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YOUNGHAN BAE, MARTIJN KOOL AND HYEONJUN PARK

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Counting surfaces on Calabi–Yau 4-folds I: Foundations

YOUNGHAN BAE, MARTIJN KOOL AND HYEONJUN PARK

This is the first in a series of papers on counting surfaces on Calabi–Yau 4-folds. To complement the Hilbert scheme of 2-dimensional subschemes, we introduce *two* types of moduli spaces of stable pairs. We show that all three moduli spaces are related by GIT wall-crossing and parametrize stable objects in the bounded derived category.

We construct *reduced* Oh–Thomas virtual cycles on the moduli spaces via Kiem–Li cosection localization and prove that they are deformation invariant along Hodge loci. As an application, we show that the variational Hodge conjecture holds for any family of Calabi–Yau 4-folds supporting a nonzero reduced virtual cycle.

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1 Introduction

1.1 Background

Counting curves on a space X is one of the central topics in modern enumerative geometry. It is a rich subject with links to many other fields of mathematics as well as physics (string theory). Kontsevich [1995] introduced the concept of *hidden smoothness* of moduli spaces of stable maps in order to define algebraic Gromov–Witten (GW) invariants of X. The mathematical foundation of its virtual cycles was established by [Li and Tian 1998; Behrend and Fantechi 1997]. When X is a 3-*fold*, an alternative enumeration can be carried out using Grothendieck's Hilbert schemes. This change of perspective represents the difference between viewing curves on X as being parametrized or cut out by equations. Thomas [2000] then defined virtual cycles on Hilbert schemes of a 3-fold X, which yields Donaldson–Thomas (DT) invariants[1] of X. A remarkable connection between GW and DT invariants for 3-folds was discovered by

[1] Thomas [2000] defines DT invariants for stable sheaves of any rank on Fano and Calabi–Yau 3-folds.

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Maulik, Nekrasov, Okounkov and Pandharipande [Maulik et al. 2006]. This correspondence was further refined by Pandharipande and Thomas [2009b; 2009a; 2010] via stable pair (PT) invariants. We refer the reader to [Pandharipande and Thomas 2014] for an overview.

It was recently found that a similar relation between curve-counting invariants exists on *Calabi–Yau 4-folds* X. The structure of GW invariants of X was studied by Klemm and Pandharipande [2008]. On the sheaf side, the obstruction theories are not 2-term, and a new discovery was needed. Inspired by [Donaldson and Thomas 1998], Cao and Leung [2014] defined such a virtual cycle in special cases. The general case was subsequently established by Borisov and Joyce [2017] using methods of derived differential geometry. The algebraic counterpart was obtained by Oh and Thomas [2023]. Both constructions [Borisov and Joyce 2017; Oh and Thomas 2023] deeply rely on the derived structure of the moduli space — the shifted symplectic structures of Pantev, Toën, Vaquié and Vezzosi [Pantev et al. 2013] and the derived Darboux theorem of Brav, Bussi and Joyce [Brav et al. 2019]. Another feature of the theory is that it requires orientations. The existence of orientations was shown by Cao, Gross and Joyce [Cao et al. 2020]. An explicit GW–PT correspondence on Calabi–Yau 4-folds was conjectured by Cao, Maulik and Toda [Cao et al. 2018; 2022b; Cao and Toda 2021]. Virtual counts of points and curves on Calabi–Yau 4-folds recently attracted quite some attention. In physics, the magnificent four conjecture [Nekrasov 2020; Nekrasov and Piazzalunga 2019] provides a formula for counting points on the Calabi–Yau 4-fold $\mathbb{C}^4$. For the case of counting points on X we refer to [Cao and Kool 2018; Park 2021; Bojko 2021; Kool and Rennemo 2025] and for the case of counting curves on X we refer to [Cao and Kool 2020; Cao et al. 2022a; Cao and Toda 2022a; 2022b; Park 2021].

The next natural step in enumerative geometry is counting higher-dimensional objects, such as surfaces on Calabi–Yau 4-folds — the topic of this paper. It is not clear at present how to define virtual cycles on the stable map side (e.g., see [Alexeev 1996] for the moduli space and [Jiang 2022] for some recent developments). On the sheaf side, a natural starting point is the virtual cycle of [Borisov and Joyce 2017; Oh and Thomas 2023] on Hilbert schemes of 2-dimensional subschemes. However, we will see that for enumeration of surfaces, this virtual cycle generally does *not* suffice and has to be modified.

1.2 Counting surfaces

The aim of this project is to count surfaces on Calabi–Yau 4-folds X via sheaf theory. This goal leads to two immediate challenges:

(A) The Hilbert scheme parametrizes subschemes which are not pure-dimensional, i.e., 2-dimensional subschemes may have 0- and 1-dimensional components. Informally, Hilbert schemes allow for free roaming curves and points, thereby obscuring the contribution of genuine surfaces.

(B) A Hodge class of type $(2,2)$ on X generally does not remain of type $(2,2)$ as the complex structure of X deforms. When this happens, the virtual cycle vanishes.

The main purpose of this paper is to resolve (A) and (B).

Inspired by Pandharipande and Thomas's work [2009b] on curves on 3-folds, we settle (A) by constructing moduli spaces of stable pairs on X. We find that for 2-dimensional pairs, there is more structure

than for 1-dimensional pairs, and we obtain *two* types, which we call PT_0 and PT_1 pairs. The stable pair moduli spaces give smaller compactifications — roughly speaking, PT_0 pairs prevent free roaming points on X and PT_1 pairs prevent free roaming points *and* curves on X. The Hilbert scheme and two stable pair moduli spaces each have desirable features and should be studied together.

We overcome (B) by constructing reduced virtual cycles using the cosection localization method of [Kiem and Li 2013]. This method was generalized to the case of *isotropic cosections* by [Kiem and Park 2023]. Instead, we use a variant, *nondegenerate cosections*. For a class $\gamma \in H^2(X, \Omega_X^2) \cap H^4(X, \mathbb{Q})$, virtual cycles of moduli spaces parametrizing 2-dimensional pairs in class γ often vanish. Similar vanishing for curve-counting theories on surfaces/3-folds led to *reduced theories*, e.g., [Maulik et al. 2010; Kool and Thomas 2014]. From the perspective of curves on surfaces/3-folds X, reducing is usually needed when X has a nonzero holomorphic 2-form. However, for compact Calabi–Yau 3-folds satisfying $h^{0,1}(X) = h^{0,2}(X) = 0$, it is of no concern. As we will see, even on compact Calabi–Yau 4-folds satisfying $h^{0,1}(X) = h^{0,2}(X) = h^{0,3}(X) = 0$, reducing is necessary for most interesting surface classes γ.

1.3 Moduli spaces of stable pairs

There are three types of stability conditions on 2-dimensional pairs.

Definition 1.1 (Definition 2.1) Let X be a smooth projective variety of dimension $n \geq 2$. Let F be a 2-dimensional coherent sheaf on X, $s : \mathcal{O}_X \to F$ a section, and $Q := \operatorname{coker}(s)$. The pair (F, s) is called

(i) $\mathrm{DT}(= \mathrm{PT}_{-1})$-*stable* if s is surjective,

(ii) PT_0-*stable* if $T_0(F) = 0$ and $\dim(Q) \leq 0$,

(iii) PT_1-*stable* if $T_1(F) = 0$ and $\dim(Q) \leq 1$.

Here $T_d(F)$ denotes the maximal d-dimensional subsheaf of F.

The DT-stable pairs correspond to ideal sheaves. The PT_1-stable pairs are natural generalizations of Pandharipande and Thomas's stable pairs [2009b], which are a special case of Le Potier's coherent systems [1993; 1995]. The underlying sheaf of a PT_0-stable pair is usually *not* pure and this stability condition appears to be new. We introduce PT_0-stability as a bridge between DT- and PT_1-stability.

Suppose (F, s) is a PT_q pair on X with cokernel Q and *smooth* scheme-theoretic support S, which generally has 0-, 1-, and 2-dimensional connected components. For $q = 0$, S has no 0-dimensional connected components and Q is supported on the 1-dimensional connected components of S. For $q = 1$, S has no 0- or 1-dimensional components and the pair (F, s) can be equivalently described by a nested triple $Z \subset C \subset S$, where C is an effective divisor and Z is a 0-dimensional subscheme (Corollary 2.13). We give various characterizations of PT_q pairs with *singular* support in Section 2.

Example 1.2 (Proposition 2.6) Suppose (F, s) is a pair which is PT_0- and PT_1-stable, i.e., F is pure and Q is 0-dimensional. Then Q is a quotient sheaf of the 0-dimensional sheaf $\mathcal{E}xt_X^{n-1}(\mathcal{O}_S, K_X)$, which is supported on the locus where S is not Cohen–Macaulay.

Our first main result (Section 3) is the construction of projective moduli spaces of PT_q-stable pairs via geometric invariant theory (GIT).

Theorem 1.3 (Theorem 3.10) *Let X be a smooth projective variety and $v \in H^*(X,\mathbb{Q})$. Then, for each $q \in \{-1,0,1\}$, there exists a fine moduli space $\mathscr{P}_v^{(q)}(X)$ of PT_q-stable pairs on X with $\mathrm{ch}(F) = v$ as a projective scheme. Moreover, there exists a projective scheme $\mathcal{M}$, an action of PGL_N on $\mathcal{M}$ for some N, and an SL_N-equivariant ample line bundle $\mathcal{L}^{(q)} \in \mathrm{Pic}^{\mathrm{SL}_N}(\mathcal{M})_{\mathbb{Q}}$ for each $q \in \{-1,0,1\}$ such that*

$$\mathscr{P}_v^{(q)}(X) = [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(q)})/\mathrm{PGL}_N] = [\mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(q)})/\mathrm{PGL}_N].$$

In particular, $\mathscr{P}_v^{(q)}(X)$ is the GIT quotient $\mathcal{M}/\!\!/_{\mathcal{L}^{(q)}}\mathrm{SL}_N$.

Theorem 1.3 shows that PT_q pair moduli spaces are related by GIT wall-crossing. Since the underlying sheaves of PT_0 pairs are not *pure*, we needed finer arguments than the standard ones for pure sheaves. This approach is inspired by an analogous result of [2011] on 1-dimensional pairs.

Following [Pandharipande and Thomas 2009b], we show that the moduli space of PT_q pairs consists of open components of the moduli space of perfect complexes. Denote by $\mathscr{P}erf(X)^{\mathrm{spl}}_{\mathcal{O}_X}$ the moduli space of simple perfect complexes on X with trivial determinant.

Theorem 1.4 (Theorem 3.19, Proposition 3.28) *Let X be a smooth projective variety of dimension $n \geq 4$, $v \in H^*(X,\mathbb{Q})$, and $q \in \{-1,0,1\}$. The canonical map*

$$\mathscr{P}_v^{(q)}(X) \to \mathscr{P}erf(X)^{\mathrm{spl}}_{\mathcal{O}_X}, \quad (F,s) \mapsto I^\bullet = [\mathcal{O}_X \xrightarrow{s} F], \tag{1}$$

is an open embedding. Moreover, there is a polynomial Bridgeland stability condition $\rho^{(q)}$ such that the image of (1) is the locus of $\rho^{(q)}$-stable objects.

For $q = 1$, Theorem 1.4 was also shown by Gholampour, Jiang and Lo [Gholampour et al. 2024], which inspired the connection to the polynomial Bridgeland stability of [Bayer 2009].

1.4 Reduced virtual cycles

Let X be a *Calabi–Yau 4-fold*, i.e., a smooth projective variety of dimension 4 with trivial canonical bundle. In particular, X could be a hyperkähler 4-fold or an abelian 4-fold. Fix a trivialization $\omega : \mathcal{O}_X \xrightarrow{\cong} \Omega_X^4$.

Throughout this section, let $q \in \{-1,0,1\}$ and let $\mathscr{P} := \mathscr{P}_v^{(q)}(X)$ be the moduli space of PT_q pairs (F,s) on X with Chern character

$$\mathrm{ch}(F) = v = (0,0,\gamma,\beta,n-\gamma\cdot \mathrm{td}_2(X)) \in H^*(X,\mathbb{Q}).$$

By Theorem 1.4, we have an Oh–Thomas virtual cycle [2023]

$$[\mathscr{P}]^{\mathrm{vir}} \in A_{\mathrm{vd}}(\mathscr{P}), \quad \mathrm{vd} = n - \tfrac{1}{2}\gamma^2,$$

which maps to Borisov and Joyce's virtual cycle [2017] under the cycle class map by [Oh and Thomas $\geq$ 2026].[2] The Oh–Thomas virtual cycle is constructed from the canonical 3-term *symmetric obstruction*

[2]Here $A_*(-)$ denotes the Chow group with rational coefficients. We also use the convention that $A_i(-) = 0$ for $i \in \mathbb{Q}\setminus\mathbb{Z}$.

theory

$$\phi : \mathbb{E} := R\mathcal{H}om_{\pi}(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0[3] \to \mathbb{L}_{\mathscr{P}}, \quad \mathsf{SD} : \mathbb{E}^{\vee}[2] \cong \mathbb{E},$$

of [Huybrechts and Thomas 2010] and an *orientation*

$$o : \mathcal{O}_{\mathscr{P}} \xrightarrow{\cong} \det(\mathbb{E})$$

of [Cao et al. 2020]. Then the intrinsic normal cone is *isotropic* by the derived Darboux theorem of [Brav et al. 2019], which is necessary to define the virtual cycle. See Section 4 for the precise statements.

For a generic surface class γ, the Oh–Thomas virtual cycle $[\mathscr{P}]^{\mathrm{vir}}$ vanishes due to the following two heuristics:

(a) When deforming X, the codimension of the Hodge locus of γ in the base is often positive. Therefore deformation invariance of virtual cycles implies the vanishing.

(b) There often exist nondegenerate cosections given by semiregularity maps. They give trivial factors $\mathcal{O}$ of $\mathbb{E}$ which imply the vanishing.

We present a motivating example which reveals the above two phenomena.

Example 1.5 Let $X \subseteq \mathbb{P}^5$ be a smooth sextic hypersurface which contains a plane $S \subseteq X$. Let $v = \mathrm{ch}(\mathcal{O}_S) \in H^*(X, \mathbb{Q})$ and $q \in \{-1, 0, 1\}$.

- The virtual dimension is $-\frac{19}{2}$.
- [Carlson et al. 2009] The codimension of the Hodge locus of $\gamma = v_2 = [S]$ is 19.
- There are 19 nondegenerate cosections.

We first discuss the codimension of the Hodge locus in (a). By [Bloch 1972] (see Proposition 4.9), the obstruction to deforming the Hodge class $\gamma \in H^2(X, \Omega_X^2) \subseteq H^4(X, \mathbb{C})$ is governed by the symmetric bilinear form

$$\mathsf{B}_{\gamma} : H^1(X, T_X) \otimes H^1(X, T_X) \to \mathbb{C}, \quad \xi_1 \otimes \xi_2 \mapsto \int_X \iota_{\xi_1} \iota_{\xi_2} \gamma \cup \omega,$$

where ι denotes the contraction map. In particular, its rank

$$\rho_{\gamma} := \mathrm{rk}(\mathsf{B}_{\gamma})$$

is the codimension of the Hodge locus of γ.[3]

We now discuss the cosections in (b). By [Buchweitz and Flenner 2003], a perfect complex $[I^{\bullet}] \in \mathscr{P}$ gives rise to a *semiregularity map*

$$\mathsf{sr} : \mathrm{Ext}^2_X(I^{\bullet}, I^{\bullet})_0 \to H^3(X, \Omega_X^1) \cong H^1(X, T_X)^{\vee}, \quad e \mapsto \mathrm{tr}(\mathrm{At}(I^{\bullet}) \circ e), \tag{2}$$

[3]Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth proper morphism of smooth analytic spaces such that the fibre $\mathcal{X}_0$ over a point $0 \in \mathcal{B}$ is X^{an} and $\mathcal{B}$ is contractible. Then there exists a unique lift $\widetilde{\gamma}$ of $\gamma \in H^4(X, \mathbb{Q})$ as a section of the local system $R^4 f_* \mathbb{Q}$. The *Hodge locus* of γ is defined as the subset $\mathrm{Hd}_{\gamma} := \{b \in \mathcal{B} : \widetilde{\gamma}_b \in H^{2,2}(\mathcal{X}_b)\}$ and has a natural analytic scheme structure (see [Voisin 2013]). If the Kodaira–Spencer map $\mathsf{KS} : T_{\mathcal{B},0} \to H^1(X, T_X)$ is an isomorphism, then $\rho_{\gamma} = \dim(T_{\mathcal{B},0}) - \dim(T_{\mathrm{Hd}_{\gamma},0})$ by [Bloch 1972]. We will not use the Hodge locus in this paper except for heuristic arguments.

where $\mathrm{At}(I^\bullet)$ is the Atiyah class. This semiregularity map relates the obstructions of deforming the perfect complex $I^\bullet$ and the Hodge class $\gamma = \mathrm{ch}_2(I^\bullet)$ via the commutative triangle

$$\begin{array}{ccc} H^1(X,T_X) & \xrightarrow{\quad \mathsf{ob}=\mathsf{sr}^\vee \quad} & \mathrm{Ext}^2_X(I^\bullet,I^\bullet)_0 \\ & {\scriptstyle \mathsf{B}_\gamma} \searrow \quad \swarrow {\scriptstyle \mathsf{sr}} & \\ & H^1(X,T_X)^\vee & \end{array} \tag{3}$$

The symmetry of the above triangle given by $\mathsf{ob} = \mathsf{sr}^\vee$ is a special feature of Calabi–Yau 4-folds X and $(2,2)$-classes $\gamma \in H^2(X, \Omega_X^2)$. The semiregularity maps sr can be extended to cosections on the moduli space,

$$\mathsf{SR} : \mathbb{E}^\vee[1] = R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[2] \to H^1(X,T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}}.$$

Then $\mathsf{SR}^2 := \mathsf{SR} \circ \mathsf{SR}^\vee = \mathsf{B}_\gamma \otimes 1_{\mathscr{P}}$, which can be shown by extending the triangle (3). See Section 4 for details.

Choose a nondegenerate subspace $V \subseteq H^1(X,T_X)$ with respect to the bilinear form B_γ. Then $V \cong V^\vee$. Assume that V is *maximal*, i.e., $\dim(V) = \rho_\gamma$.

(a′) From the perspective of Hodge theory, V is a transversal slice to the Hodge locus of γ. Heuristically, we want to constrain virtual cycles to be deformation invariant only on the Hodge locus of γ.

(b′) From the perspective of cosections, V induces *nondegenerate* cosections

$$\Sigma : \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} H^1(X,T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}} \to V^\vee \otimes \mathcal{O}_{\mathscr{P}}.$$

Here nondegeneracy means that the square $\Sigma^2 := \Sigma \circ \Sigma^\vee$ is an isomorphism. In particular, we have a direct sum decomposition

$$\mathbb{E} = \mathbb{E}^{\mathrm{red}} \oplus (V \otimes \mathcal{O}_{\mathscr{P}})[1]$$

and we want to remove the trivial factors from the obstruction theory.

The second main result in this paper achieves (a′) and (b′).

Theorem 1.6 (Theorem 4.5, Proposition 4.19) *Let X be a Calabi–Yau 4-fold, $v \in H^*(X,\mathbb{Q})$, and $q \in \{-1,0,1\}$. For any maximal nondegenerate subspace $V \subseteq H^1(X,T_X)$ with respect to B_γ, the composition*

$$\phi^{\mathrm{red}} : \mathbb{E}^{\mathrm{red}} \xrightarrow{(1,0)} \mathbb{E} \xrightarrow{\phi} \mathbb{L}_{\mathscr{P}_v^{(q)}(X)}$$

is a symmetric obstruction theory such that the reduced closed substack of the intrinsic normal cone is isotropic (see Section 4 for the precise definition). *Consequently, given orientations of $\mathbb{E}$ and $H^1(X,T_X)_\gamma := H^1(X,T_X)/\ker(\mathsf{B}_\gamma)$, there exists a canonical reduced virtual cycle*

$$[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \in A_{\mathrm{rvd}}(\mathscr{P}_v^{(q)}(X)), \quad \mathrm{rvd} = n - \tfrac{1}{2}\gamma^2 + \tfrac{1}{2}\rho_\gamma,$$

which is independent of the choice of V.[4]

[4]The subspace V depends on a choice, but $H^1(X,T_X)_\gamma$ is canonically defined. These spaces are identified via the map $V \subset H^1(X,T_X) \to H^1(X,T_X)_\gamma$.

We prove that ϕ^{red} is an obstruction theory via the *algebraic twistor family* [Kool and Thomas 2014] associated to the transversal slice V and prove the isotropic condition via a variation of the *cosection localization* in [Kiem and Li 2013; Kiem and Park 2023] for the nondegenerate cosections Σ. In contrast to the *isotropic* cosections in [Kiem and Park 2023], the *nondegenerate* cosections considered in this paper allow us to handle cases where the virtual dimension $n-\frac{1}{2}\gamma^2$ is not an integer, but the reduced virtual dimension $n-\frac{1}{2}\gamma^2+\frac{1}{2}\rho_\gamma$ is an integer, which actually occurs quite often (e.g., Example 1.5).

We note that the algebraic twistor method alone does not produce a reduced virtual cycle since the isotropic condition for ϕ does not imply it for ϕ^{red}. The cosection localization method alone produces the reduced virtual cycle as a short-cut, but does not give a reduced obstruction theory. Combining the two methods enables us to directly apply all the known tools for handling Oh–Thomas virtual cycles (torus localization [Oh and Thomas 2023], cosection localization [Kiem and Park 2023], and virtual pullback [Park 2021]) to the reduced virtual cycle. We will indeed apply these tools in the sequels to this paper.

We observe that the reduced virtual cycle is closely related to semiregularity. Buchweitz and Flenner [2003] call a perfect complex $[I^\bullet]\in\mathscr{P}$ *semiregular* if the semiregularity map sr in (2) is injective.

Theorem 1.7 (Theorem 4.6) *Let X be a Calabi–Yau 4-fold, $v\in H^*(X,\mathbb{Q})$, and $q\in\{-1,0,1\}$. For any PT_q pair $[(F,s)]\in\mathscr{P}_v^{(q)}(X)$, the associated complex $I^\bullet$ is semiregular if and only if the moduli space $\mathscr{P}_v^{(q)}(X)$ is smooth of dimension $\mathrm{rvd}=n-\frac{1}{2}\gamma^2+\frac{1}{2}\rho_\gamma$ at $[I^\bullet]$.*

As a direct corollary of Theorem 1.7, if all $[(F,s)]\in\mathscr{P}_v^{(q)}(X)$ are semiregular, the reduced virtual cycle equals the fundamental cycle (for a certain choice of orientations). Thus, heuristically, we view the semiregular case as the *ideal situation*. Put differently, we view the reduced virtual cycle as the correct *virtual generalization* of the fundamental cycle in the semiregular case.

For Calabi–Yau 4-folds, Theorem 1.7 generalizes the results of [Bloch 1972] and [Buchweitz and Flenner 2003]. Indeed, it is shown in [loc. cit.] that semiregularity implies smoothness, whereas Theorem 1.7 also provides a formula for the dimension and a converse.

Example 1.8 We revisit Example 1.5. All planes $S\cong\mathbb{P}^2$ in a smooth sextic 4-fold X are semiregular and the moduli space $\mathscr{P}_v^{(-1)}(X)\cong\mathscr{P}_v^{(0)}(X)\cong\mathscr{P}_v^{(1)}(X)$, with $v=\mathrm{ch}(\mathcal{O}_S)$, consists of isolated reduced points (Corollary 4.28). Hence the reduced virtual dimension is zero and the contribution of any plane $S\subset X$ to the reduced virtual cycle is $\pm[\{S\}]$.

More generally, we give a simple description of the semiregularity map for local complete intersection surfaces. This identification is important for computing ρ_γ and the reduced virtual dimension. Determining ρ_γ is an interesting new aspect of the surface-counting theory. For example, ρ_γ is only subadditive in γ, e.g., $\rho_{d\gamma}=\rho_\gamma$ for any $d\in\mathbb{Z}_{>0}$.

Proposition 1.9 (Proposition 4.24) *Let $i:S\hookrightarrow X$ be a local complete intersection surface inside a Calabi–Yau 4-fold X. Then there exists an isomorphism*

$$\theta_S: H^1(S,N_{S/X})\to \mathrm{Ext}^2_X(I_{S/X},I_{S/X})_0$$

preserving nondegenerate symmetric bilinear forms induced by Serre duality. Under this isomorphism, the sheaf-theoretic obstruction and semiregularity maps (3) *match with the ones from the deformation theory of subschemes.*

We identify the dual of the semiregularity map (2) with the obstruction map

$$\mathrm{ob}_{S/X} : H^1(X, T_X) \to H^1(S, N_{S/X})$$

for Hilbert schemes, which is easier to determine. This leads to the following corollary:

Corollary 1.10 (Corollaries 4.28 and 4.30) (1) *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection Calabi–Yau 4-fold. Suppose $S \subset X$ is a complete intersection surface in $\mathbb{P}^N$. Then $I_{S/X}$ is semiregular. Moreover, if $v = \mathrm{ch}(\mathcal{O}_S)$ and $q \in \{-1, 0, 1\}$, then $\mathscr{P}_v^{(q)}(X)$ is smooth of dimension $\mathrm{rvd} = h^0(N_{S/X})$ at $I_{S/X}$.*

(2) *Let X be a Calabi–Yau 4-fold with nondegenerate holomorphic 2-form. Suppose $i : S \hookrightarrow X$ is a smooth lagrangian surface and $v = \mathrm{ch}(\mathcal{O}_S)$. Then $\mathrm{rvd} = h^{1,0}(S) - \frac{1}{2}\dim \mathrm{coker}(i^*)$, where $i^* : H^1(X, \Omega_X^1) \to H^1(S, \Omega_S^1)$.*

For Corollary 1.10(1) and X a sextic 4-fold, we show that rvd is often zero, but can be positive in some cases. For Corollary 1.10(2), we find that lagrangian planes are semiregular with $\mathrm{rvd} = 0$, but we also consider Schoen surfaces [2007] embedded in their Albanese 4-fold which are semiregular with $\mathrm{rvd} = 4$. Furthermore, for Corollary 1.10(2), we also consider nonsemiregular cases on hyperkähler 4-folds with lagrangian fibration and the Hilbert square $\mathrm{K3}^{[2]}$. Finally, we show that ρ_γ vanishes when $h^{2,0}(X) = 0$ and the class γ is a product of two $(1,1)$-classes. In this case, nonreduced virtual cycles contain nontrivial information. See Section 4.6 for a precise list of examples.

1.5 Deformation invariance

Let

$$f : \mathcal{X} \to \mathcal{B}$$

be a *family of Calabi–Yau 4-folds*, i.e., a smooth projective morphism of relative dimension 4 to a smooth connected affine scheme $\mathcal{B}$ such that $\omega_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{X}}$ and with connected fibres. Consider a horizontal section

$$\tilde{v} \in \bigoplus_{p=2}^{4} F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}),$$

where $F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ is the p-th filtered piece of the Hodge bundle $H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Then $\tilde{v}_b \in H^*(\mathcal{X}_b, \mathbb{C})$ is a locally constant family of cohomology classes on the fibres $\mathcal{X}_b := \mathcal{X} \times_{\mathcal{B}} \{b\}$ over $b \in \mathcal{B}$. The component $\tilde{v}_2$ induces a section $\tilde{\gamma}$ of $R^2 f_* \Omega^2_{\mathcal{X}/\mathcal{B}}$. The bilinear forms $\mathsf{B}_{\tilde{\gamma}_b}$ on $H^1(\mathcal{X}_b, T_{\mathcal{X}_b})$ for $b \in \mathcal{B}$ extend to a bilinear form $\mathsf{B}_{\tilde{\gamma}}$ on the vector bundle $R^1 f_*(T_{\mathcal{X}/\mathcal{B}})$.

Let $\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})$ be the moduli space of PT_q-stable pairs on the fibres of $f : \mathcal{X} \to \mathcal{B}$. This relative moduli space fits into the fibre diagram

$$\begin{array}{ccc} \mathscr{P}_{\tilde{v}_b}^{(q)}(\mathcal{X}_b) & \hookrightarrow & \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B}) \\ \downarrow & & \downarrow \\ \{b\} & \xrightarrow{\ i_b\ } & \mathcal{B} \end{array}$$

for each $b \in \mathcal{B}$. We have a relative obstruction theory $\phi_{\mathcal{X}/\mathcal{B}} : \mathbb{E}_{\mathcal{X}/\mathcal{B}} \to \mathbb{L}_{\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})/\mathcal{B}}$.

Our third main result (Section 5) is the deformation invariance of the reduced virtual cycle in Theorem 1.6.

Theorem 1.11 (Theorem 5.1) *Let $f : \mathcal{X} \to \mathcal{B}$ be a family of Calabi–Yau 4-folds and $\tilde{v} \in \bigoplus_p F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be a horizontal section, $\tilde{\gamma} \in \Gamma(\mathcal{B}, R^2 f_* \Omega^2_{\mathcal{X}/\mathcal{B}})$ the induced section, and $q \in \{-1, 0, 1\}$. Assume that the function $b \mapsto \rho_{\tilde{\gamma}_b}$ on $\mathcal{B}$ is constant. Given orientations*

$$\mathcal{O}_{\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})} \cong \det(\mathbb{E}_{\mathcal{X}/\mathcal{B}}), \quad \mathcal{O}_{\mathcal{B}} \cong \det(R^1 f_*(T_{\mathcal{X}/\mathcal{B}})/\ker(\mathsf{B}_{\tilde{\gamma}})),$$

there exists a cycle class

$$[\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})]^{\mathrm{red}} \in A_{\mathrm{rvd}+\dim(\mathcal{B})}(\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B}))$$

such that, for all $b \in \mathcal{B}$, we have

$$[\mathscr{P}_{\tilde{v}_b}^{(q)}(\mathcal{X}_b)]^{\mathrm{red}} = i_b^![\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})]^{\mathrm{red}} \in A_{\mathrm{rvd}}(\mathscr{P}_{\tilde{v}_b}^{(q)}(\mathcal{X}_b))$$

for the induced orientation. Here $i_b^!$ denotes the refined Gysin pullback.

In general, reduced virtual cycles given by cosection localization [Kiem and Li 2013] are deformation invariant only under an additional assumption. In Theorem 1.11, this assumption follows from the fact that the Hodge locus of $\tilde{\gamma}$ in $\mathcal{B}$ is $\mathcal{B}$ itself.

The existence of a relative orientation is required in Theorem 1.11. We conjecture (Conjecture 5.7) that a relative orientation exists after a finite étale base change of the base $\mathcal{B}$ and we prove its topological counterpart (Corollary C.2). The existence of a relative orientation is crucial for deformation invariance of numerical invariants, but is *not* needed in the main application below.

Variational Hodge conjecture The Hodge conjecture predicts that for any smooth projective variety X, all rational (p, p)-classes on X are algebraic. Grothendieck [1966a] introduced a variant of this:

Conjecture 1.12 (variational Hodge conjecture) *Let X be a smooth projective variety and γ an algebraic (p, p)-class on X. For any smooth projective morphism $f : \mathcal{X} \to \mathcal{B}$ to a smooth connected scheme $\mathcal{B}$ and a horizontal section $\tilde{v}_p$ of $F^p \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ such that $\mathcal{X}_0 \cong X$ and $(\tilde{v}_p)_0 = \gamma$ for some closed point $0 \in \mathcal{B}$, the cohomology classes $(\tilde{v}_p)_b$ are algebraic for all closed points $b \in \mathcal{B}$.*

Our fourth main result is an affirmative answer to Conjecture 1.12 when there exists a nonzero reduced virtual cycle.

Theorem 1.13 (Theorem 5.13) *Let X be a Calabi–Yau 4-fold and let γ be a $(2,2)$-class on X. If for some $v \in H^*(X,\mathbb{Q})$ with $v_2 = \gamma$, $q \in \{-1,0,1\}$, and choice of orientation on $\mathscr{P}_v^{(q)}(X)$ we have*

$$[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0 \in A_{\mathrm{rvd}}(\mathscr{P}_v^{(q)}(X)),$$

then Conjecture 1.12 holds for X and γ.

We prove Theorem 1.13 via the deformation invariance in Theorem 1.11. Here we do not require the two technical assumptions in Theorem 1.11. The function $\mathcal{B} \to \mathbb{Z}$, $b \mapsto \rho_{\widetilde{\gamma}_b}$, is lower semicontinuous and the reduced virtual cycle vanishes if $\rho_{\widetilde{\gamma}_b}$ drops at $0 \in \mathcal{B}$ (Proposition 5.5). The assumption on orientations can be removed by working with virtual *cycles* instead of *invariants*.

We view Theorem 1.13 as a *virtual* generalization of [Buchweitz and Flenner 2003] (see also [Bloch 1972]) for Calabi–Yau 4-folds. In fact, [Buchweitz and Flenner 2003] implies that Conjecture 1.12 holds if there exists a semiregular PT_q pair $I^\bullet = [\mathcal{O}_X \to F]$ on X with $\mathrm{ch}_2(F) = \gamma$. Theorem 1.13 recovers [loc. cit.] for Calabi–Yau 4-folds, because the reduced virtual cycle equals the fundamental cycle near a semiregular point (Theorem 4.6).

This application to the variational Hodge conjecture shows that it can be useful to work with the virtual cycle instead of invariants, because one can restrict to each connected component of the moduli space.

Example 1.14 (Example 5.16) We give a nonsemiregular application of Theorem 1.13. Let $X = S_1 \times S_2$ be a product of two K3 surfaces. Let $\beta_1 \in H^{1,1}(S_1)$, $\beta_2 \in H^{1,1}(S_2)$ be two irreducible effective classes with $\beta_1^2 \geq 2$ and $\beta_2^2 = 0$. Let $\gamma = \beta_1 \cup \beta_2$. Then, for any $q \in \{-1,0,1\}$, we have $\mathscr{P}_v^{(q)}(X) = |\mathcal{O}_{S_1}(\beta_1)| \times \mathbb{P}^1$ for some $v \in H^*(X,\mathbb{Q})$ with $v_2 = \gamma$. In this case, the reduced virtual cycle is $\pm h^0(\mathcal{O}_{S_1}(\beta_1)) \cdot [\mathrm{pt} \times \mathbb{P}^1]$. Therefore Conjecture 1.12 holds for such γ. This is a nonsemiregular case since the reduced virtual cycle differs from the fundamental cycle.

1.6 Moduli stack of semistable sheaves

In Appendix B, we generalize the construction of reduced virtual cycles to *Artin* stacks (which are global quotient stacks). In particular, we consider the moduli stack $\mathcal{M}_v^{H,\mathrm{ss}}(X)$ of Gieseker semistable sheaves on X (with respect to a polarization H) with Chern character $v \in H^*(X,\mathbb{Q})$, rigidified by the action of $B\mathbb{G}_m$. The Chern character v is arbitrary so this includes moduli of 2-dimensional *torsion* sheaves and moduli of *higher rank* sheaves.

Theorem 1.15 (Theorem B.6) *Let (X,H) be a polarized Calabi–Yau 4-fold and $v \in H^*(X,\mathbb{Q})$. For any choice of orientation on $\mathcal{M}_v^{H,\mathrm{ss}}(X)$, there exists a canonical reduced virtual cycle*

$$[\mathcal{M}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}} \in A_{1-\frac{1}{2}\chi(v,v)+\frac{1}{2}\rho_\gamma}(\mathcal{M}_v^{H,\mathrm{ss}}(X)),$$

where $\gamma := v_2$ and $\chi(v,v) := \int_X v^\vee \cdot v \cdot \mathrm{td}(X)$. Moreover, if $[\mathcal{M}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}} \neq 0$, then Conjecture 1.12 holds for X and γ.

We construct the reduced virtual cycle as a class in Kresch's Chow group [1999b] by extending Oh and Thomas's construction [2023] and Kiem and Li's cosection localization [2013] to Artin stacks (which are

global quotients). If there exist strictly semistable sheaves, this virtual cycle does not provide numerical invariants since there is no pushforward map for $\mathcal{M}_v^{H,\mathrm{ss}}(X) \to \mathrm{Spec}(\mathbb{C})$. Nonetheless, this virtual cycle is sufficient for applications to the variational Hodge conjecture.

There is a "fixed determinant version" of Theorem 1.15 under the assumption $v_0 \neq 0$ and $v_1 = 0$ (Theorem B.10).

1.7 Derived algebraic geometry

In Appendix A, we revisit Theorem 1.6 via derived algebraic geometry. The main result is the following:

Theorem 1.16 (Corollary A.15) *In the situation of Theorem 1.6, there exists a (-2)-shifted symplectic derived enhancement of $\mathscr{P}_v^{(q)}(X)$ inducing the reduced symmetric obstruction theory ϕ^{red}.*

We first introduce general reduction methods via (-1)-shifted *closed/exact* 1-forms (Theorem A.1). In order to apply this to the moduli space $\mathscr{P}_v^{(q)}(X)$, we show that the semiregularity maps can be enhanced to (-1)-shifted closed 1-forms (Proposition A.13). This is achieved by generalizing the integration map of [Pantev et al. 2013] (Theorem A.6).

As a byproduct, we obtain a new proof of Bloch's conjecture for surfaces on Calabi–Yau 4-folds (Remark A.18). The general case of Bloch's conjecture was shown by Pridham [2024] and Bandiera, Lepri and Manetti [Bandiera et al. 2023] (and by Iacono and Manetti [2013] when the normal bundle is extendable).

1.8 Relations to physics

Finding connections between surface counting on Calabi–Yau 4-folds and string theory is an interesting direction. We briefly point out relations to two (disjoint) parts of the physics literature:

- In M-/F-theory, the complex moduli of a Calabi–Yau 4-fold X are fixed by fixing a rational 4-cycle class called the G_4-flux. Then Gukov, Vafa and Witten [Gukov et al. 2000] introduced an associated superpotential giving rise to a symmetric bilinear form which (up to change of coordinates) coincides with B_γ. Explicit bounds on the rank ρ_γ of B_γ appear to be of interest to physicists [Braun et al. 2012; Braun and Valandro 2021; Bena et al. 2021; Graña et al. 2022; Lüst et al. 2022].[5]
- Nekrasov [2016; 2017a; 2018; 2019; 2017b] studied the moduli space of (noncommutative) instantons on origami spacetimes in $\mathbb{C}^4$. Our choice of K-theoretic insertions for toric Calabi–Yau 4-folds in the sequels (and the choice in [Cao et al. 2022a; Kool and Rennemo 2025]) are motivated by the study of SUSY Yang–Mills theory on $\mathbb{C}^4$ [Nekrasov 2020; Nekrasov and Piazzalunga 2019].

1.9 Sequels

We briefly mention the content of the sequels to this paper:

[5] We thank Grimm for pointing us to the physics literature, specifically [Braun et al. 2012].

- In part II [Bae et al. 2024], we study the $\mathrm{DT}/\mathrm{PT}_0$ correspondence. We conjecture a general correspondence and provide evidence on toric Calabi–Yau 4-folds via a vertex formalism, and on compact Calabi–Yau 4-folds via a virtual Lefschetz principle.
- In part III (in preparation), we study the $\mathrm{PT}_0/\mathrm{PT}_1$ correspondence in special cases. We consider virtual projective bundles and Weierstrass 4-folds.
- In part IV (in preparation), we compute PT_1 invariants in some examples including local surfaces and Calabi–Yau 4-folds with a nondegenerate holomorphic 2-form.

We will also further explore the link between the variational Hodge conjecture and the reduced virtual cycles.

1.10 Related work

Reduced virtual cycles for *curves* on hyperkähler 4-folds were studied by Cao, Oberdieck and Toda [Cao et al. 2024; 2022c]. For the moduli space of PT_1 pairs on X, Theorem 1.4 was independently proven in [Gholampour et al. 2024]. Correspondences for PT_q invariants in the context of Joyce's wall-crossing framework [Gross et al. 2022; Joyce 2018; 2021] will be an interesting direction to study.

There are two types of generalizations of cosection localization [Kiem and Li 2013] to Donaldson–Thomas theory of Calabi–Yau 4-folds [Borisov and Joyce 2017; Oh and Thomas 2023]. Savvas [2022] showed vanishing of Borisov–Joyce virtual cycles with a nowhere-vanishing isotropic cosection. Kiem and Park [2023] studied cosection localization for Oh–Thomas virtual cycles [2023].

1.11 Notation and conventions

We use the following notation and conventions throughout the paper:

- All schemes and algebraic stacks are assumed to be of finite type over the field $\mathbb{C}$, unless stated otherwise. A *variety* is a separated integral scheme. A *point* $x \in X$ in a scheme or an algebraic stack X means a $\mathbb{C}$-valued point.
- A *Calabi–Yau n-fold* is a smooth projective variety X of dimension n with $K_X \cong \mathcal{O}_X$.
- For a coherent sheaf F on a smooth quasiprojective variety X, we use the following notation for the various duals:
  - $F^\vee := R\mathcal{H}om_X(F, \mathcal{O}_X)$;
  - $F^D := \mathcal{E}xt^c_X(F, K_X)$, where c is the codimension of F in X.
- For a smooth projective variety X, we denote by
  - $\mathrm{Coh}(X)$ the abelian category of coherent sheaves on X;
  - $\mathscr{C}oh(X)$ the moduli stack of coherent sheaves on X;
  - $D^b_{\mathrm{coh}}(X)$ the bounded derived category of coherent sheaves on X.
- For a perfect complex $\mathbb{F}$ on a scheme $\mathcal{X}$ of tor-amplitude $(-\infty, 0]$, denote by $\mathfrak{C}(\mathbb{F}) := h^1/h^0(\mathbb{F}^\vee)$ the associated abelian cone stack [Behrend and Fantechi 1997, Proposition 2.4].

- An *orthogonal bundle* on a scheme $\mathcal{X}$ is a pair of a vector bundle E and an isomorphism $q : E \to E^\vee$ such that $q = q^\vee$. By abuse of notation, we also denote by $q : E \to \mathbb{A}^1_{\mathcal{X}}$ the corresponding quadratic function. An *orientation* of an orthogonal bundle E is an isomorphism $o : \mathcal{O}_{\mathcal{X}} \cong \det(E)$ such that $\det(q) = (-1)^{r(r-1)/2}(o \circ o^\vee)^{-1}$, where $r = \operatorname{rank}(E)$.
- A *symmetric complex* on a scheme $\mathcal{X}$ is a pair of a perfect complex $\mathbb{E}$ on $\mathcal{X}$ of tor-amplitude $[-2, 0]$ and an isomorphism $\theta : \mathbb{E}^\vee[2] \cong \mathbb{E}$ such that $\theta^\vee[2] = \theta$. An *orientation* of a symmetric complex $\mathbb{E}$ is an isomorphism $o : \mathcal{O}_{\mathcal{X}} \cong \det(\mathbb{E})$ such that $\det(\theta) = (-1)^{r(r-1)/2} o \circ o^\vee$, where $r = \operatorname{rank}(\mathbb{E})$.[6]
- For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, we denote by $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ the full cotangent complex [Illusie 1971; 1972; Olsson 2007] and by $\tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ the truncated cotangent complex.
- For any algebraic stack $\mathcal{X}$, we denote by $A_*(\mathcal{X})$ the Chow group of [Kresch 1999b] with $\mathbb{Q}$-coefficients. For any $d \in \mathbb{Q} \setminus \mathbb{Z}$, we let $A_d(\mathcal{X}) := 0$.
- For a morphism $f : \mathcal{X} \to \mathcal{Y}$ and a coherent sheaf (resp. a perfect complex) E on $\mathcal{Y}$, we denote the pullback f^*E (resp. the derived pullback Lf^*E) by $E|_{\mathcal{X}}$.

2 Stable pairs

In this section, we introduce the PT_q-stability conditions on 2-dimensional pairs for $q \in \{-1, 0, 1\}$, and present basic properties which will be used throughout the paper and its sequels. The PT_{-1}-stable pairs coincide with DT-stable pairs (i.e., ideal sheaves), PT_1-stable pairs are PT-stable pairs (i.e., Le Potier stable pairs [1993; 1995; Pandharipande and Thomas 2009b]), and PT_0-stable pair is a new intermediate notion

$$\mathrm{DT} := \mathrm{PT}_{-1} \rightsquigarrow \mathrm{PT}_0 \rightsquigarrow \mathrm{PT}_1 =: \mathrm{PT}.$$

Any PT_q pair (F, s) on a smooth variety X has an associated complex

$$I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$$

in $D^b_{\mathrm{coh}}(X)$, the bounded derived category of coherent sheaves on X. We study PT_q pairs through these associated complexes. We provide descriptions of these associated complexes under various assumptions on the singularities of the support of F. The results in this section provide some insights into the geometric meaning of PT_q-stability. They will also play an important role in the sequels to this paper.

Throughout this section, we fix a smooth projective variety X of dimension n over the complex field $\mathbb{C}$.

2.1 PT_q-stability

For any integer $q \geq -1$, consider the two full subcategories

$$\mathrm{Coh}_{\leq q}(X) := \{F \in \mathrm{Coh}(X) : \dim(F) \leq q\}, \quad \mathrm{Coh}_{\geq q+1}(X) := \{F \in \mathrm{Coh}(X) : T_q(F) = 0\}.$$

[6] This convention is slightly different from [Park 2021], in which all symmetric complexes are assumed to be equipped with an orientation.

Definition 2.1 Let F be a coherent sheaf on X with 2-dimensional support and let $s : \mathcal{O}_X \to F$ be a section. For $q \in \{-1, 0, 1\}$, we say that the pair (F, s) is PT_q*-stable* if

$$F \in \mathrm{Coh}_{\geq q+1}(X) \quad \text{and} \quad Q := \mathrm{coker}(\mathcal{O}_X \xrightarrow{s} F) \in \mathrm{Coh}_{\leq q}(X).$$

Abbreviating, we also refer to PT_q-stable pairs as PT_q *pairs*.

Example 2.2 The PT_q-stability conditions for the following two extreme cases are well known:

(DT) A pair (F, s) is PT_{-1}-stable if and only if $s : \mathcal{O}_X \to F$ is surjective. Hence PT_{-1} pairs on X correspond to 2-dimensional closed subschemes of X (which may have irreducible components of dimension 0 and 1). Thus we refer to PT_{-1}-stability as DT-stability.

(PT) A pair (F, s) is PT_1-stable if and only if F is pure and $\dim(Q) \leq 1$. Hence PT_1 pairs are exactly the 2-dimensional stable pairs in the sense of [Le Potier 1993, definition 4.2].[7] Their 1-dimensional analogs are used in [Pandharipande and Thomas 2009b; Cao et al. 2022b] (and many other places). Thus we sometimes refer to PT_1-stability as PT-stability.

We present some elementary properties of PT_q pairs. We begin with an observation on the supports.

Lemma 2.3 *Given a* PT_q *pair* (F, s) *on* X*, the scheme-theoretic support of the coherent sheaf* F *is the scheme-theoretic support of the section* s*.*

Proof The proof is analogous to [Pandharipande and Thomas 2009b, Lemma 1.6]. Let $\mathrm{Ann}(s) \subseteq \mathcal{O}_X$ be the ideal sheaf of the support of s. Away from the support of $Q = \mathrm{coker}(s)$, the sheaf F is generated by the section s. Hence $\mathrm{Ann}(s) \cdot F \subseteq F$ is set-theoretically supported in the support of Q, and thus $\mathrm{Ann}(s) \cdot F \in \mathrm{Coh}_{\leq q}(X)$. Since $F \in \mathrm{Coh}_{\geq q+1}(X)$, we obtain $\mathrm{Ann}(s) \cdot F = 0$. □

We note that the PT_q pairs have limit descriptions.

Proposition 2.4 *Let* $W \subseteq Z \subseteq X$ *be closed subschemes such that* $\mathcal{O}_Z \in \mathrm{Coh}_{\geq q+1}(X)$ *and* $\mathcal{O}_W \in \mathrm{Coh}_{\leq q}(X)$*. Suppose* W *is reduced and denote its ideal by* $\mathcal{I} \subseteq \mathcal{O}_Z$*. Then* PT_q *pairs* (F, s) *with support* Z *and cokernel* Q *satisfying* $\mathrm{Supp}(Q)^{\mathrm{red}} \subseteq W$ *are equivalent to coherent subsheaves of* $\varinjlim \mathcal{H}om_X(\mathcal{I}^r, \mathcal{O}_Z)/\mathcal{O}_Z$*.*

Proof The proof is analogous to [ibid., Proposition 1.8]. Choose a PT_q pair (F, s). By Lemma 2.3, we can form a short exact sequence

$$0 \to \mathcal{O}_Z \to F \to Q \to 0, \tag{4}$$

where Z is the support of F and Q is the cokernel of s. Since $\mathcal{O}_Z \in \mathrm{Coh}_{\geq q+1}(X)$ and $Q \in \mathrm{Coh}_{\leq q}(X)$, we have $\mathcal{H}om_X(Q, \mathcal{O}_Z) = 0$. Hence the short exact sequence (4) gives us an inclusion

$$\mathcal{H}om_X(F, \mathcal{O}_Z) \hookrightarrow \mathcal{H}om_X(\mathcal{O}_Z, \mathcal{O}_Z) = \mathcal{O}_Z,$$

[7] Equivalently, PT_1 pairs are the 2-dimensional α-stable pairs in the sense of [Le Potier 1995, definition 4.4] for sufficiently big $\alpha(t) \in \mathbb{Q}[t]$.

which is an isomorphism away from the support of Q. Hence $\mathcal{H}om_X(F,\mathcal{O}_Z)$ is an ideal such that $\mathcal{I}^r \subseteq \mathcal{H}om_X(F,\mathcal{O}_Z)$ for some integer r. Since $F \in \mathrm{Coh}_{\geq q+1}(X)$, we have

$$F \subseteq \mathcal{H}om_X(\mathcal{H}om_X(F,\mathcal{O}_Z),\mathcal{O}_Z) \subseteq \mathcal{H}om_X(\mathcal{I}^r,\mathcal{O}_Z) \subseteq \varinjlim_{r\to\infty} \mathcal{H}om_X(\mathcal{I}^r,\mathcal{O}_Z),$$

where the last inclusion map follows from $\mathcal{H}om_X(\mathcal{I}^r/\mathcal{I}^{r+1},\mathcal{O}_Z)=0$.

On the other hand, choose a coherent subsheaf $F \subseteq \varinjlim \mathcal{H}om_X(\mathcal{I}^r,\mathcal{O}_Z)$ containing $\mathcal{O}_Z$. Since F is coherent, we have $F \subseteq \mathcal{H}om_X(\mathcal{I}^r,\mathcal{O}_Z)$ for some sufficiently large r. Let $s:\mathcal{O}_X \to \mathcal{O}_Z \hookrightarrow F$ be the composition. Consider a subsheaf $G \subseteq F$ of dimension $\leq q$. By adjunction, the inclusion map $G \hookrightarrow \mathcal{H}om_X(\mathcal{I}^r,\mathcal{O}_Z)$ corresponds to a map $G\otimes\mathcal{I}^r \to \mathcal{O}_Z$, which is zero since $\mathcal{O}_Z \in \mathrm{Coh}_{\geq q+1}(X)$. Hence, $G=0$. This means that $F \in \mathrm{Coh}_{\geq q+1}(X)$. Therefore (F,s) is a PT_q-stable pair. □

Remark 2.5 Some results in this paper hold for d-dimensional pairs. For a coherent sheaf F on X with a d-dimensional support, a section $s:\mathcal{O}_X\to F$, and any $q\in\{-1,0,\dots,d-1\}$, we define the pair (F,s) to be PT_q*-stable* if

$$F \in \mathrm{Coh}_{\geq q+1}(X) \quad \text{and} \quad Q := \mathrm{coker}(\mathcal{O}_X \xrightarrow{s} F) \in \mathrm{Coh}_{\leq q}(X).$$

Then Lemma 2.3 and Proposition 2.4 hold for d-dimensional PT_q pairs as well (as is immediately obvious from the proofs). However, most parts of the paper require $d=2$, so we restrict to this case.

2.2 Special cases

Given a PT_0 pair (F,s) on an n-dimensional smooth projective variety X, we may consider the following two special cases:

(1) (F,s) is additionally DT-stable, i.e., $s:\mathcal{O}_X\to F$ is surjective.

(2) (F,s) is additionally PT_1-stable, i.e., F is pure.

Case (1) is very clear. It consists of 2-dimensional subschemes $Z\subset X$ with no (possibly embedded) 0-dimensional components. Case (2) is more interesting. It depends on the singularities of the support.

Proposition 2.6 *Let S be a* ***pure*** *2-dimensional subscheme of X. Then the following three conditions are equivalent*:

(i) *S is not Cohen–Macaulay.*

(ii) $\mathcal{E}xt_X^{n-1}(\mathcal{O}_S,K_X)\neq 0$.

(iii) *There exists a* PT_0 *pair (F,s) with support S and nonzero cokernel Q.*

Furthermore, for any $m\geq 0$, the PT_0 *pairs (F,s) with support S and $\chi(Q)=m$ are in bijective correspondence with the closed points of the Quot scheme*

$$\mathcal{Q}uot_X(\mathcal{E}xt_X^{n-1}(\mathcal{O}_S,K_X),m)$$

parametrizing length m quotients of the 0*-dimensional sheaf $\mathcal{E}xt_X^{n-1}(\mathcal{O}_S,K_X)$.*

Proof Since S is pure, $\mathscr{E}xt^i_X(\mathcal{O}_S, K_X) = 0$ for $i \neq n-2, n-1$ and $\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X)$ is 0-dimensional [Huybrechts and Lehn 2010, Propositions 1.1.6 and 1.1.10]. Let $i : S \hookrightarrow X$ denote the inclusion map. Since $R\mathcal{H}om_X(\mathcal{O}_S, K_X) = Ri_*i^!K_X$, we have $\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X) = 0$ if and only if S is Cohen–Macaulay.

For any 0-dimensional sheaf Q, we have

$$\mathrm{Ext}^1_X(Q, \mathcal{O}_S) \cong \mathrm{Hom}_X(\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X), Q^D), \tag{5}$$

where $Q^D := \mathscr{E}xt^n_X(Q, K_X)$. Hence (iii) implies (ii). On the other hand, if $\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X) \neq 0$, we can always find a closed point $P \in X$ in the support of $\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X)$ and a nonzero map $\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X) \to \mathcal{O}_P$. By (5), we find a nonsplit extension in $\mathrm{Ext}^1_X(\mathcal{O}_P^D, \mathcal{O}_S) \neq 0$, from which we obtain a PT_0 pair (F, s) with nonzero cokernel. By Lemma 2.3, the support of this PT_0 pair is S. This proves (ii) $\Longrightarrow$ (iii).

Finally, it follows from (5) that there exists a bijection between the collections

$$\left(\bigsqcup_Q \mathrm{Ext}^1_X(Q, \mathcal{O}_S)\right)\Big/\cong, \quad \left(\bigsqcup_Q \mathrm{Hom}_X(\mathscr{E}xt^{n-1}_X(\mathcal{O}_S, K_X), Q)\right)\Big/\cong,$$

where the union is over all 0-dimensional sheaves on X and we divide out by isomorphisms. This bijection is obtained by dualizing the short exact sequence $0 \to \mathcal{O}_S \to F \to Q \to 0$. Since F is pure if and only if $\mathscr{E}xt^{n-1}_X(F, K_X)$ is 0-dimensional and $\mathscr{E}xt^n_X(F, K_X) = 0$ [Huybrechts and Lehn 2010, Proposition 1.1.10], it follows that the short exact sequences with F *pure* correspond precisely to the *surjective* morphisms. □

Now consider a PT_1 pair (F, s). We can similarly consider the following two special cases:

(3) (F, s) is additionally PT_0-stable, i.e., Q is 0-dimensional.

(4) F is additionally reflexive.

Case (3) is exactly case (2) above. Case (4) is a "dual" version of case (1) above, in the sense of the following proposition:

Proposition 2.7 *Let (F, s) be a pair of a coherent sheaf F on X and a section $s : \mathcal{O}_X \to F$.*

(1) *If (F, s) is PT_0-stable and $\chi(F)$ is minimal among all PT_0 pairs with Chern characters $\mathrm{ch}_{n-2}(F)$ and $\mathrm{ch}_{n-1}(F)$, then (F, s) is DT-stable.*

(2) *If (F, s) is PT_1-stable and $\chi(F)$ is maximal among all PT_1 pairs with Chern characters $\mathrm{ch}_{n-2}(F)$ and $\mathrm{ch}_{n-1}(F)$, then F is reflexive.*

In Proposition 2.14, we will show that $\chi(F)$ is bounded below for DT and PT_0 pairs (F, s) with fixed Chern characters $\mathrm{ch}_{n-2}(F)$ and $\mathrm{ch}_{n-1}(F)$, and bounded above in the PT_1 case.

Proof (1) Form the short exact sequence

$$0 \to \mathcal{O}_Z \to F \to Q \to 0,$$

where Z is the support of F and Q is the cokernel of s. Then we have

$$\mathrm{ch}_{n-2}(\mathcal{O}_Z) = \mathrm{ch}_{n-2}(F), \quad \mathrm{ch}_{n-1}(\mathcal{O}_Z) = \mathrm{ch}_{n-1}(F) \quad \text{and} \quad \chi(\mathcal{O}_Z) = \chi(F) - \chi(Q).$$

Since $(\mathcal{O}_Z, 1 : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z)$ is a PT_0 pair and $\chi(F)$ is minimal, $Q = 0$. Therefore (F, s) is a DT pair.

(2) Consider the canonical map

$$\theta_F : F \to F^{DD}.$$

The purity of F implies $\mathscr{E}xt_X^{n-1}(F, K_X) \in \mathrm{Coh}_{\leq 0}(X)$ and $\mathscr{E}xt_X^n(F, K_X) = 0$ by [loc. cit.]. Moreover, the spectral sequence

$$\mathscr{E}xt_X^p(\mathscr{E}xt_X^{-q}(F, K_X), K_X) \Rightarrow F$$

in [ibid., Lemma 1.1.8] gives us a short exact sequence

$$0 \to F \to F^{DD} \to \mathscr{E}xt_X^{n-1}(F, K_X)^D \to 0,$$

where $\mathscr{E}xt_X^{n-1}(F, K_X)^D \in \mathrm{Coh}_{\leq 0}(X)$. Hence $(F^{DD}, \theta_F \circ s)$ is also a PT_1 pair such that

$$\chi(F^{DD}) = \chi(F) + \chi(\mathscr{E}xt_X^{n-1}(F, K_X)^D) \geq \chi(F).$$

Since $\chi(F)$ is maximal, $F = F^{DD}$. □

2.3 Associated complexes

Let X be a smooth projective variety of dimension n. Given a PT_q pair (F, s) on X, we can form an associated complex

$$I^\bullet := [\mathcal{O}_X \xrightarrow{s} F],$$

with $\mathcal{O}_X$ in degree zero. We view $I^\bullet$ as an element of $D^b_{\mathrm{coh}}(X)$, the bounded derived category of coherent sheaves on X. In this subsection, we analyze PT_q pairs in terms of the associated complex $I^\bullet$.

We first note that a PT_q pair is equivalent to its associated complex.

