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Harmonic maps into Euclidean buildings and non-Archimedean superrigidity

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We prove that harmonic maps into Euclidean buildings, which include $\mathbb{R}$-buildings, have singular sets of Hausdorff codimension 2, extending the locally finite regularity result of Gromov and Schoen. As an application, we prove superrigidity for algebraic groups over fields with non-Archimedean valuation, thereby generalizing the rank-1 p-adic superrigidity results of Gromov and Schoen and casting the Bader–Furman generalization of Margulis' higher-rank superrigidity result in a geometric setting. We also prove an existence theorem for a pluriharmonic map from a Kähler manifold to a Euclidean building.

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1 Introduction

Gromov and Schoen's celebrated result [1992] established p-adic superrigidity and the consequent arithmeticity for lattices of certain rank-1 groups. Alongside the rank-1 Archimedean superrigidity result of Corlette [1992], these findings complement Margulis' higher-rank superrigidity results [1991]. Both Corlette and Gromov–Schoen's theorems are geometric superrigidity results. In other words, they determine conditions under which an isometric action on a complete CAT(0) space has a fixed point or leaves a convex subset invariant. The proof is through harmonic map techniques, with Gromov–Schoen's proof notably involving the generalization of classical harmonic map techniques to a singular setting.

A Euclidean building X is equipped with a distance function d which makes (X, d) into a Hadamard space (i.e., a complete metric space satisfying CAT(0) triangle comparison). Euclidean buildings share similarities with Riemannian symmetric spaces of noncompact type which make them natural subjects of geometric study. Kleiner and Leeb [1997] illustrated this connection by showing that asymptotic cones of Riemannian symmetric spaces are Euclidean buildings.

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More broadly, J Tits [1986] introduced buildings to provide a geometric interpretation of a certain class of groups. Specifically, given a semisimple group G over a field endowed with a non-Archimedean valuation, Tits constructed a metric space X where G acts by isometries. Special cases are the Bruhat–Tits buildings [1972] associated with p-adic Lie groups, which are distinguished by the property that they are locally finite Euclidean buildings.

The technical achievement of [Gromov and Schoen 1992] lies in developing a harmonic map theory applicable to singular spaces which include Bruhat–Tits buildings for p-adic groups. In particular, Gromov and Schoen establish that the singular set of a harmonic map into such a building is small. This enables them to utilize nonlinear Bochner techniques developed by Siu [1980] and Corlette [1992] for maps into smooth manifolds to prove rigidity results. (See [Freidin 2019; Freidin and Zhang 2020; Zhang et al. 2019] for Bochner formulas for harmonic maps into singular spaces.) To the best knowledge of the authors, harmonic map theory remains the only known method for proving superrigidity in rank-1 cases.

When the valuation is not discrete, the work of [Gromov and Schoen 1992] no longer applies. In particular, analyzing harmonic maps into the associated Euclidean building becomes challenging due to the absence of local finiteness in the space. The main technical result of this paper is the following. When $\dim X = 1$, this theorem is due to Sun [2003].

Theorem 1.1 *If $u : \Omega \to X$ is a harmonic map from a smooth Riemannian domain with Lipschitz boundary into a Euclidean building (possibly an $\mathbb{R}$-building), then the singular set of u is a closed set of Hausdorff codimension 2.*

The *singular set* is the set of points satisfying the property that none of its neighborhoods is mapped into a single apartment. See Definition 7.1 for the precise definition. Note that in the locally finite setting, for $\dim \Omega = n$, Dees [2022] recently proved the stronger result that the singular set is in fact $(n-2)$-rectifiable.

A consequence of Theorem 1.1 is the following rigidity theorem. For lattices in groups of rank 1, this generalizes the geometric superrigidity results of Corlette [1992] and Gromov and Schoen [1992]. For lattices in groups of rank ≥ 2, this constitutes the geometric superrigidity assertion corresponding to the group-theoretic statement of Bader and Furman [2022] proved through the dynamics of semisimple Lie groups.

Theorem 1.2 *Let $\tilde{M} = G/K$ be an irreducible symmetric space of noncompact type that is not the Euclidean space, $\mathrm{SO}_0(p,1)/\mathrm{SO}(p)\times\mathrm{SO}(1)$, nor $\mathrm{SU}_0(p,1)/S(U(p)\times U(1))$. Let Γ be a lattice in G and let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a homomorphism where X is a Euclidean building (possibly an $\mathbb{R}$-building), and $\rho(\Gamma)$ does not fix a point at infinity. If the rank of $\tilde{M}$ is ≥ 2, we additionally assume that Γ is cocompact. Then $\rho(\Gamma)$ fixes a point of X.*

Next, we consider the case when the domain is a Kähler manifold. Following [Gromov and Schoen 1992, Section 7], a harmonic map from a Kähler manifold to a Euclidean building is called *pluriharmonic* if it is pluriharmonic in the usual sense away from the singular set.

Theorem 1.3 *Let $\widetilde{M}$ be the universal cover of a complete finite volume Kähler manifold (M,ω). Let $\Gamma=\pi_1(M)$, X a Euclidean building (possibly an $\mathbb{R}$-building) and $\rho:\Gamma\to \mathrm{Isom}(X)$ a group homomorphism. Then any finite-energy ρ-equivariant harmonic map $u:\widetilde{M}\to X$ is pluriharmonic. In particular if there exists a ρ-equivariant map of finite energy into X, then there exists a ρ-equivariant pluriharmonic map into X.*

Theorem 1.2 is a further extension of the development of harmonic map techniques in singular spaces to address geometric superrigidity problems. Pioneered by Gromov and Schoen [1992], these singular spaces encompass locally finite hyperbolic buildings [Daskalopoulos et al. 2011], the Weil–Petersson completion of Teichmüller space [Daskalopoulos and Mese 2021] and now, nonsimplicial Euclidean buildings.

Theorem 1.3 builds upon the work of Gromov and Schoen [1992, Section 9], which asserts the existence of pluriharmonic maps in the singular setting. A notable application is in the factorization theorems for Zariski dense representations of fundamental groups of algebraic varieties. Corlette and Simpson [2008] proved that such a representation into $\mathsf{SL}_2(\mathbb{C})$ factors through an orbicurve if it is nonrigid or not integral. The rigidity aspect can be interpreted as the statement that a representation into $\mathsf{SL}_2(\mathbb{C}(t))$ goes into a compact subgroup. The core argument in their proof involves harmonic maps to the Bruhat–Tits building (also known as the Serre tree) of $\mathsf{SL}_2(\mathbb{C}(t))$. Since the Serre tree for $\mathsf{SL}_2(\mathbb{C}(t))$ is not locally compact, Gromov–Schoen theory does not apply, and they have to make a reduction mod p to the case of representations in $\mathsf{SL}_2(\mathbb{F}_p((t)))$, where $\mathbb{F}_p$ is a finite field. Hence, this part of their paper can be greatly simplified by Theorem 1.3 or by Sun's treatment [2003] of harmonic maps into $\mathbb{R}$-trees. Our theorem can further be exploited in the generalizations to nonlocally finite buildings that have been studied by various authors; see, e.g., [Katzarkov et al. 2015, 2017]. Additionally, Theorem 1.3 can be applied to study factorization theorems for higher-rank local system.

All of these theorems expand the scope of harmonic map techniques by allowing general Euclidean building targets. An important example is the harmonic map that appears as the limit of a sequence of rescaled maps associated to a sequence of harmonic maps into a symmetric space with unbounded energy. This is an important tool in understanding the compactification of representation varieties [Wolf 1995; Daskalopoulos et al. 1998; Loftin et al. 2026].

1.1 Main ideas

Let Ω be a Lipschitz Riemannian domain and let (X,d) be a Euclidean building. We review concepts from [Gromov and Schoen 1992] that generalize analytical notions for real-valued functions. These key concepts are also used in our paper.

- *Order of a harmonic map $u:\Omega\to(X,d)$ at $x_0\in\Omega$* [Gromov and Schoen 1992, Section 2]: For a harmonic function f, the order at x_0 is the degree of the dominant homogeneous harmonic polynomial approximating $f(x)-f(x_0)$ near x_0.

- *Homogeneous degree* 1 *maps into X* [Gromov and Schoen 1992, Section 3]: These are map with the property that the restriction to a radial ray is a constant speed geodesic. Homogeneous degree 1 harmonic maps generalize affine functions.

• *Intrinsically differentiable maps into X* [Gromov and Schoen 1992, Section 5]: These are maps that can be approximated near a point by a homogeneous degree 1 harmonic error term going to zero faster than distance to the point. Intrinsically differentiable maps generalize differentiable functions.

• *Blowup maps u_σ of a harmonic map $u : \Omega \to (X, d)$ at $x_0 \in \Omega$* [Gromov and Schoen 1992, Section 3]: For $\sigma > 0$ small, restrict u to $B_\sigma(x_0)$ and rescale in the domain by a factor of σ with respect to normal coordinates centered at x_0 and rescale the distance function of X by an appropriate constant μ_σ dependent on σ to construct $u_\sigma : B_1(0) \to (X, \mu_\sigma^{-1} d)$. Blowup maps generalize the difference quotients of functions.

We now revisit the main components in the proof of Gromov–Schoen's regularity statement. The assumption that X is locally finite plays a pivotal role, presenting a challenge when extending the analysis to general Euclidean buildings. The two key components of Gromov and Schoen's proof are:

(1) *A harmonic map $u : \Omega \to (X, d)$ near $x_0 \in \Omega$ is approximated by its tangent map u_* at x_0.* For a locally finite building X, there is a neighborhood U of a point $u(x_0) \in X$ and a neighborhood V of the vertex of the tangent cone $T_{u(x_0)}X$ such that U and V are isometric. Using this fact, Gromov and Schoen can assume that the blowup maps u_σ and u_* map into a metric cone. Indeed, identifying U with V, the rescaling of the distance function is equivalent to rescaling the cone. Thus, they can assume that for all $\sigma > 0$ sufficiently small, u_σ maps into the tangent cone $T_{u(x_0)}X$. Applying Arzelà–Ascoli, they take the limit of a subsequence $u_k := u_{\sigma_k}$ to obtain a tangent map $u_* : B_1(0) \to T_{u(x_0)}X$.

(2) *The tangent map u_* at an order 1 point is effectively contained in a product space $\mathbb{R}^m \times Y$ where Y is a lower-dimensional Euclidean building.* For an order 1 point x_0 of u, a tangent map u_* of u at x_0 is a homogeneous degree 1 harmonic map and its image is a flat F, i.e., a copy of Euclidean space $\mathbb{R}^m$ isometrically and totally geodesically embedded in $T_{u(x_0)}X$. The union P_F of all apartments (i.e., all top-dimensional flats) containing F is a subbuilding of $T_{u(x_0)}X$ which is isometric to $\mathbb{R}^m \times Y$ where Y is a Euclidean building of dimension $N - m$. The simplicial structure of apartments in a locally finite building implies that u_* is then *effectively contained* in P_F [Gromov and Schoen 1992, Section 5]. See Figure 1.

Using the above ingredients, Gromov and Schoen [1992, Theorem 6.3] prove a regularity theorem by an inductive argument based on the dimension of X. The main step of the regularity theorem is [loc. cit., Theorem 5.1], which shows that, near an order 1 point, the image of a harmonic map is contained in the product space $\mathbb{R}^m \times Y$ from (2). Therefore, a harmonic map locally decomposes into two harmonic maps, one into $\mathbb{R}^m$ and another into Y. From this, they conclude that the singular set in a neighborhood of an order 1 point is of Hausdorff codimension 2 by the inductive hypothesis. Thus, by proving that the set of higher-order points is of Hausdorff codimension 2, they complete the proof of Theorem 1.1 for Bruhat–Tits buildings. Since the Gromov–Schoen theory exclusively addresses locally finite simplicial complexes, it cannot be directly applied to analyze the nonsimplicial target spaces considered in this paper.

The core of this paper involves proving the *local product structure* of a harmonic map at an order 1 point described in the above paragraph for a general Euclidean building. To do so, we capitalize on the Euclidean building structure to replace the reliance on the local finiteness and simplicial structure. An N-

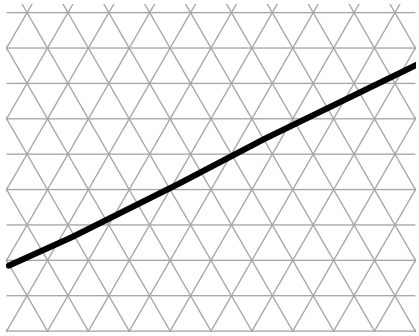

Figure 1: In the locally finite case, a homogeneous degree-1 map L is effectively contained in P_F. This diagram depicts an example when P_F is an apartment. The thin lines represent walls of an apartment and the thick line represent $F = L(\mathbb{R}^n)$. The map L is effectively contained in P_F since the preimage of walls is locally a finite set of hyperplanes. Therefore, the set of points mapping close to the complement of P_F is small since apartments only intersect along walls. For the non-locally-finite case, the situation is much more complicated. For example, the union of walls could be a dense set in an apartment.

dimensional Euclidean building is a union of apartments, i.e., isometrically embedded copies of Euclidean space $\mathbb{R}^N$. Moreover, $\mathbb{R}^N$ comes equipped with an affine Weyl group, i.e., a subgroup of the isometry group of $\mathbb{R}^N$ generated by reflections across hyperplanes, and such that its rotational part is a finite reflection group W. The group W plays an especially important role in our paper, which we highlight here.

First, because X is not necessarily locally compact, we cannot assume that the blowup maps u_σ centered at a point x_0 all have the same target space. Hence, instead of employing the conventional limit of a sequence of maps, we rely on the ultralimit $u_\omega = \omega\text{-lim}\, u_k$ (or alternatively, the Korevaar–Schoen limit u_* defined in [Korevaar and Schoen 1997, Section 3]) of blowup maps $u_k := u_{\sigma_k}$. The target space of u_ω is the ultralimit $X_\omega := \omega\text{-lim}\, X_k$, where $X_k := (X, \mu_{\sigma_k}^{-1} d)$ are the rescalings of X. By [Kleiner and Leeb 1997, Theorem 5.5.1], X_ω is also a Euclidean building of type W. If x_0 is an order 1 point, then u_ω is a homogeneous degree 1 harmonic map that can be extended to a map $L : \mathbb{R}^n \to X_\omega$ whose image is a flat F contained in an apartment A. We use the map L to "pull back" u_ω to X_k and construct a homogeneous degree 1 harmonic map L_k which is close to u_k. The subbuilding P_{F_k} associated to the flat $F_k = \text{image}(L_k)$ is isometric to a product $\mathbb{R}^m \times Y_k$ where the building structure of Y_k depends on W.

Second, because X_k is not necessarily locally finite, we cannot assume that L_k is effectively contained in the subbuilding P_{F_k}. To overcome this difficulty, we use the building structure of X that governs the way in which two apartments intersect. Indeed, the angle of intersection of two apartments is restricted by the finite reflection group W that defines the building structure of X, and hence of X_k. The main idea of this paper is to show that, using this structure, if the image of a harmonic map is not contained in P_{F_k}, there is a significant loss of energy when we project that harmonic map into P_{F_k}. Since u_k is close to L_k and $\text{image}(L_k) \subset P_{F_k}$, such a loss of energy contradicts the energy minimality of u_k. The crux of this paper is a careful analysis of the projection map $\pi_k : X_k \to P_{F_k}$ in order to derive a quantitative estimate of the energy loss when composing π_k and u_k.

1.2 Organization of this paper

Section 2 provides references for concepts most relevant to this paper. These are:

- [Gromov and Schoen 1992; Korevaar and Schoen 1993, 1997] for harmonic map theory into CAT(0) spaces,
- [Kleiner and Leeb 1997] for Euclidean buildings, and
- [Kleiner and Leeb 1997; Lytchak 2004] for ultralimits of metric spaces and Lipschitz maps.

Essential details are briefly summarized in that section. We recommend having these references readily available when reading this paper.

Section 3 discusses the subbuilding P_F of a Euclidean building X defined as a union of all apartments parallel to a given flat F. The closest-point projection map $\pi : X \to P_F$ is carefully analyzed. We apply this analysis to show that a composition $\pi \circ u$ of the projection map $\pi : X \to P_F$ with a harmonic map u into X results in a loss of energy.

Section 4 is the technical core of the paper. Expanding on the results of Section 3, we demonstrate that a harmonic map u that is close to a homogeneous degree 1 harmonic map L takes most points in the domain into the subbuilding P_F defined by the flat $F = \text{image}(L)$. Specifically, we show that "closeness in C^0" implies "closeness in measure"; i.e., the set of points not mapping to the subbuilding P_F via u can be made arbitrarily small by assuming that u is sufficiently close to a homogeneous degree 1 map L. This is analogous to key component (2) of Gromov–Schoen's proof, extended to the general setting considered here.

In Section 5, we construct a sequence of homogeneous degree 1 harmonic maps that approximates a harmonic map at a point. The argument presented in this section adapts key component (1) of Gromov–Schoen's argument to the present case.

Section 6 uses the "closeness in measure" statement of Section 4 and the approximation of blowup maps by homogeneous degree 1 harmonic maps described in Section 5 to show the local product structure of a harmonic maps u near order 1 points.

Section 7 uses the local product structure to first prove Theorem 1.1. Then the rigidity results, Theorems 1.2 and 1.3, follow from adapting the classical Bochner techniques described in [Gromov and Schoen 1992].

In the appendix we prove the technical results needed for Section 6; they are contained in [Gromov and Schoen 1992; Sun 2003], but we provide the complete proofs here as the results of this paper require that we better understand the constants, and what they depend on, that appear in their statements.

2 Preliminaries

2.1 CAT(0) spaces

A complete CAT(0) space (X, d) generalizes the notion of a Hadamard manifold. These are geodesic spaces of nonpositive curvature where curvature is defined by triangle comparison; particular examples of CAT(0) spaces include Euclidean buildings (the focus of this paper) as well as Hadamard manifolds.

We refer to [Bridson and Haefliger 1999] for a complete introduction to these spaces. For $P, Q \in X$ and $\lambda \in [0,1]$, we will denote the point on the unique geodesic segment $\overline{PQ}$ connecting P and Q at a distance $(1-\lambda)d(P,Q)$ from P by $(1-\lambda)P + \lambda Q$.

2.2 Euclidean buildings

Euclidean buildings are CAT(0) spaces with extra structure. In this paper, we use Kleiner and Leeb's notion of Euclidean buildings and refer to [Kleiner and Leeb 1997] for the precise definition. The equivalence of this notion and of that by Tits [1986] was established by A Parreau [2000]. Here, we only highlight concepts important to this paper.

Let $\mathbb{E}^N$ be an N-dimensional affine space and $\partial_{\text{Tits}}\mathbb{E}^N \simeq \mathbb{S}^{N-1}$ be its Tits boundary. Denote by $\rho : \text{Isom}(\mathbb{E}^N) \to \text{Isom}(\partial_{\text{Tits}}\mathbb{E}^N)$ the canonical homomorphism which assigns to each affine isometry its rotational part. An affine Weyl group W_{aff} is a subgroup of $\text{Isom}(\mathbb{E}^N)$ generated by reflections and whose *reflection group* $W := \rho(W_{\text{aff}}) \subset \text{Isom}(\partial_{\text{Tits}}\mathbb{E}^N)$ is finite. The pair $(\mathbb{E}^N, W_{\text{aff}})$ is then called a *Euclidean Coxeter complex*. A *wall* is a hyperplane of $\mathbb{E}^N$ which is the fixed-point set of a reflection in W_{aff}.

Let (X,d) be a CAT(0) space endowed with a structure which makes it into a Euclidean building modeled on a Coxeter complex $(\mathbb{E}^N, W_{\text{aff}})$ in the sense of [Kleiner and Leeb 1997, Section 4.1.2]. We refer to such an (X,d) (or more simply X) as a *Euclidean building of type W*.

We refer to the integer N as the *dimension* of X. As a consequence of [Kleiner and Leeb 1997, Corollary 4.6.2], there is a collection $\mathcal{A}$ of isometric embeddings $\iota : \mathbb{R}^N \to X$ satisfying the following two properties:

- Every geodesic segment, ray and line is contained in an image of an isometric embedding of the collection [Kleiner and Leeb 1997, EB3], and
- Two isometric embeddings ι_1, ι_2 of the collection are compatible in the sense that $\iota_1^{-1} \circ \iota_2$ is a restriction of an isometry in W_{aff} [Kleiner and Leeb 1997, EB4],

and such that $\mathcal{A}$ is the maximal collection satisfying the above two properties. We call $\mathcal{A}$ an *atlas*, $\iota \in \mathcal{A}$ a *chart* and the image $A := \iota(\mathbb{R}^N)$ an *apartment*.

For those less familiar with buildings, we add the following lemma which is probably well-known to the experts.

Lemma 2.1 *Let (X,d) be as in the above paragraph.*

- *If a Weyl group W'_{aff} contains W_{aff} as a subgroup, then (X,d) is also endowed with a structure of a Euclidean building modeled on a Coxeter complex $(\mathbb{E}^N, W'_{\text{aff}})$ with atlas*

$$\mathcal{A}' = \{\iota \circ w : \iota \in \mathcal{A} \text{ and } w \in W'_{\text{aff}}\}.$$

- *Let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation and define*

$$\varphi \cdot W_{\text{aff}} := \{\varphi^{-1} \circ f \circ \varphi \mid f \in W_{\text{aff}}\} \quad \text{and} \quad \varphi \cdot \mathcal{A} := \{\iota \circ \varphi \mid \iota \in \mathcal{A}\}.$$

Then (X, d) is also endowed with a structure of a Euclidean building modeled on the Coxeter complex $(\mathbb{E}^N, \varphi \cdot W_{\text{aff}})$ with atlas $\varphi \cdot \mathcal{A}$.

