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Galois reconstruction of Artin–Tate $\mathbb{R}$-motivic spectra

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We explain how to reconstruct the category of Artin–Tate $\mathbb{R}$-motivic spectra as a deformation of the purely topological C_2-equivariant stable category. The special fiber of this deformation is algebraic, and equivalent to an appropriate category of C_2-equivariant sheaves on the moduli stack of formal groups. As such, our results directly generalize the cofiber of τ philosophy of Gheorghe, Isaksen, Wang and Xu.

A key observation is that the Artin–Tate subcategory of $\mathbb{R}$-motivic spectra is easier to understand than the previously studied cellular subcategory. In particular, the Artin–Tate category contains a variant of the τ map, which is a feature conspicuously absent from the cellular category.

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1 Introduction

A striking surprise of modern computational stable homotopy theory is that the category $\mathcal{SH}(\mathbb{C})$ of $\mathbb{C}$-motivic spectra, as introduced by Morel and Voevodsky [1999], has found use not only in algebraic geometry, but also in the computation of stable homotopy groups of spheres, a question of purely topological origin. In particular, the p-completed bigraded homotopy groups of the unit in $\mathcal{SH}(\mathbb{C})$ record — in a precise sense — the Adams–Novikov spectral sequence for the sphere spectrum, including all differentials and extensions. The ultimate expression of this connection is the discovery that the *cellular* subcategory of p-complete $\mathbb{C}$-motivic spectra is equivalent to a category of *formal* Adams–Novikov

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spectral sequences, and in particular has a purely homotopy theoretic description without reference to algebraic geometry.

The first evidence that a large sector of the motivic category contains only topological information appears in the work of Thomason [1985] on algebraic K-theory and étale cohomology. As a consequence of this work, Thomason proves that for well-behaved $\mathbb{C}$-schemes X, there is an equivalence $\mathrm{K}^{\mathrm{alg}}(X)[\beta^{-1}]^\wedge_\ell \simeq \mathrm{KU}(X(\mathbb{C}))^\wedge_\ell$, where β is a certain Bott element. These ideas have been refined over the years, culminating in the following theorem.

Theorem 1.1 [Voevodsky 2002; Dugger and Isaksen 2010; Levine 2014, 2015; Gheorghe 2018; Gheorghe et al. 2021; 2022]; Pstrągowski 2023 *In $\mathcal{SH}(\mathbb{C})$ there is a map $\tau : \mathbb{S}^1 \to (\mathbb{G}_m)_p$ which enjoys the following properties*:

(i) *Generically topological: The full subcategory of τ-local objects in $\mathcal{SH}(\mathbb{C})^{\mathrm{cell}}_p$ is equivalent to Sp_p.*

(ii) *Algebraic degeneration: The cofiber of τ (often denoted by $C\tau$) is a commutative algebra, and the category of dualizable modules over $C\tau$ is equivalent to the category of p-completions of dualizable objects in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$, where $\mathcal{M}_{\mathrm{fg}}$ is the moduli stack of formal groups.*

(iii) *Galois reconstruction: There is a purely topological construction of $\mathcal{SH}(\mathbb{C})^{\mathrm{cell}}_p$.*

We summarize this situation by saying that $\mathcal{SH}(\mathbb{C})^{\mathrm{cell}}_p$ is a 1-parameter deformation of p-complete spectra with parameter τ and a purely algebraic special fiber.

Although it does not appear in the statement, understanding the algebraic cobordism spectrum, MGL, is key to proving this theorem. The importance of MGL to the study of cellular motivic spectra goes back to Voevodsky's work [2002] on the effective slice filtration. In that work, Voevodsky conjectured that the effective slice filtration of the unit over an algebraically closed field may be described in terms of the Adams–Novikov spectral sequence. Levine [2014; 2015] later proved Voevodsky's conjectures. Voevodsky's notion of "rigid homotopy groups" also foreshadowed *algebraic degeneration*. In this framework he and Rezk predicted at odd primes p that the rigid Adams spectral sequence is equivalent to the algebraic Novikov spectral sequence [Voevodsky 2002, pages 20–21], a result later proven in modern language and at all primes by Gheorghe, Wang and Xu [Gheorghe et al. 2021, Theorem 1.17], who also introduced the perspective that $\mathcal{SH}(\mathbb{C})^{\mathrm{cell}}_p$ is a 1-parameter deformation with parameter τ. The fact that the category of p-complete $\mathbb{C}$-motivic spectra is *generically topological* is strongly suggested by results of Dugger and Isaksen [2010, Section 2.6], though to the best of our knowledge the precise statement did not appear in print until [Pstrągowski 2023]. Isaksen [2019, Sections 5–6] established an isomorphism between $\pi_{*,*}C\tau$ and $\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ and observed computationally that the τ-Bockstein spectral sequence agrees with the with the Adams–Novikov spectral sequence. The commutative algebra structure on $C\tau$ was first constructed by Gheorghe [2018], and the category of modules over $C\tau$ was then identified by Gheorghe, Wang and Xu in [Gheorghe et al. 2021]. Finally, Pstrągowski [2023] and Gheorghe, Isaksen, Krause and Ricka [Gheorghe et al. 2022] independently provided two different *Galois reconstructions* of cellular p-complete $\mathbb{C}$-motivic spectra.

With the case of $\mathbb{C}$ resolved we raise the following natural question:

Question 1.2 To what extent can this theorem be extended to a general field k?

As stated, this question is too imprecise to admit a definitive answer, so we begin by refining two points of ambiguity: the appropriate subcategory of $\mathcal{SH}(k)$ one should consider and the meaning of purely topological. In our study of this question we have found that the appropriate subcategory is the category of Artin–Tate motivic spectra (defined below). Notably, over $\mathbb{R}$, if one further restricts to the subcategory of cellular objects, it becomes significantly more difficult to construct a topological model.

Definition 1.3 The category of Artin–Tate motivic spectra over k is the smallest stable full subcategory $\mathcal{SH}(k)^{\mathrm{AT}} \subset \mathcal{SH}(k)$, closed under tensor products and colimits, that contains the motives of finite étale k-algebras, $\mathbb{P}^1_k$ and $(\mathbb{P}^1_k)^{-1}$.

The phrase "purely topological" has a double meaning. On the one hand it refers to the input to the construction; it should not depend directly on the arithmetic of k, instead using only the absolute Galois group, G, together with the character[1] $G \to \widehat{\mathbb{Z}}^\times$ induced by the maximal cyclotomic extension of k. On the other hand it asks for a construction which uses only the "standard machinery of homotopy theory".

The authors are not the first to take up questions of this nature. Positselski [2011; 2014] has studied the question of Galois reconstruction for the category $\mathrm{DM}(k;\mathbb{F}_\ell)^{\mathrm{AT}}$ of mixed Artin–Tate motives with mod ℓ coefficients. With a few exceptions, he has shown that when k is a finite, local or global field, then $\mathrm{DM}(k,\mathbb{F}_\ell)^{\mathrm{AT}}$ may be viewed as a derived category of filtered discrete G-modules with restricted subquotients.

In a different direction, work of Bachmann, Elmanto and Østvær [Bachmann 2021; Bachmann et al. 2020] shows that, up to a completion, $\mathcal{SH}(S)$ is generically étale for a wide range of schemes S. In particular, Bachmann, Elmanto and Østvær show that, after a suitable completion, étale localization corresponds to inverting τ. Note that τ may not exist in the homotopy of the completed unit, so care must be taken to interpret this statement. Furthermore, Bachmann showed that, again up to completion, the étale motivic category is equivalent to the category of hypercomplete sheaves of spectra on the small étale site. Specializing to the case $S = \operatorname{Spec} k$, we find that, up to completion, the category of τ-local objects in $\mathcal{SH}(k)$ admits a description in terms of Borel G-equivariant spectra.

Previously, work of Behrens and Shah [2020] had taken up the question of when a suitable map τ exists over $\mathbb{R}$. Although there is no map $\tau : \mathbb{S}^1 \to (\mathbb{G}_m)^\wedge_2$ in $\mathcal{SH}(\mathbb{R})$, they prove that τ exists whenever a different class $\rho : \mathbb{S}^0 \to \mathbb{G}_m$ has been killed. If only ρ^2 is killed, but not ρ itself, then τ does not necessarily exist, but τ^2 does. Continuing in this way, they make sense of inverting τ in any ρ-complete situation.

In this paper we resolve Question 1.2 in the case $k = \mathbb{R}$. This begins with the observation that if one does not insist that the target of τ must be a completion of $\mathbb{G}_m$, then a suitable replacement can easily be constructed.

[1] More specifically, the p-complete category should only depend on the $\mathbb{Z}_p^\times$ component of this map.

Theorem 1.4 *In $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ there is an invertible object Q and a map $\text{त} : \mathbb{S}^1 \to Q_2$ which enjoys the following properties*:

(GT) *The full subcategory of त-local objects in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is equivalent to $\mathrm{Sp}_{C_2,i2}$.*

(AD) *The cofiber of त (denoted by $C\text{त}$) is a commutative algebra, and the category of dualizable modules over $C\text{त}$ is equivalent to the category of dualizable objects in the derived category of Mackey-functor $\mathrm{MU}_*\,\mathrm{MU}$-comodules.* (*See Section 5 for a precise definition of this category.*)

(GR) *There is a purely topological construction of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. In particular, we construct a commutative algebra $R_\bullet$ in filtered, C_2-equivariant spectra such that the category of filtered modules over $R_\bullet$ is equivalent to $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. The commutative algebra $R_\bullet$ is the even slice–décalage of the $\mathrm{MU}_{\mathbb{R}}$-Adams tower for the sphere.* (*See Section 4 for precise definitions.*)

We summarize this by saying that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is a 1-parameter deformation of C_2-equivariant stable homotopy theory with parameter त and purely algebraic special fiber.

The romanization[2] of the Devanagari letter त is "ta" and our choice of this symbol will be explained in Remark 1.34. As in the case over $\mathbb{C}$, we are also able to give an explicit formula for the homotopy groups of $C\text{त}$, though since we have not yet set up the appropriate notion of homotopy groups we will defer an explicit statement until later. The subscript $i2$ appearing in the theorem statement refers to the category of modules over the 2-completion of the unit. The reader who is worried by this departure from the norm may wish to read the section on notation and conventions before proceeding.

With the resolution of Question 1.2 in the case of $\mathbb{R}$, the authors are hopeful that we might see this question resolved in its entirety in the near future.

1.1 Examples of Artin–Tate motivic spectra

In this subsection, which is preparatory to all later material, we give a more comprehensive introduction to the category of Artin–Tate motivic spectra over a field. This begins with introducing the smaller subcategories of Artin and Tate objects, which together generate the category of Artin–Tate objects. The bulk of the subsection is spent building a collection of important examples of Artin–Tate motivic spectra. Over $\mathbb{R}$ a special role is played by the invertible objects and so we focus specific attention there. We close the section by discussing homotopy groups.

Definition 1.5 The categories of Artin, Tate and Artin–Tate motivic spectra over k are the stable, full subcategories of $\mathcal{SH}(k)$, closed under tensor products and colimits, with the following generators:

- The category of Artin motivic spectra, $\mathcal{SH}(k)^{\mathrm{A}}$, is generated by the motives of finite étale k-algebras.
- The category of Tate motivic spectra,[3] $\mathcal{SH}(k)^{\mathrm{T}}$, is generated by the motives of $\mathbb{P}^1_k$ and $(\mathbb{P}^1_k)^{-1}$.

[2] However, the reader who is not familiar with the Devanagari alphabet is advised that the pronunciation of त in the majority of South Asian languages using Devanagari is given in IPA by /t̪ə/, which is closer to "tuh" in English than "ta".

[3] This category is sometimes referred to as the cellular category following [Dugger and Isaksen 2005].

- The category of Artin–Tate motivic spectra, $\mathcal{SH}(k)^{\mathrm{AT}}$, is generated by the motives of finite étale k-algebras together with $\mathbb{P}^1_k$ and $(\mathbb{P}^1_k)^{-1}$.

Since these subcategories are closed under tensor products and colimits, they each inherit the structure of a stable, presentably symmetric monoidal category from $\mathcal{SH}(k)$. (In the sequel, we will state without remark results that are only known to be true after the characteristic of k is inverted. Readers are invited to restrict themselves to k of characteristic zero if they prefer.)

We now turn to examples of objects in each of these categories, starting with the trivial and heading towards the nontrivial.

Example 1.6 Every stable, presentably symmetric monoidal category admits a unique symmetric monoidal functor from the category of spectra [Lurie 2017, Corollary 4.8.2.19].[4] This gives us objects $X \otimes \mathbb{1}_k$ for every spectrum X. Of particular interest are the integer simplicial suspensions of the unit, $\mathbb{S}^n \otimes \mathbb{1}_k$.

Inductively applying the homotopy purity theorem [Morel and Voevodsky 1999, Theorem 3.2.23] to the decomposition $\mathbb{P}^n_k = \mathbb{A}^n_k \amalg \mathbb{P}^{n-1}_k$, we learn:

Example 1.7 [Dugger and Isaksen 2005, Example 2.12] For each n, the motivic spectrum associated to $\mathbb{P}^n_k$ is Tate.

More generally, using a Bialynicki–Birula decomposition, Wendt [2010] shows that any smooth projective variety which admits a $\mathbb{G}_m$-action whose fixed points are discrete and rational is Tate.[5]

Example 1.8 [Dugger and Isaksen 2005, Theorem 6.4; Hoyois 2015, Proposition 8.1; Bachmann and Hopkins 2020, Proposition 8.12] Each of the following commutative algebras is Tate:

$$\mathrm{MGL}, \quad \mathrm{kq}, \quad \mathrm{kgl}, \quad \mathrm{M}\mathbb{Z}, \quad \mathrm{M}\mathbb{F}_p\,.$$

Example 1.9 Essentially by definition, the Galois correspondence provides functors

$$\begin{array}{ccccc} \{\text{finite, continuous } \mathrm{Gal}(\overline{k}/k)\text{-sets}\} & \longrightarrow & \{\text{finite, étale } k\text{-algebras}\}^{\mathrm{op}} & \hookrightarrow & \mathrm{Sm}_k \\ & & \downarrow & & \downarrow \\ & & \mathcal{SH}(k)^{\mathrm{A}} & \hookrightarrow & \mathcal{SH}(k) \end{array}$$

which give our first examples of Artin objects.

Since the functor in the example above sends disjoint unions to sums there is no loss in generality if we restrict to transitive $\mathrm{Gal}(\overline{k}/k)$-sets (field extensions). In the case of $\mathbb{R}$ this gives only two objects: $\mathrm{Spec}(\mathbb{R})$, which we will temporarily denote by $\mathbb{1}_{\mathbb{R}}$ since it is the monoidal unit of the category, and $\mathrm{Spec}(\mathbb{C})$. Using the natural map we get a cofiber sequence

$$\mathrm{Spec}(\mathbb{C}) \to \mathbb{1}_{\mathbb{R}} \xrightarrow{a} \mathbb{S}^{\mathbb{C}}\,. \tag{1}$$

On the level of $\mathbb{C}$-points (with Galois action), this cofiber sequence gives the representation sphere $\mathbb{S}^{\sigma}$.

[4] Another way of saying this is that Sp is the initial object of $\mathrm{CAlg}(\mathrm{Pr}^{L,\mathrm{stab}})$.

[5] In the case where the fixed points are not rational, the authors wonder what conditions are necessary to guarantee that the motive is Artin–Tate.

Given a quadric in the plane, V, we can take its closure in $\mathbb{P}^2_k$ to obtain $\bar{V}$ which is a form of $\mathbb{P}^1_k$. If we let $x_1, \dots, x_n$ denote the points at infinity, then by the homotopy purity theorem [Morel and Voevodsky 1999, Theorem 3.2.23] we have a cofiber sequence

$$V \to \bar{V} \to \bigoplus_{i=1,\dots,n} \mathbb{P}^1_{k(x_i)}.$$

Under the assumption that V admits a rational point, $\bar{V}$ is just $\mathbb{P}^1_k$, and so we obtain our first nontrivial example of a motive which need not be Artin or Tate, but is Artin–Tate:[6]

Example 1.10 The motive of an affine quadric which admits a rational point is Artin–Tate.

Over $\mathbb{R}$ there is a particular affine quadric of interest to us. It is the object Q which appeared in the statement of Theorem 1.4.

Example 1.11 Let $Q := \{x^2 + y^2 = 1\} \subset \mathbb{A}^2_{\mathbb{R}}$, which we shall call the algebraic circle. Q has the additional property that it is a form[7] of $\mathbb{G}_m$. Its group scheme structure comes from the usual rule for multiplication of complex numbers, namely

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

In Section 2 we will construct the maps $\mathfrak{a} : \mathbb{S}^1 \to Q_p$ using our explicit understanding of this group scheme.

Proposition 1.12 [Hu 2005, Proposition 1.1] *There is an equivalence $Q \otimes \mathbb{S}^{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{R}}$. In particular, both $\mathbb{S}^{\mathbb{C}}$ and Q are invertible.*

At this point we are ready to define the appropriate notion of homotopy groups for studying $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}$. In the case of $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}$, the category has a bigraded family of compact invertible generators given by the spheres $\mathbb{S}^{p-2w} \otimes (\mathbb{P}^1_{\mathbb{C}})^{\otimes w}$. Therefore, it is typical to study objects at the level of their bigraded homotopy groups, given by

$$\pi^{\mathbb{C}}_{p,w}(X) := \pi_0 \operatorname{Map}(\mathbb{S}^{p-2w} \otimes (\mathbb{P}^1_{\mathbb{C}})^{\otimes w}, X).$$

In a similar way, $\mathcal{SH}(\mathbb{R})^{\mathrm{T}}$ has a bigraded family of compact invertible generators given by the spheres $\mathbb{S}^{p-2w} \otimes (\mathbb{P}^1_{\mathbb{R}})^{\otimes w}$, and it is typical to study objects through their bigraded homotopy groups. Using Proposition 1.12 and equation (1), the category $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}$ has a trigraded family of compact invertible generators given by the spheres $\mathbb{S}^{p-w} \otimes (\mathbb{S}^{\mathbb{C}})^{\otimes q-w} \otimes (\mathbb{P}^1_{\mathbb{R}})^{\otimes w}$, and we will study most objects through their trigraded homotopy groups.

Notation 1.13 We endow the categories Sp, Sp_{C_2}, $\mathcal{SH}(\mathbb{C})$ and $\mathcal{SH}(\mathbb{R})$ with Picard gradings by spheres as in Table 1. These grading conventions were chosen to play well with the comparison functors between these categories; see Recollection 1.17.

[6] Applying this construction as in Example 1.11, for example, yields a motive that is neither Artin nor Tate by Proposition 1.12.

[7] By a form of $\mathbb{G}_m$, we mean a group scheme over $\mathbb{R}$ whose base change to $\mathbb{C}$ is isomorphic to $\mathbb{G}_m$. By Cartier duality, forms of $\mathbb{G}_m$ are classified by rank 1 lattices with Galois action, and Q corresponds to the unique nontrivial action.

category	Picard	spheres
Sp	$\mathbb{Z}$	$\mathbb{S}^p \simeq (S^1)^{\otimes p}$
Sp_{C_2}	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{S}^{p+q\sigma} \simeq (S^1)^{\otimes p}\otimes(S^\sigma)^{\otimes q}$
$\mathcal{SH}(\mathbb{C})$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{S}^{p,w} \simeq (S^1)^{\otimes p-2w}\otimes(\mathbb{P}^1_{\mathbb{C}})^{\otimes w}$
$\mathcal{SH}(\mathbb{R})$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{S}^{p,q,w} \simeq (S^1)^{\otimes p-w}\otimes(S^{\mathbb{C}})^{\otimes q-w}\otimes(\mathbb{P}^1_{\mathbb{R}})^{\otimes w}$

Table 1

For any $p,q,w\in\mathbb{Z}$, and any $\mathbb{R}$-motivic spectrum X, we let $\pi^{\mathbb{R}}_{p,q,w}X$ denote the group of homotopy classes of maps

$$(S^1)^{\otimes p-w}\otimes(S^{\mathbb{C}})^{\otimes q-w}\otimes(\mathbb{P}^1_{\mathbb{R}})^{\otimes w}\to X.$$

Example 1.14 For convenience we record what the invertible objects considered in this section look like under this new notation:

$$\mathbb{S}^{0,0,0}\cong\mathbb{1}_{\mathbb{R}},\quad \mathbb{S}^{1,0,0}\cong S^1,\quad \mathbb{S}^{0,1,0}\cong S^{\mathbb{C}},\quad \mathbb{S}^{0,1,1}\cong\mathbb{G}_m,\quad \mathbb{S}^{1,0,1}\cong Q,\quad \mathbb{S}^{1,1,1}\cong\mathbb{P}^1_{\mathbb{R}}.$$

In the case of $\mathbb{S}^{0,0,0}$ we will often drop the indices for brevity, writing only $\mathbb{S}$. The two maps between Picard elements considered thus far, $\overline{\tau}$ and a, become

$$\overline{\tau}\in\pi^{\mathbb{R}}_{0,0,-1}\mathbb{S}_p\quad\text{and}\quad a\in\pi^{\mathbb{R}}_{0,-1,0}\mathbb{S}\,.$$

Remark 1.15 The trigraded spheres in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}$ which are Tate are precisely the spheres of the form $\mathbb{S}^{p,q,q}$. For this reason the Tate category only sees a "slice" of the total information present in the trigraded homotopy groups.

The trigraded spheres in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}$ which are Artin are precisely the spheres of the form $\mathbb{S}^{p,q,0}$. The Artin category similarly only sees a "slice" of the total homotopy groups.

1.2 Comparison functors

The two simplest ways to interrogate a category are to study specific objects and to study the network of functors which relate it to other categories. While the previous subsection provided preparatory background on specific objects, this subsection sets up the suite of functors which we will use to produce and study more general objects. In the specific case of $\mathbb{R}$ this means studying the various ways we can move between $\mathcal{SH}(\mathbb{R})$, $\mathcal{SH}(\mathbb{C})$, Sp_{C_2} and Sp.

Recollection 1.16 Given a finite, separable extension of fields $i:k\to\ell$, there are two pairs of adjunctions and one extra functor coming from the six functor formalism

$$i^*:\mathcal{SH}(k)\rightleftarrows\mathcal{SH}(\ell):i_*,\quad i_!:\mathcal{SH}(\ell)\rightleftarrows\mathcal{SH}(k):i^!\quad\text{and}\quad i_\sharp:\mathcal{SH}(\ell)\to\mathcal{SH}(k),$$

where i^* is symmetric monoidal. Since i is smooth, proper and unramified we have equivalences $i_!\simeq i_*\simeq i_\sharp$ and $i^!\simeq i^*$; see [Hoyois 2017, Theorems 6.9 and 6.18]. These equivalences tell us that

(i) i^* and i_* both commute with all limits and colimits, and

(ii) if X is a smooth ℓ-scheme, then $i_*X\simeq X$, where the second copy of X is considered as a k-scheme.

In Appendix A, as an example of the techniques showcased there, we show that for fields of characteristic zero there is an equivalence of presentably symmetric monoidal categories

$$\mathcal{SH}(\ell) \simeq \mathrm{Mod}(\mathcal{SH}(k); \mathrm{Spec}(\ell)),$$

and $i_* i^*(-) \simeq \mathrm{Spec}(\ell) \otimes -$. This equivalence tells us that no new information enters the picture when we pass to a field extension. Specializing to the case of $\mathbb{C}/\mathbb{R}$ the above equivalence becomes

$$\mathcal{SH}(\mathbb{C}) \simeq \mathrm{Mod}(\mathcal{SH}(\mathbb{R}); \mathrm{Spec}(\mathbb{C})).$$

Using the descriptions of i_* and $i_* i^*$ we can conclude that both i^* and i_* restrict to the full subcategory of Artin–Tate objects. Thus we obtain a similar description of the Artin–Tate category of a field extension:

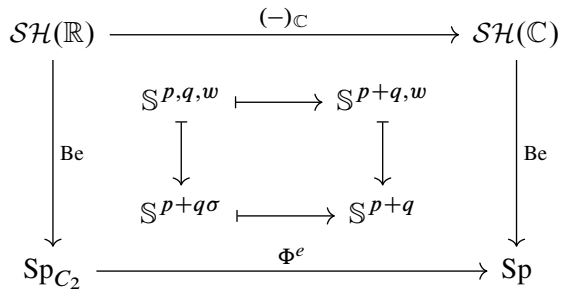
$$\mathcal{SH}(\ell)^{\mathrm{AT}} \simeq \mathrm{Mod}(\mathcal{SH}(k)^{\mathrm{AT}}; \mathrm{Spec}(\ell)).$$

Specializing to the case of $\mathbb{C}/\mathbb{R}$, we will sometimes denote $\mathrm{Spec}(\mathbb{C})$ by Ca since it is (by the definition of a) the cofiber of the map $a : \mathbb{S}^{0,-1,0} \to \mathbb{S}^{0,0,0}$. The above equivalence becomes

$$\mathcal{SH}(\mathbb{C})^{\mathrm{AT}} \simeq \mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}; Ca).$$

At this point we turn to studying the case of $\mathbb{R}$ more closely. Though some of the things we do after this point have obvious analogues in other cases, many of our key maneuvers implicitly rely on the fact that $\mathbb{R}$ has a finite (and well-understood) absolute Galois group. In particular, we will now assume that the reader is familiar with C_2-equivariant homotopy theory as in [Hill et al. 2016]. Equivariant homotopy theory first enters the picture through the Betti realization functors of [Morel and Voevodsky 1999, Section 3.3].

Recollection 1.17 There is a commutative diagram of symmetric monoidal left adjoints:



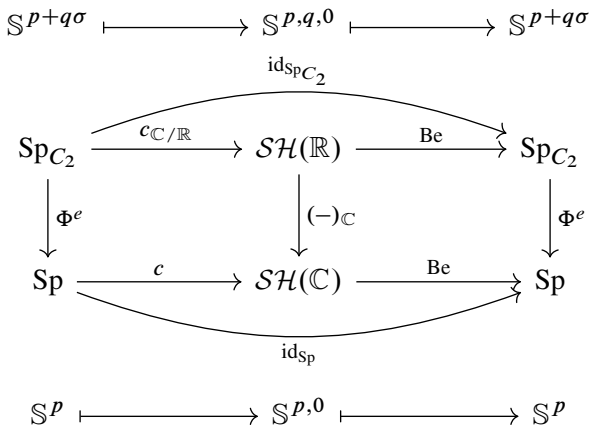
In the above diagram, $(-)_{\mathbb{C}}$ is the base change functor, Φ^e is the underlying functor and Be are the Betti realization functors, induced by the assignment $X \mapsto X(\mathbb{C})$. The inner square describing these functors on Picard objects can be verified by considering Q, $\mathbb{G}_m$ and S^1 directly.[8]

Remark 1.18 Since the Betti realization of $\mathbb{S}^{p,q,w}$ is $\mathbb{S}^{p+q\sigma}$, we think of w as recording the motivic weight and p, q as providing a copy of $\mathrm{RO}(C_2)$ in each weight. The Tate spheres are those of the form $\mathbb{S}^{p,q,q}$, so in the bigraded world the number of σ's in the $\mathrm{RO}(C_2)$-grading must equal the motivic weight. This restriction blinds one to the existence of a weight-shifting element $\overline{\tau}$ in $\mathrm{RO}(C_2)$ grading 0, which is the true analogue of the $\mathbb{C}$-motivic τ.

[8] The observation that spurred the authors to begin this project was that, upon restricting to categories of Tate objects, the induced square on Picard groups is not a pullback, but with Artin–Tate objects it is in fact a pullback.

Having discussed Betti realization, we now introduce the less well-known functors c and $c_{\mathbb{C}/\mathbb{R}}$ which provide sections of Betti realization.

Recollection 1.19 There are symmetric monoidal left adjoints c and $c_{\mathbb{C}/\mathbb{R}}$, which fit into the commutative diagram



The functor c is the unique symmetric monoidal left adjoint coming from the fact that Sp is the unit of $\mathrm{Pr}^{L,\mathrm{stab}}$; see Example 1.6. The functor $c_{\mathbb{C}/\mathbb{R}}$ is constructed in [Heller and Ormsby 2016], by beginning with the functor

$$\{\text{finite } C_2\text{-sets}\} \to \mathcal{SH}(\mathbb{R})$$

from Example 1.9 and then extending it to all of Sp_{C_2}.

Since they are symmetric monoidal left adjoints, it is easy to see that the composites $\mathrm{Be}\circ c : \mathrm{Sp} \to \mathrm{Sp}$ and $\mathrm{Be}\circ c_{\mathbb{C}/\mathbb{R}} : \mathrm{Sp}_{C_2} \to \mathrm{Sp}_{C_2}$ are equivalent to the identity. In fact, upon restricting to the appropriate target category we uncover something more interesting:

Theorem 1.20 [Levine 2014; Heller and Ormsby 2016, 2018] *The symmetric monoidal functors c and $c_{\mathbb{C}/\mathbb{R}}$ factor through the respective categories of Artin objects and provide equivalences*,

$$c : \mathrm{Sp} \xrightarrow{\simeq} \mathcal{SH}(\mathbb{C})^{\mathrm{A}} \quad \text{and} \quad c_{\mathbb{C}/\mathbb{R}} : \mathrm{Sp}_{C_2} \xrightarrow{\simeq} \mathcal{SH}(\mathbb{R})^{\mathrm{A}}.$$

Corollary 1.21 *The induced maps*,

$$\pi_p\,\mathbb{S} \xrightarrow{c} \pi^{\mathbb{C}}_{p,0}\,\mathbb{S} \xrightarrow{\mathrm{Be}} \pi_p\,\mathbb{S} \quad \text{and} \quad \pi^{C_2}_{p+q\sigma}\,\mathbb{S} \xrightarrow{c_{\mathbb{C}/\mathbb{R}}} \pi^{\mathbb{R}}_{p,q,0}\,\mathbb{S} \xrightarrow{\mathrm{Be}} \pi^{C_2}_{p+q\sigma}\,\mathbb{S},$$

are isomorphisms for all p and q.

Remark 1.22 This corollary provides an identification of $\pi^{\mathbb{R}}_{0,0,0}\,\mathbb{S}$ with the Burnside ring of C_2. It is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, with generators 1 and $[C_2]$ and the relation $[C_2]^2 = 2[C_2]$.

Morel's identification [2006] of $\pi_{n,n}\mathbb{1}_k$ in terms of Milnor–Witt K-theory provides another way of assigning names to elements of $\pi^{\mathbb{R}}_{0,0,0}\,\mathbb{S}$. In these terms $\pi^{\mathbb{R}}_{0,0,0}\,\mathbb{S}$ is generated by 1 and $\eta[-1]$, subject to the relation $(\eta[-1])^2 = -2\eta[-1]$. We denote by ρ the element $-[-1]$. The translation between the two bases described above is given by $\eta\rho = 2-[C_2]$.

Remark 1.23 When working over a general field one might think to replace the Betti realization by some variant of étale localization; see [Elmanto and Shah 2021]. The analogue of this theorem would be that the category of Artin objects is already étale local. However, in many examples (such as finite fields) one consequence of the Morel connectivity theorem [2005] is that this is not true. An important precursor to answering Question 1.2 will be producing the variant of equivariant homotopy theory which appears as the category of Artin objects.

Remark 1.24 Using the functors c and $c_{\mathbb{C}/\mathbb{R}}$ the mapping spaces in $\mathcal{SH}(\mathbb{C})$ and $\mathcal{SH}(\mathbb{R})$ can be upgraded into mapping spectra and mapping C_2-spectra, respectively.

Summary 1.25 *There are commuting diagrams*

$$\begin{array}{ccc} \mathcal{SH}(\mathbb{R}) & \xrightarrow{(-)_{\mathbb{C}}} & \mathcal{SH}(\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{Be}} & & \downarrow{\scriptstyle \mathrm{Be}} \\ \mathrm{Sp}_{C_2} & \xrightarrow{\Phi^e} & \mathrm{Sp} \end{array} \qquad \begin{array}{ccccc} & & \overset{\mathrm{id}_{\mathrm{Sp}_{C_2}}}{\frown} & & \\ \mathrm{Sp}_{C_2} & \xrightarrow{c_{\mathbb{C}/\mathbb{R}}} & \mathcal{SH}(\mathbb{R}) & \xrightarrow{\mathrm{Be}} & \mathrm{Sp}_{C_2} \\ \downarrow{\scriptstyle \Phi^e} & & \downarrow{\scriptstyle (-)_{\mathbb{C}}} & & \downarrow{\scriptstyle \Phi^e} \\ \mathrm{Sp} & \xrightarrow{c} & \mathcal{SH}(\mathbb{C}) & \xrightarrow{\mathrm{Be}} & \mathrm{Sp} \\ & & \underset{\mathrm{id}_{\mathrm{Sp}}}{\smile} & & \end{array}$$

and various functors considered in this section enjoy the following properties:

(i) *Each of* $(-)_{\mathbb{C}}$, Be, Φ^e, c *and* $c_{\mathbb{C}/\mathbb{R}}$ *is a symmetric monoidal left adjoint.*

(ii) *Both* $(-)_{\mathbb{C}}$ *and* Φ^e *are right adjoints as well.*

(iii) *All of the functors restrict to the categories of Artin–Tate objects and retain the properties listed in* (i) *and* (ii). *In later sections we will almost exclusively deal with these restrictions so we will not use distinct notation for them.*

(iv) *On Picard elements the functors in the diagrams above behave as follows*:

$$\begin{aligned} \mathrm{Be}(\mathbb{1}_{\mathbb{R}}^{p,q,w}) &\cong \mathbb{1}_{C_2}^{p+q\sigma}, \quad & \mathrm{Be}(\mathbb{1}_{\mathbb{C}}^{s,w}) &\cong \mathbb{S}^s, \quad & c_{\mathbb{C}/\mathbb{R}}(\mathbb{1}_{C_2}^{p+q\sigma}) &\cong \mathbb{1}_{\mathbb{R}}^{p,q,0}, \\ c(\mathbb{S}^s) &\cong \mathbb{1}_{\mathbb{C}}^{s,0}, \quad & (\mathbb{1}_{\mathbb{R}}^{p,q,w})_{\mathbb{C}} &\cong \mathbb{1}_{\mathbb{C}}^{p+q,w}, \quad & \Phi^e(\mathbb{1}_{C_2}^{p+q\sigma}) &\cong \mathbb{S}^{p+q}. \end{aligned}$$

Proof The only one of these which we have not already discussed is (ii). The functor $(-)_{\mathbb{C}}$ can be described as tensoring up to Ca. Since Ca is dualizable this functor commutes with all limits and colimits. Similarly, we may conclude that Φ^e commutes with all limits and colimits. □

1.3 $\mathbb{R}$-motivic spectra as a deformation

In this subsection we refine the statement of Theorem 1.4 into a sequence of precise claims, which we will verify across the remainder of the paper. We summarized both Theorem 1.1 and Theorem 1.4 by saying that the category of Artin–Tate motivic spectra is a 1-parameter deformation of a purely topological category. To start, we clarify this, first over $\mathbb{C}$ and then over $\mathbb{R}$.

(i) There is a distinguished element of the p-complete homotopy groups over $\mathbb{C}$, $\tau \in \pi_{0,-1}\,\mathbb{S}_p$, which maps to 1 under Betti realization; see [Hu et al. 2011b, Lemma 23]. (The Betti realization map $\pi^{\mathbb{C}}_{0,-1}\,\mathbb{S}_p \to \pi_0\,\mathbb{S}_p$ is an isomorphism so the latter property uniquely identifies τ.)

(ii) The Betti realization functor factors through the category of τ-local objects and provides a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}; \mathbb{S}_p[\tau^{-1}]) \simeq \mathrm{Sp}_{ip}\,.$$

The idea for this goes back to [Dugger and Isaksen 2010, Section 2.6], but was first proven in [Pstrągowski 2023].

(iii) The category $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$ can be equipped with the structure of a $\mathrm{Sp}^{\mathbf{Fil}}_{ip}$-algebra. More explicitly, this means we have a symmetric monoidal left adjoint

$$i^* : \mathrm{Sp}^{\mathbf{Fil}}_{ip} \to \mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip},$$

which sends the shift map in $\mathrm{Sp}^{\mathbf{Fil}}_{ip}$ to τ. As a consequence of this $C\tau$ acquires the structure of a commutative algebra, a fact originally proven by Gheorghe [2018].

(iv) The category of modules over $C\tau$ is equivalent to a renormalization of the derived category of even $\mathrm{BP}_*\mathrm{BP}$-comodules. More specifically we have

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}; C\tau) \simeq \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{ip}.$$

This equivalence sends $C\tau \otimes \mathbb{S}^{s,w}$ to $\Sigma^{s-2w}\omega^{\otimes w}_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$, and on the level of homotopy groups it induces an equivalence

$$\pi^{\mathbb{C}}_{s,w}(C\tau) \cong \mathrm{Ext}^{2w-s,w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

The equivalence of categories is due to Gheorghe, Wang and Xu [Gheorghe et al. 2021], though the above isomorphism of groups was first proven by Isaksen [2019, Proposition 6.2.5] and then upgraded to an isomorphism of rings with all higher structure by Gheorghe [2018].

(v) There is an equivalence of symmetric monoidal categories between $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$ and $\mathrm{Syn}^{\mathrm{even}}_{\mathrm{MU},ip}$, where the latter is the category of *even* MU-*synthetic spectra* constructed in [Pstrągowski 2023]. Notably, Pstrągowski's construction uses no algebraic geometry, and requires only knowledge of the homotopy commutative ring object MU in Sp. The comparison between these two categories proceeds via the close relationship between MGL and MU.

(vi) The adjunction i is affine in the sense that there is a commutative algebra $R^{\mathbb{C}}_\bullet$ in $\mathrm{Sp}^{\mathbf{Fil}}_{ip}$ and an equivalence of symmetric monoidal categories under $\mathrm{Sp}^{\mathbf{Fil}}_{ip}$

$$\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{ip}; R^{\mathbb{C}}_\bullet).$$

Moreover, the commutative algebra $R^{\mathbb{C}}_\bullet$ admits an explicit construction which uses no algebraic geometry. It is given by

$$R^{\mathbb{C}}_\bullet := \mathrm{Tot}^*(\tau_{\geq 2\bullet}\,\mathrm{MU}^{\otimes *+1}_p).$$

This construction uses only the commutative algebra MU in Sp, and again the comparison proceeds via the close relationship between MGL and MU. This approach is due to Gheorghe, Isaksen, Krause and Ricka [Gheorghe et al. 2022].

In fact, every aspect of the picture over $\mathbb{C}$ extends to $\mathbb{R}$ in the simplest reasonable way. The reader who has previously studied $\mathbb{R}$-motivic spectra may find this rather surprising (as the authors did), and we suggest that this highlights the primacy of the Artin–Tate category over the Tate category. Most of the results below are stated at the prime $p = 2$, where they are most interesting. The case of odd primes is discussed in Proposition 1.32 and Section 9.

(i) There is a distinguished element of the p-complete homotopy groups over $\mathbb{R}$, $\overline{\tau} \in \pi_{0,0,-1}\,\mathbb{S}_p$, which maps to 1 under Betti realization.[9] We will construct $\overline{\tau}$ in Section 2.

(ii) The Betti realization functor factors through the category of $\overline{\tau}$-local objects, and provides a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[\overline{\tau}^{-1}]) \simeq \mathrm{Sp}_{C_2,i2}\,.$$

This will be proven as the main theorem of Section 3.

(iii) The category $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ can be equipped with the structure of a $\mathrm{Sp}^{\mathbf{Fil}}_{i2}$-algebra. More explicitly, this means we have a symmetric monoidal left adjoint

$$i^* : \mathrm{Sp}^{\mathbf{Fil}}_{i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2},$$

which sends the shift map in $\mathrm{Sp}^{\mathbf{Fil}}_{i2}$ to $\overline{\tau}$. As a consequence of this $C\overline{\tau}$ acquires the structure of a commutative algebra. We may also regard $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ as a Sp_{C_2}-algebra via the functor $c_{\mathbb{C}/\mathbb{R}}$. Tensoring these two functors together we obtain a symmetric monoidal left adjoint

$$i^* : \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}.$$

These statements will be proven in Section 4 as part of Proposition 4.5.

(iv) The category of modules over $C\overline{\tau}$ is equivalent to a renormalization of the derived category of an abelian category of equivariant $\mathrm{BP}_*\mathrm{BP}$-comodules. More precisely, we have an equivalence of presentably symmetric monoidal categories

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\overline{\tau}) \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}).$$

This equivalence sends $C\overline{\tau} \otimes \mathbb{S}^{p,q,w}$ to $\Sigma^{(p-w)+(q-w)\sigma}\underline{\mathbb{Z}}_2 \otimes \omega^{\otimes w}_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$ and on the level of homotopy groups it induces an isomorphism

$$\pi^{\mathbb{R}}_{p,q,w} C\overline{\tau} \cong \bigoplus_{w+a-s=p} \mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* \otimes \pi^{C_2}_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2).$$

This will be proven as the main theorem of Section 5.

[9] As above, the Betti realization map $\pi^{\mathbb{R}}_{0,0,-1}\,\mathbb{S}_p \to \pi^{C_2}_0\,\mathbb{S}_p$ is an isomorphism so $\overline{\tau}$ is uniquely identified.

