Omega = -2\omega^2.$$

That means S^6 with this $\mathrm{SU}(3)$ structure is a *nearly Kähler* manifold. Let ∇ be the Levi-Civita connection on S^6 with respect to the round metric g. The *characteristic* $\mathrm{SU}(3)$ *connection* given by

$$\widetilde{\nabla}_u v := \nabla_u v + \tfrac{1}{2}(\nabla_u J) J v \tag{3.8}$$

satisfies $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} J = 0$ and $\mathrm{Hol}(\widetilde{\nabla}) \subset \mathrm{SU}(3)$. It turns out that $\widetilde{\nabla}$ is not torsion-free. Furthermore, it induces the following Cauchy–Riemann operator.

Definition 3.9 Let Σ be a J-holomorphic embedded curve in S^6. The *characteristic Cauchy–Riemann operator* on $N\Sigma$,

$$\overline{\partial}^N_{\widetilde{\nabla}} : C^\infty(N\Sigma) \to \Omega^{0,1}(N\Sigma),$$

is defined to be the induced Cauchy–Riemann operator from the characteristic $\mathrm{SU}(3)$ connection $\widetilde{\nabla}$, that is,

$$\overline{\partial}^N_{\widetilde{\nabla}} \xi := \tfrac{1}{2}(\widetilde{\nabla}^\perp \xi + J \circ (\widetilde{\nabla}^\perp \xi) \circ j).$$

The following lemma will help us to relate the two Cauchy–Riemann operators $\overline{\partial}^N_{\widetilde{\nabla}}$ and $\overline{\partial}^N_\nabla$.

Lemma 3.10 [McDuff and Salamon 2012, Lemma C.7.1] *For any vector fields u, v, w on S^6,*

(i) $(\nabla_u J)Jv = -J(\nabla_u J)v$ *and* $g((\nabla_u J)v, w) = -g((\nabla_u J)w, v)$,

(ii) $(\nabla_u J)v = -(\nabla_v J)u$ *and the torsion* $T_{\widetilde{\nabla}}(u, v) = \tfrac{1}{4} N_J(u, v) = (\nabla_u J)Jv$,

(iii) $3g((\nabla_u J)v, w) = d\omega(u, v, w) = 3 \operatorname{Re}\Omega(u, v, w)$.

Definition 3.11 The *multiplication map*, $\times_{S^6} : TS^6 \times TS^6 \to TS^6$, is defined by the orthogonal projection on TS^6 of the cross product in $\mathbb{R}^7$, or equivalently, for all vector fields u, v, w on S^6,

$$g(u \times_{S^6} v, w) = \operatorname{Re}\Omega(u, v, w), \quad \text{or equivalently, } u \times_{S^6} v = (\nabla_u J)v.$$

Remark 3.12 Let Σ be an oriented smooth surface in S^6 and C be the cone in $\mathbb{R}^7$ with link Σ. The following are equivalent:

(i) Σ is J-holomorphic.

(ii) C is associative.

(iii) For all $u, v \in T\Sigma$ and $w \in TS^6$, $\operatorname{Re}\Omega(u, v, w) = 0$.

(iv) For all $u, v \in T\Sigma$, $u \times_{S^6} v = 0$.

(v) For all $u \in T\Sigma$ and $v \in N\Sigma$, $u \times v \in N\Sigma$.

The following proposition is the desired relation between the above Cauchy–Riemann operators.

Proposition 3.13 *Let Σ be an embedded J-holomorphic curve in S^6. Then for all $\xi \in C^\infty(N\Sigma)$,*

$$\overline{\partial}^N_{\nabla}\xi = \overline{\partial}^N_{\widetilde{\nabla}}\xi - (\nabla^\perp J)J\xi.$$

Here $\nabla^\perp$ denotes the normal connection on $N\Sigma$, and its action on $\operatorname{End}(N\Sigma)$ is understood as the one induced from this normal connection.

Proof By Lemma 3.10, for all $\xi \in C^\infty(N\Sigma)$, we have

$$J \circ \widetilde{\nabla}\xi \circ j = J \circ \nabla\xi \circ j + \tfrac{1}{2}J(\nabla J)J\xi \circ j = J \circ \nabla\xi \circ j + \tfrac{1}{2}(\nabla J)J\xi.$$

The proposition now follows from the definitions. □

The following defines a canonical Dirac operator on a holomorphic curve Σ, which is also related to the above Cauchy–Riemann operators.

Definition 3.14 Let Σ be an embedded J-holomorphic curve in S^6. By Lemma 3.10 and Remark 3.12, the map $\gamma_\Sigma : T\Sigma \to \overline{\operatorname{End}}_{\mathbb{C}}(N\Sigma)$, given by

$$\gamma_\Sigma(f_\Sigma)(v_\Sigma) := f_\Sigma \times v_\Sigma \quad \text{for all } v_\Sigma \in N\Sigma \text{ and } f_\Sigma \in T\Sigma,$$

is a skew-symmetric J-antilinear Clifford multiplication. Moreover the normal bundle $N\Sigma$ together with the metric $g_{N\Sigma}$, Clifford multiplication γ_Σ and the metric connection $\widetilde{\nabla}^\perp := \widetilde{\nabla}^\perp_\Sigma$ is a Dirac bundle, that is, $\widetilde{\nabla}^\perp\gamma_\Sigma = 0$. The associated Dirac operator is given by

$$\boldsymbol{D}_\Sigma := \sum_{i=1}^{2} f_i \times \widetilde{\nabla}^\perp_{f_i}, \tag{3.15}$$

where $\{f_i\}$ is a local orthonormal oriented frame on Σ. Note that $\boldsymbol{D}_\Sigma$ is J-antilinear. Moreover, the map γ_Σ induces a J-antilinear isomorphism

$$\boldsymbol{\gamma}_\Sigma : \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N\Sigma) \to N\Sigma, \quad \text{defined by} \quad \boldsymbol{\gamma}_\Sigma(f_\Sigma^* \otimes v_\Sigma) = \gamma_\Sigma(f_\Sigma)(v_\Sigma).$$

Remark 3.16 The map $\boldsymbol{\gamma}_\Sigma$ fits into the following commutative diagram:

$$\begin{array}{ccc} C^\infty(N\Sigma) & \xrightarrow{\overline{\partial}^N_{\widetilde{\nabla},J}} & C^\infty(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N\Sigma)) \\ & \searrow^{\boldsymbol{D}_\Sigma} & \downarrow{\scriptstyle \boldsymbol{\gamma}_\Sigma} \\ & & C^\infty(N\Sigma) \end{array}$$

The following proposition is the desired relation between the above Cauchy–Riemann operators and the Dirac operator.

Proposition 3.17 *Let $\{f_i\}$ be a local orthonormal oriented frame on Σ. Then*

$$\boldsymbol{D}_\Sigma = \sum_{i=1}^{2} f_i \times \nabla^\perp_{f_i} - J \quad \text{and} \quad \boldsymbol{\gamma}_\Sigma \circ \overline{\partial}^N_{\nabla} = \boldsymbol{D}_\Sigma + 2J.$$

Proof First we prove that $\boldsymbol{\gamma}_\Sigma((\nabla^\perp J)J) = -2J$. For a local oriented orthonormal frame $\{f_1, f_2 = jf_1\}$ on Σ, we have $\nabla^\perp_{f_i} J = \gamma_\Sigma f_i$, by Definition 3.11 and Lemma 3.10. Therefore

$$\boldsymbol{\gamma}_\Sigma((\nabla^\perp J)J) = \sum_{i=1}^{2} f_i \times (\nabla^\perp_{f_i} J)J = \sum_{i=1}^{2} (\gamma_\Sigma f_i)^2 J = -2J.$$

Now the first equality in the proposition follows from equations (3.15) and (3.8). The second equality follows from Proposition 3.13 and Remark 3.16. □

The moduli space of holomorphic curves can be expressed locally as the zero set of the following nonlinear map.

Definition 3.18 Let Σ be a holomorphic curve in S^6 and $\Upsilon_\Sigma : V_\Sigma \to S^6$ be a tubular neighborhood map. We define $\mathcal{F} : C^\infty(V_\Sigma) \to C^\infty(N\Sigma)$ by

$$\langle \mathcal{F}(u), v \rangle_{L^2} := \int_{\Gamma_u} \iota_v(\Upsilon_\Sigma^* \operatorname{Re} \Omega),$$

where $u \in C^\infty(V_\Sigma)$ and $v \in C^\infty(N\Sigma)$. The notation v in the integrand is the extension vector field of $v \in C^\infty(N\Sigma)$ over the tubular neighborhood V_Σ as in Notation 2.25.

The proof of Theorem 1.3 requires computing the linearization of the map; the following proposition concerns this computation.

Proposition 3.19 *For $u \in C^\infty(V_\Sigma)$, we have $\mathcal{F}(u) = 0$ if and only if the graph $\Sigma_u := \Upsilon_\Sigma(\Gamma_u)$ is J-holomorphic. If $\Sigma \in \mathcal{M}^{\mathrm{hol}}$, then the linearization of $\mathcal{F}$ at zero, $d\mathcal{F}_0 : C^\infty(N\Sigma) \to C^\infty(N\Sigma)$, is given by*

$$d\mathcal{F}_0 = J\boldsymbol{D}_\Sigma - 2 = J(\boldsymbol{D}_\Sigma + 2J).$$

This is a formally self adjoint first-order elliptic operator of index 0.

Proof The first part follows from Remark 3.12. Let $\{f_1, f_2 = Jf_1\}$ be a local oriented orthonormal frame for $T\Sigma$. For $u, v \in C^\infty(N\Sigma)$,

$$\frac{d}{dt}\Big|_{t=0}\langle \mathcal{F}(tu), v\rangle_{L^2} = \frac{d}{dt}\Big|_{t=0}\int_{\Gamma_{tu}} \iota_v(\Upsilon_\Sigma^* \operatorname{Re}\Omega) = \int_\Sigma \mathcal{L}_u \iota_v(\Upsilon_\Sigma^* \operatorname{Re}\Omega).$$

This is the same as $\int_\Sigma \iota_v \mathcal{L}_u(\Upsilon_\Sigma^* \operatorname{Re}\Omega) + \iota_{[u,v]}(\Upsilon_\Sigma^* \operatorname{Re}\Omega)$. As $[u, v] = 0$ (by Remark 2.26, following Notation 2.25) and $\Sigma \in \mathcal{M}^{\mathrm{hol}}$, this is equal to

$$\begin{aligned}
&\int_\Sigma \iota_v(\Upsilon_\Sigma^* \operatorname{Re}\Omega)(\nabla_{f_1} u, f_2) + \iota_v(\Upsilon_\Sigma^* \operatorname{Re}\Omega)(f_1, \nabla_{f_2} u) + \iota_v \nabla_u(\Upsilon_\Sigma^* \operatorname{Re}\Omega) \\
&\qquad = \int_\Sigma -\langle f_2 \times \nabla^\perp_{f_1} u, v\rangle + \langle f_1 \times \nabla^\perp_{f_2} u, v\rangle - \iota_v(u \wedge \Upsilon_\Sigma^* \omega) \quad (\text{as } \nabla \operatorname{Re}\Omega = -\omega \wedge \cdot) \\
&\qquad = \int_\Sigma \langle J(f_1 \times \nabla^\perp_{f_1} u), v\rangle + \langle J(f_2 \times \nabla^\perp_{f_2} u), v\rangle - \langle u, v\rangle \\
&\qquad = \int_\Sigma \langle (J\boldsymbol{D}_\Sigma - 2)u, v\rangle.
\end{aligned}$$

Therefore $d\mathcal{F}_0 = J\boldsymbol{D}_\Sigma - 2$. Since $\boldsymbol{D}_\Sigma$ is a J-antilinear (this follows from the fact that $\widetilde{\nabla} J = 0$ and γ_Σ is J-antilinear), formally self-adjoint Dirac operator, it proves the last part of the proposition. □

Proof of Theorem 1.3 In Proposition 3.17, we see that $\boldsymbol{\gamma}_\Sigma \circ \overline{\partial}^N_\nabla = \boldsymbol{D}_\Sigma + 2J$. Extending the nonlinear map $\mathcal{F}$ to Hölder spaces, we get a smooth map

$$\mathcal{F} : C^{2,\gamma}(V_\Sigma) \to C^{1,\gamma}(N\Sigma).$$

Proposition 3.19 implies that the linearization of $\mathcal{F}$ at zero is an elliptic operator and hence is Fredholm. By the implicit function theorem applied to $\mathcal{F}$, we obtain the map ob_Σ as stated in the theorem; see [Donaldson and Kronheimer 1990, Proposition 4.2.19]. We only prove that, if $u \in C^{2,\gamma}(V_\Sigma)$ with $\mathcal{F}(u) = 0$, then $u \in C^\infty(V_\Sigma)$. To prove this, we observe that

$$0 = \boldsymbol{D}_\Sigma(\mathcal{F}(u)) = a(u, \nabla^\perp_\Sigma u)(\nabla^\perp_\Sigma)^2 u + b(u, \nabla^\perp_\Sigma u).$$

Since $a(u, \nabla^\perp_\Sigma u) \in C^{1,\gamma}$ and $b(u, \nabla^\perp_\Sigma u) \in C^{1,\gamma}$, by Schauder elliptic regularity [Joyce 2007, Theorem 1.4.2] we obtain $u \in C^{3,\gamma}$. By repeating this argument, we get higher regularity, which completes the proof of the theorem. □

3.3 The Fueter operator of an associative cone

The operator controlling the deformation theory for asymptotically conical (AC) or conically singular (CS) associatives will be an AC or CS uniformly elliptic operator, asymptotic to a conical elliptic operator (Definition 4.24). This conical operator is the Fueter operator on the asymptotic associative cone. The Fredholm theory of AC and CS uniformly elliptic operator was studied by Lockhart and McOwen [1985]; see also Karigiannis and Lotay [2020] and Marshall [2002]. It suggests that we must study the indicial roots and homogeneous kernels of this Fueter operator.

Let C be an associative cone in $\mathbb{R}^7$ with link Σ. The Fueter operator (Definition 2.32) on the cone C, $\boldsymbol{D}_C : C^\infty(NC) \to C^\infty(NC)$, is

$$\boldsymbol{D}_C = \sum_{i=1}^{3} e_i \times \nabla^{\perp}_{C,e_i},$$

where $\{e_i\}$ is a local oriented orthonormal frame on C and $\nabla^{\perp}_C$ is the normal connection induced by the Levi-Civita connection on $\mathbb{R}^7$.

Definition 3.20 (homogeneous kernel and indicial roots) For $\lambda \in \mathbb{R}$ we define the *homogeneous kernel* of rate λ by

$$V_\lambda := \{r^{\lambda-1}\nu_\Sigma \in C^\infty(NC) : \nu_\Sigma \in C^\infty(N_{S^6}\Sigma),\ \boldsymbol{D}_C(r^{\lambda-1}\nu_\Sigma) = 0\}.$$

Its dimension is denoted by

$$d_\lambda := \dim V_\lambda.$$

The set of *indicial roots* (or, critical rates) is defined by

$$\mathcal{D}_C := \{\lambda \in \mathbb{R} : d_\lambda \neq 0\}.$$

Remark 3.21

$$|r^{\lambda-1}\nu_\Sigma|_{g_{NC}} = r^\lambda |\nu_\Sigma|_{g_\Sigma}.$$

The following proposition not only studies properties of the Fueter operator but also describes the homogeneous kernel as an eigenspace of an operator on the link Σ, which will be useful later. Moreover, the almost complex structure J induces a symmetry on the homogeneous kernels.

Proposition 3.22 *Let ν_Σ be an element in $C^\infty(N_{S^6}\Sigma)$ and $\lambda \in \mathbb{R}$. Then the following hold*:

(i) $\boldsymbol{D}_C = J\partial_r + \frac{1}{r}\boldsymbol{D}_\Sigma + \frac{2}{r}J$.

(ii) $\boldsymbol{D}_C(r^{\lambda-1}\nu_\Sigma) = r^{\lambda-2}\big(\boldsymbol{D}_\Sigma\nu_\Sigma + (\lambda+1)J\nu_\Sigma\big)$.

(iii) $\boldsymbol{D}_C(r^{\lambda-1}(\log r)^j\nu_\Sigma) = r^{\lambda-2}(\log r)^j\big(\boldsymbol{D}_\Sigma\nu_\Sigma + (\lambda+1)J\nu_\Sigma\big) + j\,r^{\lambda-2}(\log r)^{j-1}J\nu_\Sigma$.

(iv) $\boldsymbol{D}_\Sigma(J\nu_\Sigma) = -J\boldsymbol{D}_\Sigma\nu_\Sigma$.

(v) $V_\lambda = \{r^{\lambda-1}\nu_\Sigma : \boldsymbol{D}_\Sigma\nu_\Sigma = -(\lambda+1)J\nu_\Sigma\} = \{r^{\lambda-1}\nu_\Sigma : (J\boldsymbol{D}_\Sigma)\nu_\Sigma = (\lambda+1)\nu_\Sigma\}$.

(vi) $JV_{-1+\lambda} = V_{-1-\lambda}$ *and* $d_{-1+\lambda} = d_{-1-\lambda}$ *for all* $\lambda \in \mathbb{R}$.

Proof We have $\nabla^\perp_C = dr \otimes \partial_r + \nabla^\perp_\Sigma$. For a local oriented orthonormal frame $\{f_i\}$ on $\Sigma \subset S^6$ and $v(r,\sigma) \in C^\infty(NC)$, applying Proposition 3.17 we compute

$$\begin{aligned}
\boldsymbol{D}_C v(r,\sigma) &= \partial_r \times \nabla^\perp_{C,\partial_r} v(r,\sigma) + \frac{1}{r^2}\sum_{i=1}^{2} f_i \times \nabla^\perp_{C,f_i} v(r,\sigma) \\
&= \partial_r \times_{S^6} \partial_r v(r,\sigma) + \frac{1}{r}\partial_r \times_{S^6} v(r,\sigma) + \frac{1}{r}\sum_{i=1}^{2} f_i \times_{S^6} \nabla^\perp_{\Sigma,f_i} v(r,\sigma) \\
&= J\partial_r v(r,\sigma) + \frac{1}{r}\boldsymbol{D}_\Sigma v(r,\sigma) + \frac{2}{r}J.
\end{aligned}$$

This proves (i). Now (ii) and (iii) follow from (i); indeed,

$$\boldsymbol{D}_C(r^{\lambda-1}(\log r)^j \nu_\Sigma) = r^{\lambda-2}(\log r)^j\big((\lambda-1)J\nu_\Sigma + \boldsymbol{D}_\Sigma\nu_\Sigma + 2J\nu_\Sigma\big) + j r^{\lambda-2}(\log r)^{j-1} J\nu_\Sigma.$$

Finally, (v) follows from (ii), and (vi) follows from (iv). And (iv) follows from the fact that $\widetilde{\nabla} J = 0$ and γ_Σ is J-antilinear. □

Although it follows from [Lockhart and McOwen 1985, equation 1.11] that a general element in the homogeneous kernel is of the form $\sum_{j=0}^{m} r^{\lambda-1}(\log r)^j \nu_{\Sigma,j}$, the following proposition implies that it must be of the form we have defined in Definition 3.20. We also see that there is a canonical one-to-one correspondence between the indicial roots and eigenvalues of the self-adjoint elliptic operator $J\boldsymbol{D}_\Sigma - 1$. In particular, they are countable, discrete and will have finite intersection with any closed bounded interval of $\mathbb{R}$. Moreover, Proposition 3.22(vi) implies that the indicial roots and homogeneous kernels are symmetric with respect to -1.

Proposition 3.23 *Let $m \in \mathbb{N}_0$ and let*

$$v(r,\sigma) = \sum_{j=0}^{m} r^{\lambda-1}(\log r)^j \nu_{\Sigma,j} \in C^\infty(NC)$$

and $\nu_{\Sigma,j} \in C^\infty(N_{S^6}\Sigma)$. If $\boldsymbol{D}_C v(r,\sigma) = 0$, then $m = 0$, and therefore $v(r,\sigma) = r^{\lambda-1}\nu_{\Sigma,0}$.

Proof If $\boldsymbol{D}_C v(r,\sigma) = 0$, then by Proposition 3.22(iii) and comparing the coefficients of $r^{\lambda-2}(\log r)^{j-1}$ for $j \geqslant 1$, we see that

$$\boldsymbol{D}_\Sigma\nu_{\Sigma,m} + (\lambda+1)J\nu_{\Sigma,m} = 0 \quad \text{and} \quad jJ\nu_{\Sigma,j} + \boldsymbol{D}_\Sigma\nu_{\Sigma,j-1} + (\lambda+1)J\nu_{\Sigma,j-1} = 0.$$

Therefore,

$$\begin{aligned}
m\|\nu_{\Sigma,m}\|^2_{L^2(\Sigma)} &= -\langle J\nu_{\Sigma,m}, \boldsymbol{D}_\Sigma\nu_{\Sigma,m-1} + (\lambda+1)J\nu_{\Sigma,m-1}\rangle_{L^2(\Sigma)} \\
&= \langle J(\boldsymbol{D}_\Sigma\nu_{\Sigma,m} + (\lambda+1)J\nu_{\Sigma,m}), \nu_{\Sigma,m-1}\rangle_{L^2(\Sigma)} = 0.
\end{aligned}$$

Here we have used Proposition 3.22(iv). The proof is completed by backwards induction starting with $j = m$. □

Whenever the above associative cone is a special Lagrangian cone, the homogeneous kernel can be expressed in a more explicit form with Hodge–de Rham operators, which in turn will help us compute or estimate lower bounds for the stability index. The following discusses this in detail.

Definition 3.24 Let L be a special Lagrangian submanifold in $\mathbb{C}^3 \subset \mathbb{R} \oplus \mathbb{C}^3$. Set $e_1 = (1,0) \in \mathbb{R} \oplus \mathbb{C}^3$. We define the isometry $\Phi_L : C^\infty(NL) \to \Omega^0(L,\mathbb{R}) \oplus \Omega^1(L,\mathbb{R})$ by

$$\Phi_L(\nu) := (\langle e_1, \nu\rangle, (e_1 \times \nu)^\flat).$$

We denote by $\check{\boldsymbol{D}}_L$ the conjugation of the Fueter operator $\boldsymbol{D}_L$ defined in Definition 2.32 under Φ_L, that is,

$$\check{\boldsymbol{D}}_L := \Phi_L \boldsymbol{D}_L \Phi_L^{-1} : \Omega^0(L,\mathbb{R}) \oplus \Omega^1(L,\mathbb{R}) \to \Omega^0(L,\mathbb{R}) \oplus \Omega^1(L,\mathbb{R}).$$

A direct computation shows that

$$\check{\boldsymbol{D}}_L = \begin{bmatrix} 0 & d^* \\ d & *d \end{bmatrix}. \tag{3.25}$$

A direct computation yields the following lemma.

Lemma 3.26 *Let C be a cone in $\mathbb{C}^3$ with link Σ, then for*

$$(f_\Sigma, h_\Sigma, \sigma_\Sigma) \in \Omega^0(\Sigma,\mathbb{R}) \oplus \Omega^0(\Sigma,\mathbb{R}) \oplus \Omega^1(\Sigma,\mathbb{R}),$$

we have:

(i) $d(r^\lambda f_\Sigma) = \lambda r^{\lambda-1} f_\Sigma\, dr + r^\lambda\, d_\Sigma f_\Sigma$.

(ii) $d(r^\lambda h_\Sigma\, dr + r^{\lambda+1}\sigma_\Sigma) = -r^\lambda\, dr \wedge d_\Sigma h_\Sigma + (\lambda+1) r^\lambda\, dr \wedge \sigma_\Sigma + r^{\lambda+1}\, d_\Sigma \sigma_\Sigma$.

(iii) $*dr = r^2 *_\Sigma 1$, $*\sigma_\Sigma = -dr \wedge *_\Sigma \sigma_\Sigma$, $*(dr \wedge \sigma_\Sigma) = *_\Sigma \sigma_\Sigma$, $*d_\Sigma \sigma_\Sigma = r^{-2}\, dr$.

(iv) $d^*(r^\lambda h_\Sigma\, dr + r^{\lambda+1}\sigma_\Sigma) = -(\lambda+2) r^{\lambda-1} h_\Sigma + r^{\lambda-1}\, d^{*_\Sigma}_\Sigma(\sigma_\Sigma)$.

(v) $*d(r^\lambda h_\Sigma\, dr + r^{\lambda+1}\sigma_\Sigma) = -r^\lambda *_\Sigma\, d_\Sigma h_\Sigma + (\lambda+1) r^\lambda *_\Sigma \sigma_\Sigma + r^{\lambda-1}(*_\Sigma\, d_\Sigma \sigma_\Sigma)\, dr$.

Proposition 3.27 *If C is a special Lagrangian cone in $\mathbb{C}^3$ whose link is Σ, then the homogeneous kernel from Definition 3.20 is*

$$V_\lambda \cong \begin{cases} \{\sigma_\Sigma \in \Omega^1(\Sigma,\mathbb{R}) : \Delta_\Sigma \sigma_\Sigma = 0\} & \text{if } \lambda = -1, \\ \{(f_\Sigma, h_\Sigma) \in \Omega^0(\Sigma,\mathbb{R}) \oplus \Omega^0(\Sigma,\mathbb{R}) : (f_\Sigma, h_\Sigma) \text{ satisfies (3.28)}\} & \text{if } \lambda \neq -1, \end{cases}$$

$$\Delta_\Sigma f_\Sigma = \lambda(\lambda+1) f_\Sigma \quad \text{and} \quad \Delta_\Sigma h_\Sigma = (\lambda+2)(\lambda+1) h_\Sigma. \tag{3.28}$$

Proof Since C is a special Lagrangian, by equation (3.25) we have

$$V_\lambda \cong \{(f_\Sigma, h_\Sigma, \sigma_\Sigma) \in \Omega^0(\Sigma,\mathbb{R}) \oplus \Omega^0(\Sigma,\mathbb{R}) \oplus \Omega^1(\Sigma,\mathbb{R}) : (f_\Sigma, h_\Sigma, \sigma_\Sigma) \text{ satisfies (3.29)}\},$$

$$\begin{cases} d^*(r^\lambda h_\Sigma\, dr + r^{\lambda+1}\sigma_\Sigma) = 0, \\ d(r^\lambda f_\Sigma) + *\, d(r^\lambda h_\Sigma\, dr + r^{\lambda+1}\sigma_\Sigma) = 0. \end{cases} \tag{3.29}$$

By Lemma 3.26 we obtain that equation (3.29) is equivalent to

$$\textbf{(3.30)}\qquad \begin{cases} (\lambda+2)h_\Sigma = d_\Sigma^{*_\Sigma}(\sigma_\Sigma), \\ \lambda f_\Sigma = -*_\Sigma d_\Sigma(\sigma_\Sigma), \\ (\lambda+1)*_\Sigma\sigma_\Sigma = -d_\Sigma f_\Sigma + *_\Sigma d_\Sigma h_\Sigma. \end{cases}$$

This yields the required proposition. □

Corollary 3.31 *If C is a special Lagrangian cone in $\mathbb{C}^3$ whose link is Σ, then*

$$V_\lambda \cong \begin{cases} \{h_\Sigma \in \Omega^0(\Sigma,\mathbb{R}) : \Delta_\Sigma h_\Sigma = (\lambda+2)(\lambda+1)h_\Sigma\} & \text{if } \lambda \in (-1,0), \\ H^0(\Sigma,\mathbb{R}) \oplus \{h_\Sigma \in \Omega^0(\Sigma,\mathbb{R}) : \Delta_\Sigma h_\Sigma = 2h_\Sigma\} & \text{if } \lambda = 0, \\ H^1(\Sigma,\mathbb{R}) & \text{if } \lambda = -1, \\ \{f_\Sigma \in \Omega^0(\Sigma,\mathbb{R}) : \Delta_\Sigma f_\Sigma = (\lambda+1)\lambda f_\Sigma\} & \text{if } \lambda \in (-2,-1). \end{cases}$$

Remark 3.32 If the associative cone C is not a 3-plane then all the translations by vectors in $\mathbb{R}^7$ yield a 7-dimensional subspace of V_0 and hence $d_0 \geqslant 7$. Indeed, there are no nontrivial translations that preserve the cone C because 0 is the unique singular point of C. Thus if C is not a 3-plane, then the lower stability index from Definition 1.7 has the lower bound

$$\tfrac{1}{2}d_{-1} \leqslant \text{s-ind}_-(C).$$

The following two examples compute the stability-indices of a pair of transverse associative planes, and the Harvey–Lawson T^2-cone.