Lemma 2.8 *Assume X is a smooth projective variety of dimension $n \geq 4$. Consider two PT_q pairs (F_1, s_1) and (F_2, s_2). Then the two associated complexes $I_1^\bullet := [\mathcal{O}_X \xrightarrow{s_1} F_1]$ and $I_2^\bullet := [\mathcal{O}_X \xrightarrow{s_2} F_2]$ are quasi-isomorphic if and only if the two PT_q pairs (F_1, s_1) and (F_2, s_2) are isomorphic.*

Proof The proof is analogous to [Pandharipande and Thomas 2009b, Proposition 1.21]. Since

$$\mathrm{Ext}_X^{\leq 1}(F_1, \mathcal{O}_X) \cong H^{\geq n-1}(X, F_1 \otimes K_X)^* = 0,$$

by $F_1 \in \mathrm{Coh}_{\leq 2}(X)$, the map

$$\mathbb{C} = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Hom}_X(I_1^\bullet, \mathcal{O}_X)$$

is an isomorphism. Therefore, any quasi-isomorphism $\phi : I_1^\bullet \to I_2^\bullet$ can be completed to an isomorphism of distinguished triangles

$$\begin{array}{ccccccc} I_1^\bullet & \longrightarrow & \mathcal{O}_X & \xrightarrow{s_1} & F_1 & \longrightarrow \\ \downarrow{\scriptstyle \phi} & & \vdots{\scriptstyle c} & & \vdots{\scriptstyle \psi} & \\ I_2^\bullet & \longrightarrow & \mathcal{O}_X & \xrightarrow{s_2} & F_2 & \longrightarrow \end{array}$$

for some dotted arrows c and ψ. Since $c \in \Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ is a nonzero constant, $c^{-1} \cdot \psi : F_1 \to F_2$ is an isomorphism of sheaves with $\psi(s_1) = s_2$. □

We show that a PT_0 pair for which the pure part of its scheme-theoretic support is Cohen–Macaulay can be expressed by a map from a DT pair to a 1-dimensional pure sheaf.

Proposition 2.9 *For any* PT_0 *pair* (F, s) *on* X *such that* $S = \mathrm{Supp}(F)^{\mathrm{pure}}$ *is Cohen–Macaulay, we have a natural quasi-isomorphism*

$$I^\bullet \cong [I_{S/X} \to T_1(F)], \tag{6}$$

where $T_1(F)$ *is zero or pure* 1*-dimensional and* $I_{S/X}$ *denotes the ideal sheaf of* S *in* X.

In particular, when $\dim(X) = 3$, *we have* $I_{S/X} = \mathcal{O}_X(-S)$ *and*

$$I^\bullet \cong [\mathcal{O}_X \to T_1(F)(S)](-S),$$

which, for $T_1(F) \neq 0$, *is a* 1*-dimensional stable pair on* X *as in* [Pandharipande and Thomas 2009b].

Proof We may assume $T_1(F) \neq 0$, because otherwise the result is immediate from Proposition 2.6. Note that $F/T_1(F)$ is a pure 2-dimensional sheaf and $T_1(F)$ is a pure 1-dimensional sheaf. Hence the support of the PT_0-stable pair $\mathcal{O}_X \to F/T_1(F)$ is also pure. Let $Z = \mathrm{Supp}(F)$, so $S = Z^{\mathrm{pure}}$. Since $T_1(\mathcal{O}_Z) = I_{S/Z}$ by definition, the support of the PT_0 pair $\mathcal{O}_X \to F/T_1(F)$ is S. By Proposition 2.6 above, we obtain the desired isomorphism $\mathcal{O}_S \cong F/T_1(F)$. Moreover, since the commutative square

$$\begin{array}{ccc} I_{S/X} & \longrightarrow & T_1(F) \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \longrightarrow & F \end{array}$$

is cartesian and cocartesian, we have the desired quasi-isomorphism (6). □

Dually, PT_1 pairs (with Cohen–Macaulay support) can be expressed by maps between 1-dimensional pure sheaves and DT pairs.

Proposition 2.10 *For any* PT_1 *pair* (F, s) *on* X *with Cohen–Macaulay support* S, *we have a natural quasi-isomorphism*

$$I^\bullet \cong \mathrm{cone}(Q[-2] \to I_{S/X}),$$

where the cokernel Q *is zero or pure* 1*-dimensional.*

Proof The quasi-isomorphism follows from the canonical distinguished triangle

$$I_{S/X} \to I^\bullet \to Q[-1] \to I_{S/X}[1].$$

Thus it suffices to assume $Q \neq 0$ and show it is pure 1-dimensional. Applying $\mathrm{Hom}_X(T_0(Q), -)$ to the short exact sequence $0 \to \mathcal{O}_S \to F \to Q \to 0$, we obtain a long exact sequence

$$\cdots \to \mathrm{Hom}_X(T_0(Q), F) \to \mathrm{Hom}_X(T_0(Q), Q) \to \mathrm{Ext}^1_X(T_0(Q), \mathcal{O}_S) \to \cdots.$$

By purity of F and Proposition 2.6, we deduce $T_0(Q) = 0$, which implies that Q is pure 1-dimensional. □

For PT_1 pairs with Gorenstein support, we also have the following characterization:

Proposition 2.11 *For any PT_1 pair (F,s) on X with Gorenstein support S and nonzero cokernel Q, we have a natural quasi-isomorphism*

$$F^\vee \otimes K_X[n-2] \cong [\mathcal{O}_S \to Q^D \otimes K_S^\vee] \otimes K_S, \tag{7}$$

where $Q^D := \mathcal{E}xt_X^{n-1}(Q, K_X)$ is a pure 1-dimensional sheaf supported on S. Moreover, this induces an equivalence between PT_1 pairs supported on S with nonzero cokernel and 1-dimensional PT pairs on S.

Proof Since S is Gorenstein, $R\mathcal{H}om_X(\mathcal{O}_S, K_X) \cong K_S[-(n-2)]$. Moreover, Q is pure 1-dimensional by Proposition 2.10. Applying the functor $R\mathcal{H}om_X(-, K_X)$ to the short exact sequence $0 \to \mathcal{O}_S \to F \to Q \to 0$, we obtain a distinguished triangle

$$F^\vee \otimes K_X[n-2] \to K_S \to Q^D \to F^\vee \otimes K_X[n-1],$$

which implies the desired quasi-isomorphism (7).

Since Q is pure 1-dimensional, we have $Q \cong Q^{DD}$, and Q^D is pure 1-dimensional as well. Moreover, F is pure; hence, the cokernel $\mathcal{E}xt_X^{n-1}(F, K_X)$ of the map $K_S \to Q^D$ is 0-dimensional. It is easy to see that all these operations are invertible, thereby establishing an equivalence. □

The equivalence in Proposition 2.11 implies that PT_1 pairs on Gorenstein (and smooth) supports have simple descriptions. Fix a 2-dimensional pure subscheme S of X.

Corollary 2.12 *Assume that S is Gorenstein. A PT_1 pair (F,s) on X with support S and nonzero cokernel is equivalent to a pair (C,t) where $C \subset S$ is a pure 1-dimensional subscheme and $t : \omega_C \twoheadrightarrow R$ is a 0-dimensional quotient.*

Proof By the previous lemma, we may view a PT_1 pair on X with support S and nonzero cokernel as a 1-dimensional PT pair on S. The support of this stable pair is a 1-dimensional pure subscheme $C \subset S$ and the stable pair is determined by a short exact sequence

$$0 \to \mathcal{O}_C \to G \to Q \to 0,$$

where G is pure 1-dimensional and Q is 0-dimensional. Applying the functor $R\mathcal{H}om_S(-, K_S)$ and using $\mathcal{E}xt_S^2(G, K_S) = 0$ (purity of G), we obtain a quotient

$$t : \omega_C = \mathcal{E}xt_S^1(\mathcal{O}_C, K_S) \twoheadrightarrow \mathcal{E}xt_S^2(Q, K_S).$$

Note that applying $R\mathcal{H}om_S(-, K_S)$ again and using that $\mathcal{O}_C^{DD} \cong \mathcal{O}_C$ and $\mathcal{O}_C^D$ is pure 1-dimensional, we can recover the original 1-dimensional PT pair on S, and hence PT_1 pair on X. □

Corollary 2.13 *Assume that S is smooth. A PT_1 pair (F,s) on X with support S and nonzero cokernel is equivalent to a pair (C,Z) of an (nonzero) effective divisor $C \subseteq S$ and a 0-dimensional subscheme $Z \subseteq C$.*

Proof Since any pure 1-dimensional subscheme C of a smooth surface S is a local complete intersection, C is Gorenstein and K_C is invertible. Hence the proof follows directly from Corollary 2.12. □

2.4 Numerical bounds

Let X be a smooth projective variety of dimension n with fixed very ample line bundle $\mathcal{O}_X(1)$. For any coherent sheaf F on X, we denote its Hilbert polynomial by

$$P_F(t) = \sum_{i \geq 0} \alpha_i(F) \cdot \frac{t^i}{i!},$$

where $\alpha_0(F) = \chi(F)$ is the Euler characteristic. In this subsection, we derive boundedness results on $\alpha_i(F)$ for PT_q pairs (F, s) which are crucial in the GIT construction in Section 3.2.

Proposition 2.14 *Fix rational numbers $\alpha_2 > 0$ and α_1.*

(1) *For each $q \in \{-1, 0, 1\}$, the set*

$$\{\alpha_1(F) : \mathrm{PT}_q \text{ pairs } (F, s) \text{ such that } \alpha_2(F) = \alpha_2\}$$

is bounded below.

(2) *The two sets*

$$\{\chi(F) : \mathrm{DT} \text{ pairs } (F, s) \text{ such that } \alpha_2(F) = \alpha_2, \alpha_1(F) = \alpha_1\},$$
$$\{\chi(F) : \mathrm{PT}_0 \text{ pairs } (F, s) \text{ such that } \alpha_2(F) = \alpha_2, \alpha_1(F) = \alpha_1\}$$

are bounded below, and the set

$$\{\chi(F) : \mathrm{PT}_1 \text{ pairs } (F, s) \text{ such that } \alpha_2(F) = \alpha_2, \alpha_1(F) = \alpha_1\}$$

is bounded above.

Proof (1) The DT case is shown in [Le Potier 1993, corollaire 2.13]. The PT_0 case follows directly from the DT case by considering the short exact sequence $0 \to \mathcal{O}_Z \to F \to Q \to 0$ since $\alpha_2(F) = \alpha_2(\mathcal{O}_Z)$ and $\alpha_1(F) = \alpha_1(\mathcal{O}_Z)$. Finally, the PT_1 case also follows directly from the DT case by considering the short exact sequence $0 \to \mathcal{O}_S \to F \to Q \to 0$, since $\alpha_2(F) = \alpha_2(\mathcal{O}_S)$ and $\alpha_1(F) \geq \alpha_1(\mathcal{O}_S)$.

(2) (DT) Choose a DT pair $(\mathcal{O}_Z, 1 : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z)$ such that $\alpha_2(\mathcal{O}_Z) = \alpha_2$, $\alpha_1(\mathcal{O}_Z) = \alpha_1$, and $\chi(\mathcal{O}_Z) < 0$. Consider the map

$$(X \setminus Z)^{[-\chi(\mathcal{O}_Z)]} \to \mathrm{Hilb}\big(X, \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t\big), \quad W \mapsto Z \sqcup W, \tag{8}$$

from the Hilbert scheme of points to the Hilbert scheme of closed subschemes with Hilbert polynomial $v = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t$. Since the map (8) is injective and the Hilbert scheme $\mathrm{Hilb}(X, v)$ is of finite type, the Euler characteristic $\chi(\mathcal{O}_Z)$ has a lower bound which only depends on X, α_2, and α_1.

(PT_0) Given a PT_0 pair (F, s), we can form a short exact sequence

$$0 \to \mathcal{O}_Z \to F \to Q \to 0,$$

where Z is the support of F and Q is the cokernel of s. Since

$$\alpha_2(F) = \alpha_2(\mathcal{O}_Z), \quad \alpha_1(F) = \alpha_1(\mathcal{O}_Z) \quad \text{and} \quad \chi(F) = \chi(\mathcal{O}_Z) + \chi(Q),$$

a lower bound in the case (DT) also gives us a lower bound for the case (PT_0).

(PT_1) Choose a PT_1 pair (F, s) and form a short exact sequence

$$0 \to \mathcal{O}_S \to F \to Q \to 0,$$

where S is the support of F and Q is the cokernel of s. By Lemma 2.15(2) below, it suffices to show that $\mu_{\min}(F)$ is bounded below. Note that there exists a surjective map $F \twoheadrightarrow F_{\min}$ to a 2-dimensional μ-semistable sheaf $F_{\min}$ such that $\mu_{\min}(F) = \mu(F_{\min})$. We can assume that $F_{\min}$ is pure. We can form an induced short exact sequence

$$0 \to \mathcal{O}_Z \to F_{\min} \to R \to 0$$

for a subscheme $Z \subseteq S$ and a quotient sheaf $Q \twoheadrightarrow R$. Note that

$$\mu_{\min}(\mathcal{O}_S) \le \mu_{\min}(\mathcal{O}_Z) \le \mu(F_{\min}) = \mu_{\min}(F)$$

(see [Huybrechts and Lehn 2010, Lemma 1.3.3]). Since S is pure, the minimal slope $\mu_{\min}(\mathcal{O}_S)$ is bounded below by [Le Potier 1993, corollaire 2.13]. Therefore $\mu_{\min}(F)$ is also bounded below. □

We need the following lemma to complete the proof of Proposition 2.14(2) for PT_1 pairs:

Lemma 2.15 *Let X be a smooth projective variety with a fixed very ample line bundle $\mathcal{O}_X(1)$ and let α_2, α_1 be rational numbers such that $\alpha_2 > 0$.*

(1) *The set of integers*

$$\{\alpha_0 : \mu\text{-stable reflexive sheaves } F \text{ on } X \text{ such that } P_F(t) = \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0\}$$

is bounded above.

(2) *For a fixed rational number C, the set of integers*

$$\{\alpha_0 : \text{pure sheaves } F \text{ on } X \text{ such that } P_F(t) = \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0 \text{ and } \mu_{\min}(F) \ge C\}$$

is bounded above.

Proof (1) The proof is similar to [Gholampour and Kool 2017, Proposition 3.6(2)]. Choose a μ-stable reflexive sheaf F with $P_F(t) = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0$. Assume that $\alpha_0 > 0$. Let Z be the support of F. Let $U \subseteq Z^{\mathrm{red}}$ be a nonempty smooth connected open subscheme of the reduced subscheme Z^{red} of Z . After shrinking U to a possibly smaller nonempty open subscheme, we may assume that $F|_U \cong \mathcal{O}_U^{\oplus r}$ is a trivial vector bundle on U.

We first claim that there is a map

$$\Phi : U^{[\alpha_0]} \to M^{\mu}\big(X, \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t\big)$$

from the Hilbert scheme of points on U to the moduli space of μ-stable pure sheaves on X such that there is a short exact sequence

$$0 \to \Phi(W) \to F \to \mathcal{O}_W \to 0 \tag{9}$$

for all $W \in U^{[\alpha_0]}$. Indeed, let $\mathcal{W} \subseteq U^{[\alpha_0]} \times U \subseteq U^{[\alpha_0]} \times X$ be the universal family. Then we have a surjective map $\mathcal{O}_{U^{[\alpha_0]}} \boxtimes F|_U \twoheadrightarrow \mathcal{O}_{U^{[\alpha_0]} \times U} \twoheadrightarrow \mathcal{O}_{\mathcal{W}}$ induced by some projection map $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U$. Hence we can form a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{O}_{U^{[\alpha_0]}} \boxtimes F \to \mathcal{O}_{\mathcal{W}} \to 0 \tag{10}$$

of sheaves on $U^{[\alpha_0]} \times X$, which are flat over $U^{[\alpha_0]}$. Hence $\mathcal{F}'$ defines the map Φ and, by taking the fibres of (10) over $W \in U^{[\alpha_0]}$, we obtain (9). From (9) and the fact that F is reflexive, it follows that $\Phi(W)$ is pure 2-dimensional and $\Phi(W)^D \cong F^D$. Moreover, $\Phi(W) \cong F \in \mathrm{Coh}_{2,1}(X) := \mathrm{Coh}_{\leq 2}(X)/\mathrm{Coh}_{\leq 0}(X)$, so $\Phi(W)$ is μ-stable.

We then claim that the map Φ is injective. Choose two 0-dimensional closed subschemes $W_1, W_2 \subseteq U$ and let $F_i = \Phi(W_i)$ for $i = 1, 2$. Assume that there is an isomorphism $\phi : F_1 \xrightarrow{\cong} F_2$ of sheaves. Then we can form a commutative square

$$\begin{array}{ccccc} F_1 & \hookrightarrow & F_1^{DD} & \xleftarrow{\cong} & F \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^{DD}} & & \vdots{\scriptstyle \psi} \\ F_2 & \hookrightarrow & F_2^{DD} & \xleftarrow{\cong} & F \end{array}$$

since $F_1^{DD} \cong F^{DD} \cong F_2^{DD}$ and F is reflexive. Since F is μ-stable, F is stable and thus simple, i.e., $\mathrm{Hom}_X(F, F) = \mathbb{C}$. Therefore, by replacing ϕ by $c \cdot \phi$ for some constant $c \in \mathbb{C}^*$, we may assume that the dotted arrow ψ is the identity map id_F. Hence $F_1 \cong F_2$ as subsheaves of F, and thus the cokernels are isomorphic: $\mathcal{O}_{W_1} \cong \mathcal{O}_{W_2}$. Therefore $W_1 = W_2$ and Φ is injective.

Since Φ is injective and $\dim(U^{[\alpha_0]}) \geq \dim(U) \cdot \alpha_0$, the Euler characteristic α_0 is bounded above by the dimension of the finite-type scheme $M^{\mu}\big(X, \frac{1}{2}\alpha_2 t^2 + \alpha_1 t\big)$. This proves (1).

(2) Case 1: We first show that

$$\big\{\alpha_0 : \mu\text{-stable pure sheaves } F \text{ on } X \text{ such that } P_F(t) = \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0\big\}$$

is bounded above. Let F be a μ-stable pure sheaf with $P_F(t) = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0$. Since F is pure, we can form a short exact sequence

$$0 \to F \to F^{DD} \to R \to 0$$

for some 0-dimensional sheaf R by [Huybrechts and Lehn 2010, Proposition 1.1.10]. Then F^{DD} is a reflexive μ-stable sheaf with $P_{F^{DD}}(t) = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t + (\alpha_0 + \chi(R))$. Since $\alpha_0(F^{DD}) = \alpha_0 + \chi(R)$ is bounded above by Lemma 2.15(1), $\alpha_0(F)$ is also bounded above.

Case 2: We then show that

$$\big\{\alpha_0 : \mu\text{-semistable pure sheaves } F \text{ on } X \text{ such that } P_F(t) = \tfrac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0\big\}$$

is bounded above. Choose a μ-semistable pure sheaf F with $P_F(t) = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0$. Consider the Jordan–Hölder filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = F$$

of F with respect to μ-stability [ibid., Theorem 1.6.7(ii)]. The length l of the filtration is bounded from above because the leading term α_2 of the Hilbert polynomial is fixed. Since F is pure, we can assume that all F_i/F_{i-1} are also pure. Since each F_i/F_{i-1} is a μ-stable pure sheaf with the slope $\mu(F_i/F_{i-1}) = \mu(F)$, case 2 follows from case 1.

Case 3: Finally, we consider the general case. Let F be a pure sheaf with $P_F(t) = \frac{1}{2}\alpha_2 t^2 + \alpha_1 t + \alpha_0$. Choose a Harder–Narasimhan filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = F$$

of F with respect to the μ-stability [ibid., Theorem 1.6.7(i)]. Since F is pure, we may assume that F_i/F_{i-1} are μ-semistable pure sheaves. If $\alpha_2(F)$, $\alpha_1(F)$ are fixed and $\mu_{\min}(F)$ is bounded below, then there are only finitely many possible $\alpha_1(F_i/F_{i-1})$ (and $\alpha_2(F_i/F_{i-1})$). Therefore

$$\alpha_0(F) = \sum_i \alpha_0(F_i/F_{i-1})$$

is also bounded above by the result in case 2. □

3 Moduli spaces

In this section, we construct moduli spaces of PT_q pairs via geometric invariant theory and prove that they are open subschemes of the moduli space of perfect complexes parametrizing polynomially Bridgeland stable objects.

3.1 Moduli stack of pairs

In this subsection, we introduce the moduli stack of all pairs and prove that the PT_q-stability conditions are bounded and open conditions.

Definition 3.1 Let X be a smooth projective variety over $\mathbb{C}$.

(1) We define the *moduli stack of pairs on X* as a 2-functor

$$\widetilde{\mathscr{P}air}(X) : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Groupoid}$$

sending a scheme T to the groupoid of (not necessarily 2-dimensional) pairs (F, s) of T-flat coherent sheaves F on $X \times T$ and sections $s \in \Gamma(X \times T, F)$. The morphisms $(F_1, s_1) \to (F_2, s_2)$ in the groupoid are isomorphisms $\phi : F_1 \to F_2$ of sheaves such that $\phi(s_1) = s_2$.

(2) We define the *moduli stack of nonzero pairs on X* as the substack

$$\mathscr{P}air(X) \subseteq \widetilde{\mathscr{P}air}(X)$$

consisting of the pairs (F, s) such that $s_t \neq 0 \in \Gamma(X, F_t)$ for all $t \in T$.

The moduli stacks $\widetilde{\mathscr{P}air}(X)$ and $\mathscr{P}air(X)$ are algebraic stacks. Moreover, they have explicit descriptions as unions of open substacks which are global quotient stacks. We will now review the proofs of these facts.

Let us first fix some notation. Let X be a smooth projective variety of dimension n with a fixed very ample line bundle $\mathcal{O}_X(1)$. Fix $v \in H^*(X,\mathbb{Q})$ and let $P_v(t) \in \mathbb{Q}[t]$ be the associated Hilbert polynomial. Choose an integer m. Fix a vector space V of dimension $P_v(m)$. Let

$$\mathcal{Q} := \mathscr{Q}uot(X, V\otimes\mathcal{O}_X(-m), v) \tag{11}$$

be the Quot scheme of quotients $\rho : V\otimes\mathcal{O}_X(-m) \twoheadrightarrow F$ with $\mathrm{ch}(F) = v$. Let

$$V\otimes\mathcal{O}_{\mathcal{Q}}\otimes\mathcal{O}_X(-m) \twoheadrightarrow \mathbb{F} \tag{12}$$

be the universal quotient on $\mathcal{Q}\times X$ and let $\pi : \mathcal{Q}\times X \to \mathcal{Q}$ be the projection map. Consider the open subscheme

$$\mathcal{Q}^\circ := \{(F,\rho)\in\mathcal{Q} : V \xrightarrow{\cong} H^0(X,F(m)) \text{ and } H^i(X,F(m)) = 0 \text{ for all } i>0\}, \tag{13}$$

where $V \to H^0(X,F(m))$ is the map induced by ρ. Then the canonical map

$$V\otimes\mathcal{O}_{\mathcal{Q}} \to R\pi_*(\mathbb{F}(m)) \tag{14}$$

induced by (12) is an isomorphism over $\mathcal{Q}^\circ$.

Proposition 3.2 *The moduli stack $\mathscr{P}air(X)$ has the following properties*:

(1) *We have a decomposition*

$$\mathscr{P}air(X) = \bigsqcup_{v\in H^*(X,\mathbb{Q})} \mathscr{P}air(X,v)$$

by disjoint open substacks $\mathscr{P}air(X,v)$ consisting of the pairs (F,s) on X with $\mathrm{ch}(F) = v \in H^*(X,\mathbb{Q})$.

(2) *We have*

$$\mathscr{P}air(X,v) = \bigcup_{m\geq 0} \mathscr{P}air(X,v)_m$$

for open substacks $\mathscr{P}air(X,v)_m$ consisting of the pairs (F,s) on X such that $F(m)$ is globally generated and $H^{>0}(X,F(m)) = 0$.

(3) *We have a global quotient stack presentation*

$$\mathscr{P}air(X,v)_m = [\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})/\mathrm{PGL}(V)],$$

where $\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F}) := \mathrm{Proj}\,\mathrm{Sym}\, h^0((R\pi_*\mathbb{F})^\vee)$ denotes the projective cone.

Consequently, the moduli stack $\mathscr{P}air(X)$ is an algebraic stack (locally of finite type).

Proof Let $\mathscr{C}oh(X)$ denote the moduli stack of coherent sheaves on X. Then

$$\mathscr{C}oh(X) = \bigsqcup_{v\in H^*(X,\mathbb{Q})} \mathscr{C}oh(X,v) \quad \text{and} \quad \mathscr{C}oh(X,v) = \bigcup_{m\geq 0} \mathscr{C}oh(X,v)_m$$

for open substacks $\mathscr{C}oh(X,v)_m \subseteq \mathscr{C}oh(X,v) \subseteq \mathscr{C}oh(X)$ such that

$$\mathscr{C}oh(X,v)(\mathbb{C}) = \{F \in \mathscr{C}oh(X)(\mathbb{C}) : \mathrm{ch}(F) = v \in H^*(X,\mathbb{Q})\},$$
$$\mathscr{C}oh(X,v)_m(\mathbb{C}) = \{F \in \mathscr{C}oh(X,v)(\mathbb{C}) : F(m) \text{ globally generated and } H^i(X,F(m)) = 0 \text{ for all } i>0\}.$$

Let $\mathbb{F}^C$ be the universal family on $\mathscr{C}oh(X)\times X$ and let $\pi^C : \mathscr{C}oh(X)\times X \to \mathscr{C}oh(X)$ be the projection map. Then $R\pi^C_*(\mathbb{F}^C(m))$ is a vector bundle over $\mathscr{C}oh(X,v)_m$ and the associated $\mathrm{GL}_{P_v(m)}$-bundle is exactly $\mathcal{Q}^\circ$. Hence we have a global quotient stack presentation (see also [Laumon and Moret-Bailly 2000, théorème 4.6.2.1])

$$\mathscr{C}oh(X,v)_m = [\mathcal{Q}^\circ/\mathrm{GL}(V)].$$

By definition, the moduli stack of pairs is the abelian cone

$$\widetilde{\mathscr{P}air}(X) = C(\mathscr{E}xt^n_{\pi^C}(\mathbb{F}^C, K_X)) = \operatorname{Spec}\operatorname{Sym}\mathscr{E}xt^n_{\pi^C}(\mathbb{F}^C, K_X)$$

associated to the coherent sheaf $\mathscr{E}xt^n_{\pi^C}(\mathbb{F}^C,K_X) = h^0((R\pi^C_*\mathbb{F}^C)^\vee)$ on $\mathscr{C}oh(X)$, and the moduli stack of nonzero pairs is the complement

$$\mathscr{P}air(X) = \widetilde{\mathscr{P}air}(X)\setminus 0_{\mathscr{C}oh(X)} \subseteq \widetilde{\mathscr{P}air}(X)$$

of the zero section $0_{\mathscr{C}oh(X)} : \mathscr{C}oh(X) \hookrightarrow \widetilde{\mathscr{P}air}(X)$. Hence, we have (1) and (2) for the open substacks $\mathscr{P}air(X,v) := \mathscr{P}air(X)\times_{\mathscr{C}oh(X)}\mathscr{C}oh(X,v)$ and $\mathscr{P}air(X,v)_m := \mathscr{P}air(X)\times_{\mathscr{C}oh(X)}\mathscr{C}oh(X,v)_m$. Moreover, we have a global quotient stack presentation

$$\mathscr{P}air(X,v)_m = \big[\big(C_{\mathcal{Q}^\circ}(\mathscr{E}xt^n_\pi(\mathbb{F},K_X))\setminus 0_{\mathcal{Q}^\circ}\big)/\mathrm{GL}(V)\big] = [\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})/\mathrm{PGL}(V)]$$

since $\mathrm{PGL}(V) = \mathrm{GL}(V)/\mathbb{G}_m$. □

Proposition 3.3 *There exists an open substack $\mathscr{P}air^{(q)}(X,v)$ of $\mathscr{P}air(X,v)$, for any $q\in\{-1,0,1\}$, such that the $\mathbb{C}$-points are given by*

$$\mathscr{P}air^{(q)}(X,v)(\mathbb{C}) = \{(F,s)\in\mathscr{P}air(X,v) : (F,s) \text{ is } \mathrm{PT}_q\text{-stable}\}.$$

Moreover, the moduli stack $\mathscr{P}air^{(q)}(X,v)$ is an algebraic space of finite type.

We will prove Proposition 3.3 through three steps. We first prove that PT_q pairs are *simple* pairs.

Lemma 3.4 *For any PT_q pair (F,s) on X, we have*

$$\mathrm{Aut}_{\mathscr{P}air(X)}(F,s) = \{1\}.$$

Proof Consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_Z \to F \to Q \to 0,$$

where Z is the support of F and Q is the cokernel of s. By the PT_q-stability condition, we have $\mathrm{Hom}_X(Q, F) = 0$. Hence we have an injective map

$$\mathrm{Hom}_X(F, F) \to \mathrm{Hom}_X(\mathcal{O}_Z, F) = \Gamma(X, F), \quad \phi \mapsto \phi(s). \tag{15}$$

Since $\mathrm{Aut}_{\mathscr{P}air(X)}(F, s)$ is the fibre of the map (15) over the point $s \in \Gamma(X, F)$, the automorphism group $\mathrm{Aut}_{\mathscr{P}air(X)}(F, s)$ is trivial. □

Secondly, we prove that the collection of PT_q pairs is *bounded*.

Lemma 3.5 *The collection of sheaves*

$$\{F \in \mathscr{C}oh(X, v)(\mathbb{C}) : (F, s) \text{ is } \mathrm{PT}_q\text{-stable for some } s \in \Gamma(X, F)\}$$

for fixed X and v is bounded in the sense of [Grothendieck 1966b, definition 1.1].

Proof The boundedness of the $\mathrm{DT} = \mathrm{PT}_{-1}$ pairs follows from [Grothendieck 1966b, théorème 2.1] and the boundedness of the $\mathrm{PT} = \mathrm{PT}_1$ pairs follows from [Le Potier 1993, théorème 4.11]. Thus it suffices to consider the PT_0 pairs.

Given a PT_0 pair (F, s) with $\mathrm{ch}(F) = v$, let Z be the support of F and Q the cokernel of s. By Proposition 2.14(2) for DT-stability, $\chi(\mathcal{O}_Z)$ is bounded below. On the other hand $\chi(\mathcal{O}_Z)$ is bounded above since $\chi(Q) \geq 0$. Moreover, $\mathrm{ch}_i(\mathcal{O}_Z) = \mathrm{ch}_i(F)$ for $i < n$. Hence there are only finitely many possible Chern characters $\mathrm{ch}(\mathcal{O}_Z)$, and the collection of all such $\mathcal{O}_Z$ is bounded by [Grothendieck 1966b, théorème 2.1]. Therefore, there exists an integer m such that, for all $\mathcal{O}_Z$ as above, we have $H^i(X, \mathcal{O}_Z(m-i)) = 0$ for all $i > 0$. For any PT_0 pair (F, s) with $\mathrm{ch}(F) = v$, we then also have $H^i(X, F(m-i)) = 0$ for all $i > 0$ (note $H^i(X, Q(m-i))$ is automatically zero since Q is 0-dimensional). By [Huybrechts and Lehn 2010, Lemma 1.7.6], we deduce the required boundedness for PT_0 pairs. □

The proof of Lemma 3.5 also proves the following lemma, which will be used in the proof of Proposition 3.17 in the next subsection:

Lemma 3.6 *The collection of sheaves*

$$\{F \in \mathscr{C}oh(X, v)(\mathbb{C}) : \mathrm{coker}(\mathcal{O}_X \xrightarrow{s} F) \in \mathrm{Coh}_{\leq 0}(X) \text{ for some } s \in \Gamma(X, F)\}$$

for fixed X and v is bounded.

Lastly, we show that the PT_q-stability conditions are *open*.

Lemma 3.7 *Let (F, s) be a T-flat family of pairs on X parametrized by a scheme T of finite type over $\mathbb{C}$. Then*

$$\{t \in T(\mathbb{C}) : \text{the fibre } (F_t, s_t) \text{ of } (F, s) \text{ is } \mathrm{PT}_q\text{-stable}\}$$

is a Zariski open subset of $T(\mathbb{C})$.

Proof We first show that the condition

$$Q_t := \operatorname{coker}(\mathcal{O}_X \xrightarrow{s_t} F_t) \in \operatorname{Coh}_{\leq q}(X)$$

is an open condition. By [Grothendieck 1966b, proposition 1.2(i)], the set $\{P_{Q_t} : t \in T(\mathbb{C})\}$ of Hilbert polynomials is finite. Hence the result follows from upper semicontinuity of Hilbert polynomials of nonflat families of coherent sheaves.

Now we consider the second condition

$$F_t \in \operatorname{Coh}_{\geq q+1}(X).$$

Firstly, if $q = -1$, then this condition is always satisfied. Secondly, if $q = 1$, this condition is open by the openness of purity [Huybrechts and Lehn 2010, Proposition 2.3.1]. Finally, assume that $q = 0$. We show that the set of the Hilbert polynomials

$$\{P_{F_t/T_0(F_t)} : t \in T(\mathbb{C})\} \tag{16}$$

is finite. It suffices to show that $\chi(T_0(F_t))$ is bounded above. Note that the short exact sequence $0 \to T_0(F_t) \to F_t \to F_t/T_0(F_t) \to 0$ induces a short exact sequence

$$0 \to \mathcal{E}xt^n_X(F_t/T_0(F_t), K_X) \to \mathcal{E}xt^n_X(F_t, K_X) \to \mathcal{E}xt^n_X(T_0(F_t), K_X) \to 0$$

of 0-dimensional sheaves by [ibid., Proposition 1.1.6]. Then we have

$$\chi(T_0(F_t)) = \chi\big(\mathcal{E}xt^n_X(T_0(F_t), K_X)\big) \leq \chi(\mathcal{E}xt^n_X(F_t, K_X)) \leq \dim_{\mathbb{C}} \operatorname{Ext}^n_X(F_t, K_X),$$

where $\pi^T : X \times T \to T$ is the projection map, $i_t : \{t\} \hookrightarrow T$ is the inclusion map, and the last inequality follows from the local-to-global spectral sequence. Since $\operatorname{Ext}^n_X(F_t, K_X)$ are the fibres of the coherent sheaf $\mathcal{E}xt^n_{\pi^T}(F, K_X)$ on a scheme T of finite type, their dimensions are bounded above.

Since, for any $t \in T(\mathbb{C})$, the relative Quot schemes

$$\mathcal{Q}uot(X \times T/T, F, P_{F_t/T_0(F_t)}) \to T$$

are proper over T, their images are closed. The claim follows from finiteness of (16). □

Proposition 3.3 follows directly from the above results:

Proof of Proposition 3.3 By Lemma 3.7, we can form $\mathscr{P}air^{(q)}(X, v)$ as an open substack of $\mathscr{P}air(X, v)$. Then Lemma 3.4 proves that $\mathscr{P}air^{(q)}(X, v)$ is an algebraic space and Lemma 3.5 proves that $\mathscr{P}air^{(q)}(X, v)$ is of finite type. □

Remark 3.8 By [Grothendieck 1966b; Le Potier 1993], the moduli spaces $\mathscr{P}air^{(q)}(X, v)$ of PT_q pairs are projective schemes for $q = -1$ and $q = 1$. In the next subsection, we will show that the space $\mathscr{P}air^{(0)}(X, v)$ is also a projective scheme.

We end this subsection with a remark on the natural derived enhancement of the moduli stack of pairs. This derived structure will not be used in the rest of the paper.

Remark 3.9 There is a derived stack $R\widetilde{\mathscr{P}air}(X)$ whose classical truncation is $\widetilde{\mathscr{P}air}(X)$ and the tangent complex at a point (F,s) is

$$\mathbb{T}_{R\widetilde{\mathscr{P}air}(X)}|_{(F,s)} = R\operatorname{Hom}_X(I^\bullet, F),$$

where $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$. We sketch the construction here. Recall that there exists a derived moduli stack $R\mathscr{C}oh(X)$ of coherent sheaves as an open substack of the derived moduli stack $R\mathscr{P}erf(X)$ of perfect complexes. Let $\mathbb{F}$ denote the universal family on $R\mathscr{C}oh(X)$ and let $\pi : R\mathscr{C}oh(X)\times X \to R\mathscr{C}oh(X)$ denote the projection map. Define the derived moduli stack of pairs as the total space

$$R\widetilde{\mathscr{P}air}(X) := \operatorname{Tot}_{R\mathscr{C}oh(X)}(R\pi_*\mathbb{F})$$

of the perfect complex $R\pi_*\mathbb{F}$ on $R\mathscr{C}oh(X)$. Moreover, the derived loop stack

$$\mathcal{L}_{R\widetilde{\mathscr{P}air}(X),(F,s)} := \operatorname{Spec}(\mathbb{C})\times^h_{(F,s),R\widetilde{\mathscr{P}air}(X),(F,s)}\operatorname{Spec}(\mathbb{C})$$

is an open substack of the fibre of the map between derived linear stacks

$$\operatorname{Tot}(R\operatorname{Hom}_X(F,F)) \xrightarrow{s} \operatorname{Tot}(R\operatorname{Hom}_X(\mathcal{O}_X,F))$$

at the point $s\in\operatorname{Hom}_X(\mathcal{O}_X,F)$. From the distinguished triangle

$$R\operatorname{Hom}_X(I^\bullet,F)[-1]\to R\operatorname{Hom}_X(F,F)\xrightarrow{s} R\operatorname{Hom}_X(\mathcal{O}_X,F),$$

the cotangent complex can be computed as

$$\mathbb{T}_{R\widetilde{\mathscr{P}air}(X)}|_{(F,s)} \cong \mathbb{T}_{\mathcal{L}_{R\widetilde{\mathscr{P}air}(X),(F,s)}}|_0[1] \cong R\operatorname{Hom}_X(I^\bullet,F).$$

3.2 Geometric invariant theory

In this subsection, we construct moduli spaces of PT_q pairs as projective schemes via geometric invariant theory. Although we largely follow [Le Potier 1993], our arguments differ since the underlying sheaves F of PT_q-stable pairs (F,s) are not *pure* for $q=-1,0$. Our approach is inspired by [Stoppa and Thomas 2011].

Theorem 3.10 *Let X be a smooth projective variety and $v\in H^*(X,\mathbb{Q})$. Then there exists a projective scheme $\mathcal{M}$, an action of PGL_N on $\mathcal{M}$ for some N, and an SL_N-equivariant ample line bundle $\mathcal{L}^{(q)}\in \mathrm{Pic}^{\mathrm{SL}_N}(\mathcal{M})_{\mathbb{Q}}$ for each $q\in\{-1,0,1\}$ such that*

$$\mathscr{P}air^{(q)}(X,v) = [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(q)})/\mathrm{PGL}_N] = [\mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(q)})/\mathrm{PGL}_N].$$

In particular, the moduli spaces $\mathscr{P}air^{(q)}(X,v)$ are the GIT quotients $\mathcal{M}/\!\!/_{\mathcal{L}^{(q)}}\mathrm{SL}_N$ and hence projective schemes.

The rest of this subsection is devoted to the proof of Theorem 3.10. By the boundedness result in Lemma 3.5, we have $\mathscr{P}air^{(q)}(X,v)\subseteq \mathscr{P}air(X,v)_m$ for sufficiently large m. Denoting the Hilbert

polynomial determined by v by $P_v(t)$, we then define

$$V := \mathbb{C}^{P_v(m)}, \quad R_m := \Gamma(X, \mathcal{O}_X(m)).$$

We use the notation from Section 3.1 (e.g., the Quot scheme $\mathcal{Q}$ in (11), the universal family (12), and the open subscheme $\mathcal{Q}^\circ$ in (13)). Moreover, by Proposition 3.2, we have a presentation of the moduli stack as a global quotient stack,

$$\mathscr{P}air(X, v)_m = [\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})/\mathrm{PGL}(V)].$$

We would like to use geometric invariant theory on $\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})$. Thus we need to consider a $\mathrm{PGL}(V)$-equivariant compactification of $\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})$ and linearizations on it.

We define the *parameter space* $\mathcal{M}$ as follows:

Definition 3.11 Define a projective scheme

$$\mathcal{M} := \overline{\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})} \subseteq \mathcal{Q} \times \mathbb{P}(V \otimes R_m^*)$$

as the closure of $\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})$ in $\mathcal{Q} \times \mathbb{P}(V \otimes R_m^*)$ with respect to the embedding

$$\begin{array}{ccc} \mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F}) & \hookrightarrow & \mathbb{P}_{\mathcal{Q}^\circ}\big(R\pi_*(\mathbb{F}(m)) \otimes R_m^*\big) \\ & & \big\uparrow \cong \\ & & \mathbb{P}_{\mathcal{Q}^\circ}(\mathcal{O}_{\mathcal{Q}^\circ} \otimes V \otimes R_m^*) \hookrightarrow \mathcal{Q} \times \mathbb{P}(V \otimes R_m^*) \end{array} \tag{17}$$

Here the first map in (17) is the closed embedding induced by the canonical map

$$\Gamma(X, \mathcal{O}_X(m)) \otimes R\pi_*\mathbb{F} \to R\pi_*(\mathbb{F}(m)),$$

the second map in (17) is an isomorphism induced by the canonical map (14), and the last map in (17) is the open embedding induced by $\mathcal{Q}^\circ \hookrightarrow \mathcal{Q}$.

Remark 3.12 The parameter space $\mathcal{M}$ in this paper is larger than the parameter space of Le Potier [1993; 1995]. Le Potier considered $\overline{\mathcal{Q}}_{\mathrm{pure}} \times \mathbb{P}(V \otimes R_m^*)$, where $\overline{\mathcal{Q}}_{\mathrm{pure}} \subseteq \mathcal{Q}$ is the closure of the open subscheme $\mathcal{Q}_{\mathrm{pure}} \subseteq \mathcal{Q}$. The difference arises from the fact that we want to realize all three moduli spaces $\mathscr{P}air^{(q)}(X, v)$ for $q = -1, 0, 1$, as GIT quotients of *the same* scheme $\mathcal{M}$ by the same group $\mathrm{PGL}(V)$ (with respect to different linearizations).

We consider the *linearizations* $\mathcal{L}_{g(t),l}$ on $\mathcal{M}$ defined as follows:

Definition 3.13 For sufficiently large l, consider the standard Grothendieck embedding into the Grassmannian

$$\mathcal{Q} \hookrightarrow \mathcal{G} := \mathscr{G}rass(V \otimes R_{l-m}, P_v(l)), \quad (F, \rho) \mapsto \big(V \otimes R_{l-m} \twoheadrightarrow \Gamma(X, F(l))\big).$$

Fix a polynomial $g(t) \in \mathbb{Q}[t]$ with positive leading coefficient. Define the $\mathrm{SL}(V)$-equivariant very ample line bundle

$$\mathcal{L}_{g(t),l} := \mathcal{O}_{\mathcal{G}}(1)|_{\mathcal{Q}} \boxtimes \mathcal{O}_{\mathbb{P}(V \otimes R_m^*)}(g(l)) \in \mathrm{Pic}^{\mathrm{SL}(V)}(\mathcal{M})_{\mathbb{Q}},$$

where $\mathcal{O}_{\mathcal{G}}(1)$ is the polarization on $\mathcal{G}$ induced by the standard Plücker embedding of $\mathcal{G}$.

We describe the $\mathcal{L}_{g(t),l}$-stable points via the Hilbert–Mumford criterion.

Lemma 3.14 *A point (F,ρ,s) in $\mathcal{Q}\times\mathbb{P}(V\otimes R_m^*)$ is $\mathcal{L}_{g(t),l}$-semistable if and only if the following two conditions are satisfied*:

(1) *For all nonzero proper subspaces $V'\subseteq V$, we have*

$$\frac{P_{F'}(l)}{\dim(V')}\geq\frac{P_F(l)-g(l)}{\dim(V)}, \tag{18}$$

where $F':=\rho(V'\otimes\mathcal{O}_X(-m))\subseteq F$.

(2) *For all nonzero proper subspaces $V'\subseteq V$ with $s\in V'\otimes R_m^*$, we have*

$$\frac{P_{F'}(l)-g(l)}{\dim(V')}\geq\frac{P_F(l)-g(l)}{\dim(V)}, \tag{19}$$

where again $F':=\rho(V'\otimes\mathcal{O}_X(-m))\subseteq F$.

Moreover, (F,ρ,s) is $\mathcal{L}_{g(t),l}$-stable if and only if the above two inequalities hold strictly.

Proof This is standard by [Huybrechts and Lehn 2010, Lemmas 4.4.3 and 4.4.4]; see also [Stoppa and Thomas 2011, Lemma 3.12]. For completeness, we briefly go through the argument.

Note that $\mathcal{Q}$ has a right $\mathrm{SL}(V)$-action and $\mathbb{P}(V\otimes R_m^*)$ has a left $\mathrm{SL}(V)$-action. Here we will use the left action convention, i.e., $\phi\cdot(F,\rho)=(F,\rho\circ\phi^{-1})$ for $\phi\in\mathrm{SL}(V)$.

Choose a 1-parameter torus $\lambda:\mathbb{G}_m\to\mathrm{SL}(V)$. Equivalently, choose a weight decomposition

$$V=\bigoplus_{i\in\mathbb{Z}}V_i,$$

in which the determinant 1 condition implies $\sum_{i\in\mathbb{Z}}i\dim(V_i)=0$. Let $V_{\leq i}:=\bigoplus_{j\leq i}V_j$, $F_{\leq i}:=\rho(V_{\leq i}\otimes\mathcal{O}_X(-m))$, $F_i:=F_{\leq i}/F_{\leq i-1}$, and $\rho_i:V_i\otimes\mathcal{O}_X(-m)\twoheadrightarrow F_i$ be the induced surjective map. Let $s=\sum_i s_i$ such that $s_i\in V_i\otimes R_m^*$ and let $i_{\max}$ be the maximal integer i such that $s_i\neq 0$.

As in [Huybrechts and Lehn 2010, Lemma 4.4.3], the limit point is

$$\lim_{t\to\infty}\lambda(t)\cdot(F,\rho,s)=\Big(\bigoplus_i F_i,\bigoplus_i\rho_i,s_{i_{\max}}\Big).$$

Since the fibre of the line bundle $\mathcal{L}_{g(t),l}$ over a point (F,ρ,s) is given as

$$\mathcal{L}_{g(t),l}|_{(F,\rho,s)}=\det\big(H^0(X,F(l))\big)\otimes\mathbb{C}\langle s\rangle^{\otimes-g(l)},$$

where $\mathbb{C}\langle s\rangle\subseteq V\otimes R_m^*$ is the subspace generated by s, we can easily compute the Hilbert–Mumford weight

$$\begin{aligned}\mu^{\lambda^{-1}}_{\mathcal{L}_{g(t),l}}(F,\rho,s)&=\sum_i(-i\cdot P_{F_i}(l)+i_{\max}\cdot g(l))\\&=\sum_i\Big((P_{F_{\leq i}}(l)-\dim(\Gamma_{\leq i})\cdot g(l))-\frac{\dim(V_{\leq i})}{\dim(V)}\cdot(P_F(l)-g(l))\Big),\end{aligned}$$

where $\Gamma_{\leq i}:=\mathbb{C}\langle s\rangle\cap(V_{\leq i}\otimes R_m^*)$.

By [Mumford et al. 1994, Theorem 2.1], the point (F,ρ,s) is semistable if and only if

$$\frac{P_{F'}(l)-\dim(\Gamma')\cdot g(l)}{\dim(V')}\geq\frac{P_F(l)-g(l)}{\dim(V)}$$

for all proper subspaces $V'\subseteq V$, where $F':=\rho(V'\otimes\mathcal{O}_X(-m))$ and $\Gamma':=\mathbb{C}\langle s\rangle\cap(V'\otimes R_m^*)$. □

We will focus on the following three types of linearizations:

Definition 3.15 Recall that $P_v(t)\in\mathbb{Q}[t]$ is the Hilbert polynomial associated to $v\in H^*(X,\mathbb{Q})$.

(DT) For any rational number $a>0$, we define a polynomial

$$g^{(-1)}(t):=P_v(t)+a\cdot t^2.$$

(PT_0) For any rational number $0<\epsilon<1$, we define a polynomial

$$g^{(0)}(t):=P_v(t)-\epsilon\cdot t.$$

(PT_1) For any rational number $0<\delta<\frac{1}{2}$, we define a polynomial

$$g^{(1)}(t):=P_v(t)-\delta\cdot t^2.$$

For each integer $q\in\{-1,0,1\}$, we define a linearization

$$\mathcal{L}^{(q)}:=\mathcal{L}_{g^{(q)}(t),l}$$

for sufficiently large l.

Remark 3.16 We list the linearizations previously considered in the literature:

(1) [Le Potier 1993] $\mathcal{O}_{\mathcal{G}}(c)\boxtimes\mathcal{O}_{\mathbb{P}(V\otimes R_m^*)}(l^d/d!)$ for $c>1$.

(2) [Le Potier 1995] $\mathcal{O}_{\mathcal{G}}(1+\alpha(m)/P_v(m))\boxtimes\mathcal{O}_{\mathbb{P}(V\otimes R_m^*)}(P_v(l))$ for $\alpha(t)\in\mathbb{Q}[t]$.

(3) [Stoppa and Thomas 2011] $\mathcal{O}_{\mathcal{G}}(c)\boxtimes\mathcal{O}_{\mathbb{P}(V\otimes R_m^*)}(l)$ for $c>0$ (for 1-dimensional pairs).

The linearizations $\mathcal{L}_{g(t),l}$ in this paper are 2-dimensional versions of the linearizations in [Stoppa and Thomas 2011], which contain the linearizations in [Le Potier 1993] as a special case. However our parameter space is larger than the one in [Le Potier 1993], as explained in Remark 3.12.

We now compare GIT stability for the line bundles $\mathcal{L}^{(q)}$ in Definition 3.15 to the PT_q-stability introduced in Definition 2.1.

Proposition 3.17 *There exists an integer M which only depends on X and v, and integers L_m which depend on X, v and m such that, for any $m\geq M$, $l\geq L_m$, and $(F,\rho,s)\in\mathcal{M}$, we have the following*:

(1) *(F,ρ,s) is $\mathcal{L}^{(q)}$-semistable $\Longrightarrow$ $(F,\rho)\in\mathcal{Q}^\circ$ and (F,s) is PT_q-stable.*

(2) *$(F,\rho)\in\mathcal{Q}^\circ$ and (F,s) is PT_q-stable $\Longrightarrow$ (F,ρ,s) is $\mathcal{L}^{(q)}$-stable.*

We will consider the three stability conditions separately. We start with PT_0-stability.

Proof of Proposition 3.17 for $\mathbf{PT_0}$-stability (1) Assume that (F,ρ,s) is $\mathcal{L}^{(0)}$-semistable. We first claim injectivity of the induced map

$$\Gamma(\rho(m)) : V \to \Gamma(X, F(m))$$

Let V' be the kernel of the above map $\Gamma(\rho(m))$. Then the induced sheaf $F' := \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ is zero. If $V' \neq 0$, then the $\mathcal{L}^{(0)}$-semistability of (F,ρ,s) implies an inequality

$$\frac{0}{\dim(V')} \geq \frac{\epsilon \cdot l}{\dim(V)}$$

by Lemma 3.14(1). This leads to a contradiction for $l > 0$. Hence $V' = 0$.

We claim that $(F,s) \in \mathscr{P}air(X,v)$. More precisely, there exists a section $\mathcal{O}_X \to F$ that fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \dashrightarrow & \Gamma(X,F) \\ \downarrow{\scriptstyle s} & & \downarrow \\ V \otimes R_m^* & \xhookrightarrow{\Gamma(\rho(m))} & \Gamma(X,F(m)) \otimes R_m^* \end{array}$$

determined by the dotted arrow. Indeed, consider the open subscheme

$$\mathcal{Q}^\star := \{(F,\rho) \in \mathcal{Q} : V \xrightarrow{\Gamma(\rho(m))} \Gamma(X,F(m)) \text{ is injective}\}$$

of $\mathcal{Q}$. Note that the restriction $\mathcal{M}|_{\mathcal{Q}^\star}$ of $\mathcal{M}$ is the closure of $\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})$ in $\mathcal{Q}^\star \times \mathbb{P}(V \otimes R_m^*)$. Hence we can form a commutative diagram

$$\begin{array}{ccc} \mathcal{M}|_{\mathcal{Q}^\star} & \overset{\exists}{\dashrightarrow} & \mathbb{P}_{\mathcal{Q}^\star}(R\pi_*\mathbb{F}) \\ \downarrow & & \downarrow \\ \mathcal{Q}^\star \otimes \mathbb{P}(V \otimes R_m^*) & \hookrightarrow & \mathbb{P}_{\mathcal{Q}^\star}\big(R\pi_*(\mathbb{F}(m)) \otimes R_m^*\big) \end{array}$$

of closed embeddings of projective schemes over $\mathcal{Q}^\star$. By the result in the previous paragraph, we have $(F,\rho,s) \in \mathcal{M}|_{\mathcal{Q}^\star}$, which proves the claim.

Next, we claim that $Q := \operatorname{coker}(\mathcal{O}_X \xrightarrow{s} F) \in \operatorname{Coh}_{\leq 0}(X)$. Let

$$V' := V \cap \Gamma(X, \mathcal{O}_Z(m)) \quad \text{in } \Gamma(X,F(m)),$$

where $\mathcal{O}_Z := \operatorname{im}(\mathcal{O}_X \xrightarrow{s} F)$. Then the induced sheaf $F' := \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ is contained in $\mathcal{O}_Z \subseteq F$. Note that we have a commutative diagram

$$\begin{array}{ccccc} V' \otimes R_m^* & \hookrightarrow & \Gamma(X,\mathcal{O}_Z(m)) \otimes R_m^* & \hookleftarrow & \Gamma(X,\mathcal{O}_Z) \\ \downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ V \otimes R_m^* & \xhookrightarrow{\rho(m)} & \Gamma(X,F(m)) \otimes R_m^* & \hookleftarrow & \Gamma(X,F) \end{array}$$

where the left square is cartesian. Hence we have $s \in V' \otimes R_m^*$. By Lemma 3.14(2), we have inequalities

$$\frac{P_{\mathcal{O}_Z}(l) - g^{(0)}(l)}{\dim(V')} \geq \frac{P_{F'}(l) - g^{(0)}(l)}{\dim(V')} \geq \frac{\epsilon \cdot l}{\dim(V)} \tag{20}$$

for sufficiently large l, where the two collections

$$\{\mathcal{O}_Z := \operatorname{im}(\mathcal{O}_X \xrightarrow{s} F)\} \quad \text{and} \quad \{F' := \operatorname{im}(V' \otimes \mathcal{O}_X(-m) \xrightarrow{\rho} F)\}$$

for fixed X, v, m are bounded by [Grothendieck 1966b, proposition 1.2(i)]. Hence the set $\{P_{\mathcal{O}_Z}(t)\}$ of Hilbert polynomials is finite. Since the leading coefficient of the left-hand side of (20), as a polynomial in l, should be positive, we can take l sufficiently large and deduce $\alpha_2(\mathcal{O}_Z) = \alpha_2(F)$ and $\alpha_1(\mathcal{O}_Z) > \alpha_1(F) - \epsilon$. Since $\epsilon < 1$, we have $\alpha_1(\mathcal{O}_Z) = \alpha_1(F)$, and thus Q is 0-dimensional.

We now show that $(F, \rho) \in \mathcal{Q}^\circ$. By Lemma 3.6, the collection of coherent sheaves F with $\operatorname{ch}(F) = v$ and $\operatorname{coker}(s) \in \operatorname{Coh}_{\leq 0}(X)$ is bounded for fixed X, v. Hence we have $H^{>0}(X, F(m)) = 0$ for sufficiently large m, as desired.

We finally claim that $F \in \operatorname{Coh}_{\geq 1}(X)$. Let $T_0(F) \subseteq F$ be the 0-dimensional torsion sheaf and let

$$V' := \Gamma(X, T_0(F)(m)) \subseteq \Gamma(X, F(m)) \xleftarrow{\cong} V,$$

where the last isomorphism follows from the result in the previous paragraph. Then the induced sheaf $F' := \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ is $T_0(F)$ since the 0-dimensional sheaf $T_0(F)$ is globally generated. If $V' \neq 0$, then

$$1 = \frac{\chi(T_0(F))}{\chi(T_0(F)(m))} = \frac{\chi(F')}{\dim(V')} \geq \frac{\epsilon \cdot l}{\dim(V)}$$

by Lemma 3.14(1). This leads to a contradiction for big enough l. Therefore $V' = 0$ and $F' = T_0(F) = 0$.

(2) Assume that $(F, \rho) \in \mathcal{Q}^\circ$ and (F, s) is PT_0-stable. We will prove the $\mathcal{L}^{(0)}$-stability of (F, ρ, s) through Lemma 3.14. Choose a proper nonzero subspace $V' \subseteq V$ and let $F' := \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ be the induced subsheaf. Then we have a commutative square

$$\begin{array}{ccc} V' & \hookrightarrow & \Gamma(X, F'(m)) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\cong} & \Gamma(X, F(m)) \end{array} \tag{21}$$

We first prove the inequality (18) in Lemma 3.14(1). Note that the collection

$$\{F' = \operatorname{im}(V' \otimes \mathcal{O}_X(-m) \xrightarrow{\rho} F) : (F, \rho) \in \mathcal{Q}^\circ \text{ and } V' \subseteq V\}$$

for fixed X, v, m is bounded by [Grothendieck 1966b, proposition 1.2(i)]. Hence the set $\{P_{F'}(t)\}$ of the corresponding Hilbert polynomials is finite. Since $F' \in \operatorname{Coh}_{\geq 1}(X)$ by PT_0-stability and $\chi(F')$ has a lower bound, independent of the choices of F, V', l, we have an inequality

$$\frac{P_{F'}(l)}{\dim(V')} > \frac{l + \chi(F')}{\dim(V)} > \frac{\epsilon \cdot l}{\dim(V)}$$

for sufficiently large l. This proves (18).

We then prove the second inequality (19) in Lemma 3.14(2) when $s \in V' \otimes R_m^*$. Indeed, consider the canonical commutative diagram

$$\begin{array}{ccccc} \Gamma(X,F') & \longrightarrow & \Gamma(X,F) & \longrightarrow & \Gamma(X,F/F') \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(X,F'(m))\otimes R_m^* & \longrightarrow & \Gamma(X,F(m))\otimes R_m^* & \longrightarrow & \Gamma(X,F/F'(m))\otimes R_m^* \end{array} \tag{22}$$

induced by the short exact sequence $0 \to F' \to F \to F/F' \to 0$. By (21), we have $s \in \Gamma(X, F'(m)) \otimes R_m^*$. By a diagram chasing argument, we can deduce that the left square in (22) is cartesian. Hence $s \in \Gamma(X, F')$, and $\mathcal{O}_Z := \mathrm{im}(s : \mathcal{O}_X \to F) \subseteq F'$ as subsheaves of F. Since (F, s) is PT_0-stable, $\alpha_2(F') = \alpha_2(F)$ and $\alpha_1(F') = \alpha_1(F)$. Since $\chi(F')$ has a lower bound, independent of the choices of F, V', l, we have

$$\frac{\epsilon \cdot l + \chi(F') - \chi(F)}{\dim(V')} > \frac{\epsilon \cdot l}{\dim(V)}$$

for sufficiently large l. This proves (19). □

By [Grothendieck 1966b; Le Potier 1993], the moduli spaces $\mathscr{P}air^{(-1)}(X, v)$ and $\mathscr{P}air^{(1)}(X, v)$ are projective schemes (see Remark 3.8). We will use these results to prove Proposition 3.17 for the DT-stability and the PT_1-stability.[8]

Proof of Proposition 3.17 for DT-stability (2) Assume that $(F, \rho) \in \mathcal{Q}^\circ$ and (F, s) is a DT pair. Choose a proper nonzero subspace $V' \subseteq V$ and let $F' = \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ be the induced subsheaf.