(v) The adjunction i is affine in the sense that there is a commutative algebra $R^{\mathbb{R}}_{\bullet}$ in $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ and an equivalence of presentably symmetric monoidal categories under $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$

$$\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; R^{\mathbb{R}}_{\bullet}).$$

Moreover, the commutative algebra $R^{\mathbb{R}}_{\bullet}$ admits an explicit construction which uses no algebraic geometry. It is given by

$$R^{\mathbb{R}}_{\bullet} := \mathrm{Tot}^*(P_{2\bullet}\,\mathrm{MU}^{\otimes *+1}_{\mathbb{R},2}),$$

where P_n is the functor which takes the n^{th} slice cover of a C_2 spectrum. This construction uses only the commutative algebra $\mathrm{MU}_{\mathbb{R}}$ in Sp_{C_2} introduced by Landweber [1967; 1968]. The comparison proceeds via understanding the close relationship between MGL and $\mathrm{MU}_{\mathbb{R}}$ over $\mathbb{R}$. This will be proven as the main theorem of Section 4.

Using the second special element of the trigraded homotopy groups of the sphere, $a \in \pi^{\mathbb{R}}_{0,-1,0}\,\mathbb{S}$, we can delve further into the structure of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. In order to do this we begin with a digression on the element a_σ in C_2-equivariant homotopy theory.

Proposition 1.26 *The category* Sp_{C_2} *can be viewed as a* 1*-parameter family with coordinate* a_σ*, special fiber* Sp *and generic fiber* Sp *in the sense that*:

- *The category of* a*-local objects can be identified with spectra, i.e., there is an equivalence of presentably symmetric monoidal categories*

$$\mathrm{Mod}(\mathrm{Sp}_{C_2}; \mathbb{S}[a_\sigma^{-1}]) \simeq \mathrm{Sp}\,.$$

- *The cofiber of* a_σ*, which we will denote by* Ca_σ*, can be endowed with a commutative algebra structure and there is an equivalence of presentably symmetric monoidal categories*

$$\mathrm{Mod}(\mathrm{Sp}_{C_2}; Ca_\sigma) \simeq \mathrm{Sp}\,.$$

- *There is a monoidal left adjoint*[10]

$$i^* : \mathrm{Sp}^{\mathbf{Gr}} \to \mathrm{Sp}_{C_2},$$

which sends $\mathbb{1}(1)$ *to* $\mathbb{S}^\sigma$*. Moreover, this adjunction is affine in the sense that we have an equivalence of categories*

$$\mathrm{Sp}_{C_2} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}; R^{C_2}_{\bullet}), \quad \textit{where } R^{C_2}_n \simeq \Sigma\mathbb{R}\mathrm{P}^{-n-1}_{-\infty}.$$

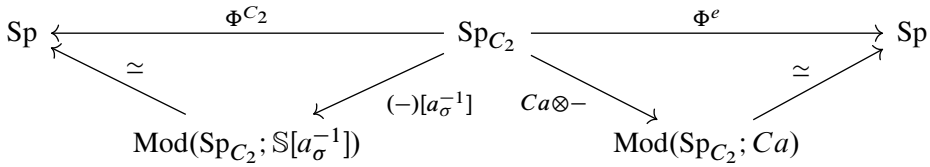
This result is likely well known to experts. We give a proof in Appendix A as Examples A.8 and A.9. Since therein the pair of functors Φ^e and Φ^{C_2} out of Sp_{C_2} become identified with modding out by a_σ

[10] The authors will return to the question of whether this functor can be upgraded to a monoidal functor in from filtered spectra in a future work.

	unspecified	$\tau = 0$	$\tau = 1$
unspecified	$\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$	$\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$	$\mathrm{Sp}_{C_2,i2}$
$a = 0$	$\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{i2}$	$\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{i2}$	Sp_{i2}
$a = 1$	?	$\mathrm{Mod}(\mathrm{Sp}; \mathbb{F}_2[u_{2\sigma}]) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$	Sp_{i2}

Table 2

and inverting a_σ, respectively, this proposition produces a diagram of symmetric monoidal left adjoints



We are now free to use the functor $c_{\mathbb{C}/\mathbb{R}}$ to push our description of Sp_{C_2} as a 1-parameter deformation into $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and obtain a description of that category as a 2-parameter deformation of Sp_{i2} with coordinates τ and a. For example, we can give Ca the structure of a commutative algebra since $c_{\mathbb{C}/\mathbb{R}}(a_\sigma) = a$. Note that by Summary 1.25 the commutative algebra structure obtained in this way is the same as the one coming from the equivalence $Ca \simeq \mathrm{Spec}(\mathbb{C})$.

While the 1-parameter deformations considered up to now had only a special and a generic fiber, when considered as a 2-parameter deformation $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ has various "limiting behaviors", which we presently study. Since each parameter can be set to either 0 or 1, or left unspecified, there are 9 total categories of interest. We summarize what we know in Table 2.

The only identification in this table which we have not yet discussed is the one in the bottom middle. However, one can easily obtain this from the identification of $C\tau$-modules and Lemma A.10:

$$\begin{aligned}
\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau[a^{-1}]) &\simeq \mathrm{Sp}_{i2} \otimes_{\mathrm{Sp}_{C_2,i2}} \mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau) \\
&\simeq \mathrm{Sp}_{i2} \otimes_{\mathrm{Sp}_{C_2,i2}} \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) \\
&\simeq \mathrm{Mod}(\mathrm{Sp}_{i2}; \Phi^{C_2}\underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) \\
&\simeq \mathrm{Mod}(\mathrm{Sp}; \mathbb{F}_2[u_{2\sigma}]) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}),
\end{aligned}$$

where the final step is just the identification of $\Phi^{C_2}\underline{\mathbb{Z}}$ with $\mathbb{F}_2[u_{2\sigma}]$ where $u_{2\sigma}$ is a polynomial generator in degree 2. As a corollary, we can give a description of the trigraded homotopy groups of several objects in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ in terms of better understood categories.

Corollary 1.27 *There are isomorphisms of rings*:

(i) $\pi^{\mathbb{R}}_{p,q,w}(Ca) \cong \pi^{\mathbb{C}}_{p+q,w}\,\mathbb{S}$,

(ii) $\pi^{\mathbb{R}}_{p,q,w}(Ca \otimes C\tau) \cong \mathrm{Ext}^{2w-p-q,w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$,

(iii) $\pi^{\mathbb{R}}_{p,q,w}(Ca[\tau^{-1}]_p) \cong \pi_{p+q}\,\mathbb{S}_p$,

(iv) $\pi^{\mathbb{R}}_{p,q,w}(C\tau) \cong \bigoplus_{w+a-s=p} \operatorname{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* \otimes \pi^{C_2}_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2)$,

(v) $\pi^{\mathbb{R}}_{p,q,w}(C\tau[a^{-1}]) \cong \bigoplus_{w+2a-s=p} \big(\mathbb{F}_2\{u^{2a}\} \otimes_{\mathbb{F}_2} \operatorname{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/2)\big)$,

(vi) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2[\tau^{-1}]) \cong \pi^{C_2}_{p,q}\,\mathbb{S}_2$,

(vii) $\pi_{p,q,w}(\mathbb{S}_2[a^{-1}, \tau^{-1}]) \cong \pi_p\,\mathbb{S}_2$.

We now turn to the category of a-local objects, only labeled "?" in the table above. It is one of the most mysterious actors in the story told in this paper and we shall devote Section 7 to its study. As we shall see, it records a deformation of the category of spectra sitting between that which corresponds to the Adams–Novikov spectral sequence and that which corresponds to the classical Adams spectral sequence. The role it plays in mediating between these two spectral sequences remains to be understood.

Since $a = c_{\mathbb{C}/\mathbb{R}}(a_\sigma)$ and $a_\sigma = \mathrm{Be}(a)$, we can construct a diagram of symmetric monoidal left adjoints

$$\begin{array}{ccc} \mathcal{SH}(\mathbb{R}) & \xrightarrow{(-)[a^{-1}]} & \mathrm{Mod}(\mathcal{SH}(\mathbb{R}); \mathbb{S}[a^{-1}]) \\ \downarrow{\scriptstyle \mathrm{Be}} & & \downarrow{\scriptstyle \mathrm{Be}} \\ \mathrm{Sp}_{C_2} & \xrightarrow{\Phi^{C_2}} & \mathrm{Sp} \end{array}$$

As the diagram shows, the category $\mathrm{Mod}(\mathcal{SH}(\mathbb{R}); \mathbb{S}_2[a^{-1}])$ can be viewed as the target category of an "$\mathbb{R}$-motivic geometric fixed points" functor. Since $\Phi^{C_2}(\mathrm{MU}_{\mathbb{R}}) \simeq \mathrm{MO}$, one might guess that the category $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$ is related to Pstrągowski's synthetic category $\mathrm{Syn}_{\mathrm{MO}}$.[11] Indeed, we construct a comparison functor.

Proposition 1.28 *There is a symmetric monoidal left adjoint*

$$\mathrm{Re}_{\mathbb{F}_2} : \mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}]) \to \mathrm{Syn}_{\mathbb{F}_2}$$

which sends $\mathbb{S}^{p,q,w}$ to $\mathbb{S}^{p,w}$.

On the other side, an interesting functor into $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$ may be constructed using the Bachmann–Hoyois motivic norm functors.

Recollection 1.29 Bachmann and Hoyois [2021] constructed symmetric monoidal norm functors along finite étale maps. These norm functors may be thought of as an indexed tensor product and in the case of $\mathbb{R}$ they fit into the following diagram showing compatibility with the Hill–Hopkins–Ravenel norm functors in equivariant homotopy theory [Bachmann and Hoyois 2021, Section 11]:

$$\begin{array}{ccc} \mathcal{SH}(\mathbb{C}) & \xrightarrow{\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}} & \mathcal{SH}(\mathbb{R}) \\ \downarrow{\scriptstyle \mathrm{Be}} & & \downarrow{\scriptstyle \mathrm{Be}} \\ \mathrm{Sp} & \xrightarrow{\mathrm{Nm}_e^{C_2}} & \mathrm{Sp}_{C_2} \end{array}$$

From [Bachmann and Hoyois 2021, Example 3.5, Lemma 4.4, Example 4.10] we can conclude that, on bigraded spheres, $\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}(\mathbb{S}^{s,w}_{\mathbb{C}}) \simeq \mathbb{S}^{s,s,2w}$. The functor $\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}$ is polynomial of degree 2 and when applied

[11] The spectrum MO is a sum of copies of $\mathbb{F}_2$, and as we shall see this implies $\mathrm{Syn}_{\mathrm{MO}} \simeq \mathrm{Syn}_{\mathbb{F}_2}$.

to a direct sum we have a formula [Bachmann and Hoyois 2021, Corollary 5.13]

$$\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(X \oplus Y) \simeq \mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(X) \oplus i_*(X \otimes Y) \oplus \mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(Y).$$

Since $\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}$ commutes with sifted colimits, it follows from the above that it restricts to a functor on the categories of Artin–Tate objects

$$\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}} : \mathcal{SH}(\mathbb{C})^{\mathrm{AT}} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}.$$

While the norm functor $\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}$ is not exact, it becomes exact after a is inverted.

Proposition 1.30 *The composite*

$$\mathcal{SH}(\mathbb{C})_{i2}^{\mathrm{AT}} \xrightarrow{\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}} \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \xrightarrow{(-)[a^{-1}]} \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; \mathbb{S}_2[a^{-1}])$$

is a symmetric monoidal left adjoint. On Picard elements this composite sends $\mathbb{S}_2^{s,w}$ to $\mathbb{S}_2^{s,*,2w}[a^{-1}]$. The map τ is sent to ন^2.

In conclusion, we have a pair of functors

$$\mathcal{SH}(\mathbb{C})_{i2}^{\mathrm{AT}} \xrightarrow{\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(-)[a^{-1}]} \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; \mathbb{S}_2[a^{-1}]) \xrightarrow{\mathrm{Re}_{\mathbb{F}_2}} \mathrm{Syn}_{\mathbb{F}_2},$$

which are each the identity on spectra after inverting ন and τ. It would be very interesting to obtain a better computational understanding of the a-local category and the behavior of these two functors. More ambitiously, we ask:

Question 1.31 Is there a good description of the full subcategory of a-local objects in $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$? What geometric information does the a-localization of a smooth projective variety X remember?

The careful reader will have noticed that we switched from p-completing when discussing $\mathbb{C}$ to 2-completing when discussing $\mathbb{R}$. The reason for this is that at an odd prime the category $\mathcal{SH}(\mathbb{R})_{ip}^{\mathrm{AT}}$ admits the following simple description in terms of $\mathcal{SH}(\mathbb{C})_{ip}^{\mathrm{AT}}$.

Proposition 1.32 *There is an equivalence of categories*

$$\mathcal{SH}(\mathbb{R})_{ip}^{\mathrm{AT}} \simeq \mathrm{Sp}_{ip} \times \mathcal{SH}(\mathbb{C})_{ip}^{\mathrm{AT}} \times \mathcal{SH}(\mathbb{C})_{ip}^{\mathrm{AT}}.$$

This will be proven in Section 9.

1.4 Potpourri

In this subsection, we collect a number of additional results about the category of Artin–Tate $\mathbb{R}$-motivic spectra that are worth highlighting.

1.4.1 Trigraded $\mathbb{R}$-motivic homology and Steenrod algebra From a computational point of view, an important first step in studying the trigraded homotopy groups of $\mathbb{R}$-motivic spectra is the computation of the trigraded homology of a point and the trigraded dual Steenrod algebra. Leaning on known bigraded computations, we make these computations in Section 3.

Proposition 1.33 *Let $\mathrm{M}\mathbb{Z}_2$ and $\mathrm{M}\mathbb{F}_2$ denote 2-adic and mod 2 motivic Eilenberg–Mac Lane spectra. As trigraded commutative rings, with $|\tau| = (0,0,-1)$, there are isomorphisms*

$$\pi^{\mathbb{R}}_{p,q,w}\mathrm{M}\mathbb{Z}_2 \cong (\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2)[\tau],$$
$$\pi^{\mathbb{R}}_{p,q,w}\mathrm{M}\mathbb{F}_2 \cong (\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2)[\tau].$$

Here, $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2$ and $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2$ are placed in degrees $(p,q,0)$

Remark 1.34 There are well-known elements $\tau \in \pi^{\mathbb{R}}_{1,-1-1}\mathrm{M}\mathbb{F}_2$ and $u_\sigma \in \pi^{C_2}_{1-\sigma}\underline{\mathbb{F}}_2$. Write $u \in \pi^{\mathbb{R}}_{1,-1,0}\mathrm{M}\mathbb{F}_2$ for the corresponding element. Then there is a relation (see Corollary 3.6)

$$\tau = \tau \cdot u, \quad \text{i.e.,} \quad \mathrm{tau} = \mathrm{ta}\cdot u.$$

This was the original motivation for our choice of the character τ.

Proposition 1.35 *The trigraded dual Steenrod algebra $\pi^{\mathbb{R}}_{*,*,*}(\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2)$ is isomorphic to*

$$(\mathrm{M}\mathbb{F}_2)_{***}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 = \tau(a\tau_{i+1} + u\xi_{i+1} + a\tau_0\xi_{i+1})).$$

Here, $|\tau_i| = (2^i, 2^i-1, 2^i-1)$ and $|\xi_i| = (2^i-1, 2^i-1, 2^i-1)$.

Remark 1.36 The reader may notice that the "negative cone" in the C_2-equivariant homology of a point appears in the above formulas. This is a feature of the Artin–Tate category, as opposed to the Tate category.

We expect that a computer could be coaxed into computing the $\mathrm{M}\mathbb{F}_2$-Adams spectral sequence for the trigraded homotopy of $C\tau$, and that the natural maps between this spectral sequence and the $\mathrm{M}\mathbb{F}_2$-Adams spectral sequence for the trigraded homotopy of the sphere would provide many of the differentials in the latter. This technique would likely be the most efficient way of computing both trigraded $\mathbb{R}$-motivic stems and (in the long-run) the bigraded C_2-equivariant homotopy groups. However, as the quad-graded nature of this computation would begin to tax our ability to visualize data it is unlikely that such a computation could be accomplished without the aid of a well-developed software suite for manipulating spectral sequence data.

1.4.2 The effective slice filtration In Section 4.2, we study Voevodsky's effective slice filtration. The effective slice filtration associates to any object $E \in \mathcal{SH}(k)$ a tower

$$\cdots \to f_2E \to f_1E \to f_0E \to f_{-1}E \to f_{-2}E \to \cdots \to E.$$

The main result of the section is the following proposition, proven as Proposition 4.8 which identifies the Betti realization of this tower.

Proposition 1.37 *The functor $i_* : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \to \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ sending E to the tower*

$$\cdots \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,1}, E) \xrightarrow{\tau} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,0}, E) \to \cdots$$

is equivalent to the functor $\mathrm{Be} \circ f_\bullet : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \to \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ taking $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ to the Betti realization of its effective slice tower.

Recollection 1.38 Voevodsky has defined the "rigid homotopy groups" [Voevodsky 2002, Definition 5.1] of an $\mathbb{R}$-motivic spectrum X to be

$$\pi^{\mathbb{R},\mathrm{rig}}_{p,q,w} X := \pi^{\mathbb{R}}_{p,q,w} s_w X,$$

where $s_w X$ is nth effective slice of X, i.e., the cofiber of $f_{n+1}X \to f_n X$.

Using the notion of rigid homotopy groups, he observed the phenomenon of *algebraic degeneration* in motivic homotopy theory. Since the associated graded of this tower also computes the homotopy groups of $X \otimes C\tau$, it follows from the proposition that $\pi^{\mathbb{R},\mathrm{rig}}_{p,q,w} X \cong \pi^{\mathbb{R}}_{p,q,w} X \otimes C\tau$ when $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. This shows that Voevodsky's ideas about algebraic degeneration line up with notions coming from the cofiber of τ (or τ) philosophy.

The proposition also implies that the so-called C_2-effective spectral sequence (see [Kong 2020]) is interchangeable with the τ-Bockstein spectral sequence.

1.4.3 The functor $\nu_{\mathbb{R}}$ In Section 6.2, we will construct a lax symmetric monoidal functor

$$\nu_{\mathbb{R}} : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2},$$

which is a section of the Betti realization functor and sends $\mathbb{S}_2^{n\rho}$ to $\mathbb{S}_2^{n,n,n}$. This functor is defined similarly to Pstrągowski's synthetic analogue functor [2023, Definition 4.3]. Unlike the section $c_{\mathbb{C}/\mathbb{R}}$ of Heller–Ormsby, the functor $\nu_{\mathbb{R}}$ is not exact. However, it is better adapted to the construction of interesting $\mathbb{R}$-motivic spectra, as the following result shows.

Proposition 1.39 *There are equivalences of commutative algebras*,

$$\nu_{\mathbb{R}}\mathbb{F}_2 \simeq \mathrm{M}\mathbb{F}_2, \quad \nu_{\mathbb{R}}\mathbb{Z}_2 \simeq \mathrm{M}\mathbb{Z}_2, \quad \nu_{\mathbb{R}}\,\mathrm{MU}_{\mathbb{R},2} \simeq \mathrm{MGL}_2, \quad \nu_{\mathbb{R}}\,\mathrm{ku}_{\mathbb{R},2} \simeq \mathrm{kgl}_2, \quad \nu_{\mathbb{R}}\,\mathrm{ko}_{C_2,2} \simeq \mathrm{kq}_2.$$

On the one hand, this proposition suggests that each of the 2-complete $\mathbb{R}$-motivic homology theories above is not particularly far from being purely topological. On the other hand, it suggests that one may profitably define $\mathbb{R}$-motivic analogues of 2-complete C_2-equivariant homology theories by applying $\nu_{\mathbb{R}}$. For example, if we had a good definition of 2-complete C_2-equivariant connective topological modular forms $\mathrm{tmf}_{C_2,2}$, then we could define a spectrum of $\mathbb{R}$-motivic modular forms as $\nu_{\mathbb{R}}\mathrm{tmf}_{C_2,2}$. Unfortunately, no such definition is currently known.

1.4.4 Further deformations One of our motivations for this project was the hope that a deformation-theoretic description of Artin–Tate $\mathbb{R}$-motivic stable homotopy theory would suggest other profitable deformations of classical stable homotopy theories.

In Section C.1, we show how a 1-parameter deformation of a stable presentably symmetric monoidal category $\mathcal{C}$ may be associated to a choice of object $E \in \mathcal{C}$ and filtration $\{\mathcal{C}_{\geq k}\}$ of $\mathcal{C}$. The case of $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$ is associated to the case where $E = \mathrm{MU}_p$, $\mathcal{C} = \mathrm{Sp}_{ip}$ and $\mathcal{C}_{\geq k}$ consists of the $2k$-connective objects. The case of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, on the other hand, is associated to the case where $E = \mathrm{MU}_{\mathbb{R},2}$, $\mathcal{C} = \mathrm{Sp}_{C_2,i2}$ and $\mathcal{C}_{\geq k}$ consists of the regular slice $2k$-connective objects. Note that while the completions are necessary for the link to motivic homotopy theory, for the purpose of studying classical homotopy theory one may just as well work with integral deformations.

This suggests several other deformations that may be profitable to study.

(i) We can take the deformation of C_{2^n}-equivariant homotopy theory with respect to the norm $\mathrm{N}_{C_2}^{C_{2^n}} \mathrm{MU}_{\mathbb{R}}$ and the even slice filtration.

(ii) At odd primes, one could try to construct the deformation associated to the hypothetical spectrum BP_{μ_p} and the even regular slice filtration.

(iii) One could take the deformation of Sp_{C_p} associated to $\underline{\mathbb{F}}_p$ and the regular slice filtration. At the prime 2, this is connected to a variant of the $\underline{\mathbb{F}}_2$-Adams spectral sequence. Dylan Wilson has suggested that this variant may be more tractable at odd primes than the $\underline{\mathbb{F}}_p$-Adams spectral sequence, considering his forthcoming work with Krishanu Sankar proving that $\underline{\mathbb{F}}_p \otimes \underline{\mathbb{F}}_p$ is not a free $\underline{\mathbb{F}}_p$-module.

(iv) The deformation of $\mathrm{Syn}_{\mathrm{MU}}$ based on $\nu \mathbb{F}_p$ and an appropriately chosen filtration would likely shed much light on the motivic Adams spectral sequence over $\mathbb{C}$. Such a category might be called a "bisynthetic".

Notations and conventions

Throughout this paper the term category will refer to an ∞-category as developed by Joyal and Lurie. In some places we use the term 1-category, which is short-hand for 1-truncated ∞-category. We will also assume the reader is familiar with higher algebra as developed in [Lurie 2017].

Throughout this paper, filtered and graded objects will be ubiquitous. We adopt the convention that a filtered object in a category $\mathcal{C}$ is a diagram of the form

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \cdots,$$

i.e., that the maps decrease the index variable. We provide a more complete introduction to filtered objects in Appendix B, where we set up notation for several standard constructions.

Let $\mathcal{C}$ denote a stable, presentably symmetric monoidal category. We will let $\mathcal{C}_p$ denote the category of p-complete objects in $\mathcal{C}$; this category acquires a symmetric monoidal structure through the completion of the symmetric monoidal structure on $\mathcal{C}$. Similarly, if X is an object of $\mathcal{C}$, we let X_p denote the p-completion of X. If $\mathcal{C}$ has a set of compact generators, then we let $\mathcal{C}_{ip}$ denote the ind-completion of the full subcategory of $\mathcal{C}$ generated under finite colimits and retracts by the p-completions of compact objects. In the situation where the unit is compact and all compact objects are dualizable, this definition admits the following simplification: $\mathcal{C}_{ip} \simeq \mathrm{Mod}(\mathcal{C}; \mathbb{1}_p)$. The tensor product can then be described as the relative tensor product over $\mathbb{1}_p$. The reason for this equivalence is that tensoring with a dualizable object commutes with limits, so for X dualizable $X_p \simeq X \otimes \mathbb{1}_p$.

Our reason for using $\mathcal{C}_{ip}$ over $\mathcal{C}_p$ is that, even if the unit in $\mathcal{C}$ is compact, the unit in $\mathcal{C}_p$ may not be compact. On the other hand, compactness of the unit in $\mathcal{C}$ implies compactness of the unit in $\mathcal{C}_{ip}$. This will make certain arguments more direct in the ip case. The reader who strongly prefers the usual notion of p-completion will be relieved to know that in convenient cases (such as those in which we work throughout this paper) $(\mathcal{C}_{ip})_p \simeq \mathcal{C}_p$. Therefore, all the main theorems above admit p-complete analogues.

When $\mathcal{C}$ is a stable category and $X, Y \in \mathcal{C}$ are objects, we will typically let $\mathrm{Map}_{\mathcal{C}}(X, Y)$ denote the spectrum of maps from X to Y. One exception is when $\mathcal{C}$ comes with a natural enrichment to C_2-spectra, such as when $\mathcal{C} = \mathcal{SH}(\mathbb{R})$. In that case, we will regard $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as a C_2-spectrum. When we want to access the underlying space of maps, we shall use the notation $\Omega^\infty \mathrm{Map}_{\mathcal{C}}(X, Y)$.

2 Constructing the element ta

The map τ and its properties are the most striking feature of the category of p-complete motives over $\mathbb{C}$. In this section we construct a map ת which plays the role of τ over $\mathbb{R}$ and verify its basic property: its Betti realization is 1.

Theorem 2.1 *For all primes p, there is a class $\text{ת} \in \pi^{\mathbb{R}}_{0,0,-1} \mathbb{S}_p$ with the following properties*:

(i) *Under Betti realization, ת goes to $1 \in \pi_0^{C_2} \mathbb{S}_p$.*

(ii) *Under base change to $\mathbb{C}$, ת goes to $\tau \in \pi^{\mathbb{C}}_{0,-1} \mathbb{S}_p$.*

Surprisingly, as with the element τ in previous work, we will not need to use any information about the construction of ת besides properties (i) and (ii) outside this section. Since the construction of ת will be completely analogous to the construction of τ over $\mathbb{C}$, we begin by recalling that construction. The construction below is inspired by [Hu et al. 2011b, Remark on page 22; Bachmann et al. 2020, Section 4].

2.1 Constructing τ over $\mathbb{C}$

We begin by fixing a compatible sequence of primitive $(p^k)^{\text{th}}$ roots of unity $\{\zeta_{p^k}\}_{k \geq 0}$. We may then construct the following diagram of varieties over $\mathbb{C}$:

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C}) \amalg \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \\ \downarrow{\scriptstyle (\zeta_{p^k}, 1)} & & \downarrow{\scriptstyle 1} \\ \mathbb{G}_m & \xrightarrow{[p^k]} & \mathbb{G}_m \end{array}$$

After passing to the associated diagram of motivic spaces we can add a third column by taking cofibers:

$$\begin{array}{ccccc} \mathrm{Spec}(\mathbb{C}) \amalg \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) & \longrightarrow & S^1 \\ \downarrow{\scriptstyle (\zeta_{p^k}, 1)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tau_k} \\ \mathbb{G}_m & \xrightarrow{[p^k]} & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m/[p^k] \end{array}$$

Using the compatibility of the chosen primitive $(p^k)^{\text{th}}$ roots of unity, the maps τ_k so defined are compatible in the sense that there are commutative diagrams

$$\begin{array}{ccc} S^1 & \xrightarrow{\mathrm{id}} & S^1 \\ \downarrow{\scriptstyle \tau_k} & & \downarrow{\scriptstyle \tau_{k-1}} \\ \mathbb{G}_m/[p^k] & \longrightarrow & \mathbb{G}_m/[p^{k-1}] \end{array}$$

Stabilizing and using the fact that $\Sigma^\infty[p^k]\colon \mathbb{S}^{1,1} \to \mathbb{S}^{1,1}$ is equivalent[12] to p^k over $\mathbb{C}$, we find that we have constructed a compatible system of elements $\tau_k \in \pi^{\mathbb{C}}_{0,-1}\,\mathbb{S}/p^k$. Passing to the limit, we obtain the element $\tau \in \pi^{\mathbb{C}}_{0,-1}\,\mathbb{S}_p$.

Proposition 2.2 *For some choice*[13] *of* $\{\zeta_{p^k}\}_{k\geq 0}$, *we have* $\mathrm{Be}(\tau) = 1$.

Proof We will show that $\mathrm{Be}(\tau) \in \pi_0\,\mathbb{S}_p = \mathbb{Z}_p$ is a p-adic unit. Then, using the action of $\mathrm{Aut}(\mu_{p^\infty}) = \mathbb{Z}_p^\times$ on systems of $(p^k)^{\text{th}}$ roots of unity, we may change τ by multiplication by any p-adic unit.

It now suffices to show that the Betti realizations of the unstable maps $\tau_k : S^1 \to \mathbb{G}_m/[p^k]$ are surjective on π_1. Since $[p^k]\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is a principal μ_{p^k}-fibration, it is classified by a map $\mathbb{C}^\times \to B\mu_{p^k}$ and we can form the following diagram:

$$\begin{array}{ccccccc}
S^0 & \longrightarrow & S^0 & \longrightarrow & * & \longrightarrow & S^1 \\
\downarrow{\scriptstyle(\zeta_{p^k},1)} & & \downarrow{\scriptstyle(\zeta_{p^k},1)} & & \downarrow & & \downarrow \\
\mu_{p^k} & \longrightarrow & \mathbb{C}^\times & \xrightarrow{[p^k]} & \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times/[p^k] \\
 & & & & \downarrow & \swarrow & \\
 & & & & B\mu_{p^k} & &
\end{array}$$

The dashed map $\mathbb{C}^\times/[p^k] \to B\mu_{p^k}$ is an isomorphism on π_1 after Betti realization, so it suffices to show that the composite map $S^1 \to \mathbb{C}^\times/[p^k] \to B\mu_{p^k}$ is surjective on Betti-realized π_1. This follows from the fact that the map $S^1 \to B\mu_{p^k}$ is adjoint to the map $S^0 \xrightarrow{(\zeta_{p^k},1)} \mu_{p^k}$. □

2.2 Constructing ϖ over $\mathbb{R}$

We now imitate the construction of τ above to construct ϖ and prove Theorem 2.1. The key point is that there are two forms of $\mathbb{G}_m$ over $\mathbb{R}$. While the roots of unity ζ_n do not lie in

$$\{\pm 1\} \subset \mathbb{G}_m(\mathbb{R}) \subseteq \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times,$$

the other form of $\mathbb{G}_m$ over $\mathbb{R}$ (which is the "algebraic circle", Q, given by $\{x^2 + y^2 = 1\}$) has

$$S^1 = Q(\mathbb{R}) \subseteq Q(\mathbb{C}) \cong \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times.$$

As such, we are free to imitate the construction of τ above with $\mathbb{G}_m$ replaced by the algebraic circle Q.

Let $\{\zeta_{p^k}\}_{k\geq 0}$ be the system of primitive $(p^k)^{\text{th}}$ roots of unity satisfying the conclusion of Proposition 2.2. As before, we form the diagram of $\mathbb{R}$-motivic spaces:

$$\begin{array}{ccccc}
\mathrm{Spec}(\mathbb{R}) \amalg \mathrm{Spec}(\mathbb{R}) & \longrightarrow & \mathrm{Spec}(\mathbb{R}) & \longrightarrow & S^1 \\
\downarrow{\scriptstyle(\zeta_{p^k},1)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varpi_k} \\
Q & \xrightarrow{[p^k]} & Q & \longrightarrow & Q/[p^k]
\end{array}$$

[12] This follows from Corollary 1.21, since the Betti realization of $[p^k]$ is clearly p^k.

[13] Tracing through the identifications, one sees that the choice of roots of unity which yields 1 is $\exp(2\pi i/p^k)$.

Before finishing the construction and proving Theorem 2.1 we need a short lemma which identifies the map $[p^k]$.

Lemma 2.3 *The map $\Sigma^\infty[p^k]:\mathbb{S}^{1,0,1}\to\mathbb{S}^{1,0,1}$ is homotopic to $p^k:\mathbb{S}^{1,0,1}\to\mathbb{S}^{1,0,1}$.*

Proof Since the map $\pi^{\mathbb{R}}_{0,0,0}\to\pi_0^{C_2}$ induced by Betti realization is an isomorphism by Corollary 1.21, it suffices to show this after Betti realization. Upon Betti realization, this is a map $\mathbb{S}^1\to\mathbb{S}^1$ which induces multiplication by p^k on both geometric fixed points and on the underlying spectrum. The map $\pi_0^{C_2}\to\mathbb{Z}\oplus\mathbb{Z}$ given by (underlying, geometric fixed points) is injective. □

Proof of Theorem 2.1 Stabilizing and applying Lemma 2.3, we obtain a compatible system of classes $\text{ता}_k\in\pi^{\mathbb{R}}_{0,0,-1}\,\mathbb{S}/p^k$ which give rise to a class $\text{ता}\in\pi^{\mathbb{R}}_{0,0,-1}\,\mathbb{S}_p$. It follows immediately from the definition that the base change of ता to $\mathbb{C}$ is τ. Since the Betti realization of τ is 1, we find that the underlying of the Betti realization of ता is 1.

In order to show that the Betti realization of ता is 1, it therefore suffices to show that it is multiplication by some p-adic integer. To do this, it suffices to do so modulo p^k for all k. Modulo p^k, the Betti realization of ता arises as the stabilization of a map $S^1\to S^1/p^k$ of C_2-equivariant spaces, and all such maps are given by multiplication by a p-adic integer. □

3 The ta-local category

In this section, we show that Betti realization identifies the category of ता-local Artin–Tate $\mathbb{R}$-motivic spectra with the category of C_2-spectra. Indeed, we know from Theorem 2.1 that $\mathrm{Be}(\text{ता})=1$, so that Be factors through the category of ता-local objects. The main theorem of this section is that the induced functor is an equivalence.

Definition 3.1 Let $\mathbb{S}_2[\text{ता}^{-1}]$ denote the commutative algebra given by

$$\mathbb{S}_2[\text{ता}^{-1}]:=\operatorname{colim}(\mathbb{S}_2^{0,0,0}\xrightarrow{\text{ता}}\mathbb{S}_2^{0,0,1}\xrightarrow{\text{ता}}\mathbb{S}_2^{0,0,2}\xrightarrow{\text{ता}}\cdots).$$

The category of modules over $\mathbb{S}_2[\text{ता}^{-1}]$ in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is equivalent[14] to the category of ता-local objects in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. Since the Betti realization of ता is 1, Betti realization factors through ता-localization providing a symmetric monoidal functor

$$\mathrm{Be}:\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2};\mathbb{S}_2[\text{ता}^{-1}])\to\mathrm{Sp}_{C_2,i2}\,.$$

Theorem 3.2 *Betti realization induces an equivalence of symmetric monoidal categories*

$$\mathrm{Be}:\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2};\mathbb{S}_2[\text{ता}^{-1}])\xrightarrow{\simeq}\mathrm{Sp}_{C_2,i2},$$

with inverse equivalence given by $Y(-):=c_{\mathbb{C}/\mathbb{R}}(-)[\text{ता}^{-1}]$. Simply put, inverting ता in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ recovers $\mathrm{Sp}_{C_2,i2}$.

[14] Here we use the fact that one may invert an element in the Picard-graded homotopy of a commutative algebra and the associated description of the module category as a localization from the proof of [Lurie 2018a, Proposition 4.3.17].

The main step in our proof of this theorem is showing that Y is fully faithful. Since $c_{\mathbb{C}/\mathbb{R}}$ is fully faithful, as proved by Heller and Ormsby (see Theorem 1.20), this reduces to studying the interaction of $\overline{\tau}$ and $c_{\mathbb{C}/\mathbb{R}}$. Our method of proof is to prove the appropriate statement at the level of $\mathrm{M}\mathbb{F}_2$ by direct computation, extend to dualizable objects by descent, then extend to all objects by colimits. This strategy requires knowledge of the trigraded homology of a point and the structure of the trigraded dual Steenrod algebra as an input. Since this is of independent interest we have separated it out as its own subsection.

3.1 The homology of a point

In this subsection we compute the trigraded homology of a point and the structure of the dual Steenrod algebra. Although we only use coarse information extracted from these computations in this paper, we hope that they are useful to readers more interested in computations. The reader who has heard that $\mathbb{R}$-motivic computations are simplified by the absence of the "negative cone" may be surprised to learn that it is present in the trigraded picture.

Proposition 3.3 *As trigraded commutative rings, with* $|\overline{\tau}| = (0,0,-1)$, *there are isomorphisms*

$$\pi^{\mathbb{R}}_{p,q,w}\mathrm{M}\mathbb{Z}_2 \cong (\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2)[\overline{\tau}] \quad \text{and} \quad \pi^{\mathbb{R}}_{p,q,w}\mathrm{M}\mathbb{F}_2 \cong (\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2)[\overline{\tau}],$$

where $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2$ *and* $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2$ *are each placed in tridegree* $(p,q,0)$.

Before we move on to the proof of Proposition 3.3, we remind the reader of the structure of $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2$ and $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2$. We also explain how to translate between the older names of elements in the bigraded homology of a point and the names arising from this proposition.

Recollection 3.4 The bigraded homotopy groups of $\underline{\mathbb{F}}_2$ are given by

$$\pi^{C_2}_{p+q\sigma}\underline{\mathbb{F}}_2 \cong \mathbb{F}_2[a_\sigma, u_\sigma] \oplus \mathbb{F}_2\left\{\frac{\theta}{a_\sigma^k u_\sigma^n} \;\middle|\; k,n \geq 0\right\},$$

where $|a_\sigma| = -\sigma$, $|u_\sigma| = 1-\sigma$ and $|\theta| = 2\sigma - 2$. The multiplicative structure is that of a trivial square-zero extension over $\mathbb{F}_2[a_\sigma, u_\sigma]$: more informally, it is given by $\theta^2 = 0$. This is pictured in Figure 1.

The bigraded homotopy groups of $\underline{\mathbb{Z}}_2$ are given by

$$\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2 \cong \mathbb{Z}_2[a_\sigma, u_{2\sigma}]/(2a_\sigma) \oplus \mathbb{Z}_2\left\{\frac{2}{u_{2\sigma}^n} \;\middle|\; n \geq 1\right\} \oplus \mathbb{F}_2\left\{\frac{\theta}{a_\sigma^k u_{2\sigma}^n} \;\middle|\; k,n \geq 0\right\},$$

where $|a_\sigma| = -\sigma$, $|u_{2\sigma}| = 2-2\sigma$ and $|\theta| = 3\sigma - 3$. Again, the term involving θ sits in a trivial square-zero extension with rest of the ring. This is pictured in Figure 2.

Recollection 3.5 Under Betti realization, $\mathrm{M}\mathbb{Z}_2$ and $\mathrm{M}\mathbb{F}_2$ are sent to $\underline{\mathbb{Z}}_2$ and $\underline{\mathbb{F}}_2$; see [Heller and Ormsby 2016]. The following are some commonly encountered homotopy elements and their Betti realizations:

- Stabilizing the inclusion of the fixed points $S^0 \to S^\sigma$, we obtain $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbb{S}$. This class maps to the corresponding class in $\underline{\mathbb{Z}}_2$ and $\underline{\mathbb{F}}_2$.

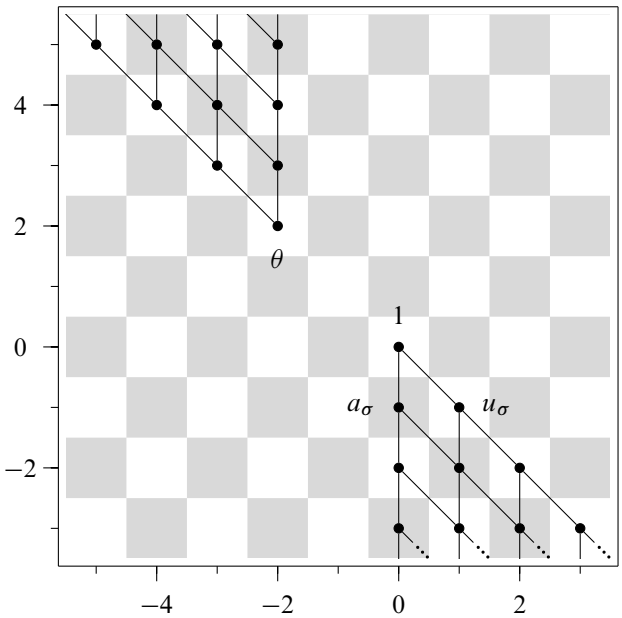


Figure 1: The C_2-equivariant $\mathbb{F}_2$-homology of a point. The bigraded homotopy groups of $\underline{\mathbb{F}}_2$. depicted on a (p,q) coordinate system. Vertical lines going down denote multiplication by a_σ. Diagonal lines going down-right denote multiplication by u_σ.

- We let $u \in \pi^{\mathbb{R}}_{1,-1,0}\mathrm{M}\mathbb{F}_2$ denote the element corresponding to $u_\sigma \in \pi^{C_2}_{1-\sigma}\underline{\mathbb{F}}_2$ under the isomorphism of Proposition 3.3.
- The class $\rho \in \pi^{\mathbb{R}}_{0,-1,-1}\mathbb{S}$ is defined to be the stabilization of the inclusion $\{\pm 1\} \hookrightarrow \mathbb{G}_m$. Under Betti realization, ρ goes to a_σ.
- The element $\tau \in \pi^{\mathbb{R}}_{1,-1,-1}\mathrm{M}\mathbb{F}_2$ corresponds to -1 under the isomorphism $\pi^{\mathbb{R}}_{1,-1,-1}\mathrm{M}\mathbb{F}_2 \cong \mu_2(\mathbb{R})$. Under Betti realization, τ goes to u_σ.