Example 3.33 (pair of transverse associative planes) Let $\Pi_\pm$ be a pair of associative planes in $\mathbb{R}^7$ with transverse intersection at the origin, that is, $\Pi_+ \cap \Pi_- = \{0\}$. Set

$$C_\times := \Pi_+ \cup \Pi_- \quad \text{and} \quad e_1 := (1,0,\dots,0) \in \mathbb{R}^7 = \mathbb{R}\oplus\mathbb{C}^3.$$

We choose a unit vector $\boldsymbol{n} \in \mathbb{R}^7$ orthogonal to both $\Pi_\pm$ so that we have an orientation-compatible splitting

$$\mathbb{R}^7 = \langle \boldsymbol{n}\rangle_{\mathbb{R}} \oplus \Pi_+ \oplus \Pi_-.$$

Here the orientations of $\Pi_\pm$ are given by the restrictions of the standard 3-form ϕ_e, and the orientation of $\langle \boldsymbol{n}\rangle_{\mathbb{R}}$ is given by $\boldsymbol{n}$. If we choose $-\boldsymbol{n}$ instead of $\boldsymbol{n}$ in the above, we interchange the role of Π_+ and Π_-. Then there exists $B \in G_2$ such that

$$\boldsymbol{n} = Be_1, \quad B\Pi_0 = \Pi_+, \quad B\Pi_\theta = \Pi_-,$$

where Π_0 and Π_θ are special Lagrangian planes in $\mathbb{C}^3$ as in Example 3.2. By Corollary 3.31 and Proposition 3.27, we have

$$V_\lambda \cong \begin{cases} 0 & \text{if } \lambda \in [-1,0)\cup(0,1), \\ (\mathbb{R}\oplus\mathbb{R})\oplus(\mathbb{R}^3\oplus\mathbb{R}^3) & \text{if } \lambda = 0, \\ (\mathbb{R}^3\oplus\mathbb{R}^5)\oplus(\mathbb{R}^3\oplus\mathbb{R}^5) & \text{if } \lambda = 1. \end{cases}$$

If H is the symmetry group of an associative 3-plane for the standard action of G_2 on $\mathbb{R}^7$, then $H \cong \mathrm{SO}(4)$. Hence, $C_\times$ is rigid as in Definition 1.9, that is, $d_1 = 2(\dim G_2 - \dim H) = 16$, and its stability index from Definition 1.5 is

$$\text{s-ind}(C_\times) = \tfrac{1}{2}d_{-1} + \sum_{-1<\lambda\leqslant 1} d_\lambda - 2(\dim G_2 - \dim H) - 7 = 0 + 8 + 16 - 16 - 7 = 1.$$

Example 3.34 (Harvey–Lawson cone) Let C_{HL} be the Harvey–Lawson special Lagrangian cone, whose link is the flat Clifford tori in $\mathbb{C}^3$, given in Example 3.3. Then by Corollary 3.31 and Proposition 3.27, and [Marshall 2002, Section 6.3.4, page 132] we have

$$V_\lambda \cong \begin{cases} \mathbb{R}^2 & \text{if } \lambda = -1, \\ 0 & \text{if } \lambda \in (-1,0)\cup(0,1), \\ \mathbb{R}\oplus\mathbb{R}^6 & \text{if } \lambda = 0, \\ \mathbb{R}^6\oplus\mathbb{R}^6 & \text{if } \lambda = 1. \end{cases}$$

If H is the symmetry group of C_{HL} for the standard action of G_2 on $\mathbb{R}^7$, then $H \cong U(1)^2$. Hence C_{HL} is rigid as in Definition 1.9, that is, $d_1 = \dim G_2 - \dim H = 12$, and its stability index from Definition 1.5 is

$$\text{s-ind}(C_{\mathrm{HL}}) = \tfrac{1}{2}d_{-1} + \sum_{-1<\lambda\leqslant 1} d_\lambda - (\dim G_2 - \dim H) - 7 = 1 + 19 - 12 - 7 = 1.$$

The following proposition provides a classification of special Lagrangian cones that attain the minimal stability index.

Proposition 3.35 *Let C be a special Lagrangian cone in $\mathbb{C}^3$ whose link Σ is connected and not a totally geodesic S^2. Then its stability index from Definition 1.5 and lower stability index from Definition 1.7 satisfy*

$$\text{s-ind}(C) \geqslant \text{s-ind}_-(C) \geqslant \tfrac{1}{2}b^1(\Sigma) \geqslant 1,$$

with equality if and only if C is C_{HL}, the Harvey–Lawson T^2-cone (up to special unitary equivalence).

Proof As Σ is not a totally geodesic S^2, C is not contained in any hyperplane [Haskins 2004a, Lemma 3.13] in $\mathbb{C}^3$. Moreover the genus of Σ is at least 1 [Haskins 2004b, Theorem 2.7]. The space of real linear functions on $\mathbb{C}^3$ induces a 6-dimensional subspace of the 2-eigenspace of Δ_Σ. Therefore, by Corollary 3.31 we have (see Remark 3.32) $d_0 \geqslant 7$ and

$$\text{s-ind}_-(C) \geqslant \tfrac{1}{2}d_{-1} = \tfrac{1}{2}b^1(\Sigma) \geqslant 1.$$

In Example 3.34, we see $\text{s-ind}(C_{\mathrm{HL}}) = 1$. If $\text{s-ind}(C) = 1$, then $b^1(\Sigma) = 1$, $d_0 = 7$ and $d_\lambda = 0$ for all $\lambda \in (-1,0)\cup(0,1)$. This implies that the first eigenvalue of Δ_Σ is 2 with multiplicity 6. Then by a theorem of Haskins [2004a, Theorem A], C is C_{HL}, the Harvey–Lawson T^2-cone (up to special unitary equivalence). □

The remainder of this section focuses on the stability index of a null-torsion holomorphic curve.

Proposition 3.36 *Let Σ be a null-torsion holomorphic curve in S^6 and let C be the cone in $\mathbb{R}^7$ with link Σ. Then the lower stability index from Definition 1.7 satisfies*

$$\text{s-ind}_-(C) > 4.$$

In particular, this inequality holds for any genus-zero holomorphic curve in S^6 except a totally geodesic S^2.

The proof of this proposition needs the following small preparation.

Definition 3.37 Let Σ be a J-holomorphic curve in S^6. The *Jacobi operator* $\mathcal{L}_\Sigma : C^\infty(N\Sigma) \to C^\infty(N\Sigma)$ is

$$\mathcal{L}_\Sigma = (\nabla^\perp_\Sigma)^* \nabla^\perp_\Sigma + \sum_{i=1}^{2} (R(f_i, \cdot) f_i)^\perp - \sum_{i,j=1}^{2} \langle \mathrm{II}(f_i, f_j), \cdot \rangle \, \mathrm{II}(f_i, f_j).$$

Proposition 3.38 *Let Σ be a J-holomorphic curve in S^6. Then the Jacobi operator $\mathcal{L}_\Sigma$ satisfies*

$$\mathcal{L}_\Sigma = (\boldsymbol{D}_\Sigma + J)(\boldsymbol{D}_\Sigma + 2J) = (J\boldsymbol{D}_\Sigma)^2 - (J\boldsymbol{D}_\Sigma) - 2.$$

Moreover, $\operatorname{spec}(\mathcal{L}_\Sigma) = \{\lambda^2 + \lambda - 2 : \lambda \in \mathcal{D}_C\}$, where $\mathcal{D}_C$ is defined in Definition 3.20, and the $\lambda^2 + \lambda - 2$ eigenspace of $\mathcal{L}_\Sigma$ has the decomposition

$$E^{\lambda^2+\lambda-2}_{\mathcal{L}_\Sigma} \cong V_{\lambda-2} \oplus V_{\lambda+1} \cong V_\lambda \oplus V_{\lambda+1}.$$

In particular, $E^{-2}_{\mathcal{L}_\Sigma} \cong V_{-1} \oplus V_0$ and $E^{0}_{\mathcal{L}_\Sigma} \cong V_0 \oplus V_1$.

Proof We denote the Jacobi operator for the cone C of Σ in $\mathbb{R}^7$ by $\mathcal{L}_C$. [Gayet 2014, Theorem 2.8, Appendix 5.3] proved that

$$\mathcal{L}_C = \boldsymbol{D}_C^2.$$

Now for all $\nu_\Sigma \in C^\infty(N_{S^6}\Sigma)$, we have $\mathcal{L}_C \nu_\Sigma = r^{-2}\mathcal{L}_\Sigma \nu_\Sigma$. Therefore by Proposition 3.22(ii) we conclude that

$$\boldsymbol{D}_C^2 = r^{-2}(\boldsymbol{D}_\Sigma + J)(\boldsymbol{D}_\Sigma + 2J).$$

This proves the first part. The remaining part follows from parts (v) and (vi) of Proposition 3.22. □

Proof of Proposition 3.36 Let $\hat{J}$ be an almost complex structure on $N\Sigma$ defined by the following exact sequence of complex vector bundles (Example 3.4):

$$0 \to (L_N, J) \xrightarrow{\alpha} (N\Sigma, \hat{J}) \xrightarrow{\beta} (L_B, -J) \to 0.$$

Madnick [2022] proved that Σ is null-torsion if and only if $\nabla^\perp \hat{J} = 0$, and in that case

$$\mathcal{L}_\Sigma = 2\bar{\partial}^*_{\nabla^\perp, \hat{J}} \bar{\partial}_{\nabla^\perp, \hat{J}} - 2,$$

where $\bar{\partial}_{\nabla^\perp, \hat{J}}$ is the Cauchy–Riemann operator induced by $\nabla^\perp$ and $\hat{J}$ on $N\Sigma$. Moreover,

$$\textbf{(3.39)} \qquad \dim E^{-2}_{\mathcal{L}_\Sigma} \geqslant \operatorname{index} \bar{\partial}_{\nabla^\perp, \hat{J}} = 2c_1(N\Sigma, \hat{J}) + 2\chi(\Sigma) = -4c_1(L_B) = 4b \geqslant 24.$$

Here $b = \text{Area}(\Sigma)/(4\pi) \geqslant 6$. By Proposition 3.38 we see that $\dim E^{-2}_{\mathcal{L}_\Sigma} = d_{-1} + d_0$ and therefore

$$\text{s-ind}_-(C) \geqslant \tfrac{1}{2}d_{-1} + d_0 - 7 \geqslant \tfrac{1}{2}(d_{-1} + d_0) - 7 \geqslant 2b - 7 \geqslant 5. \qquad \square$$

Proof of Theorem 1.10 Statement (ii) is proved in Proposition 3.36, (iii) is computed in Examples 3.33 and 3.34, (iv) follows from Proposition 3.35 and the fact that $d_0 - 7 \geqslant b^0(\Sigma) - 1$. We will now prove (i). Consider the short exact sequence (3.5). If the genus of Σ is 1, then $K_\Sigma \cong \mathcal{O}$, and therefore $\mathcal{O}(Z) \otimes L_B = L_N \otimes L_B \cong \mathcal{O}$. Assume Σ is not a null-torsion holomorphic curve. Then the zero set of III induces an effective divisor, in particular $\deg L_B \geqslant [Z]$. Therefore $\deg L_B = [Z] = 0$ and hence $L_B \cong \mathcal{O}$ and $L_N \cong \mathcal{O}$. This implies that there is a nonzero section ν_Σ of $N\Sigma$ such that $\bar{\partial}^N_{\nabla} \nu_\Sigma = 0$ and hence $d_{-1} \geqslant 2$. Moreover, considering the long exact sequence of sheaf cohomologies corresponding to equation (3.5), we obtain

$$0 \to \mathbb{C} \to H^0(\Sigma, N\Sigma) \to \mathbb{C} \xrightarrow{\text{III}} \mathbb{C} \to H^1(\Sigma, N\Sigma) \to \mathbb{C} \to 0.$$

Since $\text{III} \neq 0$, we obtain, more precisely, that $d_{-1} = 2\dim_{\mathbb{C}} H^0(\Sigma, N\Sigma) = 2$. $\square$

Remark 3.40 Let Σ be a J-holomorphic curve in S^6. The *Morse index* of the Jacobi operator $\mathcal{L}_\Sigma$ is defined by $\text{Ind}\,\mathcal{L}_\Sigma := \sum_{\delta<0} \dim E^\delta_{\mathcal{L}_\Sigma}$. Now Proposition 3.38 implies that

$$\text{Ind}\,\mathcal{L}_\Sigma = d_{-1} + 2 \sum_{-1<\lambda<0} d_\lambda + \sum_{0\leqslant\lambda<1} d_\lambda.$$

If the genus of Σ is 1 then the proof of Theorem 1.10 implies that $\text{Ind}\,\mathcal{L}_\Sigma \geqslant 9$. If Σ is a null-torsion holomorphic curve in S^6 then $\mathcal{L}_\Sigma$ does not have an eigenvalue less than -2. Therefore by equation (3.39), the Morse index of $\mathcal{L}_\Sigma$ satisfies

$$\text{Ind}\,\mathcal{L}_\Sigma \geqslant 4b + \sum_{0<\lambda<1} d_\lambda \geqslant 24.$$

Comparing the above observations with a generalized Willmore type conjecture made by [Kusner and Wang 2024, Remark 3.6(1)] we conjecture that if an associative cone C in $\mathbb{R}^7$ is not a plane then $\text{s-ind}(C) \geqslant 1$ with equality if and only if C is the Harvey–Lawson T^2-cone or a union of two special Lagrangian planes with transverse intersection at the origin. This has been confirmed in Theorem 1.10 for special Lagrangian cones in $\mathbb{C}^3$.

4 AC and CS associative submanifolds

This section introduces the definitions and examples of asymptotically conical (AC) and conically singular (CS) associative submanifolds. We also study the linear analysis for the Fueter operator, which governs the deformations of AC and CS associative submanifolds. Since these submanifolds are noncompact, we carry out the Fredholm theory in weighted function spaces.

4.1 Asymptotically conical (AC) associative submanifolds

Before defining AC associative submanifolds, we introduce the notion of a conical tubular neighborhood map of a cone in the following definition.

Definition 4.1 Let C be a cone in $\mathbb{R}^7$ with link $\Sigma \subset S^6$, that is, $C = \iota((0,\infty)\times\Sigma)$. Let

$$\Upsilon_\Sigma : V_\Sigma \subset N_{S^6}\Sigma \to U_\Sigma$$

be a tubular neighborhood map for Σ. It induces a *conical tubular neighborhood map* of C,

$$\Upsilon_C : V_C \to U_C,$$

as follows. Define $V_C \subset NC$ at (r,σ) by rV_Σ, and

$$\Upsilon_C(u(r,\sigma)) := r\,\Upsilon_\Sigma(r^{-1}u(r,\sigma)) \quad \text{and} \quad U_C := \operatorname{im}\Upsilon_C \subset \mathbb{R}^7.$$

Observe that, for each $\varepsilon > 0$, the scaling map $s_\varepsilon : \mathbb{R}^7 \to \mathbb{R}^7$, $x \mapsto \varepsilon x$ induces an action $s_{\varepsilon *}$ on NC. Moreover, Υ_C is equivariant with respect to these actions, that is,

$$\Upsilon_C(s_{\varepsilon *}u) = s_\varepsilon \Upsilon_C u.$$

We now introduce the notion of AC associative submanifolds, which are associative submanifolds in $\mathbb{R}^7$ that approach a cone at infinity in a controlled manner.

Definition 4.2 Let C be a cone in $\mathbb{R}^7$ with link Σ, let $\Sigma = \coprod_{i=1}^m \Sigma_i$ be the decomposition into components, and let C_i be the cone with link Σ_i. An oriented three-dimensional submanifold L of $\mathbb{R}^7$ is called an *asymptotically conical (AC) submanifold* with cone C and rate $\nu := (\nu_1,\dots,\nu_m)$ with $\nu_i < 1$ if there exist

- a compact submanifold with boundary K_L of L,
- a real number $R_\infty > 1$ and a diffeomorphism

$$\Psi_L : (R_\infty,\infty)\times\Sigma \to L_\infty := L\setminus K_L \subset \mathbb{R}^7$$

such that $\Psi_L - \iota$ is a section of the normal bundle NC over $\iota((R_\infty,\infty)\times\Sigma) \subset C$ lying in the tubular neighborhood V_C of C, and

$$|(\nabla^\perp_{C_i})^k(\Psi_L - \iota)| = O(r^{\nu_i - k})$$

for all $k \in \mathbb{N}\cup\{0\}$ as $r \to \infty$.

Here $\nabla^\perp_C$ is the normal connection on NC induced from Levi-Civita connection on $\mathbb{R}^7$ and $|\cdot|$ is taken with respect to the normal metric on NC and the cone metric on C. The cone C is called the *asymptotic cone* of L, and L_∞ is the *end* of L.

The above L is called an *asymptotically conical (AC) associative submanifold* if, in addition, it is associative.

Remark 4.3 Deformation theory of AC associative submanifolds has been studied by Lotay [2011]. If L is an AC associative submanifold, then C is automatically an associative cone in $\mathbb{R}^7$; see [loc. cit., Proposition 2.15]. An AC submanifold of rate $\nu < 1$ is also an AC Riemannian manifold with rate $\nu - 1$, as well as an AC submanifold of any rate ν' with $\nu \leqslant \nu' < 1$.

The following two definitions will be useful later in the context of desingularization but may be skipped for now. They explain how, given an AC associative submanifold, one can define a corresponding end-conical submanifold, over which the AC submanifold can be expressed as the graph of a decaying section inside an appropriate end-conical tubular neighborhood.

Definition 4.4 (end-conical submanifold) Let $\rho : (0, \infty) \to [0, 1]$ be a smooth function such that

$$\rho(r) = \begin{cases} 1 & \text{if } r \leqslant 1, \\ 0 & \text{if } r \geqslant 2. \end{cases} \tag{4.5}$$

For every $s > 0$, $\rho_s : (0, \infty) \to [0, 1]$ is defined by $\rho_s(r) := \rho(s^{-1}r)$. Let L be an AC submanifold with cone C as in Definition 4.2. We define an end-conical submanifold L_C, which is diffeomorphic to L:

$$L_C := K_L \cup \Upsilon_C(\rho_{R_\infty}(\Psi_L - \iota)).$$

Set $C_\infty := \iota((2R_\infty, \infty) \times \Sigma) \subset L_C$ and $K_{L_C} := L_C \backslash C_\infty$.

Definition 4.6 (end-conical tubular neighborhood map) Let L be an AC submanifold with cone C as in Definition 4.2, and Υ_C be as in Definition 4.1. We say a tubular neighborhood map

$$\Upsilon_{L_C} : V_{L_C} \to U_{L_C}$$

of L_C is end-conical if V_{L_C} and Υ_{L_C} are chosen so that they respectively agree with V_C and Υ_C on C_∞.

There is a section β in $V_{L_C} \subset NL_C$ which is zero on K_L and $\Upsilon_{L_C}(\Gamma_\beta)$ is L, and

$$|(\nabla_{C_i}^{\perp})^k \beta| = O(r^{\nu_i - k}) \quad \text{for all } k \in \mathbb{N} \cup \{0\} \text{ as } r \to \infty,$$

where $\nabla_C^{\perp}$ is the normal connection on NL_C induced from Levi-Civita connection on $\mathbb{R}^7$.

There are many examples of AC special Lagrangians in $\mathbb{C}^3$ [Joyce 2001, 2002], but we mention only two types of AC special Lagrangians in Examples 4.7 and 4.10, which are important for this article. For examples of AC associatives which are not AC special Lagrangians, see [Lotay 2011, Section 7; 2007].

Example 4.7 (Lawlor neck [Lawlor 1989; Joyce 2018, Section 4.1; Harvey 1990, pages 139–143]) Let $\Pi_0 = \mathbb{R}^3$ and Π_θ be a pair of transverse special Lagrangian planes in $\mathbb{C}^3$ as in Example 3.2, that is,

$$\Pi_0 = \mathbb{R}^3 \quad \text{and} \quad \Pi_\theta := \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot \mathbb{R}^3,$$

where $\theta_1, \theta_2, \theta_3 \in (0, \pi)$ satisfy $\theta_1 \leqslant \theta_2 \leqslant \theta_3$ and $\theta_1 + \theta_2 + \theta_3 = \pi$. Define

$$\iota^+ : \mathbb{R}^+ \times S^2 \to \Pi_\theta, \quad \iota^+(r, \sigma) = r \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot \sigma,$$
$$\iota^- : \mathbb{R}^+ \times S^2 \to \Pi_0, \quad \iota^-(r, \sigma) = r\sigma.$$

For any $A>0$ there exists a unique triple of positive real numbers (a_1,a_2,a_3) such that

$$A=\frac{4\pi}{3\cdot\sqrt{a_1a_2a_3}} \quad \text{and} \quad \theta_k=\theta_k(\infty),$$

where

$$P(x):=\frac{\prod_{j=1}^3(1+a_jx^2)-1}{x^2},$$
$$\theta_k(y)=a_k\int_{-\infty}^{y}\frac{dx}{(1+a_kx^2)\sqrt{P(x)}} \quad \text{for } y\in\mathbb{R},\ k=1,2,3.$$

The *Lawlor neck* $L_{\theta,A}$ is defined as

$$L_{\theta,A}:=\{(z_1(y)\sigma_1,z_2(y)\sigma_2,z_3(y)\sigma_3)\in\mathbb{C}^3 : y,\sigma_1,\sigma_2,\sigma_3\in\mathbb{R},\ \sigma_1^2+\sigma_2^2+\sigma_3^2=1\},$$

where for $k=1,2,3$,

$$z_k(y):=e^{i\theta_k(y)}\sqrt{a_k^{-1}+y^2}.$$

The Lawlor neck $L_{\theta,A}$ is a smooth special Lagrangian submanifold [Harvey 1990, Theorem 7.78] diffeomorphic to $\mathbb{R}\times S^2$. The map

$$\Phi_{L_{\theta,A}}:\mathbb{R}\times S^2\to L_{\theta,A},\quad (y,(\sigma_1,\sigma_2,\sigma_3))\mapsto(z_1(y)\sigma_1,z_2(y)\sigma_2,z_3(y)\sigma_3),$$

is a diffeomorphism. Define $\Phi^{\pm}_{L_{\theta,A}}:\mathbb{R}^+\times S^2\to L_{\theta,A}$ by $\Phi^{\pm}_{L_{\theta,A}}(r,\sigma)=\Phi_{L_{\theta,A}}(\pm r,\sigma)$. For every $\varepsilon>0$, we observe that the rescaled Lawlor neck satisfies

$$\varepsilon L_{\theta,A}=L_{\theta,\varepsilon^3A}.$$

Therefore, we will always use "Lawlor neck" to mean $L_{\theta,A}$ with the normalization $A=1$, if not mentioned otherwise.

We define $\xi^-\in C^\infty(N\Pi_{0|\Pi_0\setminus\{0\}})$ and $\xi^+:=\operatorname{diag}(e^{i\theta_1},e^{i\theta_2},e^{i\theta_3})\cdot\xi^-\in C^\infty(N\Pi_{\theta|\Pi_\theta\setminus\{0\}})$ as

$$\xi^-(x_1,x_2,x_3):=r^{-3}(ix_1,ix_2,ix_3),\quad \text{where } r=\sqrt{x_1^2+x_2^2+x_3^2}. \tag{4.8}$$

Observe that $\boldsymbol{D}_{\Pi_\pm}\xi^\pm=0$ and $\xi^\pm\in V_{-2}$. The Lawlor neck $L_{\theta,A}$ is an AC special Lagrangian submanifold asymptotic to $\Pi_0\cup\Pi_\theta$ with rate $\nu=-2$. Moreover, the asymptotic normal sections can be written as

$$\Psi^{\pm}_{L_{\theta,1}}-\iota^\pm=\xi^\pm+O(r^{-4}). \tag{4.9}$$

Here $\Psi^{\pm}_{L_{\theta,1}}$ can be defined (up to normalizations) as

$$\Psi^-_{L_{\theta,1}}(\operatorname{Re}\Phi^-_{L_{\theta,1}})=\Phi^-_{L_{\theta,1}} \quad \text{and} \quad \Psi^+_{L_{\theta,1}}(\operatorname{Re}\operatorname{diag}(e^{-i\theta_1},e^{-i\theta_2},e^{-i\theta_3})\Phi^+_{L_{\theta,1}})=\Phi^+_{L_{\theta,1}}.$$

Example 4.10 (Harvey–Lawson AC special Lagrangians [Harvey and Lawson 1982, Theorem 3.1; Joyce 2018, Section 5.1]) Let $C:=C_{\mathrm{HL}}$ be the Harvey–Lawson T^2-cone as in Example 3.3, and let ι be the

inclusion map in $\mathbb{R}^7$. For any positive real number $a>0$, the *Harvey–Lawson special Lagrangian* 3-folds are defined by

$$\begin{aligned}
L_a^1 &:= \{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2 - a = |z_2|^2 = |z_3|^2, z_1z_2z_3\in[0,\infty)\},\\
L_a^2 &:= \{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2 = |z_2|^2 - a = |z_3|^2, z_1z_2z_3\in[0,\infty)\},\\
L_a^3 &:= \{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - a, z_1z_2z_3\in[0,\infty)\}.
\end{aligned}$$

Each L_a^k for $k=1,2,3$ is a smooth special Lagrangian in $\mathbb{C}^3$ [Harvey and Lawson 1982, Section III.3.A] diffeomorphic to $S^1\times\mathbb{C}$. Define the diffeomorphisms

$$\begin{aligned}
\Phi_{L_a^1} &: \mathbb{R}^+\times T^2\to L_a^1, \quad (r,e^{i\theta_1},e^{i\theta_2})\mapsto (e^{i\theta_1}\sqrt{r^2+a}, re^{i\theta_2}, re^{-i(\theta_1+\theta_2)}),\\
\Phi_{L_a^2} &: \mathbb{R}^+\times T^2\to L_a^1, \quad (r,e^{i\theta_1},e^{i\theta_2})\mapsto (re^{-i(\theta_1+\theta_2)}, e^{i\theta_1}\sqrt{r^2+a}, re^{i\theta_2}),\\
\Phi_{L_a^3} &: \mathbb{R}^+\times T^2\to L_a^1, \quad (r,e^{i\theta_1},e^{i\theta_2})\mapsto (re^{i\theta_2}, re^{-i(\theta_1+\theta_2)}, e^{i\theta_1}\sqrt{r^2+a}).
\end{aligned}$$

For every $\varepsilon>0$, we have

$$\varepsilon L_a^k = L_{\varepsilon^2 a}^k.$$

Therefore, we always assume L_a^k with the normalization $a=1$, if not mentioned otherwise. Each L_a^k is an AC special Lagrangian submanifold asymptotic to C_{HL} with rate $\nu=-1$. Moreover, the asymptotic normal sections can be written as

$$\Psi_{L_1^k} - \iota = \xi_k + O(r^{-2}), \tag{4.11}$$

where $\xi_k\in V_{-1}$ and $\xi_3 = -\xi_1-\xi_2$.

4.2 Conically singular (CS) associative submanifolds

We define conically singular (CS) associative submanifolds with singularity at m points, modeled on associative cones in $\mathbb{R}^7$. Before that we need a preferred choice of coordinate system at each of these m points, which we define first.

Definition 4.12 Let (Y,ϕ) be an almost G_2-manifold (Definition 2.3) and let p be a point in Y. A *G_2-framing* at p is a linear isomorphism $\upsilon:\mathbb{R}^7\to T_pY$ such that $\upsilon^*(\phi(p))=\phi_e$, where ϕ_e is the standard G_2 structure on $\mathbb{R}^7$. A *G_2-coordinate system* at p with G_2-framing υ is a diffeomorphism

$$\Upsilon : B_R(0)\subset\mathbb{R}^7\to U$$

for some constant $0<R<1$ and some open set U containing p in Y satisfying $\Upsilon(0)=p$ and $d\Upsilon_0=\upsilon$.

Two G_2-coordinate systems Υ_1 and Υ_2 at p are called *equivalent* if they have the same G_2-framing at p.

Definition 4.13 Let P be a compact subset of an almost G_2-manifold Y and $\mathrm{sing}(P):=\{p_1,\dots,p_m\}$ be a finite set points in P such that $\widehat{P}:=P\backslash\mathrm{sing}(P)$ is a smooth three-dimensional oriented submanifold of Y. We call $\widehat{P}$ the *smooth part* of P and $\mathrm{sing}(P)$ the *set of singular points* of P. For $i=1,\dots,m$ let

$$\Upsilon^i : B_R(0)\to U\subset Y$$

be a G_2-coordinate system with G_2-framing υ_i at p_i. Let $\{C_1, \dots, C_m\}$ be a set of cones in $\mathbb{R}^7$. Denote the link of C_i by Σ_i and the inclusion map into $\mathbb{R}^7$ by ι_i.

(1) P is said to be a *conically singular (CS) submanifold* with singularities at p_i modeled on cones C_i (with respect to the G_2-framing υ_i) and rates μ_i with $1 < \mu_i \leqslant 2$ if there exist

- a compact submanifold with boundary K_P of P,
- $\varepsilon_0 > 0$ with $2\varepsilon_0 < R$ and, for $i = 1, 2, \dots, m$, smooth embeddings

$$\Psi_P^i : (0, 2\varepsilon_0) \times \Sigma_i \to B_R(0),$$

with $\bigcup_{i=1}^m \Upsilon^i \circ \Psi_P^i : (0, 2\varepsilon_0) \times \Sigma_i \to Y$ a diffeomorphism onto $\widehat{P} \setminus K_P$, such that $\Psi_P^i - \iota_i$ is a smooth section of the normal bundle NC_i over $\iota((0, 2\varepsilon_0) \times \Sigma_i)$ which lies in V_{C_i} and

$$|(\nabla_{C_i}^{\perp})^k (\Psi_P^i - \iota_i)| = O(r^{\mu_i - k})$$

for all $k \in \mathbb{N} \cup \{0\}$ as $r \to 0$.

Here $\nabla_{C_i}^{\perp}$ is the normal connection on NC_i induced from Levi-Civita connection on $\mathbb{R}^7$ and $|\cdot|$ is taken with respect to the normal metric on NC_i and cone metric on C_i.

(2) The cone $\widetilde{C}_i := \upsilon_i(C_i) \subset T_{p_i} Y$ is said to be the *tangent cone* of P at p_i.

(3) P is said to be a *conically singular (CS) associative submanifold* if it is conically singular submanifold as above and $\widehat{P}$ is an associative submanifold of (Y, ϕ).

Remark 4.14 We make some remarks about the above definition.

(i) A CS submanifold with rates μ_i is also a CS submanifold with all rates μ_i' such that $1 < \mu_i' \leqslant \mu_i \leqslant 2$ for all $i = 1, 2, \dots, m$.

(ii) A CS submanifold with rates μ_i is a conically singular Riemannian manifold with rates $\mu_i - 1$ for the metric g_ϕ induced by ϕ. Indeed,

$$|(\nabla_{C_i}^{\perp})^k \big((\Upsilon^i \circ \Psi_P^i)^* g_\phi - g_{C_i}\big)| \lesssim \sum_{j=0}^{k+1} |(\nabla_{C_i}^{\perp})^j (\Psi_P^i - \iota_i)|.$$

(iii) The condition $\mu_i > 1$ also implies that if P is a CS associative submanifold as above, then each cone C_i is also an associative cone in $\mathbb{R}^7$. Indeed, the associator on C_i

$$|[\cdot, \cdot, \cdot]_{|C_i}| = O(r^{\mu_i - 1}) \quad \text{as } r \to 0.$$

But $|[\cdot, \cdot, \cdot]_{|C_i}|$ is dilation invariant and therefore $[\cdot, \cdot, \cdot]_{|C_i} = 0$.