We first prove the inequality (18). Note that the leading coefficient of the right-hand side of (18), as a polynomial of l, is negative. On the other hand, the leading coefficient of the left-hand side of (18) is always positive. Since the collection $\{F'\}$ for fixed X, v, m is bounded by [Grothendieck 1966b, proposition 1.2(i)], the set $\{P_{F'}(t)\}$ is finite. Hence we have (18) for sufficiently large l.

We then prove the second inequality (19) when $s \in V' \otimes R_m^*$. Indeed, we can form the commutative square (21) and the commutative diagram (22) where the left square is cartesian. Hence we have $s \in \Gamma(X, F')$. Since (F, s) is a DT pair, the coherent sheaf F is generated by the section s. Therefore $F' = F$. We claim that $V' = V$, which then leads to a contradiction. Indeed, since $s \in V' \otimes R_m^*$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} \cong \Gamma(X,\mathcal{O}_X) & \longrightarrow & \Gamma(X,F) \\ \vdots & & \downarrow \\ V'\otimes R_m^* & \hookrightarrow & \Gamma(X,F(m))\otimes R_m^* \end{array}$$

for some dotted arrow.

[8] Proposition 3.17 for the PT_1-stability is not a consequence of the results in [Le Potier 1993], since the parameter space $\mathcal{M}$ in this paper is *larger* than the parameter space in [Le Potier 1993] (see Remark 3.12).

By adjunction, we can form a commutative diagram

$$\begin{array}{ccc} R_m & \longrightarrow & \Gamma(X,F)\otimes R_m \\ \vdots & & \downarrow \\ V' & \hookrightarrow & \Gamma(X,F(m)) \end{array}$$

The collection $\{\mathcal{I}_{Z/X} : \mathrm{ch}(\mathcal{O}_Z) = v\}$ is bounded, so for m sufficiently large (with lower bound only depending on $X\, v$), we may assume that the map

$$R_m = \Gamma(X, \mathcal{O}_X(m)) \to \Gamma(X, F(m))$$

is surjective. Therefore, $V' \hookrightarrow V = \Gamma(X, F(m))$ is also surjective, which proves the claim.

(1) We first claim that

$$(F,\rho,s) \in \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(-1)})|_{\mathcal{Q}^\circ} \implies (F,s) \text{ is a DT pair}$$

for sufficiently large m and l. Indeed, choose $(F,\rho,s) \in \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(-1)})|_{\mathcal{Q}^\circ}$, and let Z be the support of s and Q be the cokernel of s. Let $V' = \Gamma(X, \mathcal{O}_Z(m)) \subseteq V \cong \Gamma(X, F(m))$ and let $F' = \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq \mathcal{O}_Z \subseteq F$. Then clearly $s \in V' \otimes R_m^*$. Thus we have the inequality (19). Also note

$$-\frac{a}{\dim(V')} < -\frac{a}{\dim(V)}. \tag{23}$$

As above, for fixed X, v, m the collection of $\{F'\}$ is bounded, so there are finitely many $\{P_{F'}(t)\}$. Taking l sufficiently large, (19) contradicts (23). Hence $V' = V$ and thus $F' = F$. Therefore $\mathcal{O}_Z = F$ and (F,s) is a DT pair.

By the results in (2) and the previous paragraph, we have open embeddings

$$[\mathcal{M}^{\mathrm{ss}}|_{\mathcal{Q}^\circ}(\mathcal{L}^{(-1)})/\mathrm{PGL}(V)] \subseteq \mathscr{P}air^{(-1)}(X,v) \subseteq [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(-1)})/\mathrm{PGL}(V)],$$

which induce an open dense embedding

$$\mathscr{P}air^{(-1)}(X,v) \hookrightarrow \mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(-1)})/\!/\mathrm{PGL}(V) \hookrightarrow \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(-1)})/\!/\mathrm{PGL}(V).$$

Since $\mathscr{P}air^{(-1)}(X,v)$ is proper and $\mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(-1)})/\!/\mathrm{PGL}(V)$ is separated, the above open embedding is an isomorphism. Hence $\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(-1)}) = \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(-1)})$ and $\mathscr{P}air^{(-1)}(X,v) = [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(-1)})/\mathrm{PGL}(V)]$. □

Proof of Proposition 3.17 for PT_1-stability (2) Assume that $(F,\rho) \in \mathcal{Q}^\circ$ and (F,s) is a PT_1 pair. Choose a proper nonzero subspace $V' \subseteq V$ and let $F' = \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq F$ be the induced sheaf.

We first prove the inequality (18). Indeed, since F is pure, the subsheaf F' is also 2-dimensional. Since $\delta < \frac{1}{2}$ and $\dim(V') < \dim(V)$, we have an inequality

$$\frac{\alpha_2(F')}{2\dim(V')} > \frac{\delta}{\dim(V)}$$

between the leading coefficients of both sides of (18), as polynomials of l. For fixed X, v, m, the collection $\{F'\}$ is bounded by [Grothendieck 1966b, proposition 1.2(i)], and hence the set $\{P_{F'}(t)\}$ is finite. Therefore we have (18) for sufficiently large l.

We then prove the inequality (19) when $s \in V' \otimes R_m^*$. Indeed, we can form the commutative square (21). We can also form the commutative diagram (22), where the left square is cartesian. Hence, $s \in \Gamma(X, F')$. Since (F, s) is a PT_1 pair, $\alpha_2(F) = \alpha_2(F')$. Since $\dim(V') < \dim(V)$, we have an inequality

$$\frac{\delta}{\dim(V')} > \frac{\delta}{\dim(V)}$$

between the leading coefficients of both sides of (19), as polynomials of l. Since the collection $\{F'\}$ is bounded, we have (19) for sufficiently large l.

(1) We will show that

$$(F, \rho, s) \in \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(1)})|_{\mathcal{Q}^\circ} \implies (F, s) \text{ is a } \mathrm{PT}_1 \text{ pair} \tag{24}$$

for sufficiently large m and l. Choose $(F, \rho, s) \in \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(1)})|_{\mathcal{Q}^\circ}$. Let $Q = \operatorname{coker}(s : \mathcal{O}_X \to F)$ be the cokernel and $\mathcal{O}_S = \operatorname{im}(s : \mathcal{O}_X \to F)$ the image.

We first claim that $Q \in \mathrm{Coh}_{\leq 1}(X)$. Let $V' = \Gamma(X, \mathcal{O}_S(m)) \subseteq \Gamma(X, F(m)) \cong V$ and let $F' = \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq \mathcal{O}_S \subseteq F$. Then clearly $s \in V' \otimes R_m^*$ and in particular $V' \neq 0$. Hence we have inequality (19). If $\alpha_2(F') < \alpha_2(F)$, then the leading term of the left-hand side of (19) is negative since $\delta < \frac{1}{2}$. Since the collection $\{F'\}$ is bounded by [Grothendieck 1966b, proposition 1.2(i)] for fixed X, v, m, this leads to a contradiction for sufficiently large l. Hence $\alpha_2(F') = \alpha_2(F)$ and thus $\alpha_2(\mathcal{O}_S) = \alpha_2(F)$. This proves the claim.

We then claim that F is pure. Consider the torsion subsheaf $TF \subseteq F$. Let $V' = \Gamma(X, TF(m)) \subseteq V$ and let $F' = \rho(V' \otimes \mathcal{O}_X(-m)) \subseteq TF \subseteq F$. If $V' \neq 0$, then we have the inequality (18). Since the degree of the left-hand side of (18), as a polynomial in l, is ≤ 1 and the degree of the right-hand side of (18) is 2, and the collection $\{F'\}$ is bounded, the inequality (18) leads to a contradiction for sufficiently large l. Hence we deduce that

$$\Gamma(X, TF(m)) = 0. \tag{25}$$

Consider the short exact sequence

$$0 \to TF \to F \to F/TF \to 0.$$

Define $\mathcal{O}_Z = \operatorname{im}(\mathcal{O}_S \to F \to F/TF)$. Then we obtain a short exact sequence

$$0 \to \mathcal{O}_Z \to F/TF \to R \to 0, \tag{26}$$

where $R = \operatorname{coker}(\mathcal{O}_Z \to F/TF) \in \mathrm{Coh}_{\leq 1}(X)$. Then both F/TF and $\mathcal{O}_Z$ are pure 2-dimensional sheaves such that $\alpha_2(F/TF) = \alpha_2(\mathcal{O}_Z) = \alpha_2(F)$. In particular, (26) defines a PT_1 pair. Moreover, $\alpha_1(F/TF)$ is bounded above by $\alpha_1(F)$ and bounded below $\alpha_1(\mathcal{O}_Z)$, which is bounded below by a constant depending

on $X, \alpha_2(F)$ but not on m by [Le Potier 1993, corollaire 2.13]. In particular, $\alpha_1(F/TF)$ takes finitely many values. By Proposition 2.14(2), $\chi(F/TF)$ is bounded above by a constant depending on X, v but not on m. Since $\chi(TF) = \chi(F) - \chi(F/TF)$, we have

$$\chi(TF) \geq C(X, v) \tag{27}$$

for a constant $C(X, v)$ depending on X, v but independent of m. By the Riemann–Roch formula, we have

$$\chi(TF(m)) = r \cdot m + \chi(TF),$$

where $r = \alpha_1(F) - \alpha_1(F/TF) \geq 0$ since $TF \in \mathrm{Coh}_{\leq 1}(X)$. Moreover, $r \geq 0$ takes finitely many values independent of m. We deduce from (25) and (27) that, for all F as above, we have $r = 0$ for sufficiently large m, which means TF is 0-dimensional. Hence TF is globally generated and thus (25) implies $TF = 0$. Therefore F is pure.

Combining the results in (2) and (24), we have open embeddings

$$[\mathcal{M}^{\mathrm{ss}}|_{\mathcal{Q}^\circ}(\mathcal{L}^{(1)})/\mathrm{PGL}(V)] \subseteq \mathscr{P}air^{(1)}(X, v) \subseteq [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(1)})/\mathrm{PGL}(V)],$$

which induce an open dense embedding

$$\mathscr{P}air^{(1)}(X, v) \hookrightarrow \mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(1)})/\!\!/\mathrm{PGL}(V) \hookrightarrow \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(1)})/\!\!/\mathrm{PGL}(V).$$

Since $\mathscr{P}air^{(1)}(X, v)$ is proper and $\mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(1)})/\!\!/\mathrm{PGL}(V)$ is separated, the above map is an isomorphism. Consequently, we have $\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(1)}) = \mathcal{M}^{\mathrm{ss}}(\mathcal{L}^{(1)})$ and $\mathscr{P}air^{(1)}(X, v) = [\mathcal{M}^{\mathrm{st}}(\mathcal{L}^{(1)})/\mathrm{PGL}(V)]$. □

Theorem 3.10 is a direct corollary of Proposition 3.17:

Proof of Theorem 3.10 Since the stacks

$$\mathscr{P}air^{(q)}(X, v), \quad [\mathcal{M}^{\mathrm{st}}(X, \mathcal{L}^{(q)})/\mathrm{PGL}(V)], \quad [\mathcal{M}^{\mathrm{ss}}(X, \mathcal{L}^{(q)})/\mathrm{PGL}(V)]$$

are open substacks of $\mathscr{P}air(X, v)_m = [\mathcal{M}|_{\mathcal{Q}^\circ}/\mathrm{PGL}(V)] = [\mathbb{P}_{\mathcal{Q}^\circ}(R\pi_*\mathbb{F})/\mathrm{PGL}(V)]$, it suffices to compare the $\mathbb{C}$-points. Proposition 3.17 completes the proof. □

We end this subsection with the following question:

Question 3.18 Does Theorem 3.10 hold for d-dimensional PT_q pairs with $d > 2$ as defined in Remark 2.5?

3.3 Moduli of complexes

In this subsection, we prove that the moduli spaces of PT_q pairs are open subschemes of the moduli space of perfect complexes. We basically follow the reasoning in [Pandharipande and Thomas 2009b], but the actual proof requires a finer argument.

Let X be a smooth projective variety and let $v \in H^*(X, \mathbb{Q})$. Recall [Inaba 2002; Lieblich 2006] that there exists a moduli space

$$\mathscr{P}erf(X, v)^{\mathrm{spl}}_{\mathcal{O}_X}$$

of simple perfect complexes $I^\bullet$ on X with $\det(I^\bullet) \cong \mathcal{O}_X$ and $\mathrm{ch}(I^\bullet) = v$, as an algebraic space.

Theorem 3.19 *Let X be a smooth projective variety of dimension $n \geq 4$, $v \in H^*(X, \mathbb{Q})$, and $q \in \{-1, 0, 1\}$. Then we have an open embedding*

$$\mathscr{P}air^{(q)}(X, v) \hookrightarrow \mathscr{P}erf(X, v')^{\mathrm{spl}}_{\mathcal{O}_X}, \quad (F, s) \mapsto I^\bullet := [\mathcal{O}_X \to F], \tag{28}$$

where $v' := 1 - v \in H^*(X, \mathbb{Q})$.

The rest of this subsection is devoted to the proof of Theorem 3.19. Throughout this subsection, we use the notation in Theorem 3.19.

To any PT_q pair (F, s) on X, we can associate a perfect complex

$$I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$$

with $\det(I^\bullet) \cong \mathcal{O}_X$. We first show that $I^\bullet$ is *simple*.

Lemma 3.20 *Given a PT_q pair (F, s) on X, we have*

$$\mathrm{Ext}^{<0}_X(I^\bullet, I^\bullet) = 0 \quad \textit{and} \quad \mathrm{Hom}_X(I^\bullet, I^\bullet) = \mathbb{C}$$

for the associated perfect complex $I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$.

Proof The proof is analogous to [Pandharipande and Thomas 2009b, Lemma 1.15]. Consider the two canonical distinguished triangles

$$I^\bullet \to \mathcal{O}_X \to F \quad \text{and} \quad \mathcal{I}_Z \to I^\bullet \to Q[-1],$$

where Z is the support of F and Q is the cokernel of s. We can form the three induced distinguished triangles

$$R\,\mathrm{Hom}_X(F, \mathcal{O}_X) \to R\,\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \to R\,\mathrm{Hom}_X(I^\bullet, \mathcal{O}_X), \tag{29}$$

$$R\,\mathrm{Hom}_X(Q, F)[1] \to R\,\mathrm{Hom}_X(I^\bullet, F) \to R\,\mathrm{Hom}_X(\mathcal{I}_Z, F), \tag{30}$$

$$R\,\mathrm{Hom}_X(I^\bullet, I^\bullet) \to R\,\mathrm{Hom}_X(I^\bullet, \mathcal{O}_X) \to R\,\mathrm{Hom}_X(I^\bullet, F). \tag{31}$$

The distinguished triangle (29) implies

$$\mathrm{Hom}_X(I^\bullet, \mathcal{O}_X) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^{<0}_X(I^\bullet, \mathcal{O}_X) = 0 \tag{32}$$

since $\mathrm{Ext}^{\leq 1}_X(F, \mathcal{O}_X) = H^{\geq n-1}(X, F)^* = 0$. On the other hand, the distinguished triangle (30) implies

$$\mathrm{Ext}^{<0}_X(I^\bullet, F) = 0 \tag{33}$$

since $\mathrm{Hom}_X(Q, F) = 0$ by PT_q-stability. Then the distinguished triangle (31) and equations (32) and (33) prove that $I^\bullet$ is simple. □

Since $\mathscr{P}air^{(q)}(X, v)$ is a *fine* moduli space, there exists a universal family

$$\mathbb{I}^\bullet = [\mathcal{O}_{\mathscr{P}air^{(q)}(X,v)\times X} \to \mathbb{F}],$$

which defines a map

$$\mathscr{P}air^{(q)}(X,v) \xrightarrow{\mathbb{I}^\bullet} \mathscr{P}erf(X,v)^{\mathrm{spl}}_{\mathcal{O}_X} \tag{34}$$

between the moduli spaces by Lemma 3.21 below.

Lemma 3.21 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth projective morphism of schemes. Let $\mathcal{F}$ be a $\mathcal{Y}$-flat coherent sheaf on $\mathcal{X}$. If $\operatorname{codim}(\operatorname{Supp}(\mathcal{F}_y)) \geq 2$ for all $y \in \mathcal{Y}$, then $\det(\mathcal{F}) \cong \mathcal{O}_{\mathcal{X}}$.*

Proof This follows directly from [Hassett and Kovács 2004, Proposition 3.5]. □

We show that the above map (34) satisfies the *infinitesimal lifting property*.

Proposition 3.22 *Let $B_0 \subseteq B$ be an nilpotent extension of 0-dimensional schemes with a unique closed point $b \in B_0$. Let $I^\bullet$ be a perfect complex on $X \times B$ with trivial determinant. Let $i : X \times B_0 \hookrightarrow X \times B$ be the inclusion map. If $I^\bullet|_{X\times B_0} \cong [\mathcal{O}_{X\times B_0} \xrightarrow{s_0} F_0]$ for a B_0-flat family (F_0, s_0) of PT_q pairs over B_0, then there exists a B-flat family (F, s) of PT_q pairs over B such that $I^\bullet \cong [\mathcal{O}_{X\times B} \xrightarrow{s} F]$ and $(F, s)|_{X\times B_0} = (F_0, s_0)$.*

Proof We first take $q = 0, 1$. The (easier) case $q = -1$ is discussed at the end of the proof. Let (F_b, s_b) be the fibre of (F_0, s_0) over the point $b \in B_0$. Since $F_b \in \mathrm{Coh}_{\geq 1}(X)$, we have $\operatorname{depth}_{\mathcal{O}_{X,x}}((F_b)_x) \geq 1$ for all closed points $x \in \operatorname{Supp}(F_b)$. By the Auslander–Buchsbaum formula, we have

$$\operatorname{pd}_{\mathcal{O}_{X,x}}((F_b)_x) \leq n - 1$$

for all $x \in X$. Then $I^\bullet|_{X\times\{b\}} \cong [\mathcal{O}_X \xrightarrow{s_b} F_b]$ has tor-amplitude $[-(n-2), 1]$, and thus we have a quasi-isomorphism with a complex of locally free sheaves

$$I^\bullet|_{X\times\{b\}} \cong [F^{-(n-2)} \to \cdots \to F^0 \to F^1]. \tag{35}$$

Since $X \times B$ is quasiprojective, $I^\bullet$ is quasi-isomorphic to a bounded complex of locally free sheaves. Using Nakayama's lemma, and the fact that a morphism between locally free sheaves is injective on the fibres if and only if it is injective with locally free cokernel, we see from (35) that there exists a quasi-isomorphism with a complex of locally free sheaves

$$I^\bullet \cong [E^{-(n-2)} \to \cdots \to E^0 \to E^1].$$

Moreover, since $I^\bullet$ has amplitude $[0, 1]$ by [Lieblich 2006, Lemma 2.1.4], we have

$$I^\bullet \cong [A \to E^1],$$

where $A = \operatorname{coker}(E^{-1} \to E^0)$ is B-flat.

We claim that the canonical map

$$h^0(I^\bullet) \to h^0(I^\bullet)^{**}$$

is injective. Indeed, let K be the kernel of the above map $h^0(I^\bullet) \to h^0(I^\bullet)^{**}$. Note that $h^0(I^\bullet)$ is a vector bundle away from the support of F_0 by [Lieblich 2006, Lemma 2.1.4]. Hence $\dim(K) \leq 2 < n$.

The support W of $h^1(I^\bullet)$ is of dimension $\leq q$. This follows from the fact that $i^*h^1(I^\bullet) \cong h^1(I_0^\bullet)$ and Nakayama's lemma. Away from W, the coherent sheaf $h^0(I^\bullet)$ is B-flat because A is B-flat. Since the ideal sheaf $h^0(I_b^\bullet)$ is torsion-free, all nonzero subsheaves of $h^0(I^\bullet)|_{X\times B\setminus W}$ are n-dimensional by Lemma 3.23 below. Therefore K is (set-theoretically) supported in W. Thus we have $\dim(K) \leq \dim(W) \leq q$.

Consider the local-to-global spectral sequence

$$H^i(X\times B, \mathscr{E}xt^j_{X\times B}(A, K\otimes\omega_X)) \Rightarrow \mathrm{Ext}^{i+j}_{X\times B}(A, K\otimes\omega_X).$$

Since $\dim(K\otimes\omega_X) \leq q \leq 1$ and there is a resolution $A \cong [E^{-(n-2)} \to \cdots \to E^0]$ of length $n-2$, we have

$$\mathrm{Ext}^n_{X\times B}(A, K\otimes\omega_X) = 0.$$

By the Serre duality $R\mathcal{H}om_\pi(K, A) \cong R\mathcal{H}om_\pi(A, K\otimes\omega_X))^\vee[-n]$, we have

$$\mathrm{Hom}_{X\times B}(K, A)^* \cong \mathrm{Ext}^n_{X\times B}(A, K\otimes\omega_X) = 0,$$

where the dual is taken as a $\Gamma(B, \mathcal{O}_B)$-module. Since $K \subseteq A$, we deduce $K = 0$. This proves the claim.

Since the rest of the proof is exactly the same as [Pandharipande and Thomas 2009b], we will only sketch the remaining arguments. By [Kollár 1990, Lemma 6.13], $h^0(I^\bullet)^{**}$ is a vector bundle away from the support of $Q = h^1(I^\bullet)$. In the proof of [loc. cit.], it is also shown that a reflexive rank 1 sheaf which is locally free outside a subset of codimension $n-1 \geq 3$ is in fact locally free everywhere. Therefore $h^0(I^\bullet)^{**}$ is locally free.

Next, we have $\det(I^\bullet) \cong \mathcal{O}_{X\times B}$ and $W \subset \mathrm{Supp}(F_0)$; thus,

$$h^0(I^\bullet)^{**}|_{(X\times B)\setminus \mathrm{Supp}(F_0)} \cong h^0(I^\bullet)|_{(X\times B)\setminus \mathrm{Supp}(F_0)} \cong \mathcal{O}_{(X\times B)\setminus \mathrm{Supp}(F_0)}.$$

Since $\mathrm{Supp}(F_0)$ has codimension $n-2 \geq 2$, we find $h^0(I^\bullet)^{**} \cong \mathcal{O}_{X\times B}$ globally and $\mathcal{I}_Z \cong h^0(I^\bullet)$ is an ideal sheaf of a closed subscheme $Z \subseteq X\times B$. The distinguished triangle

$$\mathcal{I}_Z \to I^\bullet \to Q[-1]$$

defines an element

$$\alpha \in \mathrm{Ext}^2_{X\times B}(Q, \mathcal{I}_Z) \cong \mathrm{Ext}^1_{X\times B}(Q, \mathcal{O}_Z)$$

since $\mathrm{Ext}^1_{X\times B}(Q, \mathcal{O}_{X\times B}) = \mathrm{Ext}^2_{X\times B}(Q, \mathcal{O}_{X\times B}) = 0$ (recall $q \leq 1$ and $n \geq 4$). Hence we have a short exact sequence

$$0 \to \mathcal{O}_Z \to F \to Q \to 0$$

of coherent sheaves on $X\times B$. Therefore we have an isomorphism

$$I^\bullet \cong [\mathcal{O}_{X\times B} \xrightarrow{s} F]$$

of perfect complexes. Since F and F_0 are 2-dimensional, $\mathrm{Hom}_{X\times B}(I^\bullet, \mathcal{O}_{X\times B}) \cong \Gamma(X\times B, \mathcal{O}_{X\times B}) \cong \Gamma(B, \mathcal{O}_B)$ and $\mathrm{Hom}_{X\times B_0}(I_0^\bullet, \mathcal{O}_{X\times B_0}) \cong \Gamma(B_0, \mathcal{O}_{B_0})$. As in [Pandharipande and Thomas 2009b], we therefore have $Li^*F \cong F_0$. Hence F is flat over B by [ibid., Lemma 2.5], which completes the proof.

For $q=-1$, the complex $I^\bullet$ is only quasi-isomorphic to a bounded complex of locally free sheaves $[E^{-(n-1)}\to\cdots\to E^1]$. However, this time $W=\varnothing$, so $h^0(I^\bullet)\to h^0(I^\bullet)^{**}$ is obviously injective (Lemma 3.23). The rest of the proof proceeds as in the previous paragraph. □

We need the following lemma to complete the proof of Proposition 3.22:

Lemma 3.23 *Let $(A,\mathfrak{m})$ be a local Artinian $\mathbb{C}$-algebra and let X be a quasiprojective scheme. Consider an A-flat family F of sheaves on $X\times\operatorname{Spec}(A)$ whose fibre $F_0=F\otimes_A A/\mathfrak{m}$ is a pure sheaf. Then F is also a pure sheaf, i.e., for all nonzero subsheaves $G\subseteq F$, we have $\dim(G)=\dim(F)$.*

Proof Since A is Artinian, we may take a sequence of ideals [Eisenbud 1995, Proposition 3.7]

$$0=\mathfrak{m}_N\subset\cdots\subset\mathfrak{m}_2\subset\mathfrak{m}_1=\mathfrak{m}\quad\text{such that }\ \mathfrak{m}_i/\mathfrak{m}_{i+1}\cong A/\mathfrak{m}.$$

We let $\mathfrak{m}_0=A$. Suppose we have a subsheaf $G\subset F$ such that $\dim(G)<\dim(F)$. Since F is flat over A, we have a short exact sequence

$$0\to\mathfrak{m}_{i+1}\cdot F\to\mathfrak{m}_i\cdot F\to F_0\to 0 \tag{36}$$

of sheaves on $X\times\operatorname{Spec}(A)$, where $\mathfrak{m}_i\otimes_A F=\mathfrak{m}_i\cdot F$. Consider the induced morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G\cap\mathfrak{m}\cdot F & \longrightarrow & G & \longrightarrow & \dfrac{G}{G\cap\mathfrak{m}\cdot F} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{m}\cdot F & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & 0
\end{array}$$

of short exact sequences. Note that

$$\dim\Big(\frac{G}{G\cap\mathfrak{m}\cdot F}\Big)\le\dim(G)<\dim(F)=\dim(F_0).$$

Since F_0 is pure, we have $G/(G\cap\mathfrak{m}\cdot F)=0$. Therefore we deduce that

$$G\subseteq\mathfrak{m}\cdot F.$$

Moreover, applying this argument to (36), we deduce that

$$G\subseteq\mathfrak{m}_i\cdot F\implies G\subseteq\mathfrak{m}_{i+1}\cdot F.$$

Therefore we obtain $G\subseteq\mathfrak{m}_N\cdot F=0$ by induction on i. □

Now we can simply deduce Theorem 3.19 from Lemma 2.8 and Proposition 3.22 as follows:

Proof of Theorem 3.19 By Lemma 2.8, the map

$$\mathscr{P}air^{(q)}(X,v)\xrightarrow{\mathbb{I}^\bullet}\mathscr{P}erf(X,v)^{\mathrm{spl}}_{\mathcal{O}_X}$$

in (34) is injective on $\mathbb{C}$-points. Since $\mathbb{C}$ is algebraically closed, the above map is universally injective. In order to verify that the above map is formally smooth, it is enough to check the lifting property along extensions of local Artinian $\mathbb{C}$-algebras [Stacks 2005–, Tag 02HW] which is done in Proposition 3.22.

Since $\mathscr{P}air^{(q)}(X,v)$ is of finite type, the above map is étale. Moreover, the above map is universally injective and étale; thus, it is an open embedding [Stacks 2005–, Tag 02LC]. □

Remark 3.24 The open embedding (28) in Theorem 3.19 induces a derived enhancement of the moduli space $\mathscr{P}air^{(q)}(X,v)$ of PT_q pairs. The tangent complex at $(F,s)\in\mathscr{P}air^{(q)}(X,v)$ is given as

$$\mathbb{T}|_{(F,s)} = R\,\mathrm{Hom}_X(I^\bullet, I^\bullet)_0[1]$$

where $I^\bullet=[\mathcal{O}_X \xrightarrow{s} F]$. This derived structure (of perfect complexes) is *different* from the derived structure (of pairs) in Remark 3.9.

Remark 3.25 For PT_1-stability, Theorem 3.19 was independently proved in [Gholampour et al. 2024, Proposition 2.18] using [Lo 2011, Lemma 5.12].

Remark 3.26 In the proof of Proposition 3.22 we only used the facts that $T_0(F)=0$ and $Q\in\mathrm{Coh}_{\leq 1}(X)$, and the statement can be generalized as follows. Let $\mathscr{P}erf(X)_{\mathcal{O}_X}$ be the moduli stack of universally glueable perfect complexes $I^\bullet$ on X with $\det(I^\bullet)\cong\mathcal{O}_X$, which is an algebraic stack locally of finite type over $\mathbb{C}$ [Lieblich 2006, Theorem 4.2.1]. By the proof of Lemma 3.7, we have an open substack

$$\mathscr{P}air^{[0,1]}(X) := \{(F,s)\in\mathscr{P}air(X) : T_0(F)=0 \text{ and } Q\in\mathrm{Coh}_{\leq 1}(X)\}$$

inside $\mathscr{P}air(X)$. For $q=0,1$, we have

$$\begin{array}{ccccc} \mathscr{P}air^{(q)}(X) & \hookrightarrow & \mathscr{P}air^{[0,1]}(X) & \hookrightarrow & \mathscr{P}air(X) \\ & \searrow & \downarrow{\scriptstyle \epsilon} & & \\ & & \mathscr{P}erf(X)_{\mathcal{O}_X} & & \end{array}$$

where the horizontal morphisms are open embeddings and the morphism ϵ is formally smooth by Proposition 3.22. We claim that the map

$$\epsilon : \mathscr{P}air^{[0,1]}(X) \to \mathscr{P}erf(X)_{\mathcal{O}_X}$$

is an open embedding. Indeed, it suffices to show that the induced map $\epsilon(\mathbb{C})$ on the groupoids of $\mathbb{C}$-valued points is fully faithful. By the same argument as for Lemma 2.8, the induced map $\pi_0(\epsilon(\mathbb{C}))$ on the sets of isomorphism classes is injective. By (31) (in fact, by (78)), we have an exact sequence

$$0\to \mathrm{Ext}_X^{-1}(I^\bullet,F)\to \mathrm{Hom}_X(I^\bullet,I^\bullet)_0\to \mathrm{Ext}_X^1(F,\mathcal{O}_X)=0$$

for $(F,s)\in\mathscr{P}air^{[0,1]}(X)$ and hence $\epsilon(\mathbb{C})$ preserves stabilizers. This proves the claim.

3.4 Polynomial Bridgeland stability

In this subsection, we prove that the PT_q-stability conditions can be realized as Bayer's polynomial Bridgeland stability conditions [Bridgeland 2007; Bayer 2009] on the derived category. The results of this subsection, which are inspired by [Gholampour et al. 2024], are not used elsewhere in the paper. We

use a different torsion pair than [loc. cit.], which allows us to incorporate 2-dimensional (Gieseker) stable sheaves as well.

Let X be a smooth projective variety of dimension $n \geq 4$. Let $D^b_{\mathrm{coh}}(X)$ be the bounded derived category of coherent sheaves on X. We consider the perversity function

$$p : \{0, 1, \ldots, n\} \to \mathbb{Z}, \quad p(0) = p(1) = p(2) = 0, \quad p(3) = \cdots = p(n) = -1.$$

The perversity function defines a bounded t-structure on $D^b_{\mathrm{coh}}(X)$ [Arinkin and Bezrukavnikov 2010] and this t-structure coincides with the one associated to the torsion pair [Happel et al. 1996, Corollary 2.2]

$$(\mathrm{Coh}_{\leq 2}(X), \mathrm{Coh}_{\geq 3}(X)).$$

We denote the heart by

$$\mathcal{A} := \mathcal{A}^p := \langle \mathrm{Coh}_{\leq 2}(X), \mathrm{Coh}_{\geq 3}(X)[1] \rangle.$$

The abelian category $\mathcal{A}$ consists of the objects $E^\bullet \in D^{[-1,0]}_{\mathrm{coh}}(X)$ satisfying

$$h^{-1}(E^\bullet) \in \mathrm{Coh}_{\geq 3}(X) \quad \text{and} \quad h^0(E^\bullet) \in \mathrm{Coh}_{\leq 2}(X).$$

A sequence $0 \to A^\bullet \to E^\bullet \to B^\bullet \to 0$ is a short exact sequence in $\mathcal{A}$ if and only if the associated sequence in $D^b_{\mathrm{coh}}(X)$ is a distinguished triangle. For any $E^\bullet \in \mathcal{A}$, the following sequence is exact in $\mathcal{A}$:

$$0 \to h^{-1}(E^\bullet)[1] \to E^\bullet \to h^0(E^\bullet) \to 0.$$

For any PT_q pair $I^\bullet$ on X, we have $I^\bullet[1] \in \mathcal{A}$. Throughout this subsection, we fix $v \in H^*(X, \mathbb{Q})$ such that $v_{\leq n-3} = 0$ and $v_{n-2} \neq 0$.

Denote the semiclosed upper half-plane by

$$\mathbb{H} = \{z \in \mathbb{C} : z \in \mathbb{R}_{>0} \cdot e^{i\pi\phi(z)}, 0 < \phi(z) \leq 1\},$$

where $\phi(z)$ denotes the phase of z. Denote by $\mathbb{H}^\circ \subset \mathbb{H}$ the open upper half-plane. Recall that a Bayer stability condition [2009] — also known as a polynomial Bridgeland stability condition — is determined by a *stability vector*, i.e., a sequence

$$\rho = (\rho_0, \ldots, \rho_n) \in (\mathbb{C}^*)^{n+1}$$

such that $\rho_d / \rho_{d+1} \in \mathbb{H}^\circ$ for $0 \leq d \leq n-1$ and $(-1)^{p(d)}\rho_d \in \mathbb{H}$ for all d. We fix a polarization H of X. Define the central charge $Z \colon K^0(X) \to \mathbb{C}[x]$ as

$$Z(E^\bullet)(x) = \sum_{d=0}^{n} x^d \rho_d \int_X H^d \,\mathrm{ch}(E^\bullet)\,\mathrm{td}(X)$$

for any $E^\bullet \in K^0(X)$. For $E^\bullet \in \mathcal{A}$, let ϕ be the polynomial phase function, i.e.,

$$Z(E^\bullet)(x) \in \mathbb{R}_{>0} \cdot e^{\pi i \phi(E^\bullet)(x)}, \quad x \gg 0,$$

for some continuous function germ $\phi \in (\phi_0, \phi_0+1]$ for some ϕ_0. We say $\phi < \psi$ if and only if $\phi(x) < \psi(x)$ for all $0 \ll x < +\infty$, and similarly for $\phi \leq \psi$.

An object $E^\bullet$ in $\mathcal{A}$ is called *Bayer stable* (resp. *Bayer semistable*) with respect to the stability vector ρ if, for all proper nonzero subobjects $D^\bullet \subset E^\bullet$, we have $\phi(D^\bullet) < \phi(E^\bullet)$ (resp. $\phi(D^\bullet) \leq \phi(E^\bullet)$).

Definition 3.27 Consider stability vectors $\rho^{(q)}$ for $q \in \{-1, 0, 1\}$ satisfying the following:

(37)

$\rho^{(-1)}$	$\phi(-\rho_n) > \phi(\rho_0) > \phi(\rho_1) > \phi(\rho_2)$
$\rho^{(0)}$	$\phi(\rho_0) > \phi(-\rho_n) > \phi(\rho_1) > \phi(\rho_2)$
$\rho^{(1)}$	$\phi(\rho_0) > \phi(\rho_1) > \phi(-\rho_n) > \phi(\rho_2)$

From the condition on the stability vector, we have

$$\phi(-\rho_3), \dots, \phi(-\rho_{n-1}) > \phi(-\rho_n). \tag{38}$$

The following proposition is a minor modification of [Bayer 2009, Proposition 6.1.1]:

Proposition 3.28 *Fix $v \in H^*(X, \mathbb{Q})$ such that $v_{\leq n-3} = 0$ and $v_{n-2} \neq 0$. Let $E^\bullet \in D^b_{\mathrm{coh}}(X)$ with $\mathrm{ch}(E^\bullet) = v - 1$ and $\det(E^\bullet) \cong \mathcal{O}_X$. Let $q \in \{-1, 0, 1\}$. Then $E^\bullet$ is a $\rho^{(q)}$-semistable object of $\mathcal{A}$ if and only if $E^\bullet \cong I^\bullet[1]$ for some PT_q pair (F, s), where $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$ is the associated complex. Moreover, $E^\bullet$ cannot be strictly $\rho^{(q)}$-semistable.*

Proof Let $E^\bullet$ be a $\rho^{(q)}$-semistable object of $\mathcal{A}$ satisfying the above conditions. We first show that $h^{-1}(E^\bullet)$ is a rank 1 torsion-free sheaf. Since $h^0(E^\bullet) \in \mathrm{Coh}_{\leq 2}(X)$, $h^{-1}(E^\bullet)$ has rank 1 and $h^{-1}(E^\bullet) \in \mathrm{Coh}_{\geq 3}(X)$. Suppose $h^{-1}(E^\bullet)$ has a nontrivial torsion subsheaf. Let $T \subset h^{-1}(E^\bullet)$ be the maximal torsion subsheaf. Then $0 \to T[1] \to h^{-1}(E^\bullet)[1] \to (h^{-1}(E^\bullet)/T)[1] \to 0$ is a short exact sequence in $\mathcal{A}$ and hence $T[1]$ is a subobject of $E^\bullet$ in $\mathcal{A}$. Then (38) leads to a contradiction. Therefore $h^{-1}(E^\bullet)$ is a rank 1 torsion-free sheaf with trivial determinant. Next we show that $h^0(E^\bullet) \in \mathrm{Coh}_{\leq 1}(X)$. By assumption, $h^0(E^\bullet)$ has dimension at most 2. Because $E^\bullet$ is $\rho^{(q)}$-semistable, $h^0(E^\bullet)$ has dimension at most 1; otherwise, $E^\bullet \twoheadrightarrow h^0(E^\bullet)$ destabilizes $E^\bullet$. By a standard argument (see the last two paragraphs of the proof of Proposition 3.22), we have $E^\bullet \cong I^\bullet[1]$, $I^\bullet = [\mathcal{O}_X \to F]$ for some 2-dimensional coherent sheaf F. For $q = 1$, we show that F is pure. Let $T \subset F$ be the maximal torsion subsheaf of F. Then $0 \to T \to I^\bullet[1] \to [\mathcal{O}_X \to F/T][1] \to 0$ is a short exact sequence in $\mathcal{A}$ and hence T is a subobject of $E^\bullet$. By the stability vector (37), we have $T = 0$. For $q = 0$, a similar argument shows that F has no nontrivial 0-dimensional subsheaf and $h^0(E^\bullet)$ is 0-dimensional. For $q = -1$, we have $h^0(E^\bullet) = 0$.

Conversely, suppose $E^\bullet \cong I^\bullet[1]$ for some PT_q pair (F, s), where $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$. Then $E^\bullet \in \mathcal{A}$. We prove $E^\bullet$ is a $\rho^{(q)}$-stable object. Take any short exact sequence

$$0 \to A^\bullet \to E^\bullet \to B^\bullet \to 0$$

in $\mathcal{A}$ with $A^\bullet, B^\bullet \neq 0$. Consider its long exact sequence

$$0 \to h^{-1}(A^\bullet) \to h^{-1}(E^\bullet) \to h^{-1}(B^\bullet) \to h^0(A^\bullet) \to h^0(E^\bullet) \to h^0(B^\bullet) \to 0.$$

We distinguish three cases:

Case 1: $0 \neq h^{-1}(A^\bullet) \subsetneq h^{-1}(E^\bullet)$ Then $h^{-1}(B^\bullet)$ has rank 0. If $h^{-1}(B^\bullet) \neq 0$, then it has dimension in $\{3, \dots, n-1\}$, so we obtain $\phi(E^\bullet) < \phi(B^\bullet)$ from (38). Suppose $h^{-1}(B^\bullet) = 0$; then $B^\bullet \cong h^0(B^\bullet)$ and, since $h^0(E^\bullet)$ is at most 1-dimensional, $h^0(B^\bullet)$ has dimension ≤ 1. For $q = 1$, (37) implies that $\phi(E^\bullet) < \phi(B^\bullet)$. For $q = 0$, $h^0(B^\bullet)$ is 0-dimensional, so we also find $\phi(E^\bullet) < \phi(B^\bullet)$. Finally, for $q = -1$, we have $h^0(B^\bullet) = 0$, contradicting the assumption $B^\bullet \neq 0$.

Case 2: $h^{-1}(A^\bullet) = h^{-1}(E^\bullet)$ Then $h^{-1}(B^\bullet) \subset h^0(A^\bullet)$, so $h^{-1}(B^\bullet) = 0$ and $B^\bullet \cong h^0(B^\bullet)$. Then $\phi(E^\bullet) < \phi(B^\bullet)$ follows as in Case 1.

Case 3: $h^{-1}(A^\bullet) = 0$ Then $A^\bullet \cong h^0(A^\bullet)$ has dimension ≤ 2 and hence $\mathrm{Hom}(A^\bullet, E^\bullet) \cong \mathrm{Hom}(A^\bullet, F)$. Since $A^\bullet$ is a nonzero subobject of $E^\bullet$ and (F, s) is a PT_q pair, we deduce that $A^\bullet \cong h^0(A^\bullet)$ is 2-dimensional. Then (37) implies $\phi(A^\bullet) < \phi(E^\bullet)$. □

We characterize rank 0 $\rho^{(q)}$-semistable objects in $\mathcal{A}$.

Proposition 3.29 *Let $q \in \{-1, 0, 1\}$ and $v \in H^*(X, \mathbb{Q})$ such that $v_{\leq n-3} = 0$ and $v_{n-2} \neq 0$. An object $E^\bullet \in \mathcal{A}$ with $\mathrm{ch}(E^\bullet) = v$ is $\rho^{(q)}$-stable (resp. $\rho^{(q)}$-semistable) if and only if $E^\bullet \cong F$ for some Gieseker stable (resp. semistable) sheaf F on X with $\mathrm{ch}(F) = v$.*

Proof Let $E^\bullet \in \mathcal{A}$ be a $\rho^{(q)}$-(semi)stable object with $\mathrm{ch}(E^\bullet) = v$. Since $h^{-1}(E^\bullet) \in \mathrm{Coh}_{\geq 3}(X)$, we have $h^{-1}(E^\bullet) = 0$ and hence $E^\bullet \cong h^0(E^\bullet)$. Then $h^0(E^\bullet)$ is Gieseker (semi)stable [Bayer 2009, Section 2.1]. Conversely, it is easy to show that any Gieseker (semi)stable sheaf F on X with $\mathrm{ch}(F) = v$ determines a $\rho^{(q)}$-(semi)stable object in $\mathcal{A}$ [loc. cit.]. □

3.5 Relative version

We now generalize the main results in this section to the relative setting.

We first fix some notation. Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of schemes of relative dimension n. Consider the *algebraic de Rham complex*

$$\Omega^\bullet_{\mathcal{X}/\mathcal{B}} := [\mathcal{O}_\mathcal{X} \xrightarrow{d_{\mathrm{DR}}} \Omega^1_{\mathcal{X}/\mathcal{B}} \xrightarrow{d_{\mathrm{DR}}} \Omega^2_{\mathcal{X}/\mathcal{B}} \xrightarrow{d_{\mathrm{DR}}} \cdots \xrightarrow{d_{\mathrm{DR}}} \Omega^n_{\mathcal{X}/\mathcal{B}}],$$

as in [Grothendieck 1966a; Deligne 1968]. Define the *Hodge bundles* as

$$\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) := R^k f_*(\Omega^\bullet_{\mathcal{X}/\mathcal{B}}). \tag{39}$$

When $\mathcal{B}$ is an affine scheme, we write $H^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) := \Gamma(\mathcal{B}, \mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$.

Note that there is a natural spectral sequence

$$E_1^{pq} = R^q f_* \Omega^p_{\mathcal{X}/\mathcal{B}} \Rightarrow \mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$$

given by the stupid truncations

$$\Omega^{\geq p}_{\mathcal{X}/\mathcal{B}} := [0 \to \cdots \to 0 \to \Omega^p_{\mathcal{X}/\mathcal{B}} \xrightarrow{d_{\mathrm{DR}}} \Omega^{p+1}_{\mathcal{X}/\mathcal{B}} \xrightarrow{d_{\mathrm{DR}}} \cdots \xrightarrow{d_{\mathrm{DR}}} \Omega^n_{\mathcal{X}/\mathcal{B}}].$$

By [Deligne 1968], the spectral sequence degenerates at E_1. Hence we have a filtration

$$F^p\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) := \mathrm{im}(R^{p+q}f_*(\Omega^{\geq p}_{\mathcal{X}/\mathcal{B}}) \to R^{p+q}f_*(\Omega^{\bullet}_{\mathcal{X}/\mathcal{B}}))$$

of the Hodge bundles $\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ such that

$$F^p\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \cong R^{p+q}f_*(\Omega^{\geq p}_{\mathcal{X}/\mathcal{B}}) \quad \text{and} \quad \frac{F^p\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})}{F^{p+1}\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})} \cong R^q f_*\Omega^p_{\mathcal{X}/\mathcal{B}}. \tag{40}$$

Moreover, the Hodge bundles $\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$, the filtrations $F^p\mathcal{H}^{p+q}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$, and the quotients $R^q f_*\Omega^p_{\mathcal{X}/\mathcal{B}}$ are vector bundles and stable under base change, by [ibid., théorème 5.5].

Consider the *Gauss–Manin connection*

$$\nabla : \mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \to \mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \otimes \Omega^1_{\mathcal{B}}.$$

As in [Katz and Oda 1968], the Gauss–Manin connection ∇ can be defined via the canonical filtration of $\bigwedge^p \Omega^1_{\mathcal{X}}$ given by the short exact sequence

$$0 \to f^*\Omega^1_{\mathcal{B}} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\mathcal{B}} \to 0.$$

Indeed, the first two terms of the filtration give us a map

$$\Omega^p_{\mathcal{X}/\mathcal{B}} \to \Omega^{p-1}_{\mathcal{X}/\mathcal{B}} \otimes f^*\Omega^1_{\mathcal{B}}[1].$$

Then, by gathering the above maps for all p and taking the derived pushforward $R^q f_*$, we obtain a flat connection ∇ called the Gauss–Manin connection.

Following [Bloch 1972], we define the notion of *horizontal sections* of Hodge bundles.

Definition 3.30 (see [Bloch 1972, Remark 3.9]) Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of schemes of constant relative dimension. We say that a section $\tilde{v} \in \Gamma(\mathcal{B}, \mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$ is *horizontal* if

$$\nabla(\tilde{v}) = 0 \in \Gamma(\mathcal{B}, \mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \otimes \Omega^1_{\mathcal{B}}).$$

Roughly speaking, the horizontal sections $\tilde{v}$ of $\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ are locally constant families of cohomology classes $\tilde{v}_b \in H^*(\mathcal{X}_b, \mathbb{C})$ on the fibres $\mathcal{X}_b := \mathcal{X} \times_{\mathcal{B}} \{b\}$. The following remark justifies this interpretation:

Remark 3.31 Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of schemes of constant relative dimension.

(1) By considering the associated analytic spaces, we have

$$\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})^{\mathrm{an}} \cong R^k f_*^{\mathrm{an}}\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}^{\mathrm{an}}} \quad \text{and} \quad \nabla^{\mathrm{an}} \cong \mathrm{id} \otimes d_{\mathcal{B}^{\mathrm{an}}}$$

when $\mathcal{B}$ is smooth [Katz and Oda 1968]. Thus, a section $\tilde{v}$ of $\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ is horizontal if and only if $\tilde{v}^{\mathrm{an}} = v \otimes 1$ for some $v \in \Gamma(\mathcal{B}, R^k f_*^{\mathrm{an}}\mathbb{C})$.

(2) (a) Firstly, assume that $\mathcal{B} = \mathrm{Spec}(A)$ for a local Artinian $\mathbb{C}$-algebra A. Then we have isomorphisms

$$H^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \cong H^k_{\mathrm{DR}}(X) \otimes_{\mathbb{C}} A$$

by [Bloch 1972, Proposition 3.8], where X is the fibre of $f:\mathcal{X}\to\mathcal{B}$ over the unique closed point and $H^k_{\mathrm{DR}}(X):=H^k_{\mathrm{DR}}(X/\mathrm{Spec}(\mathbb{C}))$. Then the horizontal sections are exactly the sections of the form

$$v\otimes 1\in H^k_{\mathrm{DR}}(X)\otimes_{\mathbb{C}} A$$

for some $v\in H^k_{\mathrm{DR}}(X)\cong H^k(X,\mathbb{C})$.

(b) Secondly, for general $\mathcal{B}$, a section $\tilde{v}\in\Gamma(\mathcal{B},\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$ is horizontal if and only if its pullbacks

$$\tilde{v}|_{\mathcal{A}}\in\Gamma(\mathcal{A},\mathcal{H}^k_{\mathrm{DR}}(\mathcal{X}_{\mathcal{A}}/\mathcal{A}))$$

are horizontal for all morphisms $\mathcal{A}=\mathrm{Spec}(A)\to\mathcal{B}$ from local Artinian $\mathbb{C}$-algebras A. Here $\mathcal{X}_{\mathcal{A}}:=\mathcal{X}\times_{\mathcal{B}}\mathcal{A}$. This can be shown by the Krull intersection theorem.

We have a relative moduli space of PT_q pairs on the fibres of $f:\mathcal{X}\to\mathcal{B}$.

Theorem 3.32 *Let $\mathcal{B}$ be a quasiprojective scheme and $f:\mathcal{X}\to\mathcal{B}$ a smooth projective morphism of relative dimension n. Let $\tilde{v}\in\bigoplus_p\Gamma(\mathcal{B},\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$ be a horizontal section and $q\in\{-1,0,1\}$. Let $\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},\tilde{v})$ be the moduli stack consisting of triples (b,F,s) of a morphism $b:T\to\mathcal{B}$, a T-flat coherent sheaf F on $\mathcal{X}_T:=\mathcal{X}\times_{\mathcal{B}}T$, and a section $s\in\Gamma(\mathcal{X}_T,F)$ such that, for all $t\in T(\mathbb{C})$, the pair (F_t,s_t) is a PT_q pair with $\mathrm{ch}(F_t)=\tilde{v}_t\in H^*(\mathcal{X}_t,\mathbb{C})$. Then we have the following properties*:

(1) *The canonical map*

$$\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},\tilde{v})\to\mathcal{B},\quad (b,F,s)\mapsto b,$$

is a projective morphism of schemes.

(2) *If $n\geq 4$, then the canonical map*

$$\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},\tilde{v})\to\mathscr{P}erf(\mathcal{X}/\mathcal{B})^{\mathrm{spl}}_{\mathcal{O}_{\mathcal{X}}},\quad (b,F,s)\mapsto(b,[\mathcal{O}_{\mathcal{X}_T}\xrightarrow{s}F]),$$

is an open embedding.

Proof Fix a polynomial $P(t)\in\mathbb{Q}[t]$ and let $\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},P)$ be the moduli stack of PT_q pairs on the fibres of $f:\mathcal{X}\to\mathcal{B}$ with Hilbert polynomial $P(t)$ with respect to some relative very ample line bundle. Note that we can find a closed embedding $\mathcal{X}\hookrightarrow\mathbb{P}^N_{\mathcal{B}}$ over $\mathcal{B}$ since $f:\mathcal{X}\to\mathcal{B}$ is projective and $\mathcal{B}$ is quasiprojective. Consider the canonical maps

$$\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},P)\hookrightarrow\mathscr{P}air^{(q)}(\mathbb{P}^N_{\mathcal{B}}/\mathcal{B},P)\xrightarrow{\cong}\mathscr{P}air^{(q)}(\mathbb{P}^N,P)\times\mathcal{B},$$

where the first map is a closed embedding, the second map is an isomorphism, and the last term is a projective scheme over $\mathcal{B}$ by Theorem 3.10. Indeed, as a functor over $\mathscr{P}air^{(q)}(\mathbb{P}^N_{\mathcal{B}}/\mathcal{B},P)$, $\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B},P)$ is the zero locus of the canonical map

$$\mathcal{I}_{\mathcal{X}/\mathbb{P}^N_{\mathcal{B}}}\otimes\mathbb{F}\to\mathbb{F},$$

where $\mathcal{I}_{\mathcal{X}/\mathbb{P}^N_{\mathcal{B}}}$ is the ideal sheaf of $\mathcal{X}\subseteq\mathbb{P}^N_{\mathcal{B}}$ and $(\mathbb{F},s)$ is the universal pair on $\mathscr{P}air^{(q)}(\mathbb{P}^N_{\mathcal{B}}/\mathcal{B},P)\times_{\mathcal{B}}\mathbb{P}^N_{\mathcal{B}}$. By [EGA III$_1$ 1961, Corollary 7.7.8], the map $\mathcal{I}_{\mathcal{X}/\mathbb{P}^N_{\mathcal{B}}}\otimes\mathbb{F}\to\mathbb{F}$ can be represented by the abelian cone of a coherent sheaf on $\mathscr{P}air^{(q)}(\mathbb{P}^N_{\mathcal{B}}/\mathcal{B},P)$ since $\mathbb{F}$ is flat. Hence the desired map is a closed embedding

since the abelian cone is separated over the base. Therefore we have (1) since $\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B}, \tilde{v})$ is an open and closed subscheme of the projective scheme $\mathscr{P}air^{(q)}(\mathcal{X}/\mathcal{B}, P)$ (for appropriate P).

All the arguments in Section 3.3 also works in the relative setting. This proves (2). □

We end this subsection with a technical proposition stating that the relative Chern character of a perfect complex is horizontal. This fact will be used several times in Sections 4 and 5.

Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of schemes of constant relative dimension. Let $I^\bullet$ be a perfect complex on $\mathcal{X}$. Let

$$\mathrm{At}_{\mathcal{X}/\mathcal{B}}(I^\bullet) : I^\bullet \to I^\bullet \otimes \Omega^1_{\mathcal{X}/\mathcal{B}}[1]$$

be the (relative) Atiyah class of $I^\bullet$. Let

$$\mathrm{At}^p_{\mathcal{X}/\mathcal{B}}(I^\bullet) := \mathrm{At}_{\mathcal{X}/\mathcal{B}}(I^\bullet) \wedge \cdots \wedge \mathrm{At}_{\mathcal{X}/\mathcal{B}}(I^\bullet) : I^\bullet \to I^\bullet \otimes \Omega^p_{\mathcal{X}/\mathcal{B}}[p]$$

be the induced map. We define the (relative) *Chern character* $\mathrm{ch}_p(I^\bullet)$ of $I^\bullet$ as

$$H^p(\mathcal{X}, \Omega^p_{\mathcal{X}/\mathcal{B}}) \to \Gamma(\mathcal{B}, R^p f_* \Omega^p_{\mathcal{X}/\mathcal{B}}), \quad \frac{1}{p!}\,\mathrm{tr}(\mathrm{At}^p_{\mathcal{X}/\mathcal{B}}(I^\bullet)) \mapsto \mathrm{ch}_p(I^\bullet). \tag{41}$$

The Chern character $\mathrm{ch}_p(I^\bullet)$ in (41) is horizontal. More precisely, we have the following:

Proposition 3.33 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of schemes of constant relative dimension. Let $I^\bullet$ be a perfect complex on $\mathcal{X}$. Then there exists a section*

$$\widetilde{\mathrm{ch}}_p(I^\bullet) \in \Gamma(\mathcal{B}, F^p \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$$

satisfying the following properties:

(1) *Under the projection map $F^p\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \twoheadrightarrow R^p f_* \Omega^p_{\mathcal{X}/\mathcal{B}}$ given by* (40), *the section $\widetilde{\mathrm{ch}}_p(I^\bullet)$ maps to the Chern character $\mathrm{ch}_p(I^\bullet)$ in* (41).

(2) *Under the inclusion map $F^p\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \hookrightarrow \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ given by* (40), *the section $\widetilde{\mathrm{ch}}_p(I^\bullet)$ maps to a horizontal section.*

(3) *Under the restriction map $\Gamma(\mathcal{B}, \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})) \to H^{2p}_{\mathrm{DR}}(\mathcal{X}_b/\mathrm{Spec}(\mathbb{C}))$ for any $\mathbb{C}$-point $b \in \mathcal{B}$, the section $\widetilde{\mathrm{ch}}_p(I^\bullet)$ maps to the usual Chern character $\mathrm{ch}_p(I^\bullet_b) \in H^{2p}(\mathcal{X}_b, \mathbb{C})$.*

The section $\widetilde{\mathrm{ch}}_p(I^\bullet)$ is uniquely determined by properties (2) *and* (3).

Remark 3.34 An analogous statement for the fundamental class $[\mathcal{Z}]$ of a local complete intersection $\mathcal{Z} \subseteq \mathcal{X}$ was shown in [Bloch 1972, Proposition 5.6].

We briefly summarize how we prove Proposition 3.33. We use the theory of Hochschild homology and cyclic homology; see [Loday 1992] for the case of rings and [Weibel 1996; 1997] for the case of schemes.[9] By the Hochschild–Kostant–Rosenberg (HKR) theorem [1962], we may identify the Hodge bundles, the filtrations, and the associated quotients in (39) and (40) with the periodic cyclic homology, the negative cyclic homology, and the Hochschild homology, respectively. Moreover, the Chern character in (41) can be

[9] By [Keller 1998], the cyclic homologies defined in [Weibel 1997] coincide with the cyclic homologies defined in [McCarthy 1994] via the dg category of perfect complexes.

identified with the Dennis trace map. Then the key to proving Proposition 3.33 is Goodwillie's lift [1986] of the Dennis trace map to the negative cyclic homology.[10] The proof was inspired by [Pridham 2024].

Proof of Proposition 3.33 Note that the uniqueness of the section $\widetilde{\mathrm{ch}}_p(I^\bullet)$ follows immediately from properties (2) and (3) in Proposition 3.33 and Remark 3.31(2). Hence it suffices to consider the statement locally on the base $\mathcal{B}$. Thus we may assume that $\mathcal{B}$ is an affine scheme. Then (41) is an isomorphism.

By the HKR theorem in [Weibel 1997, Theorem 3.3], we have natural isomorphisms

$$\begin{aligned} &\mathsf{HN}^i(\mathcal{X}/\mathcal{B}) \cong \bigoplus_p F^p H^{2p+i}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}), \quad \mathsf{HP}^i(\mathcal{X}/\mathcal{B}) \cong \bigoplus_p H^{2p+i}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}), \\ &\mathsf{HH}^i(\mathcal{X}/\mathcal{B}) \cong \bigoplus_p R^{p+i} f_*(\Omega^p_{\mathcal{X}/\mathcal{B}}), \qquad \mathsf{HC}^i(\mathcal{X}/\mathcal{B}) \cong \bigoplus_p \frac{H^{2p+i}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})}{F^{p+1} H^{2p+i}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})}, \end{aligned} \tag{42}$$

where HN, HP, HH, and HC denote the negative cyclic homology, the periodic cyclic homology, the Hochschild homology, and the cyclic homology, respectively, with the cohomological convention. More specifically, the isomorphisms (42) are induced by the left inverse (π in [Loday 1992, Lemma 1.3.14] and e in [Weibel 1994, Corollary 9.4.4]) of the antisymmetrization map.