Proof It is straightforward to check conditions EB1–EB4 of [Kleiner and Leeb 1997, Section 4.1.2] and note that any two charts for an apartment A only differ by a precomposition with an isometry in W_{aff}. □

Remark 2.2 Let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation. By Lemma 2.1, (X, d) is a Euclidean building of type $\varphi \cdot W$ (also often described simply as a Euclidean building of type W). Changing the building structure does not change (X, d) as a CAT(0) space and does not affect the behavior of harmonic maps into (X, d); this fact is used in Sections 3.4 and 4.1.

2.3 Tangent cones

Let $\widetilde{\angle}_x(y, z)$ denote the angle at x of the comparison triangle in $\mathbb{R}^2$. If y', z' are interior points on the segments $\overline{xy}$, $\overline{xz}$, then $\widetilde{\angle}_x(y', z') \le \widetilde{\angle}_x(y, z)$. Thus, $\lim_{y' \to x, z' \to x} \widetilde{\angle}_x(y', z')$ exists, and we denote it by $\angle_x(y, z)$ [Kleiner and Leeb 1997, Section 2.1.3].

Definition 2.3 Two geodesics c_1, c_2 emanating from a common point $x \in X$ are said to be equivalent if $\angle_x(c_1, c_2) = 0$. A *geodesic germ* at x is an equivalence class of geodesics emanating from x. The space of geodesic germs at x along with the distance function defined by $\angle_x$ is a complete metric space by [Kleiner and Leeb 1997, Lemma 4.2.2] and defines the *space of directions* $\Sigma_x X$. By Section 4.2.2 of [loc. cit.], $\Sigma_x X$ is a spherical building modeled on $(\mathbb{S}^{N-1}, W)$.

Definition 2.4 For $x \in X$, the *tangent cone* $(T_x X, d_x)$ is a metric cone over $\Sigma_x X$. Denote the vertex of $T_x X$ by O. Any element of $T_x X \setminus \mathsf{O}$ can be written as $([\gamma], t)$ where $[\gamma]$ is a geodesic germ at x and $t \in (0, \infty)$.

Lemma 2.5 *If X is a Euclidean building of type W, then the tangent cone $(T_x X, d_x)$ is a Euclidean building of type W.*

Proof Let $\text{Isom}_0(\mathbb{E}^N)$ be the stabilizer subgroup of the origin and let $W_0 = \text{Isom}_0(\mathbb{E}^N) \cap \rho^{-1}(W)$. Using [Kleiner and Leeb 1997, Lemma 4.2.3], it is straightforward to check that $T_x X$ is a Euclidean building modeled on $(\mathbb{E}^N, W_0)$ and thus $T_x X$ is a Euclidean building of type W. □

If A is an apartment of X with $x \in A$, then $T_x A$ is an apartment of $T_x X$. Conversely, every apartment of $T_x X$ is of the form $T_x A$ for an apartment A in X containing the point x [Kleiner and Leeb 1997, Lemma 4.2.3].

We remark that the tangent cone $T_x X$ is variously called the tangent cone in [Kleiner and Leeb 1997], or the geodesic cone $C_x X$ in [Lytchak 2004], where a much more general notion of "tangent cones" is defined. We will see more notions from [Lytchak 2004] in Section 3.

2.4 Harmonic maps

The theory of harmonic maps into complete CAT(0) spaces was first developed in [Gromov and Schoen 1992] and generalized in [Korevaar and Schoen 1993, 1997; Jost 1994]. We refer the reader to these

papers for a more complete introduction to harmonic maps into CAT(0) spaces. In this paper, we are only interested in the case when the target space is a Euclidean building.

These papers also introduce the Sobolev space $W^{1,2}(\Omega, X)$ of finite-energy maps from a Riemannian domain (Ω, g) to a CAT(0) space X. For a map $u \in W^{1,2}(\Omega, X)$, they generalize the energy density function $|\nabla u|^2$ and the directional energy density function $|u_*(V)|^2$ for a Lipschitz vector field V defined on Ω. We reference [Korevaar and Schoen 1993, Section 1] for precise definitions. Note that these functions depend on the domain metric g, but we suppress g from the notation of the energy density function whenever it is clear from the context which domain metric we are using. Otherwise, we write $|\nabla u|_g^2$. Furthermore, we will write

$$\left|\frac{\partial u}{\partial x_i}\right|^2$$

to denote the directional energy density with respect to the coordinate vector field $\partial/\partial x_i$ and the domain metric g, although this is denoted by $|u_*(\partial/\partial x_i)|^2$ in [Korevaar and Schoen 1993].

Definition 2.6 For $u : (M, g) \to (X, d)$ and a fixed $\Omega \subset M$, the *energy of u on Ω in the metric g* is denoted by

$$^g E^u[\Omega] := \int_\Omega |\nabla u|^2 \, d\mu_g,$$

where μ_g denotes the volume measure with respect to g. We let δ_{euc} denote the Euclidean metric and when $g = \delta_{\text{euc}}$ we remove the exponent and write $E^u[\Omega]$.

To define harmonic maps, we use the notion of the trace of u, for $u \in W^{1,2}(\Omega, X)$, as defined in [Korevaar and Schoen 1993]. We denote the space of admissible maps by

$$W_u^{1,2}(\Omega, X) := \{h \in W^{1,2}(\Omega, X) : d(u, h) \in W_0^{1,2}(\Omega)\}.$$

Definition 2.7 A map $u : \Omega \to X$ is *harmonic* if, for every $p \in \Omega$, there exists $r > 0$ such that the restriction $u|B_r(p)$ minimizes energy amongst maps in $W_u^{1,2}(B_r(p), X)$.

The existence and uniqueness of energy minimizers from Riemannian domains into CAT(0) spaces was established in [Korevaar and Schoen 1993].

2.4.1 The order function Following [Gromov and Schoen 1992], for a map $u \in W^{1,2}(\Omega, X)$, a fixed point $p \in \Omega$, and $\sigma > 0$ such that $B_\sigma(p) \subset \Omega$, we let

$$E^u(\sigma) := \int_{B_\sigma(p)} |\nabla u|^2 \, d\mu_g \quad \text{and} \quad I^u(\sigma) := \int_{\partial B_\sigma(p)} d^2(u, u(p)) \, d\Sigma_g.$$

We define the order of u at p, at scale $\sigma > 0$, by

$$\text{Ord}^u(p, \sigma) := e^{c_1 \sigma^2} \frac{\sigma E^u(\sigma)}{I^u(\sigma)}, \tag{2-1}$$

where c_1 depends on the C^2-estimates of the metric g. We are presuming here that $I^u(\sigma)$ is not zero, a fact which follows easily for harmonic maps by modifying the arguments of [Gromov and Schoen 1992, Section 2] to all CAT(0) spaces. Following these arguments further implies that for appropriately

chosen c_1, $\mathrm{Ord}^u(p,\sigma)$ is monotone nondecreasing in σ for all harmonic maps u. It therefore makes sense to define the order at a point.

Definition 2.8 Let $u:\Omega\to X$ be a harmonic map. Then the *order of u at $p\in\Omega$* is given by

$$\mathrm{Ord}^u(p):=\lim_{\sigma\to 0}\mathrm{Ord}^u(p,\sigma).$$

Since $p\mapsto \mathrm{Ord}^u(p,\sigma)$ is continuous for a fixed σ, the map $p\mapsto \mathrm{Ord}^u(p)$ is upper semicontinuous.

2.5 Tangent maps and blowup maps

We follow [Gromov and Schoen 1992, Section 3] to construct homogeneous maps associated to harmonic maps which we call *tangent maps*. Because we want to consider Euclidean buildings that are not necessarily locally compact, we also rely on the notion of *convergence in the pullback sense*, introduced in [Korevaar and Schoen 1997]. This is a generalization of the Arzelà–Ascoli theorem for a sequence of maps into the same target space. Convergence in the pullback sense is defined for a sequence of maps where the target spaces may be different and not necessarily compact. We summarize this notion in Section 2.6 below, but refer to [Korevaar and Schoen 1997, Section 3] for more details.

Consider a harmonic map $u:(\Omega,g)\to(X,d)$ and $p\in\Omega$. Choose local normal coordinates centered at p and consider the restriction $u:(B_r(0),g)\to(X,d)$. For $0<\sigma<1$, define

$$u_\sigma:(B_1(0),g_\sigma)\to(X,d_\sigma), \tag{2-2}$$

where

$$u_\sigma(x):=u(\sigma x),\quad g_\sigma(x):=g(\sigma x)\quad\text{and}\quad d_\sigma(P,Q):=\left(\frac{\sigma^{n-1}}{I^u(\sigma)}\right)^{\frac{1}{2}}d(P,Q).$$

We will refer to u_σ as the (*Gromov–Schoen*) *blowup map*. A computation involving change of variables [Gromov and Schoen 1992, Section 3] shows that these maps have uniformly bounded energy $E^{u_\sigma}(1)\le 2\alpha$, where $\alpha=\mathrm{Ord}^u(p)$ for $\sigma>0$ sufficiently small. Thus, Theorem 2.4.6 of [Korevaar and Schoen 1993] implies that these maps have uniform Lipschitz estimates in any compactly contained subsets of $B_1(0)$. By Proposition 3.7 and Theorem 3.11 of [Korevaar and Schoen 1997], for any sequence $u_k=u_{\sigma_k}$, there exists a subsequence (which we will still denote by u_k) that converges locally uniformly in the pullback sense (see Section 2.6 below) to a limit map $u_*:B_1(0)\to(X_*,d_*)$ into a CAT(0) space. Furthermore, the energy density measures and the directional energy measures of u_k converge weakly to those of u_*. Moreover, following the last paragraph in the proof of Proposition 3.3 in [Gromov and Schoen 1992], we see that u_* is a *nonconstant homogeneous map of degree α*; see Section 2.8 below. The map u_* will be referred to as a *tangent map*. The rescaling and the limit preserve the order at the center. That is, $\alpha=\mathrm{Ord}^u(p)=\mathrm{Ord}^{u_\sigma}(0)=\mathrm{Ord}^{u_*}(0)$.

2.6 Convergence in the pullback sense

We give a brief summary of the limit space construction of [Korevaar and Schoen 1997, Section 3] and explain the notion of convergence in the pullback sense.

Following the notation of [Korevaar and Schoen 1997, Section 3], let $\Omega_0 = B_1(0)$ and iteratively define $\Omega_{i+1} := \Omega_i \times \Omega_i \times [0,1]$, inclusion maps $\Omega_i \to \Omega_{i+1}$ by $x \mapsto (x,x,0)$ and $\Omega_\infty := \bigcup \Omega_i$. Next, let d_∞ be a pseudodistance function defined on $\Omega_\infty \times \Omega_\infty$, denote its restriction to $\Omega_i \times \Omega_i$ by d_i, and assume

$$d_{i+1}^2(z,(x,y,\lambda)) \le (1-\lambda)d_{i+1}^2(z,(x,x,0)) + \lambda d_{i+1}^2(z,(y,y,0)) - \lambda(1-\lambda)d_i^2(x,y) \tag{2-3}$$

for $x, y \in \Omega_i$, $z \in \Omega_{i+1}$, $\lambda \in [0,1]$. Let Z be the metric completion of the quotient metric space $\Omega_\infty/\!\sim$ of Ω_∞, where $x \sim y$ if and only if $d_\infty(x,y) = 0$. The assumption (2-3) implies that Z is a CAT(0) space.

For each element $u_k = u_{\sigma_k}$ of a sequence of blowup maps defined in Section 2.5, let $u_{k,0} = u_k$ and iteratively define $u_{k,i+1} : \Omega_{i+1} \to X_k := (X, d_k)$ from $u_{k,i} : \Omega_i \to X_k$ by setting

$$u_{k,i+1}(x,y,\lambda) = (1-\lambda)u_i(x) + \lambda u_i(y) \quad \text{(see Section 2.1)}.$$

Let $d_{k,i}$ be the pullback pseudodistance of the map $u_{k,i}$ defined on $\Omega_i \times \Omega_i$. Then $d_{k,i+1}$ and $d_{k,i}$ satisfies (2-3) by the CAT(0) triangle inequality in X_k. The pseudodistance $d_{k,\infty}$ on $\Omega_\infty \times \Omega_\infty$ is defined by setting $d_{k,\infty}|\Omega_i \times \Omega_i = d_{k,i}$. Similarly, define $u_{*,i}$, $d_{*,i}$ and $d_{*,\infty}$ starting from $u_{*,0} = u_*$. When we say u_k *converges locally uniformly in the pullback sense* to u_*, we mean that $d_{k,i}$ converges locally uniformly to the pullback pseudodistance $d_{*,i}$. In this case, X_* is (isometric to) the metric completion of the quotient metric space $\Omega_\infty/\!\sim$ of Ω_∞, where $x \sim y$ if and only if $d_{*,\infty}(x,y) = 0$. The map u_* is the composition of the inclusion $\Omega \hookrightarrow \Omega_\infty$ followed by the natural projection map of $\Omega_\infty \to \Omega_\infty/\!\sim \subset X_*$.

2.7 Ultralimits of metric spaces and maps

For details on ultrafilters and ultralimits, we refer the reader to [Kleiner and Leeb 1997, Section 2.4] and [Lytchak 2004, Section 3.3]. We only give a quick summary here:

- Let $(X_k, d_k, \star_k)$ be a sequence of pointed metric spaces. The ultralimit

  $$(X_\omega, d_\omega) := \omega\text{-lim}(X_k, d_k, \star_k)$$

  is the quotient metric space of the set of all sequences (x_k) of points $x_k \in X_k$ which satisfy $\sup\{d_k(x_k, \star_k)\} < \infty$ with respect to the pseudometric $\widetilde{d}_\omega((x_k),(y_k)) := \omega\text{-lim}(d_k(x_k,y_k))$. In other words, a point of X_ω is an equivalence class $[(x_k)]$, where $d_\omega([(x_k)],[(y_k)]) = \widetilde{d}_\omega((x_k),(y_k))$.
- Let $f_k : (\widehat{X}_k, \widehat{d}_k) \to (X_k, d_k)$ be a sequence of maps between metric spaces with a uniform local Lipschitz bound. The ultralimit

  $$f_\omega := \omega\text{-lim}\, f_k : (\widehat{X}_\omega, \widehat{d}_\omega) \to (X_\omega, d_\omega)$$

  is the locally Lipschitz map defined by $f_\omega(p) := [(f_k(p))]$.

Let $u_k := u_{\sigma_k} : B_1(0) \to (X, d_k)$ be the sequence of blowup maps converging locally uniformly to $u_* : B_1(0) \to (X_*, d_*)$ as in Section 2.5. Then there exists an isometric totally geodesic embedding $\iota : X_* \to X_\omega$ such that

$$u_\omega = \iota \circ u_*.$$

Indeed, we construct ι by first defining $\hat{\iota}:\Omega_\infty \to X_\omega$ by setting $\hat{\iota}(x)=[(u_{k,i}(x))_{k=1}^\infty]$ for $x\in\Omega_i$. Then

$$d_\omega(\hat{\iota}(x),\hat{\iota}(y)) = \omega\text{-}\lim d_k(u_{k,i}(x),u_{k,i}(y)) = \lim_{k\to\infty} d_k(u_{k,i}(x),u_{k,i}(y)) = d_{*,i}(x,y).$$

In particular, if $x\sim y$ then $\hat{\iota}(x)=\hat{\iota}(y)$. Thus, $\hat{\iota}$ descends to the quotient $\Omega_\infty/\sim$ and can be isometrically extended to its metric completion to define $\iota: X_* \to X_\omega$. (See also [Kim 2017].)

Remark 2.9 Based on the above paragraph, we conclude the u_k converges locally uniformly in the pullback sense to u_ω. Thus, we can always replace u_* by u_ω. This fact is particularly useful because, given a harmonic map u to a Euclidean building X of type W, its tangent map u_ω at $x\in\Omega$ maps into a Euclidean building $X_\omega=\omega\text{-}\lim X_k$ of type W [Kleiner and Leeb 1997, Theorem 5.1.1].

2.8 Homogeneous harmonic maps

As discussed in Section 2.5, a tangent map u_* (and hence u_ω of Remark 2.9) is a homogeneous map. In this section, we show that the image of a homogeneous degree 1 map is contained in a single apartment.

Definition 2.10 Let $v: B_r(0)\subset\mathbb{R}^n\to(X,d)$ be such that $v\in W^{1,2}(B_r(0),X)$. We say v is *homogeneous degree* α if $\mathrm{Ord}^u(0,\sigma)=\alpha$ for all $\sigma\in(0,r)$.

Remark 2.11 Since the proof of [Gromov and Schoen 1992, Lemma 3.2] holds in any NPC space, a homogeneous degree α map $v: B_r(0)\subset\mathbb{R}^n\to(X,d)$ satisfies the following properties: For all $x\in\partial B_r(0)$ and $\lambda\in[0,1)$:

- $d(v(\lambda x),v(0))=\lambda^\alpha\, d(v(x),v(0))$.
- The image of $\lambda\mapsto v(\lambda x)$ is a geodesic in X.

In [Gromov and Schoen 1992], a map satisfying these two properties is referred to as *intrinsically* homogeneous.

A *flat* (or more descriptively, an m-flat) F of X is an image of an isometric embedding $\iota_F:\mathbb{R}^m\to X$. A *wall in* X is an image of a wall in $\mathbb{R}^N$ under an atlas. A geodesic line is an example of a 1-flat and a wall is an example of an $(N-1)$-flat. Every flat is contained in an apartment [Kleiner and Leeb 1997, Proposition 4.6.1], and N-flats are precisely the apartments of X [loc. cit., Corollary 4.6.2]. The next proposition shows that the image of a homogeneous degree 1 harmonic map into X is, as expected, a flat.

Proposition 2.12 *If $L: B_1(0)\to X$ is a homogeneous degree 1 harmonic map, then there exists an $r_0\in(0,1)$, an apartment A, and an extension of $L|B_{r_0}(0)$ as a homogeneous degree 1 harmonic map $\hat{L}:\mathbb{R}^n\to A\subset X$. In particular, $F:=\hat{L}(\mathbb{R}^n)$ is a flat.*

Proof This proof roughly follows that of [Gromov and Schoen 1992, Theorem 3.1], but we need to account for the more pathological behavior of the nonlocally compact target X. Following the initial part of the proof of [Gromov and Schoen 1992, Theorem 3.1], we deduce that there exists a linear map v and an isometric and totally geodesic embedding J such that $L=J\circ v|B_1(0)$. Note that this part of their proof does not use their hypothesis that X is locally compact.

If X is a geodesic cone in Euclidean space (as assumed in [Gromov and Schoen 1992, Theorem 3.1]), we can easily extend the map J, currently defined only on $\mathscr{V} := v(B_1(0)) \subset \mathbb{R}^m$, to an isometric and totally geodesic embedding defined on all of $\mathbb{R}^m$. Since we are not assuming X is a geodesic cone, some care must be taken to prove an analogous statement.

Let $p = L(0)$. Since T_pX is a cone, we can extend $H := \log_p \circ J : \mathscr{V} \to T_pX$ to a map $\hat{H} : \mathbb{R}^m \to T_pX$. Since J is an isometric and totally geodesic embedding of $\mathscr{V}$, this means that H, and hence $\hat{H}$, defines an isometric and totally geodesic embedding. By Proposition 4.6.1 and Lemma 4.2.3 of [Kleiner and Leeb 1997], the flat $\hat{H}(\mathbb{R}^m)$ is contained in an apartment T_pA of T_pX; i.e., $\hat{H} : \mathbb{R}^m \to T_pA$. Since $\log_p$ restricted to A is an isometry with inverse $(\log_p |A)^{-1} : T_pA \to A$, we can lift $\hat{H}$ to define an isometric and totally geodesic map $\hat{J} = (\log_p |A)^{-1} \circ \hat{H} : \mathbb{R}^m \to A$ in the diagram

$$\begin{array}{ccc} & & A \\ & \overset{\hat{J}}{\nearrow} & \big\downarrow \log_p \\ \mathbb{R}^m & \xrightarrow{\hat{H}} & T_pA \end{array}$$

Note that it is not necessarily true that $\hat{J} = J|\mathscr{V}$. For example, if $L = J \circ v : (-1, 1) \to X$ is a geodesic segment with $p = L(0)$, then there is a geodesic line $\mathscr{L} : \mathbb{R} \to X$ with $T_p(\mathscr{L}(\mathbb{R})) = T_p(L(\mathbb{R}))$. Since a geodesic segment does not *uniquely* extend to a geodesic line, this does not imply $\mathscr{L}$ and L agree on $(-1, 1)$. On the other hand, $\angle_p(\mathscr{L}(t), L(t)) = 0$ for $|t| > 0$; see Section 2.3 for the definition of $\angle_p$. By [Kleiner and Leeb 1997, Lemma 4.1.2], $\mathscr{L}$ and L initially coincide; i.e., there exists $r_0 > 0$ such that $\mathscr{L}|(-r_0, r_0) = L|(-r_0, r_0)$ and $\mathscr{L}$ is an extension of $L|(-r_0, r_0)$.