(iv) Since $\Psi_P^i - \iota_i$ is a section of the normal bundle NC_i which represents P over the end, Ψ_P^i is uniquely determined by the constant ε_0, the normal bundle NC_i and the G_2-coordinate system Υ^i at p_i.

(v) The condition $\mu_i \leqslant 2$ guarantees that the above definition of a CS submanifold is independent of the choice of G_2-coordinate systems Υ^i at p_i within an equivalence class; that is, it depends only on the G_2-framings υ_i at those points. However, observe that for any other G_2-framing υ_i' at p_i, there

exists $A_i \in G_2$ such that $\upsilon_i = \upsilon_i' \circ A_i$. Therefore, the same CS submanifold is also a CS submanifold modeled on the cones $A_i(C_i) \subset \mathbb{R}^7$. As a consequence, the tangent cones are independent of the choice of G_2-framings. Whenever we say that a CS submanifold is modeled on certain cones in $\mathbb{R}^7$, an implicit framing is there.

4.3 Linear analysis on CS and AC associative submanifolds

This subsection discusses the linear analysis of the deformation operator, which will be essential for the deformation theory and desingularization results presented later. By considering appropriate weighted function spaces of sections of vector bundles over CS and AC submanifolds (where the weights represent decay rates of those sections) we obtain a Fredholm theory for CS and AC elliptic operators if the weights avoid the wall of critical weights or rates. The index of these operators changes when weights cross the wall of critical rates. This theory originally appeared in Lockhart and McOwen [1985]. A very good exposition can be found in [Marshall 2002] and [Karigiannis and Lotay 2020].

Notation 4.15 For a conically singular manifold P, let NP denote the normal bundle of $\widehat{P} = P \setminus \operatorname{sing}(P)$.

4.3.1 Weighted function spaces Let P be a CS submanifold as in Definition 4.13 and L be an AC submanifold as in Definition 4.2 asymptotic to C_i, $i = 1, \dots, m$. To treat the AC and CS together introduce the following notation.

Notation 4.16 Write

$$M := \begin{cases} P & \text{if } P \text{ is a CS submanifold,} \\ L & \text{if } L \text{ is a AC submanifold,} \end{cases} \qquad \eta := \begin{cases} \mu & \text{if } P \text{ is a CS submanifold with rate } \mu, \\ \nu & \text{if } L \text{ is a AC submanifold with rate } \nu. \end{cases}$$

Notation 4.17 We wish to define weighted Sobolev and Hölder spaces with rate $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$. We say $\lambda <, =, > \lambda'$ if for each $i = 1, 2, \dots, m$ we have $\lambda_i <, =, > \lambda_i'$, respectively. For any $s \in \mathbb{R}$, set $\lambda + s := (\lambda_1 + s, \lambda_2 + s, \dots, \lambda_m + s)$ and $|\lambda| := \sum_{i=1}^m |\lambda_i|$.

Definition 4.18 (weighted function spaces) For each $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$, a *weight function* on $\widehat{P}$ is a smooth function $w_{P,\lambda} : \widehat{P} \to (0, \infty)$ such that if $x = \Upsilon \circ \Psi_P(r, \sigma)$ with $(r, \sigma) \in (0, 2\varepsilon_0) \times \Sigma_i$, then

$$w_{P,\lambda}(x) = r^{-\lambda_i}.$$

The weight function on L, $w_{L,\lambda} : L \to (0, \infty)$, is defined by

$$w_{L,\lambda}(x) := (1 + |x|)^{-\lambda_i}.$$

Let $\lambda \in \mathbb{R}^m$ and $k \geqslant 0$, $p \geqslant 1$, $\gamma \in (0, 1)$. For a continuous section u of NM, we define the *weighted L^∞-norm* and the *weighted Hölder seminorm* respectively by

$$\|u\|_{L^\infty_{M,\lambda}} := \|w_{M,\lambda} u\|_{L^\infty(NM)} \quad \text{and} \quad [u]_{C^{0,\gamma}_{M,\lambda}} := [w_{M,\lambda-\gamma} u]_{C^{0,\gamma}(NM)}.$$

For a continuous section u of NM with k continuous derivatives, we define the *weighted C^k-norm* and the *weighted Hölder norm* respectively by

$$\|u\|_{C^k_{M,\lambda}} := \sum_{j=0}^{k} \|(\nabla^\perp_M)^j u\|_{L^\infty_{M,\lambda-j}} \quad \text{and} \quad \|u\|_{C^{k,\gamma}_{M,\lambda}} := \|u\|_{C^k_{M,\lambda}} + [(\nabla^\perp_M)^k u]_{C^{0,\gamma}_{M,\lambda-k}},$$

and the *weighted Sobolev norm* by

$$\|u\|_{W^{k,p}_{M,\lambda}} := \Big(\sum_{j=0}^{k} \int_M |w_{M,\lambda-j} (\nabla^\perp_M)^j u|^p w_{M,3}\, dV_M \Big)^{\frac{1}{p}}.$$

Here, the connection $\nabla^\perp_M$ on NM is the projection of the Levi-Civita connection induced by g_ϕ for the decomposition $TY_{|M} = TM \oplus NM$, and the $|\cdot|$ is with respect to the metric g_ϕ. Set

$$L^p_{M,\lambda} := W^{0,p}_{M,\lambda}.$$

We define the *weighted Hölder space* $C^{k,\gamma}_{M,\lambda}$ to be the space of continuous sections of NM with k continuous derivatives and finite weighted Hölder norm $\|\cdot\|_{C^{k,\gamma}_{M,\lambda}}$. This is a Banach space but not separable.

We define the *weighted Sobolev space* $W^{k,p}_{M,\lambda}$, the *weighted C^k-space* $C^k_{M,\lambda}$, the *weighted L^p-space* $L^p_{M,\lambda}$ and the *weighted L^∞-space* $L^\infty_{M,\lambda}$ to be the completion of the space of compactly supported smooth sections of NM, namely, $C^\infty_c(NM)$ with respect to the weighted Sobolev norm $\|\cdot\|_{W^{k,p}_{M,\lambda}}$, the weighted C^k-norm $\|\cdot\|_{C^k_{M,\lambda}}$, the weighted L^p-norm $\|\cdot\|_{L^p_{M,\lambda}}$ and the weighted L^∞-norm $\|\cdot\|_{L^\infty_{M,\lambda}}$ respectively. These are all separable Banach spaces. Moreover, we define the *weighted C^∞-space* $C^\infty_{M,\lambda}$ by

$$C^\infty_{M,\lambda} := \bigcap_{k=0}^{\infty} C^k_{M,\lambda}. \tag{4.19}$$

Remark 4.20 The spaces $W^{k,2}_{M,\lambda}$ are all Hilbert spaces with inner product coming from the polarization of the norm. We also note that

$$L^p(NM) = L^p_{M,-(3/p)}.$$

The L^2-inner product gives rise to the following useful result.

Proposition 4.21 *Let $k, l \geqslant 0$, $p, q > 1$ with $(1/p) + (1/q) = 1$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^m$. Then the L^2-inner product map*

$$W^{k,p}_{M,\lambda_1} \times W^{l,q}_{M,\lambda_2} \xrightarrow{\langle\cdot,\cdot\rangle_{L^2}} \mathbb{R}$$

is continuous provided $M = P$ (CS) with $\lambda_1 + \lambda_2 \geqslant -3$, or $M = L$ (AC) with $\lambda_1 + \lambda_2 \leqslant -3$. Moreover, this L^2-inner product yields a Banach space isomorphism

$$(L^p_{M,\lambda})^* \cong L^q_{M,-3-\lambda}.$$

Proof For $u \in L^p_{M,\lambda_1}$ and $v \in L^q_{M,\lambda_2}$ we consider

$$\langle u, v\rangle_{L^2} = \int_M (w_{M,\lambda_1} u w_{M,3/p})(w_{M,\lambda_2} v w_{M,3/q}) w_{M,-\lambda_1-\lambda_2-3}\,\mathrm{Vol}_M \lesssim \|u\|_{W^{k,p}_{M,\lambda_1}} \|v\|_{W^{k,q}_{M,\lambda_2}},$$

provided $M = P$ (CS) with $\lambda_1 + \lambda_2 \geqslant -3$, or $M = L$ (AC) with $\lambda_1 + \lambda_2 \leqslant -3$. This proves that the L^2-inner product map is continuous. The isomorphism stated in the proposition follows from the standard fact that $\langle \cdot, \cdot \rangle_{L^2} : L^p \times L^q \to \mathbb{R}$ is a dual pairing. □

The following result is a weighted embedding and compactness theorem that will be used in this article; see [Marshall 2002, Theorems 4.17 and 4.18].

Proposition 4.22 *Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^m$, $k, l \in \mathbb{N} \cup \{0\}$, $p, q \geqslant 1$ and $\alpha, \beta \in (0,1)$. Then*:

(i) *If $k \geqslant l$ and $k - (n/p) \geqslant l - (n/q)$, the inclusion $W^{k,p}_{M,\lambda_1} \hookrightarrow W^{l,q}_{M,\lambda_2}$ is a continuous embedding provided that, for $M = L$ (AC), either $p \leqslant q, \lambda_1 \leqslant \lambda_2$ or $p > q, \lambda_1 < \lambda_2$, and for $M = P$ (CS), either $p \leqslant q, \lambda_1 \geqslant \lambda_2$ or $p > q, \lambda_1 > \lambda_2$.*

(ii) *If $k + \alpha \geqslant l + \beta$, the inclusion $C^{k,\alpha}_{M,\lambda_1} \hookrightarrow C^{l,\beta}_{M,\lambda_2}$ is a continuous embedding provided that $\lambda_1 \leqslant \lambda_2$ for $M = L$ (AC), and $\lambda_1 \geqslant \lambda_2$ for $M = P$ (CS).*

(iii) *If $k - (n/p) \geqslant l + \alpha$, the inclusion $W^{k,p}_{M,\lambda} \hookrightarrow C^{l,\alpha}_{M,\lambda}$ is a continuous embedding.*

(iv) *All of the above embeddings are compact provided that all the inequalities in the corresponding hypotheses are strict inequalities.*

4.3.2 AC and CS elliptic operators on AC and CS associatives Let P be a CS associative submanifold as in Definition 4.13 and L be an AC associative submanifold as in Definition 4.2, asymptotic to C. We continue to use M to denote either P or L, and η as μ for CS and ν for AC submanifolds. For any associative cone C in $\mathbb{R}^7$ with link Σ, the Fueter operator $\boldsymbol{D}_C$ is a conical operator (see [Marshall 2002, Section 4.3.2] for a precise definition), that is, after the identification of $\mathbb{R}^7\backslash\{0\}$ with the cylinder $\mathbb{R} \times S^6$ by substituting $r = e^t$, we obtain that (Proposition 3.22)

$$r^2 \boldsymbol{D}_C r^{-1} = J r \partial_r + (\boldsymbol{D}_\Sigma + J) = J \partial_t + (\boldsymbol{D}_\Sigma + J) \tag{4.23}$$

is a translation-invariant operator.

Definition 4.24 Let $\boldsymbol{D}_M : C^\infty_c(NM) \to C^\infty_c(NM)$ be a first-order, formally self-adjoint elliptic operator. It is called an *asymptotically conical (AC) uniformly elliptic* operator for $M = L$ (AC), and a *conically singular (CS) uniformly elliptic* operator for $M = P$ (CS), respectively, asymptotic to the conical operators $\boldsymbol{D}_{C_i}$ over the ends of M, if the operator $r^2 \boldsymbol{D}_M r^{-1}$ is an asymptotically translation-invariant uniformly elliptic operator asymptotic to $r^2 \boldsymbol{D}_{C_i} r^{-1}$ (see [Marshall 2002, Section 4.3.2]) and, after the identifications by canonical bundle isomorphisms,

$$\boldsymbol{D}_M = \boldsymbol{D}_{C_i} + O(r^{\eta_i - 1}).$$

The operator $\boldsymbol{D}_M$ has canonical extensions to weighted function spaces and we denote them as

$$\boldsymbol{D}^{k,p}_{M,\lambda} : W^{k+1,p}_{M,\lambda} \to W^{k,p}_{M,\lambda-1} \quad \text{and} \quad \boldsymbol{D}^{k,\gamma}_{M,\lambda} : C^{k+1,\gamma}_{M,\lambda} \to C^{k,\gamma}_{M,\lambda-1}.$$

The following proposition is about elliptic regularity statements for a CS or AC uniformly elliptic operator $\boldsymbol{D}_M$ [Marshall 2002, Theorem 4.6].

Proposition 4.25 *Suppose $\lambda \in \mathbb{R}^m$, $k \geqslant 0$, $p > 1$, $\gamma \in (0,1)$. Let $u, v \in L^1_{\mathrm{loc}}$ be two locally integrable sections of NM such that u is a weak solution of the equation $\boldsymbol{D}_P u = v$.*

(i) *If $v \in C^{k,\gamma}_{M,\lambda-1}$, then $u \in C^{k+1,\gamma}_{M,\lambda}$ is a strong solution and there exists a constant $C > 0$ such that*

$$\|u\|_{C^{k+1,\gamma}_{M,\lambda}} \leqslant C(\|\boldsymbol{D}^{k,\gamma}_{M,\lambda} u\|_{C^{k,\gamma}_{M,\lambda-1}} + \|u\|_{L^\infty_{M,\lambda}}).$$

(ii) *If $v \in W^{k,p}_{M,\lambda-1}$, then $u \in W^{k+1,p}_{M,\lambda}$ is a strong solution and there exists a constant $C > 0$ such that*

$$\|u\|_{W^{k+1,p}_{M,\lambda}} \leqslant C(\|\boldsymbol{D}^{k,p}_{M,\lambda} u\|_{W^{k,p}_{M,\lambda-1}} + \|u\|_{L^\infty_{M,\lambda}}).$$

The following proposition identifies the weights for which the elliptic operators D_M are adjoints of each other with the weighted function spaces.

Proposition 4.26 *Let $k \geqslant 0$, $p, q > 1$ with $(1/p) + (1/q) = 1$, and $\lambda_1, \lambda_2 \in \mathbb{R}^m$. For all $u \in W^{k+1,p}_{M,\lambda_1}$ and $v \in W^{k+1,q}_{M,\lambda_2}$ we have $\langle \boldsymbol{D}^{k,p}_{M,\lambda_1} u, v\rangle_{L^2} = \langle u, \boldsymbol{D}^{k,q}_{M,\lambda_2} v\rangle_{L^2}$ provided that $M = P$ (CS) with $\lambda_1 + \lambda_2 \geqslant -2$, or $M = L$ (AC) with $\lambda_1 + \lambda_2 \leqslant -2$.*

Proof The result is true for $u, v \in C^\infty_c(NM)$, so the general statement follows from Proposition 4.21. □

In the following, we discuss the Fredholm theory for the operator D_M. It turns out that the Fredholm property depends on the choice of weights, and in particular, it requires that the weight does not lie on the wall of critical rates, which we define below.

Definition 4.27 (wall of critical rates) Let $\{C_1, \ldots, C_m\}$ be a set of associative cones. Set $C := (C_i)_{i=1}^m$. The set of critical rates $\mathcal{D}_C$ is defined by

$$\mathcal{D}_C := \{(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m : \lambda_i \in \mathcal{D}_{C_i} \text{ for some } i\},$$

where $\mathcal{D}_{C_i}$ is the set of all indicial roots of $\boldsymbol{D}_{C_i}$ defined in Definition 3.20. We call $\mathcal{D}_C$ the *wall of critical rates* in $\mathbb{R}^m$. We define

$$V_\lambda := \bigoplus_{i=1}^m V_{\lambda_i}, \quad d_{\lambda_i} := \dim V_{\lambda_i} \quad \text{and} \quad d_\lambda := \sum_{i=1}^m d_{\lambda_i},$$

where the V_{λ_i} are defined in Definition 3.20.

We also need the weighted function spaces over the cone, which we now define.

Definition 4.28 Let C be a connected associative cone. We can define the Banach spaces $W^{k,p}_{C,\lambda}$, $C^{k,\gamma}_{C,\lambda}$ and all others over C analogously to Section 4.3.1, replacing M by C, NM by NC and the weight function $w_{M,\lambda}$ by $w_{C,\lambda} : C \to \mathbb{R}$, where $w_{C,\lambda}(r,\sigma) = r^{-\lambda}$, $\lambda \in \mathbb{R}$.

The following key lemma, which is crucial to the Fredholm theory, follows from [Donaldson 2002, Lemma 3.1, Proposition 3.4, Section 3.3.1] and [Maz'ja and Plamenevskiĭ 1978, Theorem 5.1], using the observation made in (4.23) above.

Lemma 4.29 *Let C be a connected associative cone. The conical Fueter operator*

$$\boldsymbol{D}_C : C^{k+1,\gamma}_{C,\lambda} \to C^{k,\gamma}_{C,\lambda-1}$$

is invertible if and only if $\lambda \in \mathbb{R}\backslash\mathcal{D}_C$. Moreover, any element $u \in \ker \boldsymbol{D}_C$ has an L^2-orthogonal decomposition

$$u = \sum_{\lambda\in\mathcal{D}_C} r^{\lambda-1} u_{\Sigma,\lambda},$$

where $u_{\Sigma,\lambda}$ are λ-eigensections of $J\boldsymbol{D}_\Sigma - \mathbf{1}$, that is, $r^{\lambda-1}u_{\Sigma,\lambda} \in V_\lambda$.

We obtain the following proposition.

Proposition 4.30 *Let $\lambda \in \mathbb{R}^m$, $k \geqslant 0$, $p > 1$ and $\gamma \in (0,1)$. Then $\boldsymbol{D}^{k,p}_{M,\lambda}$ and $\boldsymbol{D}^{k,\gamma}_{M,\lambda}$ are Fredholm for all $\lambda \in \mathbb{R}^m\backslash\mathcal{D}_C$. Moreover, for all $\lambda \in \mathbb{R}^m$, $\operatorname{Ker} \boldsymbol{D}^{k,p}_{M,\lambda} = \operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda}$ is finite-dimensional and independent of k, p and γ.*

Proof The operators $\boldsymbol{D}^{k,p}_{M,\lambda}$ and $\boldsymbol{D}^{k,\gamma}_{M,\lambda}$ are Fredholm with weight $\lambda \in \mathbb{R}^m\backslash\mathcal{D}_C$ by [Donaldson 2002, Proposition 3.6, Section 3.3.1] or [Lockhart and McOwen 1985, Theorem 6.2]. Independence of k, p and γ follows from Propositions 4.25 and 4.22. □

The following lemma defines the asymptotic limit map, which maps each element in the kernel of the above AC or CS elliptic operator (with an appropriate weight) to its leading-order asymptotic term. Moreover, the lemma provides a wall-crossing formula describing how the index of the operator changes as the weight crosses a wall of critical rates.

Lemma 4.31 *Let $\lambda \in \mathbb{R}^m$ and let λ_1, λ_2 be two elements in $\mathbb{R}^m\backslash\mathcal{D}_C$ with $\lambda_1 < \lambda < \lambda_2$ for $M = L$ (AC), and $\lambda_2 < \lambda < \lambda_1$ for $M = P$ (CS), such that $|\lambda_2 - \lambda_1| \leqslant |\eta - 1|$ and there are no other indicial roots in between λ_1 and λ_2 except possibly λ. We define the set*

$$\mathcal{S}_{\lambda_2} := \{u \in C^{k+1,\gamma}_{M,\lambda_2} : \boldsymbol{D}^{k,\gamma}_{M,\lambda_2} u \in C^{k,\gamma}_{M,\lambda_1-1}\}.$$

Define a linear map $e_{M,\lambda} : V_\lambda \to W^{k+1,p}_{M,\lambda_2}$ (under the identifications of normal bundles NM and NC_i by the canonical bundle isomorphisms over the ends) by

$$e_{M,\lambda}(w) := \begin{cases} \bigoplus_{i=1}^m \rho_{\varepsilon_0} w_i & \text{if } M = P \text{ (CS)}, \\ \bigoplus_{i=1}^m (1-\rho_{R_\infty}) w_i & \text{if } M = L \text{ (AC)}. \end{cases}$$

Then there exists a unique linear map $\widetilde{i}_{M,\lambda} : \mathcal{S}_{\lambda_2} \to V_\lambda$, called the **asymptotic limit map**, *satisfying, for any $u \in \mathcal{S}_{\lambda_2}$,*

$$u - e_{M,\lambda} \circ \widetilde{i}_{M,\lambda} u \in C^{k,\gamma}_{M,\lambda_1}.$$

Moreover, the following statements hold.

(i) $\mathcal{S}_{\lambda_2} \subset C^{k+1,\gamma}_{M,\lambda}$ *and* $\operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda_2} = \operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda}$, *and* $\operatorname{Ker} \widetilde{i}_{M,\lambda} = C^{k+1,\gamma}_{M,\lambda_1}$. *Moreover,*

$$\operatorname{Ker}\{i_{M,\lambda} := \widetilde{i}_{M,\lambda} : \operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda_2} \to V_\lambda\} = \operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda_1}.$$

(ii) $\mathcal{S}_{\lambda_2} = C^{k+1,\gamma}_{M,\lambda_1} + \operatorname{im} e_{M,\lambda}$ *and the restriction of* $\boldsymbol{D}^{k,\gamma}_{M,\lambda_2}$, *denoted by*

$$\widehat{\boldsymbol{D}}^{k,\gamma}_{M,\lambda_1} : C^{k+1,\gamma}_{M,\lambda_1} + \operatorname{im} e_{M,\lambda} \to C^{k,\gamma}_{M,\lambda_1-1},$$

has the property that $\operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda_2} = \operatorname{Ker} \widehat{\boldsymbol{D}}^{k,\gamma}_{M,\lambda_1}$ *and* $\operatorname{Coker} \boldsymbol{D}^{k,\gamma}_{M,\lambda_2} \cong \operatorname{Coker} \widehat{\boldsymbol{D}}^{k,\gamma}_{M,\lambda_1}$.

(iii) *Wall-crossing formula*: $\operatorname{index} \boldsymbol{D}_{M,\lambda_2} = \operatorname{index} \boldsymbol{D}_{M,\lambda_1} + d_\lambda$.

Proof The proof of this lemma can be found in [Karigiannis and Lotay 2020, Proposition 4.21 to Corollary 4.24]. We will only show the existence of the asymptotic map $\widetilde{i}_{M,\lambda}$. Given $u \in \mathcal{S}_{\lambda_2}$ we define $\widetilde{u} \in \bigoplus_{i=1}^m C^{k+1,\gamma}(NC_i)$ by

$$\widetilde{u} := \begin{cases} \bigoplus_{i=1}^m \widetilde{u}_i := \rho_{\varepsilon_0} u & \text{if } M = P \text{ (CS)}, \\ \bigoplus_{i=1}^m (1 - \rho_{R_\infty}) u & \text{if } M = L \text{ (AC)}. \end{cases}$$

Denote any of the asymptotic cone by C. Since $\boldsymbol{D}_M = \boldsymbol{D}_C + O(r^{\eta-1})$ and $|\lambda_2 - \lambda_1| \leqslant |\eta - 1|$, we have $\boldsymbol{D}_C \widetilde{u} \in C^{k,\gamma}_{C,\lambda_1-1}$. Lemma 4.29 implies that there exists a unique $v \in C^{k+1,\gamma}_{C,\lambda_1}$ such that $\boldsymbol{D}_C(\widetilde{u} - v) = 0$. We define the asymptotic map $\widetilde{i}_{M,\lambda}$ as

$$\widetilde{i}_{M,\lambda}(u) := r^{\lambda-1}(\widetilde{u} - v)_{\Sigma,\lambda}. \tag{4.32}$$

□

Finally, the following proposition computes the index of the AC or CS elliptic operator with an appropriate weight. This is essential for determining the dimension of the moduli spaces of AC and CS associative submanifolds.

Proposition 4.33 $\operatorname{Ker} \boldsymbol{D}^{k,\gamma}_{M,\lambda}$ *is independent of* λ *in each connected component of* $\lambda \in \mathbb{R}^m \backslash \mathcal{D}_C$. *Moreover, for all* $\lambda \in \mathbb{R}^m \backslash \mathcal{D}_C$, *we have*:

(i) $\operatorname{Coker} \boldsymbol{D}_{M,\lambda} \cong \operatorname{Ker} \boldsymbol{D}_{M,-2-\lambda}$. *If* $\lambda \geqslant -\frac{1}{2}$ *for* $M = L$ *(AC) or* $\lambda \leqslant -\frac{1}{2}$ *for* $M = P$ *(CS), then* $\operatorname{Ker} \boldsymbol{D}_{M,-2-\lambda}$ *is equal to* $\operatorname{Coker} \boldsymbol{D}_{M,\lambda}$.

(ii) *If* $s > 0$ *for* $M = L$ *(AC) or* $s < 0$ *for* $M = P$ *(CS) is such that* -1 *is the only possibly critical rate in between* $-1 - s$ *and* $-1 + s$, *then*

$$\operatorname{index} \boldsymbol{D}_{M,-1+s} = \dim \operatorname{Ker} \boldsymbol{D}_{M,-1+s} - \dim \operatorname{Ker} \boldsymbol{D}_{M,-1-s} = \tfrac{1}{2} d_{-1}.$$

(iii) $\operatorname{index} \boldsymbol{D}_{L,\lambda} = -\operatorname{index} \boldsymbol{D}_{P,\lambda}$, *and*

$$\operatorname{index} \boldsymbol{D}_{L,\lambda} = \sum_{\lambda_i \geqslant -1} \left(\frac{d_{-1,i}}{2} + \sum_{\zeta_i \in \mathcal{D}_{C_i} \cap (-1,\lambda_i)} d_{\zeta_i} \right) - \sum_{\lambda_i < -1} \left(\frac{d_{-1,i}}{2} + \sum_{\zeta_i \in \mathcal{D}_{C_i} \cap (\lambda_i,-1)} d_{\zeta_i} \right).$$

Proof The first statement is a direct consequence of part (iii) of previous lemma. Now the first part of (i) follows from Propositions 4.21 and 4.30, and if λ is as in second part of (i) we use the fact that $C^{k,\gamma}_{M,-2-\lambda} \subset C^{k,\gamma}_{M,\lambda}$. To see (ii), observe from (i) and the wall-crossing formula of Lemma 4.31 that

$$\dim \operatorname{Ker} \boldsymbol{D}_{M,-1+s} - \dim \operatorname{Ker} \boldsymbol{D}_{M,-1-s} = \dim \operatorname{Ker} \boldsymbol{D}_{M,-1-s} - \dim \operatorname{Ker} \boldsymbol{D}_{M,-1+s} + d_{-1}.$$

Finally, (iii) is an immediate consequence of (ii) and the wall-crossing formula stated in Lemma 4.31. □

5 Deformations of CS associative submanifolds

In this section, we study the moduli space of conically singular (CS) associative submanifolds in two settings: first, within a fixed coclosed G_2-structure, and second, within a smoothly varying one-parameter family of coclosed G_2-structures. We prove two key results: Theorem 1.12, which describes the structure of the above moduli spaces, and Theorem 1.17, which establishes transversality and generic smoothness results.

5.1 Moduli space of CS associative submanifolds

We begin by defining the universal moduli space and the moduli space in a fixed coclosed G_2-structure of CS associative submanifolds as a set.

Definition 5.1 We define the universal space of conically singular (CS) submanifolds,

$$\mathcal{S}_{\mathrm{cs}} := \{(\phi, P) : P \text{ is a CS submanifold in } (Y, \phi), \phi \in \mathscr{P}\}.$$

The universal *moduli space* $\mathcal{M}_{\mathrm{cs}}$ of conically singular associative submanifolds is defined by

$$\mathcal{M}_{\mathrm{cs}} := \{(\phi, P) \in \mathcal{S}_{\mathrm{cs}} : P \text{ is a CS associative submanifold}\}.$$

There is a canonical projection map $\pi : \mathcal{S}_{\mathrm{cs}} \to \mathscr{P}$, and we define the *moduli space* of CS associative submanifolds in (Y, ϕ) by

$$\mathcal{M}^{\phi}_{\mathrm{cs}} := \pi^{-1}(\phi) \cap \mathcal{M}_{\mathrm{cs}}.$$

Next, we define a topology on $\mathcal{M}_{\mathrm{cs}}$, called the weighted C^∞-topology. This topology is specified by defining a basis of open sets around each of its elements. Given an element of $\mathcal{M}_{\mathrm{cs}}$, we can make a choice of an end-conical singular (ECS) submanifold (Definition 5.2) equipped with an end-conical (EC) tubular neighborhood map (see Definition 5.3) near it. We then construct a canonical family of ECS submanifolds and corresponding EC tubular neighborhood maps by varying the coclosed G_2-structures, as well as the singularity data — such as the positions of the singularities and their associated model cones. The open sets consisting of CS associative submanifolds lying inside this family of tubular neighborhoods together with coclosed G_2-structures define a basis for the weighted C^∞-topology around the given element.

Definition 5.2 (ECS submanifold) Let P be a CS submanifold of (Y, ϕ) as in Definition 4.13. Given a choice of G_2-coordinate systems Υ^i at the singular points and the other data that is used for P in Definition 4.13, we define an *end-conical singular* (ECS) submanifold P_C, which is diffeomorphic to $\widehat{P}$

but conical on the ends, as follows:

$$P_C := K_P \cup \Big(\bigcup_{i=1}^{m} (\Upsilon^i \circ \Upsilon_{C_i})((1-\rho_{\varepsilon_0})(\Psi_P^i - \iota_i)) \Big).$$

Here ρ_{ε_0} is the cut-off function defined in (4.5).