Corollary 3.6 *There are relations $\rho = \mathfrak{a}\cdot a$ in $\pi^{\mathbb{R}}_{0,-1,-1}\mathbb{S}_2$ and $\tau = \mathfrak{a}\cdot u \in \pi^{\mathbb{R}}_{1,-1,-1}\mathrm{M}\mathbb{F}_2$.*

Proof In Lemma 3.12, we will show that Betti realization induces an isomorphism $\pi^{\mathbb{R}}_{0,-1,-1}\mathbb{S}_2 \to \pi^{C_2}_{-\sigma}\mathbb{S}_2$, therefore for the first relation it suffices to note that $\mathrm{Be}(\rho) = 1\cdot a_\sigma = \mathrm{Be}(\mathfrak{a}\cdot a)$. Similarly, by Proposition 3.3 and Recollection 3.4, we know that $\pi^{\mathbb{R}}_{1,-1,-1}\mathrm{M}\mathbb{F}_2 \cong \mathbb{F}_2\{\mathfrak{a}\cdot u\}$. Since $\mathrm{Be}(\tau) = u_\sigma \neq 0$, the second relation follows. □

Assuming the trigraded homology of a point, the trigraded dual Steenrod algebra can easily by deduced from results of Voevodsky.

Theorem 3.7 *The trigraded $\mathbb{R}$-motivic dual Steenrod algebra $\pi^{\mathbb{R}}_{*,*,*}(\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2)$ is isomorphic to*

$$(\mathrm{M}\mathbb{F}_2)_{*,*,*}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 = \mathfrak{a}a\tau_{i+1} + \mathfrak{a}u\xi_{i+1} + \mathfrak{a}a\tau_0\xi_{i+1}),$$

where $|\tau_i| = (2^i, 2^i - 1, 2^i - 1)$ and $|\xi_i| = (2^i - 1, 2^i - 1, 2^i - 1)$.

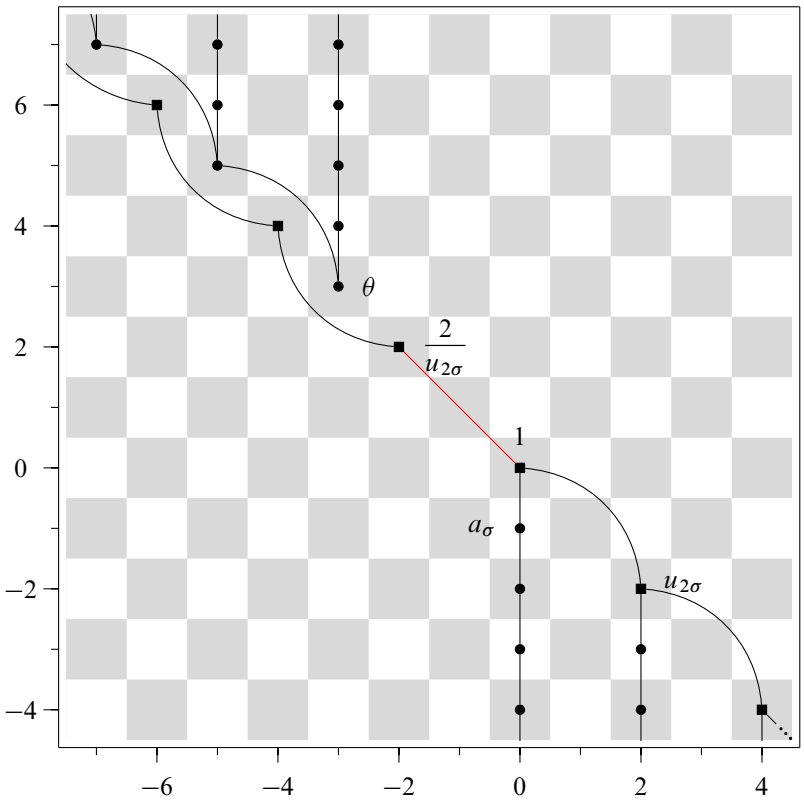


Figure 2: The C_2-equivariant integral homology of a point. The bigraded homotopy groups of $\underline{\mathbb{Z}}_2$ depicted on a (p,q) coordinate system. Squares denote copies of $\mathbb{Z}_2$ and circles denotes copies of $\mathbb{F}_2$. Vertical lines going down denote multiplication by a_σ. Diagonal lines going down-right denote multiplication by $u_{2\sigma}$. The red line indicates a $u_{2\sigma}$ multiplication that hits twice a generator. Not all $u_{2\sigma}$ multiplications are shown.

Proof We will deduce this from the classical bigraded computation of the $\mathbb{R}$-motivic Steenrod algebra. The key input is the fact that $\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2$ decomposes as a direct sum of $\mathbb{R}$-motivic spectra of the form $\Sigma^{p,q,q}\mathrm{M}\mathbb{F}_2$. This is stated as [Hoyois et al. 2017, Theorem 1.1(3)], which in the case of a characteristic-zero base field such as $\mathbb{R}$ follows from work of Voevodsky [2003b; 2010].

The decomposition of $\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2$ given in [Hoyois et al. 2017, Theorem 1.1] shows that, as a $(\mathrm{M}\mathbb{F}_2)_{*,*,*}$-module, $\pi^{\mathbb{R}}_{*,*,*}(\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2)$ is freely generated by monomials in the ξ_i and τ_i. Since each τ_i and ξ_i is represented by a map of the form $\mathbb{S}^a \otimes \mathbb{G}_m^b \to \mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2$ (with no copies of $\mathbb{S}^{\mathbb{C}}$ in the domain), we may read off the formulas for products of τ_i and ξ_i from the standard bigraded computations [Voevodsky 2003b, Theorem 12.6]. To translate the formulas into trigraded notation, we need only make use of the relations $\rho = \mathfrak{t} \cdot a$ and $\tau = \mathfrak{t} \cdot u$ of Corollary 3.6. □

We now proceed to the proof of Proposition 3.3. The main steps of the proof are split across the next several lemmas, the key input being our knowledge of the bigraded 2-complete motivic cohomology of both $\mathbb{R}$ and $\mathbb{C}$.

Lemma 3.8 *For integers $p, w \in \mathbb{Z}$, there are isomorphisms*

$$\pi^{\mathbb{R}}_{p,w,w} \mathrm{M}\mathbb{Z}_2 \cong \mathbb{Z}_2[\tau^2, \rho]/(2\rho).$$

For integers $p, q, w \in \mathbb{Z}$, there are isomorphisms

$$\pi^{\mathbb{R}}_{p,q,w}(\mathrm{Spec}(\mathbb{C}) \otimes \mathrm{M}\mathbb{Z}_2) \cong \pi^{\mathbb{C}}_{p+q,w}(\mathrm{M}\mathbb{Z}_2) \quad \text{and} \quad \pi^{\mathbb{C}}_{p,w} \mathrm{M}\mathbb{Z}_2 \cong \mathbb{Z}_2[\tau].$$

Proof The base change isomorphism is a corollary of the six functor formalism. The computations of motivic cohomology are a corollary of Voevodsky's solution [2003a] of the Bloch–Kato and Beilinson–Lichtenbaum conjectures. □

Lemma 3.9 *The groups $\pi^{\mathbb{R}}_{p,q,w} \mathrm{M}\mathbb{Z}_2$ and $\pi^{\mathbb{R}}_{p,q,w} \mathrm{M}\mathbb{F}_2$ are zero for $w > 0$.*

Proof Considering the cofiber sequence $\mathrm{M}\mathbb{Z}_2 \xrightarrow{2} \mathrm{M}\mathbb{Z}_2 \to \mathrm{M}\mathbb{F}_2$, we reduce to the case of $\mathrm{M}\mathbb{Z}_2$. Using Lemma 3.8, we see that this lemma is true for tridegrees of the form (p, w, w). For each integer n, consider the cofiber sequence

$$\mathrm{Spec}(\mathbb{C}) \otimes (\mathbb{S}^{\mathbb{C}})^{\otimes n-1} \to (\mathbb{S}^{\mathbb{C}})^{\otimes n-1} \to (\mathbb{S}^{\mathbb{C}})^{\otimes n}.$$

Tensoring this with $\mathrm{M}\mathbb{Z}_2$ and applying $\pi_{p,w,w}$, we obtain a long exact sequence

$$\pi^{\mathbb{C}}_{p+w-n+1,w} \mathrm{M}\mathbb{Z}_2 \to \pi^{\mathbb{R}}_{p,w-n+1,w} \mathrm{M}\mathbb{Z}_2 \to \pi^{\mathbb{R}}_{p,w-n,w} \mathrm{M}\mathbb{Z}_2 \to \pi^{\mathbb{C}}_{p+w-n,w} \mathrm{M}\mathbb{Z}_2.$$

When $w > 0$, Lemma 3.8 thus ensures an isomorphism between $\pi^{\mathbb{R}}_{p,w-n+1,w} \mathrm{M}\mathbb{Z}_2$ and $\pi^{\mathbb{R}}_{p,w-n,w} \mathrm{M}\mathbb{Z}_2$. As n varies, using the base case $n = 0$, we conclude that these groups always vanish. □

Lemma 3.10 *When $w \leq 0$, the Betti realization maps $\pi^{\mathbb{R}}_{p,q,w} \mathrm{M}\mathbb{Z}_2 \to \pi^{C_2}_{p+q\sigma} \underline{\mathbb{Z}}_2$ and $\pi^{\mathbb{R}}_{p,q,w} \mathrm{M}\mathbb{F}_2 \to \pi^{C_2}_{p+q\sigma} \underline{\mathbb{F}}_2$ are isomorphisms.*

Proof Considering the cofiber sequence $\mathrm{M}\mathbb{Z}_2 \xrightarrow{2} \mathrm{M}\mathbb{Z}_2 \to \mathrm{M}\mathbb{F}_2$, we reduce to the case of $\mathrm{M}\mathbb{Z}_2$. We first check that this true when $w = q$. Recollections 3.4 and 3.5 and Lemma 3.8 imply that the Betti realization map is an isomorphism for degrees of the form (p, w, w) with $w \leq 0$. As in the previous lemma we tensor the cofiber sequence

$$\mathrm{Spec}(\mathbb{C}) \otimes (\mathbb{S}^{\mathbb{C}})^{\otimes n-1} \to (\mathbb{S}^{\mathbb{C}})^{\otimes n-1} \to (\mathbb{S}^{\mathbb{C}})^{\otimes n}$$

with $\mathrm{M}\mathbb{Z}_2$ and take homotopy groups in order to spread out to other cases. Using β as notation for Betti realization we obtain maps of exact sequences of abelian groups

$$\begin{array}{ccccccc}
\pi^{\mathbb{C}}_{p+w-n+1,w} \mathrm{M}\mathbb{Z}_2 & \longrightarrow & \pi^{\mathbb{R}}_{p,w-n+1,w} \mathrm{M}\mathbb{Z}_2 & \longrightarrow & \pi^{\mathbb{R}}_{p,w-n,w} \mathrm{M}\mathbb{Z}_2 & \longrightarrow & \pi^{\mathbb{C}}_{p+w-n,w} \mathrm{M}\mathbb{Z}_2 \\
\downarrow \cong & & \downarrow \beta_{p,n-1} & & \downarrow \beta_{p,n} & & \downarrow \cong \\
\pi_{p+w-n+1} \mathbb{Z}_2 & \longrightarrow & \pi^{C_2}_{p+(w-n+1)\sigma} \underline{\mathbb{Z}}_2 & \longrightarrow & \pi^{C_2}_{p+(w-n)\sigma} \underline{\mathbb{Z}}_2 & \longrightarrow & \pi_{p+w-n} \mathbb{Z}_2
\end{array}$$

with our goal being to prove that the Betti realization maps $\beta_{p,n}$ are isomorphisms for all $p, n \in \mathbb{Z}$.

Using the 5-lemma we conclude that ($\beta_{p,n-1}$ is iso) + ($\beta_{p-1,n-1}$ is iso) implies ($\beta_{p,n}$ is iso). Similarly, the 5-lemma also implies that ($\beta_{p,n}$ is iso) + ($\beta_{p+1,n}$ is iso) implies ($\beta_{p,n-1}$ is iso). As noted above, we have already learned that $\beta_{p,0}$ is an isomorphism for all p, so we may now induct outwards from this case to conclude. □

Proof of Proposition 3.3 By Lemma 3.10, the map

$$\pi^{\mathbb{R}}_{p,q,0}\mathrm{M}\mathbb{Z}_2 \to \pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2$$

induced by Betti realization is an isomorphism. Taking the inverse, we obtain a ring map

$$\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2 \to \pi^{\mathbb{R}}_{p,q,0}\mathrm{M}\mathbb{Z}_2.$$

This extends to a ring map

$$(\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2)[\tau] \to \pi_{p,q,*}\mathrm{M}\mathbb{Z}_2.$$

It follows from Theorem 2.1 and Lemmas 3.9 and 3.10 that this map is an isomorphism. The same proof works for $\mathrm{M}\mathbb{F}_2$. □

3.2 The proof of Theorem 3.2

Recall that we are proving that Be is an equivalence on the τ-local category with inverse Y. We will check directly that Y is an equivalence of categories by showing that it is fully faithful and essentially surjective. That Be is the inverse of Y follows from the fact that $\mathrm{Be}\circ Y = \mathrm{Be}\circ c_{\mathbb{C}/\mathbb{R}}$ is the identity on $\mathrm{Sp}_{C_2,i2}$. Before we can make progress on this goal we will need a pair of lemmas.

Lemma 3.11 *The trigraded homotopy of $C\tau$ is zero in negative weights.*

Proof Our method of proof will be to apply the motivic Adams spectral sequence to $C\tau$. Since this spectral sequence converges strongly for $\mathbb{S}_2$ by [Hu et al. 2011a], it also converges strongly for $C\tau$. This spectral sequence takes the form

$$E_1^{s,t} = \pi_{t,q,w}(C\tau \otimes \mathrm{M}\mathbb{F}_2^{\otimes s+1}) \implies \pi_{t-s,q,w}C\tau.$$

It suffices therefore to check at the level of the E_1-page that $\pi_{t,q,w}(C\tau \otimes \mathrm{M}\mathbb{F}_2^{\otimes s+1}) = 0$ for $w < 0$.

For this, we use the known [Voevodsky 2003b, 2010; Hoyois et al. 2017] description of $\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2$ in $\mathcal{SH}(\mathbb{R})$. One has that $\mathrm{M}\mathbb{F}_2 \otimes \mathrm{M}\mathbb{F}_2 \simeq \oplus_{(x_i,y_i,y_i)} \Sigma^{x_i,y_i,y_i}\mathrm{M}\mathbb{F}_2$, where the x_i and y_i range over nonnegative integers. The result now follows immediately from Proposition 3.3. □

Lemma 3.12 *Let n denote a nonnegative integer. Given a pair of C_2-spectra $X, Y \in \mathrm{Sp}_{C_2,i2}$, the natural map induced by τ^n,*

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(c_{\mathbb{C}/\mathbb{R}}X, c_{\mathbb{C}/\mathbb{R}}Y) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(c_{\mathbb{C}/\mathbb{R}}X, \mathbb{S}_2^{0,0,n} \otimes c_{\mathbb{C}/\mathbb{R}}Y),$$

is an equivalence.

Proof Since $c_{\mathbb{C}/\mathbb{R}}$ commutes with colimits it will suffice to prove the proposition as X ranges through a family of compact generators of $\mathrm{Sp}_{C_2,i2}$. In particular, it will suffice to assume $X \simeq \mathbb{S}_2^{p+q\sigma}$ is a

representation sphere. Since $c_{\mathbb{C}/\mathbb{R}}X \simeq \mathbb{S}_2^{p,q,0}$ is compact, it furthermore suffices to prove the proposition as Y ranges through a family of compact generators. In particular, it suffices to assume that $Y \simeq \mathbb{S}_2^{a+b\sigma}$ is also a representation sphere.

At this point, the proposition reduces to a claim about

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}\,\mathbb{S}_2^{p+q\sigma}, \mathbb{S}_2^{0,0,n} \otimes c_{\mathbb{C}/\mathbb{R}}\,\mathbb{S}_2^{a+b\sigma}) \simeq \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(\mathbb{S}_2^{p-a,q-b,-n}, \mathbb{S}_2),$$

or in other words a claim about the trigraded stable stems $\pi^{\mathbb{R}}_{*,*,*}\,\mathbb{S}_2$. We must show that multiplication by τ^n is an isomorphism $\pi^{\mathbb{R}}_{*,*,0}\,\mathbb{S}_2 \to \pi^{\mathbb{R}}_{*,*,-n}\,\mathbb{S}_2$. Equivalently, we must check that $\tau : \pi^{\mathbb{R}}_{*,*,-n}\,\mathbb{S}_2 \to \pi^{\mathbb{R}}_{*,*,-n-1}\,\mathbb{S}_2$ is an isomorphism for each $n \geq 0$. Examining the cofiber sequence

$$\Sigma^{-1,0,0}C\tau \to \mathbb{S}_2^{0,0,-1} \xrightarrow{\tau} \mathbb{S}_2 \to C\tau,$$

we may use Lemma 3.11 to conclude. □

We are now ready to complete the proof of Theorem 3.2.

To check that Y is fully faithful, we must prove that for any pair $A, B \in \mathrm{Sp}_{C_2,i2}$ the composite

$$\mathrm{Map}_{\mathrm{Sp}_{C_2,i2}}(A, B) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}A, c_{\mathbb{C}/\mathbb{R}}B) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}A[\tau^{-1}], c_{\mathbb{C}/\mathbb{R}}B[\tau^{-1}])$$

is an equivalence. The first map is an equivalence as a consequence of the full faithfulness of $c_{\mathbb{C}/\mathbb{R}}$ as proven by Heller and Ormsby; see Theorem 1.20. The second map factors as

$$\begin{aligned}\mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}A, c_{\mathbb{C}/\mathbb{R}}B) &\xrightarrow{\simeq} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}A, c_{\mathbb{C}/\mathbb{R}}B[\tau^{-1}]) \\ &\xrightarrow{\simeq} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(c_{\mathbb{C}/\mathbb{R}}A[\tau^{-1}], c_{\mathbb{C}/\mathbb{R}}B[\tau^{-1}]),\end{aligned}$$

where first map is an equivalence as a consequence of Lemma 3.12 and the second map is an equivalence since inverting τ is a localization.

Since Y is now fully faithful and colimit-preserving, to check that it is essentially surjective it suffices to check that its image contains a family of compact generators. One such family consists of the objects $\mathbb{S}_2^{p,q,w}[\tau^{-1}]$ as p, q, w range over the integers, since $\mathbb{S}_2^{p,q,w}[\tau^{-1}] \simeq \mathbb{S}_2^{p,q,0}[\tau^{-1}] \simeq Y(\mathbb{S}_2^{p+q\sigma})$.

4 Galois reconstruction

In this section, we provide a Galois reconstruction of $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$. In other words, we show how to reconstruct $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ from C_2-equivariant homotopy theory. As in the case of $\mathbb{C}$, understanding the close connection between MGL and its Betti realization is the essential step in reconstruction. In Appendix C, we have set up a general framework for reconstruction results of this kind. We will rely heavily on the work there, so we suggest that readers familiarize themselves with the material and notation therein before proceeding.

Before we state the main theorem, let us recall the definition of the regular slice filtration of C_2-spectra.

Definition 4.1 We say that a C_2-spectrum X is regular slice n-connective if $\Phi^e X$ is n-connective and $\Phi^{C_2}X$ is $\lceil n/2 \rceil$-connective.[15]

[15] Here we work with the regular slice filtration because of its good multiplicative properties.

We let $\mathrm{Sp}_{C_2}^{\geq n}$ denote the full subcategory of Sp_{C_2} consisting of the regular slice n-connective C_2-spectra. This is a coreflective subcategory, and we let $P_n : \mathrm{Sp}_{C_2} \to \mathrm{Sp}_{C_2}^{\geq n}$ denote the right adjoint to the inclusion. We let P^n denote the n^{th} regular slice truncation functor and let P_n^n denote the n^{th} regular slice functor.

Construction 4.2 Since the functors Φ^e and Φ^{C_2} are monoidal, the hypotheses of Construction C.6 are satisfied and we may assemble the categories $\mathrm{Sp}_{C_2}^{\geq n}$ into a coreflective symmetric monoidal subcategory $\mathrm{Sp}_{C_2,\geq 0}^{\mathbf{Fil},2\mathrm{slice}} \subset \mathrm{Sp}_{C_2}^{\mathbf{Fil}}$, which consists of those $X_\bullet$ which are regular slice $2n$-connective in position n. This provides a lax symmetric monoidal connective cover functor $\tau_{\geq 0}^{2\mathrm{slice}}$, which takes the $(2n)^{\text{th}}$-slice cover at position n. The composition $\tau_{\geq 0}^{2\mathrm{slice}}(Y(-))$ is equivalent to even slice tower functor $P_{2\bullet}$ and demonstrates that this functor is lax symmetric monoidal.

The fundamental construction of the section is the following commutative algebra in filtered C_2-spectra:

$$R_\bullet := \mathrm{Tot}^*(P_{2\bullet}\,\mathrm{MU}_{\mathbb{R},2}^{\otimes *+1}).$$

Theorem 4.3 (Galois reconstruction) *There is an equivalence of presentably symmetric monoidal categories under* $\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}$,

$$\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}; R_\bullet),$$

where $\mathrm{Sp}_{C_2,i2}$ *acts on the left through* $c_{\mathbb{C}/\mathbb{R}}$.

The commutative algebra $R_\bullet$ can be called the "décalage of the $\mathrm{MU}_{\mathbb{R},2}$-Adams tower with respect to the even slice filtration". The commutative algebra $R_\bullet$ is also the image of the unit in $\mathrm{Sp}_{C_2,i2}$ under a certain lax symmetric monoidal functor.

Construction 4.4 Applying Construction C.9, we obtain a lax symmetric monoidal functor

$$\mathrm{Sh}(P_{2\bullet}; \mathrm{MU}_{\mathbb{R},2}) : \mathrm{Sp}_{C_2,i2} \to \mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}.$$

By construction, $R_\bullet \simeq \mathrm{Sh}(P_{2\bullet}; \mathrm{MU}_{\mathbb{R},2})(\mathbb{S}_2)$, so this functor factors through the category of modules over $R_\bullet$. Composing with the equivalence of Theorem 4.3, this defines a lax symmetric monoidal functor,

$$\Gamma_* : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}.$$

This functor is analogous to the functor $\Gamma_* : \mathrm{Sp}_2 \to \mathcal{SH}(\mathbb{C})_2$ studied in [Gheorghe et al. 2022], and may also be compared with the synthetic analogue functor of Pstrągowski [2023]. We will study this functor more closely in Section 6.

The proof of Theorem 4.3 will be carried out in two steps: first we will prove that there is an equivalence

$$\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}; i_*(\mathbb{S}_2))$$

for some lax symmetric monoidal functor $i_* : \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \to \mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}$. We will then construct an equivalence $i_*(\mathbb{S}_2) \simeq R_\bullet$ of commutative rings in $\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}$. The key step in the construction of this equivalence is the identification of $i_*(\mathrm{MGL}_2)$ as a commutative algebra in $\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}$. In order make this identification, we will show that i_* admits a description in terms of Voevodsky's effective slice filtration.

4.1 The filtered model

In this subsection we prove the first half of Galois reconstruction, namely that there is a filtered model for $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$.

Proposition 4.5 *There is a diagram of symmetric monoidal left adjoints*

$$\begin{array}{ccccccc}
\mathrm{Sp}_{C_2,i2} & \longrightarrow & \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2} & \xrightarrow{-\otimes i_*\mathbb{S}} & \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; i_*\mathbb{S}_2) & \xrightarrow{\mathrm{Re}} & \mathrm{Sp}_{C_2,i2} \\
\downarrow{\scriptstyle \mathrm{Id}} & & & \searrow^{i^*} & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\mathrm{Sp}_{C_2,i2} & & \xrightarrow{c_{\mathbb{C}/\mathbb{R}}} & & \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} & \xrightarrow{\mathrm{Be}} & \mathrm{Sp}_{C_2,i2}
\end{array}$$

such that $i^*(\mathbb{S}_2^{p+q\sigma}(w)) \simeq \mathbb{S}_2^{p,q,w}$.

Note that Proposition C.20 produces a diagram of this type, so in order to prove the proposition we only need to endow $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and $\mathrm{Sp}_{C_2,i2}$ with the structure of a deformation pair in the sense of Definition C.14.

Proof We begin with the diagram

$$\begin{array}{ccccc}
 & & \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} & & \\
 & \nearrow^{c_{\mathbb{C}/\mathbb{R}}} & & \searrow^{\mathrm{Be}} & \\
\mathrm{Sp}_{C_2,i2} & & \xrightarrow{\mathrm{Id}} & & \mathrm{Sp}_{C_2,i2}
\end{array}$$

In order to make i^* behave as desired on Picard elements, we pick $i_0(w) = \mathbb{S}_2^{0,0,w}$. Since $\mathrm{Be}(\mathbb{S}_2^{0,0,w}) \simeq \mathbb{S}_2^0$, this factors through the kernel of the map on Picard groups induced by Be.

To conclude, we now need to verify the two conditions in the definition of a deformation pair. The first condition is implied by Lemma 3.12. To verify the second condition, we note that the representation spheres $\mathbb{S}_2^{p+q\sigma}$ form a set of compact dualizable generators for $\mathrm{Sp}_{C_2,i2}$ and the trigraded spheres $\mathbb{S}_2^{p,q,w}$ form a set of compact dualizable generators for $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. □

Remark 4.6 Unraveling the definitions in Appendix C, we find that for $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ there is a natural identification

$$i_*(X)_n \simeq \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, X),$$

and that the natural maps

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, X) \simeq i_*(X)_n \to i_*(X)_{n-1} \simeq \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n-1}, X)$$

are induced by $\overline{\tau} : \mathbb{S}_2^{0,0,n-1} \to \mathbb{S}_2^{0,0,n}$.

The task of identifying $i_*\mathbb{S}_2$ with $R_\bullet$ will occupy us for the remainder of the section.

4.2 The effective slice filtration

In this section we relate the functor i_*, defined in the previous subsection, to the effective slice filtration of Voevodsky [2002]. We begin by recalling the definition of the effective slice filtration.

Definition 4.7 Let $\mathbf{Sm}/\mathbb{R}$ denote the category of smooth and separated $\mathbb{R}$-schemes of finite type. We let $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})_{i2}$ denote the full subcategory generated under small colimits by the collection $\{\mathbb{S}_2^{p,q,q} \otimes X_+ | X \in \mathbf{Sm}/\mathbb{R},\, p,q \in \mathbb{Z}, q \geq n\}$. We denote the right adjoint of the inclusion by $f_n : \mathcal{SH}(\mathbb{R})_{i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n}$. There are natural transformations $f_{n+1} \to f_n$ and we let s_n denote the cofiber of this map.

Since $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})_{i2}$ is a compactly generated stable subcategory, the functors f_n and s_n preserve all colimits. Moreover, the tensor product of an n-effective object with an m-effective object is $(m+n)$-effective, so that the effective slice tower functor $f_\bullet : \mathcal{SH}(\mathbb{R})_{i2} \to \mathcal{SH}(\mathbb{R})^{\mathbf{Fil}}_{i2}$ is lax symmetric monoidal.

The main result of this section is the following:

Proposition 4.8 *The lax symmetric monoidal functor $i_* : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \to \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ of the previous subsection is equivalent to the lax symmetric monoidal functor $\mathrm{Be} \circ f_\bullet : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \to \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ taking $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ to the Betti realization of its effective slice tower*

$$\cdots \to \mathrm{Be}(f_2 E) \to \mathrm{Be}(f_1 E) \to \mathrm{Be}(f_0 E) \to \mathrm{Be}(f_{-1} E) \to \mathrm{Be}(f_{-2} E) \to \cdots.$$

As a first step, we rephrase Definition 4.7 to be more natural in our trigraded context:

Lemma 4.9 *The full subcategory $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})_{i2}$ is generated under small colimits by the collection $\{\mathbb{S}_2^{p,q,w} \otimes X_+ | X \in \mathbf{Sm}/\mathbb{R},\, p,q,w \in \mathbb{Z}, w \geq n\}$. As a consequence, the suspension functors provide equivalences,*

$$\Sigma^{p,q,w} : \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \to \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n+w}.$$

Proof Since $\Sigma^{0,1,1} : \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \to \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n+1}$ is clearly an equivalence of categories, it suffices to prove the generation statement in the case that $n = 0$. Since $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq 0}$ is closed under tensor products, it suffices to show that $\mathbb{S}_2^{0,-1,0} \in \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq 0}$.

Now, $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq 0}$ is clearly closed under $\Sigma^{-1,0,0}$, so it suffices to show that $\mathbb{S}_2^{1,-1,0} \in \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq 0}$. This follows from the 2-completion of the cofiber sequence

$$\mathbb{S}^{0,0,0} \to \mathrm{Spec}\,\mathbb{C} \to \mathbb{S}^{1,-1,0},$$

which is obtained by rotating the dual of the defining cofiber sequence for $\mathbb{S}^{\mathbb{C}}$. The second statement is a clear consequence of the generation statement. □

We now define an alternative filtration on $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ which is easier to analyze; following an argument of Heard [2019], itself an adaptation of an argument of Pelaez [2013], we will prove that this filtration is in fact equivalent to the effective slice filtration.

Definition 4.10 We let $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n}$ denote the full subcategory generated under small colimits by the collection $\{\mathbb{S}_2^{p,q,w} \mid p,q,w \in \mathbb{Z}, w \geq n\}$. We let

$$f_n^{\mathrm{AT}} : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n}$$

denote the right adjoint of the inclusion. There are natural transformations $f^{\mathrm{AT}}_{n+1} \to f^{\mathrm{AT}}_n$, and we denote the cofiber by s^{AT}_n.

Since $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is a compactly generated stable subcategory, the functors f^{AT}_n and s^{AT}_n preserve colimits.

It is clear that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \cap \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n}$.

Lemma 4.11 *Given $E \in \mathcal{SH}(\mathbb{R})_{i2}$, there are natural equivalences $f_k \Sigma^{p,q,w} E \simeq \Sigma^{p,q,w} f_{k-w} E$ and $s_k \Sigma^{p,q,w} E \simeq \Sigma^{p,q,w} s_{k-w} E$. If $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, the analogous fact holds for f^{AT}_k and s^{AT}_k.*

Proof This follows directly from the fact that

$$\Sigma^{p,q,w} : \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq k} \to \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq k+w} \quad \text{and} \quad \Sigma^{p,q,w} : \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq k} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq k+w}$$

are equivalences of categories. □

Lemma 4.12 *Given $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, there are natural equivalences $f_n E \simeq f^{\mathrm{AT}}_n E$ for all $n \in \mathbb{Z}$.*

Proof It is easy to see that the categories $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})_{i2}$ and $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ define slice filtrations in the sense of [Heard 2019, Definition 2.1]. The result will therefore follow from [Heard 2019, Theorem 2.20] if we can verify three conditions. Let $\iota : \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \hookrightarrow \mathcal{SH}(\mathbb{R})_{i2}$ denote the inclusion. Then we must show that the following hold for all $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$:

(i) The natural map $\varinjlim_n \iota(f^{\mathrm{AT}}_n E) \to \iota(\varinjlim_n f^{\mathrm{AT}}_n E))$ is an equivalence.

(ii) $\iota(f^{\mathrm{AT}}_n E) \in \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n}$.

(iii) $\mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}}(X, \iota(s^{\mathrm{AT}}_n E)) \simeq 0$ for all $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n+1}$.

Condition (i) is clear from the fact that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is closed under colimits in $\mathcal{SH}(\mathbb{R})_{i2}$, and condition (ii) follows from the fact that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n} \subset \mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n}$.

To prove condition (iii), we note that, since s^{AT}_n commutes with filtered colimits and $\mathcal{SH}(\mathbb{R})^{\mathrm{eff}}_{i2,\geq n+1}$ is compactly generated, it suffices to prove the statement for generators of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n}$, namely the trigraded spheres $\mathbb{S}_2^{p,q,w}$ where $w \geq n$. Since $s^{\mathrm{AT}}_n \mathbb{S}_2^{p,q,w} \simeq \Sigma^{p,q,w} s^{\mathrm{AT}}_{n-w} \mathbb{S}_2^{0,0,0}$ by Lemma 4.11, it suffices to show this for $\mathbb{S}_2^{0,0,0}$.

This follows from the equivalence $s^{\mathrm{AT}}_n \mathbb{S}_2^{0,0,0} \simeq s_n \mathbb{S}_2^{0,0,0}$, which may be proved exactly as in [Heard 2019, Theorem 3.15]. □

We are now free to use f_n and f^{AT}_n interchangeably. The following proposition gives us the needed control over f^{AT}_n:

Proposition 4.13 *Let $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. Then $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n}$ if and only if*

$$\pi^{\mathbb{R}}_{p,q,w}(C\text{ब}\otimes E) = 0$$

for all $w < n$, i.e., if and only if $\text{ब} : \pi^{\mathbb{R}}_{p,q,w+1}E \to \pi^{\mathbb{R}}_{p,q,w}E$ is an isomorphism for all $w < n$.

Proof To show that $\pi^{\mathbb{R}}_{p,q,w}(C\text{ब}\otimes E) = 0$ for all $w < n$ if $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{AT}-\mathrm{eff}}_{i2,\geq n}$, it suffices to prove this when $E = \mathbb{S}^{p,q,w}$ for $w \geq n$, which follows from Lemma 3.12.

On the other hand, suppose that $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ satisfies $\pi^{\mathbb{R}}_{p,q,w}(C\text{ब}\otimes E) = 0$ for all $w < n$. We will show that $f^{\mathrm{AT}}_n E \to E$ is an equivalence. First, we note that it is an equivalence on $\pi^{\mathbb{R}}_{p,q,w}$ for all $w \geq n$ by definition. By assumption, $\text{ब} : \pi^{\mathbb{R}}_{p,q,w+1}E \to \pi^{\mathbb{R}}_{p,q,w}E$ is an isomorphism for all $w < n$. On the other hand, by the above $\text{ब} : \pi^{\mathbb{R}}_{p,q,w+1}(f^{\mathrm{AT}}_n E) \to \pi^{\mathbb{R}}_{p,q,w}(f^{\mathrm{AT}}_n E)$ is also an isomorphism for all $w < n$. This implies that $f^{\mathrm{AT}}_n E \to E$ in fact induces an isomorphism on all $\pi^{\mathbb{R}}_{p,q,w}$, as desired. □

Finally, we are ready to prove Proposition 4.8.

Proof of Proposition 4.8 Given a bifiltered object $X_{\bullet,*}$, we let $\mathrm{Diag}(X_{\bullet,*}) = X_{\bullet,\bullet}$ denote the filtered object obtained by restricting along the diagonal map $\mathbb{Z}^{\mathbf{Fil}} \hookrightarrow \mathbb{Z}^{\mathbf{Fil}} \times \mathbb{Z}^{\mathbf{Fil}}$. Then there is a span of lax symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{Diag} \circ i_* \circ f_\bullet & \longrightarrow & i_* \\ \downarrow & & \\ \mathrm{Be} \circ f_\bullet & & \end{array}$$

where the horizontal map is induced by the natural transformation $f_\bullet \to Y$ and the vertical map is induced by the natural transformation $i_* \to Y \circ \mathrm{Be}$. (Here as in Appendix B, Y is the functor taking an object to its constant filtered object.)

Applied to $E \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, in filtration n this span looks like

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}^{0,0,n}_2, f_n E) & \longrightarrow & \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}^{0,0,n}_2, E) \\ \downarrow & & \\ \mathrm{Be}(f_n E) & & \end{array}$$

The horizontal map is an equivalence by n-effectivity of $\mathbb{S}^{0,0,n}_2$, and the vertical map is an equivalence by Theorem 3.2 and Proposition 4.13. □

4.3 Identification of $i_*\mathrm{MGL}_2$

In this subsection, we will identify the commutative algebra in filtered C_2-spectra given by $i_*\mathrm{MGL}_2$. This requires two main inputs. The first is the description of the underlying filtered object in terms of the effective slice filtration from the previous subsection. The second is a theorem of Heard which relates the effective slice filtration of MGL to the regular slice filtration of its Betti realization $\mathrm{MU}_{\mathbb{R}}$.

Proposition 4.14 *There is an equivalence of commutative algebras in filtered C_2-spectra*

$$i_*\mathrm{MGL}_2 \simeq P_{2\bullet}\,\mathrm{MU}_{\mathbb{R},2}\,.$$

We prove this as a special case of a slightly stronger result where we allow tensor powers of MGL_2. This generalization will be useful in the sequel.

Proposition 4.15 *The objects $i_*\mathrm{MGL}_2^{\otimes k}$ are regular slice $2n$-connective in position n. This yields a natural factorization of the map to the constant filtered object*

$$\begin{array}{ccc} & & P_{2\bullet}\,\mathrm{MU}_{\mathbb{R},2}^{\otimes k} \\ & \overset{\simeq}{\nearrow} & \searrow \\ i_*\mathrm{MGL}_2^{\otimes k} & \longrightarrow & Y\,\mathrm{MU}_{\mathbb{R},2}^{\otimes k} \end{array}$$

where the indicated map is an equivalence of commutative algebras in filtered C_2-spectra.

The result of Heard that we will need is the following:

Theorem 4.16 [Heard 2019] *Under Betti realization, the slice tower of $\mathrm{MGL}^{\otimes k}$ goes to the even part of the regular slice tower of $\mathrm{MU}_{\mathbb{R}}^{\otimes k}$. More precisely, there is a commutative diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Be}(f_1\mathrm{MGL}^{\otimes k}) & \longrightarrow & \mathrm{Be}(f_0\mathrm{MGL}^{\otimes k}) & \longrightarrow & \cdots\,\mathrm{Be}(\mathrm{MGL}^{\otimes k}) \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \cdots & \longrightarrow & P_2(\mathrm{MU}_{\mathbb{R}}^{\otimes k}) & \longrightarrow & P_0(\mathrm{MU}_{\mathbb{R}}^{\otimes k}) & \longrightarrow & \cdots\,\mathrm{MU}_{\mathbb{R}}^{\otimes k} \end{array}$$

Moreover, the odd regular slices of $\mathrm{MU}_{\mathbb{R}}$ vanish.

Applying Theorem 8.1, we are able to deduce the 2-completed analogue of Theorem 4.16.

Proof For notational brevity we set $E := \mathrm{MGL}_2^{\otimes k}$. Note that it is a consequence of Theorem 8.1 that $\mathrm{Be}(E)$ is equivalent as a commutative algebra to $\mathrm{MU}_{\mathbb{R},2}^{\otimes n}$. Using Proposition 4.8, we can conclude that i_*E is given by the filtered C_2-spectrum,

$$\cdots \to \mathrm{Be}(f_2E) \to \mathrm{Be}(f_1E) \to \mathrm{Be}(f_0E) \to \mathrm{Be}(f_{-1}E) \to \mathrm{Be}(f_{-2}E) \to \cdots.$$

By Theorem 4.16, this is equivalent to the tower

$$\cdots \to P_4\mathrm{Be}(E) \to P_2\mathrm{Be}(E) \to P_0\mathrm{Be}(E) \to P_{-2}\mathrm{Be}(E) \to P_{-4}\mathrm{Be}(E) \to \cdots.$$

Now, consider the natural map of commutative algebras $i_*E \to Y(\mathrm{Be}(E))$. Looking at the explicit description of i_*E we can conclude it lies in the coreflective subcategory $\mathrm{Sp}^{\mathbf{Fil},2\mathrm{slice}}_{C_2,\geq 0}$. Thus we obtain a diagram of commutative algebras

$$\begin{array}{ccc} & & P_{2\bullet}(\mathrm{Be}(E)) \\ & \overset{\simeq}{\nearrow} & \searrow \\ i_*E & \longrightarrow & Y(\mathrm{Be}(E)) \end{array}$$

where the first map is an equivalence by the above. □

4.4 Identification of $i_*(\mathbb{S}_2)$

In order to finish the proof of Theorem 4.3, we need to prove the following proposition:

Proposition 4.17 *There is an equivalence of commutative algebras in* $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ *between* $i_*\,\mathbb{S}_2$ *and* $R_\bullet$.

Proof Recall that cb is the functor which sends a commutative algebra A to the cosimplicial commutative algebra given by its cobar complex $A^{\otimes(*+1)}$. Consider the natural map

$$\mathrm{cb}(\mathrm{MGL}_2) \to \mathrm{cb}(\mathrm{MGL}_2[\tau^{-1}])$$

in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. After applying i_*, we may use $i_*((-)[\tau^{-1}]) \simeq Y(\mathrm{Be}(-))$ and the equivalence $\mathrm{Be}(\mathrm{MGL}_2) \simeq \mathrm{MU}_{\mathbb{R},2}$ to obtain a map

$$i_*\mathrm{cb}(\mathrm{MGL}_2) \to Y(\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})).$$

Applying Proposition 4.15, we obtain a factorization

$$i_*\mathrm{cb}(\mathrm{MGL}_2) \xrightarrow{\simeq} \tau^{2\mathrm{slice}}_{\geq 0} Y(\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})) \to Y(\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})).$$

Taking totalizations, we obtain

$$\begin{aligned} i_*(\mathbb{S}_2) &\simeq i_*((\mathbb{S}_2)^\wedge_{\mathrm{MGL}_2}) \simeq i_*(\mathrm{Tot}(\mathrm{cb}(\mathrm{MGL}_2))) \simeq \mathrm{Tot}(i_*(\mathrm{cb}(\mathrm{MGL}_2))) \\ &\simeq \mathrm{Tot}(\tau^{2\mathrm{slice}}_{\geq 0} Y(\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))) \simeq \mathrm{Sh}(P_{2\bullet};\mathrm{MU}_{\mathbb{R},2})(\mathbb{1}) \simeq R_\bullet, \end{aligned}$$

where the first equivalence is the MGL_2-completeness[16] of $\mathbb{S}_2$ and the final equivalence is the definition of $R_\bullet$. □

We close with a simple corollary which we will make use of in Section 6.