The relative Chern character $\mathrm{ch}_{\mathcal{X}/\mathcal{B}}(I^\bullet) := (\mathrm{ch}_p(I^\bullet))_p$ in (41) comes from the absolute Chern character. More precisely, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \xrightarrow{\mathrm{ch}_{\mathcal{X}}(I^\bullet)} & \bigoplus_p \mathbb{L}^p_{\mathcal{X}}[p] \\ \| & & \downarrow \\ \mathcal{O}_{\mathcal{X}} & \xrightarrow[\mathrm{ch}_{\mathcal{X}/\mathcal{B}}(I^\bullet)]{} & \bigoplus_p \Omega^p_{\mathcal{X}/\mathcal{B}}[p] \end{array} \tag{43}$$

in the derived category of $\mathcal{X}$, where $\mathbb{L}_{\mathcal{X}}$ denotes Illusie's cotangent complex [1971; 1972] of $\mathcal{X}$, $\mathbb{L}^p_{\mathcal{X}} := \bigwedge^p \mathbb{L}_{\mathcal{X}} := \mathrm{Sym}^p(\mathbb{L}_{\mathcal{X}}[1])[-p]$ denotes the derived wedge product of $\mathbb{L}_{\mathcal{X}}$, and $\mathrm{ch}_{\mathcal{X}}(I^\bullet)$ denotes the absolute Chern character defined as the trace of the exponential of the absolute Atiyah class $\mathrm{At}_{\mathcal{X}}(I^\bullet) : I^\bullet \to I^\bullet \otimes \mathbb{L}_{\mathcal{X}}[1]$ [Illusie 1971; 1972].

Under the HKR isomorphism, the Chern character $\mathrm{ch}_{\mathcal{X}}(I^\bullet)$ can be identified with the Dennis trace $\mathrm{ch}^D(I^\bullet)$. More precisely, we have a commutative diagram

$$\begin{array}{ccc} & \mathcal{O}_{\mathcal{X}} & \\ {\scriptstyle \mathrm{ch}^D(I^\bullet)} \swarrow & & \searrow {\scriptstyle \mathrm{ch}_{\mathcal{X}}(I^\bullet)} \\ L\widetilde{\Delta}^*\widetilde{\Delta}_*\mathcal{O}_{\mathcal{X}} & \xrightarrow[\cong]{} & \bigoplus_p \mathbb{L}^p_{\mathcal{X}}[p] \\ \downarrow & & \downarrow \\ L\Delta^*\Delta_*\mathcal{O}_{\mathcal{X}} & \xrightarrow[\cong]{\mathrm{HKR}} & \bigoplus_p \Omega^p_{\mathcal{X}/\mathcal{B}}[p] \end{array} \tag{44}$$

[10] In the language of [Pantev et al. 2013], Chern characters lift to p-shifted *closed* p-forms [Toën and Vezzosi 2011; 2015]. By [Hoyois et al. 2021], the Chern character of [Toën and Vezzosi 2015] matches with the classical Dennis trace.

in the derived category of $\mathcal{X}$, where $\widetilde{\Delta}:\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ and $\Delta:\mathcal{X}\to\mathcal{X}\times_{\mathcal{B}}\mathcal{X}$ are the diagonal maps. The horizontal isomorphisms are given by the exponentials of the universal Atiyah classes as in [Buchweitz and Flenner 2008, Theorem 6.2.1] (see [Markarian 2009; Căldăraru 2005] for the smooth case). The hypercohomology of the left column of (44) computes the Hochschild homology,

$$\mathbb{H}^i(\mathcal{X}, L\widetilde{\Delta}^*\widetilde{\Delta}_*\mathcal{O}_{\mathcal{X}})\cong \mathsf{HH}^i(\mathcal{X}),\quad \mathbb{H}^i(\mathcal{X}, L\Delta^*\Delta_*\mathcal{O}_{\mathcal{X}})\cong \mathsf{HH}^i(\mathcal{X}/\mathcal{B}),$$

by [Yekutieli 2002, Proposition 3.3]. The lower horizontal isomorphism in (44) is compatible with the corresponding isomorphism in (42), by the arguments in [Căldăraru 2005, Proposition 4.4] since $\mathcal{X}\to\mathcal{B}$ is smooth. The map $\mathrm{ch}^D(I^\bullet)$ in (44) is the Euler map of $I^\bullet$ in the sense of [Bressler et al. 1997, 4.3], which agrees with the Dennis trace map for the dg category $\mathrm{Perf}(\mathcal{X})$ [McCarthy 1994], as described in [Bressler et al. 1997, 4.5, 4.7].

As explained in [Weibel 1997, 3.8], the Dennis trace map ch^D can be extended to the Goodwillie Chern character ch^G [1986] via descent [Weibel and Geller 1991]:

(45)
$$\begin{array}{ccccc} & & K^0(\mathcal{X}) & & \\ & \swarrow^{\mathrm{ch}^D} & \downarrow{\scriptstyle \mathrm{ch}^G} & \searrow & \\ \mathsf{HH}^0(\mathcal{X}) & \longleftarrow & \mathsf{HN}^0(\mathcal{X}) & \longrightarrow & \mathsf{HP}^0(\mathcal{X}) \end{array}$$

Consider the canonical commutative diagram

(46)
$$\begin{array}{ccccc} \mathsf{HH}^0(\mathcal{X}) & \longleftarrow & \mathsf{HN}^0(\mathcal{X}) & \longrightarrow & \mathsf{HP}^0(\mathcal{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{HH}^0(\mathcal{X}/\mathcal{B}) & \twoheadleftarrow & \mathsf{HN}^0(\mathcal{X}/\mathcal{B}) & \hookrightarrow & \mathsf{HP}^0(\mathcal{X}/\mathcal{B}) \\ \cong\downarrow{\scriptstyle \mathrm{HKR}} & & \cong\downarrow{\scriptstyle \mathrm{HKR}} & & \cong\downarrow{\scriptstyle \mathrm{HKR}} \\ \bigoplus_p R^pf_*\Omega^p_{\mathcal{X}/\mathcal{B}} & \twoheadleftarrow & \bigoplus_p F^pH^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) & \hookrightarrow & \bigoplus_p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \end{array}$$

Denote by

$$\widetilde{\mathrm{ch}}_p(I^\bullet)\in F^pH^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) \tag{47}$$

the image of $\mathrm{ch}^G(I^\bullet)\in HN_0(\mathcal{X})$ under the vertical maps in (46). We now show that $\widetilde{\mathrm{ch}}_p(I^\bullet)$ in (47) satisfies the three properties in Proposition 3.33 separately:

(1) Clearly $\widetilde{\mathrm{ch}}_p(I^\bullet)$ maps to the Chern character $\mathrm{ch}_p(I^\bullet)$ in (41). This follows from the commutative diagrams (43), (44), (45), and (46).

(2) We now claim that $\widetilde{\mathrm{ch}}_p(I^\bullet)$ is horizontal. Indeed, we may assume that $\mathcal{B}=\mathrm{Spec}(A)$ for a local Artinian $\mathbb{C}$-algebra A (Remark 3.31). Let $X:=\mathcal{X}_0$ be the fibre over the $\mathbb{C}$-point $0\in\mathcal{B}$. By the Feigin–Tsygan theorem [1985, Theorem 5] (see also [Emmanouil 1995, Theorem 2.2; Weibel 1997, Theorem 3.4]), we

have a commutative square with bijective horizontal arrows

$$\begin{array}{ccc} \mathsf{HP}^0(\mathcal{X}) & \xrightarrow{\cong} & \bigoplus_p \mathbb{H}^{2p}(\widehat{\mathcal{Y}}, \widehat{\Omega}^\bullet_{\mathcal{Y}}) \\ \downarrow & & \downarrow \\ \mathsf{HP}^0(\mathcal{X}/\mathcal{B}) & \xrightarrow{\cong} & \bigoplus_p \mathbb{H}^{2p}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{B}}) \end{array} \tag{48}$$

where $\bigoplus_p \mathbb{H}^{2p}(\widehat{\mathcal{Y}}, \widehat{\Omega}^\bullet_{\mathcal{Y}})$ is Hartshorne's algebraic de Rham cohomology [1975] given by an embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ into a smooth scheme $\mathcal{Y}$. Here the lower isomorphism in (48) coincides with the corresponding isomorphism in (42) since they both are given by the same left inverse (μ in [Emmanouil 1995] and e in [Weibel 1997]) of the antisymmetrization map. By [Hartshorne 1975, Theorem IV.1.1] (and [Deligne 1968, lemme 5.5.3]), we have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{H}^{2p}(\widehat{\mathcal{Y}}, \widehat{\Omega}^\bullet_{\mathcal{Y}}) & \xrightarrow{\cong} & \mathbb{H}^{2p}(\widehat{\mathcal{Y}}^{\mathrm{an}}, \widehat{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}}) & \xleftarrow{\cong} & H^{2p}(\mathcal{X}^{\mathrm{an}}, \mathbb{C}) & = & H^{2p}(X^{\mathrm{an}}, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^{2p}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/\mathcal{B}}) & \xrightarrow{\cong} & \mathbb{H}^{2p}(\mathcal{X}^{\mathrm{an}}, \Omega^\bullet_{\mathcal{X}^{\mathrm{an}}/\mathcal{B}^{\mathrm{an}}}) & \xleftarrow{\cong} & H^{2p}(\mathcal{X}^{\mathrm{an}}, A) & = & H^{2p}(X^{\mathrm{an}}, A) \end{array}$$

where the last vertical map is induced by the canonical map of constant sheaves $\mathbb{C} \to A$ on X^{an}. Hence the image of the vertical map consists of the horizontal sections (see Remark 3.31), which proves the claim.

(3) We finally claim that the restriction $\widetilde{\mathrm{ch}}_p(I^\bullet)$ to the fibre of a point $b \in \mathcal{B}$ is the usual Chern character. Indeed, we may assume that $\mathcal{B} = \mathrm{Spec}(\mathbb{C})$ since Chern characters are compatible with base change. Then the claim follows from [Weibel 1997, Proposition 3.8.1]. □

4 Virtual cycles

In this section, we construct reduced virtual cycles for the moduli spaces of PT_q pairs on Calabi–Yau 4-folds. We use Kiem and Li's cosection localization technique [2013; Kiem and Park 2023] to reduce the Oh–Thomas virtual cycles [2023] on the moduli spaces. The semiregularity map of [Buchweitz and Flenner 2003] will provide the necessary cosections.

Throughout this section, we fix a (smooth projective) Calabi–Yau 4-fold X and a Calabi–Yau 4-form $\omega : \mathcal{O}_X \xrightarrow{\cong} \Omega^4_X$.

4.1 Nonreduced virtual cycles

In this preliminary subsection, we construct *nonreduced* virtual cycles for the moduli spaces of stable pairs. We review the [Oh and Thomas 2023] construction of virtual cycles for the reader's convenience. Instead of following [loc. cit.] directly, we will use the setup introduced in [Park 2021].

Let $\mathscr{P} := \mathscr{P}_v^{(q)}(X)$ be the moduli space of PT_q pairs (F, s) on X satisfying

$$\mathrm{ch}(F) = v = (0, 0, \gamma, \beta, n - \gamma \cdot \mathrm{td}_2(X)) \in H^*(X, \mathbb{Q}).$$

By Theorem 3.19, we have an open embedding

$$\mathscr{P} \hookrightarrow \mathscr{P}erf(X,v)^{\mathrm{spl}}_{\mathcal{O}_X}$$

into the moduli space of perfect complexes with fixed trivial determinant. Thus, by [Oh and Thomas 2023], we can construct an Oh–Thomas virtual cycle on $\mathscr{P}$.

Theorem 4.1 [Oh and Thomas 2023] *There exists a virtual cycle*

$$[\mathscr{P}]^{\mathrm{vir}} \in A_{\mathrm{vd}}(\mathscr{P})$$

of virtual dimension

$$\mathrm{vd} = n - \tfrac{1}{2}\gamma^2,$$

which depends[11] *on the choice of an orientation* (50) *below.*

We call the virtual cycle $[\mathscr{P}]^{\mathrm{vir}}$ in Theorem 4.1 the *nonreduced virtual cycle*. We use this terminology since we will construct a canonical *reduced virtual cycle* $[\mathscr{P}]^{\mathrm{red}}$ for any Calabi–Yau 4-fold X and a surface class γ in the next subsection (see Theorem 4.5).

There are three key ingredients for defining the Oh–Thomas virtual cycle (in the language of [Park 2021]):

(1) Symmetric obstruction theory: The moduli space $\mathscr{P}$ has a (3-term) *symmetric obstruction theory*

$$\phi : \mathbb{E} := R\mathcal{H}om_{\pi}(\mathbb{I}^{\bullet},\mathbb{I}^{\bullet})_0[3] \xrightarrow{\mathrm{At}(\mathbb{I}^{\bullet})} \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}, \quad \theta : \mathbb{E}^{\vee}[2] \xrightarrow{\mathsf{SD}} \mathbb{E} \tag{49}$$

induced by the Atiyah class $\mathrm{At}(\mathbb{I}^{\bullet})$ of the universal complex $\mathbb{I}^{\bullet}$ and the relative Serre duality for the projection map $\pi : \mathscr{P} \times X \to \mathscr{P}$. Here ϕ and θ satisfy the following properties:

(a) $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ is an *obstruction theory* [Behrend and Fantechi 1997], i.e., $h^0(\phi)$ is bijective and $h^{-1}(\phi)$ is surjective.

(b) $\mathbb{E}$ is a *symmetric complex* (of tor-amplitude $[-2,0]$), i.e., θ is an isomorphism and $\theta = \theta^{\vee}[2]$.

The symmetric obstruction theory (49) is constructed by [Huybrechts and Thomas 2010].[12]

(2) Orientation: There exists an *orientation* for the virtual cotangent complex $\mathbb{E}$, i.e., an isomorphism of line bundles

$$o : \mathcal{O}_{\mathscr{P}} \xrightarrow{\cong} \det(\mathbb{E}) \tag{50}$$

such that $o^2 = (-1)^{(2\mathrm{vd})(2\mathrm{vd}-1)/2} \cdot \det(\theta) : \det(\mathbb{E}^{\vee}) \to \det(\mathbb{E})$. The existence of an orientation was shown by [Cao et al. 2020].

[11] A different choice of orientation changes the *signs* of the virtual cycles on the connected components.

[12] Alternatively, we can also use the derived enhancement of $\mathscr{P}erf(X)^{\mathrm{spl}}_{\mathcal{O}_X}$ in [Toën and Vaquié 2007; Schürg et al. 2015].

(3) Isotropic condition: The intrinsic normal cone $\mathfrak{C}_{\mathscr{P}}$ is *isotropic* in the virtual normal cone $\mathfrak{C}(\mathbb{E}) := h^1/h^0(\mathbb{E}^\vee)$ with respect to the canonical quadratic function $\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1$ (defined in Remark 4.2 below), i.e., the restriction

$$\mathfrak{q}_{\mathbb{E}}|_{\mathfrak{C}_{\mathscr{P}}} : \mathfrak{C}_{\mathscr{P}} \hookrightarrow \mathfrak{C}(\mathbb{E}) \xrightarrow{\mathfrak{q}_{\mathbb{E}}} \mathbb{A}^1_{\mathscr{P}}$$

vanishes. This isotropic condition can be shown by the Darboux theorem of [Brav et al. 2019] for the (-2)-shifted symplectic derived enhancement; see [Pantev et al. 2013] and Schürg, Toën and Vezzosi [Schürg et al. 2015]. Indeed, by [Park 2021, Lemma 1.11], the arguments in [Oh and Thomas 2023, Proposition 4.3] proves the isotropic condition.[13]

We recall the definition of the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ from [Park 2021].

Remark 4.2 Given a symmetric complex $\mathbb{E}$ on a quasiprojective scheme $\mathcal{X}$, there is a canonical quadratic function

$$\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_{\mathcal{X}}$$

induced by the symmetric form of $\mathbb{E}$. Indeed, if $\mathbb{E} = [B \to E \to B^\vee]$ is a symmetric resolution, then $\mathfrak{C}(\mathbb{E}) = [C(D)/B]$ for $D = \operatorname{coker}(B \to E)$, and the restriction $\mathfrak{q}_E|_{C(D)} : C(D) \hookrightarrow E \xrightarrow{\mathfrak{q}_E} \mathbb{A}^1$ of the quadratic function $\mathfrak{q}_E$ of E is B-invariant. The quadratic function $\mathfrak{q}_{\mathbb{E}}$ is defined via descent of $\mathfrak{q}_E|_{C(D)}$, which is independent of the choice of the symmetric resolution. See [Park 2021, Proposition 1.7] for details.

Note that, from the above data, we can form a commutative diagram

(51)
$$\begin{array}{ccccc} & & \mathfrak{Q}(\mathbb{E}) & \longrightarrow & \mathscr{P} \\ & \nearrow & \downarrow & & \downarrow 0 \\ \mathfrak{C}_{\mathscr{P}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{q}_{\mathbb{E}}} & \mathbb{A}^1_{\mathscr{P}} \end{array}$$

where $\mathfrak{Q}(\mathbb{E})$ is the *quadratic cone stack* associated to the symmetric complex $\mathbb{E}$, defined as the zero locus of the quadratic function $\mathfrak{q}_{\mathbb{E}}$ on $\mathfrak{C}(\mathbb{E})$.

We can construct the Oh–Thomas virtual cycle $[\mathscr{P}]^{\mathrm{vir}}$ from the closed embedding $\mathfrak{C}_{\mathscr{P}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ in (51).

Proof of Theorem 4.1 We first construct a *square root Gysin pullback*

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(\mathscr{P})$$

for the zero section $0_{\mathfrak{Q}(\mathbb{E})} : \mathscr{P} \to \mathfrak{Q}(\mathbb{E})$. Note that we can find a symmetric resolution

$$\mathbb{E} \cong [B \xrightarrow{d} E \cong E^\vee \xrightarrow{d^\vee} B^\vee] \tag{52}$$

for some special orthogonal bundle E and a vector bundle B. Indeed, this can be shown by the proof of [Oh and Thomas 2023, Proposition 4.1]; see also [Park 2021, Proposition 1.3]. Here an orientation of $\mathbb{E}$

[13] One does not need the full proof of [Oh and Thomas 2023, Proposition 4.3], but only the part involving MacPherson's graph construction [Fulton 1998, Remark 5.11].

induces an orientation of E via the canonical isomorphism

$$\det(\mathbb{E}) \cong \det(B^\vee) \otimes \det(E^\vee)^\vee \otimes \det(B) \cong \det(E)$$

of line bundles. By considering the stupid truncation

$$[0 \to E^\vee \xrightarrow{d^\vee} B^\vee] \to [B \xrightarrow{d} E \cong E^\vee \xrightarrow{d^\vee} B^\vee],$$

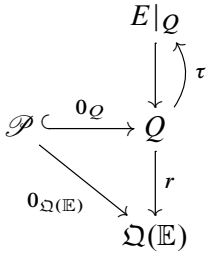
we can form a fibre diagram

$$\begin{array}{ccccc} Q & & \hookrightarrow & & E \\ \downarrow{\scriptstyle r} & & & & \downarrow \\ \mathfrak{Q}(\mathbb{E}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \hookrightarrow & [E/B] \end{array} \tag{53}$$

for some cone Q. The horizontal maps in (53) are closed embeddings and the vertical maps in (53) are B-torsors. Then Q is an isotropic subcone of E since

$$\mathfrak{q}_E|_Q = \mathfrak{q}_{\mathfrak{C}(\mathbb{E})}|_Q = 0$$

by [Park 2021, Proposition 1.7(3)]. Consider a factorization



of the zero section $0_{\mathfrak{Q}(\mathbb{E})}$, where τ is the tautological section induced by the embedding $Q \hookrightarrow E$ in (53). Note that the zero section $0_Q : \mathscr{P} \hookrightarrow Q$ is the zero locus of the tautological section $\tau \in \Gamma(Q, E|_Q)$. We define the square root Gysin pullback as the composition

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} := \sqrt{e}(E|_Q, \tau) \circ r^* : A_*(\mathfrak{Q}(\mathbb{E})) \to A_{*+b}(Q) \to A_{*+b-e/2}(\mathscr{P}), \tag{54}$$

where $b := \mathrm{rk}(B)$, $e := \mathrm{rk}(E)$, $\mathrm{vd} = b - \frac{1}{2}e$, and $\sqrt{e}(E|_Q, \tau)$ denotes Oh and Thomas's localization [2023, Definition 3.2] of Edidin and Graham's square root Euler class [1995].[14] The arguments in [Oh and Thomas 2023, pages 31–34] show that the square root Gysin pullback in (54) is independent of the choice of the symmetric resolution (52).

We then define the Oh–Thomas virtual cycle as

$$[\mathscr{P}]^{\mathrm{vir}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}[\mathfrak{C}_{\mathscr{P}}] \in A_{\mathrm{vd}}(\mathscr{P}), \tag{55}$$

where $[\mathfrak{C}_{\mathscr{P}}] \in A_0(\mathfrak{Q}(\mathbb{E}))$ is the fundamental class of the intrinsic normal cone with respect to the closed embedding $\mathfrak{C}_{\mathscr{P}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ in (51).

[14] See also [Kiem and Park 2023, Definition 4.1] for an alternative construction of the localized square root Euler class $\sqrt{e}(E|_C, \tau)$ using the blowup method.

We can easily compute the virtual dimension via the Hirzebruch–Riemann–Roch formula [Fulton 1998, Corollary 15.2.1]. Indeed, we have

$$\begin{aligned}2\cdot \mathrm{vd} &= \mathrm{rank}(R\,\mathrm{Hom}_X(I^\bullet, I^\bullet)_0[1])\\ &= -\int_X \mathrm{ch}(I^\bullet)^\vee \,\mathrm{ch}(I^\bullet)\,\mathrm{td}(X) + \int_X \mathrm{td}(X)\\ &= -(-2n + 2\,\mathrm{td}_2(X)\gamma + \mathrm{td}_4(X) - 2\,\mathrm{td}_2(X)\gamma + \gamma^2) + \mathrm{td}_4(X)\\ &= 2n - \gamma^2.\end{aligned}$$

□

We provide a simple analogy between the above description (55) of the Oh–Thomas virtual cycles [2023] and the Behrend–Fantechi virtual cycles [1997].

Remark 4.3 Fix an arbitrary (separated) Deligne–Mumford stack $\mathcal{X}$.

Recall that a perfect obstruction theory $\psi : \mathbb{K} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ of tor-amplitude $[-1, 0]$ is equivalent to a closed embedding of cone stacks[15] $\mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{C}(\mathbb{K})$ for some vector bundle stack $\mathfrak{C}(\mathbb{K})$. Kresch [1999b] constructed a Gysin pullback

$$0^!_{\mathfrak{C}(\mathbb{K})} : A_*(\mathfrak{C}(\mathbb{K})) \to A_*(\mathcal{X})$$

for any perfect complex $\mathbb{K}$ of tor-amplitude $[-1, 0]$. In [Behrend and Fantechi 1997; Kresch 1999b], the Behrend–Fantechi virtual cycle for a closed embedding $\mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{C}(\mathbb{K})$ was defined as

$$[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{BF}} := 0^!_{\mathfrak{C}(\mathbb{K})}[\mathfrak{C}_{\mathcal{X}}].$$

Analogously, a symmetric obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ of tor-amplitude $[-2, 0]$ satisfying the isotropic condition is equivalent to a closed embedding of cone stacks $\mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ for some quadratic cone stack $\mathfrak{Q}(\mathbb{E})$ associated to a symmetric complex $\mathbb{E}$. Note that we can define the square root Gysin pullback

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(\mathcal{X})$$

for any symmetric complex $\mathbb{E}$ of tor-amplitude $[-2, 0]$ and an orientation $\mathcal{O}_{\mathcal{X}} \xrightarrow{\cong} \det(\mathbb{E})$ as in the proof of Theorem 4.1.[16] Thus the Oh–Thomas virtual cycle for a closed embedding $\mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ and an orientation can be defined as

$$[\mathcal{X}]^{\mathrm{vir}}_{\mathrm{OT}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}[\mathfrak{C}_{\mathcal{X}}].$$

4.2 Reduced virtual cycles

Recall that the standard curve-counting invariants (GW/PT) on K3 surfaces vanish due to the following two reasons:

[15] A morphism of cone stacks is always assumed to be $\mathbb{A}^1$-equivariant as in [Behrend and Fantechi 1997, Definition 1.8].

[16] The proof of Theorem 4.1 works when $\mathcal{X}$ is a quasiprojective scheme. Using the Kimura sequence for Artin stacks [Bae and Park ≥ 2026], this can be generalized to the case when $\mathcal{X}$ is a separated Deligne–Mumford stack; see [Park 2021, Appendix A].

(1) [Kiem and Li 2013] There exists a surjective cosection.

(2) [Kool and Thomas 2014; Maulik et al. 2010] The codimension of the Noether–Lefschetz locus is positive.

To obtain nonzero invariants, one should consider *reduced* virtual cycles.

An analogous phenomenon occurs when we want to count surfaces on Calabi–Yau 4-folds. Given a surface class γ, there are as many natural cosections as the codimension ρ_γ of the Hodge locus. Thus, in a generic situation, the nonreduced virtual cycle in Theorem 4.1 vanishes and we want to construct a *reduced* virtual cycle. This can be achieved by the DT4 version of the cosection localization technique developed in [Kiem and Park 2023]. In this subsection, we present the main results on the reduced virtual cycles and we provide the proofs in the subsequent subsections.

We first introduce an important bilinear form B_γ on $H^1(X,T_X)$ associated to each surface class γ on X. We also fix some notation.

Definition 4.4 Let $\gamma\in H^2(X,\Omega_X^2)$ be a (2,2)-class on X.

(1) We define a symmetric bilinear form B_γ on $H^1(X,T_X)$ as

$$\mathsf{B}_\gamma : H^1(X,T_X)\otimes H^1(X,T_X)\to\mathbb{C},\quad \xi_1\otimes\xi_2\mapsto\int_X \iota_{\xi_1}\iota_{\xi_2}\gamma\cup\omega,$$

where ι_{ξ_1} and ι_{ξ_2} denote the contraction maps and ω is the (chosen) Calabi–Yau 4-form.[17]

(2) We denote the rank of the bilinear form B_γ by

$$\rho_\gamma := \operatorname{rank}(\mathsf{B}_\gamma).$$

(3) For the induced nondegenerate quadratic space of dimension ρ_γ, we write

$$H^1(X,T_X)_\gamma := \frac{H^1(X,T_X)}{\ker(\mathsf{B}_\gamma)}.$$

The bilinear form B_γ is closely related to the Hodge theory of the (2,2)-class

$$\gamma\in H^2(X,\Omega_X^2)\subseteq H^4(X,\mathbb{C}).$$

Essentially, the rank ρ_γ is equal to the codimension of the Hodge locus of γ [Bloch 1972] (see note 3).

We now state our main result.

Theorem 4.5 *Let X be a Calabi–Yau 4-fold, $v=(0,0,\gamma,\beta,n-\gamma\cdot\mathrm{td}_2(X))\in H^*(X,\mathbb{Q})$, and $q\in\{-1,0,1\}$. Then there exists a canonical reduced virtual cycle*

$$[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}}\in A_{\mathrm{rvd}}(\mathscr{P}_v^{(q)}(X))$$

of reduced virtual dimension

$$\mathrm{rvd}=n-\tfrac{1}{2}\gamma^2+\tfrac{1}{2}\rho_\gamma$$

[17] For $\xi_1\in H^i(X,T_X)$ and $\xi_2\in H^j(X,T_X)$, we have $\iota_{\xi_2}\iota_{\xi_1}=(-1)^{(1+i)(1+j)}\iota_{\xi_1}\iota_{\xi_2}$ (see for example [Brav et al. 2019, Lemma 2.5]).

depending on the choices of orientation of the symmetric complex $\mathbb{E}$ *and the quadratic space* $H^1(X, T_X)_\gamma$. *Moreover,* $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}}$ *is deformation invariant in the sense of Theorem 5.1.*

The reduced virtual cycle $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}}$ has a relation to the notion of *semiregularity* of perfect complexes $I^\bullet$ associated to pairs $[(F,s)] \in \mathscr{P}_v^{(q)}(X)$ (Definition 4.8(2)).

Theorem 4.6 *Let* X *be a Calabi–Yau 4-fold,* $v = (0,0,\gamma,\beta, n - \gamma\cdot \mathrm{td}_2(X)) \in H^*(X,\mathbb{Q})$, *and* $q \in \{-1,0,1\}$. *For any* PT_q *pair* $[(F,s)] \in \mathscr{P}_v^{(q)}(X)$, *the associated perfect complex* $I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$ *is semiregular if and only if* $\mathscr{P}_v^{(q)}(X)$ *is smooth of dimension* $\mathrm{rvd} = n - \frac{1}{2}\gamma^2 + \frac{1}{2}\rho_\gamma$ *at* $[(F,s)]$.

By Theorem 4.6, if all points in $\mathscr{P}_v^{(q)}(X)$ are semiregular, then the space is smooth and the reduced virtual cycle is equal to the fundamental cycle

$$[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} = [\mathscr{P}_v^{(q)}(X)] \in A_{\mathrm{rvd}}(\mathscr{P}_v^{(q)}(X)).$$

Remark 4.7 The role of ρ_γ in the surface-counting theory is an interesting new feature compared to known enumerative geometry. Indeed, ρ_γ is not a topological datum of $\gamma \in H^4(X)$, but rather it is a Hodge-theoretic datum of $\gamma \in H^{2,2}(X)$. In particular, the rank of B_γ is not additive in γ in general, but it is only subadditive, i.e., $\rho_{\gamma_1+\gamma_2} \le \rho_{\gamma_1} + \rho_{\gamma_2}$. For example, ρ_γ remains the same when taking multiples of γ.

We now explain the main ideas in the proof of Theorem 4.5. We will consider three different, but interrelated, approaches to the reduced theory:

(1) Cosection localization: In Section 4.4, we will prove Theorem 4.5 using the *cone reduction* lemma of [Kiem and Li 2013]. The *semiregularity map* of [Buchweitz and Flenner 2003] will provide the necessary cosections. This cosection localization method is the simplest way to construct the reduced virtual cycle.

(2) Algebraic twistor family: In Section 4.5, we will construct a *reduced obstruction theory* using the algebraic twistor family introduced in [Kool and Thomas 2014]. The main idea is to consider an infinitesimal deformation of X such that the Hodge locus of γ is just the origin. The algebraic twistor family method alone does not provide a reduced virtual cycle without the cone reduction lemma, due to the isotropic condition. See Remark 4.22.

(3) Reduction by (-1)-shifted 1-forms: In Appendix A, we will construct a *reduced* (-2)*-shifted symplectic derived enhancement*. The crucial part is that the cosections arising from the semiregularity map have natural extensions to (-1)-shifted *closed* 1-forms.

4.3 Semiregularity map

In this subsection, we revisit the *semiregularity map* of [Buchweitz and Flenner 2003] and show that the key commutative triangle in [loc. cit.] is *symmetric* for Calabi–Yau 4-folds.

We first recall the definition of the semiregularity map.[18]

[18] Here we consider the fixed-determinant version of the semiregularity map. For strict Calabi–Yau 4-folds, this is identical to the 2-semiregularity map in [Buchweitz and Flenner 2003]. Also we use a different sign convention than [loc. cit.].

Definition 4.8 (cf. [Bloch 1972; Buchweitz and Flenner 2003]) Let $I^\bullet$ be a perfect complex on X.

(1) We define the *semiregularity map* by

$$\mathsf{sr} : \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0 \to H^3(X, \Omega^1_X), \quad e \mapsto \mathrm{tr}(\mathrm{At}(I^\bullet) \circ e), \tag{56}$$

where $\mathrm{At}(I^\bullet) : I^\bullet \to I^\bullet \otimes \Omega^1_X[1]$ denotes the Atiyah class of $I^\bullet$.

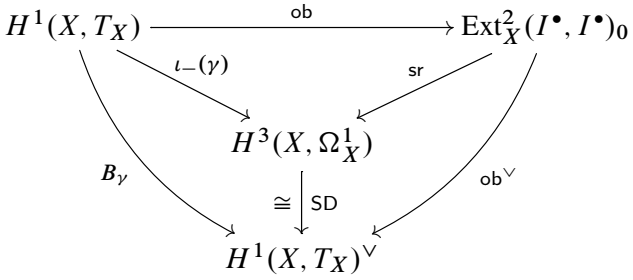
(2) We say that $I^\bullet$ is *semiregular* if sr is injective.

The semiregularity map connects the obstruction to deforming a perfect complex $I^\bullet$ and its Chern character $\mathrm{ch}(I^\bullet)$ by the key commutative triangle in [Buchweitz and Flenner 2003, Corollary 4.3]. For Calabi–Yau 4-folds, this commutative triangle has a *symmetric* description.

Proposition 4.9 *Let $I^\bullet$ be a perfect complex on X such that $\mathrm{ch}_1(I^\bullet) = 0 \in H^1(X, \Omega^1_X)$ and $\mathrm{ch}_2(I^\bullet) = \gamma \in H^2(X, \Omega^2_X)$. Then the obstruction map*

$$\mathsf{ob} : H^1(X, T_X) \to \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0, \quad \xi \mapsto \iota_\xi \,\mathrm{At}(I^\bullet), \tag{57}$$

fits into the commutative diagram

(58)


where the vertical isomorphism SD is induced by the Serre duality pairing

$$H^3(X, \Omega^1_X) \otimes H^1(X, T_X) \to \mathbb{C}, \quad \alpha \otimes \xi \mapsto \int_X \iota_\xi(\alpha) \cup \omega,$$

and we identify $(\mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0)^\vee = \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0$ via the Yoneda pairing

$$\mathsf{Y} : \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0 \otimes \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0 \to \mathbb{C}, \quad u \otimes v \mapsto \int_X \mathrm{tr}(u \circ v) \cup \omega.$$

Proof Note that the obstruction map ob in (57) is well defined since

$$\mathrm{tr} \circ \mathsf{ob}(\xi) = \iota_\xi \,\mathrm{ch}_1(I^\bullet) = 0.$$

The commutativity of the three triangles in (58) can be shown as follows:

(1) Top triangle: We have $\mathsf{sr} \circ \mathsf{ob} = \iota_-(\gamma)$ by [Buchweitz and Flenner 2003, Corollary 4.3].

(2) Left triangle: For any $\xi_1, \xi_2 \in H^1(X, T_X)$, we have

$$\mathsf{SD}(\iota_{\xi_1}(\gamma))(\xi_2) = \int_X \iota_{\xi_2}\iota_{\xi_1}\gamma \cup \omega = \int_X \iota_{\xi_1}\iota_{\xi_2}\gamma \cup \omega = \mathsf{B}_\gamma(\xi_1, \xi_2).$$

(3) Right triangle: For any $\xi \in H^1(X, T_X)$ and $e \in \mathrm{Ext}^2_X(I^\bullet, I^\bullet)$, we have

$$\mathsf{ob}^\vee(e)(\xi) = \int_X \mathrm{tr}(\iota_\xi \mathrm{At}(I^\bullet) \circ e) \cup \omega = \int_X \iota_\xi \mathrm{tr}(\mathrm{At}(I^\bullet) \circ e) \cup \omega = \mathsf{SD}(\mathsf{sr}(e))(\xi). \qquad \square$$

The commutative triangle (58) says that the map of vector spaces

$$\mathsf{ob} = \mathsf{sr}^\vee : (H^1(X, T_X), \mathsf{B}_\gamma) \to (\mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0, \mathsf{Y})$$

is compatible with the corresponding symmetric bilinear forms. Equivalently, we may write $\mathsf{B}_\gamma = \mathsf{ob}^*(\mathsf{Y})$.

In the following subsections, we will develop the reduced theory with respect to a nondegenerate subspace $V \subseteq H^1(X, T_X)$. We provide a simple observation on such nondegenerate subspaces.

Corollary 4.10 *Let $I^\bullet$ be a perfect complex on X with $\mathrm{ch}_1(I^\bullet) = 0$ and $\mathrm{ch}_2(I^\bullet) = \gamma$. Choose a nondegenerate subspace*

$$V \subseteq H^1(X, T_X)$$

with respect to the bilinear form B_γ. Then the restriction

$$(\mathsf{ob} = \mathsf{sr}^\vee)|_V : (V, \mathsf{B}_\gamma|_V) \hookrightarrow (\mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0, \mathsf{Y})$$

is an injective map of nondegenerate quadratic spaces.

Proof If $\mathsf{ob}(v) = 0$ for some $v \in V$, then

$$\mathsf{B}_\gamma(v, -) = \int_X \mathrm{tr}(\mathsf{ob}(v) \circ \mathsf{ob}(-)) = 0 \in H^1(X, T_X)^\vee.$$

Thus the nondegeneracy of $\mathsf{B}_\gamma|_V$ implies that $v = 0$. $\square$

For the reader's convenience, we review how the maps ob and B_γ in the commutative triangle (58) determine the obstructions to deforming the perfect complex $I^\bullet$ and the Hodge class $\gamma \in H^2(X, \Omega^2_X)$, respectively. For simplicity, we only consider the first-order deformations here. We refer to [Bloch 1972, Proposition 4.2; Illusie 1971, proposition IV.3.1.5] (cf. [Huybrechts and Thomas 2010, Corollary 3.4]) for the general case.

Lemma 4.11 *Consider a first-order deformation of X, i.e., a fibre diagram*

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \{o\} & \hookrightarrow & \mathcal{A} \end{array}$$

where $f : \mathcal{X} \to \mathcal{A}$ is a smooth projective morphism and $\mathcal{A}$ is a square-zero extension of $\mathrm{Spec}(\mathbb{C})$. *Let $o \in \mathcal{A}$ be the unique point.*

(1) *Given a $(2,2)$-class $\gamma \in H^2(X, \Omega_X^2) \subseteq H^4(X, \mathbb{C})$, there exists a horizontal section $\widetilde{\gamma} \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{A})$ such that $\widetilde{\gamma}_o = \gamma$ if and only if the composition*

$$T_{o,\mathcal{A}} \xrightarrow{\mathsf{KS}_{\mathcal{X}/\mathcal{A}}} H^1(X, T_X) \xrightarrow{\mathsf{B}_\gamma} H^1(X, T_X)^\vee$$

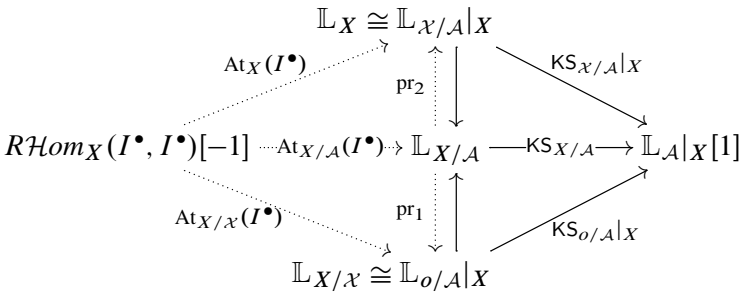
vanishes.

(2) *Given a perfect complex $I^\bullet$ on $\mathcal{X}$ with $\mathrm{ch}_1(I^\bullet) = 0$, there exists a perfect complex $\widetilde{I}^\bullet$ on $\mathcal{X}$ with $\det(\widetilde{I}^\bullet) \cong \mathcal{O}_{\mathcal{X}}$ such that $\widetilde{I}^\bullet_o \cong I^\bullet$ if and only if the composition*

$$T_{o,\mathcal{A}} \xrightarrow{\mathsf{KS}_{\mathcal{X}/\mathcal{A}}} H^1(X, T_X) \xrightarrow{\mathsf{ob}} \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0 \tag{59}$$

vanishes.

Proof (1) follows directly from [Bloch 1972, Proposition 4.2]. We would like to deduce (2) from [Illusie 1971, proposition IV.3.1.5] (or [Huybrechts and Thomas 2010, Corollary 3.4]). Consider a diagram



where the left two triangles commute with the dotted arrows and the right two triangles commute with the solid arrows. Moreover, the composition of the middle arrows is zero because $\mathrm{At}_{X/\mathcal{A}}(I^\bullet)$ factors through $\mathrm{At}_X(I^\bullet)$. Hence the total big square commutes (up to sign). Therefore, the map (59) corresponds to the composition

$$I^\bullet \xrightarrow{\mathrm{At}_{X/\mathcal{X}}(I^\bullet)} I^\bullet \otimes \mathbb{L}_{X/\mathcal{X}}[1] \to I^\bullet \otimes \mathcal{I}_{X/\mathcal{X}}[2] = I^\bullet \otimes \mathcal{I}_{o/\mathcal{A}}[2]$$

via adjunction. □

4.4 Cosection localization

In this subsection, we construct the *reduced virtual cycle* and prove Theorem 4.5 via the *cosection localization* technique of [Kiem and Li 2013], as in [Kiem and Park 2023]. We also prove Theorem 4.6 as a corollary.

Let $\mathscr{P} := \mathscr{P}_v^{(q)}(X)$ be the moduli space of PT_q pairs on X with fixed Chern character

$$v = (0, 0, \gamma, \beta, n - \gamma \cdot \mathrm{td}_2(X)) \in H^*(X, \mathbb{Q}).$$

Let $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ be the standard symmetric obstruction theory of $\mathscr{P}$ in (49).

We first observe that the semiregularity maps provide *cosections*.

Proposition 4.12 *There exists a natural map*

$$\mathsf{SR} : \mathbb{E}^\vee[1] \to H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}}$$

satisfying the following properties:

(1) *The fibre of the map* SR *over* $[(F,s)] \in \mathscr{P}$ *is the composition*[19]

$$\mathsf{SR}|_{(F,s)} : R\operatorname{Hom}_X(I^\bullet, I^\bullet)_0[2] \to \operatorname{Ext}^2_X(I^\bullet, I^\bullet)_0 \xrightarrow{\mathsf{sr}} H^1(X, T_X)^\vee,$$

where sr *is the semiregularity map* (*Definition 4.8*(1)) *for the associated complex* $I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$.

(2) *The square of the map* SR *is*

$$\mathsf{SR}^2 = \mathsf{B}_\gamma \otimes 1 : H^1(X, T_X) \otimes \mathcal{O}_{\mathscr{P}} \to H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}}.$$

Here, we identified $H^3(X, \Omega^1_X) = H^1(X, T_X)^\vee$ *via Serre duality and the Calabi–Yau 4-form, as in Proposition 4.9.*

Proof Let $\mathbb{I}^\bullet$ denote the universal complex. Consider the map

$$R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[2] \xrightarrow{\mathrm{At}_{\mathscr{P}\times X/\mathscr{P}}(\mathbb{I}^\bullet)} R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \Omega^1_X)[3] \xrightarrow{\mathrm{tr}^3} H^3(X, \Omega^1_X) \otimes \mathcal{O}_{\mathscr{P}},$$

where the third trace map is defined as (see note 19)

$$\mathrm{tr}^3 : R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \Omega^1_X)[3] \xrightarrow{\mathrm{tr}} R\Gamma(X, \Omega^1_X) \otimes \mathcal{O}_{\mathscr{P}}[3] \to H^3(X, \Omega^1_X) \otimes \mathcal{O}_{\mathscr{P}}.$$

The map SR is then defined by composing the above with the isomorphism $H^3(X, \Omega^1_X) \cong H^1(X, T_X)^\vee$. The functoriality of the Atiyah classes ensures that the fibres of SR are the semiregularity maps. This proves (1).

Next, we claim that the square

$$\mathsf{SR}^2 : H^1(X, T_X) \otimes \mathcal{O}_{\mathscr{P}} \xrightarrow{\mathsf{SR}^\vee} \mathbb{E}[-1] \cong \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}}$$

is given by contraction with the Chern character

$$\mathrm{ch}_2(\mathbb{I}^\bullet) := \tfrac{1}{2}\operatorname{tr}(\mathrm{At}^2(\mathbb{I}^\bullet)) \in H^2(X, \Omega^2_X) \otimes \Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}).$$

Statement (2) then follows because Proposition 3.33 implies that the Chern character $\mathrm{ch}_2(\mathbb{I}^\bullet)$ is a horizontal section, i.e.,

$$\mathrm{ch}_2(\mathbb{I}^\bullet) = \gamma \otimes 1.$$

In order to prove the claim, it suffices to take hypercohomology $\mathbb{H}^0(\mathscr{P}, -)$ of the composition SR^2 and show that this equals contraction with $\mathrm{ch}_2(\mathbb{I}^\bullet)$. By [Buchweitz and Flenner 2003, Proposition 4.2],

[19] The first map is given by the canonical quasi-isomorphism $C^\bullet \cong \bigoplus_i h^i(C^\bullet)[-i]$ for any complex of *vector spaces* [Gelfand and Manin 2003, Proposition III.2.4].

the triangle

$$\begin{array}{ccc} H^1(\mathscr{P}\times X, T_{\mathscr{P}\times X/\mathscr{P}}) & \longrightarrow & \mathrm{Ext}^2_{\mathscr{P}\times X}(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0 \\ & {\scriptstyle \iota_{\mathrm{ch}_2(\mathbb{I}^\bullet)}}\searrow \quad \swarrow & \\ & H^3(\mathscr{P}\times X, \Omega^1_{\mathscr{P}\times X/\mathscr{P}}) & \end{array}$$

commutes, where the top map is induced by $\mathsf{SR}^\vee$ and the right map by SR. Using the decomposition

$$H^m(\mathscr{P}\times X, T_{\mathscr{P}\times X/\mathscr{P}}) = \bigoplus_{i+j=m} H^i(X,T_X)\otimes H^j(\mathscr{P},\mathcal{O}_{\mathscr{P}})$$

(see note 19) and similarly for $\Omega^1_{\mathscr{P}\times X/\mathscr{P}}$, we obtain the commutative diagram

$$\begin{array}{ccc} H^1(X,T_X)\otimes H^0(\mathscr{P},\mathcal{O}_{\mathscr{P}}) & \xrightarrow{\mathbb{H}^0(\mathscr{P},\mathsf{SR}^\vee)} & \mathrm{Ext}^2_{\mathscr{P}\times X}(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0 \\ & {\scriptstyle \iota_{\mathrm{ch}_2(\mathbb{I}^\bullet)}}\searrow \quad \swarrow{\scriptstyle \mathbb{H}^0(\mathscr{P},\mathsf{SR})} & \\ & H^3(X,\Omega^1_X)\otimes H^0(\mathscr{P},\mathcal{O}_{\mathscr{P}}) & \end{array}$$

as desired. □

Remark 4.13 The map SR in Proposition 4.12 is the collection of all cosections associated to $(3,1)$-forms in [Kiem and Park 2023, Example 9.2]. More precisely, if we choose a $(3,1)$-form $\xi \in H^1(X,T_X) \cong H^1(X,\Omega^3_X)$, then the composition

$$\mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} H^1(X,T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}} \xrightarrow{\xi} \mathcal{O}_{\mathscr{P}}$$

is exactly the cosection σ^ξ associated to the $(3,1)$-form ξ considered in [Kiem and Park 2023].

We sometimes call the symmetric complex $\mathbb{E}$ in the obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ the *virtual cotangent complex*. Following this perspective, we define the *reduced virtual cotangent complex* $\mathbb{E}^{\mathrm{red}}_V$ as follows:

Definition 4.14 Choose a nondegenerate subspace $V \subseteq H^1(X,T_X)$ with respect to the bilinear form B_γ.

(1) We denote by

$$\mathsf{SR}_V : \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} H^1(X,T_X)^\vee \otimes \mathcal{O}_{\mathscr{P}} \twoheadrightarrow V^\vee \otimes \mathcal{O}_{\mathscr{P}}$$

the composition. Then the square

$$\mathsf{SR}^2_V = \mathsf{B}_\gamma|_V \otimes 1 : V\otimes \mathcal{O}_{\mathscr{P}} \xrightarrow{\mathsf{SR}^\vee_V} \mathbb{E}[-1] \cong \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}_V} V^\vee \otimes \mathcal{O}_{\mathscr{P}}$$

is an isomorphism by Proposition 4.12(2).

(2) We define $\mathbb{E}^{\mathrm{red}}_V := \mathrm{cone}(\mathsf{SR}^\vee_V[1])$. Since the natural map $\mathsf{SR}^\vee_V[1]$ has a left inverse, we have a splitting

$$\mathbb{E} \cong \mathbb{E}^{\mathrm{red}}_V \oplus (V\otimes \mathcal{O}_{\mathscr{P}}[1]). \tag{60}$$

Moreover, $(\mathbb{E}^{\mathrm{red}}_V)^\vee[2] \to \mathbb{E}^\vee[2] \xrightarrow{\theta} \mathbb{E} \to \mathbb{E}^{\mathrm{red}}_V$ induces a symmetric form on $\mathbb{E}^{\mathrm{red}}_V$ and (60) is a splitting of symmetric complexes.

(3) We denote by

$$\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}}) \subseteq \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}})$$

the quadratic cone stack induced by the above symmetric form.

The main result in this subsection is the following *cone reduction* property:

Proposition 4.15 *Given a nondegenerate subspace $V \subseteq H^1(X, T_X)$ with respect to B_γ, there exists a commutative diagram*

$$\begin{array}{ccccc} & & \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) \\ & \nearrow & \downarrow & & \downarrow \\ (\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{Q}(\mathbb{E}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}) \end{array}$$

for a unique dotted arrow, where the square is cartesian. Here the closed embedding $(\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is given by (51).[20]

Proof We will form a commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) & \longrightarrow & \mathscr{P} \\ & \nearrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} \\ (\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{Q}(\mathbb{E}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{l}_{\mathsf{SR}_V}} & V|_{\mathscr{P}}^{\vee} \\ & & \downarrow & & \downarrow{\scriptstyle \mathfrak{q}_{\mathbb{E}}} & & \\ & & \mathscr{P} & \xrightarrow{0} & \mathbb{A}^1_{\mathscr{P}} & & \end{array} \tag{61}$$

and claim that all the squares are cartesian. Here, $\mathfrak{l}_{\mathsf{SR}_V} := \mathfrak{C}(\mathsf{SR}_V^{\vee}[1])$ is the linear function induced by $\mathsf{SR}_V^{\vee} : V \otimes \mathcal{O}_{\mathscr{P}} \to \mathbb{E}[-1]$ and $\mathfrak{q}_{\mathbb{E}}$ is the quadratic function induced by $\theta : \mathcal{O}_{\mathscr{P}} \to (\mathbb{E} \otimes \mathbb{E})[-2]$. Indeed, the lower square in (61) is cartesian since the quadratic cone stack $\mathfrak{Q}(\mathbb{E})$ is defined as the zero locus of the quadratic function $\mathfrak{q}_{\mathbb{E}}$. The upper right square in (61) is cartesian since $\mathbb{E}_V^{\mathrm{red}}$ is defined as the cone of $\mathsf{SR}_V^{\vee}[1] : V \otimes \mathcal{O}_{\mathscr{P}}[1] \to \mathbb{E}$; see the proof of Lemma 4.16. The middle upper square is cartesian since (60) is a direct sum decomposition of *symmetric complexes*. More precisely, we have

$$\mathfrak{q}_{\mathbb{E}} = \mathfrak{q}_{\mathbb{E}_V^{\mathrm{red}}} \circ p_1 + \mathfrak{q}_V|_{\mathscr{P}} \circ p_2 : \mathfrak{C}(\mathbb{E}) = \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) \times_{\mathscr{P}} V|_{\mathscr{P}}^{\vee} \to \mathbb{A}^1_{\mathscr{P}}, \tag{62}$$

where p_1, p_2 are the projection maps and $\mathfrak{q}_V : V \times V \to \mathbb{C}$ denotes the quadratic function induced by $\mathsf{B}_\gamma|_V$. Then (62) implies that the triangle

$$\begin{array}{ccc} \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) & & \\ \downarrow & \searrow^{\mathfrak{q}_{\mathbb{E}_V^{\mathrm{red}}}} & \\ \mathfrak{C}(\mathbb{E}) & \xrightarrow[\mathfrak{q}_{\mathbb{E}}]{} & \mathbb{A}^1_{\mathscr{P}} \end{array}$$

[20]When we use "red" as a subscript, it denotes the associated reduced substack.

commutes. Hence, the middle upper square in (61) is cartesian, as claimed. Finally, by Kiem and Li's cone reduction lemma [2013] in the form of Lemma 4.16 below, we can find the dotted arrow that fits into the commutative diagram (61). □

We rephrase Kiem and Li's cone reduction lemma as follows. This lemma is required to complete the proof of Proposition 4.15.

Lemma 4.16 (cf. [Kiem and Li 2013, Proposition 4.3]) *Let $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ be an obstruction theory for an arbitrary Deligne–Mumford stack $\mathcal{X}$. Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a map for a vector bundle F. Then there exists a commutative diagram*

(63)
$$\begin{array}{ccccc} & & \mathfrak{C}(\mathbb{E}_\Sigma) & \longrightarrow & \mathcal{X} \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle 0_F} \\ (\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{l}_\Sigma} & F \end{array}$$

with a unique dotted arrow, where the square is cartesian. Here the closed embedding $(\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} \hookrightarrow \mathfrak{C}(\mathbb{E})$ is induced by the obstruction theory ϕ, the linear function $\mathfrak{l}_\Sigma := \mathfrak{C}(\Sigma^\vee[1]) : \mathfrak{C}(\mathbb{E}) \to F$ is induced by the map Σ, and the perfect complex $\mathbb{E}_\Sigma$ is the cone of the map $\Sigma^\vee[1] : F^\vee[1] \to \mathbb{E}$.

Proof Note that the statement is local on $\mathcal{X}$. Thus we may assume that $\mathcal{X}$ is a quasiprojective scheme and $F = \mathcal{O}_{\mathcal{X}}^{\oplus r}$ is a trivial vector bundle. By the resolution property of $\mathcal{X}$, we can find a chain complex of vector bundles that represents $\mathbb{E}$,

$$\mathbb{E} \cong [E^{-e} \to \cdots \to E^{-2} \xrightarrow{d^{-1}} E^{-1} \xrightarrow{d^0} E^0],$$

and a morphism of chain complexes that represents Σ,

$$\begin{array}{ccccccccccc} \mathbb{E}^\vee[1] & & & E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots \longrightarrow & E_e \\ \downarrow{\scriptstyle \Sigma} & \cong & & \downarrow & & \downarrow{\scriptstyle \widetilde{\Sigma}} & & \downarrow & & \downarrow \\ F & & & 0 & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 \end{array}$$

where we denoted $E_i := (E^{-i})^\vee$. Then we also have

$$\mathbb{E}_\Sigma \cong [E^{-e} \to \cdots \to E^{-2} \oplus F^\vee \xrightarrow{(d^{-1}, \widetilde{\Sigma}^\vee)} E^{-1} \xrightarrow{d^0} E^0].$$

Firstly, we will show that the square in (63) is cartesian. Indeed, note that the cokernels

$$D := \mathrm{coker}(d^{-1} : E^{-2} \to E^{-1}), \quad D_{\widetilde{\Sigma}} := \mathrm{coker}((d^{-1}, \widetilde{\Sigma}^\vee) : E^{-2} \oplus F^\vee \to E^{-1})$$

fit into a right exact sequence of coherent sheaves

$$F^\vee \to D \to D_{\widetilde{\Sigma}} \to 0.$$

By taking the associated abelian cones, we obtain a cartesian square

$$\begin{array}{ccc} C(D_{\widetilde{\Sigma}}) & \longrightarrow & \mathbb{A}^1_{\mathcal{X}} \\ \downarrow & & \downarrow{\scriptstyle 0_F} \\ C(D) & \xrightarrow{\widetilde{\Sigma}|_{C(D)}} & F \end{array}$$

Since $\mathfrak{C}(\mathbb{E}) = [C(D)/E_0]$ and $\mathfrak{C}(\mathbb{E}_\Sigma) = [C(D_{\widetilde{\Sigma}})/E_0]$, we obtain the desired cartesian square via descent.

Secondly, we will find a dotted arrow that fits into the commutative diagram (63). The uniqueness is obvious. Consider the induced (2-term) perfect obstruction theory

$$\phi^\tau : \mathbb{E}^\tau := [E^{-1} \xrightarrow{d^0} E^0] \xrightarrow{t} \mathbb{E} \xrightarrow{\phi} \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$$

and an induced cosection Σ^τ that fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{E}^\vee[1] & \xrightarrow{\Sigma} & F \\ {\scriptstyle t^\vee[1]}\downarrow & \nearrow_{\Sigma^\tau} & \\ (\mathbb{E}^\tau)^\vee[1] & & \end{array}$$

Since F is a trivial vector bundle, we may write

$$\Sigma^\tau = (\sigma_1, \ldots, \sigma_r) : (\mathbb{E}^\tau)^\vee[1] \to F \cong \mathcal{O}_{\mathcal{X}}^{\oplus r}.$$

Then $\sigma_i : (\mathbb{E}^\tau)^\vee[1] \to \mathcal{O}_{\mathcal{X}}$ are cosections for the perfect obstruction theory ϕ^τ. Hence, by [Kiem and Li 2013, Proposition 4.3], we have

$$(\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} \subseteq \mathfrak{C}((\mathbb{E}^\tau)_{\sigma_i})$$

as substacks of $\mathfrak{C}(\mathbb{E}^\tau)$, where $(\mathbb{E}^\tau)_{\sigma_i} := \mathrm{cone}(\sigma_i^\vee[1])$. Here $\mathfrak{C}((\mathbb{E}^\tau)_{\sigma_i})$ are set-theoretically the kernel cone stacks in [Kiem and Li 2013]. Hence we have a commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{C}(\mathbb{E}_\Sigma) & \hookrightarrow & \bigcap_i \mathfrak{C}((\mathbb{E}^\tau)_{\sigma_i}) & \longrightarrow & \mathcal{X} \\ & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow{\scriptstyle 0_F} \\ (\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{C}(t)} & \mathfrak{C}(\mathbb{E}^\tau) & \xrightarrow{\mathfrak{t}_{\Sigma^\tau}} & F \\ & & & \searrow & & \nearrow_{\mathfrak{t}_\Sigma} & \end{array}$$

where the squares are cartesian by the result in the previous paragraph. Hence the dotted arrow in the above diagram gives us the desired dotted arrow in (63). □

Now Theorem 4.5 is a direct corollary of Proposition 4.15.

Proof of Theorem 4.5 We first construct a reduced virtual cycle with respect to a maximal nondegenerate subspace $V \subseteq H^1(X, T_X)$. Here, "maximal" means that $\dim(V) = \mathrm{rank}(\mathsf{B}_\gamma) = \rho_\gamma$. Thus the composition

$$V \hookrightarrow H^1(X, T_X) \twoheadrightarrow H^1(X, T_X)_\gamma \tag{64}$$

is an isomorphism.

As in the proof of Theorem 4.1 (see also Remark 4.3), we can construct a square root Gysin pullback

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})}} : A_*(\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})) \to A_*(\mathscr{P})$$

for the symmetric complex $\mathbb{E}_V^{\mathrm{red}}$. Here the orientation of $\mathbb{E}_V^{\mathrm{red}}$ is determined by the orientations of $\mathbb{E}$ and $H^1(X,T_X)_\gamma$ via the canonical isomorphisms

$$\det(\mathbb{E}) \cong \det(\mathbb{E}_V^{\mathrm{red}}) \otimes \det(V) \cong \det(\mathbb{E}_V^{\mathrm{red}}) \otimes \det(H^1(X,T_X)_\gamma)$$

of line bundles, induced by the decomposition (60) and the isomorphism (64).