Set

$$C_{i,\varepsilon_0} := \iota((0,\varepsilon_0)\times\Sigma_i) \quad \text{and} \quad K_{P_C} := P_C \backslash \bigcup_{i=1}^{m} \Upsilon^i(C_{i,\varepsilon_0}).$$

Definition 5.3 (EC tubular neighborhood map) Let P be a CS submanifold as in Definition 4.13. Let P_C be a choice of an ECS submanifold as in Definition 5.2. A tubular neighborhood map of P_C,

$$\Upsilon_{P_C} : V_{P_C} \to U_{P_C},$$

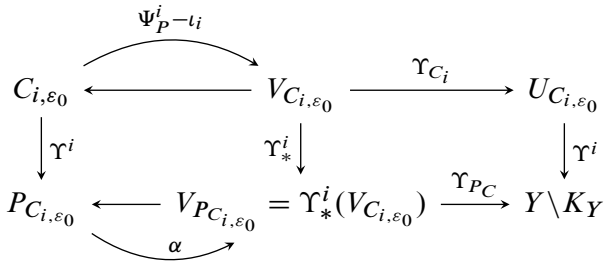
is called *end-conical (EC)* if V_{P_C} and Υ_{P_C} agree with $\Upsilon_*(V_C)$ and $\Upsilon \circ \Upsilon_C \circ \Upsilon_*^{-1}$ on $\Upsilon\big(\iota((0,\varepsilon_0)\times\Sigma)\big)$, respectively. Here, $\Upsilon_C : V_C \to U_C$ is a conical tubular neighborhood map of C as in Definition 4.1

Given a choice of an ECS submanifold P_C and a choice of an EC tubular neighborhood map Υ_{P_C}, there is a section α in $V_{P_C} \subset N P_C$ which is zero on K_P and $\Upsilon_*^i \circ (\Psi_P^i - \iota_i) = \alpha \circ \Upsilon^i$ such that $\Upsilon_{P_C}(\Gamma_\alpha)$ is P.

Remark 5.4 In the above framework, set

$$V_{C_{i,\varepsilon_0}} := V_{C_i|C_{i,\varepsilon_0}}, \quad U_{C_{i,\varepsilon_0}} := U_{C_i|C_{i,\varepsilon_0}}, \quad P_{C_{i,\varepsilon_0}} := P_{C_{|\Upsilon^i(C_{i,\varepsilon_0})}} \quad \text{and} \quad V_{P_{C_{i,\varepsilon_0}}} := V_{P_{C_{|\Upsilon^i(C_{i,\varepsilon_0})}}}.$$

The following commutative diagram helps us to keep track of the definitions above:



Definition 5.5 (CS submanifolds after varying G_2-structure and singularity data) Let $P \in \mathcal{M}_{\mathrm{cs}}^{\phi}$ be a CS associative submanifold with singularities at p_i modeled on cones C_i, $i = 1,\dots,m$ as in Definition 4.13. Let Σ_i be the link of C_i. Let P_C be a choice of an ECS submanifold as in Definition 5.2, and let $\Upsilon_{P_C} : V_{P_C} \to U_{P_C}$ be a choice of an EC tubular neighborhood map of P_C as in Definition 5.3. Since the singular points p_i and associative cones C_i are allowed to vary, we need to also vary the P_C and Υ_{P_C}. This is done as follows. By Theorem 1.3, there are tubular neighborhood maps $\Upsilon_{\Sigma_i} : V_{\Sigma_i} \to U_{\Sigma_i}$ of Σ_i and obstruction maps $\mathrm{ob}_{\Sigma_i} : \mathcal{I}_{\Sigma_i} \to \mathcal{O}_{\Sigma_i}$. Let U_ϕ and U_{p_i} be sufficiently small neighborhoods of ϕ in $\mathscr{P}$ and p_i in $\Upsilon^i\big(B(0,\varepsilon_0)\big) \subset Y$, respectively, where Υ^i is a G_2-coordinate system from Definition 4.13. Set

$$\textbf{(5.6)} \qquad U_{\tau_0} := U_\phi \times \Big(\prod_{i=1}^{m} U_{p_i} \Big) \times \Big(\prod_{i=1}^{m} \mathcal{I}_{\Sigma_i} \Big) \quad \text{and} \quad \tau_0 := (\phi, p_1, \dots, p_m, 0, \dots, 0) \in U_{\tau_0}.$$

The open set U_{τ_0} essentially parametrizes nearby coclosed G_2-structures together with nearby singularity data (positions and model cones).

For the parameter τ_0, we already have that P_C is an ECS submanifold and Υ_{P_C} is an EC tubular neighborhood map. Our goal is to construct a canonical smoothly varying (in the parameter $\tau \in U_{\tau_0}$) family of ECS submanifolds P_C^τ with G_2-structure, and the singularities and model cones are moved according to the data encoded in τ. Furthermore, we aim to construct an EC tubular neighborhood map $\Upsilon_{P_C^\tau}$, which will serve to define the weighted C^∞-topology.

To achieve this, we construct a *canonical smooth family of diffeomorphisms* (introduced in the next Definition 5.8),

$$\Phi : U_{\tau_0} \to \prod_{i=1}^{m} \mathrm{Diff}(B(0,R)), \quad \tau := \big(\phi_\tau, (p_i^\tau), (\xi_i^\tau)\big) \mapsto (\Phi_i^\tau), \tag{5.7}$$

satisfying the following conditions:

(i) $\Upsilon_i^\tau := \Upsilon^i \circ \Phi_i^\tau$ defines a G_2-coordinate system at p_i^τ for the coclosed G_2-structure ϕ_τ, where ϕ_τ and p_i^τ are the data encoded in τ.

(ii) $\Phi_i^{\tau_0}$ is the identity on $B(0,R)$, and for all $\tau \in U_{\tau_0}$, Φ_i^τ is equal to the identity outside $B(0,2\varepsilon_0)$.

With this in place, we define for each $\tau \in U_{\tau_0}$ an ECS submanifold P_C^τ, and an associated EC tubular neighborhood map $\Upsilon_{P_C^\tau}$, by

$$P_C^\tau := \Big(\bigcup_{i=1}^{m} \Upsilon_i^\tau \circ (\Upsilon^i)^{-1}(P_C)\Big) \cup K_P \quad \text{and} \quad \Upsilon_{P_C^\tau} := \Big(\bigcup_{i=1}^{m} \Upsilon_i^\tau \circ (\Upsilon^i)^{-1} \circ \Upsilon_{P_C}\Big) \cup \Upsilon_{P_{C|K_P}} : V_{P_C} \to Y.$$

Note that P_C^τ has singularities located at p_i^τ, modeled on associative cones C_i^τ whose links are $\Sigma_i^\tau := \Upsilon_{\Sigma_i}(\xi_i^\tau)$, where $\xi_i^\tau \in \mathcal{I}_{\Sigma_i}$ represents the infinitesimal deformation of Σ_i encoded in τ. Moreover, any CS submanifold in (Y,ϕ) lying inside U_{P_C} with the same asymptotic data as P is mapped, via $\Upsilon_{P_C^\tau} \circ \Upsilon_{P_C}^{-1}$, to a CS submanifold in (Y,ϕ_τ) with asymptotic data matching that of P_C^τ, and vice versa.

Definition 5.8 (canonical smooth family of diffeomorphisms) The time-1 flows of the following family of vector fields in (5.11), parametrized by U_{τ_0}, defines the required smooth family of diffeomorphisms Φ in (5.7).

To define (5.11), we first construct a family of vector fields parametrized by $\mathcal{I}_{\Sigma_i}$ as follows. Given an element $\xi_i \in \mathcal{I}_{\Sigma_i}$, we use the extension map $\widetilde{\bullet}$ from Definition 2.24 to obtain a vector field $\widetilde{\xi_i}$ on V_{Σ_i}. Applying the differential of the tubular neighborhood map Υ_{Σ_i}, we obtain the vector field $d\Upsilon_{\Sigma_i}(\widetilde{\xi_i})$ on U_{Σ_i}. Next, we extend this to a global vector field on S^6 by multiplying with a cut-off function supported in a neighborhood of U_{Σ_i}. Finally, we obtain a vector field v_{ξ_i} supported on $B(0,2\varepsilon_0) \subset B(0,R)$ by radially translating and multiplying by the cut-off function ρ_{ε_0} defined in (4.5). In summary, the construction can be expressed as

$$\mathcal{I}_{\Sigma_i} \xrightarrow{\widetilde{\bullet}} \mathrm{Vect}(V_{\Sigma_i}) \xrightarrow{d\Upsilon_{\Sigma_i}} \mathrm{Vect}(U_{\Sigma_i}) \hookrightarrow \mathrm{Vect}(S^6) \xrightarrow{\rho_{\varepsilon_0}\cdot} \mathrm{Vect}(B(0,R)). \tag{5.9}$$

The time-1 flows of the vector fields constructed above move the holomorphic curves and their corresponding associative cones near the origin but fix the positions of the singularities.

Next, we construct another smooth family of vector fields parametrized by $U_\phi \times U_{p_i}$ as follows. Given a pair $(\phi', p_i') \in U_\phi \times U_{p_i}$, we use the G_2-coordinate system Υ^i to pull back the G_2-structures ϕ at p_i and ϕ' at p_i'. The resulting pullbacks $(\Upsilon^i)^*\phi(p_i)$ and $(\Upsilon^i)^*\phi'(p_i')$ are two G_2-structures on $\mathbb{R}^7$, where the first can be mapped to the second by an element of $\mathrm{GL}_7(\mathbb{R})$. This element can be taken to be the exponential of a matrix $A_{\phi',p_i'} \in M_7(\mathbb{R})$. However, this choice is not unique, since composing with an element of G_2 yields another valid representative in $\mathrm{GL}_7(\mathbb{R})$. To resolve this ambiguity, we fix a smooth family of matrices $A_{\phi',p_i'} \in M_7(\mathbb{R})$ satisfying the initial condition $A_{\phi,p_i} = 0$. We interpret $A_{\phi',p_i'}$ as a linear vector field on $\mathbb{R}^7$ and hence consider its restriction to the ball $B(0,R)$. Translating by $(\Upsilon^i)^{-1}(p_i')$ and after that multiplying by the cut-off function ρ_{ε_0}, we obtain a vector field $v_{\phi',p_i'}$ supported in $B(0,2\varepsilon_0)$. In summary, this defines the smooth family

$$U_\phi \times U_{p_i} \to \mathrm{Vect}(B(0,R)), \quad (\phi', p_i') \mapsto v_{\phi',p_i'}, \tag{5.10}$$

where

$$v_{\phi',p_i'}(x) := \rho_{\varepsilon_0}\big(A_{\phi',p_i'}(x) + (\Upsilon^i)^{-1}(p_i')\big).$$

The time-1 flows of this family vary both the G_2-structure and the positions of the singularities together with the model associative cones.

By summing the smooth families of vector fields from equations (5.9) and (5.10), we obtain the desired smooth family of vector fields

$$U_{\tau_0} \to \prod_{i=1}^{m} \mathrm{Vect}(B(0,R)). \tag{5.11}$$

Definition 5.12 (weighted C^∞-topology) We now define the weighted C^∞-topology on the moduli space $\mathcal{M}_{\mathrm{cs}}$ by specifying a local basis around each element $(\phi, P) \in \mathcal{M}_{\mathrm{cs}}$. Given such an element, we make a choice of an ECS submanifold P_C as in Definition 5.2, and a choice of an EC tubular neighborhood map of P_C as in Definition 5.3, and consider a neighborhood U_{τ_0} as in (5.6), small enough so that, for all $\tau \in U_{\tau_0}$, the critical rates greater than 1 of the associative cones C_i^τ remain uniformly bounded away from 1. This allows us to choose a decay rate $\mu = (\mu_1, \dots, \mu_m)$ such that

$$2 > \mu_i > 1 \qquad \text{and} \qquad (1,\mu_i) \cap \mathcal{D}_{C_i^\tau} = \varnothing \quad \text{for all } \tau \in U_{\tau_0}. \tag{5.13}$$

This technical assumption is explained in Remark 5.14 below. For now, fix such a μ. The local basis of the topology is then given by all subsets of the form

$$\mathcal{V}^\mu_{\phi,P} := \{(\phi_\tau, \Upsilon_{P_C^\tau}(\Gamma_u)) : u \in \mathcal{V}_{P_C,\mu}, \tau \in \mathcal{V}_{\tau_0}\} \cap \mathcal{M}_{\mathrm{cs}},$$

where $\mathcal{V}_{\tau_0} \subset U_{\tau_0}$ and $\mathcal{V}_{P_C,\mu} \subset C^\infty_\mu(V_{P_C}) := C^\infty_{P_C,\mu} \cap C^\infty(V_{P_C})$ are open subsets. Here $C^\infty_{P_C,\mu}$ is the space of smooth normal vector fields of order $O(r^\mu)$ on P_C as defined in (4.19).

Remark 5.14 The construction of P_C and the tubular neighborhood map Υ_{P_C} involves several choices; however, the weighted C^∞-topology is independent of these choices.

At first glance, the above definition may appear to depend on the choice of the decay rate μ satisfying (5.13). However, for any other choice μ' satisfying (5.13), μ_i' must lie in the same connected component of $(1,2)\backslash\mathcal{D}_{C_i^\tau}$ as μ_i. Consequently, by Remark 5.34, we have $\mathcal{V}^{\mu}_{\phi,P} = \mathcal{V}^{\mu'}_{\phi,P}$ and the above definition does not depend on the choice.

Another advantage of choosing each μ_i close to 1 is to avoid crossing walls of critical rates. If μ were to cross such a wall, the corresponding set $\mathcal{V}^{\mu}_{\phi,P}$ — parametrizing CS associative submanifolds with stronger decay — would become strictly smaller, yet still open in $\mathcal{M}_{\mathrm{cs}}$, which is undesirable. In principle, one could define moduli subspaces by restricting to CS associative submanifolds with stronger decay rates; these would inherit the subspace topology from the weighted C^∞-topology introduced above. However, we do not explore this direction in the present article.

To understand the local structure of the moduli space $\mathcal{M}_{\mathrm{cs}}$ of CS associative submanifolds, we define below a nonlinear map whose zero set locally models $\mathcal{M}_{\mathrm{cs}}$. Let $P \in \mathcal{M}^{\phi}_{\mathrm{cs}}$ be a CS associative submanifold. We make a choice of an ECS submanifold P_C as in Definition 5.2, and a choice of an EC tubular neighborhood map of P_C as in Definition 5.3. There is a *canonical* bundle isomorphism from Definition 2.27,

$$\Theta^C_P : NP_C \to NP. \tag{5.15}$$

Choose U_{τ_0} and μ as in Definition 5.12. For each $\tau \in U_{\tau_0}$, the ECS submanifold P_C^τ and the associated EC tubular neighborhood map $\Upsilon_{P_C^\tau}$ be as in Definition 5.5.

Definition 5.16 Define $\mathfrak{F} : C^\infty_\mu(V_{P_C}) \times U_{\tau_0} \to C^\infty(NP_C)$ by

$$\langle \mathfrak{F}(u,\tau), w\rangle_{L^2} := \int_{\Gamma_u} \iota_w \Upsilon^*_{P_C^\tau}\psi_\tau \quad \text{for all } u \in C^\infty_\mu(V_{P_C}), \tau \in U_{\tau_0}, w \in C^\infty_c(NP_C).$$

The notation w in the integrand is the extension vector field of $w \in C^\infty(NP_C)$ in the tubular neighborhood as in Notation 2.25. The L^2 inner product we choose here comes from the canonical bundle isomorphism $\Theta^C_P : NP_C \to NP$ as in (5.15) and the induced metric on NP of g_ϕ.

Remark 5.17 The CS associative P is represented by a section $\alpha \in C^\infty_\mu(V_{P_C})$ from Definition 5.3, and therefore $\mathfrak{F}(\alpha,\tau_0) = 0$. Moreover, for $u \in C^\infty_\mu(V_{P_C})$ and $\tau \in U_{\tau_0}$ we have $\Upsilon_{P_C^\tau}(\Gamma_u) \in \mathcal{M}^{\phi_\tau}_{\mathrm{cs}}$ if and only if $\mathfrak{F}(u,\tau) = 0$.

To express the moduli space $\mathcal{M}^{\phi}_{\mathrm{cs}}$ locally as the zero set of a map (Kuranishi map) between finite-dimensional spaces, and to establish transversality results, we need to analyze the Fredholm property of the linearization of the nonlinear map introduced above. This is the focus of the following discussion.

Proposition 5.18 *For $\tau \in U_{\tau_0}$, the linearization of $\mathfrak{F}_\tau := \mathfrak{F}(\cdot,\tau)$ at $u \in C^\infty_{P_C,\mu}(V_{P_C})$,*

$$d\mathfrak{F}_{\tau|u} : C^\infty_{P_C,\mu} \to C^\infty_{P_C,\mu-1},$$

is given for all $v \in C^\infty_{P_C,\mu}$ and $w \in C^\infty_c(NP)$ by

$$\langle d\mathfrak{F}_{\tau|u}(v), w\rangle = \int_{\Gamma_u} \iota_w \mathcal{L}_v(\Upsilon^*_{P_C^\tau}\psi_\tau).$$

This is the same as

$$\int_{\Gamma_u} \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \langle [e_2, e_3, \nabla_{e_1} v], w\rangle + \iota_w \nabla_v (\Upsilon^*_{P_C^\tau} \psi_\tau) + \iota_{\nabla_w v} (\Upsilon^*_{P_C^\tau} \psi_\tau),$$

where $\{e_1, e_2, e_3\}$ is a local orthonormal frame for $T\Gamma_u$, and the associator $[\cdot,\cdot,\cdot]$ is defined with respect to $\Upsilon^*_{P_C{}^\tau}\phi_\tau$.

Proof For a family $\{u + tv \in C^\infty_\mu(V_{P_C}) : |t| \ll 1\}$, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathcal{F}_{u+tv}(w) = \frac{d}{dt}\Big|_{t=0} \int_{\Gamma_{u+tv}} \iota_w(\Upsilon^*_{P_C^\tau}\psi_\tau) &= \int_{\Gamma_u} \mathcal{L}_v \iota_w (\Upsilon^*_{P_C^\tau}\psi_\tau)\\ &= \int_{\Gamma_u} \iota_w \mathcal{L}_v(\Upsilon^*_{P_C^\tau}\psi_\tau) + \iota_{[v,w]}(\Upsilon^*_{P_C^\tau}\psi_\tau).\end{aligned}$$

As $[v, w] = 0$ (Notation 2.25), this is further equal to

$$\int_{\Gamma_u} \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \iota_w(\Upsilon^*_{P_C^\tau}\psi_\tau)(\nabla_{e_1} v, e_2, e_3) + \int_{\Gamma_u} \iota_{\nabla_w v}(\Upsilon^*_{P_C^\tau}\psi_\tau) + \iota_w \nabla_v(\Upsilon^*_{P_C^\tau}\psi_\tau). \qquad \square$$

Corollary 5.19 *Let $u \in C^\infty_\mu(V_{P_C})$ and $\tau \in U_{\tau_0}$. If $\Upsilon_{P_C^\tau}(\Gamma_u) \in \mathcal{M}_{\mathrm{cs}}$, a CS associative submanifold, then $d\mathfrak{F}_{\tau|_u}$ is given by*

$$\langle d\mathfrak{F}_{\tau|_u}(v), w\rangle_{L^2} = \int_{\Gamma_u} \left\langle \sum_{\substack{\text{cyclic}\\ \text{permutations}}} [e_2, e_3, \nabla_{e_1} v], w \right\rangle + \int_{\Gamma_u} \iota_w \nabla_v (\Upsilon^*_{P_C^\tau}\psi_\tau)$$

for all $v \in C^\infty_{P_C,\mu}$ and $w \in C^\infty_c(NP_C)$. $\square$

Proposition 5.20 *For $\tau \in U_{\tau_0}$, the linearization $d\mathcal{F}_{\tau|_u}$ of $\mathfrak{F}_\tau$ at $u \in C^\infty_{P_C,\mu}(V_{P_C})$ is a formally self-adjoint first-order differential operator.*

Proof For all $v, w \in C^\infty_c(NP_C)$, the difference $d\mathcal{F}_{\tau|_u}(v)(w) - d\mathcal{F}_{\tau|_u}(w)(v)$ is

$$\int_{\Gamma_u} \mathcal{L}_v \iota_w(\Upsilon^*_{P_C^\tau}\psi_\tau) - \mathcal{L}_w \iota_v(\Upsilon^*_{P_C^\tau}\psi_\tau) = \int_{\Gamma_u} \iota_w \mathcal{L}_v(\Upsilon^*_{P_C^\tau}\psi_\tau) + \iota_{[v,w]}(\Upsilon^*_{P_C^\tau}\psi_\tau) - \mathcal{L}_w \iota_v(\Upsilon^*_{P_C^\tau}\psi_\tau).$$

As $[v, w] = 0$ and $d\psi_\tau = 0$, this is equal to

$$\int_{\Gamma_u} \iota_w \iota_v(\Upsilon^*_{P_C^\tau} d\psi_\tau) - d(\iota_w \iota_v(\Upsilon^*_{P_C^\tau}\psi_\tau)) = 0. \qquad \square$$

Definition 5.21 We define the differential operator $\mathfrak{L}_P : C^\infty_c(NP_C) \to C^\infty_c(NP_C)$ by

$$\mathfrak{L}_P u := d\mathfrak{F}_{\tau_0|_\alpha}(u),$$

where the CS associative P is represented by $\alpha \in C^\infty_\mu(V_{P_C})$ as in Definition 5.3.

Proposition 5.22 *We have*

$$\mathfrak{L}_P = (\Theta_P^C)^{-1} \circ \boldsymbol{D}_P \circ \Theta_P^C,$$

where $\boldsymbol{D}_P : C_c^\infty(NP) \to C_c^\infty(NP)$ is the Fueter operator defined in Definition 2.32, and the canonical bundle isomorphism $\Theta_P^C : NP_C \to NP$ is as in (5.15).

Proof For all $v, w \in C_c^\infty(NP_C)$,

$$\begin{aligned}\langle \mathfrak{L}_P v, w\rangle_{L^2} = \langle \Theta_P^C \mathfrak{L}_P v, \Theta_P^C w\rangle_{L^2(NP)} &= \int_{\Gamma_\alpha} \Big\langle \sum_{\substack{\text{cyclic}\\ \text{permutations}}} [e_2, e_3, \nabla^\perp_{e_1} v], w \Big\rangle + \int_{\Gamma_\alpha} \iota_w \nabla_v (\Upsilon^*_{P_C} \psi) \\ &= \int_P \Big\langle \sum_{\substack{\text{cyclic}\\ \text{permutations}}} [e_2, e_3, \nabla^\perp_{e_1} (\Theta_P^C v)], \Theta_P^C w \Big\rangle + \int_P \iota_{\Theta_P^C w} \nabla_{\Theta_P^C v} \psi \\ &= \langle \boldsymbol{D}_P \Theta_P^C v, \Theta_P^C w\rangle_{L^2(NP)}.\end{aligned}$$

The second-to-last equality holds because $w - \Theta_P^C w \in TP$ and $v - \Theta_P^C v \in TP$, and $[\cdot,\cdot,\cdot]_{|P} = 0$. □

Proposition 5.23 *The operator $\mathfrak{L}_P$ is a conically singular uniformly elliptic operator asymptotic to the conical operators $\boldsymbol{D}_{C_i}$.*

Proof Since $\boldsymbol{D}_P$ is elliptic and $\mathfrak{L}_P = (\Theta_P^C)^{-1} \circ \boldsymbol{D}_P \circ \Theta_P^C$ by Proposition 5.22, $\mathfrak{L}_P$ is elliptic.

Without loss of generality we assume that $m = 1$ and $C = C_i$. It remains to prove that $\mathfrak{L}_P$ is CS asymptotic to $\boldsymbol{D}_C$. We substitute below $r = e^t$ and denote $C_t := (t, \infty) \times \Sigma$. For all $v, w \in C_c^\infty(NC_t)$,

$$\begin{aligned}\langle r^2 d\mathfrak{F}_{\tau_0|_0} \Upsilon_*(r^{-1} v), \Upsilon_* w\rangle_{L^2} &= \int_{C_t} r \iota_w \mathcal{L}_v (\Upsilon_C^* \Upsilon^* \psi) \\ &= \int_{C_t} r \iota_w \mathcal{L}_v (\Upsilon_C^* \psi_0) + O(e^{-t}) \|v\|_{W^{1,2}(NC)} \|w\|_{L^2(NC)}.\end{aligned}$$

By Proposition 5.32(i), we get

$$\langle \Upsilon^* \mathfrak{L}_P v, w\rangle_{L^2(NC)} = \langle \boldsymbol{D}_C v, w\rangle_{L^2(NC)} + O(r^{\mu-1}) \|v\|_{W^{1,2}(NC)} \|w\|_{L^2(NC)}.$$

This completes the proof of the proposition. □

Propositions 5.23 and 5.22 imply that $\boldsymbol{D}_P$ is also a CS uniformly elliptic operator and asymptotic to the conical operator $\boldsymbol{D}_C$. Thus $\mathfrak{L}_P$ and $\boldsymbol{D}_P$ has canonical extensions to weighted function spaces and this is the content of the following definition.

Definition 5.24 Let P be a CS associative submanifold and $\boldsymbol{D}_P : C_c^\infty(NP) \to C_c^\infty(NP)$ be the Fueter operator defined in Definition 2.32. Proposition 5.23 implies that it is a conically singular uniformly elliptic operator. Therefore it has canonical extensions

$$\boldsymbol{D}^{k,p}_{P,\lambda} : W^{k+1,p}_{P,\lambda} \to W^{k,p}_{P,\lambda-1} \quad \text{and} \quad \boldsymbol{D}^{k,\gamma}_{P,\lambda} : C^{k+1,\gamma}_{P,\lambda} \to C^{k,\gamma}_{P,\lambda-1}.$$

Similarly we have canonical extensions of the operator $\mathfrak{L}_P$ to weighted function spaces,

$$\mathfrak{L}^{k,p}_{P,\lambda} : W^{k+1,p}_{P_C,\lambda} \to W^{k,p}_{P_C,\lambda-1} \quad \text{and} \quad \mathfrak{L}^{k,\gamma}_{P,\lambda} : C^{k+1,\gamma}_{P_C,\lambda} \to C^{k,\gamma}_{P_C,\lambda-1}.$$

Definition 5.25 (asymptotic limit map) We define the *asymptotic limit map*

$$i_{P,\lambda} : \operatorname{Ker} \boldsymbol{D}_{P,\lambda} \to V_\lambda$$

to be the map $i_{P,\lambda}$ defined in Lemma 4.31 for $\boldsymbol{D}_P$.

The following proposition computes the linearization of the nonlinear map $\mathfrak{F}$ with respect to variations in the singularity data — that is, the effect of moving the singular points and deforming their associated model cones.

Proposition 5.26 *The linearization of $\mathfrak{F}(\alpha, \phi, \cdot)$ at $(p_1, \dots, p_m, 0, \dots, 0)$,*

$$L_1 : \bigoplus_{i=1}^{m} (\mathbb{R}^7 \oplus V_{1,i}) \to C^\infty_{P_C,\mu-1},$$

is of the form

$$L_1(\hat{p}_i, \xi_i) = \mathfrak{L}_P\big(\rho_{\varepsilon_0} \Upsilon^i_*(\hat{p}_i^\perp + \xi_i)\big) + O(r^{\mu-1}).$$

Here, $\mathbb{R}^7$ is identified with the tangent space of U_{p_i} at p_i by the framing υ_i, and $V_{1,i}$ is the homogeneous kernel of the cone C_i whose link is Σ_i, at rate $\lambda = 1$ in the sense of Definition 3.20. The latter is identified with the tangent space of $\mathcal{I}_{\Sigma_i}$ at 0, which is again $T_{\Sigma_i}\mathcal{M}^{\mathrm{hol}}$. The vector $\hat{p}_i \in \mathbb{R}^7$ projects to a normal vector field $\hat{p}_i^\perp$ on C_i, and $\xi_i \in V_{1,i}$ is already a normal vector field on C_i.

Proof Without loss of generality, we assume that $m = 1$ and $C = C_i$. Suppose $\xi \in V_1$ and for each small t, set $\tau_t := (\phi, p, t\xi)$. For all $w \in C^\infty_c(NP_C)$, we have

$$\langle L_1(0,\xi), w\rangle_{L^2} = \int_{\Gamma_\alpha} \frac{d}{dt}\Big|_{t=0} \iota_w(\Upsilon^*_{P_C^{\tau_t}}\psi) = \int_{\Gamma_\alpha} \iota_w \mathcal{L}_{(\rho_{\varepsilon_0}\xi)}(\Upsilon^*_{P_C}\psi).$$

Suppose $\hat{p} \in \mathbb{R}^7$ and for each small t, set $\tau_t := (\phi, \Upsilon(t\hat{p}), 0)$. For all $w \in C^\infty_c(NP_C)$, we have

$$\begin{aligned}\langle L_1(\hat{p}, 0), w\rangle_{L^2} &= \int_{\Gamma_\alpha} \frac{d}{dt}\Big|_{t=0} \iota_w(\Upsilon^*_{P_C^{\tau_t}}\psi) \\ &= \int_{\Gamma_\alpha} \iota_w \mathcal{L}_{(\rho_{\varepsilon_0}\hat{p}^\perp)}(\Upsilon^*_{P_C}\psi) + O(r^{\mu-1})\|w\|_{L^2}.\end{aligned}$$

The proposition now follows from Definition 5.21 of $\mathfrak{L}_P$. □

Remark 5.27 The expression for L_1 in Proposition 5.26 is given only up to terms of order $O(r^{\mu-1})$ near the singularities. This imprecision poses no issue for the purposes of this article, as the explicit form of L_1 is not used except in Lemma 7.3. In that lemma, the ambiguity is inconsequential.