For the general case, we will prove that $\hat{J}$ agrees with J in a small neighborhood of the origin; that is, $\hat{J}$ is an extension of $J|v(B_{r_0}(0))$ for $r_0 > 0$ sufficiently small:

$$\begin{array}{ccc} & & A \cap B_\epsilon(p) \\ & \overset{\hat{J}=J}{\nearrow} & \big\downarrow \log_p \\ v(B_{r_0}(0)) & \xrightarrow{\hat{H}=H} & T_pA \cap B_\epsilon(\mathsf{O}) \end{array}$$

Following the arguments of [Kleiner and Leeb 1997, Section 4.4] regarding Weyl cones in X, we pick finitely many points in $\Sigma_p(\hat{J}(\mathscr{V})) = \Sigma_p(J(\mathscr{V}))$ whose convex hull is $\Sigma_p(\hat{J}(\mathscr{V})) = \Sigma_p(J(\mathscr{V}))$. Then the convex hull of the corresponding segments is the convex set $\hat{J}(\mathscr{V}) \cap J(\mathscr{V})$ and is a neighborhood $\mathscr{U}$ of p in $\hat{J}(\mathscr{V})$ and in $J(\mathscr{V})$. Choosing $r_0 > 0$ such that $v(B_{r_0}(0)) \subset \mathscr{U}$, we have that $\hat{J}|v(B_{r_0}(0)) = J|v(B_{r_0}(0))$. In other words, $\hat{L} = \hat{J} \circ v : \mathbb{R}^n \to X$ is a homogeneous degree 1 harmonic extension of $L|B_{r_0}(0) = J \circ v|B_{r_0}(0)$. □

2.9 (X, A, L)-triples

Many of our arguments use sequences of homogeneous degree 1 harmonic maps $L_k : \mathbb{R}^n \to (X_k, d_k)$ with different target spaces. We introduce a notion which relates these maps to a fixed homogeneous degree 1 harmonic map.

Definition 2.13 Let X be a Euclidean building of type W with the dimension of X at least 2, let A be an apartment of X and let $L : \mathbb{R}^n \to X$ be a homogeneous degree 1 harmonic map with $F := L(\mathbb{R}^n) \subseteq A$.

Now consider another Euclidean building X' of type W, an apartment A' of X', a homogeneous degree 1 harmonic map $L' : \mathbb{R}^n \to X'$ with $F' = L'(\mathbb{R}^n) \subseteq A'$. If there exists an isometry $\phi : A \to A'$ compatible with W (in the sense that if $\iota_1 : \mathbb{R}^N \to A$ and $\iota_2 : \mathbb{R}^N \to A'$ are charts in $\mathscr{A}$ and $\mathscr{A}'$, respectively, then $\iota_2^{-1} \circ \phi \circ \iota_1 : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry with rotational part in W) and $L' = \phi \circ L$, then we say that (X', A', L') is an (X, A, L)-*triple*.

Lemma 2.14 *If (X', A', L') is an (X, A, L)-triple then for every $\Omega \subset B_1(0)$,*

$$E^{L'}[\Omega] = E^{L}[\Omega]. \tag{2-4}$$

For all $i = 1, \dots, n$,

$$\int_\Omega \left| \frac{\partial L'}{\partial x_i} \right|^2 d\mu_0 = \int_\Omega \left| \frac{\partial L}{\partial x_i} \right|^2 d\mu_0, \tag{2-5}$$

where μ_0 is the n-dimensional Lebesgue measure.

Proof From the definitions, $d'(L'(x), L'(y)) = d'(\phi \circ L(x), \phi \circ L(y)) = d(L(x), L(y))$ and thus the result follows immediately. □

Remark 2.15 For $\sigma \in \mathbb{R}^+$ and $\sigma^{-1}X$ denoting $(X, \sigma^{-1}d)$, $(\sigma^{-1}X, \sigma^{-1}A, \sigma^{-1}L(\sigma x))$ is an (X, A, L)-triple.

2.10 Notation and conventions

We denote the C^2 distance between two metrics g, h by

$$\|g - h\|_{C^2(B_1(0))} := \max_{i,j,k,l=1,\dots,n} \sup_{B_1(0)} \left(|g_{ij} - h_{ij}| + \left| \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial h_{ij}}{\partial x^k} \right| + \left| \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} \right| \right).$$

It will often be convenient to work in the Euclidean coordinate system on a single normal coordinate chart of a Riemannian manifold M. Recall that if σ is sufficiently small, $(g_\sigma)_{ij}$ and $(\delta_{\text{euc}})_{ij}$ are close in the sense that the C^2 norm

$$\|g_\sigma - \delta_{\text{euc}}\|_{C^2(B_1(0))} \to 0 \quad \text{as } \sigma \to 0.$$

We will let

- μ_g denote the volume measure with respect to g, and
- μ_0^k denote the k-dimensional Lebesgue measure,

and we will suppress the k when $k = n$.

3 Projection into the subbuilding defined by a flat

In this section, we investigate the closest-point projection map $\pi : X \to P_F$ from a building to a subbuilding defined by a flat F. The goal is to quantify the "loss of energy" when we compose a harmonic map with π

(Proposition 3.16). In other words, we show that if $u : B_1(0) \to X$ is a harmonic map and $u(x) \notin P_F$, then $|\nabla(\pi \circ u)|^2(x)$ is less than $|\nabla u|^2(x)$ by a controlled amount.

Throughout this section, (X, d) is a Euclidean building of type W with dimension at least 2. An m-flat is a copy of Euclidean space $\mathbb{R}^m$ isometrically and totally geodesically embedded in X. We fix an m-flat F.

3.1 Subbuilding defined by a flat

A flat $F' \subset X$ is *parallel to* F if the Hausdorff distance between F and F' is bounded. Let P_F be the union of all flats parallel to the flat F.

Lemma 3.1 [Kleiner and Leeb 1997, Proposition 4.8.1] *The union P_F is a convex subbuilding and splits isometrically as*

$$P_F \simeq F \times Y, \tag{3-1}$$

where Y is itself a Euclidean building.

Since P_F is convex, we can define the closest-point projection map

$$\pi : X \to P_F.$$

Remark 3.2 In what follows, we need to maintain the building structure of X when we consider the subbuilding P_F. That is, rather than consider the canonical building structure (where P_F is thick [Kleiner and Leeb 1997, Proposition 4.9.2]), we continue to view P_F as a building of type W and preserve all walls of X which are contained in P_F.

Let $x \in X$ and $x_0 := \pi(x) \in P_F$. We write $x_0 = (f, y_0)$ and, using decomposition (3-1),

$$T_{x_0} P_F \simeq \mathbb{R}^m \times T_{y_0} Y.$$

This product structure is important in our analysis of the energy of the composition $\pi \circ u$ of the projection with a harmonic map. In particular, the product structure distinguishes the walls of $T_{x_0} P_F$ into two categories.

Definition 3.3 A wall $T_{x_0} H$ of $T_{x_0} P_F$ *contains* $T_{x_0} F$ if $T_{x_0} H \simeq \mathbb{R}^m \times T_{y_0} H_Y$, where $T_{y_0} H_Y$ is a wall of $T_{y_0} Y$. Otherwise, we say that $T_{x_0} H$ *does not contain* $T_{x_0} F$. (In view of Remark 3.2, not every wall of $T_{x_0} P_F$ must contain $T_{x_0} F$.)

The main idea in the proof of the "loss of energy" of $\pi \circ u$ is that, if $x_0 := \pi \circ u(p) \neq u(p)$, then the image of the blowup map of $\pi \circ u$ is contained in a wall that is transverse to $T_{x_0} F$ (a wall that does not contain $T_{x_0} F$) where F is the image of the homogeneous approximation L of u at p. Since the energy density of u at p agrees with the energy density of L, this implies the loss of energy at p. The goal of this section is to make this idea precise.

3.2 Differentiable maps

In this section, we consider the differentials of u and $\pi \circ u$ in the sense of Lytchak [2004] and Korevaar and Schoen [1993].

Given a sequence (ϵ_j) of positive numbers converging to zero and $x \in X$, we follow the notation of [Lytchak 2004] and let

$$(X_x^{(\epsilon_j)}, d_x^{(\epsilon_j)}) := \omega\text{-lim}\left(X, \frac{1}{\epsilon_j} d, x\right). \tag{3-2}$$

Thus, a point $Q = [(x_j)] \in X_x^{(\epsilon_j)}$ is an equivalence class of a sequence of points (x_j) in X with $(1/\epsilon_j) d(x_j, x) < C$ for some $C > 0$. Two sequences (x_j) and (y_j) are equivalent if the pseudodistance

$$d_x^{(\epsilon_j)}((x_j), (y_j)) := \omega\text{-}\lim \frac{1}{\epsilon_j} d(x_j, y_j) = 0.$$

Let $\mathbb{O}$ denote the vertex point of $T_{x_0}X$. Recall that $T_{x_0}X$ is isometrically embedded in $X_{x_0}^{(\epsilon_j)}$ as follows: For $Q_0 = ([\gamma_0], t_0) \in T_{x_0}X \setminus \{\mathbb{O}\}$, we view Q_0 as an element of $X_{x_0}^{\epsilon_j}$ by

$$([\gamma_0], t_0) \mapsto [(\gamma_0(\epsilon_j t_0)].$$

Here $[\gamma_0]$ denotes the equivalence class of a space of directions emanating from x_0. That is, given two arclength parametrizations γ_1, γ_2 of geodesics emanating from x_0,

$$\gamma_1 \sim \gamma_2 \iff \angle_{x_0}(\gamma_1, \gamma_2) = 0.$$

Given a sequence $\epsilon_j \to 0$, the blowup of a Lipschitz map $f : \Omega \to X$ at p with $x_0 = f(p)$ is

$$f_p^{(\epsilon_j)} : T_p\Omega \to X_{x_0}^{(\epsilon_j)} \quad \text{defined by} \quad f_p^{(\epsilon_j)}(v) = [(f(\exp_p(\epsilon_j v))] \in X_{x_0}^{(\epsilon_j)}.$$

Definition 3.4 We say $f : \Omega \to X$ is (*Lytchak*) *differentiable at* p if, for $f(p) = x_0$,

$$f_p^{(\epsilon_j)}(v) := [(f(\exp_p(\epsilon_j v))] \in T_{x_0}X \quad \text{for all } v \in T_p\Omega,$$

and $f_p^{(\epsilon_j)}(v)$ is independent of the choice of sequence $\epsilon_j \to 0$ [Lytchak 2004, Definition 7.1]. In this case, we denote the differential of f at p by

$$D_p f : T_p\Omega \to T_{x_0}X.$$

Thus, if $f_p^{(\epsilon_j)}(v) = Q_0 = ([\gamma_0], t_0)$, then by definition

$$\lim_{\epsilon_j \to 0} \frac{d(f(\exp_p(\epsilon_j v)), \gamma_0(\epsilon_j t_0))}{\epsilon_j} = 0, \tag{3-3}$$

and if f is differentiable at p then the independence in $\epsilon_j \to 0$ implies further that

$$\lim_{\tau \to 0} \frac{d(f(\exp_p(\tau v)), \gamma_0(\tau t_0))}{\tau} = 0. \tag{3-4}$$

Let $u : (B_1(0), g) \to (X, d)$ be a harmonic map. By [Korevaar and Schoen 1993, Theorem 2.4.6], u is locally Lipschitz continuous. Thus, [Lytchak 2004, Theorem 1.6] implies that the set

$$U' := \{p \in B_1(0) \mid D_p u \text{ and } D_p(\pi \circ u) \text{ exist}\}$$

is of full measure.

Since $u \in W^{1,2}(B_1(0), X)$ and $\pi \circ u \in W^{1,2}(B_1(0), P_F)$, there are associated generalized pullback metrics by [Korevaar and Schoen 1993, Theorem 2.3.2]. More precisely (replacing π by $\mathfrak{p}$ in [Korevaar and Schoen 1993, Theorem 2.3.2] since π is already used here for the projection map),

$$\mathfrak{p}^v : \Gamma(TB_1(0)) \times \Gamma(TB_1(0)) \to L^1(B_1(0)) \quad \text{defined by} \quad \mathfrak{p}^v(Z, W) = \tfrac{1}{4}|v_*(Z+W)|^2 - \tfrac{1}{4}|v_*(Z-W)|^2$$

is symmetric, bilinear, nonnegative, and tensorial for $v = u$ and $v = \pi \circ u$. For $p \in B_1(0)$, let $\{\partial_i^p\}$ be the coordinate vector fields with respect to normal coordinates centered at p. Define

$$U := \big\{p \in U' \mid \mathfrak{p}^u_p(\partial_i^p, \partial_i^p) = d_x^2(D_p u(\partial_i^p), \mathbb{O}_x) \text{ and } \mathfrak{p}^{\pi\circ u}_p(\partial_i^p, \partial_i^p) = d_{x_0}^2(D_p(\pi \circ u)(\partial_i^p), \mathbb{O}_{x_0}) \text{ for all } i = 1, \dots, n\big\},$$

where $x = u(p)$, $x_0 = \pi \circ u(p)$ and d_x, d_{x_0} and $\mathbb{O}_x$, $\mathbb{O}_{x_0}$ denote the distance functions and the origins of $T_x X$, $T_{x_0} P_F$, respectively. By [Korevaar and Schoen 1993, Lemma 1.9.5], U is full measure in U' and thus in $B_1(0)$.

The inner product structure defined by $\mathfrak{p}^{\pi\circ u}$ implies that the map $D_p(\pi \circ u)$ has the same pullback distance function as a linear map[1] $\ell : \mathbb{R}^n \to \mathbb{R}^n$. Thus, by the same argument as Proposition 2.12, the image of $D_p(\pi \circ u)$ is a flat of $T_{x_0} P_F$.

3.3 Differential of the projection into P_F

We choose a point $p \in U$, so in particular both u and $\pi \circ u$ are (Lytchak) differentiable at p. Denote the differentials of u and $\pi \circ u$ at p as

$$D_p u : T_p B_1(0) \to T_x X \quad \text{and} \quad D_p(\pi \circ u) : T_p B_1(0) \simeq \mathbb{R}^n \to T_{x_0} P_F.$$

Since $p \in U$, by the discussion in the previous subsection, the differentials $D_p u$ and $D_p(\pi \circ u)$ can be viewed as linear maps from $\mathbb{R}^n$. Consequently,

$$D_p(\pi \circ u)(\mathbb{R}^n) := F_{x_0}$$

is a flat and contained in some apartment of $T_{x_0} P_F$. Similarly, $F_x := D_p u(\mathbb{R}^n) \subset T_x X$ is a flat of $T_x X$.

Let

$$\Pi : T_{x_0} X \to T_{x_0} P_F$$

[1] The matrix $A = (\mathfrak{p}^{\pi\circ u}_{ij})$ is a symmetric matrix and has a decomposition $A = Q^t D Q = (\sqrt{D}Q)^t \sqrt{D} Q$, where Q is an orthogonal matrix, D is a diagonal matrix with eigenvalues on the diagonal entries, $\sqrt{D}$ is the diagonal matrix with square root of the eigenvalues on the diagonal entries, and t means transpose. The matrix $\sqrt{D}Q$ defines a linear map $\ell : \mathbb{R}^n \to \mathbb{R}^n$ such the pullback metric $\ell^* \delta$ is $\mathfrak{p}^{\pi\circ u}$.

be the closest-point projection map. We will prove (Lemma 3.8 below) that

$$F_{x_0} \subset \Pi(T_{x_0}X).$$

Lemma 3.5 *Denote by $\partial T_{x_0}P_F$ the topological boundary of $T_{x_0}P_F$ as a subset of $T_{x_0}X$. Then*

$$F_{x_0} \subset \partial T_{x_0}P_F.$$

In particular, for any $Q_0 \in F_{x_0}$,

$$T_{Q_0}(T_{x_0}X) \setminus T_{Q_0}(T_{x_0}P_F) \neq \varnothing.$$

Proof For an arbitrarily chosen $Q_0 \in F_{x_0}$, we will show that $Q_0 \in \partial T_{x_0}P_F$. If $Q_0 = O_{T_{x_0}X}$, the vertex of the cone $T_{x_0}X$, then let $Q_\epsilon = ([\sigma_0], \epsilon)$ where $\sigma_0(s)$ is the arclength parametrization of a geodesic from x_0 to x. Then $Q_\epsilon \notin T_{x_0}P_F$ and $Q_\epsilon \to Q_0$ which proves $Q_0 \in \partial T_{x_0}P_F$.

We will henceforth assume $Q_0 \neq O_{T_{x_0}X}$. Let $v \in \mathbb{R}^n \simeq T_pB_1(0)$ be such that

$$D_pu(v) = Q = ([\gamma], t) \in T_xX \quad \text{and} \quad D_p(\pi \circ u)(v) = Q_0 = ([\gamma_0], t_0) \in T_{x_0}P_F,$$

where γ (resp. γ_0) is an arclength parametrization of a geodesic ray with $\gamma(0) = x$ (resp. $\gamma_0(0) = x_0$).

The assumption that u (resp. $\pi \circ u$) is Lytchak differentiable at p implies that

$$\lim_{\tau\to 0^+} \frac{d(u(\exp_p(\tau v)), \gamma(t\tau))}{\tau} = 0 \quad \text{and} \quad \lim_{\tau\to 0^+} \frac{d(\pi \circ u(\exp_p(\tau v)), \gamma_0(t_0\tau))}{\tau} = 0.$$

Since π is 1-Lipschitz, the above equalities together with the triangle inequality imply

$$\lim_{\tau\to 0^+} \frac{d(\pi \circ \gamma(t\tau), \gamma_0(t_0\tau))}{\tau} = 0. \tag{3-5}$$

Among other things, (3-5) implies that $\lim_{\tau\to 0} \angle_{x_0}(\overline{x_0\, \pi \circ \gamma(t\tau)}, \gamma_0) = 0$.

Let $\sigma_0(s)$ be the arclength parametrization of the geodesic from x_0 to x.

Claim 3.6 $$\angle_{x_0}(\gamma_0, \sigma_0) = \tfrac{1}{2}\pi.$$

Proof Assume for the purposes of a contradiction that $\angle_{x_0}(\sigma_0, \gamma_0) > \frac{1}{2}\pi$; and in particular there exists $\epsilon > 0$ such that $\angle_{x_0}(\sigma_0, \gamma_0) = \frac{1}{2}\pi + 3\epsilon$. Let $p_\tau := \gamma(\tau t)$. By the continuity of angles [Bridson and Haefliger 1999, Proposition 3.3], for sufficiently small τ we have that $\angle_{x_0}(\overline{x_0 p_\tau}, \gamma_0^+) > \frac{1}{2}\pi + 2\epsilon$. Moreover, by (3-5) we have, for sufficiently small τ, that $\angle_{x_0}(\gamma_0, \overline{x_0\pi(p_\tau)}) < \epsilon$. The triangle inequality for angles then implies that for such τ,

$$\angle_{x_0}(\overline{x_0 p_\tau}, \overline{x_0\pi(p_\tau)}) > \tfrac{1}{2}\pi + \epsilon. \tag{3-6}$$

Now, consider the geodesic triangle $\Delta(p_\tau, \pi(p_\tau), x_0)$ and the corresponding Euclidean comparison triangle $\Delta_E(A, B, C)$. Because Euclidean comparison angles are greater than their CAT(0) counterparts, (3-6) implies that

$$\angle_C(\overline{CA}, \overline{CB}) > \tfrac{1}{2}\pi + \epsilon. \tag{3-7}$$

In particular, this means that the side opposite C (the side $\overline{AB}$ corresponding to $\overline{p_\tau \pi(p_\tau)}$ in the original) is the *longest* side of the Euclidean comparison triangle. In particular its length is greater than the length of $\overline{AC}$ (which is equal to the length $\overline{x_0 p_\tau}$ in the original). Hence, we have that

$$d(p_\tau, \pi(p_\tau)) > d(p_\tau, x_0)$$

which contradicts the minimality of the projection. □

Let α be an apartment in $T_{x_0}X$ containing $Q_0 = ([\gamma_0], t_0)$ and $Q_1 = ([\sigma_0], 1)$ and A be an apartment of X such that $T_{x_0}A = \alpha$. Then there exists $\tau_1, s_0 > 0$ such that $\sigma_0(s) \in A$ for $s \in [0, s_0]$ and $\gamma_0(\tau) \in A$ for $\tau \in [0, \tau_1]$. Identify A with $\mathbb{R}^n$ with standard coordinates denoted by $(x^1, \dots, x^n)$ such that x_0 identified with the origin $(0, \dots, 0)$ of $\mathbb{R}^n$.