Corollary 4.18 *There is an equivalence* $\mathrm{MGL}_2 \simeq \Gamma_*(\mathrm{MU}_{\mathbb{R},2})$ *of commutative algebras in* $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$.

Proof By Theorem 4.3, it suffices to show that there is an equivalence

$$i_*\mathrm{MGL}_2 \simeq \mathrm{Sh}(P_{2\bullet};\mathrm{MU}_{\mathbb{R},2})(\mathrm{MU}_{\mathbb{R},2})$$

of commutative algebras over $i_*\,\mathbb{S}_2$. Unraveling the construction of the equivalence

$$i_*\,\mathbb{S}_2 \simeq \mathrm{Sh}(P_{2\bullet};\mathrm{MU}_{\mathbb{R},2})(\mathbb{S}_2),$$

we can build a diagram of commutative algebras in $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$

$$\begin{array}{ccccc} i_*\,\mathbb{S}_2 & \xrightarrow{\simeq} & \mathrm{Sh}(P_{2\bullet};\mathrm{MU}_{\mathbb{R},2})(\mathbb{S}_2) & \longrightarrow & \mathrm{Sh}(P_{2\bullet};\mathrm{MU}_{\mathbb{R},2})(\mathrm{MU}_{\mathbb{R},2}) \\ \downarrow & & \downarrow & \swarrow_{\simeq} & \\ i_*\mathrm{MGL}_2 & \xrightarrow{\simeq} & P_{2\bullet}\,\mathrm{MU}_{\mathbb{R},2} & & \end{array}$$

where the dashed equivalence comes from Example C.10. □

[16] One argument for this is that $\mathrm{M}\mathbb{F}_2$-completeness implies MGL_2-completeness and $\mathbb{S}_2$ is $\mathrm{M}\mathbb{F}_2$-complete by [Hu et al. 2011a].

5 Modules over the cofiber of ta

In this section, we study the category of $C\text{ন}$-modules. Our main theorem states that this category admits an explicit, algebraic description. Below, we use $\mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$ to denote Ind of the thick subcategory of $\mathrm{MU}_*\mathrm{MU}$-comodules generated by even shifts of the unit. For more details on this category, see Definition 5.14.

Theorem 5.1 *There is an equivalence of presentably symmetric monoidal categories under* $\mathrm{Sp}_{C_2,i2}$,

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\text{ন}) \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}}), \tag{2}$$

where $\mathrm{Sp}_{C_2,i2}$ *acts on the left-hand side through* $c_{\mathbb{C}/\mathbb{R}}$ *and on the left factor of the right-hand side. On Picard elements this equivalence takes*

$$C\text{ন} \otimes \mathbb{S}_2^{p,q,w} \mapsto \Sigma^{(p-w)+(q-w)\sigma} \underline{\mathbb{Z}}_2 \otimes \omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}^{\otimes w}.$$

As a consequence of this equivalence of categories, there is an isomorphism of trigraded groups:[17]

$$\pi^{\mathbb{R}}_{p,q,w}(C\text{ন}) \cong \bigoplus_{w+a-s=p} \mathrm{Ext}^{s,2w}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_* \otimes \pi^{C_2}_{a+(q-w)\sigma} \underline{\mathbb{Z}}_2).$$

Over $\mathbb{C}$ the corresponding result is the main theorem of [Gheorghe et al. 2021]. Upon tensoring with $\mathrm{Spec}(\mathbb{C})$, our theorem recovers theirs.

Notation 5.2 In the above theorem and throughout the section, we follow the convention of writing $\mathcal{C} \otimes_R \mathcal{D}$ for the tensor product of two presentable R-linear categories (instead of $\mathcal{C} \otimes_{\mathrm{Mod}_R} \mathcal{D}$).

Remark 5.3 In the statement of the above theorem, we use the $\mathbb{Z}$-linear structure on $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2)$ coming from the symmetric monoidal functor

$$(_) \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}}_2 : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2),$$

where $_ : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}(\mathrm{Sp}_{C_2}; \underline{\mathbb{Z}})$ is the symmetric monoidal functor discussed in Remark 5.11 below.

By the final proof in this section, the ring structure on the trigraded homotopy groups of $C\text{ন}$ can be computed according to the following recipe. First, write down the usual dga cobar complex computing $\mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$. Then, apply the functor $A \mapsto \underline{A}$ taking an abelian group to the constant C_2-equivariant Mackey functor with value at that group. Finally, take $\mathrm{RO}(C_2)$ graded homotopy and compute the homology ring of the resulting dga.

Remark 5.4 We will prove in Lemma 5.16 that the category $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$ admits a t-structure with heart the category of comodules in C_2-Mackey functors over the Hopf algebroid $\big(\underline{\pi}^{C_2}_{*\rho}(\mathrm{MU}_{\mathbb{R},2}), \underline{\pi}^{C_2}_{*\rho}(\mathrm{MU}_{\mathbb{R},2} \otimes \mathrm{MU}_{\mathbb{R},2})\big)$.

We may therefore view any such comodule as an element of $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$.

[17] The tensor product may be moved outside the Ext, but then it must be taken in the derived sense.

In fact, we are able to explicitly describe the trigraded homotopy groups of $Y \otimes C\tau$ for a more general collection of Y than just trigraded spheres.

Definition 5.5 We say that $X \in \mathrm{Sp}_{C_2,i2}$ is $\mathrm{MU}_{\mathbb{R},2}$*-projective* if $\mathrm{MU}_{\mathbb{R},2} \otimes X$ is a retract (as an $\mathrm{MU}_{\mathbb{R},2}$-module) of $\bigoplus_\alpha \Sigma^{n_\alpha \rho} \mathrm{MU}_{\mathbb{R},2}$ for some set of integers n_α.

Examples of $\mathrm{MU}_{\mathbb{R},2}$-projective X include $\mathbb{S}_2^0$ and $\mathrm{MU}_{\mathbb{R},2}$, as well as any object of $\mathrm{Sp}_{C_2,i2}$ built out of cells of the form $\mathbb{S}_2^{n\rho}$.

Theorem 5.6 *If $X \in \mathrm{Sp}_{C_2,i2}$ is $\mathrm{MU}_{\mathbb{R}}$-projective, then under the equivalence of Theorem 5.1, $C\tau \otimes \Gamma_*(X)$ corresponds to the $\underline{\pi}_{*\rho}^{C_2}(\mathrm{MU}_{\mathbb{R},2} \otimes \mathrm{MU}_{\mathbb{R},2})$-comodule $\underline{\pi}_{*\rho}^{C_2}(\mathrm{MU}_{\mathbb{R},2} \otimes X)$.*

Moreover, there is an isomorphism of trigraded groups,

$$\pi^{\mathbb{R}}_{p,q,w}(\Gamma_*(X) \otimes C\tau) \cong \bigoplus_{w+a-s=p} \mathrm{Ext}^{s,2w}_{(\mathrm{MU}_2)_* \mathrm{MU}_2}((\mathrm{MU}_2)_*, (\mathrm{MU}_2)_*(\Phi^e(X)) \otimes_{\mathbb{Z}_2} \pi^{C_2}_{a+(q-w)\sigma} \underline{\mathbb{Z}}_2).$$

The proofs of Theorems 5.1 and 5.6 will be quite long, and correctly handling the symmetric monoidal structures involved requires us to take a rather circuitous route. For this reason, before proceeding we provide a sketch of the argument.

The proof of the equivalence

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau) \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$$

has three main steps and one significant subtlety. First, we produce symmetric monoidal functors into each side of equation (2) from $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$. Second, we show that there are commutative algebras R_1 and R_2 in $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ such that the left-hand side is equivalent to $\mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; R_1)$ and the right-hand side is equivalent to $\mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; R_2)$. Third, we examine R_1 and R_2 directly and find that they are in fact equivalent.

The functor into $C\tau$-modules from $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ is the composite of the functor i_* from Section 4 with $C\tau \otimes -$. The functor into the right-hand side is more delicate to construct. A first guess would be to use the equivalence $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2} \cong \mathrm{Sp}_{C_2,i2} \otimes \mathrm{Sp}^{\mathbf{Fil}}$, and tensor the pair of maps

$$\mathrm{Sp}_{C_2,i2} \to \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \quad \text{and} \quad \mathrm{Sp}^{\mathbf{Fil}} \to \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; \mathbb{Z}) \xrightarrow{\mathbf{Gr}} \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}; \mathbb{Z}) \to \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}}).$$

In fact, this does not produce the correct functor; in $C\tau$-modules the periodicity class v_1 lives in C_2-degree ρ, but the given tensor product functor puts v_1 in C_2-degree 0. In order to fix this, we twist by a functor

$$\mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2}; \underline{\mathbb{Z}}_2) \xrightarrow{\mathrm{tw}^\rho} \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2}; \underline{\mathbb{Z}}_2)$$

that has the effect of tensoring with $\mathbb{S}^{n\rho}$ on the n^{th} graded piece. The presence of this twist, and its interaction with the symmetric monoidal structure, is the main subtlety of the argument.

The second step is straightforward, and demonstrates the power of higher algebra in proving "Koszul duality" statements as a corollary of Barr–Beck–Lurie monadicity. At this point we have two commutative algebras R_1 and R_2, and all we need to do is show they are equivalent.

The third step relies on special properties of R_1 and R_2. Both commutative algebras come equipped with a preferred presentation as a totalization of a cosimplicial diagram of commutative algebras. What we do is show that these cosimplicial diagrams are levelwise equivalent. Each cosimplicial diagram takes values in the heart of a t-structure, and in particular is determined by 1-categorical data. Thus, it is genuinely possible to write down a comparison map by hand and check that it is an equivalence, without concern for the potential infinity of higher coherence data.

(i) In Section 5.1, we review the categories which appear in the right-hand side of Theorem 5.1. We also construct a t-structure on the right-hand side of Theorem 5.1, which will play an important role in our argument.

(ii) In Section 5.2, we collect some material on twisted t-structures on categories of graded and filtered objects, as well as on twisting isomorphisms between them.

(iii) In Section 5.3, we relate twisted t-structures to the slice filtration in C_2-equivariant homotopy theory. This finishes the construction of the comparison functors.

(iv) In Section 5.4, we use Koszul duality to reduce the proof of Theorems 5.1 and 5.6 to understanding a specific pair of commutative algebras.

(v) In Section 5.5, we prove Lemma 5.40, which is the key result that allows us to compare the commutative algebras R_1 and R_2 above. Using this lemma, we then complete the proof of Theorems 5.1 and 5.6.

5.1 Categories of interest

In this subsection we set notation for working in the various categories of interest. In particular, we aim for a practical understanding of the category $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2};\underline{\mathbb{Z}}_2)\otimes_{\mathbb{Z}}\mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$ that appears on the right-hand side of Theorem 5.1. Proceeding from inside out, we start by fixing notation for the category of $\mathbb{Z}$-modules:

Definition 5.7 Let $\mathrm{Ab}^{\heartsuit}$ denote the abelian category of discrete abelian groups. Let Ab denote the category of $\mathbb{Z}$-modules with its standard t-structure, so that $\mathrm{Ab}^{\heartsuit}$ is the heart of Ab.

Next, we describe the C_2-equivariant analogue of Ab:

Definition 5.8 Let $M(C_2)^{\heartsuit}$ denote the abelian category of discrete Mackey functors for the group C_2, which may be explicitly described as follows. A Mackey functor $B\in M(C_2)^{\heartsuit}$ consists of

- abelian groups $B(C_2)$ and $B(*)$,
- an involution $\sigma : B(C_2)\to B(C_2)$,
- homomorphisms $r : B(*)\to B(C_2)$ and $t : B(C_2)\to B(*)$ which satisfy the relations $\sigma\circ r = r$, $t\circ\sigma = t$ and $r\circ t = 1+\sigma$.

We will sometimes write $[C_2]$ for the endomorphism $t \circ r$ of $B(*)$. We let $M(C_2)$ denote the unbounded derived category of $M(C_2)^\heartsuit$ equipped with its standard t-structure.[18]

In our setting, $M(C_2)$ arises naturally because it is equivalent to the category of $\underline{\pi}_0^{C_2}\,\mathbb{S}$-modules in Sp_{C_2}; see [Patchkoria et al. 2022, Theorem 5.10]. This description equips $M(C_2)$ with a symmetric monoidal structure compatible with the t-structure. The induced symmetric monoidal structure on $M(C_2)^\heartsuit$ is known as the box product and admits an explicit description as in [Loyd 2017].

Example 5.9 Given a discrete abelian group A, consider the constant Mackey functor $\underline{A} \in M(C_2)^\heartsuit$. This Mackey functor is determined by $\underline{A}(C_2) = \underline{A}(*) = A$, $r = \sigma = \mathrm{id}_A$ and $t = 2 \cdot \mathrm{id}_A$. This construction provides an exact colimit-preserving functor

$$_ : \mathrm{Ab}^\heartsuit \to M(C_2)^\heartsuit.$$

Examining the explicit description of the box product in [Loyd 2017], we find that _ acquires the structure of a (nonunital) tensor-product preserving lax symmetric monoidal functor. In particular, we obtain a commutative algebra $\underline{\mathbb{Z}}$ in $M(C_2)^\heartsuit$.

Definition 5.10 Let $\underline{\mathrm{Ab}}$ denote the symmetric monoidal category of $\underline{\mathbb{Z}}$-modules in $M(C_2)$ (or, equivalently, Sp_{C_2}), equipped with its standard t-structure. This is equivalent to the derived category of $\mathrm{Mod}(M(C_2)^\heartsuit; \underline{\mathbb{Z}})$; in particular, we have $\underline{\mathrm{Ab}}^\heartsuit = \mathrm{Mod}(M(C_2)^\heartsuit; \underline{\mathbb{Z}})$.

Remark 5.11 Since $\mathbb{Z}$ is the unit of $\mathrm{Ab}^\heartsuit$, we learn that _ factors through an exact colimit-preserving symmetric monoidal functor

$$_ : \mathrm{Ab}^\heartsuit \to \underline{\mathrm{Ab}}^\heartsuit.$$

Since _ is exact, colimit-preserving and symmetric monoidal it extends uniquely to a colimit-preserving symmetric monoidal functor $_ : \mathrm{Ab} \to \underline{\mathrm{Ab}}$ of derived categories. Moreover, the pair of functors $A \to A(*)$ and $A \to A(C_2)$ extend to limit-preserving functors on the level of the derived category. Since this pair of functors is jointly conservative, we can use the fact that $\underline{A}(*) \cong A$ and $\underline{A}(C_2) \cong A$ to conclude that _ preserves limits as well.

Before proceeding to explicitly describe $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$, we record a few lemmas for later use.

Lemma 5.12 *The functor $_ : \mathrm{Ab}^\heartsuit \to \underline{\mathrm{Ab}}^\heartsuit$ is fully faithful and left adjoint to the functor $B \mapsto B(*)$. Its essential image is therefore a coreflective subcategory of $\underline{\mathrm{Ab}}^\heartsuit$, with coreflector given by the counit map $\underline{B(*)} \to B$. In particular, for any B in the essential image of _, there are canonical isomorphisms*

$$B \cong \underline{B(*)} \cong \underline{B(C_2)}.$$

Proof Clear. □

Lemma 5.13 *The forgetful functor $\underline{\mathrm{Ab}}^\heartsuit \to M(C_2)^\heartsuit$ is fully faithful, with image spanned by the Mackey functors B for which $[C_2] = 2$ as endomorphisms of $B(*)$.*

[18] As defined, for example, in [Lurie 2017, Definition 1.3.5.8].

Proof The unit of $M(C_2)^{\heartsuit}$ is the Burnside Mackey functor $\underline{\mathcal{A}}$, whose values are given by $\underline{\mathcal{A}}(C_2) = \mathbb{Z}$ and $\underline{\mathcal{A}}(*) = \mathbb{Z}[\![C_2]\!]/([C_2]^2 - 2[C_2])$. As a commutative algebra in $M(C_2)^{\heartsuit}$, $\underline{\mathbb{Z}}$ is the quotient of $\underline{\mathcal{A}}$ by the relation $[C_2] = 2$. The result follows. □

The following definition and construction sum up what we need to know about categories of quasicoherent and ind-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$:

Definition 5.14 We let $\mathcal{M}_{\mathrm{fg}}$ denote the moduli stack of formal groups and let $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ denote the category of quasicoherent sheaves on $\mathcal{M}_{\mathrm{fg}}$. It is equivalent to the derived category of evenly graded MU_* MU-comodules; see [Goerss 2008, Remarks 2.38 and 3.14].

Let $\mathcal{D} \subset \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ denote the thick subcategory generated by the sheaves $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}^{\otimes k}$ for $k \in \mathbb{Z}$, where $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$ is the sheaf of invariant differentials on the universal formal group $\mathbb{G}/\mathcal{M}_{\mathrm{fg}}$. We define $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) := \mathrm{Ind}(\mathcal{D})$. The category $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ is equivalent to Hovey's category of evenly graded stable comodules $\mathrm{Stable}_{\mathrm{MU}_*\,\mathrm{MU}}$; see [Barthel et al. 2018, Remark 4.30].

Both $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ and $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ are naturally stable presentably symmetric monoidal categories and come equipped with compatible t-structures whose hearts are equivalent to the 1-category of MU_* MU-comodules.

Construction 5.15 Since $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ is equivalent to the derived category of a Grothendieck abelian category, it is equipped with the structure of a colimit-preserving symmetric monoidal functor $\mathrm{Ab} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. Letting $\mathrm{Ab}^{\mathrm{fin}}$ denote the full subcategory of compact objects, this restricts to an exact symmetric monoidal functor $\mathrm{Ab}^{\mathrm{fin}} \to \mathcal{D}$, where $\mathcal{D}$ is defined as in Definition 5.14. Taking Ind and using the fact that Ab is compactly generated, we obtain a colimit-preserving symmetric monoidal functor

$$\mathrm{Ab} \simeq \mathrm{Ind}(\mathrm{Ab}^{\mathrm{fin}}) \to \mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}).$$

Using this functor and the functor $\mathrm{Ab} \xrightarrow{\underline{\ }} \underline{\mathrm{Ab}} \xrightarrow{-\otimes_{\underline{\mathbb{Z}}}\underline{\mathbb{Z}}_2} \underline{\mathrm{Ab}}_{i2}$, we are able to define the stable presentably symmetric monoidal category

$$\underline{\mathrm{Ab}}_{i2} \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}),$$

which appeared in Theorem 5.1 with slightly different notation.

The core of our understanding of $\underline{\mathrm{Ab}}_{i2} \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ rests on the following lemma, where we put a t-structure on this category whose heart we can describe explicitly.

Lemma 5.16 *There exists a t-structure on $\underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ with the following properties*:

(i) *It is compatible with the symmetric monoidal structure, accessible, compatible with filtered colimits and right complete.*

(ii) *The natural functor*

$$\underline{\mathrm{Ab}}_{i2,\geq 0} \otimes_{\mathrm{Ab}_{\geq 0}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0} \to \underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$$

is fully faithful with essential image the connective objects.

(iii) *The natural functor*

$$\underline{\mathrm{Ab}}_{i2}^{\heartsuit} \otimes_{\mathrm{Ab}^{\heartsuit}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit} \to \underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$$

is fully faithful with essential image the heart of $\underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$.

Using the explicit description of $\underline{\mathrm{Ab}}^{\heartsuit}$ given by Definition 5.8 and Lemma 5.13 and the fact that $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}$ is equivalent to the category of evenly graded $\mathrm{MU}_*\,\mathrm{MU}$-comodules, we obtain the following description of

$$\underline{\mathrm{Ab}}_{i2}^{\heartsuit} \otimes_{\mathrm{Ab}^{\heartsuit}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}.$$

This is the category of pairs of $(\mathrm{MU}_2)_*\,\mathrm{MU}_2$-comodules $A(C_2)$ and $A(*)$ equipped with following structures (all of which are compatible with the comodule structure):

- maps $r : A(*) \to A(C_2)$ and $t : A(C_2) \to A(*)$, along with an involution $\sigma : A(C_2) \to A(C_2)$,
- satisfying the relations $\sigma \circ r = r$, $t \circ \sigma = t$, $t \circ r = 2$ and $r \circ t = 1 + \sigma$.

In other words, $\underline{\mathrm{Ab}}_{i2}^{\heartsuit} \otimes_{\mathrm{Ab}^{\heartsuit}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}$ is the category of evenly graded comodules over the Hopf algebroid $(\underline{(\mathrm{MU}_2)_*}, \underline{(\mathrm{MU}_2)_*\,\mathrm{MU}_2})$ in graded C_2-Mackey functors.

Remark 5.17 Using the fact that

$$(\underline{(\mathrm{MU}_{\mathbb{R},2})_{*\rho}}, \underline{(\mathrm{MU}_{\mathbb{R},2})_{*\rho}\,\mathrm{MU}_{\mathbb{R},2}}) \cong (\underline{(\mathrm{MU}_2)_{2*}}, \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2})$$

(see [Hu and Kriz 2001, Theorems 2.25 and 2.28]), this category may equally well be described as the category of graded comodules over the Hopf algebroid $(\underline{(\mathrm{MU}_{\mathbb{R},2})_{*\rho}}, \underline{(\mathrm{MU}_{\mathbb{R},2})_{*\rho}\,\mathrm{MU}_{\mathbb{R},2}})$ in graded C_2-Mackey functors.

The proof of Lemma 5.16 is not difficult, but it will rely on material from [Lurie 2018b, Appendix C], which we presently recount. We begin with a simple method for producing t-structures.

Construction 5.18 Given a pointed presentable category $\mathcal{C}$, we let $\mathrm{Sp} \otimes \mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})$ denote the category of spectrum objects in $\mathcal{C}$. It is a stable presentable category.

The category $\mathrm{Sp} \otimes \mathcal{C}$ admits a t-structure with $(\mathrm{Sp} \otimes \mathcal{C})_{\geq 0}$ equal to the full subcategory generated by the essential image of the functor $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp} \otimes \mathcal{C}$ under extensions and colimits [Lurie 2017, Proposition 1.4.3.4 and Remark 1.4.3.5]. This t-structure is accessible and right complete.

Note that if $\mathcal{C}$ is endowed with the structure of a presentably symmetric monoidal category, then $\mathrm{Sp} \otimes \mathcal{C}$ naturally inherits this structure and the t-structure defined above is compatible with it.

In fact, the following lemma, which follows from [Lurie 2018b, Remark C.3.1.5], shows that this construction is the universal way to produce a (well-behaved) t-structure on a presentable stable category.

Lemma 5.19 *Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0})$ is a stable presentably symmetric monoidal category with compatible t-structure. Suppose that the t-structure is accessible, compatible with filtered colimits and right complete.*

Then there is a natural equivalence of symmetric monoidal categories with compatible t-structure

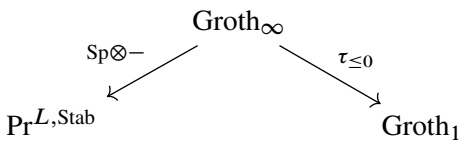
$$(\mathcal{C}, \mathcal{C}_{\geq 0}) \simeq (\mathrm{Sp} \otimes \mathcal{C}_{\geq 0}, (\mathrm{Sp} \otimes \mathcal{C}_{\geq 0})_{\geq 0}).$$

In conclusion, we may as well work with the categories of connective objects. For Ab, $\underline{\mathrm{Ab}}_{i2}$, and $\operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})$, these are all Grothendieck prestable categories in the sense of [Lurie 2018b, Definition C.1.4.2].

Recollection 5.20 We let Groth_∞ denote the category of Grothendieck prestable categories [Lurie 2018b, Definition C.3.0.5]. By [Lurie 2018b, Theorem C.4.2.1], Groth_∞ is a symmetric monoidal category with tensor product given by the usual tensor product of presentable categories and unit $\mathrm{Sp}_{\geq 0}$.

Moreover, let Groth_1 denote the category of Grothendieck abelian 1-categories. It is a symmetric monoidal category with tensor product given by the usual tensor product of presentable categories and unit $\mathrm{Ab}^{\heartsuit}$ [Lurie 2018b, Corollary C.5.4.19]. There is a functor $\tau_{\leq 0} : \mathrm{Groth}_\infty \to \mathrm{Groth}_1$ sending a Grothendieck prestable category to its subcategory of discrete objects. By [Lurie 2018b, Remark C.5.4.20], the functor $\tau_{\leq 0}$ is symmetric monoidal.

We summarize this in the following span of symmetric monoidal categories:



Proof of Lemma 5.16 Using Lemma 5.19 and [Lurie 2018b, Remark C.4.2.3], we have

$$\mathrm{Sp} \otimes (\underline{\mathrm{Ab}}_{i2,\geq 0} \otimes_{\mathrm{Ab}_{\geq 0}} \operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0}) \simeq \underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}}).$$

Moreover, since $\underline{\mathrm{Ab}}_{i2,\geq 0} \otimes_{\mathrm{Ab}_{\geq 0}} \operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0}$ is prestable, the Σ^∞ functor identifies it with

$$\big(\mathrm{Sp} \otimes (\underline{\mathrm{Ab}}_{i2,\geq 0} \otimes_{\mathrm{Ab}_{\geq 0}} \operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0})\big)_{\geq 0}.$$

This immediately implies the first two properties, with the exception of compatibility with filtered colimits. This follows from the fact that $\underline{\mathrm{Ab}}_{i2,\geq 0} \otimes_{\mathrm{Ab}_{\geq 0}} \operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0}$ is compactly generated, which holds because each one of $\underline{\mathrm{Ab}}_{i2,\geq 0}$, $\mathrm{Ab}_{\geq 0}$ and $\operatorname{Ind}\operatorname{Coh}(\mathcal{M}_{\mathrm{fg}})_{\geq 0}$ is compactly generated [Lurie 2018b, Corollary C.6.2.3].

The third property follows from the fact that $\mathcal{C}^{\heartsuit} = \tau_{\leq 0}(\mathcal{C}_{\geq 0})$ and the fact that $\tau_{\leq 0} : \mathrm{Groth}_\infty \to \mathrm{Groth}_1$ is symmetric monoidal. □

5.2 Twistings, I: t-structures on filtered objects

We begin this subsection with a discussion of several natural t-structures on categories of graded and filtered objects, obtained by twisting a given t-structure by a Picard element. Although these constructions are simple, they have the effect of modifying the symmetric monoidal structure on the heart of a category. Next, we explicitly describe this modification at the level of the heart, where it admits a description in terms of an Euler characteristic. Read another way, there is an Euler characteristic obstruction to producing symmetric monoidal twisting functors. We close the subsection by producing symmetric monoidal twisting functors whenever this Euler characteristic obstruction vanishes.

Definition 5.21 For the remainder of this subsection, we fix the following data: a stable, presentably symmetric monoidal category $\mathcal{C}$ equipped with a compatible t-structure[19] and a Picard element $\mathcal{L}$.

We then define the $\mathcal{L}$-twisted t-structure on $\mathcal{C}^{\mathbf{Gr}}$ and $\mathcal{C}^{\mathbf{Fil}}$ as follows:

- An object $X_\bullet$ of $\mathcal{C}^{\mathbf{Gr}}$ is ≥ 0 in the $\mathcal{L}$-twisted t-structure if, for all n, $\mathcal{L}^{\otimes -n} \otimes X_n$ is ≥ 0 in the original t-structure.
- Similarly, an object $X_\bullet$ of $\mathcal{C}^{\mathbf{Fil}}$ is ≥ 0 in the $\mathcal{L}$-twisted t-structure if $\mathcal{L}^{\otimes -n} \otimes X_n$ is ≥ 0 in the original t-structure for all n.

We denote the hearts of these t-structures by $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ and $\mathcal{C}^{\mathbf{Fil},\mathcal{L},\heartsuit}$, respectively. We denote the i^{th} $\mathcal{L}$-twisted t-structure homotopy objects by $\underline{\pi}_i^{\mathcal{L},\mathbf{Gr}}$ and $\underline{\pi}_i^{\mathcal{L},\mathbf{Fil}}$, respectively. When $\mathcal{L} = \mathbb{1}$, we frequently omit it from the notation.

We summarize the basic properties of this definition in the following lemma.

Lemma 5.22 *In the graded case*:

(i) *The $\mathcal{L}$-twisted t-structure on $\mathcal{C}^{\mathbf{Gr}}$ is compatible with the symmetric monoidal structure.*

(ii) *An object $X_\bullet$ is < 0 in the $\mathcal{L}$-twisted t-structure if and only if, for all n, $\mathcal{L}^{\otimes -n} \otimes X_n$ is < 0 in the original t-structure.*

(iii) *The t-structure homotopy groups are determined by the formula*

$$(\underline{\pi}_0^{\mathcal{L},\mathbf{Gr}} X_\bullet)_n \cong \mathcal{L}^{\otimes n} \underline{\pi}_0(\mathcal{L}^{\otimes -n} \otimes X_n).$$

Assuming that $\mathcal{L} \geq 0$, we obtain similar results in the filtered case:

(i′) *The $\mathcal{L}$-twisted t-structure on $\mathcal{C}^{\mathbf{Fil}}$ is compatible with the symmetric monoidal structure.*

(ii′) *An object $X_\bullet$ is < 0 in the $\mathcal{L}$-twisted t-structure if and only if $\mathcal{L}^{\otimes -n} \otimes X_n < 0$.*

(iii′) *The t-structure homotopy groups are given the by following formula:*

$$(\underline{\pi}_0^{\mathcal{L},\mathbf{Fil}} X_\bullet)_n \cong \mathcal{L}^{\otimes n} \underline{\pi}_0(\mathcal{L}^{\otimes -n} \otimes X_n).$$

Under the stronger assumption that $\mathcal{L} \geq 1$, the functor $\mathcal{C}^{\mathbf{Gr}} \to \mathcal{C}^{\mathbf{Fil}}$, which is right adjoint to the associated graded functor, restricts to an equivalence of symmetric monoidal 1-categories

$$\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit} \simeq \mathcal{C}^{\mathbf{Fil},\mathcal{L},\heartsuit}.$$

Notation 5.23 We write $- \otimes^{\heartsuit,\mathcal{L}} -$ for the tensor product induced on either $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ or $\mathcal{C}^{\mathbf{Fil},\mathcal{L},\heartsuit}$. In the case that $\mathcal{L} = \mathbb{1}$, we simply write $- \otimes^{\heartsuit} -$.

Proof The statements (i), (ii) and (iii) are all clear. We therefore restrict ourselves to the filtered case for the rest of the proof.

[19] Here, compatible means that a tensor product of objects which are ≥ 0 is ≥ 0 itself, and that the unit is ≥ 0.

From the expression of the tensor product as a Day convolution, we observe that a tensor product of connective objects has n^{th} term presented as a colimit over a diagram of connective objects tensored with $\mathcal{L}$ at least n times. Since $\mathcal{L} \geq 0$ we may conclude that (i′) holds.

The expression for the homotopy objects in (iii′) follows from (ii′) in a straightforward way. We now prove (ii′). Suppose that $X_\bullet$ is an object such that $\mathcal{L}^{\otimes -n} \otimes X_n < 0$. We want to show that $X_\bullet < 0$, i.e., that it receives no maps from $Y_\bullet \geq 0$. Using the condition that $\mathcal{L} \geq 0$, we learn that $[Y_n, X_m] = 0$ for all $n \geq m$, which is enough to imply that $[Y_\bullet, X_\bullet] = 0$.

Suppose that $X_\bullet < 0$; then we would like to show that $\mathcal{L}^{\otimes -n} \otimes X_n < 0$. Associated to every $Y \geq 0$ in $\mathcal{C}$ we can consider the filtered object $Y \otimes \mathcal{L}^{\otimes n}(n)$, which is depicted below:

$$\cdots \to 0 \to 0 \to Y \otimes \mathcal{L}^{\otimes n} \xrightarrow{\mathrm{id}} Y \otimes \mathcal{L}^{\otimes n} \xrightarrow{\mathrm{id}} \cdots.$$

Here, the first nonzero object occurs at position n. Using the assumption that $\mathcal{L} \geq 0$, we can conclude that $Y \otimes \mathcal{L}^{\otimes n}(n) \geq 0$. Now, on mapping out of this object we have

$$* \simeq \Omega^\infty \operatorname{Map}_{\mathcal{C}^{\mathbf{Fil}}}(Y \otimes \mathcal{L}^{\otimes n}(n), X) \simeq \Omega^\infty \operatorname{Map}_{\mathcal{C}}(Y \otimes \mathcal{L}^{\otimes n}, X_n),$$

which implies that $\mathcal{L}^{\otimes -n} \otimes X_n < 0$ for all n, as desired.

We now proceed to prove the final statement. Let $\tau : \mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}}(-1) \to \mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}}$ denote the shift map in $\mathcal{C}^{\mathbf{Fil}}$. Then $C\tau$ admits an $\mathbb{E}_\infty$-algebra structure in $\mathcal{C}^{\mathbf{Fil}}$ such that there is an equivalence of symmetric monoidal categories $\mathcal{C}^{\mathbf{Gr}} \simeq \operatorname{Mod}(\mathcal{C}^{\mathbf{Fil}}; C\tau)$; see Lemma B.5. Using the assumption that $\mathcal{L} \geq 0$, we learn that $\mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}}$ and $C\tau$ are ≥ 0. As a consequence, we find that $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ may be identified with $\operatorname{Mod}(\mathcal{C}^{\mathbf{Fil},\mathcal{L},\heartsuit}; \underline{\pi}_0^{\mathcal{L}} C\tau)$.

The proposition will therefore follow if we prove that $\underline{\pi}_0^{\mathcal{L}} \mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}} \to \underline{\pi}_0^{\mathcal{L}} C\tau$ is an equivalence. By Lemma 5.22, we find that

$$(\underline{\pi}_0^{\mathcal{L},\mathbf{Fil}} \mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}})_n \cong \begin{cases} \mathcal{L}^{\otimes n} \pi_0(\mathcal{L}^{\otimes -n}) & \text{if } n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{L} \geq 1$ by assumption, we find for $n < 0$ that $\mathcal{L}^{\otimes n} \pi_0 \mathcal{L}^{\otimes -n} \cong 0$, whereas for $n = 0$ we just get $\pi_0 \mathbb{1}_{\mathcal{C}}$.

On the other hand, we have that

$$(\underline{\pi}_0^{\mathcal{L},\mathbf{Fil}} C\tau)_n \cong \begin{cases} \pi_0 \mathbb{1}_{\mathcal{C}} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from this that the map $\underline{\pi}_0^{\mathcal{L},\mathbf{Fil}} \mathbb{1}_{\mathcal{C}^{\mathbf{Fil}}} \to \underline{\pi}_0^{\mathcal{L},\mathbf{Fil}} C\tau$ is an equivalence, as desired. □

We can now profitably define the twist functors.

Construction 5.24 Given a symmetric monoidal Grothendieck abelian 1-category[20] $\mathcal{A}$ and an invertible object $a \in \mathcal{A}$, we can construct a monoidal functor $i_a : \mathbb{Z} \to \mathcal{A}$ which sends 1 to a. If the swap map

[20] By symmetric monoidal Grothendieck abelian 1-category, we mean a commutative algebra object of the symmetric monoidal category Groth_1 discussed in Recollection 5.20. In other words, we assume that the tensor product commutes with colimits in each variable.

$s_{a,a} : a \otimes a \xrightarrow{\cong} a \otimes a$ is the identity, then we can make this functor symmetric monoidal. Tensoring up with $\mathrm{Ab}^{\heartsuit}$, we obtain a monoidal (or symmetric monoidal) functor

$$i_a : \mathrm{Ab}^{\mathbf{Gr},\heartsuit} \xrightarrow{\mathbb{1}(1)\mapsto a} \mathcal{A}.$$

In our situation of interest, we take $\mathcal{A} = \mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ and $a = (\underline{\pi}_0\mathbb{1}) \otimes \mathcal{L}(1)$. Then we can define the functor $\mathrm{tw}^{\mathcal{L}}$ as the composite

$$\mathcal{C}^{\mathbf{Gr},\heartsuit} \cong \mathcal{C}^{\heartsuit} \otimes \mathrm{Ab}^{\mathbf{Gr},\heartsuit} \xrightarrow{\mathrm{Id}\otimes i_{(\underline{\pi}_0\mathbb{1})\otimes\mathcal{L}(1)}} \mathcal{C}^{\heartsuit} \otimes \mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit} \xrightarrow{\otimes} \mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}.$$

On objects this has the effect of sending $\{X_n\}$ to $\{X_n \otimes \mathcal{L}^{\otimes n}\}$. It is easy to see that this is a monoidal equivalence between $\mathcal{C}^{\mathbf{Gr},\heartsuit}$ and $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$, and is further symmetric monoidal if $s_{a,a}$ is the identity.

The above construction succinctly explains how much the symmetric monoidal structure on $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ has been twisted in terms of the quantity $s_{a,a}$. We now give a description of this quantity in terms of the following standard definition:

Definition 5.25 Given a dualizable object X, with dual $X^\vee$, unit η and counit ϵ, the trace of an endomorphism $f : X \to X$ is the element $\mathrm{tr}(f) \in \pi_0 \mathrm{End}(\mathbb{1})$ given by the composite

$$\mathbb{1} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{f\otimes X^\vee} X \otimes X^\vee \xrightarrow{s_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\epsilon} \mathbb{1}.$$

The trace of the identity is called the Euler characteristic and denoted by $\chi(X)$.

A simple diagram chase tells us that the Euler characteristic is multiplicative and defines an $\mathrm{End}(\mathbb{1})$-linear map $\mathrm{tr} : \mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$. Moreover, one can compute that the trace of the swap map $s_{X,X}$ is equal to $\chi(X)$. In the case where X is invertible the trace map is an isomorphism, and we can use this to conclude that the condition $s_{a,a} = \mathrm{Id}_{a\otimes a}$ in Construction 5.24 is equivalent to asking that $\chi(a) = 1$. Specializing further to the case $a = (\underline{\pi}_0\mathbb{1}) \otimes \mathcal{L}(1)$, we note that the diagram computing the Euler characteristic of $(\underline{\pi}_0\mathbb{1}) \otimes \mathcal{L}(1)$ is $\underline{\pi}_0^{\mathcal{L},\mathbf{Gr}}$ applied to the diagram computing the Euler characteristic of $\mathcal{L}(1)$ in the ambient category $\mathcal{C}^{\mathbf{Gr}}$. This implies that

$$\chi(a) = \chi(\mathcal{L}(1)) = \chi(\mathcal{L})\chi(\mathbb{1}(1)) = \chi(\mathcal{L}) \cdot 1.$$

Thus, we have proved:

Lemma 5.26 *The functor $\mathrm{tw}^{\mathcal{L}}$ can be made symmetric monoidal if $\chi(\mathcal{L}) = 1$.*

Conversely, if we assume that $\mathrm{tw}^{\mathcal{L}}$ is symmetric monoidal, then we can make the following Euler characteristic computation:

$$1 = \mathrm{tw}^{\mathcal{L}}(1) = \mathrm{tw}^{\mathcal{L}}(\chi(\mathbb{1}(1))) = \chi(\mathrm{tw}^{\mathcal{L}}(\mathbb{1}(1))) = \chi(\mathcal{L}(1)) = \chi(\mathcal{L}).$$

Remark 5.27 Examining the failure of $\mathrm{tw}^{\mathcal{L}}$ to be symmetric monoidal, we find that in general the symmetric monoidal structure on $\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ is twisted by the Euler characteristic $\chi(\mathcal{L})$.

Example 5.28 If we take $\mathcal{C} = \mathrm{Sp}$ with its usual t-structure, then the $\mathbb{S}^1$-twisted category $\mathrm{Sp}^{\mathbf{Gr},\mathbb{S}^1,\heartsuit}$ is equivalent to the symmetric monoidal category of graded abelian groups with the Koszul sign convention, since $\chi(\mathbb{S}^1) = -1$.

On the other hand, $\mathrm{Sp}^{\mathbf{Gr},\mathbb{S}^2,\heartsuit}$ is symmetric monoidally equivalent to the category of graded abelian groups by Lemma 5.26, since $\chi(\mathbb{S}^2) = 1$.

Lemma 5.29 *Suppose that $\mathcal{C}$ is equivalent to the derived category of its heart $\mathcal{C}^\heartsuit$ as a symmetric monoidal category with compatible t-structure. Then there exists an equivalence of monoidal categories $\mathrm{tw}^{\mathcal{L}} : \mathcal{C}^{\mathbf{Gr}} \simeq \mathcal{C}^{\mathbf{Gr}}$ making the following diagram of monoidal functors commute*:

$$\begin{array}{ccc} \mathcal{C}^{\mathbf{Gr},\heartsuit} & \xrightarrow{\mathrm{tw}^{\mathcal{L}}} & \mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\mathbf{Gr}} & \xrightarrow{\mathrm{tw}^{\mathcal{L}}} & \mathcal{C}^{\mathbf{Gr}} \end{array}$$

If $\chi(\mathcal{L}) = 1$, then the above square naturally lifts to a diagram of symmetric monoidal functors. On objects, $\mathrm{tw}^{\mathcal{L}} : \mathcal{C}^{\mathbf{Gr}} \to \mathcal{C}^{\mathbf{Gr}}$ is given by $\{X_n\} \mapsto \{X_n \otimes \mathcal{L}^{\otimes n}\}$.

Proof The assumption implies that $\mathcal{C}^{\mathbf{Gr}}$, equipped with the usual t-structure, is equivalent as a symmetric monoidal category with compatible t-structure to $\mathcal{D}(\mathcal{C}^{\mathbf{Gr},\heartsuit})$. It also implies that $\mathcal{C}^{\mathbf{Gr}}$, when equipped with the $\mathcal{L}$-twisted t-structure, is equivalent as a symmetric monoidal category with compatible t-structure to $\mathcal{D}(\mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit})$.