The operator $\mathfrak{L}_P$ from Definition 5.21 governs the associative deformations of P with fixed singularity data. By incorporating the additional operator L_1 from Proposition 5.26, we define the extended operator $\widetilde{\mathfrak{L}}_P$ in the following lemma, which is less obstructed and captures the broader class of deformations in which the singularities are permitted to move and the model cones may vary.

Definition 5.28 Let P be a CS associative as in Definition 4.13 asymptotic to C_i for $i=1,\dots,m$. Denote the link of C_i by Σ_i. Let $\Sigma_i=\bigsqcup_{j=1}^{l}\Sigma_i^j$ be the decomposition into connected components. Let $\mathcal{Z}_i^j$ be the manifold in the decomposition (1.4) that contains Σ_i^j. Set $\mathcal{Z}_i=\prod_{j=1}^{l}\mathcal{Z}_i^j$. We also denote by Σ_i the product $\prod_{j=1}^{l}\Sigma_i^j\in\mathcal{Z}_i$. Since $T_{\Sigma_i}\mathcal{Z}_i\subset T_0\mathcal{I}_{\Sigma_i}=V_{1,i}$, we define

$$\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}:C^{\infty}_{P_C,\mu}\oplus\left(\bigoplus_{i=1}^{m}(\mathbb{R}^7\oplus T_{\Sigma_i}\mathcal{Z}_i)\right)\to C^{\infty}_{P_C,\mu-1}$$

to be the restriction of the linearization of $\mathfrak{F}(\cdot,\phi,\cdot)$ at $(\alpha,\phi,p_1,\dots,p_m,0,\dots,0)$, that is,

$$\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}(u,\hat{p}_i,\xi_i)=\mathfrak{L}_P u+L_1(\hat{p}_i,\xi_i),$$

where $u\in C^{\infty}_{P_C,\mu}$, $\hat{p}_i\in\mathbb{R}^7$ and $\xi_i\in T_{\Sigma_i}\mathcal{Z}_i$. The operators $\mathfrak{L}_P$ and L_1 are from Definition 5.21 and Proposition 5.26.

The operator

$$\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}:C^{\infty}_{P,\mu}\oplus\left(\bigoplus_{i=1}^{m}(\mathbb{R}^7\oplus T_{\Sigma_i}\mathcal{Z}_i)\right)\to C^{\infty}_{P,\mu-1}$$

is the operator $\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}$ under the identification map Θ_P^C from (5.15), that is,

$$\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}(u,\hat{p}_i,\xi_i):=\Theta_P^C\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}((\Theta_P^C)^{-1}u,\hat{p}_i,\xi_i)=\boldsymbol{D}_P u+\Theta_P^C L_1(\hat{p}_i,\xi_i).$$

The following proposition computes the linearization of the nonlinear map $\mathfrak{F}$ with respect to variations of the coclosed G_2-structures.

Proposition 5.29 *The linearization of $\mathfrak{F}(\alpha,\cdot,p_1,\dots,p_m,0,\dots,0)$ at ϕ,*

$$L_2:T_\phi\mathscr{P}\to C^{\infty}_{P_C,\mu-1},$$

satisfies the following: for any $\hat{\phi}\in T_\phi\mathscr{P}=\Omega^3(Y)$ and $w\in C_c^{\infty}(NP_C)$, we have

$$\langle L_2(\hat{\phi}),w\rangle_{L^2}=\int_{\Gamma_\alpha}\iota_w(\Upsilon^*_{P_C}\hat{\psi})+\sum_{i=1}^{m}\int_{\Gamma_\alpha}\iota_w\mathcal{L}_{\Upsilon^i_*X_{\hat{\phi},i}}(\Upsilon^*_{P_C}\psi),$$

where $\hat{\psi}:=(d/dt)|_{t=0}\psi_t$ with $\psi_t=*_{\phi_t}\phi_t$ and $\phi_t:=\phi+t\hat{\phi}$, and the vector field $X_{\hat{\phi},i}:=v_{\phi+\hat{\phi},p_i}$ is from (5.10).

Proof Without loss of generality we assume that $m=1$ and $C=C_i$. Suppose $\hat{\phi}\in T_\phi\mathscr{P}$ and for each small t, set $\tau_t:=(\phi_t,p,0)$. For all $w\in C_c^{\infty}(NP_C)$, we have

$$\langle L_2(\hat{\phi}),w\rangle_{L^2}=\int_{\Gamma_\alpha}\frac{d}{dt}\Big|_{t=0}\iota_w(\Upsilon^*_{P_C^{\tau_t}}\psi_t)=\int_{\Gamma_\alpha}\iota_w(\Upsilon^*_{P_C}\hat{\psi})+\int_{\Gamma_\alpha}\iota_w\mathcal{L}_{\Upsilon_*X_{\hat{\phi}}}(\Upsilon^*_{P_C}\psi).$$

The proposition now follows from Definition 5.21 of $\mathfrak{L}_P$. □

The following definition introduces the one-parameter moduli space of CS associative submanifolds, along with the corresponding linear operators used to describe the local structure of this moduli space.

Definition 5.30 Let $\mathscr{P}$ be the space of paths $\boldsymbol{\phi}:[0,1]\to\mathscr{P}$ that are smooth as a section over $[0,1]\times Y$. Equip $\mathscr{P}$ with the C^∞-topology. Define the one-parameter moduli space of CS associatives by the fiber product

$$\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}} := [0,1]\times_{\mathscr{P}}\mathcal{M}_{\mathrm{cs}}.$$

If $\boldsymbol{\phi}\in\mathscr{P}$ and $(t_0,P)\in\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$, we set

$$\widehat{\phi} := \frac{d}{dt}\Big|_{t=t_0}\phi_t, \quad f_P := L_2(\widehat{\phi})\in C^\infty(NP_C) \quad\text{and}\quad \widehat{f}_P := \Theta_P^C f_P\in C^\infty(NP),$$

where L_2 and Θ_P^C are as in Proposition 5.29 and equation (5.15), respectively. We define

$$\overline{\mathfrak{L}}_{P,\mu,\mathcal{Z}}:\mathbb{R}\oplus C^\infty_{P_C,\mu}\oplus\Big(\bigoplus_{i=1}^m(\mathbb{R}^7\oplus T_{\Sigma_i}\mathcal{Z}_i)\Big)\to C^\infty_{P_C,\mu-1},\quad \overline{\mathfrak{L}}_{P,\mu,\mathcal{Z}}(t,\widetilde{u})=\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}\widetilde{u}+tf_P,$$

where $\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}$ is defined in Definition 5.28.

The operator

$$\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}:\mathbb{R}\oplus C^\infty_{P,\mu}\oplus\Big(\bigoplus_{i=1}^m(\mathbb{R}^7\oplus T_{\Sigma_i}\mathcal{Z}_i)\Big)\to C^\infty_{P,\mu-1}$$

is the operator $\overline{\mathfrak{L}}_{P,\mu,\mathcal{Z}}$ under the identification map Θ_P^C, that is,

$$\bar{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}(t,\widetilde{u})=\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}\widetilde{u}+t\,\widehat{f}_P,$$

where $\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ is again defined in Definition 5.28

The final ingredient we need before proving Theorem 1.12 about the local structure of the moduli space is a quadratic estimate, which we now proceed to establish.

Definition 5.31 For all $\tau\in U_{\tau_0}$, the nonlinear map $Q_\tau:C^\infty_\mu(V_{P_C})\to C^\infty(NP_C)$ is defined by

$$Q_\tau := \mathfrak{F}_\tau - d\mathfrak{F}_{\tau|_0} - \mathfrak{F}_\tau(0).$$

Proposition 5.32 *For all $\tau\in U_{\tau_0}$ and $u,v\in C^\infty_{P_C,\mu}(V_{P_C})$, and $\eta\in C^\infty_{P_C,\mu}$, we have*

(i) $|d\mathfrak{F}_{\tau|_u}(\eta)-d\mathfrak{F}_{\tau|_v}(\eta)|\lesssim(w_{P_C,1}|u-v|+|\nabla^\perp(u-v)|)(w_{P_C,1}|\eta|+|\nabla^\perp\eta|)$,

(ii) $|Q_\tau(u)-Q_\tau(v)|\lesssim(w_{P_C,1}|u|+|\nabla^\perp u|+w_{P_C,1}|v|+|\nabla^\perp v|)(w_{P_C,1}|u-v|+|\nabla^\perp(u-v)|)$,

(iii) $\|Q_\tau(u)\|_{C^0_{P_C,\mu-1}}\lesssim\|Q_\tau(u)\|_{C^0_{P_C,2\mu-2}}\lesssim\|u\|^2_{C^1_{P_C,\mu}}$.

Proof Since τ is fixed we abuse notation and denote ψ_τ by ψ. For all $w\in C^\infty_c(NP_C)$ and $\eta\in C^\infty_{P_C,\mu}$, we write

$$\langle d\mathfrak{F}_{\tau|_u}(\eta)-d\mathfrak{F}_{\tau|_v}(\eta),w\rangle=\int_0^1\frac{d}{dt}\big(d\mathcal{F}_{\tau|_{tu+(1-t)v}}(\eta)(w)\big)\,dt,$$

and using Proposition 5.18, this becomes

$$\int_0^1\Big(\frac{d}{dt}\int_{\Gamma_{tu+(1-t)v}}L_{\eta\lrcorner w}(\Upsilon^*_{P_C^\tau}\psi)\Big)\,dt=\int_0^1\int_{\Gamma_{tu+(1-t)v}}L_{(u-v)}L_{\eta\lrcorner w}(\Upsilon^*_{P_C^\tau}\psi)\,dt.$$

As $[u-v,w]=0$ and $[\eta,w]=0$, the last expression is the same as

$$\int_0^1 \int_{\Gamma_{tu+(1-t)v}} \iota_w L_{(u-v)} L_\eta (\Upsilon^*_{P_C^\tau} \psi)\, dt.$$

The required estimate in (i) now follows from Lemma A.1. The estimate in (ii) follows from (i) after writing

$$Q_\tau(u) - Q_\tau(v) = \int_0^1 dQ_{\tau_{|tu+(1-t)v}}(u-v)\, dt = \int_0^1 \big(d\mathfrak{F}_{\tau_{|tu+(1-t)v}}(u-v) - d\mathfrak{F}_{\tau_{|0}}(u-v)\big)\, dt.$$

Finally, (iii) follows from (ii). Indeed, substituting $v=0$ we have

$$w_{P_C,2\mu-2}|Q_\xi(u)| \lesssim w_{P_C,2\mu-2}(w_{P_C,1}|u| + |\nabla^\perp u|)^2 \lesssim (w_{P_C,\mu}|u| + w_{P_C,\mu-1}|\nabla^\perp u|)^2.$$

Since $\mu>1$, it follows $w_{P_C,\mu-1}|Q_\xi(u)| \lesssim w_{P_C,2\mu-2}|Q_\xi(u)|$. This completes the proof. □

Proposition 5.33 *For all $\tau \in U_{\tau_0}$ and $u, v \in C^{k+1,\gamma}_{P_C,\mu}(V_{P_C})$, we have*

$$\|Q_\tau(u) - Q_\tau(v)\|_{C^{k,\gamma}_{P_C,\mu-1}} \lesssim \|Q_\tau(u) - Q_\tau(v)\|_{C^{k,\gamma}_{P_C,2\mu-2}} \lesssim \|u-v\|_{C^{k+1,\gamma}_{P_C,\mu}} \big(\|u\|_{C^{k+1,\gamma}_{P_C,\mu}} + \|v\|_{C^{k+1,\gamma}_{P_C,\mu}}\big).$$

Proof Since τ is fixed we abuse notation and again denote ψ_τ by ψ. With the above notation and appropriate product operation $\cdot$, one can express $\mathcal{L}_u\mathcal{L}_v\psi$ formally as a quadratic polynomial

$$\mathcal{L}_u\mathcal{L}_v\psi = O(f_1)\cdot u\cdot v + O(f_2)\cdot(u\cdot\nabla^\perp v + v\cdot\nabla^\perp u) + \psi\cdot\nabla^\perp u\cdot\nabla^\perp v,$$

where $O(f_1) := \psi\cdot\nabla B + \psi\cdot B\cdot B + B\cdot\nabla\psi + \nabla^2\psi + R\cdot\psi$ and $O(f_2) := \nabla\psi + B\cdot\psi$. With this observation and a similar computations as in Lemma A.1 one can prove the proposition. □

Proof of Theorem 1.12 By Propositions 5.33, 5.18 and 5.26 we conclude that the extension of the map $\mathcal{F}_\phi := \mathcal{F}(\cdot,\phi,\cdot)$ to weighted Hölder spaces

$$\mathfrak{F}_{\phi,\mathcal{Z}} : C^{2,\gamma}_{P_C,\mu}(V_{P_C}) \times \prod_{i=1}^m U_{p_i} \times \prod_{i=1}^m \mathcal{Z}_i \cap \mathcal{I}_{\Sigma_i} \to C^{1,\gamma}_{P_C,\mu-1}$$

is a well-defined smooth map. Here the rate μ is chosen as in Definition 5.12. By Proposition 4.30 we have that the linearization of $\mathfrak{F}_{\phi,\mathcal{Z}}$ at $(\alpha, p_1,\dots,p_m,0,\dots,0)$, $\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}}$ is a Fredholm operator. Moreover, Proposition 4.33 implies that

$$\operatorname{index}\widetilde{\mathfrak{L}}_{P,\mu,\mathcal{Z}} = -\sum_{i=1}^m \Big(\frac{d_{-1,i}}{2} + \sum_{\lambda_i\in\mathcal{D}_{C_i}\cap(-1,1]} d_{\lambda_i}\Big) + \sum_{i=1}^m (7+\dim\mathcal{Z}_i) = -\sum_{i=1}^m \text{s-ind}(C_i).$$

Applying the implicit function theorem to $\mathfrak{F}_{\phi,\mathcal{Z}}$ and the elliptic regularity from Proposition 4.25 we obtain the existence of $\mathrm{ob}_{P,\mathcal{Z}}$ in (i) (see [Donaldson and Kronheimer 1990, Proposition 4.2.19]) of the theorem, except the following. If $u\in C^{2,\gamma}_{P_C,\mu}(V_{P_C})$ with $\mathfrak{F}_{\phi_0,\mathcal{Z}}(u,\xi)=0$ for some $\xi\in\prod_{i=1}^m U_{p_i}\times\prod_{i=1}^m \mathcal{Z}_i\cap\mathcal{I}_{\Sigma_i}$, then $u\in C^{\infty}_{P_C,\mu}(V_{P_C})$. To prove this, we observe

$$0 = \mathfrak{L}_P\mathfrak{F}_{\phi,\mathcal{Z}}(u,\xi) = A_\xi(u,\nabla^\perp_{P_C}u)(\nabla^\perp_{P_C})^2u + B_\xi(u,\nabla^\perp_{P_C}u).$$

Since $A_\xi(u, \nabla^\perp_{P_C} u)$ and $B_\xi(u, \nabla^\perp_{P_C} u)$ are in $C^{1,\gamma}_{P_C,\mu}$, by a weighted version of Schauder elliptic regularity for second-order operator (similar to Proposition 4.25) we obtain $u \in C^{3,\gamma}_{P_C,\mu}$. By repeating this argument we get higher regularity. This completes the proof of (i).

The proof of (ii) is similar to (i). Indeed, we replace $\mathfrak{F}_{\phi,\mathcal{Z}}$ by $\mathfrak{F}_{\boldsymbol{\phi},\mathcal{Z}}$ as follows. There exists an interval $I \subset [0,1]$ containing t_0 such that $\boldsymbol{\phi}(I) \subset U_{\phi_{t_0}}$, which was defined in Definition 5.12. We define

$$\mathfrak{F}_{\boldsymbol{\phi},\mathcal{Z}} : I \times C^{2,\gamma}_{P_C,\mu}(V_{P_C}) \times \prod_{i=1}^m U_{p_i} \times \prod_{i=1}^m \mathcal{Z}_i \cap \mathcal{I}_{\Sigma_i} \to C^{1,\gamma}_{P_C,\mu-1}$$

by $\mathfrak{F}_{\boldsymbol{\phi},\mathcal{Z}}(t,\cdot) := \mathfrak{F}_{\phi_t,\mathcal{Z}}$, $t \in I$. The linearization of $\mathfrak{F}_{\boldsymbol{\phi},\mathcal{Z}}$ is $\bar{\mathfrak{L}}_{P,\mu,\mathcal{Z}}$ as in Definition 5.30. The remaining proof is left to the reader. □

Remark 5.34 If μ' were another choice in Definition 5.12 distinct from μ, then $u \in C^\infty_{P_C,\mu}(V_{P_C})$ satisfying $\mathfrak{F}_\tau(u) = 0$ implies $u \in C^\infty_{P_C,\mu'}(V_{P_C})$. To see this we write $d\mathfrak{F}_{\tau|0}(u) = -Q_\tau(u) - \mathfrak{F}_\tau(0)$. By Proposition 5.32, $d\mathfrak{F}_{\tau|0}$ is also a CS uniformly elliptic operator asymptotic to $\boldsymbol{D}_C$. Also we have $Q_\tau(u) \in C^{k,\gamma}_{P_C,2\mu-2}$ and $\mathfrak{F}_\tau(0) \in C^{k,\gamma}_{P_C,1}$. Therefore by Lemma 4.31 we can conclude that $u \in C^\infty_{P_C,\mu'}(V_{P_C})$.

5.2 Proof of genericity results: Floer's C_ε-space and Taubes' trick

Proof of Theorem 1.17 We prove that $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},\mathcal{Z}}$ is comeager in $\mathscr{P}$ in Lemma 5.39. The proof of the fact that $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},\mathcal{Z}}$ is comeager in $\mathscr{P}$ is similar. Then (i) and (ii) in Theorem 1.17 are direct consequences of (i) and (ii) in Theorem 1.12. □

To prove that $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},\mathcal{Z}}$ is comeager in $\mathscr{P}$ in Lemma 5.39, we would like to use the Sard–Smale theorem applied to the map

$$\mathfrak{F}_{\mathcal{Z}} : W^{k+1,p}_{P_C,\mu}(V_{P_C}) \times U_\phi \times \prod_{i=1}^m U_{p_i} \times \prod_{i=1}^m \mathcal{Z}_i \cap \mathcal{I}_{\Sigma_i} \to W^{k,p}_{P_C,\mu-1}.$$

We have chosen Sobolev spaces rather than Hölder spaces as the former is separable. The reader may observe that all the analysis in Section 5.1 will also go through with Sobolev spaces. A serious problem here is that $U_\phi \subset \mathscr{P}$ is not a Banach manifold. The standard way to deal with this is to consider $\mathscr{P}_k$, the space of all C^k coclosed G_2-structures on Y, to enlarge U_ϕ to a Banach manifold. This comes with a drawback, that the map $\mathfrak{F}$ will have only finitely many derivatives and extra effort is required to check exactly how many in order to state the regularity of the moduli space. To avoid all this we will instead use Floer's C_ε-space $\mathscr{P}_\varepsilon \subset \mathscr{P}$ of coclosed G_2-structures. Since the Sard–Smale theorem yields the genericity results in $\mathscr{P}_\varepsilon$ instead of $\mathscr{P}$, we use Taubes' trick of exhausting the moduli spaces by countably many compact subsets (Definition 5.35) to conclude the genericity results in $\mathscr{P}$. For more details on this idea, see [Wendl 2021].

Definition 5.35 For every $N, m \in \mathbb{N}$ and $\mathcal{Z} = \prod_{i=1}^m \mathcal{Z}_i$, we define $\mathcal{M}^\phi_{\mathrm{cs},N,\mathcal{Z}} \subset \mathcal{M}^\phi_{\mathrm{cs},\mathcal{Z}}$ to be the set of all CS associative submanifolds P in (Y,ϕ) with singularities at p_i, cones C_i with links $\Sigma_i \in \mathcal{Z}_i$, rates $\mu_i \in [1+1/N, 2]$, G_2-coordinate systems $\Upsilon^i : B(0,R) \to Y$ and embeddings $\Upsilon^i_P : (0, 2\varepsilon_0) \times \Sigma_i \to B(0,R)$ for $i = 1, \ldots, m$ satisfying:

- $1 > R \geqslant 4\varepsilon_0 \geqslant 1/N$. $|\nabla^k \mathrm{II}_{\Sigma_i}| \leqslant N$, $\mathrm{Vol}(\Sigma_i) \leqslant N$ for all $k = 0, \dots 3$,
- $d_{\mathcal{M}^{\mathrm{hol}}}(\Sigma_i, \overline{\mathcal{Z}}_i \setminus \mathcal{Z}_i) \geqslant 1/N$.
- $|\nabla^k \mathrm{II}_{K_P}| \leqslant N$ for all $k = 0, \dots 3$ and $\mathrm{Vol}(P) \leqslant N$.
- $|\nabla^k \Upsilon^i| \leqslant N$ and $|(\nabla^{\perp}_C)^k(\Psi^i_P - \iota)| \leqslant N r^{\mu_i - k}$ for all $k = 0, \dots 3$.
- $N|x_1 - x_2| \geqslant d_{\Sigma_i}(x_1, x_2)$ for all $x_1, x_2 \in \Sigma_i$, and $N d_Y(p_1, p_2) \geqslant d_P(p_1, p_2)$ for all $p_1, p_2 \in P$.

We define $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}} \subset \mathscr{P}$ to be the set of all $\phi \in \mathscr{P}$ for which every $P \in \mathcal{M}^{\phi}_{\mathrm{cs},N,\mathcal{Z}}$ has the property that $\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ is surjective.

Remark 5.36 Obviously,

$$\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},\mathcal{Z}} = \bigcap_{N \in \mathbb{N}} \mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}} \subset \mathscr{P}.$$

The next lemma about convergence of CS associative submanifolds can be deduced from [Shen 1995].

Lemma 5.37 *Let $\phi_n \in \mathscr{P}$ be a sequence of G_2-structures on Y converging to ϕ in $\mathscr{P}$ with C^∞-topology. Fix $N \in \mathbb{N}$ and $\mathcal{Z}$, and let $P_n \in \mathcal{M}^{\phi_n}_{\mathrm{cs},N,\mathcal{Z}}$ be a sequence of CS associative submanifolds. Then there is a subsequence of P_n converging to a CS associative $P \in \mathcal{M}^{\phi}_{\mathrm{cs},N,\mathcal{Z}}$ with weighted C^∞-topology defined in Definition 5.12.*

Lemma 5.38 *For every $N \in \mathbb{N}$ and $\mathcal{Z}$, $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}}$ is open in $\mathscr{P}$.*

Proof The complement of $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}}$ is closed. Indeed, it follows from Lemma 5.37 and the fact that surjectivity is an open condition. □

Lemma 5.39 *For every $N \in \mathbb{N}$ and $\mathcal{Z}$, $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}}$ is dense in $\mathscr{P}$. Hence $\mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},\mathcal{Z}}$ is comeager in $\mathscr{P}$.*

To prove Lemma 5.39 we use Floer's C_ε-space.

Definition 5.40 Let $E \to M$ be a vector bundle over a compact Riemannian manifold M with or without boundary. For each integer $k \geqslant 0$, we denote by $C^k(E)$ the Banach space of C^k-sections of E. Let $\mathcal{E}$ be the set of all sequences $(\varepsilon_k)_{k=0}^{\infty}$ of positive real numbers with $\varepsilon_k \to 0$. For each $\varepsilon \in \mathcal{E}$, *Floer's C_ε-space* of sections of E is defined by

$$C_\varepsilon(E) := \left\{ s \in C^\infty(E) : \|s\|_{C_\varepsilon} := \sum_{k=0}^{\infty} \varepsilon_k \|s\|_{C^k} < \infty \right\}.$$

A pre-order $\prec$ on $\mathcal{E}$ is defined by

$$\varepsilon \prec \varepsilon' \quad \text{if and only if} \quad \limsup_{k \to \infty} \frac{\varepsilon_k}{\varepsilon'_k} < \infty.$$

Remark 5.41 As M is compact, different C^k-norms for different smooth atlases of M are equivalent, but the norm on Floer's C_ε-space is an infinite sum, therefore it might not be equivalent. But this is not a big concern, as no statement in any theorem will mention this space; it will only be used inside the proofs and where we can fix an atlas.

Lemma 5.42 [Wendl 2016, Appendix B] *Floer's C_ε-spaces $C_\varepsilon(E)$ have the following properties*:

(i) *$C_\varepsilon(E)$ is a separable Banach space and the inclusion $C_\varepsilon(E) \hookrightarrow C^\infty(E)$ is continuous. Moreover, $C_{\varepsilon'}(E) \hookrightarrow C_\varepsilon(E)$ is a continuous embedding if $\varepsilon \prec \varepsilon'$.*

(ii) *For every countable subset $\boldsymbol{Q}$ of $C^\infty(E)$ there exists an $\varepsilon_0 \in \mathcal{E}$ such that $\boldsymbol{Q} \subset C_\varepsilon(E)$ for all $\varepsilon \prec \varepsilon_0$. In particular,*

$$C^\infty(E) = \bigcup_{\varepsilon \in \mathcal{E}} C_\varepsilon(E).$$

To prove Lemma 5.39, we will also require the following lemma. It essentially asserts the existence of a finite-dimensional subfamily of $\mathscr{P}$ in which the CS associative submanifold P is unobstructed, so that the moduli space of CS associatives in a neighborhood of P within this subfamily forms a smooth manifold. Furthermore, by Sard's theorem, for a generic member of this subfamily, the CS associatives in this neighborhood of P are all unobstructed.

Lemma 5.43 [Doan and Walpuski 2019, Proposition A.2] *Let P be a CS associative in (Y, ϕ) with singularities at p_i and rates $\mu_i \in (1,2] \backslash \mathcal{D}_{C_i}$. Then for all $w \in \ker \mathfrak{L}_{P,-2-\mu} \cong \operatorname{coker} \mathfrak{L}_{P,\mu}$, there exists a 3-form $\widehat{\phi} \in T_\phi \mathscr{P}$ supported away from p such that*

$$\langle L_2(\widehat{\phi}), w \rangle_{L^2} \neq 0, \quad \text{where } L_2 \text{ is as defined in Proposition 5.29.}$$

Proof By unique continuation, there exists an open set U in the interior of K_P on which w does not vanish identically. Let V be a tubular neighborhood of U in Y. Choose a real-valued smooth function f supported in V so that $df(w) \geqslant 0$ on U and there exists a point in U where $df(w) > 0$. Let $\Theta \in \Omega^3(Y)$ be an extension of the 3-form vol_U and $\iota_w d\Theta_{|V} = 0$. Then there exists a 3-form $\widehat{\phi} \in T_\phi \mathscr{P}$ supported in V such that $\widehat{\psi} = d(f\Theta) \in \Omega^4(Y)$, where $\widehat{\psi}$ is as in Proposition 5.29. By Proposition 5.29, we have

$$\langle L_2(\widehat{\phi}), w \rangle_{L^2} = \int_{\Gamma_\alpha} \iota_w (\Upsilon_{P_C}^* \widehat{\psi}) = \int_U df(w) \operatorname{vol}_U > 0. \qquad \square$$

Proof of Lemma 5.39 For each $\varepsilon \in \mathcal{E}$, let $\Omega^3_\varepsilon(Y)$ be Floer's C_ε-space of smooth 3-forms on Y. We define

$$\mathscr{P}_\varepsilon := \mathscr{P} \cap \Omega^3_\varepsilon(Y) \quad \text{and} \quad \mathcal{M}_{\mathrm{cs},\varepsilon,\mathcal{Z}} := \{(\phi, P) \in \mathcal{M}_{\mathrm{cs},\mathcal{Z}} : \phi \in \mathscr{P}_\varepsilon, P \in \mathcal{M}^\phi_{\mathrm{cs},\mathcal{Z}}\}.$$

Suppose $(\phi, P) \in \mathcal{M}_{\mathrm{cs},\varepsilon,\mathcal{Z}}$. Let p_i for $i = 1, \ldots, m$ be the singular points of P. Choose U_{τ_0} and μ as in Definition 5.12. Set

$$U_{\tau_0,\varepsilon,\mathcal{Z}} := (U_\phi \cap \mathscr{P}_\varepsilon) \times \prod_{i=1}^m U_{p_i} \times \prod_{i=1}^m \mathcal{Z}_i.$$

Let $\mathfrak{F}_{\varepsilon,\mathcal{Z}} : W^{2,p}_\mu(V_{P_C}) \times U_{\tau_0,\varepsilon,\mathcal{Z}} \to W^{1,p}_{P_C,\mu-1}$ be the restriction of $\mathfrak{F}$. We define

$$\mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}} := \{(\phi, P) \in \mathcal{M}_{\mathrm{cs},\varepsilon,\mathcal{Z}} : d\mathfrak{F}_{\varepsilon,\mathcal{Z}|\tau_0} \text{ is surjective}\}.$$

By the implicit function theorem [McDuff and Salamon 2012, Theorem A.3.3] we obtain that $\mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}}$ is a separable Banach manifold. Moreover, by the Sard–Smale theorem [loc. cit., Lemma A.3.6], we see that the canonical projection map $\pi_{\varepsilon,\mathcal{Z}} : \mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}} \to \mathscr{P}_\varepsilon$ is a Fredholm map. Therefore, again by [loc. cit., Theorem A.3.3], we conclude that there exists a comeager subset $\mathscr{P}^{\mathrm{reg}}_{\varepsilon,\mathrm{cs},\mathcal{Z}} \subset \mathscr{P}_\varepsilon$ having the property that for each $\phi \in \mathscr{P}^{\mathrm{reg}}_{\varepsilon,\mathrm{cs},\mathcal{Z}}$ the linear operator $\widetilde{\boldsymbol{D}}_{P,\mu,\mathcal{Z}}$ is surjective for all P with $(\phi, P) \in \mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}}$.