By Claim 3.6, we can assume that γ_0 (resp. σ_0) parametrizes the initial segment of the positive x^1-axis (resp. x^2-axis) in $\mathbb{R}^n$. Let

$$c_\epsilon : [0, \infty) \to \mathbb{R}^n \simeq A, \quad c_\epsilon(\tau) = \left(\frac{1}{\sqrt{1+\epsilon^2}}, \frac{\epsilon}{\sqrt{1+\epsilon^2}}, 0, \dots, 0 \right), \tag{3-8}$$

be a parametrization of a ray in A. Define $Q_\epsilon := ([c_\epsilon], t_0)$. Then

$$Q_\epsilon \to Q_0 \quad \text{as } \epsilon \to 0 \text{ in } T_{x_0}X.$$

We are done once we prove

$$Q_\epsilon \notin T_{x_0}P_F \quad \text{for } \epsilon > 0 \text{ sufficiently small.}$$

To prove this statement, it is sufficient to show that $c_\epsilon(\tau) \notin P_F$ for $\tau \in [0, \tau_\epsilon]$. On the contrary, assume that $c_\epsilon(\tau) \in P_F$ for some $\tau_\star > 0$. By the convexity of P_F and the fact that $c_\epsilon(0) = x_0 \in P_F$, this also implies that $c_\epsilon(\tau) \in P_F$ for all $\tau \in [0, \tau_\star]$. Fix $\tau_1 > 0$ small. By Claim 3.6, $\angle_{x_0}(c_\epsilon, \sigma_0) < \frac{1}{2}\pi$. Thus, the closest point to $\sigma_0(\tau_1)$ on the ray $c_\epsilon([0, \infty))$ is not x_0. This contradicts $\pi(\sigma_0(\tau_1)) = \pi(x) = x_0$, thereby proving that $c_\epsilon(\tau) \notin P_F$. □

Lemma 3.7 *Let $Q_0 \in F_{x_0}$ and let O_{Q_0} be the vertex of the cone $T := T_{Q_0}(T_{x_0}X)$ and let $C := T_{Q_0}(T_{x_0}P_F) \subset T$. Denote by $\Pi_C : T \to C$ the closest-point projection map. Then there exists $w_0 \in T \setminus C$ such that $\Pi_C(w_0) = O_{Q_0}$.*

Proof By Lemma 3.5, there exists $z_1 \in T \setminus C$. Let $c_1 := \Pi_C(z_1)$ and write

$$z_t = (1-t)O_{Q_0} + t z_1 \quad \text{and} \quad c_t = (1-t)O_{Q_0} + t c_1.$$

Claim 1 $$\Pi_C(z_t) = c_t \quad \text{for all } t \in (0, 1).$$

Proof Assume on the contrary that there exists $t \in (0, 1)$ such that $\Pi_C(z_t) = p \neq c_t$. Since T is a cone, there is a unique geodesic ray from O_{Q_0} going through p. Let $\overline{p} \in T$ be the point on this geodesic ray with $t(O_{Q_0}, \overline{p}) = d(O_{Q_0}, p)$. Since distances in the cone scale linearly along radial geodesics, we have

$$t\, d(z_1, \overline{p}) = d(z_t, p).$$

Since $p \in C$ and C is a cone, $\overline{p} \in C$. Thus, the fact that $\Pi_C(z_t) = p \neq c_t$ and $c_1 = \Pi_C(z_1)$ implies

$$d(z_t, p) < d(z_t, c_t) = t\,d(z_1, c_1) \leq t\,d(z_1, \overline{p}) = d(z_t, p).$$

The above inequality is strict since $p \neq c_t$. This contradiction proves the claim. □

Since $T_{x_0}X$ is a Euclidean building, its tangent cone at any point is again a Euclidean building, and any two points lie in a common apartment. Thus, we can choose an apartment A_1 of T containing z_1 and c_1. Let $t \mapsto w_t$ be the unique geodesic line in A_1 with $w_1 = z_1$ and parallel to the geodesic $t \mapsto c_t$. In other words, $\overline{c_t w_t}$ is a geodesic extension of $\overline{c_t z_t}$ for each $t \in (0, 1)$.

Claim 2 $$\Pi_C(w_t) = c_t \quad \text{for } t \in (0, 1].$$

Proof On the contrary, assume $\Pi_C(w_t) = \gamma_t \neq c_t$ for $t \in (0, 1]$. Since $\Pi_C(z_t) = c_t$, there exists a unique point $w_\star \in \overline{w_t z_t} \cap \overline{w_t \gamma_t}$ such that $\angle_{w_\star}(\overline{w_\star \gamma_t}, \overline{w_\star z_t}) > 0$. Note that $\Pi_C(w_\star) = \gamma_t$ so in particular, while $w_\star$ might equal w_t, $w_\star \neq z_t$. Then the geodesic quadrilateral defined by $c_t, \gamma_t, w_\star, z_t$ has sum of interior angles $> 2\pi$. Indeed, $\angle_{c_t}(\overline{c_t \gamma_t}, \overline{c_t z_t})$, $\angle_{\gamma_t}(\overline{\gamma_t c_t}, \overline{\gamma_t w_\star}) \geq \frac{1}{2}\pi$ by the characterization of closest-point projections in CAT(0) spaces, $\angle_{z_t}(\overline{z_t c_t}, \overline{z_t w_\star}) = \pi$ since z_t is a point on the geodesic $\overline{c_t w_\star}$, and $\angle_{w_\star}(\overline{w_\star z_t}, \overline{w_\star \gamma_t}) > 0$. This is a contradiction since quadrilaterals in CAT(0) spaces has sum of interior angles $\leq 2\pi$. □

We are now in position to finish the proof of the lemma. Indeed, by the continuity of Π_C, we have

$$O_{Q_0} = \lim_{t \to 0} c_t = \lim_{t \to 0} \Pi_C(w_t) = \Pi_C\Big(\lim_{t \to 0} w_t\Big) = \Pi_C(w_0).$$ □

Lemma 3.8 *For $Q_0 \in F_{x_0}$, there exists $\widehat{Q} \in T_{x_0}X$ such that $\Pi(\widehat{Q}) = Q_0$. In other words, $F_{x_0} \subset \Pi(T_{x_0}X)$.*

Proof For $w_0 \in T_{Q_0}(T_{x_0}X) \setminus T_{Q_0}(T_{x_0}P_F)$ of Lemma 3.7, write $w_0 = ([\overline{Q_0 \widehat{Q}}], t_0)$. The fact that $\Pi_C(w_0) = O_{Q_0}$ implies

$$\angle_{Q_0}(\overline{Q_0 \widehat{Q}}, \overline{Q_0 Q'}) \geq \tfrac{1}{2}\pi \quad \text{for any } Q' \in T_{x_0}P_F.$$

By the characterization of closest-point projections in CAT(0) spaces, this implies $\Pi(\widehat{Q}) = Q_0$. □

We now prove two technical lemmas about projection maps into conical Euclidean buildings. Lemma 3.11 is crucial for establishing the energy loss, as it provides a description of the image of the blowup of $\pi \circ u$ at p (Proposition 3.12).

Let B be a cone over a spherical building with vertex O and let C be a subbuilding. Furthermore, suppose that C is the set of all flats in B parallel to a given flat $\mathfrak{f}$. That is,

$$C = P_{\mathfrak{f}} \simeq \mathfrak{f} \times C', \quad \text{where } \mathfrak{f} \text{ is a flat.}$$

As in Remark 3.2, we preserve the walls of B in C which the canonical building structure would remove. We have the following analog of Definition 3.3.

Definition 3.9 Let $\mathfrak{h}$ be a wall of C. We say $\mathfrak{h}$ *contains* $\mathfrak{f}$ if it is of the form $\mathfrak{f} \times \mathfrak{h}'$, where $\mathfrak{h}'$ is a wall of C'. Otherwise, we say $\mathfrak{h}$ *does not contain* $\mathfrak{f}$.

Lemma 3.10 *If $C \neq B$, then there exists a wall $\mathfrak{h}$ of C that does not contain $\mathfrak{f}$ and $O \in \mathfrak{h}$.*

Proof Let $\mathfrak{a}_0$ be an apartment of C and $\mathfrak{a}_1$ be an apartment of B such that $\mathfrak{a}_1 \not\subset C$. By [Kleiner and Leeb 1997, Corollary 4.4.6],

$$\mathfrak{a}_0 \cap \mathfrak{a}_1 = \bigcap_{i=1}^{I} \mathfrak{a}_{0i}^+,$$

where $\mathfrak{a}_{01}^+, \dots, \mathfrak{a}_{0I}^+$ are half-apartments of $\mathfrak{a}_0$. If $\mathfrak{f} \subset \mathfrak{a}_{0i}^+$ for all $i = 1, \dots, I$, then $\mathfrak{f} \subset \mathfrak{a}_1$, which would imply that $\mathfrak{a}_1$ is an apartment of C, a contradiction. Thus, there exists a half-apartment $\mathfrak{a}_{0i}^+$ such that $\mathfrak{f} \not\subset \mathfrak{a}_{0i}^+$, which implies that the wall $\mathfrak{h} := \partial \mathfrak{a}_{0i}^+$ does not contain $\mathfrak{f}$. Lastly, $O \in \mathfrak{h}$ since C is a cone. □

Lemma 3.11 *Let $\pi_C : B \to C$ be the closest-point projection map. If $C \neq B$, $Q \in B \setminus C$ and $Q_0 = \pi_C(Q)$, then there exists a wall $\mathfrak{h}$ of C that does not contain $\mathfrak{f}$ such that $Q_0 \in \mathfrak{h}$.*

Proof Let $\widetilde{C} = T_{Q_0} C$, $\widetilde{B} = T_{Q_0} B$, and $\widetilde{\mathfrak{f}} = T_{Q_0}\mathfrak{f}$. We observe that the geodesic germ $\overline{Q\,Q_0}$ cannot lie in $T_{Q_0}C$ (because Q_0 is the closest point of C to Q), and therefore $\widetilde{C} \neq \widetilde{B}$. Applying Lemma 3.10, we conclude that there is a wall $\widetilde{\mathfrak{h}}$ of $\widetilde{C}$ which does not contain $\widetilde{\mathfrak{f}}$, and $O_{Q_0} \in \widetilde{\mathfrak{h}}$, where O_{Q_0} is the vertex of the tangent cone $T_{Q_0}C$.

It follows from [Kleiner and Leeb 1997, Lemma 4.2.3] and the Euclidean building structure for tangent cones that $\widetilde{\mathfrak{h}}$ is a wall of $\widetilde{C}$ if and only if $\widetilde{\mathfrak{h}} = T_{Q_0}\mathfrak{h}$ for a wall $\mathfrak{h}$ of C containing Q_0. Hence, there is a wall $\mathfrak{h}$ of C that does not contain $\mathfrak{f}$ with $Q_0 \in \mathfrak{h}$. □

Using the above lemma, we demonstrate that the full image of the differential map $D_p(\pi \circ u)$ is contained in a single wall which does not contain $T_{x_0}F$.

Proposition 3.12 *Let $p \in U$, $x = u(p)$ and $x_0 = \pi \circ u(p)$. If $x \neq x_0$, there exists a wall $T_{x_0}H$ that does not contain $T_{x_0}F$ such that F_{x_0}, the image of $D_p(\pi \circ u)$, is contained in $T_{x_0}H$.*

Proof Let $Q_0 \in D_p(\pi \circ u)(\mathbb{R}^n)$. By Lemma 3.8, there exists a $\widehat{Q} \in T_{x_0}X$ such that $\Pi(\widehat{Q}) = Q_0$ (and $\widehat{Q} \neq Q_0$).

Set $B = T_{x_0}X$ and $C = T_{x_0}P_F$. By [Appenzeller et al. 2025, Corollary 3.6], $T_{x_0}P_F = P_{T_{x_0}F}$ inside $T_{x_0}X$, so C is the collection of all flats parallel to $\mathfrak{f} = T_{x_0}F \subset T_{x_0}X$. Applying Lemma 3.11 with this B and C, there exists a wall $T_{x_0}H_{Q_0}$ containing Q_0 but not containing $T_{x_0}F$. Now, since the image of $D_p(\pi \circ u)$ is a flat, $D_p(\pi \circ u)(\mathbb{R}^n)$ lies in a single apartment $T_{x_0}A \subset T_{x_0}P_F$. If there did not exist a single wall containing the full image of $D_p(\pi \circ u)$, then the linear sum of those subspaces would be all of $T_{x_0}A$. The proposition follows. □

3.4 Loss of energy

Finally, we demonstrate that at points p where $u, \pi \circ u$ are differentiable and $u(p) \neq \pi \circ u(p)$, projecting into F results in a quantitative energy loss (Proposition 3.16). Before we give a proof of the loss of energy we prove a technical result crucial for establishing the loss of energy statement. Specifically, we demonstrate an upper bound on a component function of the differential of π_F (Lemma 3.15 below).

For $x_0 = (f, y_0) \in P_F$, we denote the projection to the first component by

$$\pi_F : P_F \simeq F \times Y \to F, \quad \pi_F(x_0) = \pi_F(f, y_0) = f.$$

Fixing an identification $F \simeq \mathbb{R}^m$, we can also view π_F as an $\mathbb{R}^m$-valued map

$$\pi_F : P_F \to \mathbb{R}^m.$$

We denote the i-th component function by

$$\pi_F^i : P_F \to \mathbb{R}.$$

Let A be an apartment of P_F. Then $A \simeq F \times Y_A$ where Y_A is an apartment of Y. Let $\iota : \mathbb{R}^N \to A$ be a chart. Let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation such that

$$\begin{aligned} &\iota \circ \varphi(\mathbb{R}^m \times \{(0, \dots, 0)\}) = F \simeq \mathbb{R}^m \text{ is the identity map of } \mathbb{R}^m, \\ &\iota \circ \varphi(\{(0, \dots, 0) \times \mathbb{R}^{N-m}\}) = Y_A. \end{aligned} \tag{3-9}$$

Then $\iota_A := \iota \circ \varphi$ is a chart of the building X of type $\varphi \cdot W$ (Remark 2.2). The chart ι_A induces the natural identifications

$$T_f F \simeq \mathbb{R}^m \simeq \mathbb{R}^m \times \{(0, \dots, 0)\} \subset \mathbb{R}^N \simeq T_{x_0} A, T_{x_0} P_F \simeq \mathbb{R}^m \times T_{y_0} Y. \tag{3-10}$$

With the above identifications, the restriction $\pi_F | A \to F$ is simply the projection to the first m components of $\mathbb{R}^N$; i.e.,

$$(x^1, \dots, x^m, x^{m+1}, \dots, x^N) \mapsto (x^1, \dots, x^m, 0, \dots, 0). \tag{3-11}$$

Furthermore, the (classical) differential $d(\pi_F | A) : T_{x_0} A \simeq \mathbb{R}^N \to T_{\pi_F(x_0)} F \simeq \mathbb{R}^m$ of a projection map $\pi_F | A$ is of course also given by (3-11). Similarly, the restriction $\pi_F^i | A$ and its (classical) differential $d(\pi_F^i | A)$ are both given by

$$(x^1, \dots, x^m, x^{m+1}, \dots, x^N) \mapsto x^i. \tag{3-12}$$

The following lemma shows (3-11) and (3-12) also describe the restriction to $T_{x_0} A$ of the blowup maps

$$(\pi_F)_{x_0}^{(\epsilon_j)} : {P_F}_{x_0}^{(\epsilon_j)} \to (\mathbb{R}^m)_{\pi_F(x_0)}^{(\epsilon_j)} \simeq \mathbb{R}^m \quad \text{and} \quad (\pi_F^i)_{x_0}^{(\epsilon_j)} : {P_F}_{x_0}^{(\epsilon_j)} \to \mathbb{R}_{\pi_F^i(x_0)}^{(\epsilon_j)} \simeq \mathbb{R},$$

by proving that blowing up and taking the restriction commute; i.e.,

$$(\pi_F)_{x_0}^{(\epsilon_j)} | T_{x_0} A = d(\pi_F | A) \quad \text{and} \quad (\pi_F^i)_{x_0}^{(\epsilon_j)} | T_{x_0} A = d(\pi_F^i | A).$$

Lemma 3.13 *With the identification $T_{x_0} A \simeq \mathbb{R}^N$ of (3-10), the restriction $(\pi_F)_{x_0}^{(\epsilon_j)} | T_{x_0} A$ is given by (3-11). Similarly, the restriction of $(\pi_F^i)_{x_0}^{(\epsilon_j)} | T_{x_0} A$ is given by (3-12).*

Proof For $([\gamma],t)\in T_{x_0}A$, the geodesic germ $[\gamma]$ is represented by a geodesic contained in A [Kleiner and Leeb 1997, Lemma 4.1.2]. View $([\gamma],t)$ as a point in $P_F{}_{x_0}^{(\epsilon_j)}$ under the identification defined by the exponential map, $([\gamma],t)=[(\gamma(t\epsilon_j))]$. Since $(\gamma(t\epsilon_j))$ is a sequence of points in A,

$$(\pi_F)_{x_0}^{(\epsilon_j)}([\gamma],t)=[(\pi_F(\gamma(t\epsilon_j)))]=[(\pi_F|A(\gamma(t\epsilon_j)))]=(\pi_F|A)_{x_0}^{(\epsilon_j)}([\gamma],t).$$

Since Lytchak's notion of differentials [2004] agrees with the classical differentials on $\mathbb{R}^N$, we conclude $(\pi_F)_{x_0}^{(\epsilon_j)}|T_{x_0}A=(\pi_F|A)_{x_0}^{(\epsilon_j)}=d_{x_0}(\pi_F|A)$. The second assertion follows from a similar argument. □

Let $T_{x_0}H$ be a wall of $T_{x_0}P_F$. Choose an apartment $T_{x_0}A$ of $T_{x_0}P_F$ containing $T_{x_0}H$ and use the identification (3-10) to define

$$\theta(H,i):=\min_{Q\in T_{x_0}H\setminus\{\mathbb{O}_{x_0}\}}\angle_{\mathbb{O}_{x_0}}(Q,\vec{e}_i),$$

where $\mathbb{O}_{x_0}$ denotes the origin of the tangent cone $T_{x_0}P_F$ and $\vec{e}_1,\dots,\vec{e}_m$ are the standard basis of $\mathbb{R}^m\simeq F$. We then have the following:

- For a wall $T_{x_0}H$ that contains $T_{x_0}F$, we have $\theta(H,i)=0$ for all $i\in\{1,\dots,m\}$.
- For a wall $T_{x_0}H$ that does not contain $T_{x_0}F$,

$$0<\Theta(H)\le\tfrac{1}{2}\pi,\quad\text{where }\Theta(H):=\max_{i=1,\dots,m}\theta(H,i),$$

Remark 3.14 Although the quantity $\Theta(H)$ was defined by choosing an apartment $T_{x_0}A$ that contains $T_{x_0}H$ and the identification (3-10) induced by the chart $\iota_A:\mathbb{R}^n\to A$, it does not depend on these choices. The choice of a chart ι_A resulted from the choice of the orthogonal transformation φ satisfying (3-9). Any other choice of an orthogonal transformation which also satisfies (3-9) does not change the angle used to define $\theta(H,i)$. Furthermore, let $T_{x_0}A'$ be another apartment that contains $T_{x_0}H$ and $\iota':\mathbb{R}^N\to A'$ be a chart in $\varphi\cdot\mathscr{A}$. This implies that $\iota'^{-1}\circ\iota_A:\iota_A^{-1}(A\cap A')\to\iota'^{-1}(A\cap A')$ is a restriction of $w'\in\varphi\cdot W_{\mathrm{aff}}$. Thus, $\iota_{A'}:=\iota'\circ w'$ is a chart such that

$$\iota_{A'}^{-1}\circ\iota_A|\iota_A^{-1}(A\cap A')\to\iota_{A'}^{-1}(A\cap A')\quad\text{is the identity map.}\tag{3-13}$$

In particular, for each apartment A' such that $T_{x_0}H$ is contained in $T_{x_0}A'$, there exists a unique chart satisfying (3-13) that induces the same identification as (3-10) with A replaced by A'.

Lemma 3.15 *There exists $\theta_0\in\left(0,\frac{1}{2}\pi\right]$ with the following property: For $x_0\in P_F$ and a wall $T_{x_0}H$ that does not contain $T_{x_0}F$, there exists $i_H\in\{1,\dots,m\}$ such that*

$$|(\pi_F^{i_H})_{x_0}^{(\epsilon_j)}(Q)|^2\le\cos^2\theta_0\cdot d_{x_0}^2(Q,\mathbb{O}_{x_0})\quad\text{for all }Q\in T_{x_0}H.$$

Here, $\mathbb{O}_{x_0}$ denotes the origin of the tangent cone $T_{x_0}P_F$, and d_{x_0} denotes the distance function on $T_{x_0}P_F$.

Proof Since the walls through any point $x_0\in P_F$ are determined by the finite reflection group $\varphi\cdot W$, there are only a finite number of possible values for $\theta(H,i)$. Thus, there exists $\theta_0\in\left(0,\frac{1}{2}\pi\right]$, which can be chosen independently of $x_0\in P_F$, such that

$$\theta_0\le\Theta(H)\le\tfrac{1}{2}\pi\quad\text{for any wall }T_{x_0}H\text{ that does not contain }T_{x_0}F.$$

We emphasize that θ_0 is determined only by F and W. Choosing an apartment $T_{x_0}A$ containing $T_{x_0}H$, and using the identification (3-10), for $i_H := \operatorname{argmax}_{i=1,\dots,m}\theta(H,i)$,

$$|\operatorname{proj}_{\vec{e}_{i_H}} Q| = |Q\cdot\vec{e}_{i_H}| \le |\cos\Theta(H)||Q| \le \cos\theta_0\cdot|Q| \quad \text{for all } Q\in T_{x_0}H\subset\mathbb{R}^N.$$

Thus, the result follows from Lemma 3.13. □

Finally, we can demonstrate the quantitative energy loss.