It follows that the equivalence of monoidal 1-categories $\mathrm{tw}^{\mathcal{L}} : \mathcal{C}^{\mathbf{Gr},\heartsuit} \to \mathcal{C}^{\mathbf{Gr},\mathcal{L},\heartsuit}$ of Lemma 5.26 determines a compatible equivalence of monoidal categories $\mathrm{tw}^{\mathcal{L}} : \mathcal{C}^{\mathbf{Gr}} \to \mathcal{C}^{\mathbf{Gr}}$, via taking derived categories. If $\chi(\mathcal{L}) = 1$, these are equivalences of symmetric monoidal categories by Lemma 5.26. □

5.3 Twistings, II: The slice filtration

In this short subsection, we specialize the material from the previous subsection to the case of interest. This means we look at $\mathrm{Sp}_{C_2,i2}$ with Picard element $\mathbb{S}^\rho_2$. In order to connect this with $\mathrm{MU}_{\mathbb{R},2}$ and the material from Section 4, we relate the ρ-twisted t-structure to the regular slice filtration.

Example 5.30 If we take $\mathcal{C} = \mathrm{Sp}_{C_2}$, then the following Euler characteristic computations provide the three possible nontrivial twists on graded Mackey functors:

$$\chi(\mathbb{S}^\sigma) = 1 - [C_2], \quad \chi(\mathbb{S}^1) = -1, \quad \chi(\mathbb{S}^\rho) = [C_2] - 1.$$

To verify this, one uses the compatibility of the Euler characteristic with the symmetric monoidal functors Φ^e and Φ^{C_2}.

Example 5.31 Write $\underline{\mathrm{Ab}}^{\mathbf{Gr},\rho,\heartsuit}_{i2}$ for $\underline{\mathrm{Ab}}^{\mathbf{Gr},\Sigma^\rho\underline{\mathbb{Z}}_2,\heartsuit}_{i2}$. Then, since the image of $\chi(\mathbb{S}^\rho)$ in $\pi_0\underline{\mathbb{Z}}_2$ is 1, there is a symmetric monoidal equivalence

$$\mathrm{tw}^\rho : \underline{\mathrm{Ab}}^{\mathbf{Gr},\heartsuit}_{i2} \xrightarrow{\simeq} \underline{\mathrm{Ab}}^{\mathbf{Gr},\rho,\heartsuit}_{i2},$$

by Lemma 5.26. Applying Lemma 5.29, we can then upgrade this to a diagram of symmetric monoidal functors

$$\begin{array}{ccc} \underline{\mathrm{Ab}}_{i2}^{\mathbf{Gr},\heartsuit} & \xrightarrow{\mathrm{tw}^{\rho}} & \underline{\mathrm{Ab}}_{i2}^{\mathbf{Gr},\rho,\heartsuit} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Ab}}_{i2}^{\mathbf{Gr}} & \xrightarrow{\mathrm{tw}^{\rho}} & \underline{\mathrm{Ab}}_{i2}^{\mathbf{Gr}} \end{array}$$

Convention 5.32 From this point on, our category $\mathcal{C}$ will be one of $\mathrm{Sp}_{C_2,i2}$, $M(C_2)_{i2}$, $\underline{\mathrm{Ab}}_{i2}$ or Ab_{i2} with its usual t-structure, and the Picard element we work with will be $\mathbb{S}_2^{\rho}$, $\Sigma^{\rho}\underline{\pi}_0^{C_2}\mathbb{S}_2$, $\Sigma^{\rho}\underline{\mathbb{Z}}_2$ or $\Sigma^2\mathbb{Z}_2$, respectively. In order to reduce the notational burden, we denote the former three Picard objects by ρ and the final Picard object by 2 in superscripts. So, in the example of $\mathrm{Sp}_{C_2,i2}$, we will write $\tau_{\geq 0}^{\rho}$ for the connective cover, $\underline{\pi}_0^{\rho}$ for the 0^{th} homotopy object and $\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil},\rho,\heartsuit}$ for the heart of $\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}$.

This key result of this section is the following:

Lemma 5.33 *Let $E \in \mathrm{Sp}_{C_2,i2}$. Then*

$$\tau_{\geq 0}^{\rho}Y(E) \simeq (\cdots \to P_4E \to P_2E \to P_0E \to P_{-2}E \to P_{-4}E \to \cdots),$$

where Y is the functor that sends an object to the associated constant filtered object. If we further suppose that $\underline{\pi}_{n\rho-1}^{C_2}E = 0$ for all n, there is a natural equivalence

$$\mathbf{Gr}(\tau_{\geq 0}^{\rho}Y(E)) \simeq \underline{\pi}_0^{\rho}Y(E) \cong \{\Sigma^{n\rho}\underline{\pi}_{n\rho}^{C_2}E\}.$$

Finally, if $\underline{\pi}_{n\rho}E$ is a constant Mackey functor for all n, then $\underline{\pi}_0^{\rho}Y(E)$ factors through the full subcategory

$$\underline{\mathrm{Ab}}_{i2}^{\mathbf{Gr},\rho,\heartsuit} \subset M(C_2)_{i2}^{\mathbf{Gr},\rho,\heartsuit} \simeq \mathrm{Sp}_{C_2,i2}^{\mathbf{Fil},\rho,\heartsuit}.$$

Proof This lemma has three statements. For the first statement, it suffices to note the following:

(i) A C_2-spectrum is t-structure 0-connective if and only if it is regular slice 0-connective.

(ii) A C_2-spectrum E is regular slice $2n$-connective if and only if $E \otimes \mathbb{S}^{-n\rho}$ is regular slice 0-connective.

For the second statement, since $\mathbb{S}_2^{\rho} \geq 1$, we can use the final statement of Lemma 5.22 to obtain a canonical map

$$\mathbf{Gr}(\tau_{\geq 0}^{\rho}Y(E)) \to \underline{\pi}_0^{\rho}Y(E).$$

On the one hand, we have $\underline{\pi}_0^{\rho}Y(E) \cong \{\Sigma^{n\rho}\underline{\pi}_{n\rho}^{C_2}E\}$ by Lemma 5.22(3'). Since by assumption the odd slices of E vanish, we learn that the $(2n)^{\text{th}}$ slice of E is equivalent to $\Sigma^{n\rho}\underline{\pi}_{n\rho}^{C_2}E$ and that there are cofiber sequences

$$P_{2n+2}E \to P_{2n}E \to \Sigma^{n\rho}\underline{\pi}_{n\rho}^{C_2}E.$$

The desired equivalence now follows from the first statement.

The third statement now follows from the fact that any discrete C_2-Mackey functor which is constant admits a (necessarily unique) $\underline{\mathbb{Z}}$-module structure. □

5.4 Reductions

In this subsection, we express each side of the equivalence (2) of Theorem 5.1 as the category of modules over some commutative algebra in $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$. Almost everything in this subsection is a standard application of Lurie's theory of higher algebra. We have extracted the necessary material in Appendix A, where we present it in a digested form.

We begin with the left-hand side of (2).

Lemma 5.34 *The cofiber of* $\tau : \mathbb{S}_2^{0,0,-1} \to \mathbb{S}_2^{0,0,0}$ *is a commutative algebra in* $\mathcal{SH}(\mathbb{R})_{i2}$. *Furthermore, there is an equivalence of symmetric monoidal categories*

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2}; \mathbf{Gr}(R_\bullet)),$$

where $R_\bullet = \mathrm{Tot}^*(P_{2\bullet}\,\mathrm{MU}^{\otimes *+1}_{\mathbb{R},2})$.

Proof Recall that the shift map $\mathbb{S}_2(-1) \to \mathbb{S}_2$ in $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ is denoted by τ. Tensoring the equivalence of Example A.5 with $\mathrm{Sp}_{C_2,i2}$ and applying Lemma A.10, we see that there is an equivalence of presentably symmetric monoidal categories

$$\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; C\tau).$$

Under the symmetric monoidal functor $i^* : \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ constructed in Proposition 4.5, the commutative algebra $C\tau$ maps to $C\tau$. Therefore, using Lemma A.10, we have

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau) \simeq \mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2} \otimes_{\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}} \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}.$$

Using Theorem 4.3, there is an equivalence of presentably symmetric monoidal categories under $\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}$ between $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and $\mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; R_\bullet)$. Tensoring down along the associated graded ring map and using Lemma A.10, we obtain equivalences

$$\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2} \otimes_{\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}} \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \simeq \mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2} \otimes_{\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}} \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; R_\bullet) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2}; \mathbf{Gr}(R_\bullet)). \qquad \square$$

Now we proceed to the right-hand side of (2).

Lemma 5.35 *The Euler characteristic of* $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}} \in \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}$ *is equal to* 1. *As a consequence, Construction 5.24 provides a symmetric monoidal left adjoint*

$$p^* : \mathrm{Ab}^{\mathbf{Gr},\heartsuit} \to \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}$$

which sends $\mathbb{Z}(1)$ *to* $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$.

Proof Let L denote the Lazard ring. Then the flat cover $\mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fg}}$ determines a pullback map $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) \to \mathrm{Mod}_L$. This determines an injective pullback map $\pi_0\,\mathrm{End}(\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}) \to \pi_0\,\mathrm{End}(L)$. Since the pullback of $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$ is equivalent to L we learn that $\chi(\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}) = 1$. $\square$

Since p^* sends a family of compact dualizable objects (the powers of $\mathbb{Z}(1)$) to a family of compact dualizable generators (the powers of $\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$), we may apply Proposition A.4 to obtain the following lemma:

Lemma 5.36 *There is an equivalence of presentably symmetric monoidal categories*

$$\mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq \mathrm{Mod}(\mathrm{Ab}^{\mathbf{Gr}}; p_*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}).$$

Using Lemmas 5.36 and A.10, we obtain the following corollary:

Corollary 5.37 *There is an equivalence of presentably symmetric monoidal categories*

$$\underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq \mathrm{Mod}(\underline{\mathrm{Ab}}^{\mathbf{Gr}}_{i2}; \underline{p_*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}} \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}}_2).$$

5.5 The main lemma

In this subsection, we prove our main lemma, Lemma 5.40, and use it to prove Theorems 5.1 and 5.6. This lemma gives us an explicit formula for $\mathbf{Gr}(i_*(\Gamma_*(-)))$ on a restricted class of objects. Once we have this formula the remaining work is relatively easy. We begin with a definition and a couple of useful lemmas.

Definition 5.38
- Let $\mathrm{Sp}^{\mathrm{proj}}_{C_2,i2}$ denote the full subcategory of $E \in \mathrm{Sp}_{C_2,i2}$ for which $\mathrm{MU}_{\mathbb{R},2} \otimes E$ is a retract of a sum of pure suspensions of $\mathrm{MU}_{\mathbb{R},2}$ (a suspension $\Sigma^{a+b\sigma}\,\mathrm{MU}_{\mathbb{R},2}$ is said to be pure if $a = b$).
- Let $\mathrm{Sp}^{\mathrm{proj}}_{i2}$ denote the full subcategory of $E \in \mathrm{Sp}_{i2}$ for which $\mathrm{MU}_2 \otimes E$ is a retract of a sum of even suspensions of MU_2.

Since the underlying spectrum of $\mathrm{MU}_{\mathbb{R},2}$ is MU_2, the underlying spectrum of an object of $\mathrm{Sp}^{\mathrm{proj}}_{C_2,i2}$ is contained in $\mathrm{Sp}^{\mathrm{proj}}_{i2}$. Note that both $\mathrm{Sp}^{\mathrm{proj}}_{C_2,i2}$ and $\mathrm{Sp}^{\mathrm{proj}}_{i2}$ contain units and are closed under tensor product, so each inherits the structure of a symmetric monoidal category.

Lemma 5.39 *Suppose that $E \in \mathrm{Sp}^{\mathrm{proj}}_{C_2,i2}$. Then*

$$\underline{(\mathrm{MU}_{\mathbb{R},2})^{C_2}_{n\rho}(E)} \cong \underline{(\mathrm{MU}_2)_{2*}(E)}.$$

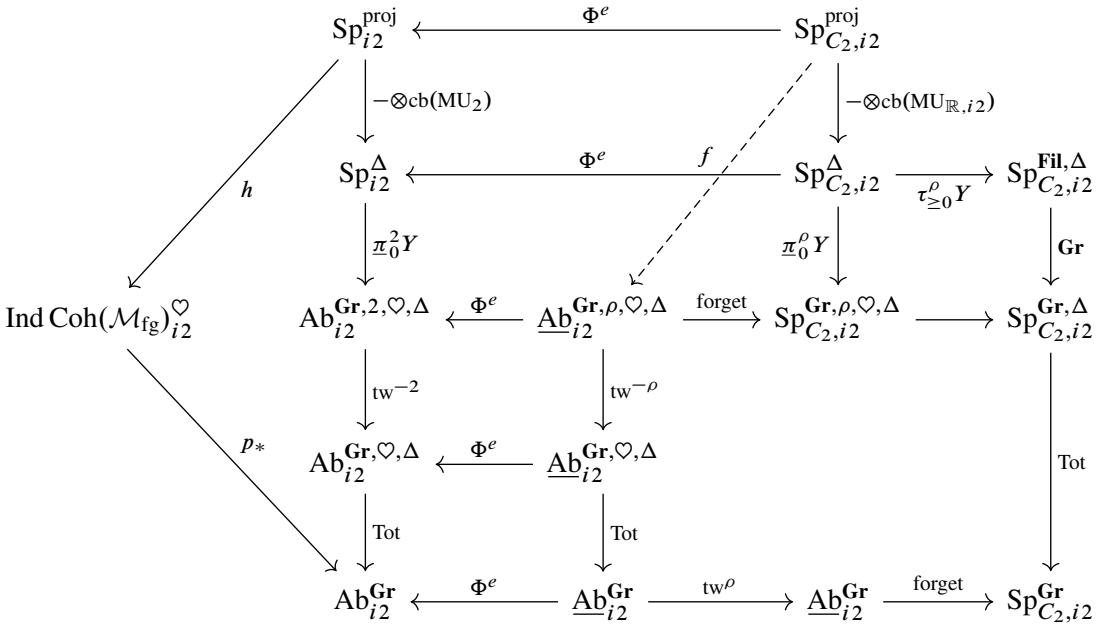
As a consequence, $\underline{(\mathrm{MU}_{\mathbb{R},2})^{C_2}_{n\rho}(E)}$ acquires the structure of a $\underline{\mathbb{Z}}_2$-module.

Proof By definition of $\mathrm{Sp}^{\mathrm{proj}}_{C_2,i2}$, it suffices to note that this is true for $E = \mathbb{S}_2$, which follows from [Hu and Kriz 2001, Theorem 2.28]. □

Lemma 5.40 *Let h denote the symmetric monoidal functor $h : \mathrm{Sp}^{\mathrm{proj}}_{i2} \to \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}_{i2}$ which sends a spectrum to its associated $((\mathrm{MU}_2)_*, (\mathrm{MU}_2)_*\,\mathrm{MU}_2)$-comodule. There is a commutative diagram of lax symmetric monoidal functors*

$$\begin{array}{ccccccccc}
\mathrm{Sp}^{\mathrm{proj}}_{C_2,i2} & & & \xrightarrow{\quad i_*\circ\Gamma_*\quad} & & & & & \mathrm{Sp}^{\mathbf{Fil}}_{C_2,i2} \\
\downarrow{\scriptstyle \Phi^e} & & & & & & & & \downarrow{\scriptstyle \mathbf{Gr}} \\
\mathrm{Sp}^{\mathrm{proj}}_{i2} & \xrightarrow{h} & \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})^{\heartsuit}_{i2} & \xrightarrow{p_*} & \mathrm{Ab}^{\mathbf{Gr}}_{i2} & \xrightarrow{\ -\ } & \underline{\mathrm{Ab}}^{\mathbf{Gr}}_{i2} & \xrightarrow{\mathrm{tw}^{\rho}} & \mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2}
\end{array}$$

Proof Our argument will rest on the existence of the following commuting diagram of lax symmetric monoidal functors:



The squares formed by the underlying functor, Φ^e, commute since underlying commutes with limits and colimits, the underlying of $\mathbb{S}^{\rho}_2$ is $\mathbb{S}^2_2$ and the underlying of $\mathrm{MU}_{\mathbb{R},2}$ is MU_2. In grid position $(4,3)$ we have implicitly made use of the equivalence $\mathrm{Sp}^{\mathbf{Fil},\rho,\heartsuit,\Delta}_{C_2} \simeq \mathrm{Sp}^{\mathbf{Gr},\rho,\heartsuit,\Delta}_{C_2}$ to identify the target of $\underline{\pi}^{\rho}_0 Y$ with the latter. The upper-right square commutes by Lemma 5.33. The dashed arrow, labeled f, is unique (and lax symmetric monoidal) if it exists since the forgetful functor is fully faithful. The dashed arrow then exists by Lemma 5.39. The large bottom-right square commutes because tw^{ρ} commutes with Tot. Finally, the commutativity of the left-hand triangle is a corollary of our ability to identify the E_2-page of the Adams–Novikov spectral sequence; see [Adams 1974].

Using the fact that $\mathbf{Gr}$ and Tot commute, we have equivalences

$$\mathbf{Gr} \circ i_* \circ \Gamma_* \simeq \mathrm{Tot}\,\mathbf{Gr}\big(\tau^{\rho}_{\geq 0} Y(- \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))\big) \simeq \mathrm{tw}^{\rho}(\mathrm{Tot}(\mathrm{tw}^{-\rho}(f(-)))).$$

By Lemma 5.39, the functor f (and its composition with the twist) lands in the full subcategory of the target which is spanned, in each grading and cosimplicial degree, by $\underline{\mathbb{Z}}_2$-modules for which the restriction map r is an equivalence. Thus, we have an equivalence of lax symmetric monoidal functors

$$\mathrm{tw}^{\rho}(\mathrm{Tot}(\mathrm{tw}^{-\rho}(f(-)))) \simeq \mathrm{tw}^{\rho}(\mathrm{Tot}(\underline{\Phi^e \mathrm{tw}^{-\rho}(f(-))})).$$

Since _ commutes with limits we then have further equivalences,

$$\begin{aligned}\mathrm{tw}^{\rho}(\mathrm{Tot}(\underline{\Phi^e \mathrm{tw}^{-\rho}(f(-))})) &\simeq \mathrm{tw}^{\rho}(\underline{\mathrm{Tot}(\Phi^e \mathrm{tw}^{-\rho}(f(-)))}) \\ &\simeq \mathrm{tw}^{\rho}(\underline{p_* h \Phi^e(-)}).\end{aligned}$$

□

Finally, we are able to prove our main theorems:

Proof of Theorems 5.1 and 5.6 (part 1) We have a chain of symmetric monoidal equivalences:

$$\begin{aligned}\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tilde{\tau}) &\xrightarrow[5.34]{\simeq} \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}_{C_2,i2}; \mathbf{Gr}(R_\bullet))\\ &\xrightarrow[5.40]{\simeq} \mathrm{Mod}(\underline{\mathrm{Ab}}^{\mathbf{Gr}}_{i2}; \mathrm{tw}^{\rho}(\underline{p_*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}}))\\ &\xrightarrow[\mathrm{tw}^{-\rho}]{\simeq} \mathrm{Mod}(\underline{\mathrm{Ab}}^{\mathbf{Gr}}_{i2}; \underline{p_*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}})\\ &\xrightarrow[5.37]{\simeq} \underline{\mathrm{Ab}}_{i2} \otimes_{\mathrm{Ab}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}).\end{aligned}$$

The claim about the $\mathrm{Sp}_{C_2,i2}$-algebra structure follows once we know $\mathrm{Sp}_{C_2,i2}$ acts through the ambient category at each step. The key points here are that the equivalence from Lemma 5.40 was induced by an equivalence of algebras, and that the twist functor is an $\underline{\mathrm{Ab}}_{i2}$-algebra map.

Using the $\mathrm{Sp}_{C_2,i2}$-linearity, the claim about Picard elements reduces to tracking $C\tilde{\tau}\otimes\mathbb{S}_2^{0,0,1}$ around the diagram. Under the first equivalence, $C\tilde{\tau}\otimes\mathbb{S}_2^{0,0,1}$ goes to $\mathbb{1}(1)$. The twist by $-\rho$ sends this to $\Sigma^{-\rho}\mathbb{1}(1)$. The final equivalence sends $\mathbb{1}(1)$ to $\underline{\mathbb{Z}}_2\otimes\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$. Altogether we learn that $C\tilde{\tau}\otimes\mathbb{S}_2^{0,0,1}$ is sent to $\Sigma^{-\rho}\underline{\mathbb{Z}}_2\otimes\omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$, as desired.

Now suppose that $X\in\mathrm{Sp}^{\mathrm{proj}}_{C_2}$. Then, using Lemma 5.40, we have

$$\mathbf{Gr}(i_*(\Gamma_* X)) \simeq \mathrm{tw}^{\rho}(\underline{p_* h\Phi^e(X)}).$$

Continuing along the sequence of equivalences above, we see that this object is sent to $\underline{\mathbb{Z}}_2\otimes h(\Phi^e(X))$, as desired. □

In order to calculate the homotopy groups of $C\tilde{\tau}$-modules, we begin with a simple lemma.

Lemma 5.41 *Given any $A\in\mathrm{Ab}$, we have*

$$\pi^{C_2}_{p+q\sigma}\underline{A} \cong \bigoplus_a \pi_{p-a}(A\otimes\pi_{a+q\sigma}\underline{\mathbb{Z}}).$$

Proof Consider the composite $\mathrm{Ab}\to\underline{\mathrm{Ab}}\to\mathrm{Ab}^{\mathbf{Gr}}$ given by $A\mapsto\underline{A}\mapsto\{\mathrm{Map}(\mathbb{S}^{q\sigma},\underline{A})\}$. This composite is colimit-preserving, hence is equivalent to $A\mapsto\{A\otimes\mathrm{Map}(\mathbb{S}^{q\sigma},\underline{\mathbb{Z}})\}$. Now, we have

$$\mathrm{Map}(\mathbb{S}^{q\sigma},\underline{\mathbb{Z}}) \simeq \bigoplus_a \Sigma^a\pi_{a+q\sigma}\underline{\mathbb{Z}}$$

because it is a $\mathbb{Z}$-module, and the result follows by applying π_p. □

Proof of Theorems 5.1 and 5.6 (part 2) We now turn to our assertions about homotopy groups. Tracing through the various equivalences, we find that for X which is $\mathrm{MU}_{\mathbb{R},2}$-projective

$$\pi^{\mathbb{R}}_{p,q,w}(C\tilde{\tau}\otimes X) \cong \pi^{C_2}_{(p-w)+(q-w)\sigma}((\underline{\mathbb{Z}}_2\otimes_{\mathbb{Z}_2} p_* h\Phi^e(X))_w).$$

We can commute taking the w^{th} component past the tensor product and then apply Lemma 5.41 and the fact that p_*h computes $\mathrm{Ext}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}$ to conclude that

$$\begin{aligned}\pi^{\mathbb{R}}_{p,q,w}(C\bar{\tau}\otimes X) &\cong \pi^{C_2}_{(p-w)+(q-w)\sigma}(\underline{\mathbb{Z}}_2\otimes_{\mathbb{Z}_2}(p_*h\Phi^e(X))_w)\\ &\cong \bigoplus_a \pi_{p-w-a}(\pi_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2\otimes_{\mathbb{Z}_2}\mathrm{Ext}^{*,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*,(\mathrm{MU}_2)_*\Phi^e(X)))\\ &\cong \bigoplus_a \pi_{p-w-a}(\mathrm{Ext}^{*,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*,\pi_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2\otimes_{\mathbb{Z}_2}(\mathrm{MU}_2)_*\Phi^e(X)))\\ &\cong \bigoplus_a \mathrm{Ext}^{a+w-p,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*,\pi_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2\otimes_{\mathbb{Z}_2}(\mathrm{MU}_2)_*\Phi^e(X)).\end{aligned}$$

Since $\pi^{C_2}_{a+q\sigma}\underline{\mathbb{Z}}_2$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_2$, and $(\mathrm{MU}_2)_*X$ is torsion-free (since it is a projective $(\mathrm{MU}_2)_*$-module), the tensor product inside the Ext can be taken in a 1-categorical sense. □

6 The Chow t-structure and $\nu_{\mathbb{R}}$

In this section, we discuss the Chow t-structure on $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and how it can be used to define an interesting lax symmetric monoidal functor

$$\nu_{\mathbb{R}} : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}.$$

In the context of $\mathrm{DM}(k)$, the Chow t-structure was first studied[21] by Bondarko [2010]. In the context of modules over the cofiber of τ in $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}$, it was studied by Gheorghe, Wang and Xu in [Gheorghe et al. 2021]. In the context of $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}$ it was studied by Pstrągowski [2023] and by Gheorghe, Isaksen, Krause and Ricka [Gheorghe et al. 2022]. The Chow t-structure on $\mathcal{SH}(k)$ is the subject of forthcoming work of Bachmann, Kong, Wang and Xu, who we thank for carefully explaining their results to us. Among their results is a determination of the heart of the Chow t-structure in full generality. Since this work has not yet appeared,[22] we determine the heart when restricted to $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ below.

In Section 6.1, we define the Chow t-structure on $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, determine its heart and give a formula for the t-structure homotopy objects. In Section 6.2, we use the Chow t-structure to define the functor

$$\nu_{\mathbb{R}} : \mathrm{Sp}_{C_2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}.$$

We prove several basic properties of this functor, compare it with the functor Γ_* defined in Section 4, and show that several Artin–Tate $\mathbb{R}$-motivic spectra may be recovered from their Betti realizations via $\nu_{\mathbb{R}}$.

[21] The publication history of this definition is somewhat complicated. In the published version of [Bondarko 2010], the Chow t-structure is only conjectured, not shown, to exist. However, in a later version of [Bondarko 2010] which appeared on the arXiv, Bondarko gave a construction of the Chow t-structure.

[22] In the time since the first version of this work was posted, the theorems of Bachmann, Kong, Wang and Xu appeared in [Bachmann et al. 2022].

6.1 The Chow t-structure

In this section, we define the Chow t-structure on Artin–Tate $\mathbb{R}$-motivic spectra, compute its heart, and describe the t-structure homotopy objects, which we denote by $\pi_k^{\mathrm{C}-\heartsuit}$.

Definition 6.1 By [Lurie 2017, Proposition 1.2.1.16], we can construct a t-structure on $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ with

- connective part $(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}})_{\geq 0}^{\mathrm{C}}$ generated under colimits and extensions by
$$\{\mathbb{S}_2^{n+k_1,n+k_2,n} \mid k_1 \geq 0 \text{ and } k_1+k_2 \geq 0\},$$
- (-1)-coconnective part $(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}})_{<0}^{\mathrm{C}}$ spanned by those $X \in \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ with $\pi_{n+k_1,n+k_2,n}^{\mathbb{R}} X = 0$ for all $n, k_1, k_2 \in \mathbb{Z}$ satisfying $k_1 \geq 0$ and $k_1 + k_2 \geq 0$.

We call this t-structure the *Chow t-structure* on $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$. Moreover, let $\tau_{\geq 0}^{\mathrm{C}} : \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \to (\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}})_{\geq 0}^{\mathrm{C}}$ denote the connective cover with respect to the Chow t-structure, and denote the homotopy object functors by $\pi_n^{\mathrm{C}-\heartsuit} : \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}$.

The following result is immediate from the definition:

Proposition 6.2 *The Chow t-structure on $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ is compatible with the symmetric monoidal structure, right complete and compatible with filtered colimits. Moreover, $\Sigma^{n,n,n}$ restricts to an equivalence*
$$\Sigma^{n,n,n} : (\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}})_{\geq 0}^{\mathrm{C}} \to (\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}})_{\geq 0}^{\mathrm{C}}.$$

The main two theorems that we prove in this section are as follows:

Theorem 6.3 *Let $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}$ denote the heart of $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ with respect to the Chow t-structure. Then there is a symmetric monoidal equivalence of categories*
$$\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit} \simeq \mathrm{Comod}(\underline{\mathrm{Ab}}_{i2}^{\heartsuit}; \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}),$$
where $\mathrm{Comod}(\underline{\mathrm{Ab}}^{\heartsuit}; \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2})$ denotes the category of evenly graded $(\underline{(\mathrm{MU}_2)_*}, \underline{(\mathrm{MU}_2)_*\,\mathrm{MU}_2})$-comodules in C_2-Mackey functors.

Theorem 6.4 *Given any $X \in \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ and $n, k \in \mathbb{Z}$, there is an isomorphism*
$$(\pi_k^{\mathrm{C}-\heartsuit} X)_n \cong \underline{(\mathrm{MGL}_2)}_{n+k,n,n}(X).$$

Recollection 6.5 In Theorem 5.1, we constructed a symmetric monoidal equivalence
$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\overline{\mathbb{R}}) \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}).$$
Moreover, in Lemma 5.16 we equipped $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ with a t-structure whose heart is identified with $\mathrm{Comod}(\underline{\mathrm{Ab}}^{\heartsuit}; \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2})$. We call the induced t-structure on $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\overline{\mathbb{R}})$ the *tensor t-structure* and write $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\overline{\mathbb{R}})^{\otimes-\heartsuit}$ for the heart.

To prove Theorem 6.3, it will suffice to prove the following:

Proposition 6.6 *The tensor t-structure on $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau)$ is induced by the Chow t-structure on $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$: a $C\tau$-module X is tensor (co)connective if and only if its underlying Artin–Tate $\mathbb{R}$-motivic spectrum is Chow (co)connective.*

Moreover, the induced symmetric monoidal functor

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau)^{\otimes -\heartsuit} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2}$$

is an equivalence of categories.

First, we need a lemma:

Lemma 6.7 *We have $C\tau \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2}$ and the unit map $\mathbb{S}^{0,0,0}_2 \to C\tau$ induces an equivalence $\pi_0^{\mathrm{C}-\heartsuit}\mathbb{S}^{0,0,0}_2 \simeq C\tau$.*

Proof First, we note that $\mathbb{S}^{0,0,-1}_2 \geq 1$ in the Chow t-structure. Since

$$\mathbb{S}^{0,0,-1}_2 \simeq \Sigma\, \mathbb{S}^{0,1,0}_2 \otimes \mathbb{S}^{-1,-1,-1}_2,$$

it suffices to show that $\mathbb{S}^{0,1,0}_2 \geq 0$. This follows immediately from the cofiber sequence $\Sigma^\infty_+ \mathrm{Spec}(\mathbb{C})_2 \to \mathbb{S}^{0,0,0}_2 \to \mathbb{S}^{0,1,0}_2$.

Combining this fact with the cofiber sequence $\mathbb{S}^{0,0,-1}_2 \xrightarrow{\tau} \mathbb{S}^{0,0,0}_2 \to C\tau$, we find that $C\tau \geq 0$ and that $\mathbb{S}^{0,0,0}_2 \to C\tau$ induces an equivalence on $\pi_0^{\mathrm{C}-\heartsuit}$.

To conclude, it suffices to show that $C\tau \leq 0$. By definition, we must show that $\pi^{\mathbb{R}}_{n+k_1,n+k_2,n} C\tau = 0$ for all n, k_1, k_2 satisfying $k_1 > 0$ and $k_1 + k_2 > 0$. This follows directly from Theorem 10.1(ii). □

We note down the following corollary for later use:

Corollary 6.8 *Suppose that $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is a filtered colimit of Artin–Tate $\mathbb{R}$-motivic spectra that each admit a finite cell structure with all cells of the form $\mathbb{S}^{n,n,n}_2$. Then $X \otimes C\tau \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2}$ and $X \to X \otimes C\tau$ induces an equivalence $\pi_0^{\mathrm{C}-\heartsuit} X \to X \otimes C\tau$.*

Proof It is clear that the collection of X which satisfy the conclusions of the corollary is closed under filtered colimits and extensions, so it suffices to assume that $X \simeq \mathbb{S}^{n,n,n}_2$ Since $\Sigma^{n,n,n}$ is an automorphism of the Chow t-structure, we may reduce to the case of $X \simeq \mathbb{S}^{0,0,0}_2$, which is precisely Lemma 6.7. □

Proof of Proposition 6.6 We first show that the tensor t-structure is induced by the Chow t-structure. We begin by showing that the connective part of the tensor t-structure on $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\tau)$ is generated under colimits and extensions by

$$\{\Sigma^{n+k_1,n+k_2,n} C\tau \mid k_1 \geq 0 \text{ and } k_1 + k_2 \geq 0\}.$$

The t-structure on $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$ has connective part generated under colimits and extensions by $\omega^{\otimes n}_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}$ for $n \in \mathbb{Z}$. On the other hand, the t-structure on $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2)$ has connective part generated under colimits and extensions by $\Sigma^{k_1+k_2\sigma}\underline{\mathbb{Z}}_2$, where $k_1 \geq 0$ and $k_1 + k_2 \geq 0$.

By definition, it follows that the connective part of the induced t-structure on

$$\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})$$

is generated under colimits and extensions by the tensor products $\Sigma^{k_1+k_2\sigma}\underline{\mathbb{Z}}_2 \otimes \omega_{\mathbb{G}/\mathcal{M}_{\mathrm{fg}}}^{\otimes n}$, where $k_1 \geq 0$ and $k_1+k_2 \geq 0$. Under the equivalence of Theorem 5.1, these correspond to $\Sigma^{n+k_1,n+k_2,n}C\text{ত}$, as desired.

It follows directly from the above identification of the tensor connective category that a $C\text{ত}$-module is tensor coconnective if and only if it is Chow coconnective. By Chow connectivity of $C\text{ত}$, which follows from Lemma 6.7, it also follows that a $C\text{ত}$-module is Chow connective if it is tensor connective. On the other hand, if X is Chow connective, then $X \otimes C\text{ত}$ is clearly tensor connective. Using the equivalence $X \otimes C\text{ত} \simeq X \oplus \Sigma^{1,0,-1}X$ (since X is a $C\text{ত}$-module), we see that X itself is also tensor connective.

As a consequence of the above, we obtain a symmetric monoidal functor

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\text{ত})^{\otimes-\heartsuit} \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}.$$

Since the tensor t-structure is induced by the Chow t-structure, the fact that $C\text{ত}$ lies in the Chow heart from Lemma 6.7 implies that the above functor factors through a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\text{ত})^{\otimes-\heartsuit} \simeq \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}; C\text{ত}).$$

Finally, the equivalence $\pi_0^{\mathrm{C}-\heartsuit}\mathbb{S}_2^{0,0,0} \simeq C\text{ত}$ of Lemma 6.7 implies that $C\text{ত}$ is the unit of $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}$, so that the forgetful functor

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}; C\text{ত}) \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT},\mathrm{C}-\heartsuit}$$

is a symmetric monoidal equivalence. □

We now move on to the proof of Theorem 6.4. First we need to recall some basic facts about cell structures on finite approximations of MGL.

Definition 6.9 Let $\mathrm{Gr}_k(\mathbb{A}^{n+k})$ denote the Grassmannian scheme of k-planes in $\mathbb{A}^{n+k}$, and let γ denote the tautological k-dimensional vector bundle over $\mathrm{Gr}_k(\mathbb{A}^{n+k})$.

We let $\mathrm{MGL}(n,k)$ denote the Thom spectrum of the virtual bundle $\gamma - k\cdot\mathrm{triv}$. Then there is a canonical map $\mathrm{MGL}(n_1,k_1) \to \mathrm{MGL}(n_2,k_2)$ whenever $n_2 \geq n_1$ and $k_2 \geq k_1$, and we have $\varinjlim \mathrm{MGL}(n,n) \simeq \mathrm{MGL}$.

Proposition 6.10 [Dugger and Isaksen 2005] *The $\mathbb{R}$-motivic spectrum $\mathrm{MGL}(n,k)$ admits a finite cell structure with cells of the form $\mathbb{S}^{m,m,m}$.*

As a consequence, $\mathrm{MGL}(n,k)_2$, its Spanier–Whitehead dual $\mathbb{D}(\mathrm{MGL}(n,k)_2)$, and MGL_2 are all Chow connective.

Proof The cell structure is a consequence of [Dugger and Isaksen 2005]. It follows from Theorem 8.1 and Proposition 8.2(i) that there are equivalences

$$\mathrm{MGL}(n,k) \otimes \mathbb{S}_2 \simeq \mathrm{MGL}(n,k)_2, \quad \mathbb{D}(\mathrm{MGL}(n,k)) \otimes \mathbb{S}_2 \simeq \mathbb{D}(\mathrm{MGL}(n,k)_2), \quad \mathrm{MGL} \otimes \mathbb{S}_2 \simeq \mathrm{MGL}_2.$$

Therefore these spectra continue to have the same cell structure after 2-completion, from which the statements about Chow connectivity follow immediately. □

Proposition 6.11 *The map $\mathrm{MGL}_2 \to \mathrm{MGL}_2 \otimes C\text{ত}$ induces an equivalence $\pi_0^{\mathrm{C}-\heartsuit}\mathrm{MGL}_2 \simeq \mathrm{MGL}_2 \otimes C\text{ত}$, and the corresponding $\underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$-comodule is $\underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$.*

Proof The first part of the statement follows directly from Corollary 6.8 and Proposition 6.10. On the other hand, Corollary 4.18 states that $\mathrm{MGL}_2 \simeq \Gamma_*(\mathrm{MU}_{\mathbb{R},2})$, so that Theorem 5.6 and Proposition 6.6 together imply that $\mathrm{MGL}_2 \otimes C\bar{\tau}$ corresponds to $\underline{\pi}^{C_2}_{*\rho}(\mathrm{MU}_{\mathbb{R},2} \otimes \mathrm{MU}_{\mathbb{R},2}) \cong \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$. This last isomorphism is a consequence of Lemma 5.39. □

Notation 6.12 We let $\otimes^{\heartsuit}$ denote the symmetric monoidal structure on

$$\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2} \simeq \mathrm{Comod}(\underline{\mathrm{Ab}}^{\heartsuit}_2; \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}).$$

Given a graded $\underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$-comodule M_*, we let $M[n]_*$ denote the shift with $M[n]_{k+n} = M_k$.

Lemma 6.13 *Let M_* be a graded $\underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$-comodule in C_2-Mackey functors, viewed as a element of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2}$. Then $M_* \otimes \mathrm{MGL}_2$ lies in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},\mathrm{C}-\heartsuit}_{i2}$ and corresponds to $M_* \otimes^{\heartsuit} \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}$.*

Proof It follows from Proposition 6.10 that $M_* \otimes \mathrm{MGL}_2$ is Chow connective. As a consequence, Proposition 6.11 implies that there are equivalences

$$\pi_0^{\mathrm{C}-\heartsuit}(M_* \otimes \mathrm{MGL}_2) \simeq M_* \otimes^{\heartsuit} \pi_0^{\mathrm{C}-\heartsuit}\mathrm{MGL}_2 \simeq M_* \otimes^{\heartsuit} \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}.$$

It therefore suffices to show that $M_* \otimes \mathrm{MGL}_2$ is Chow coconnective. For this, we compute for $n, k_1, k_2 \in \mathbb{Z}$ with $k_1 > 0$ and $k_1 + k_2 > 0$:

$$\begin{aligned}
\pi^{\mathbb{R}}_{n+k_1,n+k_2,n}(M_* \otimes \mathrm{MGL}_2) &\cong [\mathbb{S}_2^{n+k_1,n+k_2,n}, M_* \otimes \mathrm{MGL}_2]_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}} \\
&\cong \varinjlim [\mathbb{S}_2^{n+k_1,n+k_2,n}, M_* \otimes \mathrm{MGL}(m,m)_2]_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}} \\
&\cong \varinjlim [\mathbb{S}_2^{n+k_1,n+k_2,n} \otimes \mathbb{D}(\mathrm{MGL}(m,m)_2), M_*]_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}} \\
&\cong 0,
\end{aligned}$$

since $\mathbb{D}(\mathrm{MGL}(m,m)_2)$ is Chow connective by Proposition 6.10. It follows that $M_* \otimes \mathrm{MGL}_2$ is Chow coconnective, as desired. □

Lemma 6.14 *Suppose that X is Chow connective. Then the map $X \otimes \mathrm{MGL}_2 \to \pi_0^{\mathrm{C}-\heartsuit}(X) \otimes \mathrm{MGL}_2$ induces an equivalence on $\underline{\pi}^{\mathbb{R}}_{n-k_1,n-k_2,n}$ for all $n, k_1, k_2 \in \mathbb{Z}$ with $k_1 \geq 0$ and $k_1 + k_2 \geq 0$.*

Proof It suffices to show that $\underline{\pi}^{\mathbb{R}}_{n-k_1,n-k_2,n} Y \otimes \mathrm{MGL}_2 = 0$ for all Chow 1-connective Y. Since this condition is closed under filtered colimits, suspensions and extensions, it is closed under all colimits and extensions. It therefore suffices to assume that $Y = \mathbb{S}_2^{m+a_1,m+a_2,m}$ for $m, a_1, a_2 \in \mathbb{Z}$ satisfying $a_1 > 0$ and $a_1 + a_2 > 0$. In other words, we must show that $\underline{\pi}^{\mathbb{R}}_{n-k_1,n-k_2,n}\mathrm{MGL}_2$ for all $n, k_1, k_2 \in \mathbb{Z}$ now satisfying $k_1 > 0$ and $k_1 + k_2 > 0$.