With this preparation we are now ready to prove the lemma. Suppose $\phi_0 \in \mathscr{P}$. We can choose ε sufficiently small so that $\phi_0 \in \mathscr{P}_\varepsilon$. Since $\mathscr{P}^{\mathrm{reg}}_{\varepsilon,\mathrm{cs},\mathcal{Z}}$ is dense in $\mathscr{P}_\varepsilon$, there exists a sequence $\phi_n \in \mathscr{P}^{\mathrm{reg}}_{\varepsilon,\mathrm{cs},\mathcal{Z}}$ converging to ϕ_0 in C_ε-topology and hence in C^∞-topology. Our claim is that $\phi_n \in \mathscr{P}^{\mathrm{reg}}_{\mathrm{cs},N,\mathcal{Z}}$ for all n sufficiently large. If not, then there is a subsequence $P_n \in \mathcal{M}^{\phi_n}_{\mathrm{cs},N,\mathcal{Z}}$ such that $\widetilde{\boldsymbol{D}}_{P_n,\mu_n,\mathcal{Z}}$ is not surjective. By Lemma 5.37 we obtain that P_n converges to a CS associative P in (Y,ϕ_0). Hence, by Lemmas 5.43 and 5.42, $(\phi_0, P) \in \mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}}$ for all sufficiently small ε. Therefore $(\phi_n, P_n) \in \mathcal{M}^{\mathrm{reg}}_{\mathrm{cs},\varepsilon,\mathcal{Z}}$ for all n sufficiently large, which contradicts the fact that $\widetilde{\boldsymbol{D}}_{P_n,\mu_n,\mathcal{Z}}$ is not surjective. □

6 Desingularizations of CS associative submanifolds

Let (Y,ϕ) be a coclosed G_2-manifold. We will glue a conically singular (CS) associative submanifold P in (Y,ϕ) with singularity at one point, and rescaled asymptotically conical (AC) associative submanifolds εL in $\mathbb{R}^7$ with $\varepsilon > 0$, to construct closed approximate associative submanifolds P_ε. We have seen in Theorem 2.35 that for generic coclosed G_2 structures the moduli space of closed associative submanifolds is 0-dimensional and therefore we cannot expect to deform the P_ε to associative submanifolds $\widetilde{P}_\varepsilon$ (desingularization) in (Y,ϕ). In this section we will see that under some hypothesis we can do this in a one-parameter family of coclosed G_2-structures.[3]

6.1 Approximate associative desingularizations

Let P be a CS associative submanifold of a coclosed G_2-manifold (Y,ϕ) with a singularity at p, rate $\mu \in (1,2]$ and cone C as in Definition 4.13. Let L be an AC associative submanifold in $\mathbb{R}^7$ with the same cone C and rate $\nu < 1$ as in Definition 4.2. The real numbers R, R_∞ and ε_0 are as in Definitions 4.2 and 4.13. Let P_C, NP_C, K_{P_C} be as in Definition 5.2 and L_C, NL_C, K_{L_C} be as in Definition 4.4.

Let $\boldsymbol{\phi} \in \mathscr{P}$ be a path of coclosed G_2-structures and $t_0 \in (0,1)$ such that $\boldsymbol{\phi}(t_0) = \phi$. Set $\phi_t := \boldsymbol{\phi}(t)$. Let U_{τ_0} be as in (5.6). Let $T > 0$ be sufficiently small such that for each $t \in (t_0 - T, t_0 + T)$ we have

$$\tau_t := (\phi_t, p, 0) \in U_{\tau_0}.$$

Let Υ^{τ_t}, $P_C^{\tau_t}$ and $\Upsilon_{P_C^{\tau_t}}$ be as in Definition 5.5. We first glue $P_C^{\tau_t}$ and εL_C to obtain a submanifold $P_{\varepsilon,t,C}$ in (Y,ϕ_t), along with a tubular neighborhood of it, inside which the approximate associative desingularization will be defined.

Definition 6.1 ($P_{\varepsilon,t,C}$:= gluing of $P_C^{\tau_t}$ and εL_C) For any sufficiently small $\varepsilon > 0$, we define a closed 3-dimensional submanifold $P_{\varepsilon,t,C}$ of Y as follows. For a real number $0 < q < 1$ we define a real number, $\delta := (\varepsilon R_\infty)^q$. Then for sufficiently small ε, we have

$$\varepsilon R_\infty < 2\varepsilon R_\infty < \delta < 2\delta < \delta^{\frac{1}{2}} < 2\delta^{\frac{1}{2}} < \varepsilon_0 < 2\varepsilon_0 < R.$$

[3] One can generalize this by using similar analysis to multiple isolated points and glue that many asymptotically conical associative submanifolds in $\mathbb{R}^7$, but in that case one needs to desingularize in a higher-dimensional parameter space of coclosed G_2 structures, which we will not cover in this article.

Let $A(a,b)$ denote the annulus $\{x\in\mathbb{R}^7 : a<|x|<b\}$. For any $t\in(t_0-T,t_0+T)$, we define

$$P_{\varepsilon,t,C} := P^+_{\varepsilon,t,C}\cup P^-_{\varepsilon,t,C},$$

where

$$P^+_{\varepsilon,t,C} := \Upsilon^{\tau_t}\big(\varepsilon K_{L_C}\cup(C\cap A(\varepsilon R_\infty,2\delta))\big) \quad\text{and}\quad P^-_{\varepsilon,t,C} := \Upsilon^{\tau_t}(C\cap A(\delta,2\varepsilon_0))\cup K_{P_C}.$$

Here $(\Upsilon^{\tau_t})^{-1}(P^+_{\varepsilon,t,C})\subset\varepsilon L_C$ and $P^-_{\varepsilon,t,C}\subset P_C^{\tau_t}$.

Notation 6.2 If $t=t_0$, then we will from now on omit the subscript t.

Definition 6.3 (tubular neighborhood map of $P_{\varepsilon,t,C}$) The normal bundles $N(\varepsilon L_C)$ and NP_C can be glued to get a normal bundle

$$NP_{\varepsilon,C} = NP^+_{\varepsilon,C}\cup NP^-_{\varepsilon,C}.$$

The scaling $s_\varepsilon:\mathbb{R}^7\to\mathbb{R}^7$ given by $x\mapsto\varepsilon x$ induces an isomorphism $s_{\varepsilon*}:NL_C\to N(\varepsilon L_C)$ defined by

$$s_{\varepsilon*}\nu(x) := \varepsilon\nu(\varepsilon^{-1}x).$$

We have a tubular neighborhood map

$$\Upsilon_{P_{\varepsilon,t,C}} := \Upsilon_{P^+_{\varepsilon,t,C}}\cup\Upsilon_{P^-_{\varepsilon,t,C}} : V_{P_{\varepsilon,C}}\to Y,$$

where $(\Upsilon^{\tau_t})^{-1}\circ\Upsilon_{P^+_{\varepsilon,t,C}}\circ\Upsilon^{\tau_t}_*$ is the restriction of $s_\varepsilon\circ\Upsilon_{L_C}\circ s_{\varepsilon*}^{-1}$, and $\Upsilon_{P^-_{\varepsilon,t,C}}$ is the restriction of $\Upsilon_{P_C^{\tau_t}}$.

Definition 6.4 (approximate associative desingularization) Let $\alpha\in C^\infty(NP_C)$ and $\beta\in C^\infty(NL_C)$ be as in Definitions 5.3 and 4.6, representing P and L, respectively.

We define the *approximate associative desingularization* $P^1_{\varepsilon,t}$ by

$$P^1_{\varepsilon,t} := \Upsilon_{P_{\varepsilon,t,C}}(\alpha^1_\varepsilon),$$

where $\alpha^1_\varepsilon\in C^\infty(V_{P_{\varepsilon,C}})$ is

$$\alpha^1_\varepsilon := \rho_\delta\Upsilon_*(s_{\varepsilon*}\beta)+(1-\rho_\delta)\alpha. \tag{6.5}$$

Definition 6.6 (improved approximate associative desingularization) Suppose that $\alpha\in C^\infty(NP_C)$ and $\beta\in C^\infty(NL_C)$ are as in Definition 6.4. Suppose there exists $\lambda_0<\nu$ such that

(i) $(\lambda_0,\nu]\cap\mathcal{D}=\{\lambda_1,\dots,\lambda_l\}$ with $\lambda_1<\cdots<\lambda_l$,

(ii) there exist $s_0>0$ as well as $\alpha_i\in\ker\mathfrak{L}_{P,\lambda_i}$ and $\beta_i\in V_{\lambda_i}$ for $i=1,\dots,l$ such that, for all $k\in\mathbb{N}\cup\{0\}$,

$$\Big|(\nabla^\perp_C)^k\Big(\beta-\sum_{i=1}^l\beta_i\Big)\Big| = O(r^{\max\{\lambda_0,2\nu-1\}-k}) \quad\text{as } r\to\infty,$$
$$|(\nabla^\perp_C)^k(\Upsilon^*_{P_C}\alpha_i-\beta_i)| = O(r^{\lambda_i+s_0-k}) \quad\text{as } r\to 0.$$

We then define the *improved approximate associative desingularization* $P^2_{\varepsilon,t}$ by

$$P^2_{\varepsilon,t} := \Upsilon_{P_{\varepsilon,t,C}}(\alpha^2_\varepsilon),$$

where

$$\alpha_\varepsilon^2 := \rho_\delta \Upsilon_*(s_{\varepsilon *}\beta) + (1-\rho_\delta)\Big(\sum_{i=1}^{l} \varepsilon^{1-\lambda_i}\alpha_i + \alpha\Big). \tag{6.7}$$

Notation 6.8 We use the notation

$$P_{\varepsilon,t} := P^1_{\varepsilon,t} \text{ or } P^2_{\varepsilon,t} \quad \text{and} \quad \alpha_\varepsilon := \alpha^1_\varepsilon \text{ or } \alpha^2_\varepsilon.$$

Set

$$\nu_0 := \max\{\lambda_0, 2\nu-1\} \quad \text{and} \quad \mathfrak{c}_q := \min\{(1-q)(1-\nu), q(\mu-1)\}.$$

Note that

$$\varepsilon^{\mathfrak{c}_q} \lesssim \max\{\varepsilon^{1-\nu}\delta^{\nu-1}, \delta^{\mu-1}\} \lesssim \varepsilon^{\mathfrak{c}_q}.$$

Since the above approximate associative desingularization is not associative, we aim to find a genuine associative submanifold near it. In other words, our goal is to find a zero of the following nonlinear map.

Definition 6.9 The map $\widetilde{\mathfrak{F}}_\varepsilon : C^\infty(V_{P_{\varepsilon,C}}) \times (t_0-T, t_0+T) \to C^\infty(NP_{\varepsilon,C})$ is defined by

$$\langle \widetilde{\mathfrak{F}}_\varepsilon(u,t), w\rangle_{L^2} := \int_{\Gamma_u} \iota_w \Upsilon^*_{P_{\varepsilon,t,C}} \psi_t,$$

where $t \in (t_0-T, t_0+T)$, $u \in C^\infty(V_{P_{\varepsilon,C}})$ and $w \in C^\infty(NP_{\varepsilon,C})$. The notation w in the integrand is the extension vector field of $w \in C^\infty(NP_{\varepsilon,C})$ in the tubular neighborhood as in Notation 2.25.

Notation 6.10 We use the notation

$$\mathfrak{F}_{\varepsilon,t} := \widetilde{\mathfrak{F}}_\varepsilon(\cdot, t).$$

If $t = t_0$, we omit the subscript t.

Definition 6.11 The map $\widetilde{\mathfrak{F}}_\varepsilon : C^\infty(V_{P_{\varepsilon,C}}) \times (t_0-T, t_0+T) \to C^\infty(NP_{\varepsilon,C})$ can be written as

$$\widetilde{\mathfrak{F}}_\varepsilon(u,t) := \mathfrak{L}_{P_\varepsilon} u + (t-t_0) f_\varepsilon + \widetilde{Q}_\varepsilon(u,t) + e_\varepsilon,$$

where

$$e_\varepsilon := \mathfrak{F}_\varepsilon(\alpha_\varepsilon) \in C^\infty(NP_{\varepsilon,C}) \quad \text{and} \quad f_\varepsilon := \frac{d\mathfrak{F}_{\varepsilon,t}(\alpha_\varepsilon)}{dt}\Big|_{t=t_0} \in C^\infty(NP_{\varepsilon,C}),$$

and $\mathfrak{L}_{P_\varepsilon}$ is the *linearization* of $\mathfrak{F}_\varepsilon$ at $\alpha_\varepsilon \in C^\infty(V_{P_{\varepsilon,C}})$, that is,

$$\mathfrak{L}_{P_\varepsilon} := d\mathfrak{F}_{\varepsilon|\alpha_\varepsilon} : C^\infty(NP_{\varepsilon,C}) \to C^\infty(NP_{\varepsilon,C}).$$

The following proposition concerns the self-adjointness of $\mathfrak{L}_{P_\varepsilon}$, which will be crucial when we attempt to invert it. The obstruction to this inversion can be expressed in terms of (approximate) kernel elements, which leads to the hypothesis in the desingularization theorem.

Proposition 6.12 *$\mathfrak{L}_{P_\varepsilon}$ is a formally self-adjoint elliptic operator for all sufficiently small ε.*

To prove this, we need the following facts about AC associative submanifolds, which have already been observed in the case of CS associative submanifolds.

Definition 6.13 Let $\beta \in C^\infty(NL_C)$ be as in Definition 4.6 representing the AC associative L. We define the differential operator $\mathfrak{L}_L : C^\infty_c(NL_C) \to C^\infty_c(NL_C)$ by

$$\mathfrak{L}_L u := d\mathfrak{F}^{\mathrm{AC}}_{|\beta}(u),$$

where the map $\mathfrak{F}^{\mathrm{AC}} : C^\infty_{L,\nu}(V_{L_C}) \to C^\infty(NL_C)$ is defined by

$$\langle \mathfrak{F}^{\mathrm{AC}}(u), w\rangle_{L^2} := \int_{\Gamma_u} \iota_w \Upsilon^*_{L_C} \psi_e \quad \text{for } u \in C^\infty C^\infty_{L,\nu}(V_{L_C}) \text{ and } w \in C^\infty_c(NL_C).$$

Remark 6.14 The operator $\mathfrak{L}_L$ is an AC uniformly elliptic operator asymptotic to $\boldsymbol{D}_C$. Moreover, there is an identification map $\Theta^C_L : NL_C \to NL$ similar to Θ^C_P (equation (5.15), Definition 5.24) such that

$$\boldsymbol{D}_L = \Theta^C_L \mathfrak{L}_L (\Theta^C_L)^{-1}.$$

Proof of Proposition 6.12 That $\mathfrak{L}_{P_\varepsilon}$ is a formally self-adjoint operator follows from the same arguments as in the proof of Proposition 5.20. It remains to prove that $\mathfrak{L}_{P_\varepsilon}$ is an elliptic operator. We denote the restrictions of $\mathfrak{L}_{P_\varepsilon}$ over $P^\pm_{\varepsilon,C}$ by $\mathfrak{L}_{P^\pm_\varepsilon}$. By Proposition 5.32 we obtain that

$$\mathfrak{L}_{P^-_\varepsilon} = d\mathfrak{F}_{\tau_0|\alpha} + O(\varepsilon^{\mathfrak{c}_q}) = \mathfrak{L}_P + O(\varepsilon^{\mathfrak{c}_q}).$$

Similar to Proposition 5.32 for AC associatives in $\mathbb{R}^7$ and the fact that $\Upsilon^*\psi = (\Upsilon^*\psi)(0) + O(r)$, we get

$$\mathfrak{L}_{P^+_\varepsilon} = \Upsilon_* \mathfrak{L}_{\varepsilon L} \Upsilon_*^{-1} + O(\varepsilon^{\mathfrak{c}_q}) = \varepsilon^{-1}(\Upsilon_* s_{\varepsilon *}) \mathfrak{L}_L (\Upsilon_* s_{\varepsilon *})^{-1} + O(\varepsilon^{\mathfrak{c}_q}).$$

Since $\mathfrak{L}_P$ and $\mathfrak{L}_L$ are elliptic, $\mathfrak{L}_{P_\varepsilon}$ is an elliptic operator for all sufficiently small ε. □

6.2 Weighted function spaces and estimates

In this subsection, we introduce the weighted function spaces on $P_{\varepsilon,C}$, inspired by the corresponding function spaces on its CS and AC sides. These spaces will be used to prove the desingularization theorem. The theorem requires error estimates, linear estimates, and quadratic estimates with these weighted function spaces — all of which are established in this subsection.

Definition 6.15 For each $l \in \mathbb{R}$, a *weight function* $w_{\varepsilon,l} : P_{\varepsilon,C} \to (0,\infty)$ is any smooth function on $P_{\varepsilon,C}$ such that if $x = \Upsilon(r,\sigma) \in \Upsilon(B(0,R))$, then

$$w_{\varepsilon,l}(x) = (\varepsilon + r)^{-l}.$$

Let $k \geqslant 0$ and $\gamma \in (0,1)$. For a continuous section u of $NP_{\varepsilon,C}$, we define the *weighted L^∞-norm* and the *weighted Hölder seminorm*, respectively, by

$$\|u\|_{L^\infty_{\varepsilon,l}} := \|w_{\varepsilon,l} u\|_{L^\infty(NP_{\varepsilon,C})} \quad \text{and} \quad [u]_{C^{0,\gamma}_{\varepsilon,l}} := [w_{\varepsilon,l-\gamma} u]_{C^{0,\gamma}(NP_{\varepsilon,C})}.$$

For a continuous section u of $NP_{\varepsilon,C}$ with k continuous derivatives, we define the *weighted C^k norm* and the *weighted Hölder norm*, respectively, by

$$\|u\|_{C^k_{\varepsilon,l}} := \sum_{j=0}^k \|(\nabla^\perp_{P_{\varepsilon,C}})^j u\|_{L^\infty_{\varepsilon,l-j}} \quad \text{and} \quad \|u\|_{C^{k,\gamma}_{\varepsilon,l}} := \|u\|_{C^k_{\varepsilon,l}} + [(\nabla^\perp_{P_{\varepsilon,C}})^k u]_{C^{0,\gamma}_{\varepsilon,l-k}}.$$

We define the *weighted Hölder space* $C^{k,\gamma}_{\varepsilon,l}$, the *weighted* C^k*-space* $C^k_{\varepsilon,l}$ and the *weighted* L^∞*-space* $L^\infty_{\varepsilon,l}$ to be the $C^{k,\gamma}$-, C^k- and L^∞-spaces with the weighted Hölder norm $\|\cdot\|_{C^{k,\gamma}_{\varepsilon,l}}$, weighted C^k-norm $\|\cdot\|_{C^k_{\varepsilon,l}}$ and weighted L^∞-norm $\|\cdot\|_{L^\infty_{\varepsilon,l}}$, respectively.

Proposition 6.16 (Schauder estimate) *There exists a constant $C>0$ such that for any sufficiently small $\varepsilon>0$, and for all $u\in C^{k+1,\gamma}_{\varepsilon,l}$, we have*

$$\|u\|_{C^{k+1,\gamma}_{\varepsilon,l}} \leqslant C(\|\mathfrak{L}_{P_\varepsilon}u\|_{C^{k,\gamma}_{\varepsilon,l-1}} + \|u\|_{L^\infty_{\varepsilon,l}}).$$

Proof For $u\in C^{k+1,\gamma}_{\varepsilon,l}$, define $u_\pm$ by restricting u over $P^\pm_{\varepsilon,C,}$. Using the Schauder estimates in Propositions 4.25 and 5.32 we obtain

$$\|u_-\|_{C^{k+1,\gamma}_{\varepsilon,l}} \lesssim \|u_-\|_{C^{k+1,\gamma}_{P,l}} \lesssim \|\mathfrak{L}_P u_-\|_{C^{k,\gamma}_{P,l-1}} + \|u_-\|_{L^\infty_{P,l}} \lesssim \|\mathfrak{L}_{P_\varepsilon}u_-\|_{C^{k,\gamma}_{\varepsilon,l-1}} + \varepsilon^{\mathfrak{c}_q}\|u_-\|_{C^{k+1,\gamma}_{\varepsilon,l}} + \|u_-\|_{L^\infty_{\varepsilon,l}}.$$

Similarly,

$$\begin{aligned}
\|u_+\|_{C^{k+1,\gamma}_{\varepsilon,l}} &\lesssim \varepsilon^{-l+1}\|s_{\varepsilon*}^{-1}\Upsilon_*^{-1}u_+\|_{C^{k+1,\gamma}_{L,l}}\\
&\lesssim \varepsilon^{-l+1}\|\mathfrak{L}_L s_{\varepsilon*}^{-1}\Upsilon_*^{-1}u_+\|_{C^{k,\gamma}_{L,l-1}} + \varepsilon^{-l+1}\|s_{\varepsilon*}^{-1}\Upsilon_*^{-1}u_+\|_{L^\infty_{L,l}}\\
&\lesssim \varepsilon^{-1}\|\Upsilon_* s_{\varepsilon*}\mathfrak{L}_L s_{\varepsilon*}^{-1}\Upsilon_*^{-1}u_+\|_{C^{k,\gamma}_{\varepsilon,l-1}} + \|u_+\|_{L^\infty_{\varepsilon,l}}\\
&\lesssim \|\mathfrak{L}_{P_\varepsilon}u_+\|_{C^{k,\gamma}_{\varepsilon,l-1}} + \varepsilon^{\mathfrak{c}_q}\|u_+\|_{C^{k+1,\gamma}_{\varepsilon,l}} + \|u_+\|_{L^\infty_{\varepsilon,l}}.
\end{aligned}$$

□

Proposition 6.17 (error estimate) *The error $e_\varepsilon=\mathfrak{F}_\varepsilon(\alpha_\varepsilon)$ can be estimated as*

$$\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\|_{C^{k,\gamma}_{\varepsilon,-1}} \lesssim \begin{cases} \delta^\mu & \text{if } \alpha_\varepsilon=\alpha^1_\varepsilon \text{ and } 0<q\leqslant\dfrac{1-\nu}{\mu-\nu}n,\\[2mm] \delta^\mu+\varepsilon^{1-\nu_0}\delta^{\nu_0}+\varepsilon^{2(1-\nu)} & \text{if } \alpha_\varepsilon=\alpha^2_\varepsilon \text{ and } \dfrac{\nu-\nu_0}{\nu+\mu-1+s_0-\nu_0}\leqslant q<1, \end{cases}$$

where $\nu_0=\max\{\lambda_0,2\nu-1\}$ and $\delta=\varepsilon^q$.

Proof Let ϕ_e be the standard G_2-structure on $\mathbb{R}^7$ and C be a cone. Since $|\phi-\Upsilon_*\phi_e|=O(r)$ on $B(0,R)$, over $P_{\varepsilon,C}\cap\Upsilon(B(0,R))$ we have

$$|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)| \lesssim |\mathfrak{F}_\varepsilon(\alpha_\varepsilon)-\mathfrak{F}_\varepsilon(\alpha_\varepsilon,\Upsilon_*\phi_e)| \lesssim |\mathfrak{F}_\varepsilon(\alpha_\varepsilon,\Upsilon_*\phi_e)|+r.$$

Case 1: Suppose $\alpha_\varepsilon=\alpha^1_\varepsilon$. Then $\alpha_\varepsilon=\alpha$ over $P_{\varepsilon,C}\setminus\Upsilon(B(0,2\delta))$, so $\mathfrak{F}_\varepsilon(\alpha_\varepsilon)=0$. Over $P_{\varepsilon,C}\cap\Upsilon(\overline{B(0,\delta)})$ we have $\alpha_\varepsilon=\Upsilon_*(s_{\varepsilon*}\beta)$ and therefore $\mathfrak{F}_\varepsilon(\alpha_\varepsilon,\Upsilon_*\phi_e)=0$. Over $P_{\varepsilon,C}\cap\Upsilon(A(\delta,2\delta))$ we have

$$|\mathfrak{F}_\varepsilon(\alpha_\varepsilon,\Upsilon_*\phi_e)| \lesssim |\nabla\alpha_\varepsilon| \lesssim r^{\mu-1}+\varepsilon^{1-\nu}r^{\nu-1}.$$

Here the last inequality uses the assumption that $0<q\leqslant(1-\nu)/(\mu-\nu)$, where $\delta=(\varepsilon R_\infty)^q$. From these estimates we conclude that $\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\|_{C^0_{\varepsilon,-1}}\lesssim\delta^\mu$. A similar computation with Hölder seminorm and higher derivatives will prove that $\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\|_{C^{k,\gamma}_{\varepsilon,-1}}\lesssim\delta^\mu$.

Case 2: Suppose $\alpha_\varepsilon=\alpha^2_\varepsilon$. Over $P_{\varepsilon,C}\setminus\Upsilon(B(0,2\delta))$ we have

$$|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)| \lesssim \left(r^{-1}\left|\sum_{i=1}^l\varepsilon^{1-\lambda_i}\alpha_i\right|+\left|\sum_{i=1}^l\varepsilon^{1-\lambda_i}\nabla\alpha_i\right|\right)^2 \lesssim \sum_{i=1}^l(\varepsilon^{-1}r)^{2(\lambda_i-1)} \lesssim (\varepsilon^{-1}r)^{2(\nu-1)}.$$

Over $P_{\varepsilon,C} \cap \Upsilon(A(\delta, 2\delta))$ we have $\mathfrak{F}_\varepsilon(\alpha_\varepsilon, \Upsilon_*\phi_e) = \boldsymbol{D}_C(\alpha_\varepsilon) + Q_C(\alpha_\varepsilon)$. The term $|\boldsymbol{D}_C(\alpha_\varepsilon)|$ can be estimated by

$$\begin{aligned}|\boldsymbol{D}_C(\alpha_\varepsilon)| &\lesssim |\nabla\rho_\delta|\Big|s_{\varepsilon*}\beta - \sum_{i=1}^{l}\varepsilon^{1-\lambda_i}\alpha_i - \alpha\Big| + |\boldsymbol{D}_C(s_{\varepsilon*}\beta)| + \Big|\boldsymbol{D}_C\Big(\sum_{i=1}^{l}\varepsilon^{1-\lambda_i}\alpha_i + \alpha\Big)\Big| \\ &\lesssim (\varepsilon^{-1}r)^{\nu_0-1} + r^{\mu-1} + (\varepsilon^{-1}r)^{2(\nu-1)} + r^{\mu-1}\sum_{i=1}^{l}\varepsilon^{1-\lambda_i}r^{\lambda_i+s_0-1} + r^{2(\mu-1)}.\end{aligned}$$

If $q \geqslant (\nu-\nu_0)/(\nu+\mu-1+s_0-\nu_0)$ then we have

$$\varepsilon^{1-\lambda_i}r^{\lambda_i+\mu+s_0-2} \lesssim \varepsilon^{1-\nu}r^{\nu+\mu+s_0-2} \lesssim (\varepsilon^{-1}r)^{\nu_0-1}.$$

The term $|Q_C(\alpha_\varepsilon)|$ can be estimated as

$$|Q_C(\alpha_\varepsilon)| \lesssim (r^{-1}|\alpha_\varepsilon| + |\nabla\alpha_\varepsilon|)^2 \lesssim (\varepsilon^{-1}r)^{2(\nu-1)} + r^{2(\mu-1)}.$$

Hence,

$$\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\|_{C^0_{\varepsilon,-1}} \lesssim \delta^\mu + \varepsilon^{1-\nu_0}\delta^{\nu_0} + \varepsilon^{2(1-\nu)}.$$

A similar computation with Hölder seminorm and higher derivatives will also prove that

$$\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\|_{C^{k,\gamma}_{\varepsilon,-1}} \lesssim \delta^\mu + \varepsilon^{1-\nu_0}\delta^{\nu_0} + \varepsilon^{2(1-\nu)}. \qquad \square$$

Proposition 6.18 (quadratic estimate) *For any sufficiently small $\varepsilon > 0$ there exists $0 < T_\varepsilon < T$ such that for all $u, v \in C^\infty(V_{P_{\varepsilon,C}})$, $\eta \in C^\infty(NP_{\varepsilon,C})$ and $t_1, t_2 \in (t_0 - T_\varepsilon, t_0 + T_\varepsilon)$, the following estimates hold.*

(i) $|d\mathfrak{F}_{\varepsilon|u}(\eta) - d\mathfrak{F}_{\varepsilon|v}(\eta)| \lesssim (w_{\varepsilon,1}|u-v| + w_{\varepsilon,0}|\nabla^\perp(u-v)|)(w_{\varepsilon,1}|\eta| + |w_{\varepsilon,0}\nabla^\perp\eta|)$.

(ii) $\|Q_\varepsilon(u) - Q_\varepsilon(v)\|_{C^{k,\gamma}_{\varepsilon,-1}} \lesssim \varepsilon^{-1}\|u-v\|_{C^{k+1,\gamma}_{\varepsilon,0}}\big(\|u-\alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + \|v-\alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}}\big)$.

(iii) $\|\widetilde{Q}_\varepsilon(u,t_1) - \widetilde{Q}_\varepsilon(v,t_2)\|_{C^{k,\gamma}_{\varepsilon,-1}}$
$\lesssim \varepsilon^{-1}\big(\|u-v\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t_1-t_2|\big)\big(\|u-\alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + \|v-\alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t_1-t_0| + |t_2-t_0|\big)$.