Proposition 3.16 *Let $u^i := \pi_F^i\circ\pi\circ u : B_1(0)\to\mathbb{R}$. There exists $\theta_0\in\left(0,\frac{1}{2}\pi\right]$ with the following property: For every $p\in U$ such that $u(p)\notin P_F$, there exists $i\in\{1,\dots,m\}$ such that*

$$\left|\frac{\partial u^i}{\partial x^i}\right|^2(p) \le \cos^2\theta_0\cdot\left|\frac{\partial u}{\partial x^i}\right|^2(p).$$

Proof Choose normal coordinates centered at $p\in U$ and let $x=u(p)$, $x_0=\pi\circ u(p)$. By Proposition 3.12, we may choose a wall $T_{x_0}H$ that does not contain $T_{x_0}F$ such that $D_p(\pi\circ u)(\mathbb{R}^n)\subset T_{x_0}H$. Let θ_0 and $i=i_H$ be from Lemma 3.15. By the chain rule for blown-up maps, we have

$$D_p u^i = (\pi_F^i)_{x_0}^{(\epsilon_j)}\circ D_p(\pi\circ u).$$

Thus, Lemma 3.15 implies

$$\left|\frac{\partial u^i}{\partial x^i}\right|^2(p) = \left|(\pi_F^i)_{x_0}^{(\epsilon_j)}\circ D_p(\pi\circ u)\left(\frac{\partial}{\partial x^i}\right)\right|^2 \le \cos^2\theta_0\cdot d_{x_0}^2\left(D_p(\pi\circ u)\left(\frac{\partial}{\partial x^i}\right),\mathbb{C}_{x_0}\right).$$

Since the projection map $\pi: X\to P_F$ is distance nonincreasing,

$$d_{x_0}^2\left(D_p(\pi\circ u)\left(\frac{\partial}{\partial x^i}\right),\mathbb{C}_{x_0}\right) \le d_x^2\left(D_p u\left(\frac{\partial}{\partial x^i}\right),\mathbb{C}_x\right) = \left|\frac{\partial u}{\partial x^i}\right|^2(p).$$

The desired inequality follows from combining the above two inequalities. □

4 Closeness in measure

In this section, we establish the cornerstone of the proof for Theorem 1.1. The statement asserts that if a harmonic map into X is sufficiently close to a homogeneous degree 1 harmonic map, then the image of the harmonic map mostly lies in a subbuilding defined by the homogeneous degree 1 map. The precise statement we need is contained in Proposition 4.1. In the proof of the regularity theorem of Gromov and Schoen [1992, Theorem 5.1], this assertion follows from the fact that, for the locally finite case, a homogeneous degree 1 map L is *effectively contained* in the subbuilding P_F where $F=L(\mathbb{R}^n)$.

However, we cannot expect this property to be true in a general Euclidean building. Instead, we take advantage of the observation in the previous section that a projection into P_F results in a loss of energy. We use this to prove the "closeness in measure" assertion of Proposition 4.1.

Proposition 4.1 *Fix $E_0>0$, $r_0\in(0,1)$ and a homogeneous degree 1 harmonic map*

$$L:\mathbb{R}^n\to A\subset X$$

where X is of type W, the dimension of X is at least 2, and A is an apartment of X. For every $\epsilon > 0$, there exists $\eta = \eta(\epsilon, E_0, r_0, X, A, L) > 0$ such that the following holds: Let (X', A', L') be an (X, A, L)-triple and let $u : (B_1(0), g) \to (X', d)$ be a harmonic map with ${}^g E^u[B_1(0)] \le E_0$. If $\sup_{B_{r_0}(0)} d(u, L') < \eta$ and $\|g - \delta_{\mathrm{euc}}\|_{C^2} < \eta$, then

$$\mu_0\{p \in B_{r_0}(0) \mid u(p) \notin P_{F'}\} < \epsilon,$$

where $P_{F'}$ denotes the union of all flats of X' parallel to $F' := L'(\mathbb{R}^n)$.

We will prove Proposition 4.1 by contradiction. Therefore, we assume that there exists $\epsilon > 0$, a sequence of (X, A, L)-triples (X_k, A_k, L_k), and a sequence of harmonic maps $u_k : (B_1(0), g_k) \to (X_k, d_k)$ satisfying

- ${}^{g_k} E^{u_k}[B_1(0)] \le E_0$,
- $\sup_{B_{r_0}(0)} d(u_k, L_k) < \dfrac{1}{k}$,
- $\|g_k - \delta_{\mathrm{euc}}\|_{C^2} < \dfrac{1}{k}$,

and such that

$$\mu_0\{q \in B_{r_0}(0) \mid u_k(q) \notin P_{F_k}\} > \epsilon, \tag{4-1}$$

where P_{F_k} is the union of all flats of X_k parallel to $F_k := L_k(\mathbb{R}^n) \subset A_k$. Before we finish the proof of Proposition 4.1 (Section 4.2), we prove some preliminary lemmas (Lemmas 4.2 and 4.4) regarding the convergence in measure of the directional energies of the harmonic maps u_k.

4.1 Convergence in measure

Using an appropriate chart to identify $A \simeq \mathbb{R}^N$, we view L as a linear map $L : \mathbb{R}^n \to \mathbb{R}^N$. Furthermore, by the singular value decomposition, after an appropriate orthogonal change of coordinates $\psi : \mathbb{R}^n \to \mathbb{R}^n$ and $\varphi : \mathbb{R}^N \to \mathbb{R}^N$, we can express $\varphi \circ L \circ \psi : \mathbb{R}^n \to \mathbb{R}^N$ by an $(N \times n)$-matrix

$$\begin{bmatrix} \Lambda & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}.$$

Here, Λ is a diagonal $(m \times m)$-square matrix with diagonal entries $\lambda_1 \ge \cdots \ge \lambda_m > 0$ and the $\boldsymbol{O}$s represents zero matrices (of appropriate sizes). In particular,

$$F = L(\mathbb{R}^n) \simeq \mathbb{R}^m \simeq \mathbb{R}^m \times \{(0, \ldots, 0)\} \subset \mathbb{R}^N.$$

We will proceed using these new coordinates on $\mathbb{R}^n$ and $\mathbb{R}^N$ and changing the building structure of (X, d) to the one of type $\varphi \cdot W$ (Remark 2.2). By an abuse of notation, we denote $\varphi \circ L \circ \psi$ again by L.

Recall from Definition 2.13 that there exists an isometry $\phi_k : A \to A_k$, compatible with the building structure, such that $L_k = \phi_k \circ L$. In particular, $F_k = L_k(\mathbb{R}^n) = \phi_k \circ L(\mathbb{R}^n)$. We use ϕ_k to identify

$$F_k \simeq F \simeq \mathbb{R}^m. \tag{4-2}$$

Under this identification via ϕ_k,

$$L_k \equiv L. \tag{4-3}$$

In particular, the i-th coordinate function L_k^i of L_k and the i-th coordinate function L^i of L are the same function, and we have

$$\frac{\partial L_k^i}{\partial x^i} = \frac{\partial L^i}{\partial x^i} = \lambda_i \quad \text{for } i = 1, \dots, m,$$
$$\frac{\partial L_k^j}{\partial x^i} = \frac{\partial L^j}{\partial x^i} = 0 \quad \text{for } i \neq j \text{ or } i = j = m+1, \dots, n.$$

In particular, for $i = 1, \dots, m$,

$$\left| \frac{\partial L_k}{\partial x^i} \right|^2 = \left| \frac{\partial L}{\partial x^i} \right|^2 = \lambda_i^2. \tag{4-4}$$

Lemma 4.2 *Fix $\tau > 0$ and $i = 1, \dots, m$.*

$$\text{If} \quad \Upsilon_k(\tau, i) := \left\{ p \in B_{r_0}(0) \;\middle|\; (1+\tau)^2 \lambda_i^2 \leq \left| \frac{\partial u_k}{\partial x^i} \right|^2 (p) \right\}, \quad \text{then} \quad \lim_{k \to \infty} \mu_0(\Upsilon_k(\tau, i)) = 0.$$

Proof Let $\epsilon_0 > 0$ be given. We invoke [Korevaar and Schoen 1993, last paragraph of the proof of Theorem 2.4.6] which states that $|\partial u_k / \partial x^i|^2$ is essentially subharmonic (Remark 4.3 below). By a standard argument, we obtain a mean-value inequality of the form

$$\left| \frac{\partial u_k}{\partial x^i} \right|^2 (p) \leq \frac{c}{\mu_{g_k}(B_r(p))} \int_{B_r(p)} \left| \frac{\partial u_k}{\partial x^i} \right|^2 d\mu_{g_k}, \tag{4-5}$$

where the constant $c = c(r, g_k)$ depends on the C^2-closeness of g_k to the Euclidean metric in $B_r(p)$. More precisely, $c = 1$ if the domain metric is Euclidean, and we can choose $r > 0$ sufficiently close to 0 and $K_0 \in \mathbb{N}$ sufficiently large such that

$$c < (1+\tau) \quad \text{for all } k \geq K_0.$$

Assume further that $r > 0$ is so small that $\mu_0(B_{r_0}(0) \setminus B_{r_0 - r}(0)) < \frac{1}{2}\epsilon_0$. Then for all $p \in B_{r_0 - r}(0)$, $B_r(p) \subset B_{r_0}(0)$. Define

$$\overline{u}_k^r : B_{r_0 - r}(0) \to \mathbb{R}, \quad \overline{u}_k^r(p) = (\mu_{g_k}(B_r(p)))^{-1} \int_{B_r(p)} \left| \frac{\partial u_k}{\partial x^i} \right|^2 d\mu_{g_k}.$$

It is straightforward to show that the sequence of measures $|\partial u_k / \partial x^i|^2 \, d\mu_{g_k}$ converges weakly[2] to $\lambda_i^2 \, d\mu_0$ in $B_{r_0}(0)$. The portmanteau theorem implies (since $B_r(p)$ is a continuity set for the measure

[2]We first prove that u_k converges uniformly in the pullback sense to L in $B_{r_0}(0)$. To do so, following the notation of Section 2.6, we need to check inductively in j that the sequence of the pullback pseudodistance function of $u_{k,j}$ converges uniformly to the pullback pseudodistance function of L_j on $\Omega_j \times \Omega_j$, where $\Omega = B_{r_0}(0)$. This follows easily because (i) the pullback pseudodistance of L_k is equal to the pullback pseudodistance of L since $L_k = \phi_k \circ L$ for an isometry $\phi_k : A \to A_k$, and (ii) the difference between pullback pseudodistance of u_k and the pullback pseudodistance of L_k converges to 0 since $\sup_{B_{r_0}(0)} d_k(u_k, L_k) < 1/k$. Next, the convergence of measures follows from (4-4) and [Korevaar and Schoen 1997, Theorem 3.11].

$\lambda_i^2\, d\mu_0$ normalized to a probability measure) that

$$\lim_{k\to\infty}\int_{B_r(p)}\left|\frac{\partial u_k}{\partial x^i}\right|^2 d\mu_{g_k}=\lambda_i^2\mu_0(B_r(p)).$$

Since $\mu_{g_k}(B_r(p))\to\mu_0(B_r(p))$ as $k\to\infty$, we conclude that $\overline{u}_k^r\to\lambda_i^2$ pointwise in $B_{r_0-r}(0)$. By Egorov's theorem, there exists a set V with $\mu_0(V)<\frac{1}{2}\epsilon_0$ and $K\geq K_0$ such that $|\overline{u}_k^r(p)-\lambda_i^2|<\tau\lambda_i^2$ for all $k\geq K$ and $p\in B_{r_0-r}(0)\setminus V$. Thus, for all $k\geq K$ and $p\in B_{r_0-r}(0)\setminus V$,

$$\left|\frac{\partial u_k}{\partial x^i}\right|^2(p)\leq c\overline{u}_k^r(p)\leq c(1+\tau)\lambda_i^2\leq(1+\tau)^2\lambda_i^2.$$

In other words, $\Upsilon_k(\tau,i)\cap B_{r_0-r}(0)\subset V$ for all $k\geq K$. Therefore,

$$\mu_0(\Upsilon_k(\tau,i))\leq\mu_0(V)+\mu_0(B_{r_0}(0)\setminus B_{r_0-r}(0))<\epsilon_0\quad\text{for all } k\geq K.$$ □

Remark 4.3 If the domain metric is Euclidean, the weak subharmonicity of $|\partial u_k/\partial x^i|^2$ follows from [Korevaar and Schoen 1993, Remark 2.4.3]. Indeed, if $u:B_1(0)\to X$ is a harmonic map with respect to the Euclidean metric on $B_1(0)$, then

$$\int_{B_1(0)}\nabla d^2(u,u_{sw})\cdot\nabla\eta\, d\mu_0\geq 0$$

for a constant vector w, $s\in\mathbb{R}$ and $\eta\in C_c^\infty(B_1(0))$, where $u_{sw}(x)=u(x+sw)$. We let $w=\partial/\partial x^i$ and divide by s^2 and let $s\to 0$ to prove $|\partial u/\partial x^i|^2$ is weakly subharmonic. If the metric g on $B_1(0)$ is not Euclidean, we follow the proof of [Korevaar and Schoen 1993, Theorem 2.4.6]. We now set $u_{sw}(x)=u(\overline{x}(x,s))$, where $\overline{x}(x,s)$ is the one-parameter family of flows defined by the vector w and the metric g. We observe that the constant C that appears in the (subharmonicity) inequality [Korevaar and Schoen 1993, (2.4.xxv)],

$$\int_{B_1(0)}|\nabla u|^2(\triangle\eta+C|\nabla\eta|+C\eta)\,d\mu_g\geq 0,$$

is due to the difference of the measures $(g^{ij}\,d\mu_g)_{sw}$ and $(g^{ij}\,d\mu_g)_{-sw}$ to $g^{ij}\,d\mu_g$, and hence $C\to 0$ as $\|g-\delta_{\text{euc}}\|_{C^2}\to 0$. The standard technique of letting η be radially symmetric test functions in the above inequality proves the mean value inequality (4-5), where c depends on C and hence on the metric g, and $c\to 1$ as $\|g-\delta_{\text{euc}}\|_{C^2}\to 0$.

For each k, denote the closest-point projection map from X_k to P_{F_k} by

$$\pi_k:X_k\to P_{F_k}.$$

Following the setup of Section 3.4, denote the projection onto the first component of $P_{F_k}\simeq F_k\times Y_k$ by

$$\pi_{F_k}:P_{F_k}\to\mathbb{R}^m\simeq F_k$$

and let $\pi_{F_k}^i$ be the i-th component function of π_{F_k}. Finally, define

$$u_k^i:=\pi_{F_k}^i\circ\pi_k\circ u_k.$$

Lemma 4.4 *Fix $\delta > 0$ and $i \in \{1, \dots, m\}$.*

$$\text{If } \Theta_k(\delta, i) := \left\{ p \in B_{r_0}(0) \;\middle|\; \left| \frac{\partial u_k^i}{\partial x^i}(p) \right|^2 \le (1-\delta)^2 \lambda_i^2 \right\}, \text{ then } \lim_{k\to\infty} \mu_0(\Theta_k(\delta, i)) = 0.$$

Proof We consider the case $i = 1$ as all other cases follow similarly. On the contrary, assume there exists a subsequence of $k \to \infty$ (which we will still denote as k by an abuse of notation) such that $\lim_{k\to\infty} \mu_0(\Theta_k(\delta, 1)) \ge \beta > 0$.

Let $\boldsymbol{B} \subset \mathbb{R}^{n-1} \simeq \{0\} \times \mathbb{R}^{n-1}$ be the ball of radius r_0 centered at the origin. For $p \in B_{r_0}(0)$, write $p = (p_1, \overline{p})$ where $\overline{p} := (p_2, \dots, p_n) \in \boldsymbol{B}$. Use the projection of $B_{r_0}(0) \to \boldsymbol{B}$, $(p_1, \overline{p}) \mapsto \overline{p}$ to view $B_{r_0}(0)$ as a fiber bundle over $\boldsymbol{B}$ with intervals as fibers. More precisely, to each $\overline{p} \in \boldsymbol{B}$ we associate an interval $I_{\overline{p}} := (-\rho(\overline{p}), \rho(\overline{p})) \subset \mathbb{R}$, where $\rho(\overline{p}) = \sqrt{(3/4)^2 - |\overline{p}|^2}$.

For $\overline{p} \in \boldsymbol{B}$, define a subset $\theta_k(\overline{p})$ of the interval $I_{\overline{p}}$ by

$$\theta_k(\overline{p}) := \left\{ p_1 \in I_{\overline{p}} \;\middle|\; \left| \frac{\partial u_k^1}{\partial x^1}(p_1, \overline{p}) \right| \le (1-\delta)\lambda_1 \right\}.$$

Recalling that μ_0^k denotes the k-dimensional Lebesgue measure, define a subset $\mathcal{A}_k$ of the base space $\boldsymbol{B}$ by

$$\mathcal{A}_k := \left\{ \overline{p} \in \boldsymbol{B} \;\middle|\; \mu_0^1(\theta_k(\overline{p})) > \frac{\beta}{4\omega} \right\},$$

where $\omega := \mu_0^{n-1}(\boldsymbol{B})$. Now suppose that $\liminf_{k\to\infty} \mathcal{A}_k = 0$. Then there exists a subsequence (again labeled by k) such that $\lim_{k\to\infty} \mu_0^{n-1}(\mathcal{A}_k^c) = \omega$. (The superscript c denotes the complement of a set.)

Then Fubini's theorem implies

$$\begin{aligned} \frac{\beta}{4} = \lim_{k\to\infty} \mu_0^{n-1}(\mathcal{A}_k^c) \cdot \frac{\beta}{4\omega} \ge \lim_{k\to\infty} \int_{\mathcal{A}_k^c} \mu_0^1(\theta_k(\overline{p})) d\mu_0^{n-1} &= \lim_{k\to\infty} \int_{\boldsymbol{B}} \mu_0^1(\theta_k(\overline{p})) d\mu_0^{n-1} \\ &= \lim_{k\to\infty} \mu_0(\Theta_k(\delta, 1)) \ge \beta, \end{aligned}$$

a contradiction. Thus, $\liminf_{k\to\infty} \mu_0^{n-1}(\mathcal{A}_k) > 0$.

For $\kappa, \tau > 0$ to be chosen later, define a subset $\mathcal{B}_k$ of the base $\boldsymbol{B}$ by

$$\mathcal{B}_k := \left\{ \overline{p} \in \boldsymbol{B} \;\middle|\; \left| \frac{\partial u_k^1}{\partial x^1}(p_1, \overline{p}) \right| \le (1+\tau)\lambda_1 \text{ for } p_1 \in I_{\overline{p}} \text{ except on a subset of measure} < \kappa \right\}.$$

Lemma 4.2 implies that $\lim_{k\to\infty} \mu_0^{n-1}(\mathcal{B}_k) = \omega$. Thus, there exists K with the property that $\mathcal{A}_k \cap \mathcal{B}_k \neq \varnothing$ for all $k > K$. Choose $\overline{p}_k \in \mathcal{A}_k \cap \mathcal{B}_k$ for each k.

Define a subset b_k of the interval $I_{\overline{p}_k}$ by

$$b_k := \left\{ t \in I_{\overline{p}_k} \;\middle|\; \left| \frac{\partial u_k^1}{\partial x^1}(t, \overline{p}_k) \right| \le (1+\tau)\lambda_1 \right\}.$$

For ease of notation, let

$$f_k(t) := u_k^1(t, \overline{p}_k).$$

The inequality $|f_k'(t)|^2 \le |\partial u_k/\partial t|^2(t, \bar{p}_k)$ and the uniform Lipschitz bounds on u_k in $B_{r_0}(0)$ imply there exists $C > 0$ such that

$$|f_k'(t)| \le C \quad \text{for all } t \in I_{\bar{p}_k} = (-\rho(\bar{p}_k), \rho(\bar{p}_k)).$$

Since $\bar{p}_k \in \mathscr{A}_k \cap \mathscr{B}_k$,

$$\mu_0^1(\theta_k(\bar{p}_k)) > \frac{\beta}{4\omega} \quad \text{and} \quad \mu_0^1(b_k^c) < \kappa.$$

Therefore, for $k > K$,

$$\begin{aligned}
f_k(\rho(\bar{p}_k)) - f_k(-\rho(\bar{p}_k)) &= \int_{-\rho(\bar{p}_k)}^{\rho(\bar{p}_k)} f_k'(t)\,dt \le \int_{-\rho(\bar{p}_k)}^{\rho(\bar{p}_k)} |f_k'(t)|\,dt \\
&= \int_{\theta_k(\bar{p}_k)} |f_k'(t)|\,dt + \int_{\theta_k(\bar{p}_k)^c \cap b_\kappa} |f_k'(t)|\,dt + \int_{\theta_k(\bar{p}_k)^c \cap b_\kappa^c} |f_k'(t)|\,dt \\
&\le \mu_0^1(\theta_k(\bar{p}_k))(1-\delta)\lambda_1 + \mu_0^1(\theta_k(\bar{p}_k)^c)(1+\tau)\lambda_1 + C\mu_0^1(b_\kappa^c) \\
&= 2\rho(\bar{p}_k)\lambda_1 - \mu_0^1(\theta_k(\bar{p}_k))\delta\lambda_1 + \mu_0^1(\theta_k(\bar{p}_k)^c)\tau\lambda_1 + C\mu_0^1(b_\kappa^c) \\
&< 2\rho(\bar{p}_k)\lambda_1 - \frac{\beta\delta\lambda_1}{4\omega} + \mu_0^1(\theta_k(\bar{p}_k)^c)\tau\lambda_1 + C\kappa.
\end{aligned}$$

Thus, by choosing $\kappa, \tau > 0$ sufficiently small (depending only on $\beta, \delta, \lambda_1, \omega, C$), we conclude

$$f_k(\rho(\bar{p}_k)) - f_k(-\rho(\bar{p}_k)) < 2\rho(\bar{p}_k)\lambda_1 - \frac{\beta\delta\lambda_1}{8\omega} \quad \text{for all } k > K. \tag{4-6}$$

On the other hand, for any $p \in B_{r_0}(0)$, we can view $u_k^1(p)$, $L^1(p)$ as points in $\mathbb{R} \simeq \mathbb{R} \times \{0, \dots, 0\} \subset \mathbb{R}^N \simeq A_k \subset P_{F_k}$. Under this identification,

$$|L^1(p) - u_k^1(p)| = d_k(L^1(p), u_k^1(p)).$$

Since $\sup_{B_{r_0}(0)} d_k(L^1(p), u_k^1(p)) \le \sup_{B_{r_0}(0)} d_k(L_k(p), u_k(p)) < 1/k$, we conclude

$$\lim_{k\to\infty} \sup_{p \in B_{r_0}(0)} |L^1(p) - u_k^1(p)| = 0.$$

In particular, this implies $f_k(\rho(\bar{p}_k)) - f_k(-\rho(\bar{p}_k)) \to 2\rho(\bar{p}_k)\lambda_1$ as $k \to \infty$, contradicting (4-6). □

4.2 Completion of the proof of Proposition 4.1

We are now in position to contradict inequality (4-1) and finish the proof.