By Proposition 4.14, there is an isomorphism

$$\underline{\pi}^{\mathbb{R}}_{n-k_1,n-k_2,n}\mathrm{MGL}_2 \cong \underline{\pi}^{C_2}_{n-k_1+(n-k_2)\sigma} P_{2n}\,\mathrm{MU}_{\mathbb{R},2}\,.$$

This is zero because $\Phi^e(\mathbb{S}_2^{n-k_1+(n-k_2)\sigma}) \simeq \mathbb{S}_2^{2n-k_1-k_2}$ is of dimension $< 2n$ and $\Phi^{C_2}(\mathbb{S}_2^{n-k_1+(n-k_2)\sigma}) \simeq \mathbb{S}_2^{n-k_1}$ is of dimension $< n$. □

Proof of Theorem 6.4 It suffices to prove this when $k = 0$, from which we may take topological (de)suspensions to obtain the result in general. For $k = 0$, we have

$$\begin{aligned}
\pi_0^{\mathrm{C}-\heartsuit}(X)_n &\cong \underline{\mathrm{Hom}}_{\underline{(\mathrm{MU}_2)_{2*}}\text{-mod}}(\underline{(\mathrm{MU}_2)_{2*}}[n], \pi_0^{\mathrm{C}-\heartsuit}(X)) \\
&\cong \underline{\mathrm{Hom}}_{\underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}\text{-comod}}(\underline{(\mathrm{MU}_2)_{2*}}[n], \pi_0^{\mathrm{C}-\heartsuit}(X) \otimes^{\heartsuit} \underline{(\mathrm{MU}_2)_{2*}\,\mathrm{MU}_2}) \\
&\cong \underline{\pi}_0^{C_2}\,\mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(\mathbb{S}_2^{n,n,n}, \pi_0^{\mathrm{C}-\heartsuit}(X) \otimes \mathrm{MGL}_2) \\
&\cong \underline{\pi}_0^{C_2}\,\mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(\mathbb{S}_2^{n,n,n}, X \otimes \mathrm{MGL}_2) \\
&\cong \underline{(\mathrm{MGL}_2)}^{\mathbb{R}}_{n,n,n}(X),
\end{aligned}$$

where the third and fourth isomorphisms follow from Lemmas 6.13 and 6.14, respectively. □

Finally, we record the following proposition for use in Section 6.2.

Proposition 6.15 *An Artin–Tate $\mathbb{R}$-motivic spectrum $X \in \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ is left complete with respect to the Chow t-structure if and only if it is MGL_2-local.*

The proof will make use of the following lemma:

Lemma 6.16 *The functors $X \mapsto \underline{\pi}^{\mathbb{R}}_{n+k,n,n} X$ are jointly conservative on $\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$.*

Proof Since the functors $X \mapsto \mathrm{Map}_{\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}}(\mathbb{S}_2^{n,n,n}, X)$ are jointly conservative, it suffices to note that the functors $Y \mapsto \underline{\pi}_k^{C_2} Y$ are jointly conservative on Sp_{C_2}. □

Proof of Proposition 6.15 Since the Chow t-structure is right complete, it is equivalent to show that $X \otimes \mathrm{MGL}_2 \simeq 0$ if and only if $\pi_n^{\mathrm{C}-\heartsuit} X = 0$ for all n. This is an immediate consequence of Theorem 6.4 and Lemma 6.16. □

6.2 The functor $\nu_{\mathbb{R}}$

Definition 6.17 We define a lax symmetric monoidal functor

$$\nu_{\mathbb{R}} : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$$

to be the composite

$$\mathrm{Sp}_{C_2,i2} \xrightarrow{\mathrm{Be}^{-1}} \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; \mathbb{S}_2[\mathfrak{r}^{-1}]) \xrightarrow{\tau^{\mathrm{C}}_{\geq 0}} \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}.$$

One of the main results of this section is the following theorem, which summarizes the basic properties of $\nu_{\mathbb{R}}$:

Theorem 6.18 *Let $X, Y, Z \in \mathrm{Sp}_{C_2,i2}$. The lax symmetric monoidal functor $\nu_{\mathbb{R}} : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}$ satisfies the following properties*:

(i) *There is a natural equivalence $\mathrm{Be}(\nu_{\mathbb{R}}(X)) \simeq X$.*

(ii) *The functor $X \mapsto \nu_{\mathbb{R}}(X)$ commutes with filtered colimits.*

(iii) *Given any $k \in \mathbb{Z}$, there is a natural equivalence $\nu_{\mathbb{R}}(\Sigma^{k\rho} X) \simeq \Sigma^{k,k,k} \nu_{\mathbb{R}}(X)$.*

(iv) *Suppose that $X \to Y \to Z$ is a cofiber sequence. Then*

$$\nu_{\mathbb{R}}(X) \to \nu_{\mathbb{R}}(Y) \to \nu_{\mathbb{R}}(Z)$$

is a cofiber sequence if and only if $\underline{\pi}^{C_2}_{n\rho-1}(X \otimes \mathrm{MU}_{\mathbb{R},2}) \to \underline{\pi}^{C_2}_{n\rho-1}(Y \otimes \mathrm{MU}_{\mathbb{R},2})$ is a monomorphism for all n.

(v) *Suppose that X is a filtered colimit of C_2-spectra that admit finite cell structures with all cells of the form $\mathbb{S}^{n\rho}_2$ for $n \in \mathbb{Z}$. Then, for all Y, the natural map*

$$\nu_{\mathbb{R}}(X) \otimes \nu_{\mathbb{R}}(Y) \to \nu_{\mathbb{R}}(X \otimes Y)$$

is an equivalence.

(vi) *The C_2-spectrum X is $\mathrm{MU}_{\mathbb{R},2}$-local if and only if $\nu_{\mathbb{R}}(X)$ is MGL_2-local.*

We also compare $\nu_{\mathbb{R}}$ to the functor $\Gamma_* : \mathrm{Sp}_{C_2,i2} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ constructed in Construction 4.4.

Theorem 6.19 *For $X \in \mathrm{Sp}_{C_2,i2}$, there is a natural equivalence*

$$\Gamma_*(X) \simeq \nu_{\mathbb{R}}(X)^{\wedge}_{\mathrm{MGL}_2}.$$

As evidence that the functor $\nu_{\mathbb{R}}$ is a good way to produce Artin–Tate $\mathbb{R}$-motivic spectra, we prove that several $\mathbb{R}$-motivic spectra of interest may be recovered from their Betti realizations via $\nu_{\mathbb{R}}$:

Theorem 6.20 *There are natural equivalences of commutative rings in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$:*

$$\mathrm{MGL}_2 \simeq \nu_{\mathbb{R}}(\mathrm{MU}_{\mathbb{R},2}), \quad \mathrm{M}\mathbb{F}_2 \simeq \nu_{\mathbb{R}}(\underline{\mathbb{F}}_2), \quad \mathrm{M}\mathbb{Z}_2 \simeq \nu_{\mathbb{R}}(\underline{\mathbb{Z}}_2), \quad \mathrm{kgl}_2 \simeq \nu_{\mathbb{R}}(\mathrm{ku}_{\mathbb{R},2}), \quad \mathrm{kq}_2 \simeq \nu_{\mathbb{R}}(\mathrm{ko}_{C_2,2}).$$

Remark 6.21 In light of Theorem 6.20, it would be reasonable to define an $\mathbb{R}$-motivic spectrum of motivic modular forms as $\nu_{\mathbb{R}}(\mathrm{tmf}_{C_2,2})$, assuming that one had a suitable C_2-equivariant spectrum of connective topological modular forms $\mathrm{tmf}_{C_2,2}$. Since no such equivariant spectrum has yet been constructed, we leave this to future work.

Remark 6.22 Since all of the Artin–Tate $\mathbb{R}$-motivic spectra in question are MGL_2-complete, in light of Theorem 6.19 we may replace $\nu_{\mathbb{R}}$ in Theorem 6.20 by Γ_*.

We now begin with the proof of Theorem 6.18.

Proof of Theorem 6.18 (i) Equivalently, we are to show that $\nu_{\mathbb{R}}(X)[\tau^{-1}] \simeq \mathrm{Be}^{-1}(X)$. Using the cofiber sequence $\nu_{\mathbb{R}}(X) \to \mathrm{Be}^{-1}(X) \to \tau^{\mathrm{C}}_{\leq -1}(\mathrm{Be}^{-1}(X))$, we find that the cofiber of

$$\nu_{\mathbb{R}}(X)[\tau^{-1}] \to \mathrm{Be}^{-1}(X)[\tau^{-1}] \simeq \mathrm{Be}^{-1}(X)$$

may be expressed as $\varinjlim \Sigma^{0,0,m}\tau^{\mathrm{C}}_{\leq -1}(\mathrm{Be}^{-1}(X))$.

Now, we know that

$$\pi^{\mathbb{R}}_{n+k_1,n+k_2,n}(\tau^{\mathrm{C}}_{\leq -1}(\mathrm{Be}^{-1}(X))) \cong 0$$

for all $n, k_1, k_2 \in \mathbb{Z}$ with $k_1 > 0$ and $k_1 + k_2 > 0$. It follows that

$$\pi^{\mathbb{R}}_{n+k_1,n+k_2,n}(\Sigma^{0,0,m}\tau^{\mathrm{C}}_{\leq -1}(\mathrm{Be}^{-1}(X))) \cong 0$$

for all $n, k_1, k_2 \in \mathbb{Z}$ with $k_1 > -m$ and $k_1 + k_2 > -2m$. This implies that $\varinjlim \Sigma^{0,0,m} \tau^{\mathrm{C}}_{\leq -1}(\mathrm{Be}^{-1}(X))$ is null, as desired.

(ii) This is immediate from the fact that the Chow t-structure is compatible with filtered colimits.

(iii) It is clear that there is a natural equivalence $\mathrm{Be}^{-1}(\Sigma^{k\rho} X) \simeq \Sigma^{k,k,k} \mathrm{Be}^{-1}(X)$. Moreover, since $\Sigma^{m,m,m}$ induces an automorphism of the Chow t-structure, we find further that $\tau^{\mathrm{C}}_{\geq 0}$ commutes with $\Sigma^{m,m,m}$. Combining these facts, we obtain the desired result.

(iv) It is clear that $X \to Y \to Z$ is a cofiber sequence if and only if $\mathrm{Be}^{-1}(X) \to \mathrm{Be}^{-1}(Y) \to \mathrm{Be}^{-1}(Z)$ is. Applying a general fact about t-structures, we see that

$$\tau^{\mathrm{C}}_{\geq 0}(\mathrm{Be}^{-1}(X)) \to \tau^{\mathrm{C}}_{\geq 0}(\mathrm{Be}^{-1}(Y)) \to \tau^{\mathrm{C}}_{\geq 0}(\mathrm{Be}^{-1}(Z))$$

is a cofiber sequence if and only if $\pi^{\mathrm{C}-\heartsuit}_{-1}(\mathrm{Be}^{-1}(X)) \to \pi^{\mathrm{C}-\heartsuit}_{-1}(\mathrm{Be}^{-1}(Y))$ is a monomorphism. The result then follows from Theorem 6.4 and the isomorphism $\underline{\pi}^{\mathbb{R}}_{p,q,w}(\mathrm{MGL}_2 \otimes \mathrm{Be}^{-1}(X)) \cong \underline{\pi}^{C_2}_{p+q\sigma}(\mathrm{MU}_{\mathbb{R},2} \otimes X)$.

(v) By compatibility with filtered colimits, we may assume that X admits a finite cell structure with all cells of the form $\mathbb{S}_2^{n\rho}$. Let us write

$$\mathbb{S}_2^{n_1\rho} \simeq X_1 \to X_2 \to \cdots \to X_k = X$$

with $X_i / X_{i-1} \simeq \mathbb{S}_2^{n_i \rho}$. Then it is easy to prove that $\underline{\pi}^{C_2}_{n\rho-1}(X_i \otimes \mathrm{MU}_{\mathbb{R},2}) = 0$ for all n and i by induction, so that by (iii) and (iv) there are cofiber sequences

$$\nu_{\mathbb{R}}(X_{i-1}) \to \nu_{\mathbb{R}}(X_i) \to \mathbb{S}_2^{n_i, n_i, n_i}\,.$$

It follows that $\mathbb{D}(\nu_{\mathbb{R}}(X))$ is Chow connective. To show that the natural map $\nu_{\mathbb{R}}(X) \otimes \nu_{\mathbb{R}}(Y) \to \nu_{\mathbb{R}}(X \otimes Y)$ is an equivalence, it suffices to show that it induces an equivalence on $\Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z, -)$ for all Chow connective Z. For such Z, we have

$$\begin{aligned}
\Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z, \nu_{\mathbb{R}}(X) \otimes \nu_{\mathbb{R}}(Y)) &\simeq \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z \otimes \mathbb{D}(\nu_{\mathbb{R}}(X)), \nu_{\mathbb{R}}(Y)) \\
&\simeq \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z \otimes \mathbb{D}(\nu_{\mathbb{R}}(X)), \mathrm{Be}^{-1}(Y)) \\
&\simeq \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z, \nu_{\mathbb{R}}(X) \otimes \mathrm{Be}^{-1}(Y)) \\
&\simeq \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z, \mathrm{Be}^{-1}(X) \otimes \mathrm{Be}^{-1}(Y)) \\
&\simeq \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Z, \nu_{\mathbb{R}}(X \otimes Y)),
\end{aligned}$$

as desired. We have used Chow connectivity of $\mathbb{D}(\nu_{\mathbb{R}}(X))$ in the second equivalence and (i) in the fourth.

(vi) It is clear that X is $\mathrm{MU}_{\mathbb{R},2}$-local if and only if $\mathrm{Be}^{-1}(X)$ is MGL_2-local. By Proposition 6.15, it suffices to show that $\mathrm{Be}^{-1}(X)$ is left complete with respect to the Chow t-structure if and only if $\nu_{\mathbb{R}}(X) = \tau^{\mathrm{C}}_{\geq 0}(\mathrm{Be}^{-1}(X))$ is. This follows from the fact that left completeness is invariant under taking connective cover. □

We now move on to the proof of Theorem 6.19. We begin by studying the interaction of the Chow t-structure with MGL_2.

Proposition 6.23 *Let $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$. Then the following statements hold*:

(i) *If X is Chow connective, then*

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X)$$

is regular slice $2n$-connective for all n.

(ii) *There is a natural equivalence*

$$\mathrm{MGL}_2 \otimes (\tau^{\mathrm{C}}_{\geq 0} X) \simeq \tau^{\mathrm{C}}_{\geq 0}(\mathrm{MGL}_2 \otimes X).$$

(iii) *The natural map*

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \tau^{\mathrm{C}}_{\geq 0} X) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X)$$

induced by the counit $\tau^{\mathrm{C}}_{\geq 0} X \to X$ is equivalent to the canonical map

$$P_{2n} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X).$$

Proof We begin by proving (i). The full subcategory of X for which $\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X)$ is regular slice $2n$-connective is closed under colimits and extensions, so it suffices to prove this for $X = \mathbb{S}_2^{m+k_1,m+k_2,m}$ for all $m, k_1, k_2 \in \mathbb{Z}$ where $k_1 \geq 0$ and $k_1 + k_2 \geq 0$.

Now, we have

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \mathbb{S}_2^{m+k_1,m+k_2,m}) \simeq \Sigma^{m+k_1+(m+k_2)\sigma} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n-m}, \mathrm{MGL}_2).$$

Since $\Sigma^{m+k_1+(m+k_2)\sigma}$ of a regular slice $2(n-m)$-connective C_2-spectrum is regular slice $2n$-connective, this reduces us to the case where $X = \mathbb{S}_2^{0,0,0}$, where this follows from Proposition 4.14.

We now prove (ii). Since MGL_2 is Chow connective, there is a natural map

$$(\tau^{\mathrm{C}}_{\geq 0} X) \otimes \mathrm{MGL}_2 \to \tau^{\mathrm{C}}_{\geq 0}(X \otimes \mathrm{MGL}_2).$$

To show that this is an equivalence, it suffices to show that

$$\Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y, (\tau^{\mathrm{C}}_{\geq 0} X) \otimes \mathrm{MGL}_2) \to \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y, \tau^{\mathrm{C}}_{\geq 0}(X \otimes \mathrm{MGL}_2))$$

is an equivalence for all Y compact and Chow connective. This is equivalent to the map

$$\Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y, (\tau^{\mathrm{C}}_{\geq 0} X) \otimes \mathrm{MGL}_2) \to \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y, X \otimes \mathrm{MGL}_2).$$

Now, using compactness of Y, the identification $\mathrm{MGL}_2 \simeq \varinjlim_k \mathrm{MGL}(k,k)_2$, and duality, we may rewrite this map as

$$\varinjlim_k \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y \otimes \mathbb{D}(\mathrm{MGL}(k,k)_2), (\tau^{\mathrm{C}}_{\geq 0} X)) \to \varinjlim_k \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(Y \otimes \mathbb{D}(\mathrm{MGL}(k,k)_2), X).$$

This is an equivalence by Proposition 6.10.

Finally, we prove (iii). By (i), the map

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \tau^{\mathrm{C}}_{\geq 0} X) \to \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X)$$

factors through a map

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \tau^{\mathrm{C}}_{\geq 0} X) \to P_{2n}\,\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X).$$

Since both sides are regular slice $2n$-connective, it suffices to show that this map is an equivalence after applying $\Omega^\infty \Sigma^{-n\rho}$. Making some basic manipulations, this is equivalent to showing that

$$\Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{n,n,n}, \mathrm{MGL}_2 \otimes \tau^{\mathrm{C}}_{\geq 0} X) \to \Omega^\infty \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{n,n,n}, \mathrm{MGL}_2 \otimes X)$$

is an equivalence. This is equivalence by (ii) and the Chow connectivity of $\mathbb{S}_2^{n,n,n}$. □

We are now ready to prove Theorem 6.19.

Proof of Theorem 6.19 In this proof, we freely use the notation of Section 4 and Section C.2, in particular the functors i_*, $\tau^{2\mathrm{slice}}_{\geq 0}$ and cb.

Let X be a C_2-spectrum. By Proposition 6.23(ii), there are equivalences

$$\nu_{\mathbb{R}}(X)^\wedge_{\mathrm{MGL}_2} \simeq \mathrm{Tot}^\bullet(\nu_{\mathbb{R}}(X) \otimes \mathrm{cb}(\mathrm{MGL}_2)) \simeq \mathrm{Tot}^\bullet(\nu_{\mathbb{R}}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))).$$

Applying i_* to the map $\nu_{\mathbb{R}}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})) \to \mathrm{Be}^{-1}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))$ of $\mathrm{cb}(\mathrm{MGL}_2)$-modules in cosimplicial Artin–Tate $\mathbb{R}$-motivic spectra, we obtain a map

$$i_*(\nu_{\mathbb{R}}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))) \to X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})$$

of $i_*(\mathrm{cb}(\mathrm{MGL}_2)) \simeq \tau^{2\mathrm{slice}}_{\geq 0}\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})$-modules in cosimplicial filtered C_2-spectra, where the target is constant in the filtered direction.

By Proposition 6.23(iii), this factors through an equivalence

$$i_*(\nu_{\mathbb{R}}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))) \xrightarrow{\sim} \tau^{2\mathrm{slice}}_{\geq 0}(X \otimes \mathrm{cb}(\mathrm{MU}_{\mathbb{R},2}))$$

of $\tau^{2\mathrm{slice}}_{\geq 0}\mathrm{cb}(\mathrm{MU}_{\mathbb{R},2})$-modules. Totalizing, we obtain an equivalence

$$i_*(\nu_{\mathbb{R}}(X)^\wedge_{\mathrm{MGL}_2}) \simeq i_*(\Gamma_*(X))$$

of $R_\bullet$-modules. Finally, translating back along the equivalence i_*, we obtain the desired equivalence

$$\nu_{\mathbb{R}}(X)^\wedge_{\mathrm{MGL}_2} \simeq \Gamma_*(X) \quad \text{in } \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}.$$ □

We now move on to the proof of Theorem 6.20. We will prove Theorem 6.20 as an application of a general criterion for there to be an MGL_2-local equivalence $X \simeq \nu_{\mathbb{R}}(\mathrm{Be}(X))$. This will be based on:

Definition 6.24 We say that $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is *slice simple* if, for all $n \in \mathbb{Z}$, the natural map

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, X) \to \mathrm{Be}(X)$$

factors through an equivalence

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, X) \simeq P_{2n}\mathrm{Be}(X).$$

Proposition 6.25 *Let $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}$, and let L_{MGL_2} denote MGL_2-localization. Then $L_{\mathrm{MGL}_2}X \simeq L_{\mathrm{MGL}_2}\nu_{\mathbb{R}}(\mathrm{Be}(X))$ if and only if $\mathrm{MGL}_2 \otimes X$ is slice simple.*

If X is a commutative algebra, then the equivalence $L_{\mathrm{MGL}_2}X \simeq L_{\mathrm{MGL}_2}\nu_{\mathbb{R}}(\mathrm{Be}(X))$ is one of commutative algebras.

Proof Suppose that $L_{\mathrm{MGL}_2}X \simeq L_{\mathrm{MGL}_2}\nu_{\mathbb{R}}(\mathrm{Be}(X))$, so that $\mathrm{MGL}_2 \otimes X \simeq \mathrm{MGL}_2 \otimes \nu_{\mathbb{R}}(\mathrm{Be}(X))$. Then

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes X) \simeq \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \nu_{\mathbb{R}}(\mathrm{Be}(X))$$

as C_2-spectra over $\mathrm{Be}(X)$. By Proposition 6.23(iii), the map

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \nu_{\mathbb{R}}(\mathrm{Be}(X)) \to \mathrm{Be}(\mathrm{MGL}_2 \otimes X)$$

factors through an equivalence

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, \mathrm{MGL}_2 \otimes \nu_{\mathbb{R}}(\mathrm{Be}(X)) \simeq P_{2n}\mathrm{Be}(\mathrm{MGL}_2 \otimes X),$$

so that $\mathrm{MGL}_2 \otimes X$ is slice simple.

Now suppose that $\mathrm{MGL}_2 \otimes X$ is slice simple, and consider the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & X[\mathfrak{T}^{-1}] \\ \uparrow & & \uparrow \\ \tau^{\mathrm{C}}_{\geq 0}X & \longrightarrow & \tau^{\mathrm{C}}_{\geq 0}(X[\mathfrak{T}^{-1}]) \end{array}$$

If X is a commutative algebra, then this is a diagram of commutative algebras. Applying

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, - \otimes \mathrm{MGL}_2),$$

we obtain the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, X \otimes \mathrm{MGL}_2) & \longrightarrow & \mathrm{Be}(X) \otimes \mathrm{MU}_{\mathbb{R},2} \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, (\tau^{\mathrm{C}}_{\geq 0}X) \otimes \mathrm{MGL}_2) & \longrightarrow & \mathrm{Map}_{\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}}(\mathbb{S}_2^{0,0,n}, (\tau^{\mathrm{C}}_{\geq 0}(X[\mathfrak{T}^{-1}])) \otimes \mathrm{MGL}_2) \end{array}$$

Finally, applying Proposition 6.23(iii) and slice simplicity of $\mathrm{MGL}_2 \otimes X$, we may identify this diagram with

$$\begin{array}{ccc} P_{2n}(\mathrm{Be}(X) \otimes \mathrm{MU}_{\mathbb{R},2}) & \longrightarrow & \mathrm{Be}(X) \otimes \mathrm{MU}_{\mathbb{R},2} \\ \uparrow & & \uparrow \\ P_{2n}(\mathrm{Be}(X) \otimes \mathrm{MU}_{\mathbb{R},2}) & \longrightarrow & P_{2n}(\mathrm{Be}(X) \otimes \mathrm{MU}_{\mathbb{R},2}) \end{array}$$

In conclusion, we find that the maps $\tau^{\mathrm{C}}_{\geq 0}X \to X$ and $\tau^{\mathrm{C}}_{\geq 0}X \to \tau^{\mathrm{C}}_{\geq 0}(X[\mathfrak{T}^{-1}]) \simeq \nu_{\mathbb{R}}(\mathrm{Be}(X))$ are MGL_2-equivalences, as desired. □

To prove Theorem 6.20, we now need to show that the $\mathbb{R}$-motivic spectra in question are slice simple after being tensored with MGL_2. To this end, we record the following simple proposition:

Proposition 6.26 *The collection of slice simple Artin–Tate $\mathbb{R}$-motivic spectra is closed under direct sums, retracts, and pure suspensions $\Sigma^{n,n,n}$.*

Proof of Theorem 6.20 By Proposition 6.25, it suffices to show that $\mathrm{MGL}_2 \otimes E$ is slice simple for $E = \mathrm{MGL}_2, \mathrm{M}\mathbb{F}_2, \mathrm{M}\mathbb{Z}_2, \mathrm{kgl}_2, \mathrm{kq}_2$. Indeed, each of these E is MGL_2-local, and $\nu_{\mathbb{R}}(\mathrm{Be}(E))$ is also MGL_2-local by Theorem 6.18(vi), since $\mathrm{Be}(E)$ is $\mathrm{MU}_{\mathbb{R},2}$-local.

Now, for $E = \mathrm{MGL}_2, \mathrm{M}\mathbb{F}_2, \mathrm{M}\mathbb{Z}_2, \mathrm{kgl}_2$, the tensor product $\mathrm{MGL}_2 \otimes E$ is a direct sum of pure suspensions of E, so it suffices to show that E itself is slice simple. For MGL_2, this follows from Proposition 4.14. For $\mathrm{M}\mathbb{Z}_2$ and $\mathrm{M}\mathbb{F}_2$, it follows from Lemmas 3.9 and 3.10. For kgl_2, it follows from Proposition 4.8 and [Heard 2019, Theorem 1.1].

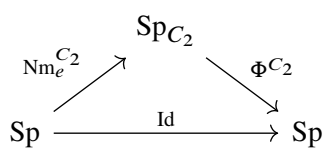
Finally, for the case of kq_2, we use the equivalence $\mathrm{kq}_2 \otimes C\eta \simeq \mathrm{kgl}_2$ and the fact that $\eta = 0$ in $\pi^{\mathbb{R}}_{0,1,1}\mathrm{MGL}_2$ to deduce that $\mathrm{MGL}_2 \otimes \mathrm{kgl}_2 \simeq \mathrm{MGL}_2 \otimes \mathrm{kq}_2 \otimes C\eta \simeq \mathrm{MGL}_2 \otimes \mathrm{kq}_2 \oplus \Sigma^{1,1,1}\mathrm{MGL}_2 \otimes \mathrm{kq}_2$. It follows that $\mathrm{MGL}_2 \otimes \mathrm{kq}_2$ is a retract of $\mathrm{MGL}_2 \otimes \mathrm{kgl}_2$, so that $\mathrm{MGL}_2 \otimes \mathrm{kq}_2$ is slice simple. □

7 The a-local category

In this section, we study the a-localized category, $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$. The authors find this category to be the most enduring mystery in our study of Artin–Tate motivic spectra over $\mathbb{R}$. Some of the behavior of this category is the subject of Questions 7.5 and 7.7.

In the category of C_2-equivariant spectra, inverting the class $a_\sigma \in \pi^{C_2}_{-\sigma}\mathbb{S}$ corresponds to taking geometric fixed points. As a consequence, since $a = c_{\mathbb{C}/\mathbb{R}}(a_\sigma)$, we may interpret $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$ as the natural target for some kind of "motivic geometric fixed points" operation. Using $\overline{\mathfrak{n}}$, we may also view the category $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$ as a deformation of Sp_{i2} with algebraic special fiber. Since $\Phi^{C_2}\mathrm{MU}_{\mathbb{R},2} \simeq \mathrm{MO}$, one might imagine that this degeneration is related to $\mathrm{Syn}_{\mathrm{MO}} \simeq \mathrm{Syn}_{\mathbb{F}_2}$. In Proposition 7.6, we will construct a comparison functor between $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$ and $\mathrm{Syn}_{\mathbb{F}_2}$. However, it is not an equivalence.

Another feature of the equivariant setting is the existence of the diagram



This suggests we study the composite $\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}(-)[a^{-1}]$. Although the norm functor itself is not exact, in Lemma 7.3 we observe that this composite preserves all colimits. However, it is not an equivalence.

Altogether, we conclude that the a-local category sits between the even BP-synthetic category and the $\mathbb{F}_2$-synthetic category in a nontrivial way. In fact, although we are able to identify the three algebraic categories which arise as the special fibers, the maps between them are surprisingly nontrivial.

7.1 The a-local category as a deformation

The symmetric monoidal category $\mathrm{Mod}(\mathcal{SH}(\mathbb{R}); \mathbb{S}[a^{-1}, \text{ब}^{-1}])$ is equivalent to Sp by [Bachmann 2018] and Corollary 3.6. As a consequence, there is an equivalence[23]

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}, \text{ब}^{-1}]) \simeq \mathrm{Sp}_{i2}\,.$$

On the other hand, the following is an immediate corollary of Theorem 5.1.

Corollary 7.1 *There is an equivalence of symmetric monoidal categories*

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; C\text{ब}[a^{-1}]) \simeq \mathrm{Mod}(\mathrm{Sp}_{i2}; \Phi^{C_2}\underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}).$$

On homotopy groups, this induces an isomorphism[24]

$$\pi^{\mathbb{R}}_{p,q,w}(C\text{ब}[a^{-1}]) \cong \bigoplus_{w+2b-s=p} \left(\mathbb{F}_2\{u^{2b}\} \otimes_{\mathbb{F}_2} \mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/2)\right).$$

Recollection 7.2 There is an isomorphism $\pi_*\Phi^{C_2}\underline{\mathbb{Z}}_2 \cong \mathbb{F}_2[y]$, where $|y| = 2$. Viewing $\Phi^{C_2}\underline{\mathbb{Z}}_2$ as $\underline{\mathbb{Z}}_2[a_\sigma^{-1}]$, y may be identified with $u_{2\sigma}/a_\sigma^2 = u_\sigma^2/a_\sigma^2$.

The presence of the nonnilpotent element u in the homotopy of $C\text{ब}[a^{-1}]$ prevents this category from being equivalent to either a BP-synthetic category or an $\mathbb{F}_2$-synthetic category.

The simplest way to gain computational access to the a-local category seems to be the ब-Bockstein spectral sequence.

7.2 The a-local norm

Equivariantly, the composition of the norm functor with geometric fixed points is the identity. As we shall see, over $\mathbb{R}$ the situation is not so straightforward.

Lemma 7.3 *The composite*

$$\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{i2} \xrightarrow{\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}} \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2} \xrightarrow{(-)[a^{-1}]} \mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}])$$

is a symmetric monoidal left adjoint. On Picard elements this composite sends $\mathbb{S}_2^{s,w}$ to $\mathbb{S}_2^{s,*,2w}[a^{-1}]$. The map τ is sent to ब^2.

Proof The norm functor $\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}$ is symmetric monoidal and commutes with sifted colimits [Bachmann and Hoyois 2021, Proposition 4.5]. Since we compose it with a symmetric monoidal left adjoint, in order to prove the first claim we need only show the composite preserves binary sums. From [Bachmann and Hoyois 2021, Corollary 5.13], we have a formula

$$\mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}(X \oplus Y) \simeq \mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}(X) \oplus i_*(X \otimes Y) \oplus \mathrm{Nm}^{\mathbb{R}}_{\mathbb{C}}(Y),$$

[23] Of course, this can also be deduced directly from Theorem 3.2 and the equivalence $\mathrm{Mod}(\mathrm{Sp}_{C_2,i2}; \mathbb{S}_2[a_\sigma^{-1}]) \simeq \mathrm{Sp}_{i2}$.

[24] Since $|a| = (0, -1, 0)$, the trigraded homotopy groups of an a-local $\mathbb{R}$-motivic spectrum are periodic in the q degree. This is reflected by the fact that the given formula has no dependence on q.

where i is the inclusion $\mathbb{R} \to \mathbb{C}$. Since the functor i_* lands in modules over the cofiber of a, the middle term vanishes upon inverting a, as desired. The claim about Picard elements is just a restatement of what we already know about Picard elements from Recollection 1.29.

Now we examine $\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(\tau) \in \pi_{0,0,-2}^{\mathbb{R}} \mathbb{S}_2$. Since the invert $\overline{\tau}$ map

$$\pi_{0,0,-2}^{\mathbb{R}} \mathbb{S}_2 \to \pi_{0,0,-2}^{\mathbb{R}} \mathbb{S}_2[\overline{\tau}^{-1}] \cong \pi_{0,0}^{C_2} \mathbb{S}_2$$

is an isomorphism which sends $\overline{\tau}$ to 1, it suffices to note that

$$\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(\tau)[\overline{\tau}^{-1}] = \mathrm{Nm}_{e}^{C_2}((\tau)[\tau^{-1}]) = \mathrm{Nm}_{e}^{C_2}(1) = 1. \qquad \square$$

Remark 7.4 Despite the fact that upon inverting $\overline{\tau}$ the functor $\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}[a^{-1}]$ becomes the identity on Sp_{i2}, for degree reasons the class η in $\pi_{1,1}^{\mathbb{C}} \mathbb{S}_2$ cannot map to the class $1 \otimes \alpha_1$ in the homotopy of $C\overline{\tau}[a^{-1}]$, which one might expect detects η.

Question 7.5 Is there an equivalence $\mathrm{Nm}_{\mathbb{C}}^{\mathbb{R}}(C\tau) \simeq C\overline{\tau}^2$ of commutative algebras? Postcomposing with the quotient map $C\overline{\tau}^2 \to C\overline{\tau}$, one obtains functors

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{C})_{ip}^{\mathrm{AT}}; C\tau) \to \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\overline{\tau}^2[a^{-1}]) \to \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; C\overline{\tau}[a^{-1}]).$$

Can the composite functor above be identified with the composite functor below?

$$\begin{array}{ccc} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{i2} & \longrightarrow & \mathrm{Mod}(\mathrm{Ab}; \Phi^{C_2}\underline{\mathbb{Z}}_2) \otimes_{\mathbb{Z}_2} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{i2} \\ \downarrow{\scriptstyle \mathbb{F}_2 \otimes -} & & \uparrow \\ \mathrm{Vect}_{\mathbb{F}_2} \otimes_{\mathbb{Z}_2} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{i2} & \xrightarrow{\text{Frobenius}} & \mathrm{Vect}_{\mathbb{F}_2} \otimes_{\mathbb{Z}_2} \mathrm{Ind}\,\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}})_{i2} \end{array}$$

7.3 Mapping down to $\mathbb{F}_2$-synthetic spectra

In this section, we will construct a realization functor from the a-local category to the category of $\mathbb{F}_2$-synthetic spectra. In some ways, this functor is even more surprising than the norm functor. While the norm functor seems to double degrees on the special fiber, Question 7.7 suggests that inverting a cuts degrees in half.

Proposition 7.6 *There is a symmetric monoidal left adjoint*

$$\mathrm{Re}_{\mathbb{F}_2} : \mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; \mathbb{S}_2[a^{-1}]) \to \mathrm{Syn}_{\mathbb{F}_2, i\tau},$$

which sends $\mathbb{S}_2^{p,q,w}$ to $\mathbb{S}_\tau^{p,w}$ and $\overline{\tau}$ to τ.

Proof We begin by noting that under the equivalence

$$\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}} \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2,i2}^{\mathbf{Fil}}; R_\bullet) \simeq \mathrm{Mod}(\mathrm{Sp}_{C_2}^{\mathbf{Fil}}; R_\bullet)$$

from Theorem 4.3, inverting a becomes levelwise geometric fixed points. Thus, $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})_{i2}^{\mathrm{AT}}; \mathbb{S}_2[a^{-1}])$ is equivalent to modules over the commutative algebra $\Phi^{C_2} R_\bullet$ obtained by applying Φ^{C_2} levelwise.

Now recall that

$$R_\bullet := \mathrm{Tot}^*(P_{2\bullet}\,\mathrm{MU}^{*+1}_{\mathbb{R},2}).$$

We may then construct a chain of comparison maps

$$\Phi^{C_2}R_\bullet \xrightarrow{\simeq} \Phi^{C_2}\mathrm{Tot}^*(P_{2\bullet}\,\mathrm{MU}^{*+1}_{\mathbb{R},2}) \to \mathrm{Tot}^*\Phi^{C_2}(P_{2\bullet}\,\mathrm{MU}^{*+1}_{\mathbb{R},2}) \to \mathrm{Tot}^*\tau_{\geq\bullet}(\Phi^{C_2}\,\mathrm{MU}^{*+1}_{\mathbb{R},2}) \xrightarrow{\simeq} \mathrm{Tot}^*\tau_{\geq\bullet}(\mathrm{MO}^{*+1}),$$

where the key step is a use of the fact that the geometric fixed points of a regular slice $2n$-connective C_2-spectrum are n-connective. This is used to produce the map across the line-break.

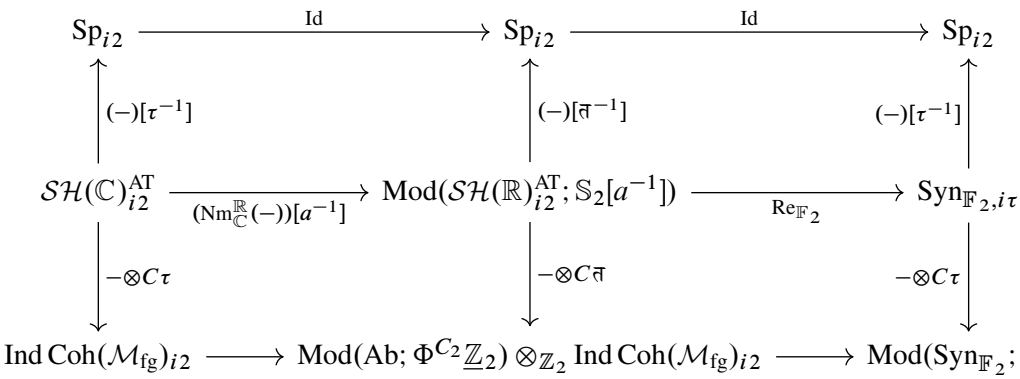
In Proposition C.22, we produced a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; \mathrm{Tot}^*\tau_{\geq\bullet}(\mathrm{MO}^{*+1})) \simeq \mathrm{Syn}^{\mathrm{cell}}_{\mathrm{MO},i\tau}.$$

Since the category of MO-finite-projective spectra is equivalent to the category of $\mathbb{F}_2$-finite-projective spectra, and a map $X \to Y$ of spectra is an MO_*-surjection if and only if it's an $(\mathbb{F}_2)_*$-surjection, there is a symmetric monoidal equivalence $\mathrm{Syn}_{\mathrm{MO}} \simeq \mathrm{Syn}_{\mathbb{F}_2}$. Finally, we can use the fact that $\mathrm{Syn}^{\mathrm{cell}}_{\mathbb{F}_2} \simeq \mathrm{Syn}_{\mathbb{F}_2}$ to drop the decoration. Altogether, base-change along this ring map produces a symmetric monoidal left adjoint

$$\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}; \mathbb{S}_2[a^{-1}]) \to \mathrm{Syn}_{\mathbb{F}_2,i\tau}. \qquad \square$$

Much of what was said in this section can be summarized with the existence of the following diagram of symmetric monoidal left adjoints:



The bottom-right corner can be described in terms of comodules over the dual Steenrod algebra [Pstrągowski 2023, Section 4.5]. The maps from the left to the right are comparison maps from the Adams–Novikov to the Adams spectral sequence.

Question 7.7 Is the map from the bottom middle to the bottom right induced by the map of Hopf algebroids $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}) \to (\mathbb{F}_2, \mathcal{A})$ which sends t_i to ζ_i? Note that this map is *not* the usual Thom reduction map that sends t_i to ζ_i^2.

8 Completions

In this paper, correctly handling a variety of different types of completion has been a key point. The purpose of this section is to give a single uniform source of information on completeness questions and how we handle them.

In Section 8.1, we show that 2-completion agrees with tensoring with $\mathbb{S}_2$ for some important $\mathbb{R}$-motivic spectra. Since these $\mathbb{R}$-motivic spectra are cellular, this implies that their 2-completions in $\mathcal{SH}(\mathbb{R})$ lie in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and that their cell decompositions carry over unchanged to the 2-completion. In the case of MGL, this is an important technical point in the rest of the paper.

In Section 8.2, we record the fact that every dualizable object of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is $\overline{\eta}$-complete.

Finally, in Section 8.3, we make some remarks about a-completion.

8.1 2 completion is sometimes i2 completion

The main goal of this section is to prove the following theorem:

Theorem 8.1 *There are natural equivalences in $\mathcal{SH}(\mathbb{R})$:*

$$\mathrm{MGL}\otimes\mathbb{S}_2\to\mathrm{MGL}_2,\quad \mathrm{H}\mathbb{Z}\otimes\mathbb{S}_2\to\mathrm{H}\mathbb{Z}_2,\quad \mathrm{kgl}\otimes\mathbb{S}_2\to\mathrm{kgl}_2,\quad \mathrm{kq}\otimes\mathbb{S}_2\to\mathrm{kq}_2.$$

The proof for MGL, H$\mathbb{Z}$ and kgl will be completely independent of the rest of this paper. On the other hand, the proof for kq will use results from Sections 2 through 4.

We begin with some basic facts about the situation. In the following, we will make use of the homotopy t-structure on the category of $\mathcal{SH}(\mathbb{R})$ of $\mathbb{R}$-motivic spectra. We refer to the reader to [Hoyois 2015, Section 2.1] for basic facts about the homotopy t-structure.

Proposition 8.2 *Given $X\in\mathcal{SH}(\mathbb{R})$, the following statements are true*:

(i) *If X is dualizable, then*

$$X\otimes\mathbb{S}_2\to X_2$$

is an equivalence.

(ii) *Suppose that $\varinjlim X(n)\simeq X$ and that the connectivity of the maps $X(n)\to X$ tend to infinity with respect to the homotopy t-structure. Then if*

$$X(n)\otimes\mathbb{S}_2\to X(n)_2$$

is an equivalence for all n, we also learn that

$$X\otimes\mathbb{S}_2\to X_2$$

is an equivalence.