By the cone reduction property in Proposition 4.15, we may consider the fundamental class of the intrinsic normal cone $\mathfrak{C}_{\mathscr{P}}$ as a cycle class in $\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})$,

$$[\mathfrak{C}_{\mathscr{P}}] \in A_0(\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})).$$

Indeed, this can be defined as the pushforward of the fundamental class $[\mathfrak{C}_{\mathscr{P}}] \in A_*((\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}}) = A_*(\mathfrak{C}_{\mathscr{P}})$ under the closed embedding $(\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})$.

Analogously to the definition of the nonreduced virtual cycle in (55), we define the *reduced virtual cycle* as

$$[\mathscr{P}]_V^{\mathrm{red}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})}}[\mathfrak{C}_{\mathscr{P}}] \in A_{\mathrm{rvd}}(\mathscr{P}). \tag{65}$$

Then the reduced virtual dimension is

$$\mathrm{rvd} = \mathrm{vd} + \tfrac{1}{2}\dim(V) = n - \tfrac{1}{2}\gamma^2 + \tfrac{1}{2}\rho_\gamma.$$

We now claim that the reduced virtual cycle $[\mathscr{P}]_V^{\mathrm{red}}$ in (65) is independent of the choice of a maximal nondegenerate subspace $V \subseteq H^1(X,T_X)$. Note that a choice of a maximal isotropic subspace V of $H^1(X,T_X)$ is equivalent to a choice of a section of the surjection $H^1(X,T_X) \twoheadrightarrow H^1(X,T_X)_\gamma$. Thus, if we fix one splitting

$$H^1(X,T_X) = H^1(X,T_X)_\gamma \oplus \ker(\mathsf{B}_\gamma),$$

then any other splitting corresponds to a map $H^1(X,T_X)_\gamma \to \ker(\mathsf{B}_\gamma)$. Therefore, any two splittings are connected via a family of splittings parametrized by $\mathbb{A}^1$. A deformation argument proves the claim.[21] □

Remark 4.17 We view the cosection localization (Proposition 4.15) in this paper as an *orthogonal* version to the one in [Kiem and Park 2023]. Indeed, there are two types of cosection $\Sigma : \mathbb{E}^\vee[1] \to F$ in DT4 theory:

(1) *isotropic* cosections, i.e., $\Sigma^2 = 0$;

(2) *nondegenerate* cosections, i.e., Σ^2 is an isomorphism.

In [loc. cit.], cosection localization for the first case is studied, whereas we introduced the second case in this paper.

[21] Generally, we need to be careful when we apply the cone reduction in the relative setting. But, in this case, the map $\mathscr{P} \times \mathbb{A}^1 \to \mathbb{A}^1$ is flat, so we can detect the cone reduction property fibrewise.

In the second case, if the orthogonal bundle (F, Σ^2) is of even rank and has a maximal isotropic subbundle, then we can reduce to the first case. Then both approaches give the same reduced virtual cycle.

However, for surfaces on Calabi–Yau 4-folds, there are *many* examples (e.g., Example 1.5) that have an *odd* number of cosections. Therefore we indeed need cosection localization as in the second case for counting surfaces.

Remark 4.18 In general, the cone reduction property in Proposition 4.15 does not imply the existence of a reduced obstruction theory; see [Maulik et al. 2010, Appendix A]. However, in our situation for $\mathscr{P}_v^{(q)}(X)$, we will see in the next subsection that there indeed exists a reduced obstruction theory.

We now prove Theorem 4.6. The implication $\Longrightarrow$ can be shown using the infinitesimal lifting criterion as in [Bloch 1972, Theorem 7.3]. We provide an alternative proof, more in line with DT4 theory, using the cone reduction property (Proposition 4.15).

Proof of Theorem 4.6 ($\Longrightarrow$) Let $[(F, s)] \in \mathscr{P}$ be a PT_q pair for which the associated perfect complex $I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$ is semiregular.

Choose a maximal nondegenerate subspace $V \subseteq H^1(X, T_X)$ with respect to B_γ. We first claim that the restriction of the obstruction map ob in (57) to V,

$$\mathsf{ob}|_V : V \hookrightarrow H^1(X, T_X) \twoheadrightarrow \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0,$$

is an isomorphism. Indeed, by Corollary 4.10, the above map $\mathsf{ob}|_V$ is injective. On the other hand, the semiregularity of $I^\bullet$ implies that the obstruction map $\mathsf{ob} = \mathsf{sr}^\vee : H^1(X, T_X) \to \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0$ is surjective. Hence we have

$$\dim(V) = \rho_\gamma = \mathrm{rank}(\mathsf{B}_\gamma) = \mathrm{rank}(\mathsf{ob}^*(\mathsf{Y})) = \mathrm{rank}(\mathsf{Y}) = \dim_{\mathbb{C}} \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0.$$

Therefore the above map $\mathsf{ob}|_V$ is bijective. This proves the claim.

By the derived Darboux theorem [Brav et al. 2019, Theorem 5.18(ii)], we can find a diagram

$$\begin{array}{ccc} & & E \\ & & \big\downarrow \big) t \\ U(t) & \hookrightarrow & U \end{array}$$

where U is a smooth affine scheme, E is an orthogonal bundle over U, $t \in \Gamma(U, E)$ is an isotropic section, and $U(t)$ is the zero locus of t in U, with a Zariski open embedding $U(t) \hookrightarrow \mathscr{P}$ whose image contains $[I^\bullet] \in \mathscr{P}$. Also, the standard symmetric obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ in (49) can be written as

$$\begin{array}{ccc} \mathbb{E}|_{U(t)} & & T_U|_{U(t)} \xrightarrow{dt} E|^\vee_{U(t)} \xrightarrow{dt} \Omega_U|_{U(t)} \\ \big\downarrow \phi & = & \big\downarrow \qquad\quad \big\downarrow t \qquad\quad \big\| \\ \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}|_{U(t)} & & 0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_U|_{U(t)} \end{array}$$

where $\mathcal{I} = \mathcal{I}_{U(t)/U}$ is the ideal sheaf of $U(t) \subseteq U$.

Moreover, by choosing a minimal Darboux chart (in the sense of [Brav et al. 2019, Definition 2.13]), we can further assume that the fibre of the map $dt : T_U|_{U(t)} \to E|_{U(t)}^\vee$ at $[I^\bullet] \in U(t)$ is zero.

Consider the map SR_V in Definition 4.14(1). Then we have a diagram

$$\begin{array}{ccccc} \Omega_U|_{U(t)} & \longrightarrow & \mathbb{E}|_{U(t)} & \longrightarrow & [T_U|_{U(t)} \to E|_{U(t)}^\vee \to 0] \\ & \searrow & \downarrow{\scriptstyle \mathsf{SR}_V} & & \\ & & V^\vee \otimes \mathcal{O}_{U(t)}[1] & & \end{array}$$

where the last term is the stupid truncation and the dotted arrow is the composition. Since $U(t)$ is affine, the dotted arrow is zero and we obtain a morphism of cochain complexes

$$\begin{array}{ccc} \mathbb{E}|_{U(t)} & & T_U|_{U(t)} \xrightarrow{dt} E|_{U(t)}^\vee \xrightarrow{dt} \Omega_U|_{U(t)} \\ \downarrow{\scriptstyle \mathsf{SR}_V} & = & \downarrow \qquad \downarrow{\scriptstyle \widetilde{\mathsf{SR}}_V} \qquad \downarrow \\ V^\vee \otimes \mathcal{O}_{U(t)}[1] & & 0 \longrightarrow V^\vee \otimes \mathcal{O}_{U(t)} \longrightarrow 0 \end{array}$$

for some map $\widetilde{\mathsf{SR}}_V$. By Proposition 4.12(1) and the fact that the fibre of dt at $[(F,s)]$ is zero, the fibre of the map $\widetilde{\mathsf{SR}}_V$ over $[(F,s)] \in \mathscr{P}$ can be identified with the composition

$$\widetilde{\mathsf{SR}}_V|_{(F,s)} : \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \xrightarrow{\mathsf{sr}} H^1(X, T_X)^\vee \twoheadrightarrow V^\vee. \tag{66}$$

Since $\mathsf{ob}|_V$ is an isomorphism, Proposition 4.9 implies that the map (66) is an isomorphism. Therefore, by shrinking U further, we may assume that the map $\widetilde{\mathsf{SR}}_V$ is an isomorphism.

Consider $U(t)$ as the zero section $U(t) \subset E|_{U(t)}$. By the cone reduction property in Proposition 4.15, we have

$$\left[\frac{(C_{U(t)/U})_{\mathrm{red}}}{T_U|_{U(t)}}\right] = (\mathfrak{C}_{U(t)})_{\mathrm{red}} \subseteq \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) = \left[\frac{U(t)}{T_U|_{U(t)}}\right]$$

as substacks of $\mathfrak{C}(\mathbb{E}) \subseteq [E|_{U(t)}/T_U|_{U(t)}]$ since $\ker(\widetilde{\mathsf{SR}}_V) = 0$. Therefore, we have

$$(C_{U(t)/U})_{\mathrm{red}} = U(t).$$

Equivalently, the ideal $\bigoplus_{n>0} \mathcal{I}^n/\mathcal{I}^{n+1} \subseteq \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ is nilpotent. Hence, we have $\mathcal{I}^n/\mathcal{I}^{n+1} = 0$ for some n, and Nakayama's lemma implies $(1-i)\cdot\mathcal{I}^n = 0$ for some $i \in \mathcal{I}$. Since U is integral and $U(t)$ is nonempty, we have $\mathcal{I} = 0$ and thus $t = 0$. Therefore, $U(t) = U$ is smooth of dimension

$$\dim(U) = \big(\dim(U) - \tfrac{1}{2}\mathrm{rank}(E)\big) + \tfrac{1}{2}\mathrm{rank}(E) = \mathrm{vd} + \tfrac{1}{2}\rho_\gamma = \mathrm{rvd}.$$

This completes the proof of ($\Longrightarrow$) since $U(t)$ is an open neighbourhood of $[I^\bullet] \in \mathscr{P}$.

($\Longleftarrow$) Assume that $\mathscr{P}$ is smooth of dimension rvd at $[I^\bullet] \in \mathscr{P}$. Choose a maximal nondegenerate subspace V of $H^1(X, T_X)$ with respect to B_γ. By Corollary 4.10, the restriction of the obstruction map to V,

$$\mathsf{ob}|_V : V \hookrightarrow H^1(X, T_X) \xrightarrow{\mathsf{ob}} \mathrm{Ext}_X^2(I^\bullet, I^\bullet)_0,$$

is injective. On the other hand, since $\mathscr{P}$ is smooth at $[I^\bullet] \in \mathscr{P}$, we have

$$\dim(V) = \rho_\gamma = 2 \cdot (\mathrm{rvd} - \mathrm{vd}) = 2 \cdot \big(\mathrm{ext}^1 - \big(\mathrm{ext}^1 - \tfrac{1}{2}\,\mathrm{ext}^2\big)\big) = \mathrm{ext}^2,$$

where $\mathrm{ext}^1 := \dim_{\mathbb{C}} \mathrm{Ext}^1_X(I^\bullet, I^\bullet)_0$ and $\mathrm{ext}^2 := \dim_{\mathbb{C}} \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0$. Hence the above map $\mathsf{ob}|_V$ is bijective. Then the obstruction map ob is surjective, and its dual sr is injective (Proposition 4.9). Therefore $[I^\bullet] \in \mathscr{P}$ is semiregular. □

4.5 Algebraic twistor family

In this subsection, we construct a *reduced obstruction theory* via the *algebraic twistor family* introduced in [Kool and Thomas 2014].

We use the notation from the previous subsections. Let $\mathscr{P}_v^{(q)}(X)$ be the moduli space of PT_q pairs (F, s) on a Calabi–Yau 4-fold X with fixed cohomological Chern character

$$\mathrm{ch}(F) = v = (0, 0, \gamma, \beta, n - \gamma \cdot \mathrm{td}_2(X)) \in H^*(X, \mathbb{Q}).$$

The main result in this subsection is the following:

Proposition 4.19 *Given a nondegenerate subspace $V \subseteq H^1(X, T_X)$ with respect to B_γ, the composition*

$$\phi_V^{\mathrm{red}} : \mathbb{E}_V^{\mathrm{red}} \to \mathbb{E} \xrightarrow{\phi} \tau^{\geq -1} \mathbb{L}_{\mathscr{P}_v^{(q)}(X)} \tag{67}$$

is also a symmetric obstruction theory, i.e., $h^0(\phi_V^{\mathrm{red}})$ is bijective and $h^{-1}(\phi_V^{\mathrm{red}})$ is surjective. Here the first map in (67) *is given by the decomposition* (60) *and the second map is the standard symmetric obstruction theory in* (49).

The rest of this subsection is devoted to the proof of Proposition 4.19. By the deformation theory of algebraic varieties, any subspace $V \subseteq H^1(X, T_X)$ corresponds to a fibre diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X}_V \\ \downarrow & & \downarrow f_V \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathcal{A}_V \end{array}$$

where $\mathcal{X}_V \to \mathcal{A}_V$ is a smooth projective morphism over the spectrum of the local Artinian $\mathbb{C}$-algebra $A_V := \mathcal{O}_V/\mathfrak{m}^2 = \mathbb{C} \oplus V^\vee$, where $\mathfrak{m} \subset \mathcal{O}_V$ denotes the maximal ideal of functions vanishing at the origin. Here $\Omega^1_{\mathcal{A}_V} = V^\vee \otimes \mathcal{O}_{\mathcal{A}_V}$, and the (restriction) of the (classical) Kodaira–Spencer map

$$\mathsf{KS}_{\mathcal{X}_V/\mathcal{A}_V}|_X : \Omega^1_X \to V^\vee \otimes \mathcal{O}_X[1]$$

corresponds to the given inclusion $V \subseteq H^1(X, T_X)$ via adjunction. Let

$$\tilde{v} = v \otimes 1 \in H^*_{\mathrm{DR}}(\mathcal{X}_V/\mathcal{A}_V) \cong H^*_{\mathrm{DR}}(X) \otimes_{\mathbb{C}} A_V$$

be the unique horizontal lift of $v \in H^*_{\mathrm{DR}}(X)$ (see Section 3.5 for the definitions of the de Rham cohomology and horizontal sections). Form a fibre diagram

$$\begin{array}{ccc} \mathscr{P}_v^{(q)}(X) & \hookrightarrow & \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathcal{A}_V \end{array} \tag{68}$$

where $\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V)$ is the relative moduli space of PT_q pairs on the fibres of $\mathcal{X}_V \to \mathcal{A}_V$. The key observation is the following:

Proposition 4.20 *If $V \subseteq H^1(X, T_X)$ is a nondegenerate subspace, then the canonical inclusion map*

$$\mathscr{P}_v^{(q)}(X) \hookrightarrow \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V)$$

is an isomorphism of schemes.

Proof The proof is similar to [Kool and Thomas 2014, Lemma 2.1 and Proposition 2.2]. We first claim that the *Hodge locus* of the horizontal section

$$\gamma \otimes 1 \in H^4_{\mathrm{DR}}(\mathcal{X}_V/\mathcal{A}_V) \cong H^4_{\mathrm{DR}}(X) \otimes A_V$$

is just the unique closed point in $\mathcal{A}_V$. More precisely, the scheme-theoretic zero locus of the induced section

$$\overline{\gamma \otimes 1} \in \frac{H^4_{\mathrm{DR}}(\mathcal{X}_V/\mathcal{A}_V)}{F^2 H^4_{\mathrm{DR}}(\mathcal{X}_V/\mathcal{A}_V)}$$

is the closed point in $\mathcal{A}_V$. Indeed, choose a subspace $W \subseteq V$ and let

$$\mathcal{A}_W := \mathrm{Spec}(\mathbb{C} \oplus W^\vee) \subseteq \mathcal{A}_V$$

be the corresponding subscheme. Assume that the section $\overline{\gamma \otimes 1}$ vanishes on $\mathcal{A}_W$, i.e.,

$$\overline{\gamma \otimes 1}|_{\mathcal{A}_W} = 0 \in \frac{H^4_{\mathrm{DR}}(\mathcal{X}_W/\mathcal{A}_W)}{F^2 H^4_{\mathrm{DR}}(\mathcal{X}_W/\mathcal{A}_W)}.$$

Equivalently, the unique horizontal lift of γ over $\mathcal{A}_W$ is a Hodge class.

By [Bloch 1972, Proposition 4.2] (see Lemma 4.11(1)), we deduce that the composition

$$\mathsf{B}_\gamma|_W : W \hookrightarrow H^1(X, T_X) \xrightarrow{\iota_-(\gamma)} H^3(X, \Omega_X^1) \cong H^1(X, T_X)^\vee$$

is zero. Hence we have

$$W \subseteq V \cap \ker(\mathsf{B}_\gamma) = 0$$

by the nondegeneracy of V. This proves the claim.

We will now show that any morphism

$$t : T \to \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V) \tag{69}$$

from an affine scheme T factors through $\mathscr{P}_v^{(q)}(X)$. Indeed, let

$$I^\bullet = [\mathcal{O}_{\mathcal{X}_T} \xrightarrow{s} F]$$

be the family of PT_q pairs that corresponds to the map (69), where

$$\begin{array}{ccc} \mathcal{X}_T & \longrightarrow & \mathcal{X}_V \\ \downarrow & & \downarrow \\ T & \xrightarrow{t} & \mathcal{A}_V \end{array}$$

is the induced fibre diagram. By Proposition 3.33, we have

$$t^*(\gamma \otimes 1) = \widetilde{\mathrm{ch}_2(I^\bullet)} \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}_T/T),$$

where $\widetilde{\mathrm{ch}_2(I^\bullet)} \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}_T/T)$ is the unique horizontal lift of the Chern character $\mathrm{ch}_2(I^\bullet) := \frac{1}{2}\operatorname{tr}(\mathrm{At}^2(I^\bullet)) \in H^2(\mathcal{X}_T, \Omega^2_{\mathcal{X}_T/T})$. Thus we have

$$t^*(\overline{\gamma \otimes 1}) = 0 \in \frac{H^4_{\mathrm{DR}}(\mathcal{X}_T/T)}{F^2 H^4_{\mathrm{DR}}(\mathcal{X}_T/T)}.$$

Hence the map $t : T \to \mathcal{A}_V$ factors through the zero locus of $\overline{\gamma \otimes 1}$, which is the unique closed point in $\mathcal{A}_V$ by the result in the previous paragraph. Now the fibre diagram (68) proves the desired property.

Finally, since the map $\mathscr{P}_v^{(q)}(X) \hookrightarrow \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V)$ is a closed embedding, the result in the previous paragraph proves that the map is an isomorphism. □

By Proposition 4.20, the reduced obstruction theory of $\mathscr{P}_v^{(q)}(X)$ can be obtained as the obstruction theory of $\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V)$.

Proof of Proposition 4.19 By Proposition 4.20, the fibre diagram (68) can be expressed as

$$\begin{array}{ccc} & & \mathcal{A}_V \\ & \nearrow & \uparrow\downarrow \\ \mathscr{P}_v^{(q)}(X) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

where the two triangles commute. Hence we have a canonical direct sum decomposition of the cotangent complexes

$$\mathbb{L}_{\mathscr{P}_v^{(q)}(X)} \rightleftarrows \mathbb{L}_{\mathscr{P}_v^{(q)}(X)/\mathcal{A}_V} \rightleftarrows \mathbb{L}_{\mathcal{A}_V}|_{\mathscr{P}_v^{(q)}(X)}[1], \tag{70}$$

where the upper arrows and the lower arrows are the canonical distinguished triangles (see [Qu 2018, Lemma 2.10]). Let

$$\phi_{\mathcal{X}_V/\mathcal{A}_V} : \mathbb{E}_{\mathcal{X}_V/\mathcal{A}_V} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)/\mathcal{A}_V}$$

be the standard *relative* obstruction theory for $\mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}_V/\mathcal{A}_V) \to \mathcal{A}_V$, where $\mathbb{E}_{\mathcal{X}_V/\mathcal{A}_V} = \mathbb{E}$ by Proposition 4.20. We will consider the induced *absolute* obstruction theory. Indeed, form a morphism of distinguished

triangles

$$\begin{array}{ccccc} \mathbb{E}_V^{\mathrm{red}} & \longrightarrow & \mathbb{E} & \xrightarrow{\mathsf{SR}_V[1]} & V^\vee \otimes \mathcal{O}_{\mathscr{P}_v^{(q)}(X)}[1] \\ \downarrow{\scriptstyle \phi_V^{\mathrm{red}}} & & \downarrow{\scriptstyle \phi_{\mathcal{X}_V/\mathcal{A}_V}} & & \| \\ \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)} & \longrightarrow & \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)/\mathcal{A}_V} & \xrightarrow{\mathsf{KS}} & \Omega^1_{\mathcal{A}_V}|_{\mathscr{P}_v^{(q)}(X)}[1] \end{array} \tag{71}$$

for some dotted arrow ϕ_V^{red}, where the upper distinguished triangle is induced by (60), the lower distinguished triangle is the truncation of (70), and the right square commutes by Lemma 4.21 below. By the long exact sequence associated to (71), we can deduce that the map $\phi_V^{\mathrm{red}} : \mathbb{E}_V^{\mathrm{red}} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)}$ is an obstruction theory, in the sense of [Behrend and Fantechi 1997]. Moreover, since the lower distinguished triangle in (71) splits, the obstruction theory $\phi_V^{\mathrm{red}} : \mathbb{E}_V^{\mathrm{red}} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)}$ is uniquely determined by the diagram (71). Thus the map ϕ_V^{red} in (71) is equal to the composition (67) since the composition

$$\mathbb{E} = \mathbb{E}_{\mathcal{X}_V/\mathcal{A}_V} \xrightarrow{\phi_{\mathcal{X}_V/\mathcal{A}_V}} \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)/\mathcal{A}_V} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)}$$

is the original (nonreduced) obstruction theory for $\mathscr{P}_v^{(q)}(X)$. □

We need the following compatibility result between the Kodaira–Spencer maps to complete the proof of Proposition 4.19. Here we consider a more general version than needed for Proposition 4.19. This will be used later, in the proof of deformation invariance in Theorem 5.1.

Lemma 4.21 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of relative dimension 4 over a quasi-projective scheme $\mathcal{B}$. Let $\mathscr{P} := \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B})$ be the relative moduli space of PT_q pairs on the fibres of $f : \mathcal{X} \to \mathcal{B}$. Let*

$$\begin{array}{ccc} \mathscr{P} \times_{\mathcal{B}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ \mathscr{P} := \mathscr{P}_{\tilde{v}}^{(q)}(\mathcal{X}/\mathcal{B}) & \longrightarrow & \mathcal{B} \end{array}$$

be a fibre diagram. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{E}_{\mathcal{X}/\mathcal{B}} := R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega_{\mathcal{X}/\mathcal{B}})_0[3] & \xrightarrow{\mathsf{SR}_{\mathcal{X}/\mathcal{B}}} & Rf_*(\Omega^1_{\mathcal{X}/\mathcal{B}} \otimes \omega_{\mathcal{X}/\mathcal{B}})|_{\mathscr{P}}[4] \\ \downarrow{\scriptstyle \phi_{\mathcal{X}/\mathcal{B}}} & & \downarrow{\scriptstyle -\mathsf{KS}_{\mathcal{X}/\mathcal{B}}} \\ \mathbb{L}_{\mathscr{P}/\mathcal{B}} & \xrightarrow{\mathsf{KS}_{\mathscr{P}/\mathcal{B}}} & \mathbb{L}_{\mathcal{B}}|_{\mathscr{P}}[1] \end{array} \tag{72}$$

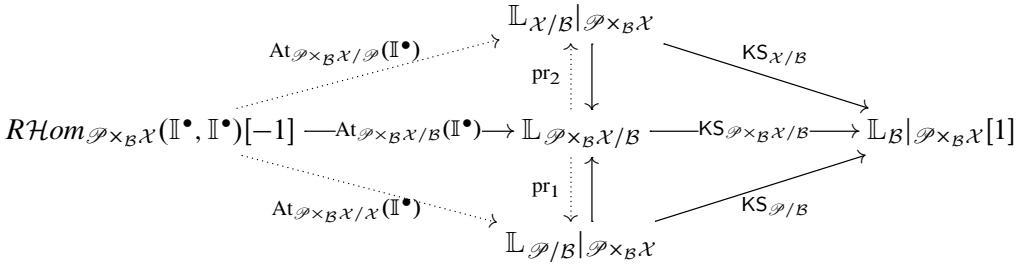
where the relative obstruction theory $\phi_{\mathcal{X}/\mathcal{B}}$ and relative cosection $\mathsf{SR}_{\mathcal{X}/\mathcal{B}}$ are given by the two components

$$\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{X}}(\mathbb{I}^\bullet) : \mathbb{I}^\bullet \xrightarrow{\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}(\mathbb{I}^\bullet)} \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}[1] \xrightarrow{\mathrm{pr}_1} \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathscr{P}/\mathcal{B}}[1],$$

$$\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathscr{P}}(\mathbb{I}^\bullet) : \mathbb{I}^\bullet \xrightarrow{\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}(\mathbb{I}^\bullet)} \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}[1] \xrightarrow{\mathrm{pr}_2} \mathbb{I}^\bullet \otimes \Omega^1_{\mathcal{X}/\mathcal{B}}[1]$$

of the Atiyah class $\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}(\mathbb{I}^\bullet)$ *of the universal complex* $\mathbb{I}^\bullet$*. For* $\phi_{\mathcal{X}/\mathcal{B}}$*, we use the adjunction* $R\pi_*(-)\dashv L\pi^*(-)\otimes\omega_{\mathcal{X}/\mathcal{B}}[4]$.

Proof The proof is similar to that of Lemma 4.11. Consider the diagram

(73)


where the left two triangles commute with the dotted arrows, and the right two triangles commute with the solid arrows. Since $\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}(\mathbb{I}^\bullet)$ factors through $\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}}(\mathbb{I}^\bullet)$, the composition of the two horizontal arrows in (73) vanishes:

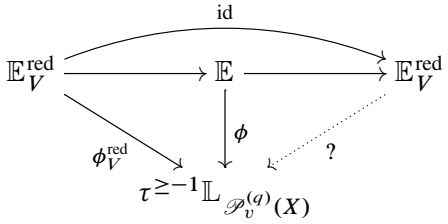
$$\mathsf{KS}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}\circ\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{B}}(\mathbb{I}^\bullet)=0.$$

We deduce that the big square in (73) commutes (up to sign)

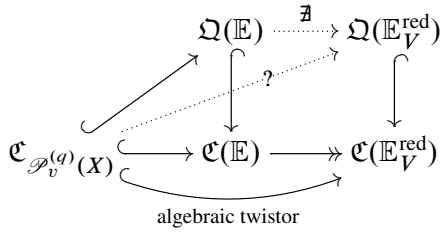
$$\mathsf{KS}_{\mathcal{X}/\mathcal{B}}\circ\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathscr{P}}(\mathbb{I}^\bullet)=-\mathsf{KS}_{\mathscr{P}/\mathcal{B}}\circ\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathcal{X}}(\mathbb{I}^\bullet). \tag{74}$$

The commutativity of (72) follows from (74) by adjunction. □

Remark 4.22 The main limitation of the algebraic twistor family approach in this subsection is that the reduced obstruction theory is given as a *composition*, but not as a *factorization*:

(75)


In classical reduced theory, e.g., [Kool and Thomas 2014], this was sufficient to construct reduced Behrend–Fantechi virtual cycles [1997]. However, in DT4 theory, this is not sufficient to construct reduced Oh–Thomas virtual cycles [2023] since we need to know whether the reduced obstruction theory ϕ_V^{red} satisfies the *isotropic condition*



Thus we additionally need the cone reduction property in the previous subsection (Proposition 4.15) to construct a reduced virtual cycle from ϕ_V^{red}.

To conclude, the cone reduction property in Proposition 4.15 and the reduced obstruction theory in Proposition 4.19 give us a commutative diagram

$$\begin{array}{ccc} (\mathfrak{C}_{\mathscr{P}_v^{(q)}(X)})_{\mathrm{red}} & \xhookrightarrow{\text{cone reduction}} & \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}}) \\ \downarrow & & \downarrow \\ \mathfrak{C}_{\mathscr{P}_v^{(q)}(X)} & \xhookrightarrow[\text{algebraic twistor}]{} & \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) \end{array} \tag{76}$$

This proves Theorem 1.7.

In Appendix A, we will construct a diagonal arrow $\mathfrak{C}_{\mathscr{P}_v^{(q)}(X)} \hookrightarrow \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}})$ in commutative diagram (76) using derived algebraic geometry; see Remark A.16.

Remark 4.23 By Proposition 4.20, the standard derived enhancement on $\mathscr{P}_v^{(q)}(\mathcal{X}_V/\mathcal{A}_V)$ gives us a *reduced derived enhancement* on $\mathscr{P}_v^{(q)}(X)$. However, this derived enhancement is not necessarily homotopically finitely presented over $\mathrm{Spec}(\mathbb{C})$ and does not give us the reduced virtual cycle in Theorem 4.5. In Appendix A, we will construct a different reduced derived enhancement on $\mathscr{P}_v^{(q)}(X)$ that has a (-2)-shifted symplectic structure and induces the desired reduced virtual cycle.

4.6 Reduced virtual dimension

In this subsection, we present various methods to compute the number ρ_γ and the reduced virtual dimension.

Local complete intersection surfaces Let X be a Calabi–Yau 4-fold. We take $i : S \hookrightarrow X$ to be a *local complete intersection* (lci) surface, and $v = \mathrm{ch}(\mathcal{O}_S) = (0, 0, \gamma, \beta, \chi(\mathcal{O}_S) - \gamma \cdot \mathrm{td}_2(X)) \in H^*(X)$. Our goal is to make the key diagram from Proposition 4.9 more explicit and determine the reduced virtual dimension $\mathrm{rvd} = \chi(\mathcal{O}_S) - \frac{1}{2}\gamma^2 + \frac{1}{2}\rho_\gamma$. The normal sheaf $N_{S/X}$ is locally free because S is lci. Since X has trivial canonical bundle, we have

$$K_S \cong N_{S/X} \wedge N_{S/X}$$

and the Serre duality pairing gives a nondegenerate symmetric bilinear pairing

$$\mathsf{SD}_S : H^1(S, N_{S/X}) \cong H^1(S, N_{S/X})^\vee.$$

The space $H^1(S, N_{S/X})$ maps to the obstruction space of $I_{S/X}$. From the short exact sequence

$$0 \to I_{S/X} \to \mathcal{O}_X \to \mathcal{O}_S \to 0, \tag{77}$$

we get an exact triangle

$$R\,\mathrm{Hom}_X(I_{S/X}, \mathcal{O}_S) \to R\,\mathrm{Hom}_X(I_{S/X}, I_{S/X})[1] \to R\,\mathrm{Hom}_X(I_{S/X}, \mathcal{O}_X)[1].$$

By taking cones of maps from $R\Gamma(X,\mathcal{O}_X)[1]$ to the second and the third factor, we get an exact triangle

$$R\operatorname{Hom}_X(I_{S/X},\mathcal{O}_S)\to R\operatorname{Hom}_X(I_{S/X},I_{S/X})_0[1]\to R\operatorname{Hom}_X(\mathcal{O}_S,\mathcal{O}_X)[2]. \tag{78}$$

Consider the map

$$\theta_S: H^1(S,N_{S/X})\to \operatorname{Ext}^1_X(I_{S/X},\mathcal{O}_S)\to \operatorname{Ext}^2_X(I_{S/X},I_{S/X})_0, \tag{79}$$

where the first map is defined by adjunction.

We give local descriptions of the obstruction map (57) and the semiregularity map (56) for local complete intersection surfaces. The map defined by

$$\mathsf{ob}_{S/X}: H^1(X,T_X)\to H^1(S,T_X|_S)\to H^1(S,N_{S/X})$$

is the obstruction map for deforming $S\subset X$ along a deformation of X [Bloch 1972, Proposition 2.6]. Let $d_S:\mathcal{O}_S\to I_{S/X}[1]$ be the map induced by (77). Let $\alpha:\operatorname{Ext}^1_X(I_{S/X},\mathcal{O}_S)\to H^3(X,\Omega^1_X)$ be the map defined by the trace of

$$I_{S/X}\to\mathcal{O}_S[1]\xrightarrow{\mathrm{At}(\mathcal{O}_S)}\mathcal{O}_S\otimes\Omega^1_X[2]\xrightarrow{d_S\otimes\mathrm{id}} I_{S/X}\otimes\Omega^1_X[3].$$

The local semiregularity map is defined by

$$\mathsf{sr}_{S/X}: H^1(S,N_{S/X})\to \operatorname{Ext}^1_X(I_{S/X},\mathcal{O}_S)\xrightarrow{\alpha} H^3(X,\Omega^1_X). \tag{80}$$

The main result of this subsection consists of the following compatibilities:

Proposition 4.24 *Let $i:S\hookrightarrow X$ be a local complete intersection surface inside a Calabi–Yau 4-fold X.*

(1) *The map* (79) *is an isomorphism between two vector spaces with nondegenerate symmetric bilinear forms*

$$\theta_S:(H^1(S,N_{S/X}),\mathsf{SD}_S)\xrightarrow{\cong}(\operatorname{Ext}^2_X(I_{S/X},I_{S/X})_0,\mathsf{Y}).$$

(2) *The following diagram commutes*:

$$\begin{array}{ccccc} H^1(X,T_X) & \xrightarrow{\mathsf{ob}_{S/X}} & H^1(S,N_{S/X}) & \xrightarrow{\mathsf{sr}_{S/X}} & H^3(X,\Omega^1_X)\\ & \searrow^{\mathsf{ob}} & \cong\downarrow\theta_S\ (2') & \nearrow_{\mathsf{sr}}\ (2'') & \\ & & \operatorname{Ext}^2_X(I_{S/X},I_{S/X})_0 & & \end{array}$$

Proof To simplify the notation, denote $I:=I_{S/X}$. We first prove that the map θ_S is an isomorphism. The compatibility of the two bilinear forms will be proved at the end. The exact triangle (78) gives a long exact sequence

$$\begin{aligned} 0\to \underbrace{\operatorname{Ext}^2_X(\mathcal{O}_S,\mathcal{O}_X)}_{\cong H^0(S,K_S)}\xrightarrow{\text{(i)}}\operatorname{Ext}^1_X(I,\mathcal{O}_S)\xrightarrow{\text{(ii)}}\operatorname{Ext}^2_X(I,I)_0\to \underbrace{\operatorname{Ext}^3_X(\mathcal{O}_S,\mathcal{O}_X)}_{\cong H^1(S,K_S)}\xrightarrow{\text{(iii)}}\operatorname{Ext}^2_X(I,\mathcal{O}_S)\\ \xrightarrow{\text{(iv)}}\operatorname{Ext}^3_X(I,I)_0\to\operatorname{Ext}^4_X(\mathcal{O}_S,\mathcal{O}_X)\xrightarrow{\text{(v)}}\operatorname{Ext}^3_X(I,\mathcal{O}_S)\to 0 \end{aligned} \tag{81}$$

since $\mathrm{Hom}_X(I,\mathcal{O}_S)\to \mathrm{Ext}^1_X(I,I)_0$ is an isomorphism (Theorem 3.19). The map (v) is Serre dual to

$$\mathrm{Ext}^1_X(\mathcal{O}_S,I)\xrightarrow{(\mathrm{v}')}\mathrm{Hom}_X(\mathcal{O}_X,\mathcal{O}_S).$$

Applying $\mathscr{E}xt^\bullet_X(\mathcal{O}_S,-)$ to (77), we have

$$\mathcal{H}om_X(\mathcal{O}_S,I)=0,\quad \mathscr{E}xt^1_X(\mathcal{O}_S,I)\cong\mathcal{H}om_X(\mathcal{O}_S,\mathcal{O}_S)$$

since $\mathcal{H}om_X(\mathcal{O}_S,\mathcal{O}_X)=\mathscr{E}xt^1_X(\mathcal{O}_S,\mathcal{O}_X)=0$. Therefore (v′) can be identified with the map induced by the isomorphism $\mathcal{H}om_X(\mathcal{O}_S,\mathcal{O}_S)\cong\mathcal{H}om_X(\mathcal{O}_X,\mathcal{O}_S)$, so (iv) is surjective.

Consider the local-to-global spectral sequence

$$H^p(X,\mathscr{E}xt^q_X(I,\mathcal{O}_S))\Rightarrow \mathrm{Ext}^{p+q}_X(I,\mathcal{O}_S).$$

Since S is lci, there are isomorphisms

$$\mathscr{E}xt^q_X(I,\mathcal{O}_S)\cong\mathscr{E}xt^{q+1}_X(\mathcal{O}_S,\mathcal{O}_S)\cong\textstyle\bigwedge^{q+1}N_{S/X}\quad\text{for all } q\geq 0$$

by [SGA 6 1971, proposition 2.5]. Since $\mathscr{E}xt^q_X(I,\mathcal{O}_S)=0$ for all $q\geq 2$, the local-to-global spectral sequence collapses to the long exact sequence

$$\begin{aligned}(82)\qquad 0\to H^1(S,N_{S/X})\to \mathrm{Ext}^1_X(I,\mathcal{O}_S)&\xrightarrow{(\mathrm{a})}\underbrace{H^0(S,N_{S/X}\wedge N_{S/X})}_{H^0(S,K_S)}\\ \to H^2(S,N_{S/X})\to \mathrm{Ext}^2_X(I,\mathcal{O}_S)&\xrightarrow{(\mathrm{b})}\underbrace{H^1(S,N_{S/X}\wedge N_{S/X})}_{H^1(S,K_S)}\to 0.\end{aligned}$$

Combined with (81), we obtain morphisms

$$H^0(S,K_S)\xrightarrow{(\mathrm{i})}\mathrm{Ext}^1_X(I,\mathcal{O}_S)\xrightarrow{(\mathrm{a})}H^0(S,K_S),\quad H^1(S,K_S)\xrightarrow{(\mathrm{iii})}\mathrm{Ext}^2_X(I,\mathcal{O}_S)\xrightarrow{(\mathrm{b})}H^1(S,K_S).$$

We show that the composition is an isomorphism. Since $\mathcal{O}_S$ is Cohen–Macaulay, $\mathscr{E}xt^q_X(\mathcal{O}_S,\mathcal{O}_X)=0$ for $q=3,4$ (Proposition 2.6). Therefore, $H^p(X,\mathscr{E}xt^2_X(\mathcal{O}_S,\mathcal{O}_X))\cong\mathrm{Ext}^{p+2}_X(\mathcal{O}_S,\mathcal{O}_X)$ for all p. By functoriality of the local-to-global spectral sequence, the map (a) $\circ$ (i) is induced by

$$\begin{array}{ccc}\mathrm{Ext}^2_X(\mathcal{O}_S,\mathcal{O}_X) & \xrightarrow{\quad(\mathrm{i})\quad} & \mathrm{Ext}^1_X(I,\mathcal{O}_S)\\ \uparrow\cong & & \downarrow(\mathrm{a})\\ H^0(X,\mathscr{E}xt^2_X(\mathcal{O}_S,\mathcal{O}_X)) & \xrightarrow{(\star)} & H^0(X,\mathscr{E}xt^2_X(\mathcal{O}_S,\mathcal{O}_S))\end{array}$$

where the map $(\star)$ is induced by applying $\mathscr{E}xt^\bullet_X(\mathcal{O}_S,-)$ to (77). By the fundamental local isomorphism, $(\star)$ is induced by

$$\mathscr{E}xt^2_X(\mathcal{O}_S,\mathcal{O}_X)\cong K_S\otimes i^*\mathcal{O}_X\cong\mathscr{E}xt^2_X(\mathcal{O}_S,\mathcal{O}_S).$$

Hence the composition is an isomorphism. Similar arguments show that the second composition is also an isomorphism. Consequently, (81) and (82) split into two short exact sequences and hence the composition

$$\theta_S : H^1(S, N_{S/X}) \hookrightarrow \mathrm{Ext}^1_X(I, \mathcal{O}_S) \xrightarrow{\text{(ii)}} \mathrm{Ext}^2_X(I, I)_0$$

is an isomorphism.

The commutativity $(2'')$ follows from the compatibility of $\mathrm{At}(I)$ and $\mathrm{At}(\mathcal{O}_S)$ along d_S. To prove the commutativity $(2')$, we first show that the following diagram commutes:

$$\begin{array}{ccc} & & \mathrm{Ext}^2_X(\mathcal{O}_S, \mathcal{O}_S) \\ & \overset{\mathsf{ob}(\mathcal{O}_S)}{\nearrow} & \uparrow{\scriptstyle \theta'_S} \\ H^1(X, T_X) & \xrightarrow{\mathrm{ob}_{S/X}} & H^1(S, N_{S/X}) \end{array} \tag{83}$$

Here the map θ'_S is obtained similarly to (79). Consider the composition

$$\theta''_S : H^1(S, N_{S/X}) \xrightarrow{\cong} \mathrm{Ext}^1_S(I/I^2, \mathcal{O}_S) \to \mathrm{Ext}^1_X(i_*I/I^2, i_*\mathcal{O}_S) \to \mathrm{Ext}^2_X(\mathcal{O}_S, \mathcal{O}_S),$$

where the second map is the pushforward and the third map is induced by the exact sequence

$$0 \to I/I^2 \to \mathcal{O}_X/I^2 \to \mathcal{O}_S \to 0. \tag{84}$$

The map θ'_S can be identified with θ''_S by comparing (77) and (84). Therefore the commutativity of (83) follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_S & \xrightarrow{e_S} & I/I^2[1] \\ \| & & \downarrow{\scriptstyle \delta} \\ \mathcal{O}_S & \xrightarrow{\mathrm{At}(\mathcal{O}_S)} & \mathcal{O}_S \otimes \Omega^1_X[1] \end{array}$$

where e_S is induced from (84) and δ is defined by the conormal sequence. By the naturality of Atiyah classes, the following diagram commutes:

$$\begin{array}{ccc} H^1(X, T_X) & \xrightarrow{\mathsf{ob}} & \mathrm{Ext}^2_X(I, I)_0 \\ \downarrow{\scriptstyle \mathsf{ob}(\mathcal{O}_S)} & & \downarrow{\scriptstyle d_S} \\ \mathrm{Ext}^2_X(\mathcal{O}_S, \mathcal{O}_S) & \xrightarrow{d_S} & \mathrm{Ext}^3_X(\mathcal{O}_S, I) \end{array}$$

The right vertical map is injective because it is dual to (ii). Therefore, it suffices to check $(2')$ after postcomposing with d_S. Since (83) commutes, this follows from the diagram

$$\begin{array}{ccc} H^1(S, N_{S/X}) & \xrightarrow{\theta_S} & \mathrm{Ext}^2_X(I, I)_0 \\ \downarrow{\scriptstyle \theta'_S} & & \downarrow{\scriptstyle d_S} \\ \mathrm{Ext}^2_X(\mathcal{O}_S, \mathcal{O}_S) & \xrightarrow{d_S} & \mathrm{Ext}^3_X(\mathcal{O}_S, I) \end{array}$$

Finally, we prove that the map θ_S preserves pairings. We first show that θ'_S preserves pairings. Let $i^!$ be the right adjoint of i_*. Then θ'_S decomposes as

$$\mathrm{Ext}^1_S(I/I^2,\mathcal{O}_S)\to \mathrm{Ext}^1_S(I/I^2,i^!i_*\mathcal{O}_S)\xrightarrow{\cong}\mathrm{Ext}^1_X(i_*I/I^2,i_*\mathcal{O}_S)\xrightarrow{e_S}\mathrm{Ext}^2_X(i_*\mathcal{O}_S,i_*\mathcal{O}_S),$$

where the first map is the adjoint, the second map is Grothendieck duality, and the third map is induced by (84). By the functoriality of Serre duality, we are reduced to checking commutativity of the diagram

$$\begin{array}{ccc}\mathrm{Ext}^1_S(I/I^2,i^!i_*\mathcal{O}_S) & \xrightarrow{\cong} & \mathrm{Ext}^1_X(i_*I/I^2,i_*\mathcal{O}_S)\\ \downarrow{\scriptstyle \mathsf{SD}_S} & & \downarrow{\scriptstyle \mathsf{SD}_X}\\ \mathrm{Ext}^3_S(Li^*i_*\mathcal{O}_S,I/I^2)^\vee & \xleftarrow{\cong} & \mathrm{Ext}^3_X(i_*\mathcal{O}_S,i_*I/I^2)^\vee\end{array}$$

where the bottom isomorphism is induced by the adjunction $Li^*\dashv i_*$. Since S is lci, we have $i^!=\mathsf{SD}_S\circ Li^*\circ\mathsf{SD}_X^{-1}$. Therefore the above diagram commutes and θ'_S preserves pairings. The pullbacks of quadratic forms on $\mathrm{Ext}^2_X(I,I)_0$ and $\mathrm{Ext}^2_X(\mathcal{O}_S,\mathcal{O}_S)$ to $\mathrm{Ext}^1_X(I,\mathcal{O}_S)$ coincide because $\mathrm{tr}(d_S\circ u\circ d_S\circ u)=\mathrm{tr}(u\circ d_S\circ u\circ d_S)$ for $u\in\mathrm{Ext}^1_X(I,\mathcal{O}_S)$. Since θ_S and θ'_S both factor through $\mathrm{Ext}^1_X(I,\mathcal{O}_S)$, the compatibility of quadratic forms under pullback along θ_S follows from compatibility under pullback along θ'_S. □

Remark 4.25 The map (80) coincides with Bloch's semiregularity map [Buchweitz and Flenner 2003, Proposition 8.2]. Furthermore, we have a map

$$\mathsf{sr}'=\mathrm{tr}(\mathrm{At}(\mathcal{O}_S)\circ -):\mathrm{Ext}^2_X(\mathcal{O}_S,\mathcal{O}_S)\to H^3(X,\Omega^1_X).$$

By Proposition 4.24(2″), compatibility of Atiyah classes, and cyclicity of trace, the following diagram commutes:

$$\begin{array}{ccc}\mathrm{Ext}^2_X(\mathcal{O}_S,\mathcal{O}_S) & & \\ {\scriptstyle\theta'_S}\uparrow & \searrow^{\mathsf{sr}'} & \\ H^1(S,N_{S/X}) & \xrightarrow{\mathsf{sr}_{S/X}} & H^3(X,\Omega^1_X)\end{array}$$

The following simple observation is useful for computing the reduced virtual dimension:

Remark 4.26 Let J be a complex, invertible, symmetric $m\times m$ matrix. In general, a complex $m\times n$ matrix A does not satisfy $\mathrm{rk}(A^TJA)=\mathrm{rk}(A)$. This equality holds if and only if J is nondegenerate on $\mathrm{im}(A)$.

We now compute the reduced virtual dimension when $v=\mathrm{ch}(\mathcal{O}_S)$ for some lci surfaces.

Corollary 4.27 *Let $i:S\hookrightarrow X$ be a local complete intersection surface inside a Calabi–Yau 4-fold X. If $v=\mathrm{ch}(\mathcal{O}_S)$, then*

$$\mathrm{rvd}=\tfrac{1}{2}\chi(N_{S/X})+\tfrac{1}{2}\,\mathrm{rk}(\mathsf{ob}^\vee_{S/X}\circ\mathsf{SD}_S\circ\mathsf{ob}_{S/X}).$$

In particular, when $\mathsf{ob}_{S/X}$ is surjective, i.e., S is semiregular, we have

$$\mathrm{rvd}=h^0(S,N_{S/X}).$$

Proof By the Hirzebruch–Riemann–Roch theorem, we have

$$\chi(N_{S/X}) = \int_S \operatorname{ch}(N_{S/X})\operatorname{td}(S) = 2\chi(\mathcal{O}_S) - \int_S c_2(N_{S/X}) = 2\chi(\mathcal{O}_S) - [S]^2,$$

where we used $c_1(N_{S/X}) = c_1\big(\bigwedge^2 N_{S/X}\big) = c_1(K_S)$ and the self-intersection formula $i^*i_*[S] = c_2(N_{S/X})$. Thus the formula for the reduced virtual dimension follows from Proposition 4.24. In the semiregular case, we have $\operatorname{rk}(\mathsf{ob}_{S/X}^\vee \circ \mathsf{SD}_S \circ \mathsf{ob}_{S/X}) = \operatorname{rk}(\mathsf{ob}_{S/X}) = h^1(N_{S/X})$ (Remark 4.26). The result follows from the fact that $h^0(N_{S/X}) = h^2(N_{S/X})$ (see also Theorem 4.6). □

Complete intersection Calabi–Yau 4-folds The following result is inspired by [Steenbrink 1987]:

Corollary 4.28 *Let $X \subset \mathbb{P}^N$ be a complete intersection Calabi–Yau 4-fold defined by a regular section of $E_1 = \bigoplus_{i=1}^{N-4} \mathcal{O}_{\mathbb{P}^N}(e_i)$ with $e_i \geq 2$. Suppose $S \subset X$ is a complete intersection surface of $\mathbb{P}^N$ defined by a regular section of $E_2 = \bigoplus_{j=1}^{N-2} \mathcal{O}_{\mathbb{P}^N}(d_j)$ with $d_j \geq 1$. Then the ideal sheaf $I_{S/X}$ is semiregular. Moreover, if $v = \operatorname{ch}(\mathcal{O}_S)$, then*

$$\operatorname{rvd} = \dim\big(\ker(H^0(E_2|_S) \to H^0(E_1|_S))\big),$$

where the map is induced by the morphism $E_2|_S = N_{S/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}|_S = E_1|_S$.

Proof By the short exact sequence

$$0 \to N_{S/X} \to N_{S/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}|_S \to 0 \tag{85}$$

and Proposition 4.24, it is enough to show that $\mathsf{ob}_{S/X}$ is surjective. We note that the following diagram commutes:

$$\begin{array}{ccccc} H^0(\mathbb{P}^N, E_1) & \longrightarrow & H^0(S, E_1|_S) & \longrightarrow & H^1(S, N_{S/X}) \\ & \searrow & \uparrow & & \uparrow{\scriptstyle \mathsf{ob}_{S/X}} \\ & & H^0(X, E_1|_X) & \longrightarrow & H^1(X, T_X) \end{array}$$

Therefore, it suffices to show the two top horizontal maps are surjective. Consider a Koszul resolution $\bigwedge^{\bullet+1} E_2^\vee \to I_{S/\mathbb{P}^N}$ and the associated spectral sequence [Stacks 2005–, Tag 07A9]

$$\operatorname{Ext}^p_{\mathbb{P}^N}\big(E_1 \otimes \textstyle\bigwedge^{q+1} E_2^\vee, K_{\mathbb{P}^N}\big) \Rightarrow \operatorname{Ext}^{p+q}_{\mathbb{P}^N}(E_1 \otimes I_{S/\mathbb{P}^N}, K_{\mathbb{P}^N}).$$

Since, on line bundles, $H^p(\mathbb{P}^N, -)$ is nonzero only when $p = 0$ or N, this spectral sequence degenerates and we have $0 = \operatorname{Ext}^{N-1}_{\mathbb{P}^N}(E_1 \otimes I_{S/\mathbb{P}^N}, K_{\mathbb{P}^N}) \cong H^1(\mathbb{P}^N, E_1 \otimes I_{S/\mathbb{P}^N})^\vee$. Therefore the first horizontal map is surjective. Similarly, we have $H^2(\mathbb{P}^N, E_2 \otimes I_{S/\mathbb{P}^N}) = 0$. Consequently, $H^1(S, E_2|_S) = H^1(S, N_{S/\mathbb{P}^N}) = 0$ and hence the second horizontal map is surjective. Therefore, $\mathsf{ob}_{S/X} = (\mathsf{sr}_{S/X})^\vee$ is surjective and $I_{S/X}$ is semiregular. □

Corollary 4.28 allows us to compute rvd and ρ_γ for semiregular cases.

Example 4.29 Suppose $X := X_{e_1,\dots,e_{N-4}} \subset \mathbb{P}^N$ is a complete intersection Calabi–Yau 4-fold. The possible cases are X_6, $X_{2,5}$, $X_{3,4}$, $X_{2,2,4}$, $X_{2,3,3}$, $X_{2,2,2,3}$, $X_{2,2,2,2,2}$. Consider complete intersection

surfaces $S \subset \mathbb{P}^N$ of type $(d_1, \dots, d_{N-2})$ such that $S \subset X$. By Corollary 4.28, the ideal sheaf $I_{S/X}$ is semiregular.

(1) Suppose for all i we have

$$-d_i + \sum_{j=1}^{N-2} d_j < N + 1. \tag{86}$$

Then we claim $S \subset X$ is rigid, i.e., $h^0(N_{S/X}) = 0$, so, in particular, rvd $= 0$. This can be seen by showing that $H^0(S, N_{S/X})^\vee \cong H^2(S, N_{S/X}) = 0$ as follows. Using (85) it suffices to show that $H^1(S, E_1|_S) = H^2(S, E_2|_S) = 0$. This follows by showing $H^2(\mathbb{P}^N, E_1 \otimes I_{S/X}) = H^3(\mathbb{P}^N, E_2 \otimes I_{S/X}) = 0$ by a Koszul calculation as in the proof of Corollary 4.28. For example, for $N = 5$ all solutions to (86) are (up to permutations)

$$(1,1,1), \quad (1,1,2), \quad (1,1,3), \quad (1,2,2); \qquad (1,2,3), \quad (2,2,2), \quad (1,1,4); \qquad (2,2,3).$$

In the first four, S is del Pezzo. In the next three, S is K3. In the last one, S is of general type. Using the formula rvd $= \chi(\mathcal{O}_S) - \frac{1}{2}\gamma^2 + \frac{1}{2}\rho_\gamma$, and determining $\chi(\mathcal{O}_S)$, γ^2 in terms of d_1, d_2, d_3, this produces the following values for ρ_γ in the above cases:

$$19, \quad 32, \quad 37, \quad 54; \qquad 62, \quad 92, \quad 32; \qquad 106.$$

(2) Suppose $X = X_6$ is defined by a homogeneous polynomial $f \in R := \mathbb{C}[x_0, \dots, x_5]$ of degree 6 and $S \subset X$ is defined by homogeneous polynomials (s_1, s_2, s_3) of degrees (d_1, d_2, d_3); then $f = s_1 t_1 + s_2 t_2 + s_3 t_3$ for some t_1, t_2, t_3. Using the Koszul resolution of $I_{S/\mathbb{P}^5}$ as in proof of Corollary 4.28, it is easy to see that $H^1(\mathbb{P}^5, I_{S/\mathbb{P}^5}(n)) = 0$ for all n. If the ideal $(s_1, s_2, s_3) \subset R$ is saturated, then $H^0(S, N_{S/X})$ is the kernel of the map

$$H^0(E_2|_S) \cong \bigoplus_{i=1}^{3} (R/(s_1, s_2, s_3))_{(d_i)} \to H^0(E_1|_S) \cong (R/(s_1, s_2, s_3))_{(6)},$$
$$(u_1, u_2, u_3) \mapsto u_1 t_1 + u_2 t_2 + u_3 t_3,$$

where $(-)_{(n)}$ denotes the degree n part. This can be implemented in `Macaulay2`. For example, for the Fermat sextic and certain S of degrees $(1,3,3)$, $(2,3,3)$, $(3,3,3)$, we obtain rvd $= h^0(N_{S/X}) = 1, 1, 3$, respectively. Using the formula rvd $= \chi(\mathcal{O}_S) - \frac{1}{2}\gamma^2 + \frac{1}{2}\rho_\gamma$, we obtain that ρ_γ is 71, 122, 141, respectively.

The value of ρ_γ can also be computed by Griffiths' residue calculus [1968a; 1968b]. In the cases of the sextic 4-fold listed above, our values for ρ_γ reproduce [Movasati 2021, Table 17.3]. Corollary 4.28 gives an algebraic method for determining ρ_γ.

Holomorphic symplectic 4-folds When a Calabi–Yau 4-fold X has a nondegenerate holomorphic 2-form, the rank ρ_γ of B_γ is easy to determine. The following computation is inspired by [Voisin 1992]:

Corollary 4.30 *Suppose X has a nondegenerate holomorphic 2-form $\sigma \in H^0(X, \Omega_X^2)$. Let $i : S \hookrightarrow X$ be a (possibly disconnected) smooth lagrangian surface, i.e., $i^*\sigma = 0$. If $v = \mathrm{ch}(\mathcal{O}_S)$, then*

$$\mathrm{rvd} = h^{1,0}(S) - \tfrac{1}{2}\dim \mathrm{coker}(i^*),$$

where $i^* : H^1(X, \Omega_X^1) \to H^1(S, \Omega_S^1)$.

Proof We prove the case when S is connected. The disconnected case then follows easily. Since S is a lagrangian, we have an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{S/X}^* & \longrightarrow & \Omega_X^1|_S & \longrightarrow & \Omega_S^1 & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow\sigma & & \cong\downarrow & & \\ 0 & \longrightarrow & T_S & \longrightarrow & T_X|_S & \longrightarrow & N_{S/X} & \longrightarrow & 0 \end{array}$$

Therefore, $\mathsf{ob}_{S/X}$ can be identified with i^*. Under this identification, SD_S corresponds to the cup product. By the Hodge index theorem, the cup product is nondegenerate on the image of i^*. Therefore, Remark 4.26 and Corollary 4.27 imply that the reduced virtual dimension equals

$$\mathrm{rvd} = \tfrac{1}{2}\chi(\Omega_S^1) + \tfrac{1}{2}\,\mathrm{rk}(i^*). \qquad \square$$

Example 4.31 Suppose X has a nondegenerate holomorphic 2-form. Consider a lagrangian plane $\mathbb{P}^2 \cong S \subset X$ and let $v = \mathrm{ch}(\mathcal{O}_S)$. Since i^* is nontrivial and $h^{1,1}(S) = 1$, S is semiregular. Moreover, $N_{S/X} \cong \Omega_S^1$ implies that S is rigid and $\mathrm{rvd} = 0$. When $\mathscr{P}_v^{(q)}(X)$ only contains lagrangian planes, $\deg([\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}})$ is a (signed) count of lagrangian planes in X in class v.

Example 4.32 Schoen [2007] constructed minimal surfaces S of general type related to the Hodge conjecture for abelian 4-folds. These surfaces satisfy $h^{1,0}(S) = 4$, $h^{2,0}(S) = 5$, and $h^{1,1}(S) = 12$. Let S be a Schoen surface; then its Albanese variety $X = \mathrm{Alb}(S)$ is an abelian 4-fold. The Albanese map $i : S \to X$ is an embedding of S as a lagrangian surface (for some nondegenerate $\sigma \in H^0(X, \Omega_X^2)$) [Ciliberto et al. 2015], and the cohomology class $[S] \in H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ is not the product of divisor classes on X [Schoen 2007]. Let $v = \mathrm{ch}(\mathcal{O}_S)$. From the constructions in [loc. cit.], one can see that there exist semiregular Schoen surfaces $S \subset X$. Then i^* is surjective. Therefore, $\mathrm{rvd} = 4$ and $\rho_{[S]} = 12$. We will revisit this example in future work.

Remark 4.33 Combining Theorem 4.6 and Corollary 4.30, we obtain a numerical criterion for a smooth lagrangian surface $i : S \hookrightarrow X$ to be nonsemiregular. Consider $i^* : H^{1,1}(X) \to H^{1,1}(S)$. Indeed, if $\mathrm{rk}(i^*)$ is *odd* or $b_1(S) < \dim \mathrm{coker}(i^*)$, then S is not semiregular.