We apply Proposition 3.16 with u replaced by u_k and U replaced by an analogous set U_k defined by u_k. Thus, there exists $\theta_0 \in \left(0, \frac{1}{2}\pi\right]$ (independent of k) such that for all $p \in U_k \cap \{q \in B_{r_0}(0) \mid u_k(q) \notin P_{F_k}\}$, there exists $i \in \{1, \dots, m\}$ such that

$$\left|\frac{\partial u_k^i}{\partial x^i}\right|^2(p) \le \cos^2\theta_0 \cdot \left|\frac{\partial u_k}{\partial x^i}\right|^2(p).$$

For $\tau > 0$, the above inequality implies

$$\left|\frac{\partial u_k^i}{\partial x^i}\right|^2(p) \le \cos^2\theta_0(1+\tau)^2\lambda_i^2 \quad \text{or} \quad (1+\tau)^2\lambda_i^2 \le \left|\frac{\partial u_k}{\partial x^i}\right|^2(p).$$

In other words,

$$U_k \cap \{q \in B_{r_0}(0) \mid u_k(q) \notin P_{F_k}\} \subset \bigcup_{i=1}^{m} \Theta_k(\delta, i) \cup \Upsilon_k(\tau, i),$$

where $\tau > 0$ satisfying $\cos^2\theta_0(1+\tau)^2 < 1$ defines $\Upsilon_k(\tau, i)$ (Lemma 4.2), and $\delta > 0$ satisfying $(1-\delta)^2 = \cos^2\theta_0(1+\tau)^2 < 1$ defines $\Theta_k(\delta, i)$ (Lemma 4.4). For a large enough k, Lemmas 4.2 and 4.4 imply that

$$\mu_0\big(U_k \cap \{q \in B_{r_0}(0) \mid u_k(q) \notin P_{F_k}\}\big) < \epsilon.$$

Since U_k is of full measure by [Lytchak 2004, Theorem 1.6], we conclude

$$\mu_0\big(\{q \in B_{r_0}(0) \mid u_k(q) \notin P_{F_k}\}\big) < \epsilon,$$

which contradicts (4-1) and completes the proof of Proposition 4.1.

5 Homogeneous approximations

In a locally finite Euclidean building, any point has a neighborhood that can be isometrically and totally geodesically embedded into the tangent cone at that point. Thus, one can assume that u locally maps into a cone, and the sequence of blowup maps (defined in Section 2.5) all have the *same* conical target space. Invoking the Arzelà–Ascoli theorem, a subsequence of blowup maps converges to a tangent map, also into the same conical target space. Hence one can approximate the sequence of blowup maps (and hence u) by a *single* homogeneous degree 1 map.

As we are not presuming that X is locally finite, we cannot assume that u maps into a cone. Thus, the blowup maps and tangent maps have different target spaces. The goal of this section is to construct a *sequence* of homogeneous degree 1 maps corresponding to a sequence of blowup maps such that the target space of the corresponding maps agree.

The main result of this section is Proposition 5.1. For simplicity, we use the following notation: For the Gromov–Schoen blowup maps $u_\sigma : (B_1(0), g_\sigma) \to (X, d_\sigma)$ defined by (2-2) and a sequence $\sigma_k \to 0$, we let

$$u_k = u_{\sigma_k}, \quad g_k = g_{\sigma_k}, \quad d_k = d_{\sigma_k} \quad \text{and} \quad X_k = (X, d_k).$$

Proposition 5.1 *Let $u : (B_1(0), g) \to (X, d)$ be a harmonic map where X is of type W and the dimension of X is at least 2. For $p \in M$ with $\mathrm{Ord}^u(p) = 1$, let u_σ be the blowup maps at p defined by (2-2). Then there exists*

- *a sequence $\sigma_k \to 0$,*
- *a sequence of homogeneous degree 1 harmonic maps $L_k : \mathbb{R}^n \to A_k \subset X_k$, where A_k is an apartment in X_k, and*
- *$r_0 \in (0, 1)$*

such that the following properties are satisfied:

- (X_k, A_k, L_k) *is an* $(X_\omega, A_\omega, L_\omega)$*-triple where* $(X_\omega, d_\omega) = \omega\text{-lim}\,(X, d_k, \star_k)$ *and*

$$L_\omega = \omega\text{-lim}\, L_k : \mathbb{R}^n \to A_\omega \subset X_\omega.$$

  Here, $\star_k = u_k(0) = u(p)$ *and* A_ω *is an apartment of the building* X_ω*.*

- *The energy density measures and the directional energy density measures of* u_k *converge to that of* u_ω *in* $B_{r_0}(0)$*.*

- $\lim\limits_{k\to\infty} \sup\limits_{x\in B_{r_0}(0)} d_k(u_k(x), L_k(x)) = 0.$

Proof As explained in Section 2.5, a subsequence of the u_k (which we still denote by u_k) converges locally uniformly to a tangent map $u_* : B_1(0) \to (X_*, d_*)$. Let $(X_\omega, d_\omega) := \omega\text{-lim}\,(X, d_k, \star_k)$ and $u_\omega := \omega\text{-lim}\, u_k$. By Section 2.6, we can assume $u_* = u_\omega$. Thus, the (directional) energy density measures of u_k converge to those of u_ω, and u_ω is a nonconstant homogeneous degree 1 harmonic map.

By [Kleiner and Leeb 1997, Theorem 5.1.1], X_ω is a Euclidean building of type W. By Proposition 2.12, there exists $r_0 \in (0, 1)$ and an apartment $A_\omega \subset X_\omega$ such that $u_\omega | B_{r_0}(0)$ can be extended as a homogeneous degree 1 harmonic map $L_\omega : \mathbb{R}^n \to A_\omega \subset X_\omega$. Consequently, we obtain assertion (ii) about the convergence of energies in $B_{r_0}(0)$, where $u_\omega = L_\omega$.

A chart $\iota_\omega = [(\iota_1, \iota_2, \dots)] : \mathbb{R}^N \to A_\omega$ is an ultralimit of the sequence of charts $\iota_k : \mathbb{R}^N \to A_k$ where each A_k is an apartment of the building X_k [Kleiner and Leeb 1997, Proof of Theorem 5.1.1]. Let

$$L_k := \iota_k \circ \iota_\omega^{-1} \circ L_\omega : \mathbb{R}^n \to A_k \subset X_k. \tag{5-1}$$

For $x \in \mathbb{R}^n$, choose $y \in \mathbb{R}^N$ satisfying $L_\omega(x) = \iota_\omega(y)$. Then

$$[(L_k(x))] = [(\iota_k \circ \iota_\omega^{-1} \circ L_\omega(x))] = [(\iota_k(y))] = \iota_\omega(y) = L_\omega(x).$$

Thus, $\omega\text{-lim}\, L_k = L_\omega$. The homogeneity of L_k follows immediately from the definition since ι_k and ι_ω are isometric embeddings. To see that (X_k, A_k, L_k) is a $(X_\omega, A_\omega, L_\omega)$-triple, note that the isometry $\phi : A_\omega \to A_k$ is given simply by $\phi = \iota_k \circ \iota_\omega^{-1}$. This completes the proof of (i). Finally, to prove (iii), we take a further subsequence according to Lemma 5.2 below. □

Lemma 5.2 *There exists a subsequence* (k_j) *such that*

$$\lim_{j\to\infty} \sup_{B_{r_0}(0)} d_{k_j}(u_{k_j}(x), L_{k_j}(x)) = 0.$$

Proof By the definition of L_k (cf (5-1)), the uniform energy bound for u_k, and [Korevaar and Schoen 1993, Theorem 2.4.6], there exists an $M > 0$ independent of $k \in \mathbb{N}$ such that for all $x, y \in B_{r_0}(0)$,

$$d_k(L_k(x), L_k(y)) \le M|x - y| \qquad \text{and} \qquad d_k(u_k(x), u_k(y)) \le M|x - y|.$$

For each $j \in \mathbb{N}$, let $\Omega_j := \{x_1, x_2, \dots, x_{N_j}\} \subset B_{r_0}(0)$ be a finite set such that, for all $x \in B_{r_0}(0)$, there exists $x_\alpha \in \Omega_j$ with $|x - x_\alpha| < 1/(3Mj)$. Define

$$S_\alpha := \left\{ k \in \mathbb{N} \;\middle|\; d_k(u_k(x_\alpha), L_k(x_\alpha)) < \frac{1}{3j} \right\} \subset \mathbb{N}.$$

As

$$0 = d_\omega(u_\omega(x_\alpha), u_\omega(x_\alpha)) = d_\omega(u_\omega(x_\alpha), L_\omega(x_\alpha)) = \omega\text{-lim}\, d_k(u_k(x_\alpha), L_k(x_\alpha)),$$

we see that $\omega(S_\alpha) = 1$ for each $\alpha \in \{1, \dots, N_j\}$. (For more information on the ultrafilter ω, consult [Kleiner and Leeb 1997, Section 2.4.1].) Therefore

$$\omega\left(\bigcap_{\alpha=1}^{N_j} S_\alpha \right) = 1.$$

Choose $k_j \in \bigcap_{\alpha=1}^{N_j} S_\alpha$ inductively such that $k_1 \geq 1$ and $k_{j+1} > k_j$.

For $x \in B_{r_0}(0)$, choose $x_\alpha \in \Omega_j$ such that $|x - x_\alpha| < 1/(3Mj)$. Then

$$\begin{aligned} d_{k_j}(u_{k_j}(x), L_{k_j}(x)) &\leq d_{k_j}(u_{k_j}(x), u_{k_j}(x_\alpha)) + d_{k_j}(u_{k_j}(x_\alpha), L_{k_j}(x_\alpha)) + d_{k_j}(L_{k_j}(x), L_{k_j}(x_\alpha)) \\ &< M|x - x_\alpha| + \frac{1}{3j} + M|x - x_\alpha| \leq \frac{1}{j}. \end{aligned}$$

This verifies that $\sup_{B_{r_0}(0)} d_{k_j}(u_{k_j}(x), L_{k_j}(x)) \to 0$ as $j \to \infty$. □

6 Local product structure

We are now ready to prove the local product structure of a harmonic map at an order 1 point, as stated in Theorem 6.4. This is proven in [Gromov and Schoen 1992, Theorem 5.1] for locally finite Euclidean buildings, and the proof here closely follows their proof. There is a key difference — the use of Proposition 4.1.

Definition 6.1 For a map $u : B_1(0) \to X$, given $B_\sigma(x) \subset B_1(0)$, the *remainder* $R^u(x, \sigma)$ is

$$R^u(x, \sigma) = \inf_L \sup_{B_\sigma(x)} d(u(y), L(y)),$$

where the infimum is taken over homogeneous degree 1 maps about x.

Definition 6.2 Fix a Riemannian domain Ω and a finite reflection group W. Consider the collection of maps

$$A_{\Omega,W} := \{u : \Omega \to X \mid u \text{ is a harmonic map, } X \text{ is a Euclidean building of type } W\}.$$

Let $K \subset \Omega$ be a compact subset. The class $A_{\Omega,W}$ is a *K-intrinsically differentiable class* if there are constants $\sigma_0, c > 0$ and $\beta \in (0, 1]$, depending only on K, Ω, W, such that for any $u \in A_{\Omega,W}$, any $x \in K$, and for all $0 < \sigma < \sigma_0$ such that $B_\sigma(x) \subset \Omega$,

$$R^u(x, \sigma) \leq c\sigma^{1+\beta} R^u(x, \sigma_0).$$

Definition 6.3 A Euclidean building X of type W is *essentially regular* if, for every Riemannian domain Ω, and every compact $K \subset \Omega$, the class $A_{\Omega,W}$ is K-intrinsically differentiable.

Theorem 6.4 *Assume $W_{\mathrm{aff}} = \rho^{-1}(W)$. Fix $E_0 > 0$, $r_0 \in (0,1)$ and a homogeneous degree 1 map*

$$L : \mathbb{R}^n \to A \subset X$$

where $\dim X \geq 2$ and A is an apartment of X. Then there exists $\delta_0 = \delta_0(E_0, r_0, X, A, L) > 0$ with the following property: Let (X', A', L') be an (X, A, L)-triple and let $P_{F'} = F' \times Y$ be the union of all flats parallel to $F' := L'(\mathbb{R}^n) \simeq \mathbb{R}^m$. Let $u : (B_1(0), g) \to X'$ be a finite-energy harmonic map with ${}^g E^u[B_1(0)] < E_0$. If

- *$P_{F'}$ is essentially regular,*
- *$\|g - \delta_{\mathrm{euc}}\|_{C^2(B_1(0))} < \delta_0$, and*
- *$\sup_{B_{r_0}(0)} d(u, L') < \delta_0$,*

then $u(B_{r_0/4}(0)) \subset P_{F'}$.

Remark 6.5 The condition that $W_{\mathrm{aff}} = \rho^{-1}(W)$ is a technical one, which we introduce here for convenience. Given any building X of type W, we can simply enlarge the group W_{aff} to satisfy this hypothesis. (Recall Lemma 2.1.)

The condition on W_{aff} is not natural when thinking about a single building, since in principle X may no longer be thick. But in the proof below, we need to replace L' by a homogeneous degree 1 map L_0 which agrees with u at a particular point, and we need to find an apartment A_0 such that (X', A_0, L_0) is an (X, A, L)-triple to invoke Proposition 4.1. This is straightforward to accomplish when $W_{\mathrm{aff}} = \rho^{-1}(W)$.

Proof We will choose an $\epsilon_1 > 0$ toward the conclusion of the proof, but for the moment demand only that $100\epsilon_1 < \mu_0(B_{\frac{1}{4}r_0}(0))$. Now choose $\eta = \eta(\epsilon_1, E_0, r_0, X, A, L) > 0$ as in Proposition 4.1. Choose $\delta_0 > 0$ such that $2(1 + 4\theta^{-1})\delta_0 \leq \eta$, where $\theta \in \left(0, \frac{1}{4}\right]$ will be chosen later. This bound implies, in particular, that

$$\mu_0\{x \in B_{\frac{1}{2}r_0}(0) \mid u(x) \notin P_{F'}\} < \epsilon_1 < \mu_0(B_{\frac{1}{4}r_0}(0)).$$

Hence, there *are* points of $B_{\frac{1}{2}r_0}(0)$ with $u(x) \in P_{F'}$.

Let $x_0 \in B_{\frac{1}{2}r_0}(0)$ be one such point. By the third bullet point, $d(u(x_0), L'(x_0)) < \delta_0$. Hence, there exists a flat F_0 parallel to F' such that $d(F', F_0) < \delta_0$ and $u(x_0) \in F_0$. Using the product structure $F' \times Y \simeq P_{F'}$ of Lemma 3.1, let $L'(x_0) = (0, y') \in F' \times \{y'\} \simeq F'$ and let $t \in F'$ and $y_0 \in Y$ be such that $u(x_0) = (t, y_0) \in F' \times \{y_0\} \simeq F_0$. Let A_Y be an apartment of Y containing y' and y_0. Thus, F' and F_0 are both contained in the apartment $F' \times A_Y \simeq A_0$ of $P_{F'}$. Let $\tau : A_0 \to A_0$ be a translation which takes $(0, y')$ to (t, y_0) and define $L_0 = \tau \circ L'$. Then $L_0(x_0) = u(x_0)$ and $\sup_{B_{\frac{1}{2}r_0}(x_0)} d(L_0, u) < 2\delta_0$.

Following the ideas in Remark 3.14, let $\iota_{A'} : \mathbb{R}^N \to A'$ be a chart in $\varphi \cdot \mathscr{A}$, where $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal transformation such that $\iota_{A'}(\mathbb{R}^m \times \{(0, \dots, 0)\}) = F'$. Let ι_{A_0} denote a chart of $\varphi \cdot \mathscr{A}$ such that the restriction of $\iota_{A_0} \circ \iota_{A'}^{-1}$ to $\mathbb{R}^m$ is the identity map. Let $\widehat{\phi} : A \to A_0$ be defined by $\widehat{\phi} := \tau \circ \iota_{A_0} \circ \iota_{A'}^{-1} \circ \phi$.

(Here $\phi : A \to A'$ is as in Definition 2.13.) Since $W_{\mathrm{aff}} = \rho^{-1}(W)$, $\widehat{\phi}$ satisfies the necessary conditions which make (X', A_0, L_0) an (X, A, L)-triple.

Fix normal coordinates centered at x_0. For a map $f : B_{\frac{1}{2}r_0}(x_0) \to (X', d')$, define

$$ {}^i f : (B_{\frac{1}{2}r_0}(0), g_i) \to (X', d_i),$$

where ${}^i f(x) := f(\theta^i x)$, $g_i(x) = \theta^{-i} g(\theta^i x)$ and $d_i := \theta^{-i} d$. The uniform Lipschitz bounds on u on $B_{\frac{1}{2}r_0}(x_0)$, which depend only on ${}^g E^u[B_1(0)]$, imply uniform energy bounds on ${}^{g_i}E^{{}^i u}[B_{\frac{1}{2}r_0}(0)]$.

Our inductive claim is as follows. For $\theta > 0$ sufficiently small and for each $i \in \mathbb{Z}_{\geq 0}$:

(1) There exists $\delta_i > 0$ such that

$$\sup_{B_{\frac{1}{4}r_0}(0)} d_i({}^i u, {}^i L_0) \leq 2\delta_i \leq 2\left(1 + 2\theta^{-1}\sum_{j=0}^{i-1} 2^{-j}\right)\delta_0 \leq 2(1 + 4\theta^{-1})\delta_0,$$

where for $i = 0$ we presume that $\sum_{j=0}^{-1} 2^{-j} = 0$.

(2) There exists a homogeneous degree 1 map

$$M_i : (B_{\frac{1}{4}r_0}(0), g_i) \to (P_{F'}, d_i) \quad \text{such that} \quad \sup_{B_{\frac{1}{4}r_0}(0)} d_i({}^i u, M_i) = D_i \leq 2^{-i+1}\delta_0.$$

For the base case, $i = 0$, we set $M_0 = L_0$ and observe that with $2\delta_0$ in the first inequality, $D_0 = 2\delta_0$ in the second, the claimed bounds hold.

We now suppose that these inequalities hold for some i and argue that they hold for $i + 1$. First, we observe $u(x_0) = {}^i u(0) = {}^i L_0(0)$, so that

$$d_i({}^i L_0(0), M_i(0)) \leq D_i.$$

While the triangle inequality implies

$$\max_{\partial B_{\frac{1}{4}r_0}(0)} d_i({}^i L_0, M_i) \leq 2\delta_i + D_i,$$

together with the homogeneity of ${}^i L_0, M_i$ we have

$$\sup_{B_{\frac{1}{4}\theta r_0}(0)} d_i({}^i L_0, M_i) \leq 2\theta\delta_i + D_i.$$

Combining this with the assumed bounds on $d_i({}^i u, M_i)$, we see that

$$\sup_{B_{\frac{1}{4}\theta r_0}(0)} d_i({}^i u, {}^i L_0) \leq 2\theta\delta_i + 2D_i.$$

In particular, considering the rescaled distance d_{i+1} on $B_{\frac{1}{2}r_0}(0)$, we have that

$$\sup_{B_{\frac{1}{4}r_0}(0)} d_{i+1}({}^{i+1}u, {}^{i+1}L_0) \leq 2\delta_i + 2\theta^{-1}D_i =: 2\delta_{i+1}.$$

The assumed bounds on δ_i, D_i immediately imply the needed bound on δ_{i+1}.