Proof The proofs are exactly like Propositions 5.32 and 5.33. An important observation here is that one can express $\mathcal{L}_u\mathcal{L}_v\psi$ over $V_{P_{\varepsilon,C}}$ formally as a quadratic polynomial

$$\mathcal{L}_u\mathcal{L}_v\psi = O(f_{1,\varepsilon})\cdot u\cdot v + O(f_{2,\varepsilon})\cdot(u\cdot\nabla^\perp v + v\cdot\nabla^\perp u) + \psi\cdot\nabla^\perp u\cdot\nabla^\perp v,$$

where $O(f_{1,\varepsilon}) = O(w_{\varepsilon,1})$ and $O(f_{2,\varepsilon}) = O(w_{\varepsilon,0})$. Finally, to see (ii) we write $Q_\varepsilon(u) - Q_\varepsilon(v)$ as

$$\int_0^1 dQ_{\varepsilon|tu+(1-t)v}(u-v)\,dt = \int_0^1 \big(d\mathfrak{F}_{\varepsilon|tu+(1-t)v}(u-v) - d\mathfrak{F}_{\varepsilon|\alpha_\varepsilon}(u-v)\big)\,dt,$$

and (ii) follows from (i). To prove (iii) we only observe that $\widetilde{Q}_\varepsilon(u,t_1)-\widetilde{Q}_\varepsilon(v,t_2)$ is equal to

$$\begin{aligned}
&\int_0^1 \big(d\widetilde{\mathfrak{F}}_{\varepsilon|(tu+(1-t)v,tt_1+(1-t)t_2)}(u-v,0)-d\widetilde{\mathfrak{F}}_{\varepsilon|(\alpha_\varepsilon,tt_1+(1-t)t_2)}(u-v,0)\big)\,dt\\
&\quad+\int_0^1 \big(d\widetilde{\mathfrak{F}}_{\varepsilon|(\alpha_\varepsilon,tt_1+(1-t)t_2)}(u-v,0)-d\widetilde{\mathfrak{F}}_{\varepsilon|(\alpha_\varepsilon,t_0)}(u-v,0)\big)\,dt\\
&\quad+\int_0^1 \big(d\widetilde{\mathfrak{F}}_{\varepsilon|(tu+(1-t)v,tt_1+(1-t)t_2)}(0,t_1-t_2)-d\widetilde{\mathfrak{F}}_{\varepsilon|(tu+(1-t)v,t_0)}(0,t_1-t_2)\big)\,dt\\
&\quad+\int_0^1 \big(d\widetilde{\mathfrak{F}}_{\varepsilon|(tu+(1-t)v,t_0)}(0,t_1-t_2)-d\widetilde{\mathfrak{F}}_{\varepsilon|(\alpha_\varepsilon,t_0)}(0,t_1-t_2)\big)\,dt. \qquad \square
\end{aligned}$$

The remaining estimates are the linear estimates, namely, establishing a uniform lower bound for the linearization operator on suitable weighted function spaces that depend on the scaling parameter ε. In general, this is hopeless unless we restrict to the complement of the following approximate kernel.

Definition 6.19 Denote the asymptotic limit maps (Lemma 4.31) $i_{P_C,0}$ and $i_{L_C,0}$ by i_P and i_L respectively. We define

(i) the *matching kernel* $\mathcal{K}^{\mathfrak{m}}$ by

$$\mathcal{K}^{\mathfrak{m}} := \{(u_L,u_P)\in \ker \mathfrak{L}_{L,0}\times \ker \mathfrak{L}_{P,0} : i_L(u_L)=i_P(u_P)\},$$

(ii) the *approximate kernel* of $\mathfrak{L}_{P_\varepsilon}$ by

$$\mathcal{K}^{\mathfrak{m}}_\varepsilon := \{\rho_{\delta/2}\Upsilon_*(s_{\varepsilon *}u_L)+(1-\rho_{2\delta})u_P : (u_L,u_P)\in\mathcal{K}^{\mathfrak{m}}\},$$

(iii) $\mathcal{X}^{k+1,\gamma}_\varepsilon := \big\{u\in C^{k+1,\gamma}_{\varepsilon,0} : \langle(\Upsilon_* s_{\varepsilon *})^{-1}u,u_L\rangle_{L^2_{K_L}}=\langle u,u_P\rangle_{L^2_{K_P}}=0 \text{ for all } (u_L,u_P)\in\mathcal{K}^{\mathfrak{m}}\big\}$.

Proposition 6.20 (linear estimate, I) *Fix a real number $\omega>0$. For any sufficiently small $\varepsilon>0$, there exists a constant $C_\omega>0$ independent of ε such that for all $u\in\mathcal{X}^{k+1,\gamma}_\varepsilon$, we have*

$$\|u\|_{C^{k+1,\gamma}_{\varepsilon,0}}\leqslant C_\omega\varepsilon^{-\omega}\|\mathfrak{L}_{P_\varepsilon}u\|_{C^{k,\gamma}_{\varepsilon,-1}}.$$

Proof By the Schauder estimate in Proposition 6.16 it is enough to prove that

$$\|u\|_{L^\infty_{\varepsilon,0}}\leqslant C_\omega\varepsilon^{-\omega}\|\mathfrak{L}_{P_\varepsilon}u\|_{C^{k,\gamma}_{\varepsilon,-1}}.$$

We prove this by contradiction. If this is not true then there exist a decreasing sequence $\varepsilon_n\to 0$ as $n\to\infty$ and u_n in $\mathcal{X}^{k+1,\gamma}_{\varepsilon_n}$ such that

$$\|u_n\|_{L^\infty_{\varepsilon_n,0}}=1 \qquad\text{and}\qquad \varepsilon_n^{-\omega}\|\mathfrak{L}_{P_{\varepsilon_n}}u_n\|_{C^{k,\gamma}_{\varepsilon_n,-1}}\to 0 \quad\text{as } n\to\infty.$$

Denote the restrictions of u_n over $P^{\pm}_{\varepsilon_n,C}$ by $u_n^{\pm}$. We define

$$u_{n,P}:=u_n^- \quad\text{and}\quad u_{n,L}:=\varepsilon_n(\Upsilon_* s_{\varepsilon_n *})^{-1}u_n^+.$$

By the Schauder estimate in Proposition 6.16 $\|u_n\|_{C^{k+1,\gamma}_{\varepsilon_n,0}}$ is bounded, and hence $\|u_{n,P}\|_{C^{k+1,\gamma}_{P,0}}$ and $\|u_{n,L}\|_{C^{k+1,\gamma}_{L,0}}$ are also bounded. The Arzelà–Ascoli theorem implies that there exist subsequences, which we again call $u_{n,P}$ and $u_{n,L}$, and there exist u_P in $C^{k+1,\gamma/2}_{P,0}$ and u_L in $C^{k+1,\gamma/2}_{L,0}$ such that

$$\mathfrak{L}_P u_P = 0, \quad \mathfrak{L}_L u_L = 0,$$
$$\|u_{n,P} - u_P\|_{C^{k+1,\gamma/2}_{P,\mathrm{loc}}} \to 0 \quad \text{and} \quad \|u_{n,L} - u_L\|_{C^{k+1,\gamma/2}_{L,\mathrm{loc}}} \to 0 \quad \text{as } n \to \infty.$$

Moreover, by elliptic regularity in Proposition 4.25 we get $u_P \in C^{k+1,\gamma}_{P,0}$ and $u_L \in C^{k+1,\gamma}_{L,0}$. By taking further subsequences if necessary we prove the following, which leads to the contradiction:

(i) $\|u_{n,P} - u_P\|_{C^{k+1,\gamma}_{P^-_{\varepsilon_n,C},0}} \to 0$ as $n \to \infty$.

(ii) $\|u_{n,L} - u_L\|_{C^{k+1,\gamma}_{L_{\varepsilon_n^{-1}\delta_n,C},0}} \to 0$ as $n \to \infty$, where $L_{\varepsilon_n^{-1}\delta_n,C} := s_{\varepsilon_n}^{-1}\Upsilon^{-1}P^+_{\varepsilon_n,C} \subset L_C$.

(iii) $u_P = 0$ and $u_L = 0$. Hence,

$$\|u_n\|_{L^\infty_{\varepsilon_n,0}} \leqslant \|u_n^+\|_{L^\infty_{P^+_{\varepsilon_n,C}}} + \|u_n^-\|_{L^\infty_{P^-_{\varepsilon_n,C}}} \to 0 \quad \text{as } n \to \infty.$$

To prove (i), we fix a real number p with $1 < p < 1/q$. For sufficiently small $s > 0$ we have

$$\begin{aligned}
\|\mathfrak{L}_P((1-\rho_{\delta_n^p})u_n)\|_{C^{k,\gamma}_{P,-s-1}} &\lesssim (\varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q})\|u_n\|_{C^{k+1,\gamma}_{0,\varepsilon n}} + \|\mathfrak{L}_{P_{\varepsilon_n}}(1-\rho_{\delta_n^p})u_n\|_{C^{k,\gamma}_{\varepsilon n,-s-1}} \\
&\lesssim (\varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps})\|u_n\|_{C^{k+1,\gamma}_{0,\varepsilon n}} + \|\mathfrak{L}_{P_{\varepsilon_n}}u_n\|_{C^{k,\gamma}_{\varepsilon n,-s-1}} \\
&\lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \|\mathfrak{L}_{P_{\varepsilon_n}}u_n\|_{C^{k,\gamma}_{\varepsilon n,-1}} \lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \varepsilon_n^{\omega}.
\end{aligned}$$

Therefore there exists $v_{n,P} \in \ker \mathfrak{L}_{P,-s} = \ker \mathfrak{L}_{P,0}$ satisfying

$$\|(1-\rho_{\delta_n^p})u_n - v_{n,P}\|_{C^{k+1,\gamma}_{P,-s}} \lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \varepsilon_n^{\omega},$$

and hence

$$\|u_{n,P} - v_{n,P}\|_{C^{k+1,\gamma}_{P^-_{\varepsilon_n,C},0}} \lesssim (\varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + +\delta_n^{ps} + \varepsilon_n^{\omega})\delta_n^{-s} \to 0 \quad \text{as } n \to \infty.$$

As $\ker \mathfrak{L}_{P,0}$ is finite-dimensional, the norms $\|\cdot\|_{C^{k+1,\gamma/2}_{K_P}}$ and $\|\cdot\|_{C^{k+1,\gamma}_{P,0}}$ are equivalent. Taking a further subsequence yields $\|v_{n,P} - u_P\|_{C^{k+1,\gamma/2}_{K_P}} \to 0$ and hence $\|v_{n,P} - u_P\|_{C^{k+1,\gamma}_{P,0}} \to 0$ as $n \to \infty$.

To prove (ii), we define $\widetilde{u}_{n,L} := \varepsilon_n(\Upsilon_* s_{\varepsilon_n *})^{-1}(\rho_{\delta_n^{1/2}}u_n)$. For sufficiently small $s > 0$ we have

$$\begin{aligned}
\|\mathfrak{L}_L\widetilde{u}_{n,L}\|_{C^{k,\gamma}_{L,s-1}} &\lesssim \varepsilon_n^{s-1}\|\Upsilon_* s_{\varepsilon_n *}\mathfrak{L}_L s_{\varepsilon_n *}^{-1}\Upsilon_*^{-1}(\rho_{\delta_n^{1/2}}u_n)\|_{C^{k,\gamma}_{\varepsilon,s-1}} \\
&\lesssim \varepsilon_n^s(\varepsilon_n^{\mathfrak{c}q} + \delta_n^{(\mu-1)/2})\|\rho_{\delta_n^{1/2}}u_n\|_{C^{k+1,\gamma}_{\varepsilon n,s}} + \varepsilon_n^s\|\mathfrak{L}_{P_{\varepsilon_n}}(\rho_{\delta_n^{1/2}}u_n)\|_{C^{k,\gamma}_{\varepsilon n,s-1}} \\
&\lesssim \varepsilon_n^s\delta_n^{-s/2}\|u_n\|_{C^{k+1,\gamma}_{\varepsilon_n,0}} + \varepsilon_n^s\delta_n^{-s/2}\|\mathfrak{L}_{P_{\varepsilon_n}}u_n\|_{C^{k,\gamma}_{\varepsilon_n,-1}} \lesssim \varepsilon_n^s\delta_n^{-s/2} + \varepsilon_n^s\delta_n^{-s/2}\varepsilon_n^{\omega}.
\end{aligned}$$

Then as in (i) there exists $v_{n,L} \in \ker \mathfrak{L}_{L,s} = \ker \mathfrak{L}_{L,0}$ satisfying

$$\|u_{n,L} - v_{n,L}\|_{C^{k+1,\gamma}_{L,0}} \lesssim (\varepsilon_n^{-1}\delta_n)^s(\varepsilon_n^s\delta_n^{-s/2} + \varepsilon_n^s\delta_n^{-s/2}\varepsilon_n^{\omega}) \lesssim \delta_n^{s/2} + \delta_n^{s/2}\varepsilon_n^{\omega} \to 0 \quad \text{as } n \to \infty.$$

Again as in (i), by taking a further subsequence if necessary, we get $\|v_{n,L}-u_L\|_{C^{k+1,\gamma/2}_{K_L}}\to 0$ and $\|v_{n,L}-u_L\|_{C^{k+1,\gamma}_{L,0}}\to 0$ as $n\to\infty$.

It remains to prove (iii). For each $\sigma\in\Sigma$, we have

$$i_P u_P(\sigma)=\Upsilon_*^{-1}\lim_{n\to\infty}u_{n,P}\big(\Upsilon\big(\tfrac{3}{2}\delta_n,\sigma\big)\big)=\lim_{n\to\infty}u_{n,L}\big(\tfrac{3}{2}\varepsilon_n^{-1}\delta_n,\sigma\big)=i_L u_L(\sigma).$$

Since $u_n\in\mathcal{X}^{k+1,\gamma}_{\varepsilon_n}$ therefore as $n\to\infty$ we get

$$\|u_P\|_{L^2_{K_P}}=\langle u_P-u_{n,P},u_P\rangle_{L^2_{K_P}}\lesssim\|u_{n,P}-u_P\|_{L^\infty_{P,0}}\|u_P\|_{L^\infty_{P,0}}\to 0,$$

and hence $u_P=0$. Similarly we get $u_L=0$. □

From the above linear estimate, we observe that $\mathfrak{L}_{P_\varepsilon}$ may not be surjective, as it is self-adjoint, and the matching kernel is never trivial due to the scaling freedom on L. This is why we did not choose to perform the desingularization in a fixed G_2-structure, but rather along a path of G_2-structures. To proceed, we require a corresponding estimate for the linearization operator along this path, and we aim to obtain a uniform lower bound on its restriction to the complement of the corresponding extended matching kernel, which we define next.

Definition 6.21 Given P a CS associative submanifold with respect to a coclosed G_2-structure ϕ and L an AC associative submanifold in $\mathbb{R}^7$ with the same asymptotic cone, and given $\boldsymbol{\phi}:[0,1]\to\mathscr{P}$ a path of coclosed G_2-structures with $\boldsymbol{\phi}(t_0)=\phi$ for some $t_0\in(0,1)$, we define the *extended matching kernel*

$$\widetilde{\mathcal{K}}^{\mathrm{m}}:=\big\{(u_L,u_P,t)\in\ker\mathfrak{L}_{L,0}\oplus C^{k+1,\gamma}_{P,0}\oplus\mathbb{R}:\mathfrak{L}_P u_P+tf_P=0,\,i_L u_L=i_P u_P\big\},$$

where f_P is defined in Definition 5.30. Note that the matching kernel satisfies $\mathcal{K}^{\mathrm{m}}\subset\widetilde{\mathcal{K}}^{\mathrm{m}}$.

Proposition 6.22 (linear estimate, II) *If the extended matching kernel $\widetilde{\mathcal{K}}^{\mathrm{m}}$ is equal to the matching kernel $\mathcal{K}^{\mathrm{m}}$ and $\omega>0$ is any real number, then for any sufficiently small $\varepsilon>0$ there exists a constant $\widetilde{C}_\omega>0$ independent of ε such that for all $u\in\mathcal{X}^{k+1,\gamma}_\varepsilon$ and $t\in\mathbb{R}$, we have*

$$\|u\|_{C^{k+1,\gamma}_{\varepsilon,0}}+|t|\leqslant\widetilde{C}_\omega\varepsilon^{-\omega}\|\mathfrak{L}_{P_\varepsilon}u+tf_\varepsilon\|_{C^{k,\gamma}_{\varepsilon,-1}}.$$

Proof Let $f_\varepsilon^\pm$ be the restrictions of f_ε over $P^\pm_{\varepsilon,C}$. Then, similar to the proof of Proposition 6.17, we obtain that

$$\|f_\varepsilon^- - f_P\|_{C^{k,\gamma}_{P^-_{\varepsilon,C},-1}}\lesssim\varepsilon^{\mathfrak{c}_q}\quad\text{and}\quad\|f_\varepsilon^+\|_{C^{k,\gamma}_{P^+_{\varepsilon,C},-1}}\lesssim\varepsilon^{\mathfrak{c}_q}.$$

Claim 1 *For all $u\in\mathcal{X}^{k+1,\gamma}_\varepsilon$ with $\|u\|_{C^{k+1,\gamma}_{\varepsilon,0}}\leqslant 1$, we have*

$$1\leqslant\widetilde{C}_\omega\varepsilon^{-\omega}\|\mathfrak{L}_{P_\varepsilon}u-f_\varepsilon\|_{C^{k,\gamma}_{\varepsilon,-1}}.$$

We will prove this by contradiction. Suppose it is not true. Then there exist sequences $\varepsilon_n \to 0$ and $u_n \in \mathcal{X}^{k+1,\gamma}_{\varepsilon_n}$ such that

$$\|u_n\|_{C^{k+1,\gamma}_{\varepsilon_n,0}} \leqslant 1 \qquad \text{and} \qquad \varepsilon_n^{-\omega}\|\mathfrak{L}_{P_{\varepsilon_n}} u_n - f_{\varepsilon_n}\|_{C^{k,\gamma}_{\varepsilon_n,-1}} \to 0 \quad \text{as } n \to \infty.$$

After defining $u_{n,P}$ and $u_{n,L}$ as in Proposition 6.20, a similar argument as in there yields a smooth section $u_P \in C^{k+1,\gamma}_{P,0}$ and $u_L \in C^{k+1,\gamma}_{L,0}$ such that

$$\mathfrak{L}_P u_P = f_P \qquad \text{and} \qquad \|u_{n,P} - u_P\|_{C^{k+1,\gamma/2}_{P,\mathrm{loc}}} \to 0 \quad \text{as } n \to \infty,$$
$$\mathfrak{L}_L u_L = 0 \qquad \text{and} \qquad \|u_{n,L} - u_L\|_{C^{k+1,\gamma/2}_{L,\mathrm{loc}}} \to 0 \quad \text{as } n \to \infty.$$

Moreover, we can prove, as in Proposition 6.20, the following:

(i) $\|u_{n,P} - u_P\|_{C^{k+1,\gamma}_{P^-_{\varepsilon_n,C},0}} \to 0$ as $n \to \infty$,

(ii) $\|u_{n,L} - u_L\|_{C^{k+1,\gamma}_{L_{\varepsilon_n^{-1}\delta_n,C},0}} \to 0$ as $n \to \infty$, where $L_{\varepsilon_n^{-1}\delta_n,C} := s_{\varepsilon_n}^{-1}\Upsilon^{-1}P^+_{\varepsilon_n,C} \subset L_C$,

(iii) $i_P u_P = i_L u_L$. This contradicts the assumption.

To prove (i), we fix a real number p with $1 < p < 1/q$. For sufficiently small $s > 0$ we have

$$\begin{aligned}
\|\mathfrak{L}_P(1-\rho_{\delta_n^p})(u_n-u_P)\|_{C^{k,\gamma}_{P,-s-1}} &\lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \|\mathfrak{L}_{P_{\varepsilon_n}}(1-\rho_{\delta_n^p})u_n - \mathfrak{L}_P(1-\rho_{\delta_n^p})u_P)\|_{C^{k,\gamma}_{\varepsilon_n,-s-1}} \\
&\lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \|(1-\rho_{\delta_n^p})(\mathfrak{L}_{P_{\varepsilon_n}} u_n - f_P)\|_{C^{k,\gamma}_{\varepsilon_n,-s-1}} \\
&\lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \|\mathfrak{L}_{P_{\varepsilon_n}} u_n - f_{\varepsilon_n}\|_{C^{k,\gamma}_{\varepsilon_n,-1}} + \|(1-\rho_{\delta_n^p})(f_{\varepsilon_n} - f_P)\|_{C^{k,\gamma}_{\varepsilon_n,-1}} \\
&\lesssim \varepsilon_n^{\mathfrak{c}pq} + \varepsilon_n^{\mathfrak{c}q} + \delta_n^{ps} + \varepsilon_n^{\omega}.
\end{aligned}$$

Now (i) follows as in Proposition 6.20(i). Since $\delta_n^{s/2}\|\rho_{\delta_n^{1/2}} f_{\varepsilon_n}\|_{C^{k,\gamma}_{\varepsilon_n,-1}} \to 0$ as $n \to \infty$, (ii) follows from Proposition 6.20(ii). Again the proof of (iii) is also similar to Proposition 6.20(iii).

Claim 2 *For all $u \in \mathcal{X}^{k+1,\gamma}_{\varepsilon}$ and $t \in \mathbb{R}$ with $\|u\|_{C^{k+1,\gamma}_{\varepsilon,0}} = 1$ and $|t| \leqslant 1$ we have*

$$1 \leqslant \widetilde{C}_\omega \varepsilon^{-\omega}\|\mathfrak{L}_{P_\varepsilon} u - t f_\varepsilon\|_{C^{k,\gamma}_{\varepsilon,-1}}.$$

Suppose it is not true. Then there exist sequences $\varepsilon_n \to 0$ and $u_n \in \mathcal{X}^{k+1,\gamma}_{\varepsilon_n}$, $t_n \in \mathbb{R}$ such that $\|u_n\|_{C^{k+1,\gamma}_{\varepsilon_n,0}} = 1$, $t_n \to t_\infty$, and $\varepsilon_n^{-\omega}\|\mathfrak{L}_{P_{\varepsilon_n}} u_n - f_{\varepsilon_n}\|_{C^{k,\gamma}_{\varepsilon_n,-1}}$ converges to 0 as $n \to \infty$. If $t_\infty \neq 0$ then similar arguments to Claim 1 replacing f_{ε_n} by $t_n f_{\varepsilon_n}$ lead to a contradiction. If $t_\infty = 0$ then similar arguments to those in Proposition 6.20 lead again to a contradiction.

Evidently, Claims 1 and 2 are enough to prove the proposition. □

6.3 Proof of the desingularization theorem

To prove the desingularization Theorem 1.22 we solve the nonlinear PDE, $\widetilde{\mathfrak{F}}_\varepsilon(u,t) = 0$, with (u,t) very close to $(\alpha_\varepsilon, 0)$. Indeed, we use the following lemma, an application of the Banach contraction principle [Donaldson and Kronheimer 1990, Lemma 7.2.23].

Lemma 6.23 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $\mathcal{X}_1 \subset \mathcal{X}$ be a Banach subspace. Let $V \subset \mathcal{X}$ be a neighborhood of $0 \in \mathcal{X}$. Let $x_0 \in V$. Let $F : V \to \mathcal{Y}$ be a smooth map of the form*[4]

$$F(x) = L(x) + Q(x) + F(x_0)$$

such that:

- *$L : \mathcal{X}_1 \to \mathcal{Y}$ is an invertible operator, and there exists a constant $c_L > 0$ such that for all $x \in \mathcal{X}_1$, $\|x\|_{\mathcal{X}} \leqslant c_L \|Lx\|_{\mathcal{Y}}$,*
- *$Q : V \to \mathcal{Y}$ is a smooth map, and there exists a constant $c_Q > 0$ such that for all $x_1, x_2 \in V$,*

$$\|Q(x_1) - Q(x_2)\|_{\mathcal{Y}} \leqslant c_Q \|x_1 - x_2\|_{\mathcal{X}} (\|x_1 - x_0\|_{\mathcal{X}} + \|x_2 - x_0\|_{\mathcal{X}}).$$

If $\|F(x_0)\|_{\mathcal{Y}} \leqslant 1/(10 c_L^2 c_Q)$ and $B(x_0, 1/(5 c_L c_Q)) \subset V$, then there exists a unique $x \in x_0 + \mathcal{X}_1$ with

$$\|x - x_0\|_{\mathcal{X}} \leqslant c_L \|F(x_0)\|_{\mathcal{Y}} \quad \text{and} \quad F(x) = 0.$$

Proof of Theorem 1.22 Let α_ε be α_ε^1 as in Definition 6.4. Let T_ε be as in Proposition 6.18. The map

$$\widetilde{\mathfrak{F}}_\varepsilon : C^{k+1,\gamma}_{\varepsilon,0}(V_{P_{\varepsilon,C}}) \times (t_0 - T_\varepsilon, t_0 + T_\varepsilon) \to C^{k,\gamma}_{\varepsilon,-1}$$

can be written as $\widetilde{\mathfrak{F}}_\varepsilon(u, t_0 + t) = \mathfrak{L}_{P_\varepsilon} u + t f_\varepsilon + \widetilde{Q}_\varepsilon(u, t_0 + t) + \mathfrak{F}_\varepsilon(\alpha_\varepsilon)$. Since the matching kernel $\mathcal{K}^{\mathrm{m}}$ is one-dimensional and the index of $\mathfrak{L}_{P_\varepsilon}$ is zero, by Propositions 6.20 and 6.22 we obtain that

$$\widetilde{L}_\varepsilon : \mathfrak{X}^{k+1,\gamma}_\varepsilon \oplus \mathbb{R} \to C^{k,\gamma}_{\varepsilon,-1},$$

defined by $\widetilde{L}_\varepsilon(u, t) := \mathfrak{L}_{P_\varepsilon} u + t f_\varepsilon$, is invertible and

$$\|u\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t| \leqslant C_{\widetilde{L}_\varepsilon} \|\widetilde{L}_\varepsilon(u, t)\|_{C^{k,\gamma}_{\varepsilon,-1}}, \quad \text{where } C_{\widetilde{L}_\varepsilon} = O(\varepsilon^{-\omega}).$$

Again by Proposition 6.18, if $t_1, t_2 \in (-T_\varepsilon, T_\varepsilon)$ then we have

$$\begin{aligned}\|\widetilde{Q}_\varepsilon(u, t_0 + t_1) - \widetilde{Q}_\varepsilon(v, t_0 + t_2)\|_{C^{k,\gamma}_{\varepsilon,-1}} \\ \leqslant C_{Q_\varepsilon}\big(\|u - v\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t_1 - t_2|\big)\big(\|u - \alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + \|v - \alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t_1| + |t_2|\big),\end{aligned}$$

where $C_{Q_\varepsilon} := O(\varepsilon^{-1})$. Since $\nu < 0$, by Proposition 6.17 we have

$$\|\mathfrak{F}_\varepsilon(\alpha_\varepsilon)\| C^{k,\gamma}_{\varepsilon,-1} \leqslant \frac{1}{10 C^2_{\widetilde{L}_\varepsilon} C_{Q_\varepsilon}} = O(\varepsilon^{1+2\omega})$$

for sufficiently small ω. By Lemma 6.23, for all sufficiently small $\varepsilon > 0$ there exist $t(\varepsilon) \in (t_0 - T, t_0 + T)$ and smooth closed associative submanifold $\widetilde{P}_{\varepsilon,t(\varepsilon)} := \Upsilon_{P_{\varepsilon,t(\varepsilon),C}}(\widetilde{\alpha}_\varepsilon)$ in $(Y, \phi_{t(\varepsilon)})$, that is, $\widetilde{\mathfrak{F}}_\varepsilon(\widetilde{\alpha}_\varepsilon, t(\varepsilon)) = 0$. Moreover, if $1/\mu < q \leqslant (1 - \nu)/(\mu - \nu)$, then

$$\|\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t(\varepsilon) - t_0| \lesssim \varepsilon^{q\mu - \omega}.$$

[4] Note that $Q(x_0) = -L(x_0)$.

Finally, we see that $\widetilde{P}_{\varepsilon,t(\varepsilon)} \to P$ in the sense of currents as $\varepsilon \to 0$. Indeed, for any 3-form $\xi \in \Omega^3(Y)$,

$$\left|\int_{\widetilde{P}_{\varepsilon,t(\varepsilon)}} \xi - \int_P \xi\right| \leqslant \left|\int_{\widetilde{P}_{\varepsilon,t(\varepsilon)}} \xi - \int_{P_{\varepsilon,t(\varepsilon)}} \xi\right| + \left|\int_{P_{\varepsilon,t(\varepsilon)}} \xi - \int_P \xi\right| \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

Remark 6.24 If there exists $\lambda_0 < \nu$ satisfying (i) and (ii) of Definition 6.6, then we can consider $\alpha_\varepsilon = \alpha_\varepsilon^2$ in the proof of Theorem 1.22. In the proof of Theorem 1.22, $(\widetilde{\alpha}_\varepsilon, t(\varepsilon))$ satisfies the following estimates:

(6.25)
$$\|\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon\|_{C^{k+1,\gamma}_{\varepsilon,0}} + |t(\varepsilon) - t_0| \lesssim \begin{cases} \varepsilon^{q\mu-\omega} & \text{if } \alpha_\varepsilon = \alpha_\varepsilon^1 \text{ and } \dfrac{1}{\mu} < q \leqslant \dfrac{1-\nu}{\mu-\nu}, \\ \varepsilon^{q\mu-\omega} + \varepsilon^{1+(q-1)\nu_0-\omega} & \text{if } \alpha_\varepsilon = \alpha_\varepsilon^2 \text{ and } \dfrac{\nu-\nu_0}{\nu+\mu-1+s_0-\nu_0} \leqslant q < 1. \end{cases}$$

Indeed, these estimates follow from Lemma 6.23 and Propositions 6.22 and 6.17.

7 Desingularizations for special cases of singularities

In this section, we apply Theorem 1.22 to desingularize a CS associative submanifold with a Harvey–Lawson T^2-cone singularity, as well as an associative submanifold with a transverse self-intersection. In other words, we prove the two transitions discussed at the beginning of this article, although the second one is established only partially.

To apply Theorem 1.22, we must verify that the hypothesis (1.23) holds in these specific cases. It turns out that there are no contributions from the CS side to the extended matching kernel, and therefore

(7.1)
$$\widetilde{\mathcal{K}}^{\mathrm{m}} = \mathcal{K}^{\mathrm{m}} \cong \ker \boldsymbol{D}_{L,-1},$$

where L is the AC associative submanifold used for the desingularization. In the cases considered here, L is special Lagrangian, so $\ker \boldsymbol{D}_{L,-1}$ is purely topological, as stated in the proposition.