For example, let $\pi : X \to B$ be a hyperkähler 4-fold with lagrangian fibration over a smooth base. The general fibre $S = F$ is a lagrangian abelian surface and the base B is isomorphic to $\mathbb{P}^2$ [Hwang 2008; Huybrechts and Xu 2022]. For $v = \mathrm{ch}(\mathcal{O}_F)$ and any $q \in \{-1, 0, 1\}$, we have a morphism

$$B \to \mathscr{P}_v^{(q)}(X), \quad P \mapsto \pi^* I_{P/B}.$$

This is a bijection on $\mathbb{C}$-points, and, using the fact that $R\pi_*\mathcal{O}_X \cong \mathcal{O}_B \oplus \Omega_B^1[-1] \oplus \Omega_B^2[-2]$ [Matsushita 2005], one can also show that it is an isomorphism on tangent spaces at $\mathbb{C}$-points. Thus, since B is smooth, the map is an isomorphism. Let $\Delta \subset B \times B$ denote the diagonal and $\pi_B : X \times B \to B$ the projection. Using again $R\pi_*\mathcal{O}_X \cong \mathcal{O}_B \oplus \Omega_B^1[-1] \oplus \Omega_B^2[-2]$, we obtain the obstruction bundle

$$\mathscr{E}xt^2_{\pi_B}(\pi^* I_\Delta, \pi^* I_\Delta)_0 \cong T_B \otimes \Omega_B^1 \cong \mathcal{E}nd(T_B).$$

The natural pairing $T_B \otimes \Omega_B^1 \to \mathcal{O}_B$ is the unique cosection (up to scaling). Since i^* is nonzero, we deduce $\operatorname{rk} i^* = 1$. A direct computation of $\operatorname{rk} i^* = 1$ can be found in [Shen and Yin 2022; Matsushita 2016]. We deduce $\operatorname{rvd} = \frac{1}{2}$, so S is not semiregular.

Example 4.34 Let $C \subset \mathrm{K3}$ be a smooth curve of genus g on a K3 surface. Then the Hilbert square $X = \mathrm{K3}^{[2]}$ is a hyperkähler 4-fold and the symmetric product $S = C^{[2]}$ is a smooth lagrangian surface. Since S is cut out transversally by a tautological section of the vector bundle $\mathcal{O}(C)^{[2]}$, we have

$$\chi(\Omega_S^1) = \int_S \operatorname{ch}(\Omega_S^1) \cdot \operatorname{td}(S) = \int_X c_2(\mathcal{O}(C)^{[2]}) \operatorname{ch}(\Omega_X^1 - \mathcal{O}(C)^{[2]\vee}) \frac{\operatorname{td}(X)}{\operatorname{td}(\mathcal{O}(C)^{[2]})}.$$

This tautological integral can be easily determined, e.g., using [Ellingsrud et al. 2001]. The result is $-\frac{1}{4}(C^2)^2 = -(g-1)^2$. Moreover, $\operatorname{rk} i^* = 1$ when $g = 0$ and $\operatorname{rk} i^* = 2$ otherwise [Voisin 1992, section 3]. By Corollary 4.30, we find

$$\operatorname{rvd} = -\tfrac{1}{2}(g-1)^2 + \begin{cases} \frac{1}{2} & \text{if } g = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This number is only nonzero for $g = 0, 1, 2$; then $\operatorname{rvd} = 0, 1, \frac{1}{2}$ respectively. In particular, for $g \geq 2$ we deduce that S is not semiregular (Remark 4.33). For $g = 0$, $S \cong \mathbb{P}^2$ is a lagrangian plane.

When X is a product of two K3 surfaces, we determine ρ_γ for diagonal classes; see also Example 5.16.

Proposition 4.35 *Let X be a product of two K3 surfaces S_1 and S_2. If $\gamma = \eta_1 \cup \eta_2$ for some nonzero $\eta_1 \in H^{1,1}(S_1)$ and $\eta_2 \in H^{1,1}(S_2)$, then we have $\rho_\gamma = 2$.*

Proof Let $\sigma_1 \in H^{2,0}(S_1)$ and $\sigma_2 \in H^{2,0}(S_2)$ be two nowhere-vanishing holomorphic 2-forms. Take $\omega = \sigma_1 \cup \sigma_2 \in H^0(X, \Omega_X^4)$. Consider the isomorphism

$$H^1(X, T_X) \cong H^1(S_1, T_{S_1}) \oplus H^1(S_2, T_{S_2}).$$

The symmetric bilinear form B_γ on $H^1(X, T_X)$ can be related to the two rank 1 maps

$$\mathsf{A}_\ell : H^1(S_\ell, T_{S_\ell}) \to \mathbb{C}, \quad \xi \mapsto \int_{S_\ell} \eta_\ell \cup \iota_\xi(\sigma_\ell) = -\int_{S_\ell} \iota_\xi(\eta_\ell) \cup \sigma_\ell, \quad \ell = 0, 1.$$

Namely, if $\xi_1 \in H^1(S_1, T_{S_1})$ and $\xi_2 \in H^1(S_2, T_{S_2})$, then

$$\mathsf{B}_\gamma(\xi_1, \xi_2) = \int_X \iota_{\xi_1}\iota_{\xi_2}(\eta_1 \cup \eta_2) \cup \sigma_1 \cup \sigma_2 = \int_{S_1} \eta_1 \cup \iota_{\xi_1}(\sigma_1) \cdot \int_{S_2} \eta_2 \cup \iota_{\xi_2}(\sigma_2) = \mathsf{A}_1(\xi_1) \cdot \mathsf{A}_2(\xi_2)$$

and, for $\xi_1, \xi_2 \in H^1(S_1, T_{S_1})$ or $\xi_1, \xi_2 \in H^1(S_2, T_{S_2})$, we have $\mathsf{B}_\gamma(\xi_1, \xi_2) = 0$. Therefore the rank of B_γ is two. □

Vanishing of ρ_γ Let X be an arbitrary Calabi–Yau 4-fold and $\gamma \in H^{2,2}(X)$. There are interesting cases for which $\rho_\gamma = 0$ and hence the nonreduced virtual cycle equals the reduced virtual cycle.

Proposition 4.36 *Assume that $H^2(X, \mathcal{O}_X) = 0$. If $\gamma = \eta_1 \cup \eta_2$ for some $(1,1)$-classes η_1 and η_2 on X, then we have $\rho_\gamma = 0$.*

Proof For $\xi \in H^1(X, T_X)$, we have $\iota_\xi(\eta_1 \cup \eta_2) = \iota_\xi(\eta_1) \cup \eta_2 + \eta_1 \cup \iota_\xi(\eta_2) = 0$ because $\iota_\xi(\eta_1), \iota_\xi(\eta_2) \in H^2(X, \mathcal{O}_X)$. Hence, $\rho_\gamma = 0$ by definition. □

Proposition 4.37 *Let $Y \subset X$ be a smooth divisor with $H^2(Y, \mathcal{O}_Y) = 0$. If $\gamma = [S]$ for some effective divisor $S \subset Y$ with $H^1(Y, \mathcal{O}_Y(S)) = 0$, then $\rho_\gamma = 0$.*

Proof By Proposition 4.24, it is enough to show that $\mathrm{ob}_{S/X}$ is zero. Consider the commutative diagram

$$\begin{array}{ccccc} & & H^1(T_Y) & \xrightarrow{\mathrm{ob}_{S/Y}} & H^1(N_{S/Y}) \\ & & \downarrow\!\!\downarrow & & \downarrow \\ \mathrm{ob}_{S/X} : H^1(T_X) & \longrightarrow & H^1(T_X|_Y) & \longrightarrow & H^1(N_{S/X}) \end{array}$$

where the left vertical map is surjective by the vanishing of $H^1(N_{Y/X}) \cong H^1(\mathcal{O}_Y(Y)) \cong H^1(K_Y) \cong H^2(\mathcal{O}_Y)^\vee$. By the vanishing of $H^1(Y, \mathcal{O}_Y(S))$, we have $H^1(N_{S/Y}) = 0$, and therefore $\mathrm{ob}_{S/X} = 0$. □

For example, the hypotheses of the previous proposition are satisfied for $\mathbb{P}^3 \cong Y \subset X$ and any effective divisor $S \subset Y$.

5 Deformation invariance

In this section, we prove the deformation invariance of reduced virtual cycles. As a corollary, we prove the variational Hodge conjecture for surface classes that carry nonzero reduced virtual cycles.

5.1 Deformation invariance

In this subsection, we prove that the reduced virtual cycles on the moduli spaces of PT_q pairs are deformation invariant along the Hodge loci.

We first fix some notation. Let $f : \mathcal{X} \to \mathcal{B}$ be a morphism of schemes satisfying the following conditions:

(1) $f : \mathcal{X} \to \mathcal{B}$ is a smooth projective morphism of relative dimension 4 with connected fibres.

(2) $\mathcal{B}$ is a smooth connected affine scheme.

(3) $\omega_{\mathcal{X}/\mathcal{B}}$ is trivial, i.e., there exists an isomorphism $w : \mathcal{O}_{\mathcal{X}} \cong \omega_{\mathcal{X}/\mathcal{B}}$.

We denote by $\mathcal{X}_b := \mathcal{X} \times_{\mathcal{B}} \{b\}$ the fibre of $f : \mathcal{X} \to \mathcal{B}$ over a point $b \in \mathcal{B}$.

Choose a horizontal section

$$\tilde{v} = (\tilde{v}_p)_{p \geq 0} \in \bigoplus_{p \geq 0} F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$$

in the sense of Definition 3.30. We may regard $\tilde{v}$ as a locally constant family of cohomology classes $\tilde{v}_b \in H^*(\mathcal{X}_b, \mathbb{C})$ by Remark 3.31. Let

$$\widetilde{\gamma} \in \Gamma(\mathcal{B}, R^2 f_* \Omega^2_{\mathcal{X}/\mathcal{B}})$$

be the family of $(2,2)$-classes induced by $\tilde{v}_2 \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$; see (40).

Consider a vector bundle

$$\mathcal{T} := R^1 f_*(T_{\mathcal{X}/\mathcal{B}})$$

on $\mathcal{B}$ with a canonical symmetric bilinear form

$$\mathsf{B}_{\widetilde{\gamma}} : \mathcal{T} \otimes \mathcal{T} \xrightarrow{\mathrm{id}\otimes\widetilde{\gamma}} \mathcal{T} \otimes \mathcal{T} \otimes R^2 f_*(\Omega^2_{\mathcal{X}/\mathcal{B}}) \to R^4 f_*(\mathcal{O}_{\mathcal{X}}) \xrightarrow{w} R^4 f_* \Omega^4_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{B}}.$$

If the function $\mathcal{B} \to \mathbb{Z}$, $b \mapsto \rho_{\widetilde{\gamma}_b}$, is constant, then we can form an induced orthogonal bundle of rank $\rho := \rho_{\widetilde{\gamma}_b}$ as

$$\mathcal{T}_{\widetilde{\gamma}} := \mathcal{T}/\ker(\mathsf{B}_{\widetilde{\gamma}}).$$

Let $\mathscr{P} := \mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})$ be the moduli space of PT_q pairs on the fibres of $f : \mathcal{X} \to \mathcal{B}$, defined in Theorem 3.32. We can form a fibre diagram

$$\begin{array}{ccc} \mathscr{P}_b & \xhookrightarrow{j_b} & \mathscr{P} \\ \downarrow & & \downarrow p \\ \{b\} & \xhookrightarrow{i_b} & \mathcal{B} \end{array} \tag{87}$$

where $\mathscr{P}_b := \mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)$ is the moduli space of PT_q pairs on the fibre $\mathcal{X}_b$. By Theorem 3.32(2) and [Huybrechts and Thomas 2010], we have a relative symmetric obstruction theory

$$\phi : \mathbb{E} \to \tau^{\geq -1} \mathbb{L}_{\mathscr{P}/\mathcal{B}} \tag{88}$$

for $p : \mathscr{P} \to \mathcal{B}$. Moreover, the induced map

$$\phi_b : \mathbb{E}_b := L j_b^* \mathbb{E} \xrightarrow{L j_b^* \phi} L j_b^* (\tau^{\geq -1} \mathbb{L}_{\mathscr{P}/\mathcal{B}}) \to \tau^{\geq -1} \mathbb{L}_{\mathscr{P}_b}$$

is the standard symmetric obstruction theory for the fibre $\mathcal{X}_b$ in (49).

Now we can state our main theorem in this section.

Theorem 5.1 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of relative dimension 4 with connected fibres to a smooth connected affine scheme $\mathcal{B}$ such that $\omega_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{X}}$. Let $\tilde{v} \in \bigoplus_p F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be a horizontal section, $\widetilde{\gamma} \in \Gamma(\mathcal{B}, R^2 f_* \Omega^2_{\mathcal{X}/\mathcal{B}})$ the section induced by $\tilde{v}_2$, and $q \in \{-1, 0, 1\}$. Assume that the function $\mathcal{B} \to \mathbb{Z}$, $b \mapsto \rho_{\widetilde{\gamma}_b}$, is constant. For all orientations*

$$o_1 : \mathcal{O}_{\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})} \cong \det(\mathbb{E}), \quad o_2 : \mathcal{O}_{\mathcal{B}} \cong \det(\mathcal{T}_{\widetilde{\gamma}})$$

of the symmetric complex $\mathbb{E}$ *on* $\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})$ *and the orthogonal bundle* $\mathcal{T}_{\tilde{\gamma}}$ *on* $\mathcal{B}$*, respectively, there exists a cycle class*

$$[\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})]^{\mathrm{red}} \in A_{\mathrm{rvd}+\dim(\mathcal{B})}(\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B}))$$

such that

$$[\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)]^{\mathrm{red}} = i_b^![\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})]^{\mathrm{red}} \in A_{\mathrm{rvd}}(\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b))$$

for all $b \in \mathcal{B}$*. Here* $i_b^! : A_*(\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})) \to A_*(\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b))$ *denotes the refined Gysin pullback* [Fulton 1998] *with respect to the fibre diagram* (87) *and* $[\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)]^{\mathrm{red}}$ *denotes the reduced virtual cycle in Theorem 4.5 with respect to the orientations induced by* o_1 *and* o_2*.*

We briefly explain how we prove Theorem 5.1. We use the cosection localization technique of [Kiem and Li 2013] as in Section 4.4. Most of the arguments are just straightforward generalizations of the proof of Theorem 4.5 to the relative setting. However there is one key difference. Kiem and Li's cone reduction lemma (cf. Lemma 4.16) fails in the relative setting. Thus, as in [Kiem and Park 2023], we need to modify the relative obstruction theory/cosection for $\mathscr{P} \to \mathcal{B}$ to an absolution obstruction theory/cosection for $\mathscr{P}$. In general, we need an additional condition on the cosection to do this. In our case, this additional condition can be deduced from the fact that the Hodge locus of $\tilde{\gamma} \in H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ in $\mathcal{B}$ is equal to $\mathcal{B}$ itself.

Proof Form a fibre diagram

$$\begin{array}{ccc} \mathscr{P} \times_{\mathcal{B}} \mathcal{X} & \xrightarrow{q} & \mathcal{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ \mathscr{P} & \xrightarrow{p} & \mathcal{B} \end{array}$$

where π and q denote the projection maps. Let $\mathbb{I}^\bullet$ denote the universal complex on $\mathscr{P} \times_{\mathcal{B}} \mathcal{X}$.

We have a relative version of the cosection in Proposition 4.12,

$$\mathsf{SR} : \mathbb{E}^\vee[1] = R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[2] \to \mathcal{T}|^\vee_{\mathscr{P}}.$$

Indeed, consider the composition

$$R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[2] \xrightarrow{\mathrm{At}_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathscr{P}}(\mathbb{I}^\bullet)} R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \Omega^1_{\mathcal{X}/\mathcal{B}})[3] \xrightarrow{\mathrm{tr}^3} (R^3 f_*\Omega^1_{\mathcal{X}/\mathcal{B}})|_{\mathscr{P}}, \tag{89}$$

where the third trace map is defined as

$$\mathrm{tr}^3 : R\mathcal{H}om_\pi(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \Omega^1_{\mathcal{X}/\mathcal{B}})[3] \xrightarrow{\mathrm{tr}} (Rf_*\Omega^1_{\mathcal{X}/\mathcal{B}}[3])|_{\mathscr{P}} \to (R^3 f_*\Omega^1_{\mathcal{X}/\mathcal{B}})|_{\mathscr{P}}.$$

Here the last arrow uses Deligne's decomposition theorem [1968, théorème 6.1],

$$Rf_*\Omega^i_{\mathcal{X}/\mathcal{B}} \cong \bigoplus_j R^j f_*\Omega^i_{\mathcal{X}/\mathcal{B}}[-j],$$

which replaces note 19 in the relative setting. The map SR is then defined by composing the above map (89) with the Serre duality isomorphism $R^3 f_*\Omega^1_{\mathcal{X}/\mathcal{B}} \cong (R^1 f_* T_{\mathcal{X}/\mathcal{B}})^\vee = \mathcal{T}^\vee$.

Note that we can form a relative version of the commutative triangle (58) in Proposition 4.12, i.e., we have a commutative triangle

$$\begin{array}{ccc} \mathcal{T}|_{\mathscr{P}} & \xrightarrow{\mathsf{SR}^\vee} & \mathbb{E}[-1] \\ & {\scriptstyle \mathsf{B}_{\widetilde{\gamma}}|_{\mathscr{P}}} \searrow \quad \swarrow {\scriptstyle \mathsf{SR}} & \\ & \mathcal{T}|_{\mathscr{P}}^\vee & \end{array} \tag{90}$$

where we identified $\mathbb{E}[-1] \cong \mathbb{E}^\vee[1]$ via the symmetric form of $\mathbb{E}$. Here we used that

$$\mathrm{ch}_2(\mathbb{I}^\bullet) = p^*(\widetilde{\gamma}) \in \Gamma(\mathscr{P}, R^2\pi_*(\Omega^2_{\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathscr{P}})) \cong \Gamma\big(\mathscr{P}, p^*(R^2 f_*(\Omega^2_{\mathcal{X}/\mathcal{B}}))\big)$$

since the sections extend to horizontal sections $\widetilde{\mathrm{ch}}_2(\mathbb{I}^\bullet)$ and $p^*(\tilde{v}_2)$, respectively, of the vector bundle $F^2\mathcal{H}^4_{\mathrm{DR}}(\mathscr{P}\times_{\mathcal{B}}\mathcal{X}/\mathscr{P})$ by Proposition 3.33 (see our convention (41)). Then the commutativity of (90) follows from [Buchweitz and Flenner 2003, Proposition 4.2] as in the proof of Proposition 4.12.

Since $\mathcal{B}$ is affine, we can find a subbundle $\mathcal{V} \subseteq \mathcal{T}$ such that the composition

$$\mathcal{V} \hookrightarrow \mathcal{T} \twoheadrightarrow \mathcal{T}_{\widetilde{\gamma}} \qquad (\text{with } \mathcal{V} \xrightarrow{\cong} \mathcal{T}_{\widetilde{\gamma}}) \tag{91}$$

is an isomorphism. Then the restriction

$$\mathsf{B}_{\widetilde{\gamma}}|_{\mathcal{V}} : \mathcal{V}\otimes\mathcal{V} \to \mathcal{O}_{\mathcal{B}}$$

is nondegenerate. Thus we may regard $\mathcal{V}$ as an orthogonal bundle over $\mathcal{B}$. Moreover, the orientation of $\mathcal{V}$ is determined by the orientation o_2 of $\mathcal{T}_{\widetilde{\gamma}}$ via the isomorphism $\mathcal{V} \cong \mathcal{T}_{\widetilde{\gamma}}$ in (91). We may view $\mathcal{V} \subseteq \mathcal{T}$ as a family of maximal nondegenerate subspaces of $H^1(\mathcal{X}_b, T_{\mathcal{X}_b})$.

As in Definition 4.14, the composition

$$\mathsf{SR}_{\mathcal{V}} : \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} \mathcal{T}|_{\mathscr{P}}^\vee \to \mathcal{V}|_{\mathscr{P}}^\vee$$

induces a direct sum decomposition of symmetric complexes

$$\mathbb{E} = \mathbb{E}_{\mathcal{V}}^{\mathrm{red}} \oplus \mathcal{V}|_{\mathscr{P}}[1].$$

More precisely, we let $\mathbb{E}_{\mathcal{V}}^{\mathrm{red}} := \mathrm{cone}(\mathsf{SR}_{\mathcal{V}}^\vee[1])$ and consider the symmetric form induced by the symmetric form of $\mathbb{E}$.

Note that the relative symmetric obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}}$ in (88) satisfies the isotropic condition. More precisely, we have

$$\mathfrak{C}_{\mathscr{P}/\mathcal{B}} \subseteq \mathfrak{Q}(\mathbb{E})$$

as substacks of $\mathfrak{C}(\mathbb{E})$. Indeed, this follows from the relative version of the Darboux theorem [Brav et al. 2019; Bouaziz and Grojnowski 2013] for the relative (-2)-shifted symplectic form for $p : \mathscr{P} \to \mathcal{B}$ in [Pantev et al. 2013] since $\widetilde{\gamma} \in F^2H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. See [Park 2024] and Remark 5.2 below.

We claim that the relative cone reduction property holds for $p : \mathscr{P} \to \mathcal{B}$. More precisely, we claim that we can form a commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{Q}(\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}) \longrightarrow \mathscr{P} \\ & \nearrow & \downarrow & & \downarrow & & \downarrow 0 \\ (\mathfrak{C}_{\mathscr{P}/\mathcal{B}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{Q}(\mathbb{E}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{C}(\mathsf{SR}_{\mathcal{V}}^{\vee}[1])} & \mathcal{V}|_{\mathscr{P}}^{\vee} \end{array} \tag{92}$$

for some dotted arrow, where the embedding $(\mathfrak{C}_{\mathscr{P}/\mathcal{B}})_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is given by the obstruction theory ϕ as in the previous paragraph.

We first modify the relative obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}}$ into an absolute one. Indeed, we can form a morphism

$$\begin{array}{ccccc} \mathbb{E}^{\mathrm{abs}} & \xrightarrow{m} & \mathbb{E} & \longrightarrow & \Omega_{\mathcal{B}}^1|_{\mathscr{P}}[1] \\ \downarrow{\scriptstyle \phi^{\mathrm{abs}}} & & \downarrow{\scriptstyle \phi} & & \| \\ \tau^{\geq -1}\mathbb{L}_{\mathscr{P}} & \longrightarrow & \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}} & \xrightarrow{\mathsf{KS}_{\mathscr{P}/\mathcal{B}}} & \Omega_{\mathcal{B}}^1|_{\mathscr{P}}[1] \end{array} \tag{93}$$

of distinguished triangles on $\mathscr{P}$, where the lower triangle is exact since $\mathcal{B}$ is smooth. The associated long exact sequence implies that $\phi^{\mathrm{abs}} : \mathbb{E}^{\mathrm{abs}} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ is an obstruction theory. Moreover, by considering the associated abelian cone stacks, we have a cartesian diagram

$$\begin{array}{ccc} \mathfrak{C}_{\mathscr{P}/\mathcal{B}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}) \\ \downarrow & & \downarrow{\scriptstyle \mathfrak{C}(m)} \\ \mathfrak{C}_{\mathscr{P}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}^{\mathrm{abs}}) \end{array} \tag{94}$$

where the horizontal arrows are closed embeddings given by the obstruction theories ϕ, ϕ^{abs}, and the vertical arrows are $T_{\mathcal{B}}$-torsors.

We then modify the relative cosection $\mathsf{SR} : \mathbb{E}^{\vee}[1] \to \mathcal{T}|_{\mathscr{P}}^{\vee}$ into an absolute one. We will find a map $\mathsf{SR}^{\mathrm{abs}} : (\mathbb{E}^{\mathrm{abs}})^{\vee}[1] \to \mathcal{T}|_{\mathscr{P}}^{\vee}$ that fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{E}^{\vee}[1] & \xrightarrow{\mathsf{SR}} & \mathcal{T}|_{\mathscr{P}}^{\vee} \\ {\scriptstyle m^{\vee}[1]}\downarrow & \nearrow{\scriptstyle \mathsf{SR}^{\mathrm{abs}}} & \\ (\mathbb{E}^{\mathrm{abs}})^{\vee}[1] & & \end{array} \tag{95}$$

We need to show that the composition

$$\mathcal{T}|_{\mathscr{P}} \xrightarrow{\mathsf{SR}^{\vee}} \mathbb{E}[-1] \xrightarrow{\phi[-1]} \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}}[-1] \xrightarrow{\mathsf{KS}_{\mathscr{P}/\mathcal{B}}} \Omega_{\mathcal{B}}^1|_{\mathscr{P}} \tag{96}$$

vanishes.

The vanishing of (96) is the key part of the proof of Theorem 5.1. By the commutative triangle (90) and the compatibility result in Lemma 4.21, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{T}|_{\mathscr{P}} & \xrightarrow{\mathsf{SR}^\vee} & \mathbb{E}[-1] & \xrightarrow{\phi[-1]} & \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}}[-1] \\ & \searrow^{\mathsf{B}_{\widetilde{\gamma}}|_{\mathscr{P}}} & \downarrow{\scriptstyle \mathsf{SR}} & & \downarrow{\scriptstyle -\mathsf{KS}_{\mathscr{P}/\mathcal{B}}} \\ & & \mathcal{T}|_{\mathscr{P}}^\vee & \xrightarrow{\mathsf{KS}_{\mathcal{X}/\mathcal{B}}|_{\mathscr{P}}} & \Omega^1_{\mathcal{B}}|_{\mathscr{P}} \end{array}$$

where $\mathsf{KS}_{\mathcal{X}/\mathcal{B}}$ denotes the Kodaira–Spencer map. Thus it suffices to show that the composition

$$\mathcal{T} \xrightarrow{\mathsf{B}_{\widetilde{\gamma}}} \mathcal{T}^\vee \xrightarrow{\mathsf{KS}_{\mathcal{X}/\mathcal{B}}} \Omega^1_{\mathcal{B}}$$

vanishes. Since $\mathcal{B}$ is smooth, it suffices to show that the composition

$$T_{\mathcal{B},b} \xrightarrow{\mathsf{KS}_{\mathcal{X}/\mathcal{B},b}} H^1(\mathcal{X}_b, T_{\mathcal{X}_b}) \xrightarrow{\mathsf{B}_{\widetilde{\gamma}_b}} H^1(\mathcal{X}_b, T_{\mathcal{X}_b})^\vee \tag{97}$$

vanishes for all $b \in \mathcal{B}$. By [Bloch 1972, Proposition 4.2] (see also Lemma 4.11(1)), the composition (97) indeed vanishes since the horizontal section $\tilde{v}_2$ is contained in $F^2 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$.

Now we can apply Kiem and Li's cone reduction lemma [2013] for the absolute versions of the obstruction theory ϕ^{abs} in (93) and cosection $\mathsf{SR}^{\mathrm{abs}}$ in (95). Indeed, by Lemma 4.16, we have a commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{C}((\mathbb{E}^{\mathrm{abs}})^{\mathrm{red}}_{\mathcal{V}}) & \longrightarrow & \mathscr{P} & & \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle 0} & & \\ (\mathfrak{C}_{\mathscr{P}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}^{\mathrm{abs}}) & \xrightarrow{\mathfrak{l}_{\mathsf{SR}^{\mathrm{abs}}}} & \mathcal{T}|_{\mathscr{P}}^\vee & \twoheadrightarrow & \mathcal{V}|_{\mathscr{P}}^\vee \end{array} \tag{98}$$

for some dotted arrow, where $\mathfrak{l}_{\mathsf{SR}^{\mathrm{abs}}} := \mathfrak{C}((\mathsf{SR}^{\mathrm{abs}})^\vee[1])$ and

$$(\mathbb{E}^{\mathrm{abs}})^{\mathrm{red}}_{\mathcal{V}} := \mathrm{cone}(\mathcal{V}|_{\mathscr{P}}[1] \to \mathcal{T}|_{\mathscr{P}}[1] \xrightarrow{(\mathsf{SR}^{\mathrm{abs}})^\vee[1]} \mathbb{E}^{\mathrm{abs}}).$$

The existence of the dotted arrow in (98) tells us that the lower horizontal composition in (98) is zero. Since the diagrams (94) and (95) are commutative, the lower horizontal composition in (92) is also zero, which gives us the desired dotted arrow in (92).

We define the relative reduced virtual cycle as

$$[\mathscr{P}]^{\mathrm{red}} := \sqrt{0^!_{\mathfrak{C}(\mathbb{E}^{\mathrm{red}}_{\mathcal{V}})}}[\mathfrak{C}_{\mathscr{P}/\mathcal{B}}] \in A_*(\mathscr{P}),$$

where the square root Gysin pullback $\sqrt{0^!_{\mathfrak{C}(\mathbb{E}^{\mathrm{red}}_{\mathcal{V}})}}$ is defined as in the proof of Theorem 4.1. By Vistoli's rational equivalence [1989, Lemma 3.16],[22] we have

$$i_b^![\mathfrak{C}_{\mathscr{P}/\mathcal{B}}] = [\mathfrak{C}_{\mathscr{P}_b}] \in A_*(\mathfrak{C}_{\mathscr{P}/\mathcal{B}}|_{\mathscr{P}_b}),$$

[22] In [Vistoli 1989], it is stated for unramified maps between DM stacks, but the result holds for arbitrary DM-type morphisms between Artin stacks. This can be deduced from the proof of [Kresch 1999a, Proposition 4] (cf. [Qu 2018, Proposition 2.5]).

where we denote $\mathscr{P}_b := \mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)$. Since Oh and Thomas's localized square root Euler classes [2023] are bivariant classes by [Kiem and Park 2023, Lemma 4.4(3)], the square root Gysin pullback $\sqrt{0^!_{\Omega(\mathbb{E}^{\mathrm{red}}_{\mathcal{V}})}}$ commutes with the ordinary Gysin pullback $i_b^!$. Therefore, we have

$$i_b^![\mathscr{P}]^{\mathrm{red}} = [\mathscr{P}_b]^{\mathrm{red}}. \qquad \square$$

Remark 5.2 The relative version of the Darboux theorem [Brav et al. 2019; Bouaziz and Grojnowski 2013] works only under an additional assumption. If $\mathbb{X}$ is a derived scheme over a scheme $\mathcal{B}$ with a (-2)-shifted symplectic form $\theta \in \mathsf{HN}^{-4}(\mathbb{X}/\mathcal{B})(2) \cong \mathcal{A}^{2,\mathrm{cl}}(\mathbb{X}/\mathcal{B}; -2)$, we have a global Darboux chart only if θ maps to zero under the canonical map

$$\mathsf{HN}^{-4}(\mathbb{X}/\mathcal{B})(2) \to \mathsf{HP}^{-4}(\mathbb{X}/\mathcal{B})(2) \tag{99}$$

to the periodic cyclic homology.

There is a simple example where there is no Darboux chart for a (-2)-shifted symplectic derived scheme. Consider the origin in the affine plane,

$$\mathbb{X} := \mathrm{Spec}(\mathbb{C}[t_1, t_2]/(t_1, t_2)) \subseteq \mathcal{B} := \mathrm{Spec}(\mathbb{C}[t_1, t_2]) \cong \mathbb{A}^2.$$

Using the quasifree resolution $(\mathbb{C}[t_1, t_2][y_1, y_2], dy_i = t_i) \xrightarrow{\mathrm{qis}} \mathbb{C}[t_1, t_2]/(t_1, t_2)$, we can find a (-2)-symplectic form

$$\theta := d_{\mathrm{DR}} y_1 \wedge d_{\mathrm{DR}} y_2 \in \mathsf{HN}^{-4}(\mathbb{X}/\mathcal{B})(2).$$

The isotropic condition fails in this case since $\mathfrak{C}_{\mathbb{X}/\mathcal{B}} = \mathfrak{C}(\mathbb{L}_{\mathbb{X}/\mathcal{B}}) = \mathbb{A}^2$ for the induced symmetric obstruction theory $\mathbb{L}_{\mathbb{X}/\mathcal{B}} \xrightarrow{\mathrm{id}} \mathbb{L}_{\mathbb{X}/\mathcal{B}}$. This implies that there is no Darboux chart. Moreover, we have $\theta \mapsto t_1 t_2 \neq 0 \in \mathsf{HP}^{-4}(\mathbb{X}/\mathcal{B})(2) \cong \mathbb{C}[\![t_1, t_2]\!]$ under the canonical map (99).

However, we still have the desired isotropic condition in the situation of Theorem 5.1. This will be shown in [Park 2024], and we briefly summarize the results here. Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of relative dimension 4 with $\omega : \mathcal{O}_{\mathcal{X}} \xrightarrow{\cong} \Omega^4_{\mathcal{X}/\mathcal{B}}$. Let $\mathcal{M}$ be any moduli space of perfect complexes on the fibres of $f : \mathcal{X} \to \mathcal{B}$. The isotropic condition can be detected infinitesimally, so we may assume that $\mathcal{B} = \mathrm{Spec}(A)$ for a local Artinian $\mathbb{C}$-algebra A. Let $\gamma = \mathrm{ch}_2(E) \in H^4(\mathcal{X}_0, \mathbb{C})$ for any $[E] \in \mathcal{M}$ and let $\widetilde{\gamma} \in H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be the horizontal lift. Then the (-2)-symplectic form on the derived enhancement of $\mathcal{M}$ [Pantev et al. 2013; Toën and Vaquié 2007] maps to zero under the map (99) if and only if

$$\widetilde{\gamma} \in F^1 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) = \mathbb{H}^4(\mathcal{X}, \Omega^{\geq 1}_{\mathcal{X}/\mathcal{B}}) \subseteq H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}) = \mathbb{H}^4(\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/\mathcal{B}}). \tag{100}$$

In the situation of Theorem 5.1, the condition (100) is satisfied since we already assumed $\widetilde{\gamma} \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Moreover, condition (100) implies the existence of a local Darboux chart which induces the desired isotropic condition.

Heuristically, the result in [Park 2024] tells us that the relative isotropic condition is satisfied along the locus in the base where the deformations of $\mathrm{ch}_2(E)$ do not have a $(0, 4)$-Hodge piece. This condition is

compatible with the gauge-theoretic point of view since the holomorphic bundles are the solutions of

$$\int_{\mathcal{X}_0} \alpha(E) \cup \omega = 0,$$

where $\alpha(E) = 2\,\mathrm{ch}_2(E)$; see [Oh and Thomas 2023, Appendix]. We thank Richard Thomas for the explanation of the gauge-theory perspective and the expectation that (100) is the necessary additional condition to obtain deformation invariance.

Consequently, we have deformation invariance of numerical invariants.

Corollary 5.3 *In the situation of Theorem 5.1, the function*

$$\mathcal{B} \to \mathbb{Q}, \quad b \mapsto \int_{[\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)]^{\mathrm{red}}} j_b^*(\star),$$

is constant for any insertion $\star \in A^{\mathrm{rvd}}(\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B}))$.

There are two technical assumptions in Theorem 5.1:

(A1) The function $\mathcal{B} \to \mathbb{Z}$, $b \mapsto \rho_{\tilde{\gamma}_b}$, is constant.

(A2) The symmetric complex $\mathbb{E}$ on $\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})$ and the orthogonal bundle $\mathcal{T}_{\tilde{\gamma}}$ on $\mathcal{B}$ are orientable.

We first discuss the assumption (A1) on $\rho_{\tilde{\gamma}_b}$.

Remark 5.4 The assumption (A1) is *generically* satisfied. Indeed, the function $\mathcal{B} \to \mathbb{Z}$, $b \mapsto \rho_{\tilde{\gamma}_b}$, is a lower semicontinuous function. Hence, if we let $\rho_{\max}$ be the maximal value, then

$$\mathcal{B}^\circ := \{b \in \mathcal{B} : \rho_{\tilde{\gamma}_b} = \rho_{\max}\}$$

is an open dense subscheme of $\mathcal{B}$. Consequently, we can form an orthogonal bundle of rank $\rho_{\max}$ on $\mathcal{B}^\circ$ as

$$\mathcal{T}_{\tilde{\gamma}} := (\mathcal{T}|_{\mathcal{B}^\circ})/\ker(\mathsf{B}_{\tilde{\gamma}}).$$

On the other hand, for the *special* cases when assumption (A1) is not satisfied, we have a vanishing result by Proposition 5.5 below.

Proposition 5.5 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of relative dimension 4 with connected fibres to a smooth connected affine scheme $\mathcal{B}$ such that $\omega_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{X}}$. Let $\tilde{v} \in \bigoplus_p F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be a horizontal section and $\tilde{\gamma} \in \Gamma(\mathcal{B}, R^2 f_* \Omega^2_{\mathcal{X}/\mathcal{B}})$ be the section induced by $\tilde{v}_2$. Let $\mathcal{B}^\circ := \{b \in \mathcal{B} : \rho_{\tilde{\gamma}_b} = \rho_{\max}\}$ as in Remark 5.4. If $b \in \mathcal{B}$ is a point such that $b \notin \mathcal{B}^\circ$, then we have*

$$[\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)]^{\mathrm{red}} = 0 \in A_*(\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b))$$

for any orientation.

Before we prove Proposition 5.5, we discuss the assumption (A2) on orientations. The following simple observation shows that the orientability of the orthogonal bundle $\mathcal{T}_{\tilde{\gamma}}$ is a minor issue:

Remark 5.6 There exists a canonical 2-fold étale cover $u : \widetilde{\mathcal{B}} \to \mathcal{B}^\circ$ such that $\mathcal{T}_{\widetilde{\mathcal{Y}}}|_{\widetilde{\mathcal{B}}}$ is orientable. Indeed, we can form a fibre diagram

$$\begin{array}{ccc} \widetilde{\mathcal{B}} & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathcal{B}^\circ & \xrightarrow{\det(\mathcal{T}_{\widetilde{\mathcal{Y}}})} & B\mu_2 \end{array}$$

where the horizontal map $\mathcal{B}^\circ \to B\mu_2$ corresponds to the pair of the line bundle $\det(\mathcal{T}_{\widetilde{\mathcal{Y}}})$ and the isomorphism $\mathcal{O}_{\mathcal{B}^\circ} \cong \det(\mathcal{T}_{\widetilde{\mathcal{Y}}})^{\otimes 2}$ induced by $\mathrm{B}_{\widetilde{\mathcal{Y}}}$. Then we have a canonical bijection

$$\{\text{orientations of } \mathcal{T}_{\widetilde{\mathcal{Y}}}\} \cong \{\text{sections of } \widetilde{\mathcal{B}} \to \mathcal{B}^\circ\}.$$

Consequently, there exists a canonical orientation of the pullback $\mathcal{T}_{\widetilde{\mathcal{Y}}}|_{\widetilde{\mathcal{B}}}$.

On the other hand, the orientability of the symmetric complex $\mathbb{E}$ is more subtle. The orientability result of [Cao et al. 2020] for the fibres does not imply the orientability for the relative case. We suggest two strategies:

(a) Use the canonical orientation on the canonical 2-fold étale cover of the *moduli space* $\mathscr{P}$ as in Remark 5.6.

(b) Find an orientation after a *base change* by a finite étale cover $\mathcal{B}' \to \mathcal{B}$.

Strategy (a) is sufficient for applications to the variational Hodge conjecture in the next subsection. However, if we want to obtain the deformation invariance of numerical invariants (Corollary 5.3), strategy (a) is not enough and we need strategy (b). We expect that the following conjecture holds:

Conjecture 5.7 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism with connected fibres of relative dimension 4 to a smooth connected affine scheme $\mathcal{B}$ such that $\omega_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{X}}$. Let $\tilde{v} \in \bigoplus_p F^p H^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be a horizontal section and let $q \in \{-1, 0, 1\}$.*

(1) *Then there exists a finite étale cover $u : \mathcal{B}' \to \mathcal{B}$ such that $\mathbb{E}|_{\mathscr{P}^{(q)}_{\tilde{v}'}(\mathcal{X}'/\mathcal{B}')}$ is orientable, where $\mathcal{X}' = \mathcal{X} \times_{\mathcal{B}} \mathcal{B}'$ and $\tilde{v}' = u^*(\tilde{v})$.*

(2) *Moreover, we can further assume that the degree of the étale cover $u : \mathcal{B}' \to \mathcal{B}$ in* (1) *divides*

$$|K^0_{\mathrm{top}}(\mathcal{X}^{\mathrm{an}}_b)_{\mathrm{tor}}|!,$$

where $K^0_{\mathrm{top}}(\mathcal{X}^{\mathrm{an}}_b)_{\mathrm{tor}} \subseteq K^0_{\mathrm{top}}(\mathcal{X}^{\mathrm{an}}_b)$ is the torsion subgroup of the complex topological K-theory of $\mathcal{X}^{\mathrm{an}}_b$, which is finite.

We will provide evidence for Conjecture 5.7 in Appendix C. Indeed, we will prove that the topological analog of Conjecture 5.7 is true (see Corollary C.2).

Remark 5.8 Assuming Conjecture 5.7, $\mathbb{E}$ is orientable in the following cases:

(1) $\mathcal{B}^{\mathrm{an}}$ is simply connected.

(2) $H^*(\mathcal{X}_b^{\mathrm{an}}, \mathbb{Z})$ has no torsion elements for some $b \in \mathcal{B}$, e.g., all complete intersection Calabi–Yau 4-folds in smooth projective toric varieties.

In case (2), we have $K^0_{\mathrm{top}}(\mathcal{X}_b^{\mathrm{an}})_{\mathrm{tor}} = 0$ by [Atiyah and Hirzebruch 1961, Corollary 2.5(i)].

Proof of Proposition 5.5 We may assume that

$$\mathcal{B}^\circ = \mathcal{B} - \{b\}.$$

Indeed, we can find a smooth connected affine curve C with a map $g : C \to \mathcal{B}$ such that $g^{-1}(\mathcal{B}^\circ) = C - \{c\}$ for some point $c \in C$. Replace $\mathcal{B}$ by C.

We claim that there exists a nondegenerate subbundle

$$\mathcal{V} \subseteq \mathcal{T}$$

of rank $\rho_{\widetilde{\gamma}_b}$, after replacing $\mathcal{B}$ by some open neighbourhood of $b \in \mathcal{B}$. Indeed, choose a maximal nondegenerate subspace $V \subseteq H^1(\mathcal{X}_b, T_{\mathcal{X}_b})$. Then we can extend this to a map $\mathcal{V} := V \otimes \mathcal{O}_{\mathcal{B}} \to \mathcal{T}$ of vector bundles.

Since its dual $\mathcal{T}^\vee \to \mathcal{V}^\vee$ is surjective over $b \in \mathcal{B}$, by shrinking $\mathcal{B}$, we may assume that $\mathcal{V}$ is a subbundle of $\mathcal{T}$. Since $\mathsf{B}_{\widetilde{\gamma}}|_{\mathcal{V}}$ is nondegenerate over $b \in \mathcal{B}$, by shrinking $\mathcal{B}$, we may assume that $\mathsf{B}_{\widetilde{\gamma}}|_{\mathcal{V}}$ is nondegenerate on $\mathcal{B}$.

Let $\widetilde{\mathscr{P}} \to \mathscr{P}$ be the canonical 2-fold étale cover induced by the line bundle $\det(\mathbb{E}) \otimes \det(\mathcal{V})^\vee$ and the symmetric forms of $\mathbb{E}$ and $\mathcal{V}$ (as in Remark 5.6). Form a fibre diagram

$$\begin{array}{ccccc}
\widetilde{\mathscr{P}}_b & \xrightarrow{j} & \widetilde{\mathscr{P}} & \xleftarrow{k} & \widetilde{\mathscr{P}}^\circ \\
{\scriptstyle t_b}\uparrow\downarrow & & \downarrow & & \downarrow \\
\mathscr{P}_b & \hookrightarrow & \mathscr{P} & \hookleftarrow & \mathscr{P}^\circ \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\{b\} & \xrightarrow{i_b} & \mathcal{B} & \hookleftarrow & \mathcal{B}^\circ
\end{array}$$

Since the symmetric complex $\mathbb{E}_b$ on the fibre $\mathscr{P}_b$ is orientable by [Cao et al. 2020] and the vector space $\mathcal{V}_b$ is always orientable, we have

$$\widetilde{\mathscr{P}}_b = \mathscr{P}_b \sqcup \mathscr{P}_b.$$

Let $t_b : \mathscr{P}_b \hookrightarrow \widetilde{\mathscr{P}}_b$ denote the inclusion to the first component.

Although the rank of $\mathsf{B}_{\widetilde{\gamma}}$ jumps at b, the rank of $\mathsf{B}_{\widetilde{\gamma}}|_{\mathcal{V}}$ does not jump, so the argument in Theorem 5.1 can indeed be used. Hence there exists a cycle class

$$[\widetilde{\mathscr{P}}]^{\mathrm{red}}_{\mathcal{V}} \in A_*(\widetilde{\mathscr{P}})$$

such that

$$i_b^![\widetilde{\mathscr{P}}]^{\mathrm{red}}_{\mathcal{V}} = \big([\mathscr{P}_b]^{\mathrm{red}}, -[\mathscr{P}_b]^{\mathrm{red}}\big) \in A_*(\widetilde{\mathscr{P}}_b) = A_*(\mathscr{P}_b) \oplus A_*(\mathscr{P}_b). \tag{101}$$

Indeed, we have a decomposition of symmetric complexes

$$\mathbb{E} \cong \mathbb{E}_{\mathcal{V}}^{\mathrm{red}} \oplus (\mathcal{V}|_{\mathscr{P}}[1])$$

for some symmetric complex $\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}$ and the relative cone reduction property

$$(\mathfrak{C}_{\mathscr{P}/\mathcal{B}})_{\mathrm{red}} \subseteq \mathfrak{Q}(\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}).$$

We can define the desired cycle class as

$$[\widetilde{\mathscr{P}}]_{\mathcal{V}}^{\mathrm{red}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}|_{\widetilde{\mathscr{P}}})}}[\mathfrak{C}_{\widetilde{\mathscr{P}}/\mathcal{B}}] \in A_*(\widetilde{\mathscr{P}})$$

since the pullback $\mathbb{E}_{\mathcal{V}}^{\mathrm{red}}|_{\widetilde{\mathscr{P}}}$ is orientable by the definition of $\widetilde{\mathscr{P}}$.

Since there are additional cosections on $\mathscr{P}^\circ$, we claim that

$$k^*[\widetilde{\mathscr{P}}]_{\mathcal{V}}^{\mathrm{red}} = 0 \in A_*(\widetilde{\mathscr{P}}^\circ). \tag{102}$$

Indeed, the orthogonal bundle $\mathcal{T}_{\widetilde{\gamma}}$ is well defined on $\mathcal{B}^\circ$, and the composition

$$\mathcal{V}|_{\mathcal{B}^\circ} \hookrightarrow \mathcal{T}|_{\mathcal{B}^\circ} \twoheadrightarrow \mathcal{T}_{\widetilde{\gamma}}$$

is injective on the fibres. Thus, by considering a section of the surjection

$$\frac{\mathcal{T}|_{\mathcal{B}^\circ}}{\mathcal{V}|_{\mathcal{B}^\circ}} \twoheadrightarrow \frac{\mathcal{T}_{\widetilde{\gamma}}}{\mathcal{V}|_{\mathcal{B}^\circ}},$$

we can find a nondegenerate subbundle $\mathcal{W}$ of $\mathcal{T}|_{\mathcal{B}^\circ}$ properly containing $\mathcal{V}|_{\mathcal{B}^\circ}$,

$$\mathcal{V}|_{\mathcal{B}^\circ} \subsetneq \mathcal{W} \subseteq \mathcal{T}|_{\mathcal{B}^\circ}.$$

Then we have $\mathcal{W} = \mathcal{V}|_{\mathcal{B}^\circ} \oplus \mathcal{U}$ for the orthogonal complement $\mathcal{U}$ of $\mathcal{V}|_{\mathcal{B}^\circ}$ in $\mathcal{W}$. Moreover, we have a decomposition of symmetric complexes

$$\mathbb{E}|_{\mathscr{P}^\circ} = \mathbb{E}_{\mathcal{W}}^{\mathrm{red}} \oplus (\mathcal{W}|_{\mathscr{P}^\circ}[1]) = \mathbb{E}_{\mathcal{W}}^{\mathrm{red}} \oplus (\mathcal{V}|_{\mathscr{P}^\circ}[1]) \oplus (\mathcal{U}|_{\mathscr{P}^\circ}[1])$$

for some symmetric complex $\mathbb{E}_{\mathcal{W}}^{\mathrm{red}}$. Hence we have

$$(\mathbb{E}_{\mathcal{V}}^{\mathrm{red}})|_{\mathscr{P}^\circ} \cong \mathbb{E}_{\mathcal{W}}^{\mathrm{red}} \oplus (\mathcal{U}|_{\mathscr{P}^\circ}[1]).$$

By shrinking $\mathcal{B}$ further, we find a section $w \in \Gamma(\mathcal{B}^\circ, \mathcal{U})$ such that w^2 is nowhere vanishing. Consequently, we can find a cosection

$$\sigma : \mathbb{E}_{\mathcal{V}}^{\mathrm{red}}|_{\mathscr{P}^\circ}[1] \to \mathcal{O}_{\mathscr{P}^\circ}$$

such that σ^2 is nowhere vanishing. Lemma 5.9 below proves the claim.

By the localization sequence

$$A_*(\widetilde{\mathscr{P}}_b) \xrightarrow{j_*} A_*(\widetilde{\mathscr{P}}) \xrightarrow{k^*} A_*(\widetilde{\mathscr{P}}^\circ) \to 0$$

and the vanishing (102), we have

$$[\widetilde{\mathscr{P}}]^{\mathrm{red}}_{\mathcal{V}} = j_*(\alpha)$$

for some $\alpha \in A_*(\widetilde{\mathscr{P}}_b)$. By the self-intersection formula, we have

$$i_b^![\widetilde{\mathscr{P}}]^{\mathrm{red}}_{\mathcal{V}} = i_b^! \circ j_*(\alpha) = e(N_{b/\mathcal{B}}) \cdot \alpha = 0.$$

Then the formula (101) proves the desired vanishing result. □

We need the following lemma to complete the proof of Proposition 5.5:

Lemma 5.9 (see [Kiem and Park 2023, Theorem 8.12]) *Let $\mathbb{E}$ be any symmetric complex on a quasiprojective scheme $\mathcal{X}$. Let $\sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_\mathcal{X}$ be a cosection such that $\sigma^2 \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is nowhere vanishing. Consider a fibre diagram*

$$\begin{array}{ccc} \mathfrak{Q}(\mathbb{E}_\sigma) & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle 0} \\ \mathfrak{Q}(\mathbb{E}) & \xrightarrow{\mathfrak{l}_\sigma} & \mathbb{A}^1_\mathcal{X} \end{array}$$

where $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1] : \mathcal{O}_\mathcal{X} \to \mathbb{E})$. Then the composition

$$A_*(\mathfrak{Q}(\mathbb{E}_\sigma)) \xrightarrow{l_*} A_*(\mathfrak{Q}(\mathbb{E})) \xrightarrow{\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}} A_*(\mathcal{X})$$

vanishes.

We omit the proof. See [Kiem and Park 2023, Section 8.3].

5.2 Variational Hodge conjecture

As an application of the deformation invariance in the previous subsection, we prove that the *variational Hodge conjecture* holds for surface classes with nonzero reduced virtual cycles.

Fix a smooth projective variety X over $\mathbb{C}$.

Definition 5.10 Let $\gamma \in H^{2p}(X, \mathbb{C})$ be a cohomology class on X.

(1) We say that γ is a *Hodge class* if

$$\gamma \in \mathrm{Hdg}^p(X) := H^{2p}(X, \mathbb{Q}) \cap H^p(X, \Omega_X^p) \subset H^{2p}(X, \mathbb{C}).$$

(2) We say that γ is an *algebraic* cohomology class if γ contained in the image of the cycle class map:

$$\gamma \in \mathrm{im}(\mathrm{cl} : A^p(X)_\mathbb{Q} \to \mathrm{Hdg}^p(X)).$$

The *Hodge conjecture* predicts that every Hodge class on X is algebraic. The *variational Hodge conjecture* [Grothendieck 1966a, note 13] (cf. [Bloch 1972]) predicts that a deformation of an algebraic cohomology class as a Hodge class is algebraic.

Thus we make the following definition:

Definition 5.11 Let X be a smooth projective variety and $\gamma \in H^{2p}(X,\mathbb{C})$ an algebraic cohomology class. We say that the *variational Hodge conjecture* holds for the pair (X,γ) if the following condition is satisfied: for any smooth projective morphism $f : \mathcal{X} \to \mathcal{B}$ to a smooth connected scheme $\mathcal{B}$ and any horizontal section $\tilde{v}_p \in \Gamma(\mathcal{B}, F^p\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$ such that $\mathcal{X}_0 \cong X$ and $(\tilde{v}_p)_0 = \gamma$ for some $0 \in \mathcal{B}$, the cohomology classes $(\tilde{v}_p)_b \in H^{2p}_{\mathrm{DR}}(\mathcal{X}_b)$ are algebraic for all $b \in \mathcal{B}$.

Remark 5.12 In some of the literature, the variational Hodge conjecture is stated for any horizontal section of the Hodge bundle $\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ instead of the filtration $F^p\mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Deligne's global invariant cycle theorem [1971, (4.1.1)] implies that the two versions are equivalent (see also [Charles and Schnell 2014, Proposition 11.3.5]).

We now state the main theorem of this subsection.

Theorem 5.13 *Let X be a Calabi–Yau 4-fold and let $\gamma \in H^2(X,\Omega^2_X)$. If*

$$[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0 \in A_{\mathrm{rvd}}(\mathscr{P}_v^{(q)}(X))$$

for some $v \in H^*(X,\mathbb{C})$ with $v_2 = \gamma$ and $q \in \{-1,0,1\}$, then the variational Hodge conjecture holds for (X,γ).

For Calabi–Yau 4-folds, we would like to regard Theorem 5.13 as a *virtual* generalization of [Bloch 1972; Buchweitz and Flenner 2003] since the semiregular case is heuristically the *ideal* case (see Theorem 4.6). In particular, for Calabi–Yau 4-folds, we recover the result of Buchweitz and Flenner as an immediate corollary:

Corollary 5.14 *Let X be a Calabi–Yau 4-fold and let $\gamma \in H^2(X,\Omega^2_X)$. If there exists a PT_q pair (F,s) on X such that $\mathrm{ch}_2(F) = \gamma$ for some $q \in \{-1,0,1\}$, and the associated complex $I^\bullet := [\mathcal{O}_X \xrightarrow{s} F]$ is semiregular, then the variational Hodge conjecture holds for (X,γ).*

Proof Theorem 4.6 provides a nonempty smooth open subset $U \subseteq \mathscr{P}_v^{(q)}(X)$ such that $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}}|_U = \pm[U] \in A_*(U)$. Hence, $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0$, and Theorem 5.13 completes the proof. □

We now prove our main theorem in this subsection.

Proof of Theorem 5.13 Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism to a smooth connected scheme $\mathcal{B}$ and $\widetilde{\gamma}$ be a horizontal section of $F^2\mathcal{H}^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Assume that $\mathcal{X}_0 \cong X$ and $\widetilde{\gamma}_0 = \gamma$ for some point $0 \in \mathcal{B}$.

Heuristically, Theorem 5.13 follows from the deformation invariance in Theorem 5.1. However, to obtain a rigorous proof, we need to overcome the following technical issues:

(1) existence of a horizontal lift $\widetilde{\beta} \in \Gamma(\mathcal{B}, F^3\mathcal{H}^6_{\mathrm{DR}}(\mathcal{X}/\mathcal{B}))$ of $\beta = v_3$;

(2) $f : \mathcal{X} \to \mathcal{B}$ may not be relatively Calabi–Yau (i.e., $\omega_{\mathcal{X}/\mathcal{B}} \cong \mathcal{O}_{\mathcal{X}}$);

(3) the function $b \mapsto \rho_{\widetilde{\gamma}_b}$ on $\mathcal{B}$ may not be constant;

(4) orientability of the orthogonal bundle $\mathcal{T}_{\widetilde{\gamma}}$ on $\mathcal{B}$;

(5) orientability of the virtual cotangent complex $\mathbb{E}$ of $\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B}) \to \mathcal{B}$;

(6) the reduced virtual cycles $[\mathscr{P}^{(q)}_{\tilde{v}_b}(\mathcal{X}_b)]^{\mathrm{red}}$ live in the Chow groups of different schemes for each $b \in \mathcal{B}$.

We first resolve issue (1). Denote by $P_v(t)$ the Hilbert polynomial determined by v with respect to some polarization on X. We will consider a small variation on the moduli space $\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})$, namely

$$\begin{aligned}\mathscr{P} &:= \mathscr{P}^{(q)}_{\tilde{\gamma},P_v}(\mathcal{X}/\mathcal{B}) \\ &= \{(F,s,b) : (F,s) \text{ is a } \mathrm{PT}_q \text{ pair on the fibre } \mathcal{X}_b \text{ of } b \in \mathcal{B} \text{ such that } \mathrm{ch}_2(F) = \tilde{\gamma}_b \text{ and } P_F(t) = P_v(t)\}.\end{aligned}$$

The basic properties of $\mathscr{P}^{(q)}_{\tilde{v}}(\mathcal{X}/\mathcal{B})$ in Theorem 3.32 also hold for $\mathscr{P}$. More precisely, the projection map

$$p : \mathscr{P} \to \mathcal{B}$$

is projective and the canonical map

$$\mathscr{P} \hookrightarrow \mathscr{P}erf(\mathcal{X}/\mathcal{B})^{\mathrm{spl}}_{\mathcal{O}_{\mathcal{X}}}$$

is an open embedding. Consequently, we have a symmetric obstruction theory

$$\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}/\mathcal{B}}$$

for $p : \mathscr{P} \to \mathcal{B}$ that satisfies the isotropic condition. To prove Theorem 5.13, it suffices to show that the projection map $p : \mathscr{P} \to \mathcal{B}$ is surjective. Since $p : \mathscr{P} \to \mathcal{B}$ is projective, it suffices to show that the image of $p : \mathscr{P} \to \mathcal{B}$ contains a nonempty open subset of $\mathcal{B}$.

Secondly, we resolve issue (2). We claim that there exists a nonempty open neighbourhood $\mathcal{B}'$ of $0 \in \mathcal{B}$ such that

$$\omega_{\mathcal{X}\times_{\mathcal{B}}\mathcal{B}'/\mathcal{B}'} \cong \mathcal{O}_{\mathcal{B}'}.$$

Indeed, by [Deligne 1968, théorème 5.5], the pushforward $f_*(\omega_{\mathcal{X}/\mathcal{B}})$ is a vector bundle that commutes with base change. Since the fibre $\mathcal{X}_0$ is Calabi–Yau, $f_*(\omega_{\mathcal{X}/\mathcal{B}})$ is a line bundle. Consider the canonical map of line bundles

$$f^* f_*(\omega_{\mathcal{X}/\mathcal{B}}) \to \omega_{\mathcal{X}/\mathcal{B}}$$

given by adjunction. Since it is an isomorphism over $0 \in \mathcal{B}$, it is an isomorphism over an open subscheme of $\mathcal{X}$ containing the fibre $\mathcal{X}_0$. Since $f : \mathcal{X} \to \mathcal{B}$ is projective, by shrinking $\mathcal{B}$ we may assume that the map $f^* f_*(\omega_{\mathcal{X}/\mathcal{B}}) \to \omega_{\mathcal{X}/\mathcal{B}}$ is an isomorphism. Moreover, by shrinking $\mathcal{B}$ again, we may assume that $f_*(\omega_{\mathcal{X}/\mathcal{B}}) \cong \mathcal{O}_{\mathcal{B}}$, and thus $\omega_{\mathcal{X}/\mathcal{B}} \cong f^* f_*(\omega_{\mathcal{X}/\mathcal{B}}) \cong \mathcal{O}_{\mathcal{X}}$. This proves the claim.