We now address point (2). Since (X', A_0, L_0) is an (X, A, L)-triple and $2\delta_i \le 2(1+4\theta^{-1})\delta_0 \le \eta$, applying Proposition 4.1,

$$\mu_0^n\{x \in B_{\frac14 r_0}(0) \mid {}^i u(x) \notin P_{F'}\} < \epsilon_1.$$

In particular, there is at least one radius $r \in \left[\frac18 r_0, \frac14 r_0\right]$ such that

$$\mu_0^{n-1}\{x \in \partial B_r(0) \mid {}^i u(x) \notin P_{F'}\} < \frac{8\epsilon_1}{r_0}.$$

Let $\pi : X' \to P_{F'}$ denote the closest-point projection, and $v : B_r(0) \to P_{F'}$ with $v = \pi \circ {}^i u$ on $\partial B_r(0)$ the energy-minimizing map. We shall first show that v is very close to ${}^i u$. Point (2) of the inductive hypothesis implies that ${}^i u$ is at distance at most D_i from $P_{F'}$ on $B_{\frac14 r_0}(0)$, and the measure estimate implies that in fact

$$\int_{\partial B_r(0)} d_i({}^i u, v)\, d\Sigma \le \frac{8\epsilon_1 D_i}{r_0}.$$

Since [Gromov and Schoen 1992, equation (2.2)] can be extended to all Euclidean buildings, we may follow the proof of [Gromov and Schoen 1992, Lemma 5.3] to see that $d_i({}^i u, v)$ is subharmonic. This implies that for some constant c_1, depending only on the domain,

$$\sup_{B_{\frac{1}{16}r_0}(0)} d_i({}^i u, v) \le \frac{c_1\epsilon_1 D_i}{r_0}. \tag{6-1}$$

Now, by hypothesis, $P_{F'}$ is essentially regular and thus, v is $B_r(0)$-intrinsically differentiable. It follows that there exists a homogeneous degree 1 map $\widetilde{M}$ with $\widetilde{M}(0) = v(0)$ such that for any homogeneous degree 1 map M and sufficiently small $\theta > 0$,

$$\sup_{B_{\frac14\theta r_0}(0)} d_i(v, \widetilde{M}) \le c_2\theta^{1+\beta} \sup_{B_{\frac{1}{16}r_0}(0)} d_i(v, M).$$

Here c_2 depends on the domains $K = \overline{B_{\frac{1}{16}r_0}(0)}$, $\Omega = B_{\frac18 r_0}(0)$, and on the total energy of v (and hence of u). We remark at this point that, although c_2 also depends on the metric on $B_{\frac18 r_0}(0)$, since these metrics are being rescaled towards the Euclidean metric, the c_2 involved will improve as the induction continues.

Applying this for $M = M_i$, and using the triangle inequality, the inductive assumption, and (6-1),

$$\begin{aligned}\sup_{B_{\frac14\theta r_0}(0)} d_i(v, \widetilde{M}) \le c_2\theta^{1+\beta} \sup_{B_{\frac{1}{16}r_0}(0)} d_i(v, M_i) &\le c_2\theta^{1+\beta} \sup_{B_{\frac{1}{16}r_0}(0)} \big(d_i(M_i, {}^i u) + d_i({}^i u, v)\big) \\ &\le c_2\theta^{1+\beta} D_i\left(1 + \frac{c_1\epsilon_1}{r_0}\right).\end{aligned}$$

Again applying the triangle inequality and (6-1),

$$\sup_{B_{\frac14\theta r_0}(0)} d_i({}^i u, \widetilde{M}) \le \left(\frac{c_1\epsilon_1}{r_0} + c_2\theta^{1+\beta} + \frac{c_1 c_2\theta^{1+\beta}\epsilon_1}{r_0}\right) D_i.$$

Now, as long as we take θ, ϵ_1 sufficiently small, we may absorb the final term into the former two, so

$$\sup_{B_{\frac14\theta r_0}(0)} d_i({}^i u, \widetilde{M}) \le 2\left(\frac{c_1\epsilon_1}{r_0} + c_2\theta^{1+\beta}\right) D_i. \tag{6-2}$$

This immediately tells us that

$$\sup_{B_{\frac{1}{4}r_0}(0)} d_{i+1}({}^{i+1}u, {}^{1}\widetilde{M}) \leq 2\theta^{-1}\left(\frac{c_1\epsilon_1}{r_0} + c_2\theta^{1+\beta}\right) D_i.$$

Setting $M_{i+1} := {}^{1}\widetilde{M}$ completes the inductive step, so long as

$$2\theta^{-1}\left(\frac{c_1\epsilon_1}{r_0} + c_2\theta^{1+\beta}\right) \leq \tfrac{1}{2}.$$

To choose θ and ϵ_1 appropriately, first observe that the constants c_1, c_2 depend on the domain (i.e., its dimension and metric — in particular, how far it is from Euclidean) and on the total energy of the map ${}^{i}u$. The dimension is constant, the metric converges to Euclidean as $i \to \infty$, and the bound on the energy of u implies that we have uniform energy bounds on the ${}^{i}u$. Hence, these constants do not depend on i, or (crucially) upon θ and ϵ_1.

We first choose $\theta \in \left(0, \frac{1}{4}\right]$ so that $2c_2\theta^{\beta} \leq \frac{1}{4}$, then decrease ϵ_1 if necessary so that $2c_1\theta^{-1}\epsilon_1 \leq \frac{1}{4}r_0$. Since, at the outset, we chose δ_0 such that $2(1+4\theta^{-1})\delta_0 < \eta$ and at each step of the induction argument, $\sup_{B_{\frac{1}{4}r_0}(0)} d({}^{i}u, {}^{i}L_0) \leq 2(1+4\theta^{-1})\delta_0$, this bound on δ_0 ensures that the inductive argument works at all stages. In particular, at each stage, we have $\mu_0\{x \in B_{\frac{1}{4}r_0}(0) \mid {}^{i}u(x) \notin P_{F'}\} < \epsilon_1$.

Finally, we show that $u(B_{\frac{1}{4}r_0}(0)) \subset P_{F'}$. Suppose, for the sake of contradiction, that there is some point $y \in B_{\frac{1}{4}r_0}(0)$ so that $u(y) \notin P_{F'}$. Recall that the open set $U = \{x \in B_{\frac{1}{4}r_0}(0) \mid u(x) \notin P_{F'}\}$ has $\mu_0(U) < \epsilon_1 < \mu_0(B_{\frac{1}{4}r_0}(0))$. Hence we may in fact choose $y \in B_{\frac{1}{4}r_0}(0)$ so that for some $0 < r < \frac{1}{4}r_0$, $B_r(y) \subseteq U$ and there exists $x_0 \in u^{-1}(P_{F'}) \cap \partial B_r(y)$. We observe that, in the limit as $\sigma \to 0$, at least half of the ball $B_\sigma(x_0)$ lies in U.

But on the other hand, by the above inductive argument, at the scale θ^i, we have that $d_i({}^{i}u, {}^{i}L_0)$ is small enough that

$$\mu_0\{x \in B_{\frac{1}{4}r_0}(0) \mid {}^{i}u(x) \notin P_{F'}\} < \epsilon_1,$$

and in particular, for i large enough, no more than $\frac{1}{4}$ of the ball $B_{r_0/(4\theta^i)}(x_0)$ lies in U. This contradiction allows us to conclude that there are no points of U in $B_{\frac{1}{4}r_0}(0)$. That is, $u(B_{\frac{1}{4}r_0}(0)) \subset P_{F'}$. □

We are now in a position to state two key ingredients in the proofs of the main theorems, namely items (1) and (2) below, which provide a gap theorem for harmonic maps into all Euclidean buildings and demonstrate that such maps have a local product structure at order 1 points. The theorem mirrors that of [Gromov and Schoen 1992, Theorem 6.3] and the proof is quite similar, though our proof applies Theorem 6.4 in place of [Gromov and Schoen 1992, Theorem 5.1]. Because of the similarities, we defer the proof to Section A.2.

Theorem 6.6 *Let X be a Euclidean building of type W. Then we have the following*:

(1) *There is a constant ϵ, depending on n, g and W, such that for a harmonic map $u : (\Omega^n, g) \to X$ and any $p \in \Omega$, either $\mathrm{Ord}^u(p) = 1$ or $\mathrm{Ord}^u(p) \geq 1 + \epsilon$.*

(2) *When* $\mathrm{Ord}^u(p)=1$, *there exists a subbuilding* $P_F \simeq F\times Y$ *and an* $r>0$ *such that* $F\simeq \mathbb{R}^m$ *for some* $m\in\{1,\dots,\min\{n,N\}\}$, $Y=Y^{N-m}$ *is a building of lower dimension, and* $u|B_r(p)$ *decomposes into two harmonic maps,* $u|B_r(p)=(u_1,u_2):B_r(p)\to P_F$, *such that* $u_1:B_r(p)\to\mathbb{R}^m$ *is a harmonic map of rank* m, *and* $u_2:B_r(p)\to Y$ *satisfies either that* $\mathrm{Ord}^{u_2}(p)\geq 1+\epsilon$, *or that* u_2 *is a constant map.*

(3) X *is essentially regular.*

7 Proof of the main theorems

Because of the analysis provided in the previous sections, the proofs of the main theorems of this paper (stated in the introduction) follow from adapting the proofs in [Gromov and Schoen 1992] for locally finite buildings. In this section, we provide the necessary adjustment to their arguments.

7.1 Proof of Theorem 1.1

Definition 7.1 Let $u:\Omega\to X$ be a harmonic map from a Riemannian domain into a Euclidean building of type W. A point $p\in\Omega$ is called a *regular point* if there exists a neighborhood U of p and an apartment A of X such that $u(U)\subset A$. Otherwise p is called a *singular point*. The *singular set* $\mathcal{S}(u)$ is the set of all singular points.

Define

$$\mathcal{S}_{>1}(u)=\{p\in\mathcal{S}(u)\mid \mathrm{Ord}^u(p)>1\},$$
$$\mathcal{S}_{=1}(u)=\{p\in\mathcal{S}(u)\mid \mathrm{Ord}^u(p)=1\},$$
$$\mathcal{S}_0(u)=\{p\in\Omega\mid \mathrm{Ord}^u(p)>1\}.$$

In particular,

$$\mathcal{S}(u)=\mathcal{S}_{=1}(u)\cup\mathcal{S}_{>1}(u)\quad\text{and}\quad \mathcal{S}_{>1}(u)\subset\mathcal{S}_0(u).$$

Lemma 7.2 *If* $n=\dim\Omega$, *then* $\mathcal{S}_0(u)$ *is a closed set with* $\dim_{\mathcal{H}}(\mathcal{S}_0(u))\leq n-2$.

Proof This follows from a slight modification of the proof in [Gromov and Schoen 1992, paragraph after Lemma 6.5]. The main modification to the argument is in fact in the invoking of the conclusion of [loc. cit., Lemma 6.5]. The conclusion of [loc. cit., Lemma 6.5] holds in this setting for the (Gromov–Schoen) blowup maps u_k and $u_\omega:=\omega\text{-}\lim u_k$ since the local uniform convergence in the pullback sense given by Remark 2.9 and properties of the order function imply that for $x_k\to x$, where $x_k\in\mathcal{S}_0(u_k)$, $\limsup_{k\to\infty}\mathrm{Ord}^{u_k}(x_k)\leq\mathrm{Ord}^{u_\omega}(x)$ and thus by Theorem 6.6(1), $x\in\mathcal{S}_0(u_\omega)$. □

Proof of Theorem 1.1 This is an easy consequence of Theorem 6.6. For the sake of completeness, we will include the proof which involves an inductive argument on the dimension of X. The one-dimensional case was proved in [Sun 2003]. Now assume that the assertion is true for dimensions less than N.

Suppose $p\in\mathcal{S}_{=1}(u)$. Then Theorem 6.6(2) asserts that there exists an $r>0$ and a lower-dimensional subbuilding (Y,d) such that for $\sigma\in(0,r]$, it holds that $u_\sigma=(u_{\sigma,1},u_{\sigma,2})$, where $u_{\sigma,1}:B_1(0)\to\mathbb{R}^m$ and $u_{\sigma,2}:B_1(0)\to(Y,d_\sigma)$. Thus, by the inductive hypothesis, there exists a $\sigma_p>0$ such that

$$\dim_{\mathcal{H}}(\mathcal{S}(u)\cap B_{\sigma_p}(p))=\dim_{\mathcal{H}}(\mathcal{S}(u_{\sigma_p}))=\dim_{\mathcal{H}}(\mathcal{S}(u_{\sigma_p,2}))\leq n-2.$$

Now cover $\mathcal{S}_{=1}(u)$ by balls $\{B_{\sigma_p}(p)\}_{p\in\mathcal{S}_{=1}(u)}$ and refine this cover to a countable subcover $\{B_{\sigma_{p_j}}(p_j)\}$. Since for all $t > n-2$, $\mathcal{H}^t(B_{\sigma_{p_j}}(p_j)) = 0$ and $\mathcal{H}^t$ is countably additive, $\mathcal{H}^t(\mathcal{S}_{=1}(u)) = 0$ for all $t > n-2$. Thus, $\dim_{\mathcal{H}}(\mathcal{S}_{=1}(u)) \le n-2$. Combined with Lemma 7.2, we conclude that $\mathcal{S}(u)$ is a closed set with $\dim_{\mathcal{H}}(\mathcal{S}(u)) \le n-2$. □

The following corollary of Theorem 1.1 will be important in the proof of the rigidity theorems.

Corollary 7.3 *For any compact subdomain Ω_1 of Ω, there is a sequence of Lipschitz functions $\{\psi_i\}$ such that $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u)\cap\overline{\Omega}_1$, $0 \le \psi_i \le 1$, $\psi_i(x) \to 1$ for all $x \in \Omega_1 \setminus \mathcal{S}_1$, and*

$$\lim_{i\to\infty}\int_\Omega |\nabla du||\nabla\psi_i|\,d\mu = 0.$$

Proof The proof follows from [Gromov and Schoen 1992, p. 227, third paragraph], adapted to the nonlocally finite case using Theorems 6.6 and 1.1. □

7.2 Proofs of Theorems 1.2 and 1.3

First we recall the definition of Finite Rank (FR) spaces given in [Korevaar and Schoen 1999].

Definition 7.4 A CAT(0) space X is an *FR-space* if there exists $\epsilon_0 > 0$ and $R_0 > 1$ such that any subset of X with diameter $D > R_0$ is contained in a ball of radius $(1-\epsilon_0)D/\sqrt{2}$.

Since every Euclidean building has finite geometric dimension, Theorem 1.3 of [Caprace and Lytchak 2010] implies that every Euclidean building is an FR-space. Korevaar and Schoen prove an existence theorem for harmonic maps to FR-spaces.

Theorem 7.5 [Korevaar and Schoen 1999, Theorem 1] *Let $\pi_1(M)$ be a fundamental group of a compact Riemannian manifold M and ρ be an isometric action of $\pi_1(M)$ on an FR-space X. Either there exists an invariant equivalence class of rays, or there exists a ρ-equivariant harmonic map $u : \widetilde{M} \to X$.*

Proposition 7.6 *Let $\widetilde{M}$, X and ρ be as in Theorem 1.2. Then there exists a ρ-equivariant harmonic map $u : \widetilde{M} \to X$.*

Proof If the rank of $\widetilde{M}$ is ≥ 2, then the assertion follows from Theorem 7.5 and the assumption that ρ does not fix a point at infinity; see also [Jost 1994, Corollary 3.6]. Moreover, it is straightforward to verify that we can replace the assumption that M is compact in Theorem 7.5 by the assumption that M is of finite volume and there exists a ρ-equivariant finite-energy locally Lipschitz map $\widetilde{M} \to X$. The existence of a finite-energy locally Lipschitz map follows from [Gromov and Schoen 1992, Lemma 8.1], noting that the proof does not need the target building to be locally finite. Thus, the assertion for the case when the rank of $\widetilde{M}$ is 1 also follows the same way. □

Proof of Theorem 1.2 The proof follows from applying the Bochner method to the map u given by Proposition 7.6. More specifically, for lattices in rank-1 groups we can follow the proofs of [Gromov and Schoen 1992, Theorems 7.2 and 7.4], using Theorem 1.1 and Corollary 7.3 in the appropriate places, to show that u is constant.

For higher-rank groups we follow the proof of [Daskalopoulos et al. 2011, Lemma 15] to show that u is totally geodesic in a neighborhood of every regular point. Then, using Theorem 1.1 and the arguments in the first part of the proof of [Daskalopoulos et al. 2011, Theorem 1], we conclude that u is totally geodesic. (Notice that part of their proof does not require any additional hypotheses on ρ.) In particular, this implies that $\mathrm{Ord}^u(p) = 1$. By Theorem 6.6, there exists a flat F and an $r > 0$ such that $u|B_r(p) = (u_1, u_2) : B_r(p) \to F \times Y$ where both u_1 and u_2 are totally geodesic and $\mathrm{Ord}^{u_2}(p) > 1$. Thus, u_2 is a constant map, which in turn implies u is a smooth, totally geodesic harmonic map into $F \subset A \simeq \mathbb{R}^N$ for some apartment $A \subset X$. It thus follows that u must be a constant map. □

Proof of Theorem 1.3 Using Proposition 7.6, Theorem 1.1 and Corollary 7.3, we can follow the proofs of [Gromov and Schoen 1992, Theorems 7.2 and 7.3]. □

Appendix Proof of Theorem 6.6

In this section we prove Theorem 6.6. We begin with a few technical lemmas, then prove essential regularity for trees and finally, in Section A.2, we prove the theorem.

The following lemma is useful for showing that complexes are essentially regular; the computations are standard but we include them for completeness.

Lemma A.1 *Suppose that there is a $\tau_0 > 0$ and a $\theta \in \left(0, \frac{1}{2}\right]$ so that for any $u \in A_{\Omega,W}$, any $x_0 \in K$, and any $\sigma \in (0, \tau_0]$,*

$$R^u(x_0, \theta\sigma) \le \tfrac{1}{2}\theta R^u(x_0, \sigma).$$

Then $A_{\Omega,W}$ is a K-intrinsically differentiable class.

Proof We will use τ_0 to be the σ_0 in the definition of intrinsic differentiability. We observe that by a quick induction argument (and the monotonicity of $R^u(x, \sigma)$ in its second argument), if $\sigma \in (\theta^n\tau_0, \theta^{n-1}\tau_0]$, then

$$R^u(x_0, \sigma) \le \frac{\theta^{n-1}}{2^{n-1}} R^u(x_0, \tau_0).$$

Choose $\beta \in (0, 1]$ such that $\frac{1}{2} = \theta^\beta$ and set $c = 2/(\theta\tau_0^{1+\beta})$. If $\sigma \in (\theta^n\tau_0, \theta^{n-1}\tau_0]$, then

$$R^u(x_0, \sigma) \le c\tau_0^{1+\beta} \frac{\theta^n}{2^n} R^u(x_0, \tau_0) = c\tau_0^{1+\beta}(\theta^n)^{1+\beta} R^u(x_0, \tau_0).$$

Then, because $\theta^n\tau_0 < \sigma$,

$$R^u(x_0, \sigma) \le c(\tau_0\theta^n)^{1+\beta} R^u(x_0, \tau_0) \le c\sigma^{1+\beta} R^u(x_0, \tau_0).$$ □

The next lemma generalizes the classical reverse Poincaré inequality.

Lemma A.2 *There exists a constant $C > 0$ such that, for a harmonic map $u : (B_1(0), g) \to X$ and $Q \in X$,*

$$\int_{B_{\frac{1}{2}}(0)} |\nabla u|^2 \, d\mu_g \le C \int_{B_1(0)} d^2(u, Q) \, d\mu_g.$$

Proof By [Gromov and Schoen 1992, Proposition 2.2],

$$2\int_{B_1(0)} |\nabla u|^2 \varphi \, d\mu_g \leq -\int_{B_1(0)} \nabla d^2(u, Q) \cdot \nabla \varphi \, d\mu_g \quad \text{for all } \varphi \in C_c^\infty(B_1(0)) \text{ with } \varphi \geq 0.$$

Let $\varphi = \eta^2$ be a radial test function with $\eta \equiv 1$ in $B_{\frac{1}{2}}(0)$, $\eta \equiv 0$ in $B_1(0) \setminus B_{\frac{3}{4}}(0)$, and $|\nabla \eta| \leq 8$. Then

$$\begin{aligned}\int_{B_1(0)} |\nabla u|^2 \eta^2 \, d\mu_g &\leq 2\int_{B_1(0)} |d(u, Q)\nabla \eta| \, |\eta \nabla d(u, Q)| \, d\mu_g \\ &\leq 2\Big(\int_{B_1(0)} d^2(u, Q)|\nabla \eta|^2 \, d\mu_g\Big)^{\frac{1}{2}} \Big(\int_{B_1(0)} |\nabla d(u, Q)|^2 \eta^2 \, d\mu_g\Big)^{\frac{1}{2}} \\ &\leq 16\Big(\int_{B_1(0)} d^2(u, Q) \, d\mu_g\Big)^{\frac{1}{2}} \Big(\int_{B_1(0)} |\nabla u|^2 \, \eta^2 d\mu_g\Big)^{\frac{1}{2}}.\end{aligned}$$

□

A.1 Trees are essentially regular

While the work of Sun [2003] and Gromov and Schoen [1992] together imply essential regularity for trees, it is important to our induction argument that the constants c and β that appear in verifying essential regularity are independent of the target tree. For this reason, we provide below more details than can be found in [Gromov and Schoen 1992, Proof of Theorem 5.5]. Given a Lipschitz Riemannian domain Ω, set

$$A_\Omega := \{u : \Omega \to T \mid u \text{ is a harmonic map to a tree } T\}.$$

Notice in this definition (cf Definition 6.2) we suppress the W from the subscript as every tree has the same finite reflection group.

Proposition A.3 *Trees are essentially regular* (*cf Definition 6.3*).