Proposition 7.2 [Marshall 2002, Section 5, Section 5.2.3, Table 5.1] *Let L be a AC special Lagrangian submanifold in $\mathbb{C}^3$. Let Σ be the link of the asymptotic cone. Then*

$$\dim \ker \boldsymbol{D}_{L,-1} = b^1(L) + b^0(\Sigma) - 1.$$

We will see that in both of the special cases considered above, $\ker \boldsymbol{D}_{L,-1}$ is one-dimensional, and the hypothesis (1.23) holds true.

7.1 Desingularizations for Harvey–Lawson T^2-cone singularity

Let $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ be a generic path of coclosed G_2-structures. Then, by Definition 1.13, any CS associative submanifold $P \in \mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ is unobstructed along this path; that is, the operator $\bar{\boldsymbol{D}}_{P,\mu,\mathbb{Z}}$ is surjective. Moreover, any such P, together with an AC associative submanifold sharing the same asymptotic cone whose homogeneous kernel at rate -1 is two-dimensional, must satisfy the matching condition in (7.1). It

follows that the stability index of the cone is 1, and P is an isolated point in the moduli space $\mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$. This is the content of the following lemma.

Lemma 7.3 *Let $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ be a path of coclosed G_2-structures and $P \in \mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$ be a CS associative with singularity at only one point. Let $L \subset \mathbb{R}^7$ be an AC associative with the same asymptotic cone C of P. If $d_{-1} = 2$, then*

$$\widetilde{\mathcal{K}}^{\mathrm{m}} = \mathcal{K}^{\mathrm{m}} \cong \ker \mathfrak{L}_{L,-1}.$$

Moreover, s-ind$(C) = 1$, $\dim \ker \mathfrak{L}_{P,-1} = 1$ *and* $\ker \mathfrak{L}_{P,-1} = \langle v_P \rangle_{\mathbb{R}}$ *such that* $\langle v_P, f_P \rangle_{L^2} \neq 0$.

Proof Since $\boldsymbol{\phi} \in \mathscr{P}_{\mathrm{cs}}^{\mathrm{reg}}$ and $P \in \mathfrak{M}_{\mathrm{cs}}^{\boldsymbol{\phi}}$, Theorem 1.17(ii) implies that the stability index of the asymptotic cone C satisfies s-ind$(C) \leqslant 1$. Since $d_{-1} = 2$, we have s-ind$(C) = 1$, $V_\lambda = 0$ for all nonzero $\lambda \in (-1, 1)$, $V_0 \cong \mathbb{R}^7$ and C is a rigid cone, that is, $V_1 \cong T_\Sigma(G_2 \cdot \Sigma)$, where Σ is the link of C. Then by Theorem 1.12(ii), we obtain that the deformation operator $\overline{\mathfrak{L}}^{k,\gamma}_{P,\mu,\mathcal{Z}}$ defined in Definition 5.30 is an isomorphism, where $\mathcal{Z}$ is the stratum in the decomposition (1.4) that contains $G_2 \cdot \Sigma$ as an open subset. Here $\mu > 1$ is chosen sufficiently close to 1, as in Definition 5.12, and therefore there are no indicial roots of C in the interval between 1 and μ. Therefore

$$\overline{\mathfrak{L}}^{k,\gamma}_{P,-1+s} : \mathbb{R} \oplus C^{k+1,\gamma}_{P_C,-1+s} \to C^{k,\gamma}_{P_C,-2+s}, \quad \overline{\mathfrak{L}}^{k,\gamma}_{P,-1+s}(t,u) := \mathfrak{L}_P u + t f_P,$$

is an isomorphism as well, for all sufficiently small $s > 0$. Indeed, the indicial roots in between -1 and μ are only 0 and 1 with the corresponding V_0 and V_1 as above, and therefore by Lemma 4.31 the above two operators have the same kernel. Since index $\mathfrak{L}_{P,-1+s} = -1$ by Proposition 4.33(ii), we have that index $\overline{\mathfrak{L}}_{P,-1+s} = 0$ and hence it is an isomorphism. This further implies that $\ker \mathfrak{L}_{P,0} \subset \ker \mathfrak{L}_{P,-1+s} = 0$, and coker $\mathfrak{L}_{P,-1+s} \cong \ker \mathfrak{L}_{P,-1} = \langle v_P \rangle_{\mathbb{R}}$ is one-dimensional with $\langle v_P, f_P \rangle_{L^2} \neq 0$. Moreover, $f_P \notin \operatorname{im} \mathfrak{L}_{P,-1+s} \supset \operatorname{im} \mathfrak{L}_{P,0}$, and hence $\widetilde{\mathcal{K}}^{\mathrm{m}} = \mathcal{K}^{\mathrm{m}} \cong \ker \mathfrak{L}_{L,-1}$. □

Proof of Theorem 1.28 We have seen in Example 3.34 that the Harvey–Lawson cone has $d_{-1} = 2$. Therefore by Lemma 7.3 and Proposition 7.2 we obtain that for each Harvey–Lawson AC special Lagrangian $L^i := L^i_1$ for $i = 1, 2, 3$ (see Example 4.10), the extended matching kernel and the matching kernel are one-dimensional. Indeed,

$$\dim \widetilde{\mathcal{K}}^{\mathrm{m}} = \dim \mathcal{K}^{\mathrm{m}} = \dim \ker \mathfrak{L}^i_{L,-1} = b^1(S^1 \times \mathbb{C}) + b^0(T^2) - 1 = 1.$$

By Theorem 1.22 we obtain that for sufficiently small $\varepsilon > 0$, there exists $t^i(\varepsilon)$ and smooth closed associative $\widetilde{P}_{\varepsilon,t^i(\varepsilon)}$ in $(Y, \phi_{t^i(\varepsilon)})$ such that $\widetilde{P}_{\varepsilon,t^i(\varepsilon)} \to P$ in the sense of currents as $\varepsilon \to 0$. $\widetilde{P}_{\varepsilon,t^i(\varepsilon)}$ is diffeomorphic to $P_{\varepsilon,t^i(\varepsilon)}$. Since L^i are obtained by Dehn filling of $C^o_{\mathrm{HL}} := C_{\mathrm{HL}} \setminus B(0,1)$ along simple closed curves μ_i as mentioned in the theorem, therefore it is diffeomorphic to the Dehn filling of P^o as required in the theorem; this was observed in [Doan and Walpuski 2019, Remark 3.6].

It remains to prove that the leading-order term of $t^i(\varepsilon) - t_0$ is of the required form if $\boldsymbol{\phi} \in \mathscr{P}^\bullet$. Let $v_P \in \ker \mathfrak{L}_{P,-1-s}$ be as in Lemma 7.3. Since $\boldsymbol{\phi} \in \mathscr{P}^\bullet$, by definition we have

$$i_{P,-1} v_P = b_1 \xi_1 + b_2 \xi_2, \quad \text{with } b_1 \neq 0,\ b_2 \neq 0,\ b_1 \neq b_2,$$

where ξ_1 and ξ_2 are as in equation (4.11) of Example 4.10. We will prove the result for $t(\varepsilon) := t^1(\varepsilon)$ only, as the others follow by similar arguments. We use the notation $\alpha_\varepsilon = \alpha_\varepsilon^1 = \rho_\delta \Upsilon_*(s_{\varepsilon *}\beta) + (1-\rho_\delta)\alpha$ and $(\widetilde{\alpha}_\varepsilon, t(\varepsilon))$ from Theorem 1.22, which satisfies

$$\widetilde{\mathfrak{F}}_\varepsilon(\widetilde{\alpha}_\varepsilon, t(\varepsilon)) = \mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon) + (t(\varepsilon)-t_0) f_\varepsilon + Q_\varepsilon(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon, t(\varepsilon)-t_0) + \mathfrak{F}_\varepsilon(\alpha_\varepsilon) = 0.$$

Define $\mathring{P}_\varepsilon := P_C \setminus \Upsilon(B(0, 2\varepsilon R_\infty))$ and $\mathring{P}_\delta := P_C \setminus \Upsilon(B(0,\delta))$. Over $\mathring{P}_\delta$,

$$\mathfrak{F}_\varepsilon(\alpha_\varepsilon) = \mathfrak{L}_P(\alpha_\varepsilon - \alpha) + Q_P(\alpha_\varepsilon - \alpha).$$

Therefore,

$$\begin{aligned} 0 &= \langle \widetilde{\mathfrak{F}}_\varepsilon(\widetilde{\alpha}_\varepsilon, t(\varepsilon)), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}} \\ &= (t(\varepsilon)-t_0)\langle f_\varepsilon, v_P\rangle_{L^2_{\mathring{P}_\varepsilon}} + \langle \mathfrak{L}_P \alpha_\varepsilon, v_P\rangle_{L^2_{\mathring{P}_\delta}} + e_\varepsilon^1 + e_\varepsilon^2 + e_\varepsilon^3 + e_\varepsilon^4 + e_\varepsilon^5, \end{aligned} \tag{7.4}$$

where

$$e_\varepsilon^1 := \langle \mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}}, \quad e_\varepsilon^2 := \langle Q_\varepsilon(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon, t(\varepsilon)-t_0), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}}, \quad e_\varepsilon^3 := \langle Q_P(\alpha_\varepsilon - \alpha), v_P\rangle_{L^2_{\mathring{P}_\delta}},$$
$$e_\varepsilon^4 = -\langle \mathfrak{L}_P(\alpha), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}} + \langle \mathfrak{L}_P(\alpha), v_P\rangle_{L^2_{\mathring{P}_\varepsilon \setminus \mathring{P}_\delta}}, \quad e_\varepsilon^5 = \langle \mathfrak{F}_\varepsilon(\alpha_\varepsilon), v_P\rangle_{L^2_{\mathring{P}_\varepsilon \setminus \mathring{P}_\delta}}.$$

By Proposition 5.20 we obtain that

$$\langle \mathfrak{L}_P \alpha_\varepsilon, v_P\rangle_{L^2_{\mathring{P}_\delta}} = \int_{\partial \mathring{P}_\delta} \iota_{\alpha_\varepsilon}\iota_{v_P} \Upsilon_{P_C}^* \psi = \int_{\partial \mathring{P}_\delta} \iota_{s_{\varepsilon *}\beta}\iota_{b_1\xi_1 + b_2\xi_2} \Upsilon^* \psi(0) + O(\varepsilon^{2+q}). \tag{7.5}$$

Now,

$$\int_{\partial \mathring{P}_\delta} \iota_{s_{\varepsilon *}\beta}\iota_{b_1\xi_1 + b_2\xi_2} \Upsilon^* \psi(0) = \int_{\partial B(0,\delta)\cap C} \iota_{s_{\varepsilon *}\xi_1}\iota_{b_1\xi_1 + b_2\xi_2} \Upsilon^* \psi(0) + O(\varepsilon^{3-q}), \tag{7.6}$$

$$\int_{\partial B(0,\delta)\cap C} \iota_{s_{\varepsilon *}\xi_1}\iota_{b_1\xi_1 + b_2\xi_2} \Upsilon^* \psi(0) = \varepsilon^2 \int_\Sigma \langle \partial_r \times \xi_1, b_1\xi_1 + b_2\xi_2\rangle = b_2\varepsilon^2 \int_\Sigma \langle \partial_r \times \xi_1, \xi_2\rangle. \tag{7.7}$$

We will now estimate the remaining terms in (7.4). Using (6.25) we obtain that

$$\begin{aligned} |e_\varepsilon^1| &= |\langle \mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon) - \mathfrak{L}_P(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}}| + |\langle \mathfrak{L}_P(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon), v_P\rangle_{L^2_{\mathring{P}_\varepsilon}}| \\ &\lesssim \varepsilon^{2+q\mu-\omega} + \int_{\partial \mathring{P}_\varepsilon} |\iota_{\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon}\iota_{v_P} \Upsilon_{P_C}^* \psi| \lesssim \varepsilon^{1+q\mu-\omega}, \\ |e_\varepsilon^2| &\lesssim \varepsilon^{2q\mu-2\omega}|\log\varepsilon| + \varepsilon^{q\mu-\omega}|t(\varepsilon)-t_0| + |t(\varepsilon)-t_0|^2|\log\varepsilon| \lesssim \varepsilon^{2q\mu-2\omega}|\log\varepsilon|. \end{aligned}$$

We obtain by a further computation that

$$|e_\varepsilon^3| \lesssim \varepsilon^{2q\mu}, \quad |e_\varepsilon^5| \lesssim \varepsilon^{3q} \quad \text{and} \quad |e_\varepsilon^4| \lesssim \int_{\partial \mathring{P}_\varepsilon} |\iota_\alpha \iota_{v_P} \Upsilon_{P_C}^* \psi| + \varepsilon^{2q\mu} \lesssim \varepsilon^{1+\mu} + \varepsilon^{2q\mu}.$$

Hence by choosing ω sufficiently small and $q > \frac{2}{3}$, we obtain

$$|e_\varepsilon^1| + |e_\varepsilon^2| + |e_\varepsilon^3| + |e_\varepsilon^4| + |e_\varepsilon^5| = o(\varepsilon^2). \tag{7.8}$$

Since $\langle f_\varepsilon, v_P\rangle_{L^2_{\mathring{P}_\varepsilon}} = \langle f_P, v_P\rangle_{L^2} + o(1)$, combining equations (7.4)–(7.8) we obtain

$$t^1(\varepsilon) = t(\varepsilon) = t_0 - \frac{cb_2}{\langle f_P, v_P\rangle_{L^2}}\varepsilon^2 + o(\varepsilon^2),$$

where $c = \int_\Sigma \langle J\xi_1, \xi_2\rangle \neq 0$. Similarly we can prove the statements for $t^2(\varepsilon)$ and $t^3(\varepsilon)$. The only thing to notice here is that $\xi_3 = -\xi_1 - \xi_2$ as in Example 4.10. □

7.2 Desingularizations for intersecting associatives

Let $\boldsymbol{\phi}$ be a path of coclosed G_2-structures. Let $P \in \mathcal{M}^{\boldsymbol{\phi}}_{\mathrm{cs}}$ be an associative submanifold in (Y, ϕ_{t_0}) with a transverse unique self-intersection. In other words, P is a conically singular associative with singularity at a unique point, which is modeled on a union of two transverse associative planes. To resolve this intersection, we glue in a Lawlor neck — an asymptotically conical (AC) special Lagrangian submanifold asymptotic to the planes (Example 4.7).

Proof of Theorem 1.35 In Example 3.33 we see that the union of two transverse associative planes has $d_{-1} = 0$ and $V_\lambda = \{0\}$ for all $\lambda \in (-1, 0)$. We glue the intersecting associative P with the AC associative L, where $B^{-1}\cdot L$ is the special Lagrangian Lawlor neck in $\mathbb{C}^3$ from Example 4.7 and $B \in G_2$ from (1.31). Since the $\boldsymbol{n}$-component (perpendicular to $B\cdot\mathbb{C}^3$) of any element in $\ker \mathfrak{L}_{L,0}$ vanishes and $\boldsymbol{\phi} \in \mathscr{P}^\dagger$ (Definition 1.32), equation (1.33) implies that the extended matching kernel and the matching kernel are the same and isomorphic to $\ker \mathfrak{L}_{L,-1}$, because $\ker \mathfrak{L}_{P,0}$ vanishes. By Proposition 7.2, we have

$$\dim \ker \mathfrak{L}_{L,-1} = b^1(S^2\times\mathbb{R}) + b^0(S^2 \sqcup S^2) - 1 = 1.$$

By Theorem 1.22, we obtain that for sufficiently small $\varepsilon > 0$, there exists $t(\varepsilon) \in (t_0 - T_\varepsilon, t_0 + T_\varepsilon)$ and a smooth closed associative $\widetilde{P}_{\varepsilon,t(\varepsilon)}$ in $(Y, \phi_{t(\varepsilon)})$ such that $\widetilde{P}_{\varepsilon,t(\varepsilon)} \to P$ in the sense of currents as $\varepsilon \to 0$. Moreover, $\widetilde{P}_{\varepsilon,t(\varepsilon)}$ are diffeomorphic to $P_{\varepsilon,t(\varepsilon)}$ and hence are diffeomorphic to the connected sums as mentioned in the theorem. □

Remark 7.9 (discussion on the leading-order term) It remains to prove that the leading-order term of $t(\varepsilon) - t_0$ is of the required form if $\boldsymbol{\phi} \in \mathscr{P}^\ddagger$; see Remark 1.36 for the reason behind this restriction. We can consider

$$\alpha_\varepsilon = \alpha^2_\varepsilon := \rho_\delta \Upsilon_*(s_{\varepsilon *}\beta) + (1-\rho_\delta)(\varepsilon^3 u_P + \alpha).$$

Here α and β represent P and L, respectively, and $u_P := (\Theta^C_P)^{-1}\hat{u}_P \in \ker \mathfrak{L}_{P,-2}$ with

$$\beta^\pm_1 = B\xi^\pm, \quad l = 1, \quad \lambda_1 = \nu = -2, \quad \lambda_0 = -4, \quad \mu = 2, \quad s_0 = 2, \quad q = \tfrac{2}{5}.$$

By abusing notation we denote $B\xi^\pm$ by simply $\xi^\pm$. We use the notation $(\widetilde{\alpha}_\varepsilon, t(\varepsilon))$ from Theorem 1.22 following Remark 6.24, which satisfies

$$\textbf{(7.10)} \qquad \widetilde{\mathfrak{F}}_\varepsilon(\widetilde{\alpha}_\varepsilon, t(\varepsilon) - t_0) = \mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon) + (t(\varepsilon) - t_0) f_\varepsilon + Q_\varepsilon(\widetilde{\alpha}_\varepsilon - \alpha_\varepsilon, t(\varepsilon)) + \mathfrak{F}_\varepsilon(\alpha_\varepsilon) = 0.$$

Define $\mathring{P}_\varepsilon := P_C \setminus \Upsilon(B(0, 2\varepsilon R_\infty))$ and $\mathring{P}_\delta := P_C \setminus \Upsilon(B(0,\delta))$. Over $\mathring{P}_\delta$,

$$\mathfrak{F}_\varepsilon(\alpha_\varepsilon) = \mathfrak{L}_P(\alpha_\varepsilon - \alpha) + Q_P(\alpha_\varepsilon - \alpha).$$

Therefore,

$$\textbf{(7.11)}\quad 0=\langle\widetilde{\mathfrak{F}}_\varepsilon(\widetilde{\alpha}_\varepsilon,t(\varepsilon)),u_P\rangle_{L^2_{\mathring{P}_\varepsilon}}=(t(\varepsilon)-t_0)\langle f_\varepsilon,u_P\rangle_{L^2_{\mathring{P}_\varepsilon}}+\langle\mathfrak{L}_P\alpha_\varepsilon,u_P\rangle_{L^2_{\mathring{P}_\delta}}+e^1_\varepsilon+e^2_\varepsilon+e^3_\varepsilon+e^4_\varepsilon+e^5_\varepsilon,$$

where

$$e^1_\varepsilon:=\langle\mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon),u_P\rangle_{L^2_{\mathring{P}_\varepsilon}},\quad e^2_\varepsilon:=\langle Q_\varepsilon(\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon,t(\varepsilon)-t_0),u_P\rangle_{L^2_{\mathring{P}_\varepsilon}},\quad e^3_\varepsilon:=\langle Q_P(\alpha_\varepsilon-\alpha),u_P\rangle_{L^2_{\mathring{P}_\delta}},$$
$$e^4_\varepsilon=-\langle\mathfrak{L}_P(\alpha),u_P\rangle_{L^2_{\mathring{P}_\varepsilon}}+\langle\mathfrak{L}_P(\alpha),u_P\rangle_{L^2_{\mathring{P}_\varepsilon\setminus\mathring{P}_\delta}},\quad e^5_\varepsilon=\langle\mathfrak{F}_\varepsilon(\alpha_\varepsilon),u_P\rangle_{L^2_{\mathring{P}_\varepsilon\setminus\mathring{P}_\delta}}.$$

By Proposition 5.20 we obtain that $\langle\mathfrak{L}_P\alpha_\varepsilon,u_P\rangle_{L^2_{\mathring{P}_\delta}}$ is

$$\textbf{(7.12)}\quad \int_{\partial\mathring{P}_\delta}\iota_{\alpha_\varepsilon}\iota_{u_P}\Upsilon^*_{P_C}\psi=\varepsilon^3\int_{\partial B_{\Pi+}(0,\delta)}\iota_{\xi+}\iota_{u_P-\xi+}\Upsilon^*\psi-\varepsilon^3\int_{\partial B_{\Pi-}(0,\delta)}\iota_{\xi-}\iota_{u_P-\xi-}\Upsilon^*\psi+o(\varepsilon^3).$$

Now,

$$\textbf{(7.13)}\quad \int_{\partial B_{\Pi\pm}(0,\delta)}\iota_{\xi\pm}\iota_{u_P-\xi\pm}\Upsilon^*\psi=\int_{\partial B_{\Pi\pm}(0,\delta)}\iota_{\xi\pm}\iota_{u_P-\xi\pm}\psi_e+o(\varepsilon),$$

$$\textbf{(7.14)}\quad \int_{\partial B_{\Pi\pm}(0,\delta)}\iota_{\xi\pm}\iota_{u_P-\xi\pm}\psi_e=(u_P-\xi^\pm)(0)\cdot\boldsymbol{n}+o(\varepsilon).$$

Define $v_P=(\Theta^C_P)^{-1}\hat{v}_P$. Then $\mathfrak{L}_Pv_P=f_P$. We observe that $\langle f_\varepsilon,u_P\rangle_{L^2_{\mathring{P}_\varepsilon}}=\langle f_P,u_P\rangle_{L^2_{\mathring{P}_\delta}}+o(1)$. Therefore, similar to equations (7.12), (7.13) and (7.14) we obtain

$$\textbf{(7.15)}\quad \langle f_P,u_P\rangle_{L^2_{\mathring{P}_\delta}}=\int_{\partial B_{\Pi+}(0,\delta)}\iota_{v_P}\iota_{\xi+}\psi_e-\int_{\partial B_{\Pi-}(0,\delta)}\iota_{v_P}\iota_{\xi-}\psi_e+o(1)=\big(v_P^+(0)-v_P^-(0)\big)\cdot\boldsymbol{n}+o(1).$$

We can further compute

$$\textbf{(7.16)}\quad |e^3_\varepsilon|+|e^4_\varepsilon|+|e^5_\varepsilon|=o(\varepsilon^3).$$

If, in addition, one had $|e^1_\varepsilon|+|e^2_\varepsilon|=o(\varepsilon^3)$, then the leading-order expansion would coincide with the expression given in (1.38). However, it is *not* immediately evident from our desingularization theorem why the estimate in (7.16) should hold. Using (6.25), we can only deduce that

$$\begin{aligned}|e^1_\varepsilon|&=|\langle\mathfrak{L}_{P_\varepsilon}(\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon)-\mathfrak{L}_P(\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon),u_P\rangle_{L^2(\mathring{P}_\varepsilon)}|+|\langle\mathfrak{L}_P(\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon),u_P\rangle_{L^2(\mathring{P}_\varepsilon)}|\\&\lesssim o(\varepsilon^3)+\left|\int_{\partial B_{\Pi+}(0,2\varepsilon R_\infty)}\iota_{\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon}\iota_{\xi+}\Upsilon^*\psi-\int_{\partial B_{\Pi-}(0,2\varepsilon R_\infty)}\iota_{\widetilde{\alpha}_\varepsilon-\alpha_\varepsilon}\iota_{\xi-}\Upsilon^*\psi\right|.\end{aligned}$$

In fact, (6.25) does not imply that the final expression above must be $o(\varepsilon^3)$. A similar issue arises for e^2_ε.

Although not obvious, a suitable adaptation of the desingularization theorem in this special case might still yield the estimate in (7.16), and could be pursued in future work.

Appendix Lemma for quadratic estimates

The following lemma provides a key pointwise estimate used in the proofs of the quadratic estimates stated in Propositions 5.32, 5.33 and 6.18.

Lemma A.1 *Let M be a 3-dimensional oriented submanifold of an almost G_2-manifold (Y,ϕ). Let $\Upsilon_M : V_M \subset NM \to Y$ be a tubular neighborhood map. There exists a constant $C>0$ such that if $w \in C^\infty(NM)$ and $u,v,s \in C^\infty(V_M)$, then over $\Gamma_s := \operatorname{graph} s \subset V_M$ we have*

$$|\iota_w \mathcal{L}_u \mathcal{L}_v \psi| \lesssim |w|\big(f_1|u||v| + f_2(|u||\nabla v| + |v||\nabla u|) + |\nabla u||\nabla v||\psi|\big),$$

with

$$f_1 := |\psi||\nabla B| + |\psi||B|^2 + |B||\nabla\psi| + |\nabla^2\psi| + |R||\psi| \quad \text{and} \quad f_2 := |\nabla\psi| + |B||\psi|,$$

where R is the Riemann curvature tensor and $B : C^\infty(NM)\times C^\infty(NM) \to C^\infty(TV_M)$ is defined by $B(u,v) := \nabla_u v$. Here u is understood as the extension of the vector field u over NM by fiberwise translation. The 4-form ψ denotes $\Upsilon_M^*\psi$, and $\mathfrak{L}$ stands for the Lie derivative.

Proof Let $\{e_1,e_2,e_3\}$ be a local orthonormal frame for $T\Gamma_s$. For a torsion-free connection we have

$$\mathcal{L}_v\psi(w,e_1,e_2,e_3) = \nabla_v\psi(w,e_1,e_2,e_3) + \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \psi(w,\nabla_{e_1}v,e_2,e_3) + \psi(\nabla_w v,e_1,e_2,e_3).$$

By definition we have

$$\mathcal{L}_u\mathcal{L}_v\psi(w,e_1,e_2,e_3) = u\big(\mathcal{L}_v\psi(w,e_1,e_2,e_3)\big) - \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \mathcal{L}_v\psi(w,[u,e_1],e_2,e_3).$$

Let's combine the above. We obtain

$$\begin{aligned}
u\big(\nabla_v\psi(w,e_1,e_2,e_3)\big) &= \nabla_u\nabla_v\psi(w,e_1,e_2,e_3) + \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \nabla_v\psi(w,\nabla_{e_1}u + [u,e_1],e_2,e_3)\\
&\qquad + \nabla_v\psi(\nabla_w u,e_1,e_2,e_3),\\
u\big(\psi(w,\nabla_{e_1}v,e_2,e_3)\big) &= \nabla_u\psi(w,\nabla_{e_1}v,e_2,e_3) + \psi(w,\nabla_u\nabla_{e_1}v,e_2,e_3) + \psi(w,\nabla_{e_1}v,\nabla_u e_2,e_3)\\
&\qquad + \psi(w,\nabla_{e_1}v,e_2,\nabla_u e_3) + \psi(\nabla_w u,\nabla_{e_1}v,e_2,e_3),\\
u\big(\psi(\nabla_w v,e_1,e_2,e_3)\big) &= \nabla_u\psi(\nabla_w v,e_1,e_2,e_3) + \sum_{\substack{\text{cyclic}\\ \text{permutations}}} \psi(\nabla_w v,\nabla_u e_1,e_2,e_3)\\
&\qquad + \psi(\nabla_w\nabla_u v + R(u,w)v,e_1,e_2,e_3),\\
\mathcal{L}_v\psi(w,[u,e_1],e_2,e_3) &= \psi(w,\nabla_{[u,e_1]}v,e_2,e_3) + \psi(\nabla_w v,[u,e_1],e_2,e_3) + \psi(w,[u,e_1],\nabla_{e_2}v,e_3)\\
&\qquad + \psi(w,[u,e_1],e_2,\nabla_{e_3}v) + \nabla_v\psi(w,[u,e_1],e_2,e_3).
\end{aligned}$$

Finally, putting things together after some cancellations we obtain

$$\begin{aligned}
|\mathcal{L}_u\mathcal{L}_v\psi(w,e_1,e_2,e_3)| \lesssim\ & |w|(|\nabla_u\nabla_v\psi| + |\nabla\psi||\nabla u||v|) + |\nabla\psi||v||\nabla_w u| + |w||\nabla\psi||u||\nabla v|\\
&+ |\psi||\nabla v||\nabla_w u| + |\nabla_w v|(|\nabla\psi||u| + |\psi||\nabla u|) + |\psi|(|\nabla_w\nabla_u v| + |R||w||u||v|)\\
&+ |w||\psi|\Big(\sum_{i=1}^3 |\nabla_{e_i}(\nabla_u v)| + |R||u||v| + |\nabla u||\nabla v|\Big).
\end{aligned}$$

Since

$$|\nabla_u \nabla_v \psi| \lesssim |u||v|(|B||\nabla\psi| + |\nabla^2\psi|),$$
$$|\nabla_{e_i}(\nabla_u v)| \lesssim |\nabla B||u||v| + |B||u||\nabla v| + |B||v||\nabla u|,$$
$$|\nabla_w(\nabla_u v)| \lesssim |w||u||v|\big(|\nabla B| + |B|^2\big),$$

we have

$$|\iota_w \mathcal{L}_u \mathcal{L}_v \psi| \lesssim f_1|w||u||v| + f_2|w|(|u||\nabla^\perp v| + |v||\nabla^\perp u|) + |w||\nabla^\perp u||\nabla^\perp v||\psi|. \qquad \square$$

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GEOMETRY & TOPOLOGY

Volume 30 Issue 4 (pages 1203–1610) 2026

Equivariant Cerf theory and perturbative SU(n) Casson invariants 1203
SHAOYUN BAI and BOYU ZHANG

Closed Ricci flows with singularities modeled on asymptotically conical shrinkers 1277
MAXWELL STOLARSKI

Deformations and desingularizations of conically singular associative submanifolds 1371
GORAPADA BERA

Integrable measure equivalence rigidity of right-angled Artin groups via quasi-isometry 1451
CAMILLE HORBEZ and JINGYIN HUANG

Entropy versus volume via Heegaard diagrams 1515
YI LIU

The canonical lamination calibrated by a cohomology class 1575
AIDAN BACKUS

Correction to the article Bridge trisections in $\mathbb{CP}^2$ and the Thom conjecture 1609
PETER LAMBERT-COLE