Next, the vanishing result in Proposition 5.5 resolves issue (3). Indeed, since $[\mathscr{P}^{(q)}_{v_0}(\mathcal{X}_0)]^{\mathrm{red}} \neq 0$ by assumption, we have $0 \in \mathcal{B}^\circ$. Replace $\mathcal{B}$ by $\mathcal{B}^\circ$. Then we may assume that the function $b \mapsto \rho_{\tilde{\gamma}_b}$ on $\mathcal{B}$ is constant.

Issue (4) is resolved by simply replacing $\mathcal{B}$ by a 2-fold étale cover $\mathcal{B}' \to \mathcal{B}$ such that $\mathcal{T}_{\tilde{\gamma}}|_{\mathcal{B}'}$ is orientable (Remark 5.6).

We resolve issue (5) via an infinitesimal trick. (Alternatively, we can also use a 2-fold étale cover of $\mathcal{P}$ as in the proof of Proposition 5.5.) Assume that $p : \mathscr{P} \to \mathcal{B}$ is not surjective. Since the image is closed,

there exists a 1-dimensional integral closed subscheme $C \subseteq \mathcal{B}$ such that $0 \in C$ and $C \not\subseteq \mathrm{im}(p : \mathscr{P} \to \mathcal{B})$. By replacing $\mathcal{B}$ by the normalization $\widetilde{C} \to C$ of C, we may assume that $\mathcal{B}$ is a smooth quasiprojective curve, and the (set-theoretic) image $\mathrm{im}(p : \mathscr{P} \to \mathcal{B})$ consists of finitely many points. By shrinking $\mathcal{B}$ further, we may assume that the image contains a single point $0 \in \mathcal{B}$. Then we have $(\mathscr{P})_{\mathrm{red}} = (\mathscr{P}_0)_{\mathrm{red}}$ and hence $\mathbb{E}$ is orientable by Lemma 5.15 below.

Finally, we explain how to avoid issue (6). We follow the situation in the previous paragraph, namely that $\mathcal{B}$ is a smooth curve and the set-theoretic image of $p : \mathscr{P} \to \mathcal{B}$ contains a single point $0 \in \mathcal{B}$. Form a fibre diagram

$$\begin{array}{ccc} \mathscr{P}_0 & \xhookrightarrow{j} & \mathscr{P} \\ \downarrow & & \downarrow p \\ \{0\} & \xhookrightarrow{i_0} & \mathcal{B} \end{array}$$

Since $\mathscr{P}_0 \hookrightarrow \mathscr{P}$ is a nilpotent thickening and the Chow groups are invariant under nilpotent thickenings, we may write

$$[\mathscr{P}]^{\mathrm{red}} = j_*(\alpha)$$

for some $\alpha \in A_*(\mathscr{P}_0)$. By Theorem 5.1 and the self-intersection formula, we have

$$[\mathscr{P}_0]^{\mathrm{red}} = i_0^![\widetilde{\mathscr{P}}]^{\mathrm{red}} = i_0^! j_*(\alpha) = e(N_{0/\mathcal{B}}) \cap \alpha = 0$$

since the normal bundle $N_{0/\mathcal{B}}$ is a trivial line bundle. This contradicts the assumption $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0$, and hence we conclude that $p : \mathscr{P} \to \mathcal{B}$ is surjective. □

We need Lemma 5.15 to complete the proof of Theorem 5.13:

Lemma 5.15 *Let $\mathcal{X} \hookrightarrow \mathcal{X}'$ be a nilpotent embedding of quasiprojective schemes. Let E be an orthogonal bundle on $\mathcal{X}'$. If $E|_{\mathcal{X}}$ is orientable, then E is orientable.*

Proof Consider the canonical short exact sequence of algebraic groups

$$1 \to \mathrm{SO}(n) \to O(n) \xrightarrow{\det} \mu_2 \to 1,$$

where n is the rank of E. Note that the orthogonal bundle E defines a function $\mathcal{X}' \xrightarrow{E} BO(n)$ to the classifying stack $BO(n) := [\mathrm{Spec}(\mathbb{C})/O(n)]$ of $O(n)$. Consider the fibre diagram

$$\begin{array}{ccccccc} U|_{\mathcal{X}} & \longrightarrow & U & \longrightarrow & B\mathrm{SO}(n) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' & \xrightarrow{E} & BO(n) & \xrightarrow{\det} & B\mu_2 \end{array}$$

Since $E|_{\mathcal{X}}$ is orientable, the principal μ_2-bundle $U|_{\mathcal{X}} \to \mathcal{X}$ is trivial. Hence, $U|_{\mathcal{X}} = U_1 \sqcup U_2$ for some open subschemes such that the compositions $U_1, U_2 \hookrightarrow U|_{\mathcal{X}} \to \mathcal{X}$ are isomorphisms. Since U and $U|_{\mathcal{X}}$ have the same Zariski topology, we can decompose $U = \widetilde{U}_1 \sqcup \widetilde{U}_2$ by disjoint open subschemes

$\widetilde{U}_1, \widetilde{U}_2$ extending U_1, U_2. Then $\widetilde{U}_1, \widetilde{U}_2 \to \mathcal{X}'$ are both finite étale bijective maps. Hence, they are both isomorphisms. This means that the μ_2-bundle $U \to \mathcal{X}'$ has a section. Hence, E is orientable. □

The following example illustrates Theorem 5.13 in some nonsemiregular cases:

Example 5.16 Let $X = S_1 \times S_2$ be a product of two K3 surfaces. Let $\beta_1 \in H^{1,1}(S_1)$, $\beta_2 \in H^{1,1}(S_2)$ be two irreducible effective curve classes with $\beta_1^2 = 2h-2$, $h \geq 2$, and $\beta_2^2 = 0$. Let $v := \mathrm{ch}(\mathcal{O}_{C_1 \times C_2})$ for any $C_1 \in |\mathcal{O}_{S_1}(\beta_1)|$ and $C_2 \in |\mathcal{O}_{S_2}(\beta_2)|$. Then we have an isomorphism

$$\phi : |\mathcal{O}_{S_1}(\beta_1)| \times \mathbb{P}^1 \to \mathscr{P}_v^{(1)}(X). \tag{103}$$

Under this isomorphism, we have

$$[\mathscr{P}_v^{(1)}(X)]^{\mathrm{red}} = h^0(\mathcal{O}_{S_1}(\beta_1)) \cdot [\mathrm{pt} \times \mathbb{P}^1]$$

for some orientation.

We briefly illustrate the argument. The proof will appear in one of the sequels to this paper. By the choice of β_2, there exists an elliptic fibration $p : S_2 \to \mathbb{P}^1$ such that β_2 is the class of a fibre. By Proposition 4.35, the reduced virtual dimension of $\mathscr{P}_v^{(1)}(X)$ is 1. Let $\mathbb{I}^\bullet = [\mathcal{O}_{|\mathcal{O}_{S_1}(\beta_1)| \times S_1} \twoheadrightarrow \mathcal{O}_{\mathcal{C}}]$ be the universal pair on $|\mathcal{O}_{S_1}(\beta_1)| \times S_1$. Consider the fibre diagram

$$\begin{array}{ccccc} & & S_1 \times S_2 & \xhookrightarrow{j} & S_1 \times \mathbb{P}^1 \times S_2 \\ & \swarrow^{\mathrm{pr}} & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}\times p} \\ S_1 & & S_1 \times \mathbb{P}^1 & \xhookrightarrow{\mathrm{id}\times\Delta} & S_1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal. Then the universal pair $j_* \mathrm{pr}^* \mathbb{I}^\bullet$ on $|\mathcal{O}_{S_1}(\beta_1)| \times \mathbb{P}^1 \times X$ induces the morphism (103). Using [Park 2021, Proposition 1.18], we can show that ϕ is an isomorphism and the reduced obstruction theory of $\mathscr{P}_v^{(1)}(X)$ reduces to a 2-term obstruction theory on $|\mathcal{O}_{S_1}(\beta_1)| \times \mathbb{P}^1$.

By Theorem 4.6, the points of $\mathscr{P}_v^{(1)}(X)$ are *not* semiregular. Since the reduced virtual cycle of $\mathscr{P}_v^{(1)}(X)$ is nonzero, Theorem 5.13 applies to the class $\gamma = \beta_1 \cup \beta_2$.

Remark 5.17 The original version of the variational Hodge conjecture by [Grothendieck 1966a] is stated for all *geometric points*, while Definition 5.11 only considers $\mathbb{C}$-points. In the situation of Theorem 5.13, since we have shown that the map $p : \mathscr{P}^{(q)}_{\widetilde{\gamma}, P_v}(\mathcal{X}/\mathcal{B}) \to \mathcal{B}$ is surjective on $\mathbb{C}$-points, it is also surjective for all geometric points. Hence we also deduce the original form of the variational Hodge conjecture.

Remark 5.18 In the situation of Theorem 5.13, we further speculate that $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0$ implies *smoothness* of the Hodge locus of γ in the moduli space of Calabi–Yau 4-folds near X.[23] We note that not much is known about singularities of the Hodge locus. At least when the scheme-theoretic Hodge locus is reduced near X and $[\mathscr{P}_v^{(q)}(X)]^{\mathrm{red}} \neq 0$, Proposition 5.5 proves smoothness near X since the dimension of the tangent space is constant near X. We hope to pursue this direction in future work.

[23] For simplicity we consider Calabi–Yau 4-folds with $h^{0,1} = h^{0,2} = h^{0,3} = 0$. Then there exists a moduli space of Calabi–Yau 4-folds as a smooth separated Deligne–Mumford stack with quasiprojective coarse moduli space by [Viehweg 1995] (see also [Kollár 2023, Corollary 8.23]) and the Bogomolov–Tian–Todorov theorem [Bogomolov 1978; Tian 1987; Todorov 1989].

Appendix A Reductions via (−1)-shifted 1-forms

In this section, we construct a reduced (−2)-shifted symplectic derived enhancement on $\mathscr{P}_v^{(q)}(X)$. In particular, this construction shows that the reduced obstruction theory constructed from the algebraic twistor family in Section 4.5 satisfies the isotropic condition.

In Section A.1, we present general reduction procedures by cosections that come from (−1)-shifted *closed/exact* 1-forms. In Section A.2, we show that the cosections on $\mathscr{P}_v^{(q)}(X)$ induced by the semi-regularity map can be enhanced to (−1)-shifted *closed* 1-forms.

We use the following conventions/notation in this section:

(1) We use the language of ∞-*categories* in the sense of [Lurie 2009; 2017].

- We write $\mathsf{Map}_{\mathcal{C}}(-,-)$ for the mapping space of an ∞-category $\mathcal{C}$.

We consider the following ∞-categories:

- Let cdga be the ∞-category of commutative differential graded $\mathbb{C}$-algebras and let $\mathsf{cdga}^{\leq 0} \subseteq \mathsf{cdga}$ be the full subcategory of nonpositively graded objects.
- Let dSt be the ∞-category of *derived stacks*, i.e., ∞-sheaves on $\mathsf{cdga}^{\leq 0}$ with respect to the étale topology.
- Let $\epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}$ (resp. $\mathsf{dg}^{\mathrm{gr}}$) be the symmetric monoidal stable ∞-category of graded mixed complexes (resp. graded complexes); see [Pantev et al. 2013]. We use the weight-shift notation $E((m)) := \bigoplus_p E(p+m)$ for any $E = \bigoplus_p E(p) \in \epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}$ and $m \in \mathbb{Z}$.

(2) We use the language of *derived Artin stacks* of [Toën and Vezzosi 2008].

- All derived Artin stacks are assumed to be *of finite type*, i.e., their classical truncation are (higher) Artin stacks of finite type.
- Let $\mathsf{D}(\mathbb{M})$ be the symmetric monoidal stable ∞-category of quasicoherent sheaves on a derived Artin stack $\mathbb{M}$. Let $f^* : \mathsf{D}(\mathbb{M}_2) \to \mathsf{D}(\mathbb{M}_1)$ be the pullback functor for a morphism $f : \mathbb{M}_1 \to \mathbb{M}_2$ of derived Artin stacks and let f_* be the right adjoint.
- A derived Artin stack $\mathbb{M}$ is *homotopically finitely presented* if the cotangent complex $\mathbb{L}_{\mathbb{M}} \in \mathsf{D}(\mathbb{M})$ is perfect (see [ibid., Proposition 2.2.2.4]).
- A derived Artin stack $\mathbb{M}$ is a *derived scheme* if the classical truncation $\mathbb{M}_{\mathrm{cl}}$ is a scheme.

(3) Let $\mathsf{DR}(-) : \mathsf{dSt}^{\mathrm{op}} \to \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}}$ be the ∞-functor of *derived de Rham complexes* of [Pantev et al. 2013], defined as follows:

- Let $\epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}}$ (resp. $\mathsf{cdga}^{\mathrm{gr}}$) be the ∞-category of graded mixed commutative differential graded $\mathbb{C}$-algebras (resp. graded commutative differential graded $\mathbb{C}$-algebras); see [Calaque et al. 2017].
- Define DR as the right Kan extension of the ∞-functor $\mathsf{DR}_{\mathrm{aff}}|_{\mathsf{cdga}^{\leq 0}} : \mathsf{cdga}^{\leq 0} \to \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}}$, where $\mathsf{DR}_{\mathrm{aff}}$ is the left adjoint of the ∞-functor $\epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}} \to \mathsf{cdga}$, $E \mapsto E(0)$.

Abusing notation, we use the same symbol $\mathsf{DR}(-)$ to denote the compositions $\mathsf{dSt}^{\mathrm{op}} \xrightarrow{\mathsf{DR}} \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}} \to \mathsf{cdga}^{\mathrm{gr}}$ and $\mathsf{dSt}^{\mathrm{op}} \xrightarrow{\mathsf{DR}} \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}} \to \mathsf{dg}^{\mathrm{gr}}$.

(4) Let $\mathsf{NC}(-) : \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}} \to \mathsf{cdga}^{\mathrm{gr}}$ (resp. $\mathsf{CC}(-) : \epsilon\text{-}\mathsf{dg}^{\mathrm{gr}} \to \mathsf{dg}^{\mathrm{gr}}$) be the ∞-functor of *weighted negative cyclic complexes* (resp. *weighted cyclic complexes*), defined as the right adjoint (resp. left adjoint) of the ∞-functor of trivial mixed structures, $\mathsf{cdga}^{\mathrm{gr}} \to \epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}}$ (resp. $\mathsf{dg}^{\mathrm{gr}} \to \epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}$), $E \mapsto (E, \epsilon = 0)$.

- Abusing notation, we use the same symbols NC, CC to denote the compositions $\mathsf{NC} \circ \mathsf{DR}$, $\mathsf{CC} \circ \mathsf{DR}$.

A.1 Three reductions of derived schemes

In this subsection, we revisit Kiem and Li's cone reduction [2013] via *derived algebraic geometry*. We will see that there are three levels of reductions for derived schemes.

We consider the hierarchy of structures

$$\{\text{derived schemes}\} \xrightarrow{\text{(a)}} \{\text{schemes with obstruction theories}\}$$
$$\xrightarrow{\text{(b)}} \{\text{schemes with closed embedding of their intrinsic normal cone into an abelian cone stack}\}$$

More precisely, the above two arrows can be given as follows:

(a) For any homotopically finitely presented derived scheme $\mathbb{X}$, there is an induced obstruction theory

$$\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$$

on the classical truncation $\mathcal{X} := \mathbb{X}_{\mathrm{cl}}$ by [Schürg et al. 2015, Proposition 1.2].

(b) For any scheme $\mathcal{X}$ with an obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$, there is an induced closed embedding

$$\iota : \mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{C}(\mathbb{E})$$

of the intrinsic normal cone $\mathfrak{C}_{\mathcal{X}}$ into the abelian cone stack $\mathfrak{C}(\mathbb{E})$.

In derived algebraic geometry, (-1)-*shifted* 1-*forms* are the natural analogs of *cosections*. We have a similar hierarchy for them:

(a) Let $\alpha : \mathcal{O}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ be a (-1)-shifted 1-form on a homotopically finitely presented derived scheme $\mathbb{X}$. Then we have an induced *cosection*

$$\sigma := \alpha|_{\mathcal{X}}^{\vee} : \mathbb{E}^{\vee}[1] \to \mathcal{O}_{\mathcal{X}}$$

for the induced obstruction theory $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_{\mathcal{X}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ on the classical truncation $\mathcal{X} := \mathbb{X}_{\mathrm{cl}}$.

(b) Let $\sigma : \mathbb{E}^{\vee}[1] \to \mathcal{O}_{\mathcal{X}}$ be a cosection for an obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ on a scheme $\mathcal{X}$. Then we have an induced *linear function*

$$\mathfrak{l}_{\sigma} := \mathfrak{C}(\sigma^{\vee}[1]) : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_{\mathcal{X}}$$

on the associated abelian cone stack $\mathfrak{C}(\mathbb{E})$.

We now state the main result in this subsection. We refer to Appendix B for the generalizations of intrinsic normal cones, obstruction theories, and abelian cone stacks to (higher) Artin stacks.

Theorem A.1 *Let $\mathbb{X}$ be a homotopically finitely presented derived Artin stack. Let $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_{\mathcal{X}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ be the induced obstruction theory on the classical truncation $\mathcal{X} := \mathbb{X}_{\mathrm{cl}}$ and let $\iota : \mathfrak{C}_{\mathcal{X}} \hookrightarrow \mathfrak{C}(\mathbb{E})$ be the induced closed embedding.*

(1) *Cone reduction* [Kiem and Li 2013]: *For any (-1)-shifted 1-form $\alpha \in \mathcal{A}^1(\mathbb{X}; -1)$, we have a commutative diagram of cone stacks*

$$\begin{array}{ccccccc} & & & & \mathfrak{C}(\mathbb{E}_\sigma) & \longrightarrow & \mathcal{X} \\ & \overset{\iota^{\mathrm{red}}}{\nearrow} & & & \downarrow & & \downarrow{\scriptstyle 0} \\ (\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}_{\mathcal{X}} & \overset{\iota}{\hookrightarrow} & \mathfrak{C}(\mathbb{E}) & \overset{\mathfrak{l}_\sigma}{\longrightarrow} & \mathbb{A}^1_{\mathcal{X}} \end{array}$$

for a unique dotted arrow, where $\sigma := \alpha|_{\mathcal{X}}^{\vee}$ and $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1])$.

(2) *Obstruction theory reduction: For any (-1)-shifted **closed** 1-form $\widetilde{\alpha} \in \mathcal{A}^{1,\mathrm{cl}}(\mathbb{X}; -1)$, we have a commutative diagram of complexes*

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{X}}[1] & \xrightarrow{\sigma^\vee[1]} & \mathbb{E} & \longrightarrow & \mathbb{E}_\sigma \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \phi^{\mathrm{red}}} & \\ & & \tau^{\geq -1}\mathbb{L}_{\mathcal{X}} & & \end{array}$$

for a unique dotted arrow, where $\alpha := \widetilde{\alpha}_0$ is the underlying (-1)-shifted 1-form, $\sigma := \alpha|_{\mathcal{X}}^{\vee}$, and $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1])$.

(3) *Derived reduction* [Schürg et al. 2015]: *For any (-1)-shifted **exact** 1-form $\widetilde{\widetilde{\alpha}} \in \mathcal{A}^{1,\mathrm{ex}}(\mathbb{X}; -1) \simeq \mathcal{A}^0(\mathbb{X}; -1)$* (see Remark A.2 below)*, we have a homotopy-commutative diagram of derived Artin stacks*

$$\begin{array}{ccccc} & & \mathbb{X}^{\mathrm{red}} & \longrightarrow & \mathbb{X} \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle 0} \\ \mathcal{X} & \longrightarrow & \mathbb{X} & \overset{\widetilde{\widetilde{\alpha}}}{\longrightarrow} & \mathbb{A}^1_{\mathbb{X}}[-1] \end{array}$$

for some $\mathbb{X}^{\mathrm{red}}$, where the square is homotopy cartesian and the triangle induces isomorphisms $\mathcal{X} \cong (\mathbb{X}^{\mathrm{red}})_{\mathrm{cl}} \cong \mathbb{X}_{\mathrm{cl}}$.

In Theorem A.1, we use the following definitions of shifted *closed*/*exact* forms:

Remark A.2 Let $\mathbb{X}$ be a homotopically finitely presented derived Artin stack.

(1) The space of *d-shifted p-forms* is defined as the mapping space

$$\mathcal{A}^p(\mathbb{X}; d) := \mathsf{Map}_{\mathsf{dg}^{\mathrm{gr}}}\big(\mathbb{C}((-p)), \mathsf{DR}(\mathbb{X})[d-p]\big) \cong \mathsf{Map}_{\mathsf{D}(\mathbb{X})}\big(\mathcal{O}_{\mathbb{X}}, \textstyle\bigwedge^p \mathbb{L}_{\mathbb{X}}[d]\big).$$

(2) The space of *d-shifted closed p-forms* is defined as the mapping space

$$\mathcal{A}^{p,\mathrm{cl}}(\mathbb{X};d) := \mathsf{Map}_{\mathsf{dg}^{\mathrm{gr}}}\big(\mathbb{C}((-p)),\mathsf{NC}(\mathbb{X})[d-p]\big) \cong \mathsf{Map}_{\epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}}\big(\mathbb{C}((-p)),\mathsf{DR}(\mathbb{X})[d-p]\big).$$

(3) The space of *d-shifted exact p-forms* is defined as

$$\mathcal{A}^{p,\mathrm{ex}}(\mathbb{X};d) := \mathsf{Map}_{\mathsf{dg}^{\mathrm{gr}}}\big(\mathbb{C}((-p+1)),\mathsf{CC}(\mathbb{X})[d-p+1]\big).$$

Then we have canonical maps of spaces

$$\mathcal{A}^{p,\mathrm{ex}}(\mathbb{X};d) \to \mathcal{A}^{p,\mathrm{cl}}(\mathbb{X};d) \xrightarrow{(-)_0} \mathcal{A}^{p}(\mathbb{X};d).$$

Before we prove Theorem A.1, we explain how the three reductions in Theorem A.1 are related.

Remark A.3 In the situation of Theorem A.1, we have the following:

(1) The obstruction theory reduction in Theorem A.1(2) is equivalent to the *scheme-theoretic cone reduction*, i.e., there exists a commutative diagram of cone stacks

$$\begin{array}{ccccc} & & \mathfrak{C}(\mathbb{E}_\sigma) & \longrightarrow & \mathfrak{X} \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle 0} \\ \mathfrak{C}_{\mathfrak{X}} & \xrightarrow{\iota} & \mathfrak{C}(\mathbb{E}) & \xrightarrow{\mathfrak{t}_\sigma} & \mathbb{A}^1_{\mathfrak{X}} \end{array}$$

for some dotted arrow. Hence the obstruction theory reduction in Theorem A.1(2) clearly implies the cone reduction in Theorem A.1(1).

(2) The derived reduction in Theorem A.1(3) implies the obstruction theory reduction in Theorem A.1(2). Indeed, the commutative diagram of derived Artin stacks induces a commutative diagram of cotangent complexes

$$\begin{array}{ccccc} \mathcal{O}_{\mathfrak{X}}[1] = \mathbb{L}_{\mathbb{X}^{\mathrm{red}}/\mathbb{X}}|_{\mathfrak{X}}[-1] & \xrightarrow{d_{\mathrm{DR}}(\widetilde{\alpha})_0} & \mathbb{L}_{\mathbb{X}}|_{\mathfrak{X}} & \longrightarrow & \mathbb{L}_{\mathbb{X}^{\mathrm{red}}}|_{\mathfrak{X}} \\ & & \downarrow & \nearrow & \\ & & \mathbb{L}_{\mathfrak{X}} & & \end{array}$$

where $\mathbb{L}_{\mathbb{X}^{\mathrm{red}}/\mathbb{X}}|_{\mathfrak{X}} = \mathbb{L}_{\mathbb{X}/\mathbb{A}^1_{\mathbb{X}}[-1]}|_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}[2]$. By composing with the canonical map $\mathbb{L}_{\mathfrak{X}} \to \tau^{\geq -1}\mathbb{L}_{\mathfrak{X}}$, we obtain the desired obstruction theory reduction.

Note that the cone reduction in Theorem A.1(1) was shown in [Kiem and Li 2013] (see Lemma 4.16) and the derived reduction in Theorem A.1(3) is trivial. (This approach was first introduced in [Schürg et al. 2015].) Thus the essential part of Theorem A.1 is the obstruction theory reduction in Theorem A.1(2). The following *derived Poincaré lemma* (which was essentially shown in [Brav et al. 2019]) shows that (-1)-shifted closed 1-forms are always exact and hence Theorem A.1(2) follows from Theorem A.1(3):

Proposition A.4 *Let $\mathbb{X}$ be a homotopically finitely presented derived Artin stack. Then the canonical map*

$$\mathcal{A}^{1,\mathrm{ex}}(\mathbb{X};-1) \to \mathcal{A}^{1,\mathrm{cl}}(\mathbb{X};-1)$$

is an equivalence.

Proof Since the statement is local, we may assume that $\mathbb{X}$ is a derived affine scheme such that the associated analytic space of the classical truncation is connected.

Let $\mathsf{PC}(\mathbb{X})$ be the cofibre of the map $B : \mathsf{CC}(\mathbb{X})((-1))[1] \to \mathsf{NC}(\mathbb{X})$ in $\mathsf{dg}^{\mathrm{gr}}$ induced by the map $\mathsf{DR}(\mathbb{X})((-1))[1] \to \mathsf{NC}(\mathbb{X})$ in [Pantev et al. 2013, Remark 1.6] via adjunction. Consider the induced morphism of cofibre sequences

$$\begin{array}{ccccc} \mathsf{NC}(\mathbb{X})(p) & \xrightarrow{I} & \mathsf{PC}(\mathbb{X})(p) & \xrightarrow{S} & \mathsf{CC}(\mathbb{X})(p-1)[2] \\ \downarrow & & \downarrow & & \| \\ \mathsf{DR}(\mathbb{X})(p) & \xrightarrow{I} & \mathsf{CC}(\mathbb{X})(p) & \xrightarrow{S} & \mathsf{CC}(\mathbb{X})(p-1)[2] \end{array} \tag{104}$$

in dg. Note that we have canonical equivalences

$$\mathsf{PC}(\mathbb{X})(p) \cong \mathsf{PC}(\mathcal{X})(p) \cong R\Gamma(\widehat{\mathcal{Y}}, \widehat{\Omega}^{\bullet}_{\mathcal{Y}})[2p] \cong R\Gamma(\mathcal{X}^{\mathrm{an}}, \mathbb{C})[2p] \tag{105}$$

in dg, where $\widehat{\Omega}^{\bullet}_{\mathcal{Y}}$ is Hartshorne's algebraic de Rham complex [1975] for an embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ into a smooth scheme $\mathcal{Y}$ and $\mathcal{X}^{\mathrm{an}}$ is the analytic space associated to $\mathcal{X}$. Based on the HKR isomorphism [Toën and Vezzosi 2011; Ben-Zvi and Nadler 2012] (see also [Hoyois 2015]), the first equivalence follows from Goodwillie's nil-invariance [1985, Theorem IV.2.1], the second equivalence follows from the Feigin–Tsygan theorem [1985, Theorem 5] (see also [Emmanouil 1995, Theorem 2.2; Weibel 1997, Theorem 3.4]), and the third equivalence follows from Hartshorne's comparison theorem [1975, Theorem IV.1.1].

We first claim that the canonical map given by the upper sequence in (104),

$$\mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{CC}(\mathbb{X})(0)[-1]) \xrightarrow{B} \mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{NC}(\mathbb{X})(1)[-2]), \tag{106}$$

is an equivalence. Indeed, since $H^k(-)$ of (105) vanishes for $p = 1$ and $k \leq -3$, it suffices to show that

$$H^{-2}(\mathsf{PC}(\mathbb{X})(1)) \xrightarrow{B} H^0(\mathsf{CC}(\mathbb{X})(0))$$

is injective. Then we have a commutative square

$$\begin{array}{ccc} H^{-2}(\mathsf{PC}(\mathbb{X})(1)) & \xrightarrow{S} & H^0(\mathsf{CC}^0(\mathbb{X})(0)) \\ \downarrow \cong & & \downarrow \\ H^{-2}(\mathsf{PC}(\mathbb{X}))(1) & \xhookrightarrow{S} & H^0(\mathsf{CC}^0(\mathbb{X})(0)) \end{array}$$

where the left vertical arrow is bijective by (105) and the lower horizontal arrow is injective by [Emmanouil 1995, Proposition 2.6]. Hence we have the desired injectivity.

Then the canonical map given by the lower sequence in (104),

$$\mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{DR}(\mathbb{X})(0)[-1]) \xrightarrow{I} \mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{CC}(\mathbb{X})(0)[-1]), \tag{107}$$

is an equivalence since $\mathsf{CC}(\mathbb{X})(-1) \cong 0$.

By combining the two equivalences (106) and (107), we can deduce that the canonical map

$$\mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{DR}(\mathbb{X})(0)[-1]) \xrightarrow{d_{\mathrm{DR}}} \mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{NC}(\mathbb{X})(1)[-2])$$

is an equivalence, as desired. □

Remark A.5 The derived Poincaré lemma also appears in [Aranha et al. 2022, Lemma 6.11]. We refer readers who are interested in a generalization of Kiem and Li's cosection localization [2013], using derived algebraic geometry and homotopical methods in intersection theory, to [Aranha et al. 2022] (we only consider the cone reduction part in this paper).

A.2 Shifted closed 1-forms on moduli of perfect complexes

In this subsection, we prove that the cosections given by the semiregularity maps can be enhanced to shifted *closed* 1-forms. This is achieved by generalizing the *integration map* of [Pantev et al. 2013] and defining the *hat product* for derived mapping stacks.

We first generalize the *integration map* in [loc. cit.] for Calabi–Yau varieties to arbitrary smooth projective varieties.

Theorem A.6 *Let X be a smooth projective variety of dimension n. Let $\mathbb{M}$ be a derived Artin stack. Then there exists a natural map*

$$\int_\pi (-) : \mathsf{DR}(X \times \mathbb{M}) \to \mathsf{DR}(\mathbb{M})((-n))$$

in ϵ-$\mathsf{dg}^{\mathrm{gr}}$, where $\pi : X \times \mathbb{M} \to \mathbb{M}$ is the projection map. Consequently, we have a natural homotopy-commutative square

$$\begin{array}{ccc} \mathcal{A}^{p,\mathrm{cl}}(X \times \mathbb{M}, i) & \xrightarrow{\int_\pi (-)} & \mathcal{A}^{p-n,\mathrm{cl}}(\mathbb{M}, i-n) \\ \downarrow & & \downarrow \\ \mathcal{A}^{p}(X \times \mathbb{M}, i) & \xrightarrow{\int_\pi (-)} & \mathcal{A}^{p-n}(\mathbb{M}, i-n) \end{array}$$

Proof By the Künneth formula in Lemma A.7 below, we have a canonical equivalence

$$\mathsf{DR}(\mathbb{M}) \otimes \mathsf{DR}(X) \xrightarrow{\cong} \mathsf{DR}(X \times \mathbb{M})$$

in ϵ-$\mathsf{cdga}^{\mathrm{gr}}$. Hence it suffices to construct a natural integration map for X,

$$\int_X (-) : \mathsf{DR}(X) \to \mathbb{C}((-n)),$$

in ϵ-$\mathsf{dg}^{\mathrm{gr}}$. Equivalently, it suffices to construct a natural map

$$\mathsf{CC}(X)(n) \to \mathbb{C}$$

in dg since we have an adjunction $\mathsf{CC}(-)(n) \dashv (-)((-n))$. The canonical equivalence

$$H^0(\mathsf{CC}(X)(n)) \cong \mathbb{H}^{2n}(X, \Omega_X^{\leq n}) \xleftarrow{\cong} H^n(X, \Omega_X^n) \xrightarrow{\cong} \mathbb{C}$$

gives us the desired map. □

We provide a Künneth formula for DR in the following lemma. This lemma is required to complete the proof of Theorem A.6.

Lemma A.7 *Let X be a smooth projective variety. Let $\mathbb{M}$ be a derived Artin stack. Then there is a canonical equivalence*

$$\mathsf{DR}(X)\otimes\mathsf{DR}(\mathbb{M})\to\mathsf{DR}(X\times\mathbb{M})$$

in ϵ-$\mathsf{cdga}^{\mathrm{gr}}$.

Proof Form a homotopy-cartesian square of derived Artin stacks

$$\begin{array}{ccc} X\times\mathbb{M} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ \mathbb{M} & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array} \tag{108}$$

Consider the canonical map

$$q^*(-)\cup\pi^*(-):\mathsf{DR}(X)\otimes\mathsf{DR}(\mathbb{M})\to\mathsf{DR}(X\times\mathbb{M}) \tag{109}$$

in ϵ-$\mathsf{cdga}^{\mathrm{gr}}$ given by the universal property of the coproduct. It suffices to show that the map (109) induces an equivalence in $\mathsf{dg}^{\mathrm{gr}}$ since the forgetful functor

$$\epsilon\text{-}\mathsf{cdga}^{\mathrm{gr}}\to\mathsf{cdga}^{\mathrm{gr}}\to\mathsf{dg}^{\mathrm{gr}}$$

is conservative.

Note that we have a canonical equivalence

$$\mathsf{DR}(\mathbb{M})\cong\bigoplus_p R\Gamma\big(\mathbb{M},\textstyle\bigwedge^p\mathbb{L}_{\mathbb{M}}[p]\big) \tag{110}$$

in $\mathsf{cdga}^{\mathrm{gr}}$ if we forget the mixed structure in $\mathsf{DR}(\mathbb{M})$ (see [Calaque et al. 2017, Remark 2.4.4]). Moreover, since we have a canonical equivalence $\mathbb{L}_{X\times\mathbb{M}}\cong q^*(\mathbb{L}_X)\oplus\pi^*(\mathbb{L}_{\mathbb{M}})$ in $\mathsf{D}(X\times\mathbb{M})$, we also have an induced canonical equivalence

$$\textstyle\bigwedge^p\mathbb{L}_{X\times\mathbb{M}}\cong\bigoplus_{p=p_1+p_2}q^*\big(\bigwedge^{p_1}\mathbb{L}_X\big)\otimes\pi^*\big(\bigwedge^{p_2}\mathbb{L}_{\mathbb{M}}\big)$$

in $\mathsf{D}(X\times\mathbb{M})$. Applying (110) for X, $\mathbb{M}$, $X\times\mathbb{M}$ to the canonical map (109), it suffices to show that the canonical map

$$R\Gamma\big(X,\textstyle\bigwedge^{p_1}\mathbb{L}_X\big)\otimes R\Gamma\big(\mathbb{M},\bigwedge^{p_2}\mathbb{L}_{\mathbb{M}}\big)\to R\Gamma\big(X\times\mathbb{M},q^*\big(\bigwedge^{p_1}\mathbb{L}_X\big)\otimes\pi^*\big(\bigwedge^{p_2}\mathbb{L}_{\mathbb{M}}\big)\big) \tag{111}$$

in dg is an equivalence.

We will factor the map (111) as follows: Firstly, consider the canonical map

$$\pi_*q^*\big(\textstyle\bigwedge^{p_1}\mathbb{L}_X\big)\otimes\bigwedge^{p_2}\mathbb{L}_{\mathbb{M}}\to\pi_*\big(q^*\big(\bigwedge^{p_1}\mathbb{L}_X\big)\otimes\pi^*\big(\bigwedge^{p_2}\mathbb{L}_{\mathbb{M}}\big)\big) \tag{112}$$

in $\mathsf{D}(\mathbb{M})$ given by the adjunction $\pi^*\dashv\pi_*$. Secondly, consider the canonical base change map

$$R\Gamma\big(X,\textstyle\bigwedge^{p_1}\mathbb{L}_X\big)\otimes\mathcal{O}_{\mathbb{M}}\to\pi_*q^*\big(\bigwedge^{p_1}\mathbb{L}_X\big) \tag{113}$$

in $\mathsf{D}(\mathbb{M})$ given by the cartesian square (108). Finally, consider the canonical map

$$R\Gamma\big(X, \textstyle\bigwedge^{p_1} \mathbb{L}_X\big) \otimes R\Gamma\big(\mathbb{M}, \bigwedge^{p_2} \mathbb{L}_M\big) \to R\Gamma\big(\mathbb{M}, R\Gamma\big(X, \bigwedge^{p_1} \mathbb{L}_X\big) \otimes \bigwedge^{p_2} \mathbb{L}_{\mathbb{M}}\big) \tag{114}$$

in dg given by the adjunction $(-) \otimes \mathcal{O}_{\mathbb{M}} \dashv R\Gamma(\mathbb{M}, -)$. Then the map (111) is the composition of (114), $R\Gamma\big(\mathbb{M}, (-) \otimes \bigwedge^{p_2} \mathbb{L}_{\mathbb{M}}\big)$ of the map (113), and $R\Gamma(\mathbb{M}, -)$ of the map (112). Hence it suffices to show that the three maps (112)–(114) are equivalences.

The map (112) is the projection formula for $\pi : X \times \mathbb{M} \to \mathbb{M}$ and the map (113) is the base change formula for the square (108). Both of these maps are equivalences by [Ben-Zvi et al. 2010, Proposition 3.10] since $X \to \mathrm{Spec}(\mathbb{C})$ is a perfect morphism, in the sense of [ibid., Definition 3.2], by [ibid., Corollary 3.23 and Proposition 3.19].

The map (114) is an equivalence since $R\Gamma\big(X, \bigwedge^{p_1} \mathbb{L}_X\big)$ is a perfect complex in dg and both the functors $(-) \otimes R\Gamma\big(\mathbb{M}, \bigwedge^{p_2} \mathbb{L}_{\mathbb{M}}\big)$, $R\Gamma\big(\mathbb{M}, (-) \otimes \bigwedge^{p_2} \mathbb{L}_{\mathbb{M}}\big)$ commute with finite colimits. □

We compare the integration map in Theorem A.6 and the one in [Pantev et al. 2013].

Remark A.8 Let X be a Calabi–Yau n-fold with fixed Calabi–Yau n-form $\omega : \mathcal{O}_X \xrightarrow{\cong} \Omega_X^n$. Let $\mathbb{M}$ be a derived Artin stack. Then the integration map in [Pantev et al. 2013] can be expressed as

$$\int_{\pi,\mathrm{PTVV}} (-) \cong \int_{\pi} ((-) \cup q^*\omega) : \mathsf{DR}(X \times \mathbb{M}) \to \mathsf{DR}(\mathbb{M})[-n], \tag{115}$$

where $q : X \times \mathbb{M} \to X$ and $\pi : X \times \mathbb{M} \to \mathbb{M}$ are the projection maps. More precisely, the right-hand side of the equivalence (115) is defined as the composition

$$\begin{aligned}\mathsf{DR}(X \times \mathbb{M}) \xrightarrow{\mathrm{id}\otimes\omega} \mathsf{DR}(X \times \mathbb{M}) \otimes \mathsf{DR}(X)((n))[-n] \xrightarrow{\mathrm{id}\otimes q^*} \mathsf{DR}(X \times \mathbb{M}) \otimes \mathsf{DR}(X \times \mathbb{M})((n))[-n] \\ \xrightarrow{\cup} \mathsf{DR}(X \times \mathbb{M})((n))[-n] \xrightarrow{\int_\pi} \mathsf{DR}(\mathbb{M})[-n],\end{aligned}$$

where $\omega \in H^0(X, \Omega_X^n) \cong \mathbb{H}^n(X, \Omega_X^{\geq n})$ is regarded as an element in

$$\pi_0 \mathsf{Map}_{\mathsf{dg}}(\mathbb{C}, \mathsf{NC}(X)(n)[-n]) \cong \pi_0 \mathsf{Map}_{\epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}}\big(\mathbb{C}, \mathsf{DR}(X)((n))[-n]\big)$$

and the cup product $\cup$ is given by the multiplicative structure of $\mathsf{DR}(X \times \mathbb{M})$.

We now explain how we can obtain the equivalence (115). Recall that the integration map in [loc. cit.] is based on the construction of the canonical map

$$\kappa : \mathsf{DR}(X \times \mathbb{M}) \to R\Gamma(X, \mathcal{O}_X) \otimes \mathsf{DR}(\mathbb{M})$$

in $\epsilon\text{-}\mathsf{dg}^{\mathrm{gr}}$. This can be reinterpreted as the composition

$$\mathsf{DR}(X \times \mathbb{M}) \xrightarrow{\cong} \mathsf{DR}(X) \otimes \mathsf{DR}(\mathbb{M}) \to R\Gamma(X, \mathcal{O}_X) \otimes \mathsf{DR}(\mathbb{M}),$$

where the first arrow is given by the Künneth formula in Lemma A.7 and the second arrow is induced by the canonical map $\mathsf{DR}(X) \to R\Gamma(X, \mathcal{O}_X)$ that corresponds to the canonical isomorphism

$\mathsf{CC}(X)(0) \xrightarrow{\cong} R\Gamma(X, \Omega_X^{\leq 0})$ via adjunction. Hence it suffices to show that

$$\int_{X,\mathrm{PTVV}} (-) \cong \int_X (-) \cup \omega : \mathsf{DR}(X) \to \mathbb{C}[-n]$$

holds in ϵ-$\mathsf{dg}^{\mathrm{gr}}$. This follows from the homotopy-commutative diagram

$$\begin{array}{ccccccccc}
\mathsf{CC}(X)(0) & \cong & R\Gamma(X, \Omega_X^{\leq 0}) & \xrightarrow{\omega} & R\Gamma(X, \Omega_X^n) & \longrightarrow & H^n(X, \Omega_X^n)[-n] & \xrightarrow{\cong} & \mathbb{C}[-n] \\
\big\downarrow{\scriptstyle (-)\cup\omega} & & \big\downarrow{\scriptstyle (-)\cup\omega} & & \big\downarrow & & \big\downarrow{\scriptstyle \cong} & & \big\| \\
\mathsf{CC}(X)(n)[-n] & \cong & R\Gamma(X, \Omega_X^{\leq n}[n]) & = & R\Gamma(X, \Omega_X^{\leq n}[n]) & \longrightarrow & \mathbb{H}^{2n}(X, \Omega_X^{\leq n})[-n] & \xrightarrow{\cong} & \mathbb{C}[-n]
\end{array}$$

in dg via adjunction.

We note that the integration map in Theorem A.6 without the closing structures can be described quite simply as follows:

Remark A.9 Let X be a smooth projective variety of dimension n. Let $\mathbb{M}$ be a derived Artin stack. Then the integration map

$$\int_\pi (-) : \mathcal{A}^p(X \times \mathbb{M}; i) \to \mathcal{A}^{p-n}(X; i-n)$$

can be described as follows: Consider the projection map

$$\bigwedge^p \mathbb{L}_{X\times\mathbb{M}} \cong \bigoplus_{p=p_1+p_2} q^*\big(\bigwedge^{p_1} \mathbb{L}_X\big) \otimes \pi^*\big(\bigwedge^{p_2} \mathbb{L}_{\mathbb{M}}\big) \to \pi^*\big(\bigwedge^{p-n} \mathbb{L}_{\mathbb{M}}\big) \otimes q^*(\Omega_X^n)$$

in $\mathsf{D}(X \times \mathbb{M})$. The adjunction $\pi_* \dashv \pi^! \cong \pi^* \otimes q^*(\Omega_X^n)[n]$ gives us a canonical map

$$\pi_*\big(\bigwedge^p \mathbb{L}_{X\times\mathbb{M}}\big) \to \bigwedge^{p-n} \mathbb{L}_{\mathbb{M}}[-n]$$

in $\mathsf{D}(\mathbb{M})$. Applying $\mathsf{Map}_{\mathsf{D}(\mathbb{M})}(\mathcal{O}_{\mathbb{M}}[-i], -)$, we obtain the above integration map.

Based on the generalized integration map in Theorem A.6, we can define the *hat products* for *derived mapping stacks*. We first collect some basic facts on derived mapping stacks.

Remark A.10 Let X be a smooth projective variety. Let $\mathbb{Y}$ be a derived Artin stack. The derived mapping stack $R\underline{\mathsf{Map}}(X, \mathbb{Y})$ is the derived stack determined by the adjunction

$$\mathsf{Map}_{\mathsf{dSt}}(\mathbb{U}, R\underline{\mathsf{Map}}(X, \mathbb{Y})) \cong \mathsf{Map}_{\mathsf{dSt}}(X \times \mathbb{U}, \mathbb{Y})$$

for any $\mathbb{U} \in \mathsf{dSt}$. By [Halpern-Leistner and Preygel 2023], $R\underline{\mathsf{Map}}(X, \mathbb{Y})$ is a derived Artin stack when $\mathbb{Y}$ has a quasiaffine diagonal or the classical truncation of $\mathbb{Y}$ is a 1-Artin stack with affine stabilizers.

The cotangent complex can be computed as follows. For simplicity, assume that $\mathbb{Y}$ is homotopically finitely presented and has quasiaffine diagonal. Let

$$\mathrm{ev} : X \times R\underline{\mathsf{Map}}(X, \mathbb{Y}) \to \mathbb{Y}$$

be the evaluation map. By taking differentials, we have a canonical map

$$\mathrm{ev}_* : \mathbb{T}_{X\times R\underline{\mathsf{Map}}(X,\mathbb{Y})} \cong q^*(T_X) \oplus \pi^*(\mathbb{T}_{R\underline{\mathsf{Map}}(X,\mathbb{Y})}) \to \mathrm{ev}^*(\mathbb{T}_{\mathbb{Y}}),$$

where π, q are the projection maps. The second component of ev_* induces an isomorphism

$$(\mathrm{ev}_*)_2 : \mathbb{T}_{R\underline{\mathsf{Map}}(X,\mathbb{Y})} \cong \pi_* \,\mathrm{ev}^*(\mathbb{T}_{\mathbb{Y}})$$

in $\mathsf{D}(R\underline{\mathsf{Map}}(X,\mathbb{Y}))$ by [ibid., Proposition 5.1.10].

We note that the first component of ev_* gives us

$$R\Gamma(X,T_X)\otimes\mathcal{O}_{R\underline{\mathsf{Map}}(X,\mathbb{Y})} \cong \pi_* q^* T_X \xrightarrow{(\mathrm{ev}_*)_1} \pi_*\,\mathrm{ev}^*(\mathbb{T}_{\mathbb{Y}}) \xleftarrow[\cong]{(\mathrm{ev}_*)_2} \mathbb{T}_{R\underline{\mathsf{Map}}(X,\mathbb{Y})},$$

and thus we obtain a canonical map

$$\widehat{=} : H^i(X,T_X) \to \mathbb{H}^i(R\underline{\mathsf{Map}}(X,\mathbb{Y}), \mathbb{T}_{R\underline{\mathsf{Map}}(X,\mathbb{Y})}) \tag{116}$$

between shifted vector fields.

We now define the *hat product* for a derived mapping stack.

Definition A.11 Let X be a smooth projective variety of dimension n. Let $\mathbb{Y}$ be a derived Artin stack. Assume that $R\underline{\mathsf{Map}}(X,\mathbb{Y})$ is representable by a derived Artin stack. We define the *hat product*

$$\widehat{-\cdot-} : \mathsf{DR}(X)\otimes\mathsf{DR}(\mathbb{Y}) \to \mathsf{DR}(R\underline{\mathsf{Map}}(X,\mathbb{Y}))((-n))$$

as the composition

$$\mathsf{DR}(X)\otimes\mathsf{DR}(\mathbb{Y}) \xrightarrow{q^*(-)\cup\mathrm{ev}^*(-)} \mathsf{DR}(X\times R\underline{\mathsf{Map}}(X,\mathbb{Y})) \xrightarrow{\int_\pi} \mathsf{DR}(R\underline{\mathsf{Map}}(X,\mathbb{Y}))((-n))$$

in ϵ-$\mathsf{dg}^{\mathrm{gr}}$, where ev is the evaluation map and π, q are the projection maps. Consequently, we also have a homotopy-commutative square

$$\begin{array}{ccc}
\mathcal{A}^{p,\mathrm{cl}}(X;i)\times\mathcal{A}^{q,\mathrm{cl}}(\mathbb{Y};j) & \xrightarrow{\widehat{-\cdot-}} & \mathcal{A}^{p+q-n,\mathrm{cl}}(R\underline{\mathsf{Map}}(X,\mathbb{Y});i+j-n) \\
\downarrow & & \downarrow \\
\mathcal{A}^{p}(X;i)\times\mathcal{A}^{q}(\mathbb{Y},j) & \xrightarrow{\widehat{-\cdot-}} & \mathcal{A}^{p+q-n}(R\underline{\mathsf{Map}}(X,\mathbb{Y});i+j-n).
\end{array}$$

We can describe the shifted symplectic forms of [Pantev et al. 2013] on the derived moduli spaces of perfect complexes in terms of the hat product.

Remark A.12 Let X be a Calabi–Yau n-fold with a fixed Calabi–Yau n-form $\omega : \mathcal{O}_X \cong \Omega_X^n$.

Let $R\mathscr{P}erf$ be the derived moduli stack of perfect complexes (of fixed tor-amplitude $[a,b]$). Then the cotangent complex can be computed as

$$\mathbb{L}_{R\mathscr{P}erf} \cong \mathbb{I}^\vee \otimes \mathbb{I}[-1]$$

for the universal complex $\mathbb{I}$ on $R\mathscr{P}erf$. Moreover, the 2-shifted nondegenerate 2-form on $R\mathscr{P}erf$ given by the duality $(\mathbb{L}_{R\mathscr{P}erf}[1])^\vee \cong \mathbb{L}_{R\mathscr{P}erf}[1]$ can be enhanced to a 2-shifted symplectic 2-form

$$\Theta \in \mathcal{A}^{2,\mathrm{cl}}(R\mathscr{P}erf; 2)$$

using the negative cyclic homology-valued second Chern character of $\mathbb{I}$ [Toën and Vezzosi 2015].

Let $R\mathscr{P}erf(X) := R\underline{\mathsf{Map}}(X, R\mathscr{P}erf)$ be the derived moduli stack of perfect complexes on X. Then

$$\theta := \widehat{\omega \cdot \Theta} = \int_\pi q^*\omega \cup \mathrm{ev}^*\,\Theta \in \mathcal{A}^{2,\mathrm{cl}}(R\mathscr{P}erf(X), 2-n)$$

is the $(2-n)$-shifted symplectic form in [Pantev et al. 2013].

We now show that the cosections associated to the semiregularity maps can be enhanced to (-1)-shifted closed 1-forms.

Proposition A.13 *Let X be a Calabi–Yau n-fold with a fixed Calabi–Yau n-form $\omega \in H^0(X, \Omega_X^n)$. For each $(n-3)$-shifted vector field $\xi \in H^{n-3}(X, T_X)$, we can form a (-1)-shifted closed 1-form*

$$\alpha_\xi \in \mathcal{A}^{1,\mathrm{cl}}(R\mathscr{P}erf(X); -1)$$

such that the induced cosection on the obstruction theory $(R\mathcal{H}om_\pi(\mathbb{I},\mathbb{I})[1])^\vee \to \mathbb{L}_{\mathscr{P}erf(X)}$ of the classical truncation $\mathscr{P}erf(X) := R\mathscr{P}erf(X)_{\mathrm{cl}}$ is equal to the composition

$$\begin{aligned} R\mathcal{H}om_\pi(\mathbb{I},\mathbb{I})[2] &\xrightarrow{\mathrm{At}(\mathbb{I})} R\mathcal{H}om_\pi(\mathbb{I}, \mathbb{I}\otimes \mathbb{L}_{X\times \mathscr{P}erf(X)})[3] \to R\mathcal{H}om_\pi(\mathbb{I}, \mathbb{I}\otimes \Omega_X^1)[3] \\ &\xrightarrow{\mathrm{tr}} R\Gamma(X, \Omega_X^1)\otimes \mathcal{O}_{\mathscr{P}erf(X)}[3] \to H^3(X,\Omega_X^1)\otimes \mathcal{O}_{\mathscr{P}erf(X)} \xrightarrow{-\int_X(-)\cup \iota_\xi\omega} \mathcal{O}_{\mathscr{P}erf(X)}, \end{aligned}$$

where $\mathbb{I}$ is the universal complex.

Proof Note that we have

$$H^{2n-4}(X, \Omega_X^{\geq n-1}) = \bigoplus_{p\geq n-1} H^{2n-4-p}(X, \Omega_X^p) \subseteq H^{2n-4}_{\mathrm{DR}}(X)$$

by the Hodge decomposition. Hence, $\iota_\xi\omega \in H^{n-3}(X, \Omega_X^{n-1})$ induces an element

$$[\iota_\xi\omega] := (\iota_\xi\omega, 0, \dots, 0) \in H^{2n-4}(X, \Omega_X^{\geq n-1}).$$

We define the (-1)-shifted closed 1-form as the following hat product in Definition A.11:

$$\alpha_\xi := \widehat{[\iota_\xi\omega]\cdot\Theta} \in \mathcal{A}^{1,\mathrm{cl}}(R\mathscr{P}erf(X; -1)).$$

It remains to compute the induced cosection of α_ξ in the classical truncation. By Lemma A.14 below, the underlying (-1)-shifted 1-form can be written as

$$(\alpha_\xi)_0 = \widehat{(\iota_\xi\omega)\cdot\Theta_0} = \iota_{\widehat{\xi}}(\widehat{\omega\cdot\Theta_0}) = \iota_{\widehat{\xi}}(\theta_0) \in \mathcal{A}^1(R\mathscr{P}erf(X); -1),$$

where $\widehat{\xi}$ is defined by (116) in Remark A.10. Here $(-)_0$ denotes the underlying form after forgetting the closing structure as in Remark A.2. By [Schürg et al. 2015, Remark A.1], we have

$$\mathrm{ev}_* = \mathrm{At}(\mathbb{I}) : \mathbb{T}_{X\times R\mathscr{P}erf(X)} \to R\mathcal{H}om(\mathbb{I},\mathbb{I})[1]$$

for the evaluation map $\mathrm{ev} : X \times R\mathscr{P}erf(X) \to R\mathscr{P}erf$. Since $\widehat{\xi}$ is defined by the first composition of ev_*, we have the desired formula by taking dual and classical truncation. □

We need the following technical lemma to complete the proof of Proposition A.13. This formula is inspired by [Vizman 2011, Lemma 8].

Lemma A.14 *Let X be a smooth projective variety of dimension n. Let $\mathbb{Y}$ be a homotopically finitely presented derived Artin stack. Assume that $R\underline{\mathsf{Map}}(X,\mathbb{Y})$ is a derived Artin stack. Then we have*

$$\iota_{\widehat{\xi}}(\widehat{\alpha\cdot\beta}) \cong -\widehat{(\iota_\xi\alpha)\cdot\beta} \in \mathcal{A}^{p+q-n-1}(R\underline{\mathsf{Map}}(X,\mathbb{Y}); i+j-n+k)$$

for $\alpha\in\mathcal{A}^p(X;i)$, $\beta\in\mathcal{A}^q(\mathbb{Y};j)$, $\xi\in H^k(X,T_X)$, where $\widehat{\xi}$ is defined by (116) in Remark A.10.

Proof For simplicity of notation, let $\mathbb{M} := R\underline{\mathsf{Map}}(X,\mathbb{Y})$ and

$$\widetilde{\xi} := (q^*\xi, -\pi^*\widehat{\xi}) \in \mathbb{H}^k(X\times\mathbb{M}, \mathbb{T}_{X\times\mathbb{M}}) = H^k(X\times\mathbb{M}, q^*\mathbb{T}_X)\oplus\mathbb{H}^k(X\times\mathbb{M},\pi^*\mathbb{T}_{\mathbb{M}}).$$

By the definition of $\widehat{\xi}$ in (116), we have

$$\mathrm{ev}_*(\widetilde{\xi}) = 0 \in \mathbb{H}^k(X\times\mathbb{M}, \mathrm{ev}^*(\mathbb{T}_{\mathbb{Y}})).$$

Firstly, we have

$$\iota_{\widetilde{\xi}}(q^*(\alpha)\cup\mathrm{ev}^*(\beta)) = (\iota_{q_*\widetilde{\xi}}q^*(\alpha))\cup\mathrm{ev}^*(\beta) \pm q^*(\alpha)\cup(\iota_{\mathrm{ev}_*\widetilde{\xi}}\mathrm{ev}^*(\beta)) = q^*(\iota_\xi(\alpha))\cup\mathrm{ev}^*(\beta)$$

in $\mathbb{H}^{i+j+k}\big(X\times\mathbb{M}, \bigwedge^{p+q-1}\mathbb{L}_{X\times\mathbb{M}}\big)$.

Secondly, we have

$$\int_\pi \iota_{\widetilde{\xi}}(q^*(\alpha)\cup\mathrm{ev}^*(\beta)) = -\iota_{\widehat{\xi}}\int_\pi(q^*(\alpha)\cup\mathrm{ev}^*(\beta))$$

in $\mathbb{H}^{i+j+k-n}\big(\mathbb{M}, \bigwedge^{p+q-n-1}\mathbb{L}_{\mathbb{M}}\big)$ (see Remark A.9).

Combining these two formulas, we have

$$\widehat{(\iota_\xi\alpha)\cdot\beta} = \int_\pi \iota_{\widetilde{\xi}}(q^*(\alpha)\cup\mathrm{ev}^*(\beta)) = -\iota_{\widehat{\xi}}\int_\pi(q^*(\alpha)\cup\mathrm{ev}^*(\beta)) = -\iota_{\widehat{\xi}}(\widehat{\alpha\cdot\beta}).$$ □

Now we have the desired reduced obstruction theory, as follows:

Corollary A.15 *Let X be a Calabi–Yau 4-fold, $v\in H^*(X,\mathbb{Q})$, and $q\in\{-1,0,1\}$. Let $\mathscr{P} := \mathscr{P}_v^{(q)}(X)$ and let $\phi:\mathbb{E}\to\tau^{\geq -1}\mathbb{L}_{\mathscr{P}}$ be the standard symmetric obstruction theory. Choose a nondegenerate subspace $V\subseteq H^1(X,T_X)$ with respect to the symmetric bilinear form B_γ for $\gamma := v_2\in H^2(X,\Omega_X^2)$. Then we*

have a reduced obstruction theory ϕ^{red} that fits into the commutative diagram

$$\begin{array}{ccccc} V\otimes\mathcal{O}_{\mathscr{P}}[1] & \xrightarrow{\mathsf{SR}_V^\vee[1]} & \mathbb{E} & \longrightarrow & \mathbb{E}_V^{\mathrm{red}} \\ & & \downarrow{\scriptstyle\phi} & \swarrow{\scriptstyle\phi^{\mathrm{red}}} & \\ & & \tau^{\geq -1}\mathbb{L}_{\mathscr{P}} & & \end{array}$$

Moreover, this reduced obstruction theory ϕ^{red} comes from a canonical (-2)-shifted symplectic derived enhancement.