Proof Let Ω be a Riemannian domain and $K \Subset \Omega$ a compact subset. Let $\tau_0 = \min\{\frac{1}{2}, d(\partial\Omega, K)\}$. We will prove that there exists $\theta \in (0, \frac{1}{2}]$ such that for any harmonic map to a tree $u : \Omega \to T$, and any ball $B_\sigma(x_0)$ for $0 < \sigma \leq \tau_0$ with $x_0 \in K$, we have

$$R^u(x_0, \theta\sigma) \leq \tfrac{1}{2}\theta R^u(x_0, \sigma).$$

An application of Lemma A.1 then implies that all trees are essentially regular, with the constants c and β depending on Ω and K, but not on the tree.

We proceed by contradiction. Assume that there is no such θ. Thus, there exists a sequence of harmonic maps $u_k : (\Omega, g) \to (T_k, d_k)$ to trees, $x_k \in K$, $\sigma_k \in (0, \tau_0]$, and $\theta_k \to 0$ such that

$$\tfrac{1}{2}\theta_k R^{u_k}(x_k, \sigma_k) < R^{u_k}(x_k, \theta_k\sigma_k).$$

Choose normal coordinates about each x_k and then rescale the maps to obtain a new sequence $v_k : (B_2(0), g_k) \to (T_k, d_k')$ by taking $v_k(x) := u_k(\sigma_k x)$ and setting $d_k' := \mu_k d_k$, where we choose μ_k so that

$$\sup_{x \in B_1(0)} d_k'(v_k(x), v_k(0)) = 1. \tag{A-1}$$

For the rescaled sequence, we have the inequality

$$\tfrac{1}{2}\theta_k R^{v_k}(0,1) < R^{v_k}(0,\theta_k). \tag{A-2}$$

Assume $\liminf_{k\to\infty} I^{v_k}\left(\frac{1}{2}\right) = 0$, and choose a subsequence v_{k_j} with $k_j \to \infty$ such that $I^{v_{k_j}}\left(\frac{1}{2}\right) \to 0$. By applying the monotonicity of $r \mapsto e^{cr} r^{-(n+1)} I^{v_{k_j}}(r)$ multiple times, we have

$$\begin{aligned}
\left(R^{v_{k_j}}(0,1)\right)^2 \le \left(\frac{2}{\theta_{k_j}} R^{v_{k_j}}(0,\theta_{k_j})\right)^2 &\le \frac{4}{\theta_{k_j}^2} \sup_{x\in B_{\theta_{k_j}}(0)} d'^2_{k_j}(v_{k_j}(x), v_{k_j}(0)) \\
&\le \frac{4e^{c\theta_{k_j}}}{\theta_{k_j}^{n+2}} \sup_{x\in B_{\theta_{k_j}}(0)} \int_{B_{\theta_{k_j}}(x)} d'^2_{k_j}(v_{k_j}, v_{k_j}(0))\, d\mu_{g_{k_j}} \\
&\le \frac{e^{c\theta_{k_j}} 2^{n+4}}{(2\theta_{k_j})^{n+2}} \int_{B_{2\theta_{k_j}}(0)} d'^2_{k_j}(v_{k_j}, v_{k_j}(0))\, d\mu_{g_{k_j}} \\
&\le \frac{e^{2c\theta_{k_j}} 2^{n+4} I^{v_{k_j}}(2\theta_{k_j})}{(2\theta_{k_j})^{n+1}} \;\le\; e^{c/2} 2^{2n+5} I^{v_{k_j}}\left(\tfrac{1}{2}\right) \to 0.
\end{aligned}$$

Thus, $R^{v_{k_j}}(0,1) \to 0$. Combined with $I^{v_{k_j}}\left(\frac{1}{2}\right) \to 0$, we have $d'_{k_j}(v_{k_j}, v_{k_j}(0)) \to 0$ uniformly in $B_1(0)$, contradicting (A-6). Therefore, $I^{v_k}\left(\frac{1}{2}\right) \ge \epsilon > 0$ for all k sufficiently large. Additionally, by Lemma A.2 and (A-6), we also have a uniform energy bound, $E^{v_k}[B_1(0)] \le E_0$. By the convergence results of [Korevaar and Schoen 1997, Section 3] and applying Remark 2.9, the sequence $v_k|B_1(0)$ converges locally uniformly in the pullback sense to a harmonic map $v_\omega : B_1(0) \to T$, where T is a tree. The uniform boundedness (A-6) and the local uniform convergence $d(v_k, v_k(0)) \to d_\omega(v_\omega, v_\omega(0))$ implies $d(v_k, v_k(0)) \to d_\omega(v_\omega, v_\omega(0))$ in $L^2(B_1(0))$. Thus, the inequality $I^{v_k}\left(\frac{1}{2}\right) \ge \epsilon$ implies that $I^{v_\omega}\left(\frac{1}{2}\right) \ge \epsilon$; in particular, v_ω is a nonconstant map. We can now follow the last paragraph of the proof of [Gromov and Schoen 1992, Theorem 5.5] with help from [Sun 2003, Theorem 3.8 and Theorem 3.9] to account for the fact that the trees considered here are not necessarily locally finite. □

A.2 Extending to higher dimension

The following lemma is useful for finding the order gap, because it provides information about the order of product maps.

Lemma A.4 *Let X, Y be complete CAT(0) spaces, and let $u = (u_1, u_2)$: $B_r(0) \subset \mathbb{R}^n \to X \times Y$ be a homogeneous harmonic map of order α in the sense that $\mathrm{Ord}^u(0,s) = \alpha$ for all $s \in (0,r)$. Then if u_1 (resp. u_2) is nonconstant, it also satisfies $\mathrm{Ord}^{u_1}(0,s) = \alpha$ for all $s \in (0,r)$.*

In the proof, we will need following characterization of homogeneous harmonic maps.

Lemma A.5 *Let X be a complete CAT(0) space. A nonconstant harmonic map $v : B_r(0) \subset \mathbb{R}^n \to X$ has $\mathrm{Ord}^v(x_0,\sigma) = \alpha$ for all $\sigma \in (0,r)$ if and only if*

(1) *for all $\lambda < 1$ and all $x \in B_r(0)$,*

$$d(v(\lambda x), v(0)) = \lambda^\alpha d(v(x), v(0)),$$

(2) *for each* $x \in B_r(0)$, *the map* $\lambda \mapsto u(\lambda x)$ *is a geodesic in* X, *and*

(3) *for all* $\lambda < 1$ *and* $\sigma \in (0, r)$, $E^v(\lambda\sigma) = \lambda^{n+2\alpha-2} E^v(\sigma)$.

Lemma A.5 follows by work similar to that in [Gromov and Schoen 1992, Section 2] for locally finite complexes, but can be extended to general complete CAT(0) spaces by the same adaptations which allowed us to define the order function.

Proof of Lemma A.4 We use the characterization of Lemma A.5, so we shall verify that (1)–(3) of that lemma hold for the maps u_1 and u_2. We write d for the metric on $X \times Y$, and d_1, d_2 for the metrics on X, Y, respectively. Moreover, throughout we presume that both u_1, u_2 are nonconstant since, for example, if u_1 is constant then u_2 must be homogeneous of order α.

We first verify (2). To see this, note that the map $\lambda \mapsto u(\lambda x)$ is a geodesic in $X \times Y$ and, for $\gamma := (\gamma_1, \gamma_2) : [0, 1] \to X \times Y$, γ is a geodesic if and only if the two factors γ_1, γ_2 are.

Since it will be useful to us in the next point, we observe at this stage that since $\gamma_i(\lambda) := u_i(\lambda x)$ is a constant-speed geodesic, if we write $\delta_i(p) = d_i(p, u_i(0))$, then

$$\delta_1(\gamma_1(\lambda))\delta_2(\gamma_2(1)) = \delta_1(\gamma_1(1))\delta_2(\gamma_2(\lambda)).$$

In particular, the ratio between $\delta_1(u_1(\lambda x))$ and $\delta_2(u_2(\lambda x))$ does not depend on λ, but only on x.

Now, we verify (1). Let $x \in B_r(0)$ and $\lambda < 1$. Since u is homogeneous harmonic, we have that

$$d(u(\lambda x), u(0)) = \lambda^\alpha d(u(x), u(0)). \tag{A-3}$$

Now, for $i, j \in \{1, 2\}$, $i \neq j$, if $u_i(x) = u_i(0) = 0$, by the previous work we see that $u_i(\lambda x) = u_i(0)$ as well. Therefore, because u maps into a product, we have that

$$d_j(u_j(\lambda x), u_j(0)) = d(u(\lambda x), u(0)) = \lambda^\alpha d(u(x), u(0)) = \lambda^\alpha d_j(u_j(x), u_j(0)),$$

and the corresponding identity clearly holds for u_i. We now assume that $u_i(x) \neq u_i(0)$ for $i \in \{1, 2\}$.

Recall that by the definition of the product metric, $d^2((p_1, p_2), u(0)) = d_1^2(p_1, u_1(0)) + d_2^2(p_2, u_2(0))$. Squaring (A-3) and expanding the result, we see that

$$d_1^2(u_1(\lambda x), u_1(0)) + d_2^2(u_2(\lambda x), u_2(0)) = \lambda^{2\alpha} d_1^2(u_1(x), u_1(0)) + \lambda^{2\alpha} d_2^2(u_2(x), u_2(0)).$$

And since

$$\frac{d_1(u_1(\lambda x), u_1(0))}{d_2(u_2(\lambda x), u_2(0))} = c$$

for c independent of λ,

$$(1 + c^2) d_1^2(u_1(\lambda x), u_1(0)) = (1 + c^2)\lambda^{2\alpha} d_1^2(u_1(x), u_1(0)).$$

It follows that for $i = 1, 2$,

$$d_i(u_i(\lambda x), u_i(0)) = \lambda^\alpha d_i(u_i(x), u_i(0)). \tag{A-4}$$

Finally we verify (3). First we note that because of (A-4) we can use triangle comparison to conclude that for $i = 1, 2$ and any $x \in B_r(0)$,

$$|\nabla u_i|^2(\lambda x) \leq \lambda^{2\alpha-2}|\nabla u_i|^2(x).$$

Integrating, we conclude that for $i = 1, 2$, and for any $\lambda < 1$ and $\sigma \in (0, r)$

$$E^{u_i}(\lambda\sigma) \leq \lambda^{n+2\alpha-2}E^{u_i}(\sigma). \tag{A-5}$$

However, we also observe that for any s,

$$E^{u}(s) = E^{u_1}(s) + E^{u_2}(s),$$

so we expand

$$E^{u}(\lambda\sigma) = E^{u_1}(\lambda\sigma) + E^{u_2}(\lambda\sigma) \leq \lambda^{n+2\alpha-2}(E^{u_1}(\sigma) + E^{u_2}(\sigma)) = \lambda^{n+2\alpha-2}E^{u}(\sigma).$$

Since the outer two quantities are equal, the inner inequality cannot be strict. Therefore, for each $i = 1, 2$,

$$E^{u_i}(\lambda\sigma) = \lambda^{n+2\alpha-2}E^{u_i}(\sigma).$$

Therefore both u_1 and u_2 are in fact homogeneous harmonic maps of order α, in the sense that $\mathrm{Ord}_{u_i}(0, s) = \alpha$ for all $s \in (0, r)$. □

Proof of Theorem 6.6 Throughout, we proceed by induction on N, the dimension of the building; we suppose at all stages that we have proved all of these results for all buildings of dimension less than N.

First, note that for trees, (1) is proven in [Sun 2003, Theorem 3.8] and the constant ϵ is independent of the tree. Now suppose $N \geq 2$. To show (1), suppose we have a sequence of harmonic maps $u_k : (\Omega, g) \to X_k$ into Euclidean buildings of type W, and $\mathrm{Ord}^{u_k}(p) = \alpha_k \to 1$ as $k \to \infty$. We replace each u_k by its homogeneous harmonic ultralimit $(u_k)_\omega$ using the (Gromov–Schoen) blowup maps and Remark 2.9. So now we have a sequence of homogeneous harmonic maps $(u_k)_\omega : \mathbb{R}^n \to (X_k)_\omega$, where $(X_k)_\omega$ are Euclidean buildings of type W, and $\mathrm{Ord}^{(u_k)_\omega}(0, 1) = \alpha_k$ by homogeneity. Now for each k, rescale the metric $d'_k := \mu_k(d_k)_\omega$ so that $\sup_{B_1(0)} d'_k((u_k)_\omega, (u_k)_\omega(0)) = 1$. Then by Lemma A.2, $E^{(u_k)_\omega}[B_{\frac{1}{2}}(0)] \leq C$.

For ease of notation, relabel the ultralimit maps as u_k and the target buildings as (X_k, d'_k). Now apply the convergence results of [Korevaar and Schoen 1997, Section 3] and Remark 2.9 to this new sequence of maps and let $u_\omega := \omega\text{-lim}\, u_k$ and $X_\omega := \omega\text{-lim}\, X_k$. Since $E^{u_k}[B_{\frac{1}{4}}(0)] \to E^{u_\omega}[B_{\frac{1}{4}}(0)]$ and $\lim_{k\to\infty} I^{u_k}\left(\frac{1}{4}\right) = I^{u_\omega}\left(\frac{1}{4}\right)$, we have $\mathrm{Ord}^{u_k}\left(0, \frac{1}{4}\right) \to \mathrm{Ord}^{u_\omega}\left(0, \frac{1}{4}\right)$. And thus, $\mathrm{Ord}^{u_\omega}\left(0, \frac{1}{4}\right) = 1$. By the monotonicity of order, for all $0 < r < 1$, $\mathrm{Ord}^{u_\omega}(0, r) = 1$ and thus u_ω is a homogeneous harmonic map of degree 1.

In particular, by Proposition 2.12, there exists $r_0 > 0$ such that $u_\omega | B_{r_0}$ can be extended to a homogeneous harmonic map $L : \mathbb{R}^n \to X_\omega$ which has image contained in some apartment A_ω of X_ω, where $A_\omega = \omega\text{-lim}\, A_k$ for some sequence of apartments $A_k \subseteq X_k$. Constructing the maps L_k as in (5-1), each (X_k, A_k, L_k) is an (X_ω, A_ω, L)-triple, and following the proof of Lemma 5.2 we observe that there exists a subsequence of the u_k such that

$$\sup_{B_{r_0}(0)} d_k(u_k, L_k) \to 0.$$

For each k, consider the subbuilding $P_{F_k} = F_k \times Y_k$ consisting of all flats parallel to $F_k := L_k(\mathbb{R}^n)$. We observe that this complex is essentially regular since $F_k \simeq \mathbb{R}^m$ and dimension$(Y_k) < N$, so Theorem 6.4 applies. In particular, for sufficiently large k, $u_k(B_{\frac{1}{4}r_0}) \subset F_k \times Y_k$. So, write $u_k|B_{\frac{1}{4}r_0}(0) = (u_{k,1}, u_{k,2})$ and observe that this is a map into a product of NPC spaces, where Y_k is a building of type W_k. Since each possible W_k that can occur is a subgroup of a restriction of the original W, there are only finitely many possible W_k, all depending on W. By Lemma A.4 we note that since u_k is homogeneous of order α_k, the maps $u_{k,1} : B_{\frac{1}{4}r_0} \to F_k$ and $u_{k,2} : B_{\frac{1}{4}r_0} \to Y_k$ are either constant, or homogeneous of order $\alpha_k > 1$. Because F_k is Euclidean, if $u_{k,1}$ is nonconstant, $\mathrm{Ord}^{u_{k,1}}(0) \geq 2$, and by the inductive hypothesis, if $u_{k,2}$ is nonconstant, $\mathrm{Ord}^{u_{k,2}}(0) \geq 1 + \epsilon_{W_k}$. Since u_k is nonconstant, $u_{k,1}$ and $u_{k,2}$ cannot both be constant, so $\mathrm{Ord}^{u_k}(0) \geq \min\{2, 1 + \epsilon_{W_k}\}$. Now, since there are only finitely many possible W_k that can occur, for any large enough k, we have $\mathrm{Ord}^{u_k}(0) \geq 1 + \epsilon_W$, where $\epsilon_W := \min\{\epsilon_{W_k}\}$. But this means that the α_k certainly cannot converge to 1.

For (2), assume that $\mathrm{Ord}^u(p) = 1$ and let u_σ be the (Gromov–Schoen) blowup maps at p defined by (2-2). Then ${}^{g_\sigma}E^{u_\sigma}(1) \leq 2$ for $\sigma > 0$ sufficiently small. Let $\sigma_k \to 0$, $r_0 \in (0,1)$, X_k, A_k, L_k, X_ω, A_ω, L_ω be given by Proposition 5.1. Let $\delta_0 = \delta_0(E_0, r_0, X_\omega, A_\omega, L_\omega) > 0$ be chosen as in Theorem 6.4. By Proposition 5.1, we can choose k such that $\sup_{x \in B_{r_0}(0)} d_k(u_k(x), L_k(x)) < \delta_0$. By the inductive hypothesis, $P_{F_k} \simeq F_k \times Y_k$ is essentially regular since Y_k is a Euclidean building of dimension N-m and $F_k \simeq \mathbb{R}^m$ for some $m \geq 1$. Thus, for this k Theorem 6.4 implies that $u_k(B_{\frac{1}{4}r_0}(0)) \subset P_{F_k} \simeq \mathbb{R}^m \times Y_k$.

Now, let $\rho := \frac{1}{4}\sigma_k r_0$. If $m = N$, then F_k is an apartment and we are done since $u|B_\rho(p) \subset F_k$. Otherwise, $u|B_\rho(p)$ decomposes as two harmonic maps $\overline{u}_1 : B_\rho(p) \to F_k \simeq \mathbb{R}^m$ and $\overline{u}_2 : B_\rho(p) \to Y_k$. Since $\overline{u}_1$ is full rank by the construction of F_k, if $\mathrm{Ord}^{\overline{u}_2}(p) \geq 1 + \epsilon$ or $\overline{u}_2$ is constant, then we are done. Alternatively, if $\mathrm{Ord}^{\overline{u}_2}(p) = 1$, then we use the inductive hypothesis again to assert that there exists an $r \in (0, \rho)$ and a subbuilding of Y_k isometric to $\mathbb{R}^j \times Z$, where $j \in \{1, \dots, \min\{n, N-m\}\}$, Z is a building of dimension $N - m - j$ and $\overline{u}_2|B_r(p)$ decomposes as $\widehat{u}_1 : B_r(p) \to \mathbb{R}^j$ and $u_2 : B_r(p) \to Z$, where $\widehat{u}_1$ is full rank and $\mathrm{Ord}^{u_2}(p) \geq 1 + \epsilon$ or u_2 is constant. Then $u|B_r(p) := (u_1, u_2) : B_r(p) \to \mathbb{R}^{m+j} \times Z$, where $u_1 = (\overline{u}_1, \widehat{u}_1) : B_r(p) \to \mathbb{R}^{m+j}$ is full rank.

For (3), we initially follow the outline of the proof for trees given in Proposition A.3. By contradiction, we again assume there exists a sequence of harmonic maps $u_k : (\Omega, g) \to (X_k, d_k)$ where each X_k is a building of dimension N of type W, $x_k \in K$, $\sigma_k \in (0, \tau_0]$, and $\theta_k \to 0$ such that

$$\tfrac{1}{2}\theta_k R^{u_k}(x_k, \sigma_k) < R^{u_k}(x_k, \theta_k \sigma_k).$$

Rescaling as in Proposition A.3, we produce a sequence $v_k : (B_2(0), g_k) \to (X_k, d_k')$ by taking $v_k(x) := u_k(\sigma_k x)$ and setting $d_k' := \mu_k d_k$, where we choose μ_k so that

$$\sup_{x \in B_1(0)} d_k'(v_k(x), v_k(0)) = 1. \tag{A-6}$$

For the rescaled sequence, we have the inequality

$$\tfrac{1}{2}\theta_k R^{v_k}(0,1) < R^{v_k}(0, \theta_k) \tag{A-7}$$

and as before, $\omega\text{-}\lim v_k = v_\omega : (B_1(0), g) \to (X_\omega, d_\omega)$ is a harmonic map, where X_ω is of type W. Now we consider two cases, depending on the order of v_ω at 0.

Case 1: Presume that $\mathrm{Ord}^{v_\omega}(0) > 1$. Then by part (1) of this same lemma, which we already know holds for buildings of dimension N, $\mathrm{Ord}^{v_\omega}(0) \geq 1+\epsilon$, where ϵ depends on W. Thus there exists a constant c such that

$$R^{v_\omega}(0,\sigma) \leq c\sigma^{1+\epsilon} R^{v_\omega}(0,1),$$

and for small enough θ depending on c, ϵ, we contradict (A-7) for k large enough.

Case 2: Presume that $\mathrm{Ord}^{v_\omega}(0) = 1$ (v_ω is not homogeneous since otherwise the contradiction to (A-7) is immediate). Then by part (2) of this same lemma, there exists a subbuilding $F \times Y$ and a radius $\sigma_0 > 0$ such that $v_\omega(B_{\sigma_0}(0)) \subset F \times Y$. Now $F \simeq \mathbb{R}^m$ and Y is a building of dimension less than N, so by induction and [Gromov and Schoen 1992, Lemma 6.1], $F \times Y$ is essentially regular. Thus there exist c, β depending on K, Ω, W such that $R^{v_\omega}(0,\sigma) \leq c\sigma^{1+\beta} R^{v_\omega}(0,\sigma_0)$. By the convergence of v_k to v_ω and the monotonicity of R in the second argument, for large enough k, depending on c and β, we get a contradiction to (A-7). □

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