Proof To prove (i), it simply suffices to note that since X is dualizable, the functor $-\otimes X$ preserves limits.

To prove (ii), we note that, since tensoring with $\mathbb{S}_2$ preserves colimits, it is equivalent to show that

$$\varinjlim X(n)_2 \to X_2$$

is an equivalence, i.e., that

$$\varinjlim(X_2/X(n)_2) \simeq \varinjlim(X/X(n))_2 \simeq 0.$$

Since $\{\mathbb{S}^{p,w,w} \otimes S_+ | S \in \mathbf{Sm}/\mathbb{R} \text{ and } p, w \in \mathbb{Z}\}$ forms a set of compact generators for $\mathcal{SH}(\mathbb{R})$, it suffices to show that

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{p,w,w} \otimes S_+, \varinjlim(X/X(n))_2) \simeq \varinjlim(\mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{p,w,w} \otimes S_+, X/X(n))_2)$$

is equivalent to 0 for all $S \in \mathbf{Sm}/\mathbb{R}$ and $p, w \in \mathbb{Z}$.

This follows from the definition of the homotopy t-structure, which implies that, for fixed $S \in \mathbf{Sm}/\mathbb{R}$ and $p, w \in \mathbb{Z}$, the connectivity of the spectrum

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{p,w,w} \otimes S_+, X/X(n))$$

goes to infinity as n goes to infinity. □

The key input to our proof of Theorem 8.1 will be the following:

Lemma 8.3 *Let* $\mathrm{MGL}(n,k)$ *denote the Thom spectrum of the bundle* $\gamma - k \cdot \mathrm{triv}$ *over* $\mathrm{Gr}_k(\mathbb{A}^{n+k})$. *Then the cofiber of the natural map* $\mathrm{MGL}(n,n) \to \mathrm{MGL}$ *is* n*-connective with respect to the homotopy* t*-structure.*

Proof This is an immediate consequence of [Hoyois 2015, Lemma 3.4]. □

Proof of Theorem 8.1 for MGL, H$\mathbb{Z}$ and kgl Since $\mathrm{MGL}(n,k)$ is dualizable, the result for MGL follows from the combination of Proposition 8.2(i)–(ii) with Lemma 8.3.

We now turn to the case of H$\mathbb{Z}$. By the Hopkins–Morel theorem [Hoyois 2015], there is an equivalence

$$\mathrm{MGL}/(a_1, a_2, \dots) \simeq \mathrm{H}\mathbb{Z}$$

for certain classes $a_i \in \pi^{\mathbb{R}}_{i,i,i}\mathrm{MGL}$. In other words,

$$\varinjlim \mathrm{MGL}/(a_1, a_2, \dots, a_k) \simeq \mathrm{H}\mathbb{Z}.$$

By the result for MGL, we know that the result holds for all $\mathrm{MGL}/(a_1, a_2, \dots, a_k)$. By Proposition 8.2(ii), it therefore suffices to show that the connectivity of

$$\mathrm{MGL}/(a_1, a_2, \dots, a_k) \to \mathrm{MGL}/(a_1, a_2, \dots) \simeq \mathrm{H}\mathbb{Z}$$

goes to infinity. To do this, it suffices to show that the connectivity of

$$\mathrm{MGL}/(a_1, a_2, \dots, a_k) \to \mathrm{MGL}/(a_1, a_2, \dots, a_k, a_{k+1})$$

goes to infinity, which follows from the cofiber sequences

$$\mathrm{MGL}/(a_1, a_2, \dots, a_k) \to \mathrm{MGL}/(a_1, a_2, \dots, a_k, a_{k+1}) \to \Sigma^{k+2,k+1,k+1}\mathrm{MGL}/(a_1, a_2, \dots, a_k).$$

Finally, we handle the case of kgl. Combining the Hopkins–Morel theorem and convergence of the effective slice towers for MGL and kgl [Hoyois 2015, Lemmas 8.10 and 8.11] with [Spitzweck 2010, Proposition 5.4], we find that there is an equivalence

$$\mathrm{MGL}/(a_2, a_3, \dots) \simeq \mathrm{kgl}.$$

We may therefore prove the result for kgl exactly as we did for $\mathrm{H}\mathbb{Z}$. □

Before we move on to the case of kq, we collect some facts that we will need:

Proposition 8.4 *The following are true*:

(i) *Suppose that $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is effective slice connective. Then the natural map*

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{0,0,n}, X) \to \mathrm{Be}(X)$$

is an equivalence for any $n \leq 0$.

(ii) *In C_2-equivariant spectra, there is a canonical equivalence*

$$\Phi^{C_2}(\mathrm{ko}_{C_2,2}) \simeq \Phi^{C_2}(\mathrm{ko}_{C_2})_2.$$

Proof Given $X \in \mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, it follows from Theorem 3.2 that

$$\mathrm{Be}(X) \simeq \varinjlim \mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{0,0,n}, X).$$

Part (i) therefore follows from Proposition 4.13.

We now prove (ii). By the cofiber sequence

$$(\mathrm{ko}_{C_2})_{hC_2} \to (\mathrm{ko}_{C_2})^{C_2} \to \Phi^{C_2}(\mathrm{ko}_{C_2})$$

and the fact that $(-)^{C_2}$ commutes with limits, it suffices to show that the canonical map

$$(\mathrm{ko}_{C_2,2})_{hC_2} \to (\mathrm{ko}_{C_2})_{hC_2,2}$$

is an equivalence. This map is equivalent to

$$\mathrm{ko}_2 \otimes \mathbb{R}\mathbb{P}^\infty_+ \to (\mathrm{ko} \otimes \mathbb{R}\mathbb{P}^\infty_+)_2,$$

so this follows from connectivity of ko and the fact that $\mathbb{R}\mathbb{P}^\infty_+$ is of finite type. □

The key input will be the following special case:

Proposition 8.5 *The map*

$$(\mathrm{kq} \otimes \mathbb{S}_2)\left[\frac{1}{2}, \frac{1}{\eta}\right] \to \mathrm{kq}_2\left[\frac{1}{2}, \frac{1}{\eta}\right]$$

is an equivalence.

This will make essential use of the following result of Bachmann.

Proposition 8.6 *The assignment $X \mapsto X(\mathbb{R})$ induces an equivalence*

$$\mathcal{SH}(\mathbb{R})\left[\frac{1}{\rho}\right] \simeq \mathrm{Sp}\,.$$

Moreover, we have

$$\mathcal{SH}(\mathbb{R})_{(2)}\left[\frac{1}{2},\frac{1}{\eta}\right] = \mathcal{SH}(\mathbb{R})_{(2)}\left[\frac{1}{2},\frac{1}{\rho}\right] \simeq \mathrm{Mod}_{\mathbb{Q}}.$$

Proof The first statement follows from [Bachmann 2018, Theorem 35 and Proposition 36]. The second statement is a consequence of the first and [Bachmann 2018, Lemma 39]. □

Given Proposition 8.6, Proposition 8.5 is reduced to the following proposition:

Proposition 8.7 *There are isomorphisms*

$$\pi^{\mathbb{R}}_{*,0,0}\left(\mathrm{kq}_{(2)}\left[\frac{1}{2},\frac{1}{\eta}\right]\right) \cong \mathbb{Q}[\beta] \quad \textit{when } |\beta| = 4,$$
$$\pi^{\mathbb{R}}_{*,0,0}\left(\mathbb{S}_2\left[\frac{1}{2},\frac{1}{\eta}\right]\right) \cong \mathbb{Q}_2,$$
$$\pi^{\mathbb{R}}_{*,0,0}\left(\mathrm{kq}_2\left[\frac{1}{2},\frac{1}{\eta}\right]\right) \cong \mathbb{Q}_2[\beta] \quad \textit{when } |\beta| = 4.$$

Proof Let $\mathrm{W}(\mathbb{R})$ denote the Witt group of the real numbers. There is an isomorphism $\mathrm{W}(\mathbb{R}) \cong \mathbb{Z}$. It follows from [Bachmann and Hopkins 2020, Section 6.3.2] that

$$\pi^{\mathbb{R}}_{*,0,0}\mathrm{kq}\left[\frac{1}{\eta}\right] \cong \mathrm{W}(\mathbb{R})[\beta] \cong \mathbb{Z}[\beta],$$

from which we deduce the first isomorphism.

By Theorem 3.2, Betti realization induces an isomorphism

$$\pi^{\mathbb{R}}_{*,*,0}(\mathbb{S}_2)\left[\frac{1}{\overline{\alpha}}\right] \cong \pi^{C_2}_{*+*\sigma}(\mathbb{S}_2).$$

As a consequence, we find

$$\pi^{\mathbb{R}}_{*,0,0}(\mathbb{S}_2)\left[\frac{1}{\rho}\right] \cong \pi^{\mathbb{R}}_{*,0,0}(\mathbb{S}_2)\left[\frac{1}{\overline{\alpha}},1/a\right] \cong \pi^{C_2}_{*}(\mathbb{S}_2)\left[\frac{1}{a\,\sigma}\right] \cong \pi_*\Phi^{C_2}(\mathbb{S}_2) \cong \pi_*\,\mathbb{S}_2\,.$$

Inverting 2 and applying Proposition 8.6, we deduce the second isomorphism.

For the third isomorphism, we make use of the sequence of equivalences

$$\mathrm{Map}_{\mathcal{SH}(\mathbb{R})}(\mathbb{S}^{0,0,n},\mathrm{kq}_2) \simeq \varprojlim \mathrm{Map}_{\mathcal{SH}(\mathbb{R})}\left(\mathbb{S}^{0,0,n},\frac{\mathrm{kq}}{2^k}\right) \simeq \varprojlim \mathrm{Be}\left(\frac{\mathrm{kq}}{2^k}\right) \simeq \mathrm{Be}(\mathrm{kq})_2 \simeq \mathrm{ko}_{C_2,2},$$

where the second equivalence follows from Proposition 8.4(i) and the fourth equivalence follows from [Kong 2020, Corollary 2.30].

As a consequence, we find that

$$\pi^{\mathbb{R}}_{*,0,0}\mathrm{kq}_2\left[\frac{1}{\rho}\right] \cong \pi^{\mathbb{R}}_{*,0,0}\mathrm{kq}_2\left[\frac{1}{\overline{\alpha}},\frac{1}{a}\right] \cong \pi^{C_2}_{*}\mathrm{ko}_{C_2,2}\left[\frac{1}{a\,\sigma}\right] \cong \pi_*\Phi^{C_2}(\mathrm{ko}_{C_2,2}) \cong \pi_*\Phi^{C_2}(\mathrm{ko}_{C_2})_2 \cong \mathbb{Z}_2[\beta],$$

where the fourth isomorphism follows from Proposition 8.4(ii) and the fifth isomorphism follows from [Guillou et al. 2020, Propositon 10.18]. Inverting 2 and applying Proposition 8.6, we obtain the result. □

Proof of Theorem 8.1 for kq We want to prove that

$$\mathrm{kq} \otimes \mathbb{S}_2 \to \mathrm{kq}_2$$

is an equivalence. It is clearly an equivalence after smashing with $C(2)$, so it suffices to show that it is an equivalence after inverting 2. Moreover, by the equivalence $\mathrm{kq} \otimes C(\eta) \simeq \mathrm{kgl}$ and the kgl case, it is also an equivalence after smashing with $C(\eta)$. It therefore suffices to show that

$$(\mathrm{kq} \otimes \mathbb{S}_2)\Big[\frac{1}{2}, \frac{1}{\eta}\Big] \to \mathrm{kq}_2\Big[\frac{1}{2}, \frac{1}{\eta}\Big]$$

is an equivalence, which is Proposition 8.5. □

8.2 Ta completion

At several points we have suggested that from a computational viewpoint the ता-Bockstein spectral sequence is probably the best way to gain access to various objects. Here we show such computations often converge to the correct answer.

Proposition 8.8 *The unit in $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is ता-complete.*

Before proving this proposition we give a corollary of it.

Corollary 8.9 *Every dualizable object of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ is ता-complete.*

Now we turn to the proof of this proposition. We begin with the following lemma.

Lemma 8.10 $\mathrm{M}\mathbb{Z}_2$ *is ता-complete.*

Proof Since the trigraded spheres are a family of compact generators it will suffice to show that

$$\varprojlim\big(\cdots \to \pi^{\mathbb{R}}_{p,q,2}(\mathrm{M}\mathbb{Z}_2) \to \pi^{\mathbb{R}}_{p,q,1}(\mathrm{M}\mathbb{Z}_2) \to \pi^{\mathbb{R}}_{p,q,0}(\mathrm{M}\mathbb{Z}_2)\big)$$

is zero and there is no $\lim^1$ term. Both of these claims follow from the fact that $\pi^{\mathbb{R}}_{p,q,w}(\mathrm{M}\mathbb{Z}_2) = 0$ for $w \geq 0$. □

Proof of Proposition 8.8 Since the $\mathrm{M}\mathbb{Z}_2$-Adams spectral sequence converges [Hu et al. 2011a] and limits of complete objects are complete we may use Lemma 8.10 to conclude. □

8.3 *a*-completion

Almost none of the objects we have encountered are a-complete. This stands in contrast to C_2-equivariant homotopy theory, where the Segal conjecture implies that the 2-completed unit is a-complete. To see explicit failures of completeness, note that if an object X is a-complete then $C\text{ता} \otimes X$ is also a-complete. On the other hand, the homotopy groups of $C\text{ता}$ contain copies of the homotopy of $\underline{\mathbb{Z}}_2$, which is not a-complete.

Example 8.11 The elements $(\theta/a^n)\otimes 1$ in the homotopy of $C\tau$ give an explicit example of non-a-completeness. If we consider the image of θ/a^n under the connecting map $\delta:\Sigma^{-1,0,1}C\tau\to\mathbb{S}$ we obtain an example of an infinitely a-divisible element in the sphere.[25]

On the other hand, it often happens that after inverting τ objects becomes a-complete.

Lemma 8.12 *On dualizable objects of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$, the following functors are equivalent: $(-)[\tau^{-1}]$, $(-)^\wedge_a[\tau^{-1}]$ and $(-)[\tau^{-1}]^\wedge_a$.*

Proof For any dualizable object X there is an n such that $\pi^{\mathbb{R}}_{p,q,w}(C\tau\otimes X)=0$ for $w\le n$. Using this we learn that on the level of trigraded homotopy groups the colimit inverting τ commutes with the inverse limit completing at a. Now, we only need to explain why $(-)[\tau^{-1}]$ is already a-complete on dualizable inputs. If we invert τ on a dualizable $i2$-complete $\mathbb{R}$-motivic spectrum we get a dualizable $i2$-complete C_2-spectrum. Thus, we're reduced to showing that dualizable $i2$-complete C_2-spectra are a_σ-complete. Using the fact that tensoring with a dualizable object commutes with limits it suffices to show this is true for the unit. The 2-complete sphere in C_2-spectra is a_σ-complete as a consequence of Lin's theorem [1980]. □

The tension this introduces is heightened when one recalls Gregersen's motivic analogue [2012] of Lin's theorem, which is proved over any field of characteristic zero. However, Gregersen's result only asserts a π_{**}-isomorphism, i.e., an equivalence after reflecting into the Tate category. In fact, a careful examination of the formula for the homotopy groups of $C\tau$ reveals that the copies of the negative cone (from which the non-a-completeness originates) all lie below the plane of Tate spheres. Therefore, upon running the τ-Bockstein spectral sequence the tridegrees (p,w,w) never receive contributions from copies of the negative cone.

9 Odd primes

In this section we show that, when working at an odd prime p, the category $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{ip}$ admits a simple description in terms of $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$. The corresponding result for Tate spectra is discussed in [Behrens and Shah 2020, pages 23–24], following results of [Bachmann 2018].

Proposition 9.1 *There is an equivalence of categories*

$$\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{ip}\simeq \mathrm{Sp}_{ip}\times\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}\times\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}.$$

We begin with a well-known decomposition of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{ip}$ into its $+$ and $-$ components. Our treatment of this is heavily inspired by [Behrens and Shah 2020, pages 23–25].

Definition 9.2 Let $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip}$ denote the category $\mathrm{Mod}(\mathcal{SH}(\mathbb{R})^{\mathrm{AT}};\mathbb{S}_{p,\eta})$.

[25] We expect that this image is nontrivial for n sufficiently large, though we don't prove it. Of course if the image is trivial for all n, then a lift of these classes to the sphere gives a different example of noncompleteness.

Lemma 9.3 *There is an equivalence of categories*

$$\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{ip} \simeq \mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip} \times \mathrm{Sp}_{ip}\,.$$

Proof Let $\epsilon = -\eta\rho - 1 \in \pi^{\mathbb{R}}_{0,0,0}\,\mathbb{S}$, which is known to be equal to the swap map

$$\epsilon = s_{\mathbb{S}^{1,1,1}} : \mathbb{S}^{1,1,1} \otimes \mathbb{S}^{1,1,1} \xrightarrow{\simeq} \mathbb{S}^{1,1,1} \otimes \mathbb{S}^{1,1,1}\,.$$

It is clear from this description that $\epsilon^2 = 1$, so that after inverting 2 we have a decomposition of the unit into ± 1-eigenspaces

$$\mathbb{S}\big[\tfrac{1}{2}\big] \simeq \mathbb{S}\big[\tfrac{1}{2}\big]^{+} \times \mathbb{S}\big[\tfrac{1}{2}\big]^{-}.$$

Now, $\epsilon = +1$ implies that $\eta\rho = -2$, so that η and ρ must be units in $\pi^{\mathbb{R}}_{*,*,*}\,\mathbb{S}[1/2]^{+}$. On the other hand, the fact that $\epsilon = s_{\mathbb{S}^{1,1,1}}$ implies that the graded ring $\pi^{\mathbb{R}}_{a,a,a}\,\mathbb{S}[1/2]^{-}$ must obey the Koszul sign rule, so that $2\eta^2 = 2\rho^2 = 0$.

As a consequence, we find that $\mathbb{S}[1/2]^{+} \simeq \mathbb{S}[1/2, 1/\eta] \simeq \mathbb{S}[1/2, 1/\rho]$ and that $\mathbb{S}[1/2]^{-} \simeq \mathbb{S}[1/2]_{\eta} \simeq \mathbb{S}[1/2]_{\rho}$. To conclude, we base change from $\mathbb{S}[1/2]$ to $\mathbb{S}_p$ and use the equivalence $\mathcal{SH}(\mathbb{R})[1/\rho] \simeq \mathrm{Sp}$ of Bachmann [2018]. □

Proposition 9.1 will now follow from showing that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip}$ splits as a product of two copies of $\mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$. This splitting will come from a decomposition of the category into two blocks (in the sense of representation theory). We begin with the following pair of lemmas.

Lemma 9.4 *In $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip}$, there is a splitting* $\mathrm{Spec}(\mathbb{C}) \simeq \mathbb{S}^{0,0,0}_{p,\eta} \oplus \mathbb{S}^{1,-1,0}_{p,\eta}$.

Proof It suffices to show that $a : \mathbb{S}^{0,-1,0}_{p,\eta} \to \mathbb{S}_{p,\eta}$ is zero. This follows from the facts that $c_{\mathbb{C}/\mathbb{R}}$ is fully faithful (see [Heller and Ormsby 2018]) and that $a_\sigma = 0$ in the C_2-equivariant category after inverting 2 and η-completing (indeed, $a_\sigma \eta = 2$). □

Lemma 9.5 *The groups $\pi^{\mathbb{R}}_{s,q,w}\,\mathbb{S}_{p,\eta}$ are zero for q odd.*

The proof of this lemma will take us farther afield, so we defer it for the moment.

Proof of Proposition 9.1. We need to show that

$$\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip} \simeq \mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip} \times \mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}.$$

Let $\mathcal{A}_{\mathrm{even}}$ (resp. $\mathcal{A}_{\mathrm{odd}}$) denote the stable full subcategory of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip}$ generated under colimits by the objects $\mathbb{S}^{s,q,w}_{p,\eta}$ for $s, w \in \mathbb{Z}$ and q even (resp. odd).

We begin by showing that if $A \in \mathcal{A}_{\mathrm{even}}$ and $B \in \mathcal{A}_{\mathrm{odd}}$, then there are no nontrivial maps between A and B. It suffices to show this for compact generators, where it follows from Lemma 9.5. From this we may conclude that $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip} \simeq \mathcal{A}_{\mathrm{even}} \times \mathcal{A}_{\mathrm{odd}}$.

Since tensoring with $\mathbb{S}^{0,1,0}$ provides an equivalence between $\mathcal{A}_{\mathrm{even}}$ and $\mathcal{A}_{\mathrm{odd}}$ it will now suffice to show that the composite

$$\mathcal{A}_{\mathrm{even}} \to \mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip} \to \mathcal{SH}(\mathbb{C})^{\mathrm{AT}}_{ip}$$

is an equivalence. Since $\mathbb{S}^{s,0,w}$ is sent to $\mathbb{S}^{s,w}$ we know this map hits a family of compact generators, so it will suffice to show that it is fully faithful. By the usual argument it will suffice to have full faithfulness on compact generators. Thus, we are reduced to showing that the map

$$\pi^{\mathbb{R}}_{s,q,w}\,\mathbb{S}_{p,\eta} \to \pi^{\mathbb{C}}_{s+q,w}\,\mathbb{S}_{p,\eta}$$

is an isomorphism for q even. Using Lemma 9.4 this map factors as

$$\pi^{\mathbb{R}}_{s,q,w}\,\mathbb{S}_{p,\eta} \hookrightarrow \pi^{\mathbb{R}}_{s,q,w}(\mathbb{S}^{0,0,0}_{p,\eta} \oplus \mathbb{S}^{1,-1,0}_{p,\eta}) \xrightarrow{\cong} \pi^{\mathbb{C}}_{s+q,w}\,\mathbb{S}_{p,\eta},$$

where the first map is the inclusion of the left summand. We conclude by noting that the relevant homotopy group of the right summand vanishes by Lemma 9.5. □

We now return to proving Lemma 9.5. The proof will be via an Adams spectral sequence argument, so we begin by computing the homology of a point. Note that $\mathrm{M}\mathbb{F}_p$ is η-complete, and so inside of $\mathcal{SH}(\mathbb{R})^{\mathrm{AT},+}_{ip}$.

Lemma 9.6 *The trigraded homotopy of $\mathrm{M}\mathbb{F}_p$ is given by*

$$\pi^{\mathbb{R}}_{s,q,w}\mathrm{M}\mathbb{F}_p \cong \mathbb{F}_p[u_{2\sigma}^{\pm}, \text{ट}], \quad \text{where } |u_{2\sigma}| = (2,-2,0).$$

Proof From the computation of the homology of a point over $\mathbb{C}$ in [Voevodsky 2003b] we may conclude that $\pi^{\mathbb{R}}_{s,q,w}(\mathrm{Spec}(\mathbb{C}) \otimes \mathrm{M}\mathbb{F}_p) \cong \mathbb{F}_p[u_\sigma^{\pm}, \text{ट}]$.[26] Using the splitting of $\mathrm{Spec}(\mathbb{C})$ we may conclude that the homology of a point over $\mathbb{R}$ is an index 2 subalgebra which contains ट (since ट is defined in the sphere). There is a unique such subalgebra. □

Lemma 9.7 *The tensor product $\mathrm{M}\mathbb{F}_p \otimes \mathrm{M}\mathbb{F}_p$ splits as a sum of trigraded suspensions of copies of $\mathrm{M}\mathbb{F}_p$ whose q-components are even.*

Proof From [Hoyois et al. 2017, Theorem 1.1(3)] (which follows from work of Voevodsky [2003b; 2010] in the case of $\mathbb{R}$, where we work) we know that $\mathrm{M}\mathbb{F}_p \otimes \mathrm{M}\mathbb{F}_p$ decomposes as a sum of copies of $\Sigma^{a,b,b}\mathrm{M}\mathbb{F}_p$. More specifically, the copies of $\mathrm{M}\mathbb{F}_p$ are indexed by monomials in the ξ_i and τ_i which live in degrees

$$|\xi_i| = (p^i - 1, p^i - 1, p^i - 1) \quad \text{and} \quad |\tau_i| = (p^i, p^i - 1, p^i - 1).$$

Since p is odd the q-component of every such monomial is even. □

Proof of Lemma 9.5. Since the motivic Adams spectral sequence for the sphere converges strongly to $\mathbb{S}_{p,\eta}$ by [Hu et al. 2011a], it will suffice to show the desired vanishing result on the E_1-page. This spectral sequence takes the form,

$$E_1^{s,t} = \pi_{t,q,w}(\mathrm{M}\mathbb{F}_2^{\otimes s+1}) \implies \pi_{t-s,q,w}\,\mathbb{S}_{p,\eta}\,.$$

Using Lemmas 9.6 and 9.7 we may conclude that the spectral sequence is zero at the E_1 page for q odd. □

[26] This is apparent as $\mathbb{F}_p[\text{ट}]$ modules, after which the algebra structure can be deduced upon inverting ट (Betti realization).

10 Examples and computations

In this section we employ technology from the rest of the paper to give example, concrete computations of trigraded homotopy groups. First, we discuss vanishing regions in the homotopy groups of objects obtained from the sphere by either killing or inverting a and τ. Then we give a detailed discussion of kq_2, which we hope will be a useful guide to computationally minded stable homotopy theorists.

10.1 Vanishing regions for trigraded homotopy

The coarsest information about the trigraded homotopy groups of Artin–Tate $\mathbb{R}$-motivic spectra comes from understanding which regions contain only zero groups. We summarize what we know in the following omnibus theorem.

Theorem 10.1 *By either coning or inverting $a \in \pi^{\mathbb{R}}_{0,-1,0}\mathbb{S}$ and $\tau \in \pi^{\mathbb{R}}_{0,0,-1}\mathbb{S}_2$, we may build 9 natural objects. The trigraded homotopy groups of these objects are concentrated in the following regions.*

(i) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2)$ *is concentrated in the region*

$$\{p+q \geq w \geq 0\} \cup \{p+q \geq 0, w \leq 0\} \cup \{p \geq 0\}.$$

(ii) $\pi^{\mathbb{R}}_{p,q,w}(C\tau)$ *is concentrated in the region*

$$\{0 \leq p \leq 2w-q, w \geq 0\} \cup \{w-q \leq p \leq w-2, w \geq 0\}.$$

(iii) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2[\tau^{-1}])$ *is periodic in the w-degree and concentrated in the region*

$$\{p \geq 0\} \cup \{p+q \geq 0\}.$$

(iv) $\pi^{\mathbb{R}}_{p,q,w}(Ca)$ *is periodic along lines of the form $(1,-1,0)$ and concentrated in the region*

$$\{0 \leq w \leq p+q\} \cup \{0 \leq p+q, w \leq 0\}.$$

(v) $\pi^{\mathbb{R}}_{p,q,w}(Ca \otimes C\tau)$ *is periodic along lines of the form $(1,-1,0)$ and concentrated in the region*

$$\{w \leq p+q \leq 2w\}.$$

(vi) $\pi^{\mathbb{R}}_{p,q,w}(Ca \otimes \mathbb{S}_2[\tau^{-1}])$ *is periodic in the w-degree, periodic along lines of the form $(1,-1,0)$ and concentrated in the region*

$$\{p+q \geq 0\}.$$

(vii) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2[a^{-1}])$ *is periodic in the q-degree and concentrated in the region*

$$\{p \geq 0\}.$$

(viii) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2[a^{-1}] \otimes C\tau)$ *is periodic in the q-degree and concentrated in the region*

$$\{p \geq 0, w \geq 0\}.$$

(ix) $\pi^{\mathbb{R}}_{p,q,w}(\mathbb{S}_2[a^{-1}, \text{ত}^{-1}])$ *is periodic in the* q*-degree, periodic in the* w*-degree and concentrated in the region*

$$\{p \geq 0\}.$$

Proof We begin with (iv), (v) and (vi), where we have killed a. In the identification of Ca-modules with the $\mathbb{C}$-motivic category, the trigraded homotopy group $\pi_{p,q,w}Ca$ is identified with the bigraded stem $\pi_{p+q,w}\,\mathbb{S}^{\mathbb{C}}_2$. Hence, $\pi_{p,q,w}Ca \cong \pi_{p+m,q-m,w}Ca$ for any integer m. That $\pi_{p,q,w}Ca$ vanishes outside the indicated regions then reduces to well-known vanishing statements for the $\mathbb{C}$-motivic stable stems, as displayed on [Gheorghe and Isaksen 2017, page 2]. Similarly, again as indicated on [Gheorghe and Isaksen 2017, page 2], (v) and (vi) are consequences of [Levine 2014, Theorem 6.7]. Analogously, statements (iii) and (ix) follow from combining Theorem 3.2 with the known vanishing regions in C_2-equivariant stable stems (for such vanishing regions, see, e.g., [Belmont et al. 2021, Figure 5]).

Upon inverting a in (i) and (ii) we recover (vii) and (viii), respectively, so it remains only to prove (i) and (ii). In fact, since Proposition 8.8 proves that $\mathbb{S}_2$ is ত-complete, we may obtain (i) from (ii) by examining the E_1-page of the ত-Bockstein spectral sequence. Finally, in order to prove (ii) we directly examine the homotopy of $C\text{ত}$. In Theorem 5.1 we computed that

$$\pi^{\mathbb{R}}_{p,q,w}(C\text{ত}) \cong \bigoplus_{w+a-s=p} \mathrm{Ext}^{s,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*, (\mathrm{MU}_2)_* \otimes_{\mathbb{Z}_2} \pi^{C_2}_{a+(q-w)\sigma}\underline{\mathbb{Z}}_2).$$

The result then follows by combining the following vanishing results:

- $\mathrm{Ext}^{s,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*, (\mathrm{MU}_2)_*)$ and $\mathrm{Ext}^{s,2w}_{(\mathrm{MU}_2)_*\,\mathrm{MU}_2}((\mathrm{MU}_2)_*, (\mathrm{MU}_2)_*/2)$ are concentrated in the region $\{0 \leq s \leq 2w\}$.
- $\pi^{C_2}_{p+q\sigma}\underline{\mathbb{Z}}_2$ is concentrated in the region $\{p \geq 0,\, p+q \leq 0\} \cup \{p \leq -2,\, p+q \geq 0\}$. □

Remark 10.2 As a corollary of the vanishing region for $C\text{ত}$ and the ত-completeness of the unit, we recover [Belmont et al. 2021, Theorem 1.1], which describes the region in which $\mathrm{Be} : \pi^{\mathbb{R}}_{p,w,w}\,\mathbb{S}_2 \to \pi^{C_2}_{p+w\sigma}\,\mathbb{S}_2$ is an isomorphism.

10.2 The homotopy of kq_2

To aid the computationally minded reader, we give below some charts of the trigraded homotopy groups of kq_2. These charts are meant to be read in concert with those of [Kong 2020, Section 6.5], and the notation here matches that in Kong's work. Indeed, all of the information within these charts is easily accessed within Kong's work, and we merely repackage it here.

We focus on the homotopy groups $\pi_{p,q,w}(\mathrm{kq}_2 \otimes C\text{ত})$ and $\pi_{p,q,w}(\mathrm{kq}_2)$ for $p = 0$ and $p = -1$. In the language of [Kong 2020], this corresponds to a focus on *coweights* 0 and -1.

Red lines denote multiplication by $a \in \pi_{0,-1,0}\,\mathbb{S}$. Black lines denote multiplication by the motivic $\eta : \mathbb{G}_m \to \mathbb{S}$, which in our grading convention lives in $\pi_{0,1,1}\,\mathbb{S}$.

In Figures 3 and 4, we depict some of the E_∞-page of the ত-Bockstein spectral sequence for $\pi_{*,*,*}\mathrm{kq}_2$. We follow the convention introduced in [Burklund et al. 2023, Section A.2] and [Burklund 2021] of

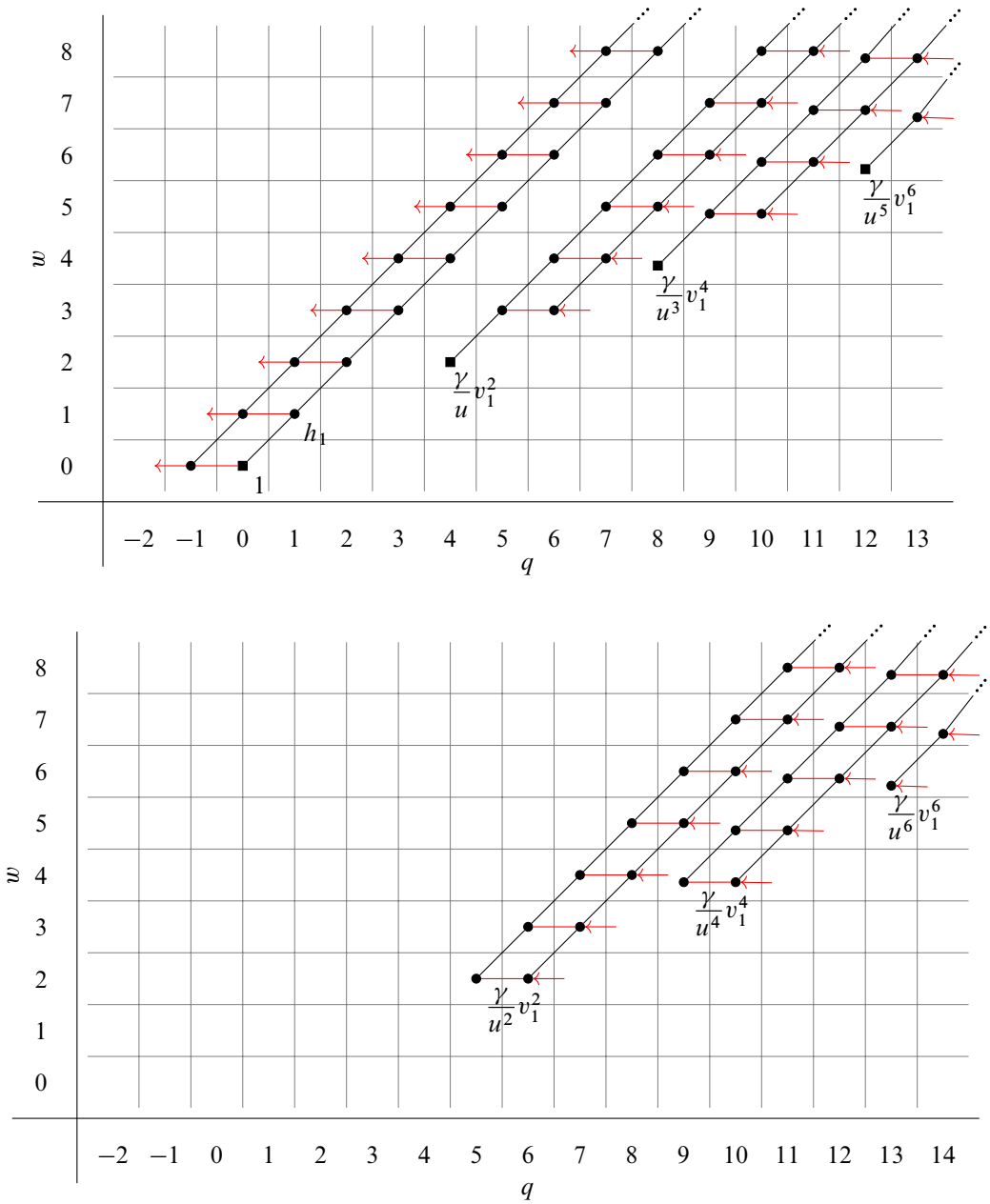


Figure 3: Charts for $\pi_{0,q,w}(C\tau \otimes \mathrm{kq}_2)$, top, and $\pi_{-1,q,w}(C\tau \otimes \mathrm{kq}_2)$, bottom.

using blue symbols to denote τ-torsion classes, while black symbols denote τ-torsion-free classes. In particular, black symbols contribute not only to the homotopy group corresponding to the box in which they appear, but also to the groups corresponding to boxes directly below where they appear. In the language of [Kong 2020], the blue dots in our charts contain information about which dots on the E_1-page of the C_2-effective spectral sequence are targets of differentials, as opposed to sources of differentials.

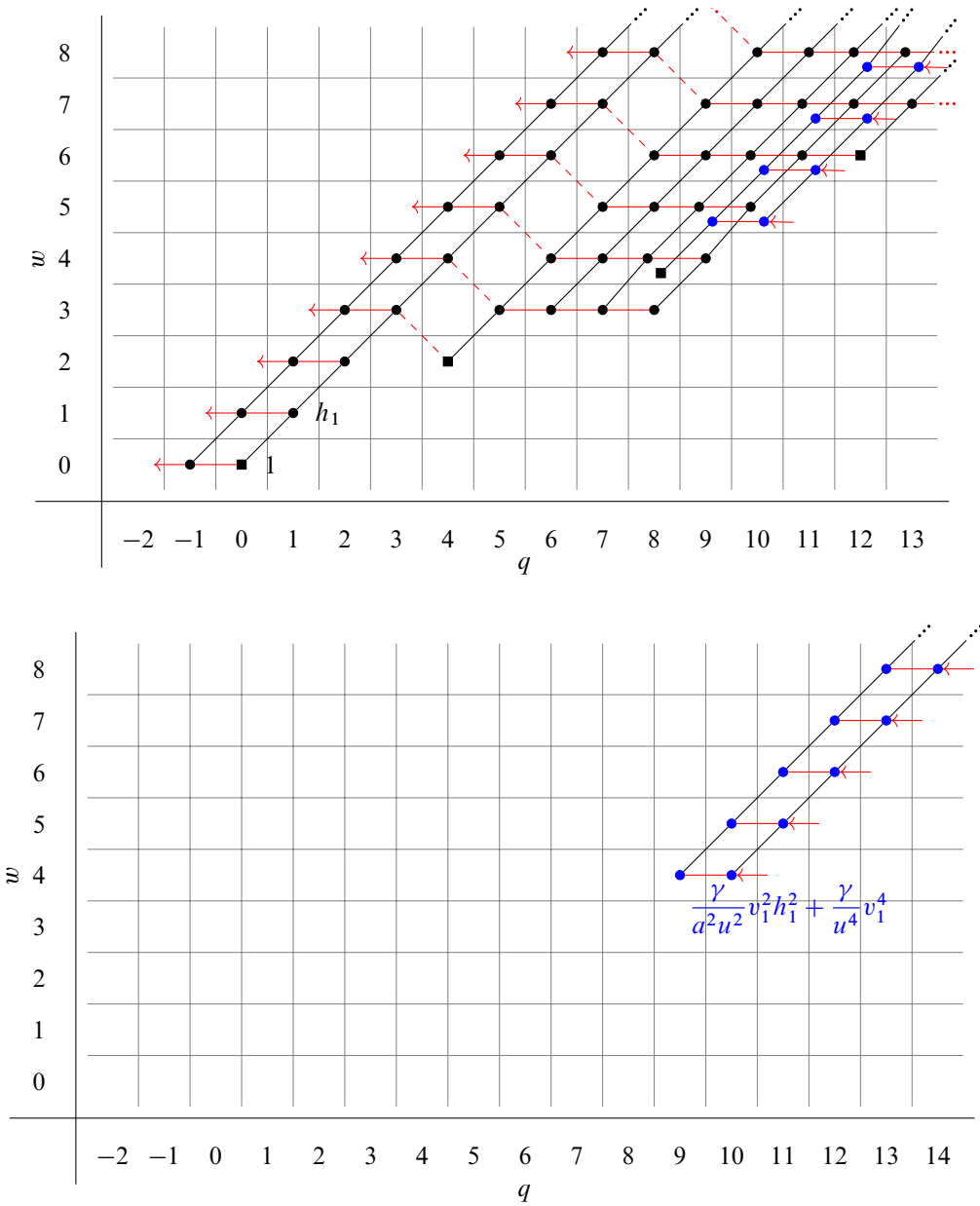


Figure 4: The E_∞-page of the $\tilde{\tau}$-Bockstein spectral sequence for $\pi_{0,q,w}(\mathrm{kq}_2)$, top, and $\pi_{-1,q,w}(\mathrm{kq}_2)$, bottom.

Remark 10.3 The groups $\pi_{-1,*,*}\mathrm{kq}_2$ and $\pi_{0,*,*}\mathrm{kq}_2$ assemble, via the multiplication by a long exact sequence, into the homotopy groups $\pi_{0,*,*}(Ca\otimes \mathrm{kq}_2)$. These groups in turn record the bigraded homotopy of the $\mathbb{C}$-motivic kq_2. The reader may note an interesting extension, of the form

$$2\mathbb{Z}_2[\tau]\oplus \tau\mathbb{Z}_2[\tau]\to \mathbb{Z}_2[\tau]\to \mathbb{F}_2,$$

which appears when computing $\pi_{0,8,*}(\mathrm{kq}_2^\wedge \otimes Ca)$. This extension is related to the orange dashed line in the final chart of [Kong 2020].

Appendix A Recollections on compact rigid generation

In this appendix we recall some useful material on compact rigid generation. Most of this material has appeared elsewhere and all of it is certainly known to experts. This appendix was mostly included for the convenience of the authors, though we do hope the reader finds it to be a concise summary of a basic technique in higher algebra.

Definition A.1 A stable, presentably monoidal category $\mathcal{C}$ is *rigidly generated* if it has a family of compact dualizable generators.[27]

Our goal will be to study some properties of monoidal left adjoints out of a rigidly generated category. This begins with the following construction.

Construction A.2 Given an $\mathbb{E}_n$-monoidal left adjoint $f^*:\mathcal{C}\to\mathcal{D}$, its right adjoint f_* is lax $\mathbb{E}_n$-monoidal. In particular, $f_*(\mathbb{1})$ is an $\mathbb{E}_n$-ring, so we obtain a factorization of f^* as an $\mathbb{E}_{n-1}$-monoidal left adjoint into

$$\mathcal{C}\xrightarrow{-\otimes f_*\mathbb{1}}\mathrm{Mod}(\mathcal{C};\, f_*\mathbb{1})\xrightarrow{g^*}\mathcal{D}.$$

We will say that the adjunction f is 0-*affine* if g is an adjoint equivalence. Note that being 0-affine is a property of the underlying monoidal functor.

The main result of this appendix is a convenient criterion for f to be 0-affine. Before we can state that result we make a preparatory definition.

Definition A.3 Given a monoidal left adjoint $f^*:\mathcal{C}\to\mathcal{D}$, we may consider the projection map,

$$X\otimes f_*Y\xrightarrow{\eta_f} f_*f^*(X\otimes f_*Y)\simeq f_*(f^*X\otimes f^*f_*Y)\xrightarrow{\epsilon_f} f_*(f^*X\otimes Y).$$

We will say that f *satisfies the projection formula at* X, if for all Y the projection map is an equivalence. If f satisfies the projection formula at X for all X, then we say that f *satisfies the projection formula*.

Proposition A.4 *Given a monoidal left adjoint $f^*:\mathcal{C}\to\mathcal{D}$ between presentable categories, the following statements are true*:

(i) *f satisfies the projection formula for dualizable objects in $\mathcal{C}$.*

(ii) *If $\mathcal{C}$ is rigidly generated and $\mathcal{D}$ has compact unit, then f_* preserves colimits.*

(iii) *If $\mathcal{C}$ is rigidly generated and f_* preserves colimits, then f satisfies the projection formula.*

(iv) *f_* is conservative if and only if the essential image of f^* contains a family of generators.*

(v) *If f satisfies the projection formula, f_* preserves colimits and f_* is conservative, then f is 0-affine.*

[27] Here and throughout this appendix we only require that an object has a one-sided dual. The side on which an object has a dual will not be important.

Our arguments follow those from [Mathew et al. 2017] fairly closely, though the hypotheses are somewhat different. Before proceeding to the proof of this proposition we give some examples to demonstrate its effectiveness.

Example A.5 In Appendix B we study the category $\mathrm{Sp}^{\mathbf{Fil}}$ of filtered spectra, where we denote the shift map $\mathbb{1}(-1) \to \mathbb{1}$ by τ. The associated graded functor $\mathbf{Gr} : \mathrm{Sp}^{\mathbf{Fil}} \to \mathrm{Sp}^{\mathbf{Gr}}$ satisfies the conditions of Proposition A.4, so we obtain an equivalence of presentably symmetric monoidal categories

$$\mathrm{Sp}^{\mathbf{Gr}} \cong \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; C\tau).$$

Here $C\tau$ acquires a commutative algebra structure from the fact that it equivalent to the image of the unit under the right adjoint of $\mathbf{Gr}$.

Example A.6 Again working with $\mathrm{Sp}^{\mathbf{Fil}}$, the realization functor $\mathrm{Re} : \mathrm{Sp}^{\mathbf{Fil}} \to \mathrm{Sp}$ satisfies the conditions of Proposition A.4, so we obtain an equivalence of presentably symmetric monoidal categories

$$\mathrm{Sp} \cong \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; \mathbb{1}[\tau^{-1}]).$$

Here $\mathbb{1}[\tau^{-1}]$ acquires a commutative algebra structure from the fact that it is equivalent to the image of the unit under Y (the right adjoint of Re).

Example A.7 Given a finite extension of characteristic zero fields ℓ/k we have a symmetric monoidal left adjoint $i^* : \mathcal{SH}(k) \to \mathcal{SH}(\ell)$. Using resolution of singularities, Nagata compactification and purity we know that both of these categories are generated by the motives of smooth projective schemes, each of which is dualizable [Röndigs and Østvær 2008]. In short, both the source and target categories are rigidly generated, with families of compact dualizable generators given by $\mathbb{S}^m \otimes (\mathbb{P}^1)^{\otimes n} \otimes X$, where X is a smooth projective scheme over the base.

In order to show that i is 0-affine we only need to show that the image of i^* contains a family of generators. Given a smooth projective ℓ-scheme X the projection formula Proposition A.4(i) gives us an equivalence

$$i_*(\mathbb{S}^m \otimes (\mathbb{P}^1)^{\otimes n} \otimes X) \simeq \mathbb{S}^m \otimes (\mathbb{P}^1)^{\otimes n} \otimes i_* X.$$

Since for field extensions one has $i_\sharp = i_*$, we learn that $i_* X = X$, where the second copy of X is considered as a k-scheme. It will now suffice to show that X is a retract of $i^* i_* X$. At the level of schemes this means looking at $X \times_{\mathrm{Spec}(\ell)} \mathrm{Spec}(\ell) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\ell)$. Using the maps $\ell \to \ell \otimes_k \ell \to \ell$ we may conclude.

Stated more explicitly, we have an equivalence of presentably symmetric monoidal categories

$$\mathcal{SH}(\ell) \cong \mathrm{Mod}(\mathcal{SH}(k); \mathrm{Spec}(\ell)).$$

Using the fact that i restricts to the subcategories of Artin–Tate objects the same argument provides an equivalence of presentably symmetric monoidal categories

$$\mathcal{SH}(\ell)^{\mathrm{AT}} \cong \mathrm{Mod}(\mathcal{SH}(k)^{\mathrm{AT}}; \mathrm{Spec}(\ell)).$$

Example A.8 The category of C_2-spectra is rigidly generated and the underlying spectrum functor, Φ^e, is essentially surjective. We may thus apply Proposition A.4 to obtain a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathrm{Sp}_{C_2}; R_1) \simeq \mathrm{Sp},$$

where R_1 is the image of $\mathbb{S}$ under the right adjoint to Φ^e. Since Φ^e can be described as homotopy fixed points (or homotopy orbits) composed with $(C_2)_+ \otimes -$, we find that its right adjoint is given by tensoring with $(C_2)_+ \simeq Ca_\sigma$. Therefore, $R_1 \simeq Ca_\sigma$.

Similarly, with Φ^e replaced by Φ^{C_2} we obtain a symmetric monoidal equivalence

$$\mathrm{Mod}(\mathrm{Sp}_{C_2}; R_2) \simeq \mathrm{Sp},$$

where R_2 is the image of $\mathbb{S}$ under the right adjoint to geometric fixed points. We can compute that $R_2 \simeq \mathbb{S}[a_\sigma^{-1}]$ using the presentation of C_2-spectra via an isotropy separation square.

Example A.9 Taking loops on the map $i : S^1 \to B\,\mathrm{Pic}(\mathrm{Sp}_{C_2})$ which sends a generator to $\mathbb{S}^\sigma$ we get a monoidal functor $i : \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp}_{C_2})$. Embedding the target into Sp_{C_2} and tensoring up to spectra we obtain a monoidal left adjoint,

$$i^* : \mathrm{Sp}^{\mathbf{Gr}} \to \mathrm{Sp}_{C_2},$$

which sends $\mathbb{S}(1)$ to $\mathbb{S}^\sigma$. Since $\mathrm{Sp}^{\mathbf{Gr}}$ is rigidly generated, the unit in Sp_{C_2} is compact and the representation spheres generate Sp_{C_2} we may apply Proposition A.4 to conclude that i is 0-affine. Stated more explicitly, we have an equivalence of presentable categories,

$$\mathrm{Sp}_{C_2} \cong \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Gr}}; i_*\,\mathbb{S}).$$

The graded ring $i_*\,\mathbb{S}$ has n^{th} term given by $\mathrm{Map}^{\mathrm{Sp}}(\mathbb{S}^{n\sigma}, \mathbb{S})$. Expressed in terms of stunted projective spaces we have $(i_*\,\mathbb{S})_n \simeq \Sigma\mathbb{R}\mathrm{P}^{-n-1}_{-\infty}$.

Proof of Proposition A.4(i)–(v) (i) Consider the following commutative diagram, which is natural in $X \in \mathcal{C}^{\mathrm{dbl}}$, $Y \in \mathcal{D}$ and $Z \in \mathcal{C}$:

$$\begin{array}{ccccc}
 & & \overset{\simeq}{\frown} & & \\
\mathrm{Map}_{\mathcal{C}}(X^\vee \otimes Z, f_*Y) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f^*(X^\vee \otimes Z), f^*f_*Y) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f^*(X^\vee \otimes Z), Y) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 & & \mathrm{Map}_{\mathcal{D}}(f^*(X^\vee) \otimes f^*Z, f^*f_*Y) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f^*(X^\vee) \otimes f^*Z, Y) \\
 & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Map}_{\mathcal{C}}(Z, X \otimes f_*Y) & & \mathrm{Map}_{\mathcal{D}}((f^*X)^\vee \otimes f^*Z, f^*f_*Y) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}((f^*X)^\vee \otimes f^*Z, Y) \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Map}_{\mathcal{D}}(f^*Z, f^*(X \otimes f_*Y)) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{D}}(f^*Z, f^*X \otimes f^*f_*Y) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f^*Z, f^*X \otimes Y)
\end{array}$$

The rectangle on the left commutes due to the compatibility of dualization with the monoidal structure on f^*. The remaining squares commute for easier reasons. Now observe that starting in the middle of

the left side and proceeding counterclockwise gives the projection map, while proceeding clockwise gives an equivalence.

(ii) Since $\mathcal{C}$ is rigidly generated it has a family of compact dualizable generators. Monoidal functors send dualizable objects to dualizable objects. Since the unit of $\mathcal{D}$ is compact we learn that f^* sends dualizable objects to compact objects. Thus, since f^* sends a family of compact generators to compact objects its right adjoint f_* preserves colimits.

(iii) The projection formula asks that the natural projection map

$$X \otimes f_* Y \to f_*(f^* X \otimes Y)$$

be an equivalence. Using the hypotheses that f_* preserves colimits and $\mathcal{C}$ is rigidly generated we can reduce to the case where X is a compact dualizable generator. We may now use Proposition A.4(i) to conclude.

(iv) Clear.

(v) We begin by showing that g^* is fully faithful. This is equivalent to showing that the unit map $X \to g_* g^* X$ is an equivalence. Applying the projection formula with $Y = \mathbb{1}$, we can conclude that this is true for induced $f_*\mathbb{1}$-modules. Since f_* preserves colimits and $\mathrm{Mod}(\mathcal{C}; f_*\mathbb{1})$ is generated by induced $f_*\mathbb{1}$-modules this is sufficient to conclude.

Now, we show that g^* is essentially surjective. By Proposition A.4(iv) we know the essential image of f^* contains a family of generators (and g^* has the same property). Using full faithfulness we can now conclude that g^* is essentially surjective. □

We close the appendix with another useful lemma.

Lemma A.10 *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are stable presentably symmetric monoidal categories, and let R be a commutative algebra in $\mathcal{C}$. Given a symmetric monoidal left adjoint $f^* : \mathcal{C} \to \mathcal{D}$, there is an equivalence of presentably symmetric monoidal categories*

$$\mathrm{Mod}(\mathcal{D}; f^* R) \simeq \mathrm{Mod}(\mathcal{C}; R) \otimes_{\mathcal{C}} \mathcal{D}\,.$$

Proof This follows from [Lurie 2017, Theorem 4.8.5.16] after unraveling the definitions. □

Example A.11 Given a stable presentably symmetric monoidal category $\mathcal{C}$ and a commutative algebra $R \in \mathcal{C}^{\mathbf{Fil}}$, this lemma provides an equivalence

$$\mathrm{Sp}^{\mathbf{Gr}} \otimes_{\mathrm{Sp}^{\mathbf{Fil}}} \mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; R) \simeq \mathcal{C}^{\mathbf{Gr}} \otimes_{\mathcal{C}^{\mathbf{Fil}}} \mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; R) \simeq \mathrm{Mod}(\mathcal{C}^{\mathbf{Gr}}; \mathbf{Gr}(R)).$$

Appendix B Recollections on filtered objects

In this appendix we give a more detailed introduction to filtered objects. The results here are well-known (see [Lurie 2015]), and we aim mainly to fix notation.

Convention B.1 In this appendix, $\mathcal{C}$ will denote a stable, presentable category. We will use $\mathbb{1}$ to denote the unit of $\mathcal{C}$ when it is monoidal.

Definition B.2 We let $\mathbb{Z}$ denote the symmetric monoidal category with underlying category given by the discrete set $\mathbb{Z}$ and symmetric monoidal structure given by addition.

Similarly, we let $\mathbb{Z}^{\mathbf{Fil}}$ denote the symmetric monoidal category with underlying category given by the poset $\mathbb{Z}$ with its order $\leq$, so that there is a unique map $n \to m$ whenever $n \leq m$, and symmetric monoidal structure given by addition.

Definition B.3 Given a category $\mathcal{C}$, we let $\mathcal{C}^{\mathbf{Fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathbf{Fil},\mathrm{op}}, \mathcal{C})$ denote the category of filtered objects in $\mathcal{C}$. Objects of $\mathcal{C}^{\mathbf{Fil}}$ are diagrams

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \cdots$$

in the category $\mathcal{C}$. We will sometimes use the notation $C_\bullet$ for an object of $\mathcal{C}^{\mathbf{Fil}}$.

- There is a natural left adjoint $c : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$, which sends C to the constant object

  $$\cdots \to 0 \to 0 \to C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} \cdots$$

  that is equal to C in nonpositive degrees and 0 in positive degrees.
- There is a natural left adjoint $Y : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$, which sends C to the constant object

  $$\cdots \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} \cdots$$

  that is equal to C in each degree.
- The functor Y admits a left adjoint $\mathrm{Re} : \mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}$, which sends $C_\bullet$ to $\varinjlim_n C_{-n}$.
- The category $\mathcal{C}^{\mathbf{Fil}}$ admits natural automorphisms $(k) : \mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}^{\mathbf{Fil}}$ that send $C_\bullet$ to $C_{\bullet-k}$.
- There is a natural transformation $\tau : (-1) \to \mathrm{Id}$, which captures the shift map in the filtration. We depict τ on cX below:

  $$\begin{array}{ccccccccccc}
  \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow & \cdots \\
  & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
  \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow & X & \longrightarrow & \cdots
  \end{array}$$

  If $\mathcal{C}$ is monoidal, then we will refer to the cofiber of $\tau : \mathbb{1}(-1) \to \mathbb{1}$ as $C\tau$.

We let $\mathcal{C}^{\mathbf{Gr}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$ denote the category of graded objects in $\mathcal{C}$. Objects of $\mathcal{C}^{\mathbf{Gr}}$ are collections $\{C_n\}_n$ of objects of $\mathcal{C}$. We will sometimes use the notation C_* for an object of $\mathcal{C}^{\mathbf{Gr}}$.

- There is a natural fully faithful left adjoint $c : \mathcal{C} \to \mathcal{C}^{\mathbf{Gr}}$, which sends C to C_* with $C_0 = C$ and $C_k = 0$ for $k \neq 0$.
- There is a natural left adjoint $\mathbf{Gr} : \mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}^{\mathbf{Gr}}$, which sends

  $$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \cdots$$

  to $\{C_n/C_{n+1}\}_n$.

If $\mathcal{C}$ is a presentably $\mathbb{E}_n$-monoidal category, then $\mathcal{C}^{\mathbf{Fil}}$ and $\mathcal{C}^{\mathbf{Gr}}$ inherit the structure of $\mathbb{E}_n$-monoidal categories under Day convolution, and the functors c, Y, Re, c and $\mathbf{Gr}$ are all $\mathbb{E}_n$-monoidal.

Remark B.4 Using the assumption that $\mathcal{C}$ is stable and presentable we can offer another description of the categories of filtered and graded objects, which is often useful in proofs.

$$\mathcal{C}^{\mathbf{Fil}} \simeq \mathrm{Sp}^{\mathbf{Fil}} \otimes \mathcal{C} \quad \text{and} \quad \mathcal{C}^{\mathbf{Gr}} \simeq \mathrm{Sp}^{\mathbf{Gr}} \otimes \mathcal{C}.$$

Since it is well-known that the analogues of c, Y, Re, c and $\mathbf{Gr}$ are symmetric monoidal in the case of spectra, the claims about $\mathbb{E}_n$ monoidality made above follow by tensoring up.

Using the fact that $\mathbb{1}(1) \otimes \mathbb{1}(-1) \simeq \mathbb{1}$, we learn that if X is a dualizable object of $\mathcal{C}$, then $c(X)(n)$ is a dualizable object of $\mathcal{C}^{\mathbf{Fil}}$. Similarly, we have that if $\{X_\alpha\}$ is a set of compact (dualizable) generators of $\mathcal{C}$, then $\{c(X_\alpha)(k)\}$ is a set of compact (dualizable) generators of $\mathcal{C}^{\mathbf{Fil}}$.

Lemma B.5 *Given an $\mathbb{E}_n$-monoidal category $\mathcal{C}$, the image of $\mathbb{1}$ under the right adjoint of* Re *is $\mathbb{1}[\tau^{-1}]$; therefore this object is an $\mathbb{E}_n$-algebra and there is an equivalence* $\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; \mathbb{1}[\tau^{-1}]) \simeq \mathcal{C}$ *of $\mathbb{E}_{n-1}$-monoidal categories. Similarly, the image of $\mathbb{1} \in \mathcal{C}^{\mathbf{Gr}}$ under the right adjoint of $\mathbf{Gr}$ is $C\tau$; therefore this object is an $\mathbb{E}_n$-algebra and there is an equivalence of $\mathbb{E}_{n-1}$-monoidal categories* $\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; C\tau) \simeq \mathcal{C}^{\mathbf{Gr}}$. *Moreover, the functors $- \otimes \mathbb{1}[\tau^{-1}]$ and $- \otimes C\tau$ are identified under these equivalences with* Re *and* $\mathbf{Gr}$, *respectively.*

Proof The case of spectra was handled in Examples A.5 and A.6. For a general $\mathcal{C}$ we just tensor the case of spectra with $\mathcal{C}$. □

Appendix C A machine for deforming homotopy theories

In this appendix, we have two goals. The first is to identify a technique which produces 1-parameter deformations in homotopy theory. This technique is essentially an elaboration of [Gheorghe et al. 2022, Definition 3.2]. The second goal is to provide a recognition criterion for 1-parameter deformations. This criterion will be applied in Section 4 of the main paper to identify $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ with the category of modules over a commutative algebra in $i2$-complete filtered C_2-spectra.

The approach to deforming categories taken here is different than that in Pstrągowski's theory [2023] of synthetic spectra, which is a theory specifically about deformations of Sp. When our construction here and Pstrągowki's construction are both defined, we will show that they are equivalent up to a minor completion issue. This in turn suggests that a more general version of Pstrągowski's approach is possible, where an arbitrary (symmetric monoidal) presentable category is deformed.

This appendix is not intended to be a definitive treatment of deformations. Instead, we view at as an illustration of a variety of elementary techniques, which in combination produce a large collection of interesting examples.

C.1 Constructing deformations

In this section we will give techniques for producing 1-parameter deformations.

Convention C.1 In this section, $\mathcal{C}$ will denote a stable presentably symmetric monoidal category. We will use $\mathbb{1}$ to denote the unit of $\mathcal{C}$, and assume the notation of Appendix B.

The deformations we produce will all be of the form

$$\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; R)$$

for some kind of algebra[28] R. Therefore, what we really do is give methods for constructing algebras in filtered objects.

We will produce commutative algebras in $\mathcal{C}^{\mathbf{Fil}}$ through the following method. Given a lax symmetric monoidal functor $F : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$, the image of the unit $F(\mathbb{1})$ is a commutative algebra in $\mathcal{C}^{\mathbf{Fil}}$. If we let L denote the composite $\mathrm{Re} \circ F$, then we have the following proposition.

Proposition C.2 *The presentably symmetric monoidal category* $\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; F(\mathbb{1}))$ *is a* 1*-parameter deformation in the sense that*:

(i) *There is a colimit-preserving symmetric monoidal functor out of* $\mathcal{C}^{\mathbf{Fil}}$ *with target* $\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; F(\mathbb{1}))$.

(ii) *The generic fiber is given by* $\mathrm{Mod}(\mathcal{C}; L(\mathbb{1}))$, *in the sense that there is an equivalences of presentably symmetric monoidal categories*

$$\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; F(\mathbb{1})[\tau^{-1}]) \simeq \mathrm{Mod}(\mathcal{C}; L(\mathbb{1})).$$

(iii) *The special fiber is given by* $\mathrm{Mod}(\mathcal{C}^{\mathbf{Gr}}; \mathbf{Gr}\, F(\mathbb{1}))$, *in the sense that there is an equivalences of presentably symmetric monoidal categories*

$$\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; F(\mathbb{1}) \otimes C\tau) \simeq \mathrm{Mod}(\mathcal{C}^{\mathbf{Gr}}; \mathbf{Gr}(F(\mathbb{1}))).$$

Moreover, *when viewed as a lax symmetric monoidal functor*, F *factors as*

$$\mathcal{C} \xrightarrow{G} \mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; F(\mathbb{1})) \to \mathcal{C}^{\mathbf{Fil}}.$$

Proof This follows immediately from Lemmas B.5 and A.10. □

We will refer to lax symmetric monoidal functors $T : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$ as *tower functors* and give several examples.

Example C.3 The functors c and Y are tower functors.

Example C.4 The functor $\tau_{\geq \bullet} : \mathrm{Sp} \to \mathrm{Sp}^{\mathbf{Fil}}$ is a tower functor.

Example C.5 If $\mathcal{C}$ has a t-structure which is compatible with the tensor product in the sense that the unit is connective and the tensor product of two connective objects is connective, then

$$\tau_{\geq \bullet} : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$$

is a tower functor. More generally we have a tower functor $\tau_{\geq m\bullet} : \mathcal{C} \to \mathcal{C}^{\mathbf{Fil}}$ for natural numbers m.

[28] Although much of the material in this appendix applies for $\mathbb{E}_n$-algebras in $\mathbb{E}_n$-monoidal categories, the extra generality was not necessary for this work.

Construction C.6 Suppose we are given a collection A of coreflective subcategories $\mathcal{C}^A_{\geq k} \subset \mathcal{C}$ such that

- if $X \in \mathcal{C}^A_{\geq k}$ and $Y \in \mathcal{C}^A_{\geq \ell}$, then $X \otimes Y \in \mathcal{C}^A_{\geq k+\ell}$,
- $\mathcal{C}^A_{\geq k+1} \subset \mathcal{C}^A_{\geq k}$ and $\mathbb{1} \in \mathcal{C}^A_{\geq 0}$.

Let $\tau_{\geq k} : \mathcal{C} \to \mathcal{C}^A_{\geq k}$ denote the right adjoints of the inclusions. Then, we can assemble these categories into a single coreflective subcategory $\mathcal{C}^{\mathbf{Fil},A}_{\geq 0}$ consisting of filtered objects

$$\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

for which $X_i \in \mathcal{C}^A_{\geq i}$. The right adjoint $\tau^A_{\geq 0} : \mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}^{\mathbf{Fil},A}_{\geq 0}$ to the inclusion is given by applying $\tau_{\geq i}$ in position i.

Our assumptions guarantee that $\mathcal{C}^{\mathbf{Fil},A}_{\geq 0}$ is closed under the tensor product and so admits a natural presentably symmetric monoidal structure, for which the inclusion $\mathcal{C}^{\mathbf{Fil},A}_{\geq 0} \subset \mathcal{C}^{\mathbf{Fil}}$ is a symmetric monoidal functor. As a consequence, we obtain a lax symmetric monoidal endofunctor

$$\tau^A_{\geq 0} : \mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}^{\mathbf{Fil}}.$$

Example C.7 If $\mathcal{C}$ is the category of G-equivariant spectra for some finite group G, then we can let $\mathcal{C}^{\mathrm{slice}}_{\geq k}$ be the regular slice k-connective G-spectra.[29] As a variant we could also take the regular slice mk-connective subcategories for some positive integer m.

Each of the above examples of tower functors can be described as a composite of Y with an appropriately chosen $\tau_{\geq 0}$. This construction can be considered a generalization of the twisted t-structures on filtered objects considered in Section 5.2. Thus, so far we have only produced "truncation type towers". We now give a construction which takes in a tower functor T and a commutative algebra in $\mathcal{C}$ and produces a new tower functor. This construction will have the effect of shearing the Adams spectral sequence based on E along the tower T. To begin, we review the monoidal properties of the cobar construction.

Construction C.8 Since the coproduct of commutative algebras in $\mathcal{C}$ is given by the tensor product, the cobar construction can be upgraded into a functor

$$\mathrm{cb} : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}^{\Delta}).$$

Construction C.9 Given a tower functor T and a commutative algebra E in $\mathcal{C}$, we define a new tower functor $\mathrm{Sh}(T; E)$, which is the composite

$$\mathcal{C} \xrightarrow{-\otimes \mathrm{cb}(E)} \mathcal{C}^{\Delta} \xrightarrow{T} \mathcal{C}^{\mathbf{Fil},\Delta} \xrightarrow{\mathrm{Tot}} \mathcal{C}^{\mathbf{Fil}}.$$

On spectra, the tower functor $\mathrm{Sh}(\tau_{\geq \bullet}; \mathbb{F}_p)$ produces the décalage of the $\mathbb{F}_p$-Adams tower on the input. More generally, given a t-structure t that is compatible with the monoidal structure, we can let $\mathcal{C}^t_{\geq n}$ be the n-connective objects. Then $\mathrm{Sh}(\tau^t_{\geq 0}; E)(X)$ captures the E-based Adams spectral sequence for the

[29] Note that we cannot use the classical slice filtration here, unless m is divisible by $|G|$, in which case it agrees with the regular one, since it is not compatible with the tensor product.

t-structure homotopy groups of X. As an analogue of the fact that the E-Adams spectral sequence for E collapses, we have the following:

Example C.10 The cosimplicial diagram $E \otimes \mathrm{cb}(E)$ admits a contracting homotopy, and therefore the totalization commutes with any functor. In particular, we obtain an equivalence of commutative algebras, $\mathrm{Sh}(T; E)(E) \simeq T(E)$.

We close with a simple lemma that lets us identify the generic fiber in certain cases.

Lemma C.11 *Suppose we are given a collection of coreflective subcategories A that satisfy the conditions of Construction C.6, together with a commutative algebra $E \in \mathcal{C}^{A}_{\geq 0}$. Then for any $X \in \mathcal{C}^{A}_{\geq k}$ there is an equivalence* $\mathrm{Re}(\mathrm{Sh}(\tau^{A}_{\geq 0}; E)) \simeq X^{\wedge}_{E}$, *where the second object is the E-nilpotent completion of X.*

Proof By construction $\tau^{A}_{\geq 0}$ comes equipped with a natural transformation $\tau^{A}_{\geq 0} \to Y$. Since $\mathrm{Sh}(Y; E) \simeq Y(-)^{\wedge}_{E}$, we have a natural transformation

$$\mathrm{Sh}(\tau^{A}_{\geq 0}; E) \to Y(X)^{\wedge}_{E}.$$

In sufficiently negative degrees the $\tau_{\geq i}$ have no effect by hypothesis, so the totalizations are levelwise equivalences. There is thus an equivalence

$$\mathrm{Re}(\mathrm{Sh}(\tau^{A}_{\geq 0}; E)) \simeq \mathrm{Re}Y(X)^{\wedge}_{E} \simeq (X)^{\wedge}_{E}. \qquad \square$$

Remark C.12 Under the assumptions of Lemma C.11, we have an equivalence

$$\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; \mathrm{Sh}(\tau^{A}_{\geq 0}; E)(\mathbb{1})[\tau^{-1}]) \simeq \mathrm{Mod}(\mathcal{C}; \mathbb{1}^{\wedge}_{E}).$$

C.2 Recognizing deformations

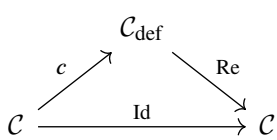
In this section, we give one answer to the following open-ended question:

Question C.13 Given a pair of presentably symmetric monoidal categories $\mathcal{C}_{\mathrm{def}}$ and $\mathcal{C}$, when can we identify $\mathcal{C}_{\mathrm{def}}$ with $\mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; R)$ for some commutative algebra $R \in \mathcal{C}^{\mathbf{Fil}}$?

This section grew out of a recognition that our original arguments in Section 4, which related $\mathcal{SH}(\mathbb{R})^{\mathrm{AT}}_{i2}$ and $\mathrm{Sp}_{C_2,i2}$, used only very general information.

Definition C.14 A deformation pair consists of the following data:

- A diagram of symmetric monoidal left adjoints



- A map of abelian groups

$$i_0 : \mathbb{Z} \to \ker\big(\pi_0 \mathrm{Pic}(\mathcal{C}_{\mathrm{def}}) \to \pi_0 \mathrm{Pic}(\mathcal{C})\big).$$

We denote the invertible object corresponding to $i_0(n)$ by $\mathbb{1}(n)$.

These data are subject to the following conditions:

- For $a \leq b$, the map on mapping spaces
$$\mathrm{Hom}_{\mathcal{C}_{\mathrm{def}}}(\mathbb{1}(a), \mathbb{1}(b)) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$$
induced by Re is an equivalence.
- The category $\mathcal{C}$ is rigidly generated, and there is a set $\{C_\alpha\}$ of compact dualizable objects in $\mathcal{C}$ with the property that $\{c(C_\alpha) \otimes \mathbb{1}(n)\}$ is a set of compact dualizable generators for $\mathcal{C}_{\mathrm{def}}$.

Example C.15 Suppose that $\mathcal{C}$ is a rigidly generated, stable, presentably symmetric monoidal category. Then it is easy to verify that $(\mathcal{C}^{\mathbf{Fil}}, \mathcal{C})$ is a deformation pair where $i_0 : \mathbb{Z} \to \mathrm{Pic}(\mathcal{C}^{\mathbf{Fil}})$ sends n to $\mathbb{1}(n)$.

Example C.16 Let E denote an Adams type homology theory, and denote Pstrągowski's category [2023] of E-synthetic spectra by Syn_E. This category is equipped with a natural notion of bigraded sphere, and we let $\mathrm{Syn}_E^{\mathrm{cell}} \subset \mathrm{Syn}_E$ denote the *cellular* subcategory[30] generated under colimits by $\mathbb{S}^{p,q}$. There is a natural realization functor $\mathrm{Syn}_E \to \mathrm{Sp}$, as well as a symmetric monoidal left adjoint $\mathrm{Sp} \to \mathrm{Syn}_E$ provided by [Lurie 2017, Corollary 4.8.2.19].

Then $(\mathrm{Syn}_E^{\mathrm{cell}}, \mathrm{Sp})$ is a deformation pair, where the map i_0 picks out the spheres $\mathbb{S}^{0,n}$. The first condition is satisfied by [Pstrągowski 2023, Corollary 4.12], and the second is satisfied since we restricted to the full subcategory generated by the bigraded spheres.

Example C.17 Another example is given by $(\mathcal{SH}(\mathbb{C})_{ip}^{\mathrm{cell}}, \mathrm{Sp}_{ip})$, where the map i_0 picks out the Tate twists. The realization functor here is Betti realization, and we set c to be the unital, colimit-preserving map in from Sp_{ip} as in Example 1.6. By restricting to cellular objects the second condition is automatically satisfied. The first condition is ultimately a corollary of the vanishing of the homotopy of $C\tau$ in positive Chow degrees. This example is discussed at length in [Gheorghe et al. 2022].

Given a deformation pair $(\mathcal{C}_{\mathrm{def}}, \mathcal{C})$, we would like to construct a symmetric monoidal left adjoint $\mathcal{C}^{\mathbf{Fil}} \to \mathcal{C}_{\mathrm{def}}$ to which we may apply Proposition A.4. We will do this in two steps:

(i) We construct a symmetric monoidal functor $i : \mathbb{Z}^{\mathbf{Fil}} \to \mathcal{C}_{\mathrm{def}}$, which sends n to $\mathbb{1}(n)$.

(ii) We tensor $i^{\mathbf{Fil}}$ up to $\mathcal{C}$ using c.

Construction C.18 Given a deformation pair $(\mathcal{C}_{\mathrm{def}}, \mathcal{C})$, we construct a square of symmetric monoidal functors,

$$\begin{array}{ccc} \mathbb{Z}^{\mathbf{Fil}} & \xrightarrow{\ i\ } & \mathcal{C}_{\mathrm{def}} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Re}} \\ * & \longrightarrow & \mathcal{C} \end{array}$$

such that $i(n) = \mathbb{1}(n)$.

[30] When E is $\mathbb{F}_p$ or MU, then $\mathrm{Syn}_E^{\mathrm{cell}} = \mathrm{Syn}_E$.

Details Let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}$ on the unit. Let $\mathcal{D}_{\mathrm{def}}$ denote the full subcategory of $\mathcal{C}_{\mathrm{def}}$ on the objects $\mathbb{1}(n)$ in the image of i_0. Since $\mathcal{D}_{\mathrm{def}}$ and $\mathcal{D}$ are closed under the tensor product they are each symmetric monoidal categories.

We now form the following diagram of symmetric monoidal categories

$$\begin{array}{ccccccc}
\mathcal{D}' & \xrightarrow{f} & \mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\mathbf{Fil}} & \xrightarrow{\mathrm{Re}\times\mathrm{Id}} & \mathcal{D} \times \mathbb{Z}^{\mathbf{Fil}} & \xrightarrow{\pi_2} & \mathbb{Z}^{\mathbf{Fil}} \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\
 & & \mathcal{D}_{\mathrm{def}} & \xrightarrow{\mathrm{Re}} & \mathcal{D} & \longrightarrow & *
\end{array}$$

where $\mathcal{D}'$ is the full subcategory of $\mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\mathbf{Fil}}$ spanned by the objects $(\mathbb{1}(n), n)$. Since $\mathcal{D}'$ is closed under the monoidal structure, it canonically inherits a symmetric monoidal structure from $\mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\mathbf{Fil}}$.

We claim that the composite $\mathcal{D}' \to \mathcal{D} \times \mathbb{Z}^{\mathbf{Fil}}$ is an equivalence. The objects of $\mathcal{D}'$ may be identified with pairs $(\mathbb{1}(n), n)$ and the objects of $\mathcal{D} \times \mathbb{Z}^{\mathbf{Fil}}$ may be identified with pairs $(\mathbb{1}, n)$. The mapping spaces are given by

$$\mathrm{Hom}_{\mathcal{D}'}((\mathbb{1}(n), n), (\mathbb{1}(m), m)) = \begin{cases} \mathrm{Hom}_{\mathcal{D}_{\mathrm{def}}}(\mathbb{1}(n), \mathbb{1}(m)) & \text{if } n \le m, \\ \varnothing & \text{if } n > m, \end{cases}$$

$$\mathrm{Hom}_{\mathcal{D}\times\mathbb{Z}^{\mathbf{Fil}}}((\mathbb{1}, n), (\mathbb{1}, m)) = \begin{cases} \mathrm{Hom}_{\mathcal{D}}(\mathbb{1}, \mathbb{1}) & \text{if } n \le m, \\ \varnothing & \text{if } n > m, \end{cases}$$

with Re giving the maps between these mapping spaces. Each of these maps of mapping spaces is an equivalence, by hypothesis.

The composition of symmetric monoidal functors

$$\mathbb{Z}^{\mathbf{Fil}} \xrightarrow{\iota} \mathcal{D} \times \mathbb{Z}^{\mathbf{Fil}} \xleftarrow{\simeq} \mathcal{D}' \to \mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\mathbf{Fil}} \to \mathcal{D}_{\mathrm{def}} \subset \mathcal{C}_{\mathrm{def}}$$

is the desired symmetric monoidal functor $\mathbb{Z}^{\mathbf{Fil}} \to \mathcal{C}_{\mathrm{def}}$, and indeed sends n to $\mathbb{1}(n)$. The square of Construction C.18 commutes, because

$$\begin{aligned} \mathrm{Re} \circ \pi_1 \circ f \circ (\mathrm{Re} \times \mathrm{Id})^{-1} \circ \iota &= \pi_1 \circ (\mathrm{Re} \times \mathrm{Id}) \circ f \circ ((\mathrm{Re} \times \mathrm{Id}) \circ f)^{-1} \circ \iota \\ &= \pi_1 \circ \iota = *. \end{aligned}$$

□

Construction C.19 Now that we have the symmetric monoidal functor $i : \mathbb{Z}^{\mathbf{Fil}} \to \mathcal{C}_{\mathrm{def}}$, we may induce it up to a symmetric monoidal left adjoint, $i^* : \mathrm{Sp}^{\mathbf{Fil}} \to \mathcal{C}_{\mathrm{def}}$. Tensoring up to $\mathcal{C}$ using c, we can use Construction C.18 to build a diagram of symmetric monoidal left adjoints

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{c} & \mathcal{C}^{\mathbf{Fil}} & \xrightarrow{\mathrm{Re}} & \mathcal{C} \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\mathcal{C} & \xrightarrow{c} & \mathcal{C}_{\mathrm{def}} & \xrightarrow{\mathrm{Re}} & \mathcal{C}
\end{array}$$

Proposition C.20 *Suppose that $(\mathcal{C}_{\mathrm{def}}, \mathcal{C})$ is a deformation pair. Then, the symmetric monoidal left adjoint i^* from Construction C.18 is 0-affine. Stated more explicitly, we have a diagram of symmetric monoidal left adjoints as shown:*

$$\begin{array}{ccccccc}
\mathcal{C} & \xrightarrow{c} & \mathcal{C}^{\mathbf{Fil}} & \xrightarrow{-\otimes i_*\mathbb{1}} & \mathrm{Mod}(\mathcal{C}^{\mathbf{Fil}}; i_*\mathbb{1}) & \xrightarrow{\mathrm{Re}} & \mathcal{C} \\
\downarrow{\scriptstyle \mathrm{Id}} & & & \searrow^{i^*} & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\mathcal{C} & & \xrightarrow{\quad c \quad} & & \mathcal{C}_{\mathrm{def}} & \xrightarrow{\mathrm{Re}} & \mathcal{C}
\end{array}$$

Proof Construction C.19 produces most of the desired diagram. The remaining claims will follow from an application of Proposition A.4.

By hypothesis, $\mathcal{C}$ and $\mathcal{C}_{\mathrm{def}}$ are both rigidly generated. Therefore, in order to apply Proposition A.4 we only need to check that the essential image of i^* contains a family of generators. Since $i^*(C_\alpha \otimes \mathbb{1}(n)) \simeq c(C_\alpha) \otimes \mathbb{1}(n)$, this is true by hypothesis. □

Remark C.21 The n^{th} piece of i_*X can be extracted by taking the $\mathcal{C}$-enriched mapping object, i.e., there is an equivalence

$$(i_*X)_n \simeq \mathrm{Map}^{\mathcal{C}}(\mathbb{1}(n), i_*X).$$

Now, since the adjunction i is $\mathcal{C}$-linear, we have an equivalence

$$\mathrm{Map}^{\mathcal{C}}(\mathbb{1}(n), i_*X) \simeq \mathrm{Map}^{\mathcal{C}}(i^*\mathbb{1}(n), X).$$

We close by considering once again the deformation context of synthetic spectra (Example C.16). Our aim will be to show that we can (nearly) identify $i_*\mathbb{1}$ with a commutative algebra produced via the constructions from the previous subsection.

Proposition C.22 *Given an Adams-type, commutative algebra E in* Sp, *there is an equivalence of presentably symmetric monoidal categories,*

$$\mathrm{Mod}(\mathrm{Syn}_E^{\mathrm{cell}}; \mathbb{1}^\wedge_\tau) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; \mathrm{Sh}(\tau_{\geq \bullet}; E)(\mathbb{S})).$$

Proof In Example C.16 we showed that $\mathrm{Syn}_E^{\mathrm{cell}}$ and Sp form a deformation pair. Using Proposition C.20 we obtain an adjunction i and an equivalence, $\mathrm{Syn}_E^{\mathrm{cell}} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathbf{Fil}}; i_*\,\mathbb{S})$. The proposition will now follow from an identification of the τ-completion of $i_*\,\mathbb{S}$.

The identification of $i_*\,\mathbb{S}$ will follow the pattern established in Section 4. We begin by identifying $i_*\nu(E^{\otimes k})$. By [Pstrągowski 2023, Proposition 4.21] we know that the n^{th} piece of this object is n-connective. Therefore, the natural comparison map $i_*\nu(E^{\otimes k}) \to Y(E^{\otimes k})$ factors as

$$i_*\nu(E^{\otimes k}) \to \tau_{\geq \bullet} E^{\otimes k} \to Y(E^{\otimes k}).$$

Examining [Pstrągowski 2023, Proposition 4.21] more closely we can actually conclude that the first map is an equivalence.

Now, we can pass to totalizations and observe the chain of equivalences:

$$(i_*\nu\,\mathbb{S})^\wedge_\tau \xrightarrow{\simeq} i_*((\nu\,\mathbb{S})^\wedge_\tau) \xrightarrow{\simeq} i_*\mathrm{Tot}^*(\mathrm{cb}(\nu E)) \xrightarrow{\simeq} i_*\mathrm{Tot}^*(\nu(\mathrm{cb}(E)))$$
$$\xrightarrow{\simeq} \mathrm{Tot}^*(i_*\nu(\mathrm{cb}(E))) \xrightarrow{\simeq} \mathrm{Tot}^*(\tau_{\geq\bullet}\mathrm{cb}(E)) \xrightarrow{\simeq} \mathrm{Sh}(\tau_{\geq\bullet}; E)(\mathbb{S}).$$

The first equivalence uses that i_* is a right adjoint. The second equivalence follows from [Burklund et al. 2023, Proposition A.11]. The third equivalence follows from [Pstrągowski 2023, Lemma 4.24], together with the assumption that E is Adams type. □

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