Proof Choose a basis $\xi_1,\dots,\xi_{\rho_\gamma}\in V$. For the first statement, by Theorem A.1(2), it suffices to show that the associated cosections $\mathsf{SR}(\xi_i):\mathbb{E}^\vee[1]\xrightarrow{\mathsf{SR}} H^1(X,T_X)^\vee\otimes\mathcal{O}_{\mathscr{P}}\xrightarrow{\xi}\mathcal{O}_{\mathscr{P}}$ can be enhanced to (-1)-shifted closed 1-forms on the standard derived enhancement $R\mathscr{P}$ of $\mathscr{P}$. By Proposition A.13, $\mathsf{SR}(\xi_i)$ are the underlying cosections of the (-1)-shifted closed 1-forms $u^*(\alpha_{\xi_i})$, where $u:R\mathscr{P}\to R\mathscr{P}erf(X)$ is given by the universal complex on $X\times R\mathscr{P}$.

For the second statement, since the classical truncation $\mathscr{P}$ is quasiprojective, by Proposition A.4, the closed 1-forms $u^*(\alpha_{\xi_i})$ can be lifted to unique exact 1-forms $\widetilde{u^*(\alpha_{\xi_i})}\in\mathcal{A}^{1,\mathrm{ex}}(R\mathscr{P};-1)=\mathcal{A}^0(R\mathscr{P};-1)$. By taking the derived zero locus of these (-1)-shifted functions, we have a reduced derived scheme $R\mathscr{P}^{\mathrm{red}}\subseteq R\mathscr{P}$ which induces the reduced obstruction theory as above. Moreover, we can pull back the (-2)-shifted symplectic form $\theta\in\mathcal{A}^{2,\mathrm{cl}}(R\mathscr{P};-2)$ via the inclusion map $R\mathscr{P}^{\mathrm{red}}\hookrightarrow R\mathscr{P}$, which is also nondegenerate. □

Remark A.16 In the situation of Corollary A.15, the existence of the reduced obstruction theory is equivalent to the scheme-theoretic cone reduction (see Remark A.3(1)). Hence we have a commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{Q}(\mathbb{E}_V^{\mathrm{red}}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}_V^{\mathrm{red}}) \\ & \nearrow{\scriptstyle\exists} & \downarrow & & \downarrow \\ \mathfrak{C}_{\mathscr{P}} & \hookrightarrow & \mathfrak{Q}(\mathbb{E}) & \hookrightarrow & \mathfrak{C}(\mathbb{E}) \\ & & \downarrow & & \downarrow{\scriptstyle \mathfrak{q}_{\mathbb{E}}} \\ & & \mathscr{P} & \xrightarrow{0} & \mathbb{A}^1_{\mathscr{P}} \end{array} \qquad \mathfrak{q}_{\mathbb{E}_V^{\mathrm{red}}}$$

In particular, the existence of a dotted arrow implies the isotropic condition.

The main difference between Corollary A.15 and Proposition 4.19 is that the reduced obstruction theory in Corollary A.15 is given as a *factorization*, whereas the one in Proposition 4.19 is given as a *composition*. However, note that, by (75), the two actually coincide.

Remark A.17 The result in this section does not replace Section 4. Corollary A.15 only replaces Kiem and Li's cone reduction lemma in Lemma 4.16. We still need the other arguments in Section 4 to define an Oh–Thomas virtual cycle from Corollary A.15. Indeed, we need to compute the square of SR to show

that SR_V is a *nondegenerate* cosection (see Remark 4.17) and hence the induced symmetric form on $\mathbb{E}_V^{\mathrm{red}}$ is nondegenerate.

Remark A.18 Corollary A.15 provides a simple alternative proof of the Bloch conjecture for surface classes on Calabi–Yau 4-folds, which was shown in general by [Pridham 2024; Bandiera et al. 2023] (and by [Iacono and Manetti 2013] when the normal bundle is extendable).

Let X be a Calabi–Yau 4-fold and $S \subseteq X$ be a local complete intersection surface. Consider an extension $\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$ of local Artinian $\mathbb{C}$-schemes such that $\mathfrak{m}_{\mathcal{A}} \cdot \mathcal{J} = 0$, where $\mathcal{J} := \mathcal{I}_{\mathcal{A}/\overline{\mathcal{A}}}$. Let $\mathcal{S} \subseteq X \times \mathcal{A}$ be an $\mathcal{A}$-flat extension of S. Let

$$\mathsf{ob}(\mathcal{S}) \in H^1(X, N_{S/X}) \otimes \mathcal{J} \cong \mathrm{Ext}^2_X(\mathcal{I}_{S/X}, \mathcal{I}_{S/X})_0 \otimes \mathcal{J}$$

be the obstruction of deforming $\mathcal{S} \subseteq X \times \mathcal{A}$ to $X \times \overline{\mathcal{A}}$ in [Bloch 1972, Proposition 2.6]. By [Bloch 1972], the obstruction $\mathsf{ob}(\mathcal{S})$ lies in the kernel of the semiregularity map

$$\mathsf{sr} : H^1(X, N_{S/X}) \cong \mathrm{Ext}^2_X(\mathcal{I}_{S/X}, \mathcal{I}_{S/X})_0 \to H^3(X, \Omega^1_X)$$

when the canonical map

$$d_{\mathrm{DR}} : \mathcal{J} \to \Omega^1_{\overline{\mathcal{A}}} \otimes \mathcal{O}_{\mathcal{A}} \tag{117}$$

is injective. The Bloch conjecture states that the same result holds without the assumption of injectivity of (117). (See Proposition 4.24 for the comparison of the obstruction/semiregularity maps for closed subschemes [Bloch 1972] and ideal sheaves [Buchweitz and Flenner 2003].)

Let $\mathscr{P} := \mathscr{P}_v^{(-1)}(X)$ be the Hilbert scheme of subschemes of Chern character $v = \mathrm{ch}(\mathcal{O}_S)$. Let $u : \mathcal{A} \to \mathscr{P} := \mathscr{P}_v^{(-1)}(X)$ be the canonical map given by $\mathcal{S}$. By [Behrend and Fantechi 1997, Theorem 4.5], the obstruction $\mathsf{ob}(\mathcal{S})$ is equivalent to the composition

$$u^*\mathbb{E} \xrightarrow{u^*\phi} u^*(\tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)}) \to \tau^{\geq -1}\mathbb{L}_{\mathcal{A}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{A}/\overline{\mathcal{A}}} \cong \mathcal{J}[1],$$

where $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathscr{P}_v^{(q)}(X)}$ is the standard obstruction theory. Instead of choosing a nondegenerate subspace V in Corollary A.15, we now consider all the cosections that come from $H^1(X, T_X)$. By the arguments in Corollary A.15, we can form a reduced obstruction theory ϕ' that fits into the diagram

$$\begin{array}{ccccc} H^1(X, T_X) \otimes \mathcal{O}_{\mathscr{P}}[1] & \xrightarrow{\mathsf{SR}^\vee[1]} & \mathbb{E} & \longrightarrow & \mathbb{E}' \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \phi'} & \\ & & \tau^{\geq -1}\mathbb{L}_{\mathscr{P}} & & \end{array}$$

Hence the obstruction $\mathsf{ob}(\mathcal{S})$ comes from the left term in the exact sequence

$$h^1(\mathbb{E}'|_{[S]}^\vee \otimes \mathcal{J}) \to h^1(\mathbb{E}|_{[S]}^\vee \otimes \mathcal{J}) \cong \mathrm{Ext}^2_X(\mathcal{I}_{S/X}, \mathcal{I}_{S/X})_0 \otimes \mathcal{J} \xrightarrow{\mathsf{sr}} H^1(X, T_X)^\vee \otimes \mathcal{J},$$

which implies $\mathsf{sr}(\mathsf{ob}(\mathcal{S})) = 0$.

Remark A.19 The results in this section can also be applied to the reduced curve-counting theory for Calabi–Yau 3-folds with holomorphic 2-forms in [Kool and Thomas 2014]. Indeed, Proposition A.13 implies that the cosections associated to the semiregularity map and a $(2,0)$-form in [loc. cit.] can be enhanced to a (-1)-shifted closed 1-form. This in particular provides reduced derived enhancements.

However, the reduced curve-counting theory for K3 surfaces in [Maulik et al. 2010] does not require the result in this section (but only Section A.1), and was already studied in [Schürg et al. 2015]. The main reason is that the cosections in [Maulik et al. 2010] do not involve the Atiyah classes and we can see directly that they come from the de Rham differential of (-1)-shifted functions.

Appendix B Moduli stacks of semistable sheaves

In this section, we generalize the results in the previous sections to the moduli *stack* of semistable sheaves. In particular, we construct reduced Oh–Thomas virtual cycles on these moduli stacks and prove that they are deformation invariant along Hodge loci.

We will use the following notation/conventions in this section:

(1) We use the language of ∞-categories [Lurie 2009; 2017] and derived algebraic geometry [Toën and Vezzosi 2008] as in Appendix A.

- An *algebraic stack* will always be a classical 1-Artin stack.

(2) We use the following concepts developed by [Aranha and Pstragowski 2024]. These are natural generalizations of analogous notions for DM stacks in [Behrend and Fantechi 1997; Manolache 2012].

- [Aranha and Pstragowski 2024, Section 3] To any algebraic stack $\mathcal{X}$ and a derived category object $\mathbb{E} \in \mathsf{D}(\mathcal{X})$, we can associate a (*higher*) *abelian cone stack*

$$\mathfrak{C}(\mathbb{E}) := \operatorname{Spec} \operatorname{Sym} \mathbb{E}[-1] : T \mapsto \mathsf{Map}_{\mathsf{D}(T)}(\mathbb{E}[-1]|_T, \mathcal{O}_T) \quad \text{on } \mathsf{Sch}.$$

  When $\tau^{\geq -1}\mathbb{E} \cong [D \to B^\vee \to S^\vee]$ for a coherent sheaf D and vector bundles B, S, we write $\mathfrak{C}(\mathbb{E}) := [C(D)/[B/S]]$.

- [ibid., Theorems 6.2 and 6.3] Let $\mathfrak{C}_{\mathcal{X}/\mathcal{Y}} \subseteq \mathfrak{C}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}})$ denote the *intrinsic normal cone* for a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$.

- An *obstruction theory*[24] is a morphism $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ in $\mathsf{D}(\mathcal{X})$ such that $h^0(\phi)$, $h^1(\phi)$ are bijective and $h^{-1}(\phi)$ is surjective. An obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ induces a closed embedding $\mathfrak{C}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \mathfrak{C}(\mathbb{E})$ by [ibid., Proposition 8.2].

(3) We consider the following generalizations of [Oh and Thomas 2023; Park 2021]:

[24] In [Aranha and Pstragowski 2024, Definition 8.1], the full cotangent complex $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ is used in the definition of an obstruction theory, but here we use the truncated cotangent complex $\tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$.

- We consider the *symmetric complexes* of tor-amplitude $[-3,1]$ as in [Park 2021, Definition C.1]. They are perfect complexes $\mathbb{E}$ on algebraic stacks $\mathcal{X}$ equipped with (-2)-shifted symmetric forms $\theta : \mathcal{O}_{\mathcal{X}} \to \mathrm{Sym}^2(\mathbb{E}[-1])$ such that $\iota(\theta) : \mathbb{E}^\vee[2] \cong \mathbb{E}$.

  Note that there is a *canonical quadratic function* $\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_{\mathcal{X}}$ associated to the symmetric form θ. We denote by $\mathfrak{Q}(\mathbb{E}) \subseteq \mathfrak{C}(\mathbb{E})$ the zero locus of $\mathfrak{q}_{\mathbb{E}}$.
- An *orientation* of a symmetric complex $\mathbb{E}$ of tor-amplitude $[-3,1]$ is an isomorphism of line bundles $o : \mathcal{O}_{\mathcal{X}} \to \det(\mathbb{E})$ such that $\det(\iota(\theta)) = (-1)^{r(r-1)/2} o \circ o^\vee$ where $r = \mathrm{rank}(\mathbb{E})$.
- A *symmetric obstruction theory* for a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a pair of a symmetric complex $\mathbb{E}$ of tor-amplitude $[-3,1]$ on $\mathcal{X}$ and an obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ for f.

B.1 Reduced virtual cycles for Artin stacks

In this subsection, we extend Oh and Thomas's virtual cycles [2023] (more generally, square root virtual pullbacks [Park 2021]) and Kiem and Li's cone reduction lemma [2013] to Artin stacks (which are global quotients). Consequently, we have reduced Oh–Thomas virtual cycles for these Artin stacks.

We first generalize *Oh–Thomas virtual cycles* [2023] (and *square root virtual pullbacks* [Park 2021]) to Artin stacks.

Theorem B.1 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ be a symmetric obstruction theory of tor-amplitude $[-3,1]$ such that $(\mathfrak{C}_{\mathcal{X}/\mathcal{Y}})_{\mathrm{red}} \subseteq \mathfrak{Q}(\mathbb{E})$ as substacks of $\mathfrak{C}(\mathbb{E})$. Let $o : \mathcal{O}_{\mathcal{X}} \to \det(\mathbb{E})$ be an orientation. Assume that $\mathcal{X}$ is a global quotient stack, i.e., there exists a quasiprojective scheme P with a linear action of a linear algebraic group G such that $\mathcal{X} \cong [P/G]$. Then there is a canonical square root virtual pullback*

$$\sqrt{f^!} : A_*(\mathcal{Y}) \to A_*(\mathcal{X}).$$

Moreover, for any cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \hookrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{i} & \mathcal{Y} \end{array}$$

where i is a regular closed embedding, we have

$$i^! \circ \sqrt{f^!} = \sqrt{(f')^!} \circ i^! : A_*(\mathcal{Y}) \to A_*(\mathcal{X}'). \tag{118}$$

Remark B.2 In the situation of Theorem B.1, when $\mathcal{Y} = \mathrm{Spec}(\mathbb{C})$, we define the *Oh–Thomas* virtual cycle as

$$[\mathcal{X}]^{\mathrm{vir}} := \sqrt{f^!}[\mathcal{Y}] \in A_*(\mathcal{X}).$$

Then the formula (118) gives us deformation invariance.

Proof of Theorem B.1 We prove Theorem B.1 in four steps.

Step 1: We first show that there exists a *symmetric resolution*

$$\mathbb{E} \cong [S \xrightarrow{e} B \xrightarrow{d} E \cong E^\vee \xrightarrow{d^\vee} B^\vee \xrightarrow{e^\vee} S^\vee] \tag{119}$$

as in [Oh and Thomas 2023, Proposition 4.1]. More precisely, (119) is an equivalence in $\mathrm{D}(\mathcal{X})$ for some vector bundles B, S and a special orthogonal bundle $(E, \mathfrak{q}_E)$ such that the symmetric form of $\mathbb{E}$ is represented by

$$\begin{array}{ccc}
\mathbb{E}^\vee[2] & & S \xrightarrow{e} B \xrightarrow{d} E \xrightarrow{d^\vee \circ \mathfrak{q}_E} B^\vee \xrightarrow{e^\vee} S^\vee \\
\downarrow{\scriptstyle \iota(\theta)} & & \| \quad\quad \| \quad\quad \downarrow{\scriptstyle \mathfrak{q}_E} \quad\quad \| \quad\quad \| \\
\mathbb{E} & & S \xrightarrow{e} B \xrightarrow{\mathfrak{q}_E \circ d} E^\vee \xrightarrow{d^\vee} B^\vee \xrightarrow{e^\vee} S^\vee
\end{array}$$

Since the global quotient stack $\mathcal{X}$ has the resolution property [Thomason 1987, Lemma 2.6], we can find a resolution $A^\bullet := [A^{-3} \xrightarrow{a^{-2}} A^{-2} \xrightarrow{a^{-1}} A^{-1} \xrightarrow{a^0} A^0 \xrightarrow{a^1} A^1] \cong \mathbb{E}$ such that $\iota(\theta) : \mathbb{E}^\vee[2] \to \mathbb{E}$ is represented by a chain map $\theta^\bullet : (A^\bullet)^\vee[2] \to A^\bullet$ satisfying $(\theta^\bullet)^\vee[2] = \theta^\bullet$. Let $S := A^{-3}$ and

$$B := \operatorname{coker}((A^1)^\vee \xrightarrow{((a^1)^\vee, \theta^{-3})} (A^0)^\vee \oplus A^{-3}).$$

Then $B^\vee = \ker(A^0 \oplus (A^{-3})^\vee \xrightarrow{(a^1, \theta^1)} A^1)$ and we have a quasi-isomorphism

$$c^\bullet : B^\bullet := [A^{-3} \xrightarrow{a^{-2}} A^{-2} \xrightarrow{a^{-1}} A^{-1} \xrightarrow{b^0} B^\vee \xrightarrow{b^1} S^\vee] \xrightarrow{\cong} A^\bullet,$$

where $b^0 := (a^0, 0)$, $b^1 := \mathrm{pr}_2$, $c^{\leq -1} := \mathrm{id}$, $c^0 := \mathrm{pr}_1$, and $c^1 := \theta^1$. Also $\theta^\bullet$ factors through a self-dual map $\tau^\bullet : (B^\bullet)^\vee[2] \to B^\bullet$. Since τ^1 is an isomorphism, we can apply [Oh and Thomas 2023, Proposition 4.1] to the middle three terms of $B^\bullet$. Let

$$E := \operatorname{coker}(B \to (A^{-1})^\vee \oplus A^{-2}) \cong \ker(A^{-1} \oplus (A^{-2})^\vee \to B^\vee) = E^\vee;$$

then we have (119), as claimed.[25]

Step 2: We then construct $\sqrt{f^!}$ as follows: Consider the fibre diagram

$$\begin{array}{ccc}
C(D) & \hookrightarrow & E \\
\downarrow & & \downarrow \\
\mathfrak{C}(\mathbb{E}) & \hookrightarrow & [E/[B/S]]
\end{array}$$

where $D := \operatorname{coker}(d : B \to E)$. The horizontal maps are closed embeddings and the vertical maps are $[B/S]$-torsors. Then we have

$$\mathfrak{q}_{\mathbb{E}}|_{C(D)} = \mathfrak{q}_E|_{C(D)}$$

since the symmetric forms of $\mathbb{E}$ and E map to the same (degenerate) symmetric form on

$$[S \xrightarrow{e} B \xrightarrow{d} E \to 0 \to 0].$$

[25] We thank Arkadij Bojko for pointing out an omission in an earlier version of Step 1.

Let $\tau \in \Gamma(C(D), E|_{C(D)})$ be the tautological section. Then we have a localized Edidin–Graham class

$$\sqrt{e}(E|_{C(D)}, \tau) : A_*(C(D)) \to A_*(\mathcal{X}) \tag{120}$$

since $C(D)$ is a global quotient stack. See [Kiem and Park 2023, Remark 3.3] for the extension of the localized Edidin–Graham class [Oh and Thomas 2023, Definition 3.2] to the equivariant Chow groups [Edidin and Graham 1998; Totaro 1999]. Here the orientation of E is induced by the orientation of $\mathbb{E}$ via the isomorphism $\det(E) \cong \det(\mathbb{E})$ induced by the symmetric resolution (119).

Consider a global factorization

(121)
$$\begin{array}{ccc} & & \mathcal{Z} \\ & \overset{g}{\nearrow} & \big\downarrow h \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

into a representable morphism g and a smooth morphism h for some algebraic stack $\mathcal{Z}$. Such a factorization exists since $\mathcal{X}$ can be embedded into a smooth algebraic stack $\mathcal{W}$ by assumption and we may take $g : \mathcal{X} \to \mathcal{Z} := \mathcal{W} \times \mathcal{Y}$. Let

$$\begin{array}{ccccc} \mathfrak{C} & \xrightarrow{s} & C & \xhookrightarrow{j} & C(D) \\ \big\downarrow r & & \big\downarrow & & \big\downarrow \\ \mathfrak{C}_{\mathcal{X}/\mathcal{Z}} & \longrightarrow & \mathfrak{C}_{\mathcal{X}/\mathcal{Y}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}) \end{array}$$

be a fibre diagram of higher Artin stacks. Then we can form maps

$$A_*(\mathcal{Y}) \xrightarrow{h^*} A_*(\mathcal{Z}) \xrightarrow{\mathrm{sp}_{\mathcal{X}/\mathcal{Z}}} A_*(\mathfrak{C}_{\mathcal{X}/\mathcal{Z}}) \xrightarrow{r^*} A_*(\mathfrak{C}) \xrightarrow{(s^*)^{-1}} A_*(C) \xrightarrow{j_*} A_*(C(D)) \tag{122}$$

between the Chow groups of algebraic stacks (= 1-Artin stacks). Here the specialization map $\mathrm{sp}_{\mathcal{X}/\mathcal{Z}}$ is well defined as in [Manolache 2012] since $g : \mathcal{X} \to \mathcal{Z}$ is representable, and the smooth pullbacks r^*, s^* are isomorphisms by [Kresch 1999b, Proposition 4.3.2] since r is a $[B/S]$-torsor and s is a $\mathrm{Tot}(\mathbb{T}_{\mathcal{Z}/\mathcal{Y}})$-torsor. By composing the two maps (120) and (122), we can define the square root virtual pullback

$$\sqrt{f^!} : A_*(\mathcal{Y}) \to A_*(\mathcal{X}). \tag{123}$$

It is easy to show that $\sqrt{f^!}$ is independent of the choice of a global factorization (121). Indeed, we just have to show that (122) is independent of (121), which can be seen by the arguments in [Kiem and Park 2021, Lemma 4.6].

Step 3: Next we show that the square root virtual pullback $\sqrt{f^!}$ in (123) is independent of the choice of the symmetric resolution (119), as in [Oh and Thomas 2023, Section 4.2]. This part is quite technical, so we will only focus on the main difference with [Oh and Thomas 2023]. By a deformation argument as in [loc. cit.] and adding an acyclic symmetric complex of the form

$$[T_1 \xrightarrow{\mathrm{id}} T_1 \to 0 \to T_1^\vee \xrightarrow{\mathrm{id}} T_1^\vee] \oplus [0 \to T_2 \xrightarrow{(1,0)} T_2 \oplus T_2^\vee \xrightarrow{(0,1)} T_2^\vee \to 0],$$

it suffices to consider the following special case. Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \hookrightarrow & N & \twoheadrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & S & \xrightarrow{e} & B & \xrightarrow{d} & E & & \end{array}$$

where the upper row is exact and the vertical arrows are injective. We will compare the two symmetric resolutions

$$\mathbb{E} \cong [S \xrightarrow{e} B \xrightarrow{d} E \xrightarrow{d^\vee} B^\vee \xrightarrow{e^\vee} S^\vee],$$
$$\mathbb{E} \cong [(S/M) \xrightarrow{e} (B/N) \xrightarrow{d} (K^\perp/K) \xrightarrow{d^\vee} (B/N)^\vee \xrightarrow{e^\vee} (S/M)^\vee].$$

Consider the fibre diagram

$$\begin{array}{ccccc} C(D) & \hookrightarrow & K^\perp & \hookrightarrow & E \\ \downarrow{\scriptstyle t} & & \downarrow & & \downarrow \\ C(G) & \hookrightarrow & K^\perp/K & & \\ \downarrow & & \downarrow & & \\ \mathfrak{C}(\mathbb{E}) & \hookrightarrow & [(K^\perp/K)/[(B/N)/(S/M)]] & \hookrightarrow & [E/[B/S]] \end{array}$$

where $G := \operatorname{coker}((B/N) \to (K^\perp/K))$. Let $\widetilde{\tau} \in \Gamma(C(G), K^\perp/K|_{C(G)})$ be the tautological section. Then the formula

$$\sqrt{e}(E|_{C(D)}, \tau) = \sqrt{e}(K^\perp/K|_{C(G)}, \widetilde{\tau}) \circ t^*$$

in [Kiem and Park 2023, Corollary 4.7] proves the claim.

Step 4: Finally, we prove (118). Consider the factorization $\mathcal{X}' \to \mathcal{Z}' \to \mathcal{Y}'$ induced by the base change of (121) along $i : \mathcal{Y}' \to \mathcal{Y}$. By Vistoli's rational equivalence [1989; Kresch 1999a] (cf. [Qu 2018, Proposition 2.5]), we have a commutative square

$$\begin{array}{ccccc} A_*(\mathcal{Z}) & & \xrightarrow{\mathrm{sp}_{\mathcal{X}/\mathcal{Z}}} & & A_*(\mathfrak{C}_{\mathcal{X}/\mathcal{Z}}) \\ \downarrow{\scriptstyle i^!} & & & & \downarrow{\scriptstyle i^!} \\ A_*(\mathcal{Z}') & \xrightarrow{\mathrm{sp}_{\mathcal{X}'/\mathcal{Z}'}} & A_*(\mathfrak{C}_{\mathcal{X}'/\mathcal{Z}'}) & \longrightarrow & A_*(\mathfrak{C}_{\mathcal{X}/\mathcal{Z}}|_{\mathcal{X}'}) \end{array}$$

Since the localized Edidin–Graham class (120) commutes with the refined Gysin pullback $i^!$ by [Kiem and Park 2023, Lemma 4.4(3)], we obtain the desired formula (118). □

Next, we generalize Kiem and Li's cone reduction lemma [2013] to Artin stacks.

Theorem B.3 *Let $\mathcal{X}$ be an algebraic stack. Let $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{X}}$ be an obstruction theory. Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a map to a vector bundle F. Then*

$$(\mathfrak{C}_{\mathcal{X}})_{\mathrm{red}} \subseteq \mathfrak{C}(\mathbb{E}_\Sigma)$$

as substacks of $\mathfrak{C}(\mathbb{E})$, where $\mathbb{E}_\Sigma := \operatorname{cone}(\Sigma^\vee[1] : F^\vee[1] \to \mathbb{E})$.

Proof Choose a smooth surjection $\widetilde{\mathcal{X}} \to \mathcal{X}$ from an affine scheme $\widetilde{\mathcal{X}}$. We claim that we can form a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathbb{E}|_{\widetilde{\mathcal{X}}} & \longrightarrow & \widetilde{\mathbb{E}} & \longrightarrow & \Omega^1_{\widetilde{\mathcal{X}}/\mathcal{X}} & \dashrightarrow & \mathbb{E}|_{\widetilde{\mathcal{X}}}[1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \widetilde{\phi}} & & \| & & \downarrow{\scriptstyle \phi[1]} \\ \tau^{\geq -1}(\mathbb{L}_{\mathcal{X}}|_{\widetilde{\mathcal{X}}}) & \longrightarrow & \tau^{\geq -1}\mathbb{L}_{\widetilde{\mathcal{X}}} & \longrightarrow & \Omega^1_{\widetilde{\mathcal{X}}/\mathcal{X}} & \longrightarrow & \tau^{\geq -1}(\mathbb{L}_{\mathcal{X}}|_{\widetilde{\mathcal{X}}})[1] \end{array} \tag{124}$$

in the underlying triangulated category $D(\mathcal{X}) := [\mathrm{D}(\mathcal{X})]$. Indeed, the dotted arrow exists since $\widetilde{\mathcal{X}}$ is affine and $\mathrm{cone}(\phi[1]) \in D(\widetilde{\mathcal{X}})$ has amplitude $(-\infty, -3]$. Then $\widetilde{\mathbb{E}}$ and $\widetilde{\phi}$ can be defined by the axioms of triangulated categories.

The long exact sequence associated to (124) ensures that $\widetilde{\phi}$ is an obstruction theory. Moreover, we can define a map $\widetilde{\Sigma}$ as the composition

$$\widetilde{\Sigma} : \widetilde{\mathbb{E}}^\vee[1] \to \mathbb{E}|_{\widetilde{\mathcal{X}}}^\vee[1] \xrightarrow{\Sigma|_{\widetilde{\mathcal{X}}}} F|_{\widetilde{\mathcal{X}}}.$$

Then we can apply Kiem and Li's cone reduction lemma (see Lemma 4.16) for the scheme $\widetilde{\mathcal{X}}$. Hence we have

$$(\mathfrak{C}_{\widetilde{\mathcal{X}}})_{\mathrm{red}} \subseteq \mathfrak{C}(\widetilde{\mathbb{E}}_{\widetilde{\Sigma}})$$

as substacks of $\mathfrak{C}(\widetilde{\mathbb{E}})$, where $\widetilde{\mathbb{E}}_{\widetilde{\Sigma}} := \mathrm{cone}(\widetilde{\Sigma}^\vee[1] : F|_{\widetilde{\mathcal{X}}}^\vee[1] \to \widetilde{\mathbb{E}})$. By (124), we have a fibre diagram

$$\begin{array}{ccccccc} (\mathfrak{C}_{\widetilde{\mathcal{X}}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}_{\widetilde{\mathcal{X}}} & \hookrightarrow & \mathfrak{C}(\widetilde{\mathbb{E}}) & \hookleftarrow & \mathfrak{C}(\widetilde{\mathbb{E}}_{\widetilde{\Sigma}}) \\ \downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow \\ (\mathfrak{C}_{\mathcal{X}}|_{\widetilde{\mathcal{X}}})_{\mathrm{red}} & \hookrightarrow & \mathfrak{C}_{\mathcal{X}}|_{\widetilde{\mathcal{X}}} & \hookrightarrow & \mathfrak{C}(\mathbb{E}|_{\widetilde{\mathcal{X}}}) & \hookleftarrow & \mathfrak{C}((\mathbb{E}_\Sigma)|_{\widetilde{\mathcal{X}}}) \end{array}$$

where the horizontal maps are closed embeddings and the vertical maps are $BT_{\widetilde{\mathcal{X}}/\mathcal{X}}$-torsors by [Aranha and Pstragowski 2024, Lemma 5.12(1) and Theorem 6.3(2)]. Hence we can deduce the desired property via descent. □

Finally, we generalize the construction of reduced Oh–Thomas virtual cycles for nondegenerate cosections in this paper (cf. [Kiem and Park 2023]) to Artin stacks.

Theorem B.4 *In the situation of Theorem B.1, consider a map $\Sigma : \mathbb{E}^\vee[1] \to F$ to a special orthogonal bundle $(F, \mathfrak{q}_F, o_F)$ on $\mathcal{X}$ such that $\Sigma^2 = \mathfrak{q}_F$. We further assume that $\mathcal{Y}$ is a smooth scheme and the composition*

$$F^\vee \xrightarrow{\Sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} (\tau^{\geq -1}\mathbb{L}_{\mathcal{X}/\mathcal{Y}})[-1] \xrightarrow{\mathsf{KS}} \Omega^1_{\mathcal{Y}}|_{\mathcal{X}} \tag{125}$$

is zero. Then there is a canonical reduced square root virtual pullback

$$\sqrt{f^!_{\mathrm{red}}} : A_*(\mathcal{Y}) \to A_*(\mathcal{X})$$

such that $\sqrt{f^!} = \sqrt{e}(F) \circ \sqrt{f^!_{\mathrm{red}}}$. *Moreover, for any cartesian square*

$$\begin{array}{ccc} \mathcal{X}' & \hookrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{i} & \mathcal{Y} \end{array}$$

where i *is a closed embedding of smooth schemes*,

$$i^! \circ \sqrt{f^!_{\mathrm{red}}} = \sqrt{(f')^!_{\mathrm{red}}} \circ i^! : A_*(\mathcal{Y}) \to A_*(\mathcal{X}'). \tag{126}$$

Proof Let $\mathbb{E} \cong \mathbb{E}^{\mathrm{red}} \oplus F[1]$ be a decomposition of symmetric complexes induced by $\Sigma^2 = \mathfrak{q}_F$. By a standard argument as in Theorem 5.1, we can modify the relative obstruction theory ϕ into an absolute obstruction theory. The vanishing of the composition (125) ensures that the relative cosection Σ can be lifted to the absolute obstruction theory. Hence we can apply the cone reduction lemma in Theorem B.3. As in Theorem 5.1, we now have a closed embedding $(\mathfrak{C}_{\mathcal{X}/\mathcal{Y}})_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}^{\mathrm{red}})$. Then we can define the reduced square root virtual pullback $\sqrt{f^!_{\mathrm{red}}}$ as in Theorem B.1 using the data $(\mathfrak{C}_{\mathcal{X}/\mathcal{Y}})_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}^{\mathrm{red}})$. Here we consider the orientation of $\mathbb{E}^{\mathrm{red}}$ induced by the canonical isomorphism $\det(\mathbb{E}^{\mathrm{red}}) \cong \det(\mathbb{E}) \otimes \det(F[1])^{\vee}$. The argument in Theorem B.1 also proves formula (126). □

Remark B.5 We hope to remove the global quotient hypothesis in Theorems B.1 and B.4. At least when $\mathcal{X}$ has reductive stabilizers and quasiaffine diagonal, the authors expect that this will not be very difficult. Indeed, we sketch the following argument. By Alper, Hall and Rydh [Alper et al. 2020], there exists an étale cover $\mathcal{X}' \to \mathcal{X}$ by a global quotient stack $\mathcal{X}'$. By the Chow lemma [Bae and Park ≥ 2026] (cf. [Laumon and Moret-Bailly 2000]), we can form a proper surjection $\mathcal{X}'' \to \mathcal{X}$ from a global quotient stack $\mathcal{X}''$ using $\mathcal{X}'$. Since we have a Kimura sequence [Aranha et al. 2022, Corollary 1.23] for the Chow groups of higher Artin stacks developed by [Khan 2019], we can reduce the situation to the global quotient case.

If this expectation works, then we can apply Theorem B.4 to any open substack of the moduli stack $\mathscr{P}erf(X)$ of a perfect complexes on a Calabi–Yau 4-fold X which has a good moduli space [Alper 2013].

B.2 Moduli stacks of semistable sheaves

In this subsection, we apply the results in the previous subsection to the moduli *stack* of sheaves.

We first consider the *rigidified* version. Let (X, H) be a polarized Calabi–Yau 4-fold. Let $\mathscr{C}oh(X)$ be the moduli stack of coherent sheaves on X. Then there exists a natural $B\mathbb{G}_m$-action on $\mathscr{C}oh(X)$. Let $\mathscr{C}oh(X)/\!/\mathbb{G}_m := [\mathscr{C}oh(X)/B\mathbb{G}_m]$ be the quotient stack.[26] Let

$$\mathcal{M}_v^{H,\mathrm{ss}}(X) \subseteq \mathscr{C}oh(X)/\!/\mathbb{G}_m$$

be the open substack of (Gieseker) H-semistable sheaves with Chern character $v \in H^*(X, \mathbb{Q})$. Here semistability is an open condition by [Huybrechts and Lehn 2010, Proposition 2.3.1].

[26] Since the $B\mathbb{G}_m$-action on $\mathscr{C}oh(X)$ is *free*, i.e., the induced $\mathbb{G}_m$-action on the stabilizers $\mathrm{Aut}(E)$ are free for all $E \in \mathscr{C}oh(X)$, we can alternatively use the *rigidification* of classical 1-Artin stacks; see [Abramovich et al. 2008, Theorem A.1].

Theorem B.6 *Let (X, H) be a polarized Calabi–Yau 4-fold and $v \in H^*(X, \mathbb{Q})$. Then, for any choice of orientation, there exists a canonical* ***reduced*** *virtual cycle*

$$[\mathcal{M}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}} \in A_{1-\frac{1}{2}\chi(v,v)+\frac{1}{2}\rho_\gamma}(\mathcal{M}_v^{H,\mathrm{ss}}(X)),$$

where $\gamma := v_2$ and $\chi(v,v) := \int_X v^\vee \cdot v \cdot \mathrm{td}(X)$. Moreover, if $[\mathcal{M}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}} \neq 0$, then the variational Hodge conjecture holds for (X, γ).

Proof We first fix some notation. Let $\mathcal{M} := \mathcal{M}_v^{H,\mathrm{ss}}(X)$ and $\lambda : \widetilde{\mathcal{M}} \to \mathcal{M}$ be the $\mathbb{G}_m$-gerbe induced by $\mathscr{C}oh(X) \to \mathscr{C}oh(X)/\!/\mathbb{G}_m$. Let $\mathcal{E}$ be the universal sheaf on $X \times \widetilde{\mathcal{M}}$ and $\pi : X \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ be the projection map. Note that we have a weight decomposition

$$D(\widetilde{\mathcal{M}}) \cong \prod_{w \in \mathbb{Z}} D(\mathcal{M})$$

by [Bergh and Schnürer 2021], where $D(\widetilde{\mathcal{M}}) := [\mathsf{D}(\widetilde{\mathcal{M}})]$ and $D(\mathcal{M}) := [\mathsf{D}(\mathcal{M})]$ are the underlying triangulated categories of the stable ∞-categories.[27] Moreover,

$$\lambda^* : D(\mathcal{M}) \to D(\widetilde{\mathcal{M}})$$

is fully faithful and the essential image consists of weight zero objects.

We describe the canonical symmetric obstruction theory on $\widetilde{\mathcal{M}}$ as follows. Let $R\mathscr{C}oh(X)$ be the derived moduli stack of coherent sheaves on X. Then the canonical map $\widetilde{\mathcal{M}} \xrightarrow{\mathcal{E}} R\mathscr{C}oh(X)$ induces an open embedding on the classical truncations. Hence we have an induced symmetric obstruction theory

$$\widetilde{\phi} : \widetilde{\mathbb{E}} := \mathbb{L}_{R\mathscr{C}oh(X)}|_{\widetilde{\mathcal{M}}} = (R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E})[1])^\vee \xrightarrow{\mathrm{At}(\mathcal{E})} \tau^{\geq -1}\mathbb{L}_{\widetilde{\mathcal{M}}},$$

which satisfies the isotropic condition by [Ben-Bassat et al. 2015] and is orientable by [Cao et al. 2020]. Since $\widetilde{\mathbb{E}} \in D(\widetilde{\mathcal{M}})$ is a weight zero object, we have $\lambda^*(\lambda_*\widetilde{\mathbb{E}}) \cong \widetilde{\mathbb{E}}$ for $\lambda_*\widetilde{\mathbb{E}} \in D(\mathcal{M})$.

We note that the canonical obstruction theory on $\mathcal{M}$ given by the standard derived enhancement is *not* symmetric. Indeed, the $B\mathbb{G}_m$-action on $\mathscr{C}oh(X)$ extends to $R\mathscr{C}oh(X)$ and the canonical map $\mathcal{M} \to [R\mathscr{C}oh(X)/B\mathbb{G}_m]$ induces an open embedding on the classical truncations. Hence we have an induced obstruction theory

$$\phi_1 : \mathbb{E}_1 := \mathbb{L}_{[R\mathscr{C}oh(X)/B\mathbb{G}_m]}|_{\mathcal{M}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{M}},$$

where the virtual cotangent complex $\mathbb{E}_1$ fits into the distinguished triangle

$$\begin{array}{ccccc}
\mathbb{L}_{[R\mathscr{C}oh(X)/B\mathbb{G}_m]}|_{\widetilde{\mathcal{M}}} & \longrightarrow & \mathbb{L}_{R\mathscr{C}oh(X)}|_{\widetilde{\mathcal{M}}} & \longrightarrow & \mathbb{L}_{B\mathbb{G}_m}|_{\widetilde{\mathcal{M}}} \\
\| & & \| & & \| \\
\lambda^*\mathbb{E}_1 & \longrightarrow & \widetilde{\mathbb{E}} & \xrightarrow{t} & \mathcal{O}_{\widetilde{\mathcal{M}}}[-1]
\end{array}$$

[27] For a 1-Artin stack, the derived category defined via descent in the context of derived algebraic geometry [Toën and Vezzosi 2008] and the derived category defined as a full subcategory of sheaves of modules on the lisse-étale site in [Laumon and Moret-Bailly 2000; Olsson 2007] coincide by [Hall and Rydh 2017, Proposition 1.3].

Here $t^\vee : \mathcal{O}_{\widetilde{\mathcal{M}}}[1] \to \widetilde{\mathbb{E}}^\vee = R\mathcal{H}om_\pi(\mathcal{E},\mathcal{E})[1]$ is given by the identity $\mathrm{id}_\mathcal{E} : \mathcal{E} \to \mathcal{E}$ since the induced $\mathbb{G}_m$-action on the derived loop space $\mathcal{L}_{R\mathscr{C}oh(X),E} \subseteq \mathrm{Tot}(R\,\mathrm{Hom}_X(E,E))$ at $[E] \in R\mathscr{C}oh(X)$ is the homothety.

We then *symmetrize* the obstruction theory ϕ_1 via the reduction operation in [Park 2021, Lemma C.2]. Indeed, since

$$\mathrm{Hom}_{D(\mathcal{M})}(\mathcal{O}_\mathcal{M}[3], \mathcal{O}_\mathcal{M}[-2]) = 0,$$

we can form a reduction diagram (in the sense of [loc. cit.])

$$\begin{array}{ccccc}
\mathcal{O}_\mathcal{M}[3] & = & \mathcal{O}_\mathcal{M}[3] & & \\
\downarrow & & \downarrow{\scriptstyle \lambda_* t^\vee[2]} & & \\
\mathbb{E}_1 & \xrightarrow{\alpha^\vee[2]} & \lambda_*\widetilde{\mathbb{E}} & \xrightarrow{\lambda_* t} & \mathcal{O}_\mathcal{M}[-1] \\
\downarrow{\scriptstyle \beta^\vee[2]} & & \downarrow{\scriptstyle \alpha} & & \| \\
\mathbb{E} & \xrightarrow{\beta} & \mathbb{E}_1^\vee[2] & \longrightarrow & \mathcal{O}_\mathcal{M}[-1]
\end{array} \tag{127}$$

in the triangulated category $D(\mathcal{M})$ where $\mathbb{E}$, is a *symmetric* complex. Since

$$\mathrm{Hom}_{D(\mathcal{M})}(\mathcal{O}_\mathcal{M}[3], \tau^{\geq -1}\mathbb{L}_\mathcal{M}) = \mathrm{Hom}_{D(\mathcal{M})}(\mathcal{O}_\mathcal{M}[4], \tau^{\geq -1}\mathbb{L}_\mathcal{M}) = 0,$$

there exists a unique symmetric obstruction theory

$$\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_\mathcal{M}$$

which fits into the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_\mathcal{M}[3] & \longrightarrow & \mathbb{E}_1 & \xrightarrow{\beta^\vee[2]} & \mathbb{E} & \longrightarrow & \mathcal{O}_\mathcal{M}[4] \\
 & & & {\scriptstyle \phi_1}\searrow & \downarrow{\scriptstyle \exists \phi} & & \\
 & & & & \tau^{\geq -1}\mathbb{L}_\mathcal{M} & &
\end{array}$$

as the dotted arrow. The isotropic condition for ϕ follows from the one for $\widetilde{\phi}$. Indeed, we can form a commutative diagram of (higher) cone stacks on $\widetilde{\mathcal{M}}$,

$$\begin{array}{ccccc}
\mathfrak{C}_{\widetilde{\mathcal{M}}} & \hookrightarrow & \mathfrak{C}(\lambda^*\mathbb{E}_1^\vee[2]) & \xrightarrow[\cong]{\mathfrak{C}(\lambda^*\alpha)} & \mathfrak{C}(\widetilde{\mathbb{E}}) \\
\downarrow & & \downarrow{\scriptstyle \mathfrak{C}(\lambda^*\beta)} & & \downarrow \\
\mathfrak{C}_\mathcal{M}|_{\widetilde{\mathcal{M}}} & \hookrightarrow & \mathfrak{C}(\lambda^*\mathbb{E}) & \xrightarrow[\cong]{} & \mathfrak{C}(\lambda^*\mathbb{E}_1)
\end{array}$$

where the horizontal maps are closed embeddings and the vertical maps are $\mathrm{Tot}(\mathbb{T}_{\widetilde{\mathcal{M}}/\mathcal{M}}) \cong B\mathbb{G}_a$-torsors. From the reduction diagram (127), we can deduce that the two quadratic functions $\mathfrak{q}_\mathbb{E}$ and $\mathfrak{q}_{\widetilde{\mathbb{E}}}$ restrict to

the same function on $\mathfrak{C}_{\widetilde{\mathcal{M}}}$. Since the composition

$$\mathfrak{C}_{\widetilde{\mathcal{M}}} \twoheadrightarrow \mathfrak{C}_{\mathcal{M}}|_{\widetilde{\mathcal{M}}} \twoheadrightarrow \mathfrak{C}_{\mathcal{M}}$$

is smooth surjective, we obtain the desired isotropic condition via descent.

Moreover, an orientation of $\mathbb{E}$ can be induced by one of $\widetilde{\mathbb{E}}$ since we have

$$\det(\mathbb{E}) \cong \lambda_* \det(\widetilde{\mathbb{E}})$$

by the reduction diagram (127).

We will now form a cosection $\mathsf{SR} : \mathbb{E}^\vee[1] \to H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}$ for the obstruction theory ϕ on $\mathcal{M}$. Firstly, define a cosection for the obstruction theory $\widetilde{\phi}$ on $\widetilde{\mathcal{M}}$ as the composition

$$\begin{aligned}\widetilde{\mathsf{SR}} : \widetilde{\mathbb{E}}^\vee[1] = R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E})[2] &\xrightarrow{\mathrm{At}(\mathcal{E})} R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E} \otimes \mathbb{L}_{X\times\widetilde{\mathcal{M}}})[3] \to R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E} \otimes \Omega_X^1)[3] \\ &\xrightarrow{\mathrm{tr}} R\Gamma(X, \Omega_X^1) \otimes \mathcal{O}_{\widetilde{\mathcal{M}}}[3] \to H^3(X, \Omega_X^1) \otimes \mathcal{O}_{\widetilde{\mathcal{M}}} \xrightarrow{\cong} H^1(X, T_X)^\vee \otimes \mathcal{O}_{\widetilde{\mathcal{M}}}.\end{aligned}$$

Then we have $\widetilde{\mathsf{SR}}^2 = \mathsf{B}_\gamma \otimes 1_{\mathcal{O}_{\widetilde{\mathcal{M}}}}$ by the argument in Proposition 4.12. Then there exists a unique cosection

$$\mathsf{SR} : \mathbb{E}^\vee[1] \to H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}$$

which fits into the commutative diagram

$$\begin{array}{ccccccc} \mathcal{O}_{\mathcal{M}}[3] & \longrightarrow & \mathbb{E}_1 & \xrightarrow{\beta^\vee[2]} & \mathbb{E} & \longrightarrow & \mathcal{O}_{\mathcal{M}}[4] \\ & & \downarrow{\scriptstyle \alpha^\vee[2]} & & \vdots{\scriptstyle \exists \mathsf{SR}} & & \\ & & \lambda_*\widetilde{\mathbb{E}} & \xrightarrow{\lambda_*\widetilde{\mathsf{SR}}} & H^1(X, T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}[1] & & \end{array}$$

since

$$\mathrm{Hom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}[3], H^1(T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}[1]) = \mathrm{Hom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}[4], H^1(T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}[1]) = 0.$$

Moreover, we have $\mathsf{SR}^2 = \lambda_*\widetilde{\mathsf{SR}}^2 = \mathsf{B}_\gamma \otimes 1_{\mathcal{O}_{\mathcal{M}}}$ by the reduction diagram (127).

Therefore, if we choose a maximal nondegenerate subspace $V \subseteq H^1(T_X)$, then we have a nondegenerate cosection $\mathsf{SR}_V : \mathbb{E}^\vee[1] \xrightarrow{\mathsf{SR}} H^1(T_X)^\vee \otimes \mathcal{O}_{\mathcal{M}}[1] \to V^\vee \otimes \mathcal{O}_{\mathcal{M}}[1]$ for the symmetric obstruction theory $\phi : \mathbb{E} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{M}}$. Hence we can apply Theorem B.4 and define a reduced Oh–Thomas virtual cycle

$$[\mathcal{M}]^{\mathrm{red}} \in A_*(\mathcal{M}),$$

where $\mathcal{M}$ is a global quotient stack [Huybrechts and Lehn 2010].

Finally, we show that $[\mathcal{M}]^{\mathrm{red}}$ is deformation invariant and $[\mathcal{M}]^{\mathrm{red}} \neq 0$ implies the variational Hodge conjecture. Let $f : \mathcal{X} \to \mathcal{B}$ be a family of Calabi–Yau 4-folds (as in Theorem 5.1). Let $\widetilde{\gamma} \in F^2 H^4_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$ be a horizontal section and let $P_v(t) \in \mathbb{Q}[t]$ be the Hilbert polynomial determined by v. Then we have a relative moduli stack $\mathcal{M}^{\mathrm{ss}}(\mathcal{X}/\mathcal{B})$ of semistable sheaves E on the fibres $\mathcal{X}_b$ of $f : \mathcal{X} \to \mathcal{B}$ over $b \in \mathcal{B}$ with $\mathrm{ch}_2(E) = \widetilde{\gamma}_b$ and $P_E(t) = P_v(t)$ as a global quotient open substack of $\mathscr{C}oh(\mathcal{X}/\mathcal{B})$ [loc. cit.]. Then all the arguments above also work for the relative setting, so we have a relative symmetric obstruction

theory $\phi_{\mathcal{X}/\mathcal{B}} : \mathbb{E}_{\mathcal{X}/\mathcal{B}} \to \tau^{\geq -1}\mathbb{L}_{\mathcal{M}(\mathcal{X}/\mathcal{B})}$ and a cosection $\mathsf{SR}_{\mathcal{X}/\mathcal{B}} : \mathbb{E}_{\mathcal{X}/\mathcal{B}}^{\vee}[1] \to (R^1 f_* T_{\mathcal{X}/\mathcal{B}})|_{\mathcal{M}(\mathcal{X}/\mathcal{B})}^{\vee}$. As in Lemma 4.21, we have an analogous commutative square

$$\begin{array}{ccc} \mathbb{E}_{\mathcal{X}/\mathcal{B}} & \xrightarrow{\mathsf{SR}_{\mathcal{X}/\mathcal{B}}} & (R^1 f_* T_{\mathcal{X}/\mathcal{B}})|_{\mathcal{M}(\mathcal{X}/\mathcal{B})}^{\vee}[1] \\ \downarrow{\scriptstyle \phi_{\mathcal{X}/\mathcal{B}}} & & \downarrow{\scriptstyle \mathsf{KS}_{\mathcal{X}/\mathcal{B}}} \\ \tau^{\geq -1}\mathbb{L}_{\mathcal{M}(\mathcal{X}/\mathcal{B})} & \xrightarrow{\mathsf{KS}_{\mathcal{M}(\mathcal{X}/\mathcal{B})/\mathcal{B}}} & \Omega^1_{\mathcal{B}}|_{\mathcal{M}(\mathcal{X}/\mathcal{B})}[1] \end{array}$$

since $\mathrm{Hom}_{\mathcal{M}(\mathcal{X}/\mathcal{B})}(\mathcal{O}_{\mathcal{M}(\mathcal{X}/\mathcal{B})}[4], \Omega^1_{\mathcal{B}}|_{\mathcal{M}(\mathcal{X}/\mathcal{B})}[1]) = 0$. Then, based on Theorem B.4, all the remaining arguments in Theorem 5.1 work in this setting. Moreover, the analog of Theorem 5.13 also works in this setting, since there is a good moduli space of $\mathcal{M}(\mathcal{X}/\mathcal{B})$ which is proper over $\mathcal{B}$. □

Remark B.7 In the proof of Theorem B.6, if we restrict to the moduli *scheme* $\mathcal{M}_v^{H,\mathrm{st}}(X)$ of stable sheaves, then the reduction in (127) is just the truncation

$$\tau^{[-2,0]}(R\mathcal{H}om_{\pi}(\mathcal{E}, \mathcal{E})[1])^{\vee},$$

which is used in [Oh and Thomas 2023; Kiem and Park 2023].

Remark B.8 Theorem B.6 also holds for the nonrigidified moduli stack $\widetilde{\mathcal{M}}_v^{H,\mathrm{ss}}(X) \subseteq \mathscr{C}oh(X)$. However, we will not use this approach since this will give us a weaker statement for the application to the variational Hodge conjecture since $[\widetilde{\mathcal{M}}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}} = \lambda^*[\mathcal{M}_v^{H,\mathrm{ss}}(X)]^{\mathrm{red}}$.

Remark B.9 Assuming the expectation in Remark B.5, the first part of Theorem B.6 holds for any 1-Artin open substack $\mathcal{M}(X) \subseteq \mathscr{P}erf(X)/\!/\mathbb{G}_m$ with good moduli space. Moreover, if we can form a 1-Artin open substack $\mathcal{M}(\mathcal{X}/\mathcal{B}) \subseteq \mathscr{P}erf(\mathcal{X}/\mathcal{B})/\!/\mathbb{G}_m$ with proper good moduli space for any family of Calabi–Yau 4-folds $\mathcal{X} \to \mathcal{B}$, then the second part of Theorem B.6 also holds.

Next, we consider the *fixed determinant* version. Let (X, H) be a polarized Calabi–Yau 4-fold. Note that there is a canonical determinant map $\det : \mathscr{C}oh(X) \to \mathscr{P}ic(X)$ to the moduli stack $\mathscr{P}ic(X)$ of line bundles on X. Let $\mathscr{C}oh(X)_{\mathcal{O}_X} := \mathscr{C}oh(X) \times_{\det, \mathscr{P}ic(X), \mathcal{O}_X} \mathrm{Spec}(\mathbb{C})$ be the fibre over the trivial line bundle. Let

$$\mathcal{M}_v^{H,\mathrm{ss}}(X)_{\mathcal{O}_X} \subseteq \mathscr{C}oh(X)_{\mathcal{O}_X}$$

be the open substack of H-semistable sheaves with Chern character $v \in H^*(X, \mathbb{Q})$.

Theorem B.10 *Let (X, H) be a polarized Calabi–Yau 4-fold and $v \in H^*(X, \mathbb{Q})$. Assume that $v_0 \neq 0$ and $v_1 = 0$. Then, for any choice of orientation, there exists a canonical* ***reduced*** *virtual cycle*

$$[\mathcal{M}_v^{H,\mathrm{ss}}(X)_{\mathcal{O}_X}]^{\mathrm{red}} \in A_{\frac{1}{2}\chi(\mathcal{O}_X) - \frac{1}{2}\chi(v,v) + \frac{1}{2}\rho_{\gamma}}(\mathcal{M}_v^{H,\mathrm{ss}}(X)_{\mathcal{O}_X}),$$

where $\gamma := v_2$. Moreover, if $[\mathcal{M}_v^{H,\mathrm{ss}}(X)_{\mathcal{O}_X}]^{\mathrm{red}} \neq 0$, then the variational Hodge conjecture holds for (X, γ).

Proof This follows from the arguments in Theorems 4.5, 5.1, and 5.13, based on Theorem B.4. This is possible by the existence of the relative moduli stack of semistable sheaves as a global quotient stack with proper good moduli space over the base. We omit the details. □

Remark B.11 Assuming the expectation in Remark B.5, we can generalize Theorem B.10 to any 1-Artin open substack of $\mathscr{P}erf(X)_{\mathcal{O}_X}$ with (proper) good moduli space, as in Remark B.9.

Appendix C Orientations for relative moduli spaces

In this subsection, we prove that family orientations exist on the topological versions of the moduli spaces. We regard this result as heuristic evidence for Conjecture 5.7.

C.1 Orientations for fibrations of topological spaces

We consider the following purely topological statement:

Proposition C.1 *Let $p:\mathcal{C}\to\mathcal{B}$ be a (locally trivial) fibration of topological spaces. Assume that the base $\mathcal{B}$ is path connected and locally path connected, and the fibre $\mathcal{C}_b:=\mathcal{C}\times_{\mathcal{B}}\{b\}$ over a point $b\in\mathcal{B}$ has finitely many connected components. Let $O\to\mathcal{C}$ be a principal $\mathbb{Z}_2$-bundle. If the fibres $O|_{\mathcal{C}_b}$ are trivial for all $b\in\mathcal{B}$, then there exists a finite-sheeted covering $u:\mathcal{B}'\to\mathcal{B}$ such that the pullback $O|_{\mathcal{C}\times_{\mathcal{B}}\mathcal{B}'}$ is trivial.*

Moreover, we can further assume that $\deg(u)=|\pi_0(\mathcal{C}_b)|!$.

Proof Fix a point $c\in\mathcal{C}$ and let $b:=p(c)\in\mathcal{B}$. Consider the fibre diagram

$$\begin{array}{ccccc} O_c & \longrightarrow & O_b & \longrightarrow & O \\ \downarrow & & \downarrow & & \downarrow \\ \{c\} & \longrightarrow & \mathcal{C}_b & \longrightarrow & \mathcal{C} \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ & & \{b\} & \longrightarrow & \mathcal{B} \end{array}$$

where $O_b:=O|_{\mathcal{C}_b}$. Since the two maps $O\to\mathcal{C}$ and $O\to\mathcal{B}$ are fibrations, there are monodromy actions

$$m:\pi_1(\mathcal{C},c)\curvearrowright\pi_0(O_c),\quad n:\pi_1(\mathcal{B},b)\curvearrowright\pi_0(O_b). \tag{128}$$

Moreover, the canonical map

$$\pi_0(O_c)\to\pi_0(O_b)$$

is compatible with the above two monodromy actions in (128). Since we have $O_b=\mathcal{C}_b\sqcup\mathcal{C}_b$ by assumption, there is a canonical bijection

$$\pi_0(O_b)\cong\pi_0(\mathcal{C}_b)\times\pi_0(O_c). \tag{129}$$

We can form a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(\mathcal{C},c) & \xrightarrow{p_*} & \pi_1(\mathcal{B},b) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ \mathrm{Aut}(\pi_0(O_c)) & \longrightarrow & \mathrm{Aut}(\pi_0(O_b)) \end{array} \tag{130}$$

where the vertical maps are given by the monodromy actions in (128), the upper horizontal map is the canonical pushforward for p, and the lower horizontal map is given by the decomposition (129).

To show that the $\mathbb{Z}_2$-bundle $O \to \mathcal{C}$ is trivial, it suffices to show that the left vertical map $m : \pi_1(\mathcal{C}, c) \to \mathrm{Aut}(\pi_0(O_c))$ in (130) is trivial. Since the lower horizontal map in (130), given by (129), is injective, it suffices to show that the right vertical map $n : \pi_1(\mathcal{B}, b) \to \mathrm{Aut}(\pi_0(O_b))$ in (130) is trivial.

Note that $\pi_0(O_b)$ is finite since $\pi_0(\mathcal{C}_b)$ is finite by assumption and we have (129). Hence the kernel of the map $n : \pi_1(\mathcal{B}, b) \to \mathrm{Aut}(\pi_0(O_b))$ is a normal subgroup of $\pi_1(\mathcal{B}, b)$ of finite index. Hence there exists a covering map

$$u : \mathcal{B}' \to \mathcal{B}$$

such that the composition

$$\pi_1(\mathcal{B}', b') \xrightarrow{u_*} \pi_1(\mathcal{B}, b) \xrightarrow{n} \mathrm{Aut}(\pi_0(O_b))$$

is trivial and the degree of u divides the order of the finite group $\mathrm{Aut}(\pi_0(O_b))$. Then the pullback $O|_{\mathcal{C}\times_{\mathcal{B}}\mathcal{B}'}$ is trivial since the commutative square (130) is compatible with base change.

Finally, replacing $\mathcal{B}'$ by its covering, we can further assume that the degree of $u : \mathcal{B}' \to \mathcal{B}$ is exactly the order of the group $\mathrm{Aut}(\pi_0(O_b))$. □

C.2 Orientations for relative topological moduli spaces

Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of smooth quasiprojective schemes. Consider the mapping space

$$\mathcal{C}(\mathcal{X}/\mathcal{B}) := \mathrm{Map}_{\mathrm{Top}_{/\mathcal{B}^{\mathrm{top}}}}(\mathcal{X}^{\mathrm{top}}, \mathcal{B}^{\mathrm{top}} \times BU \times \mathbb{Z}),$$

where $(-)^{\mathrm{top}}$ denotes the underlying topological space (with the analytic topology). Here we used the modified compact–open topology on the mapping space $\mathrm{Map}_{\mathrm{Top}/\mathcal{B}^{\mathrm{top}}}(-,-)$ which satisfies adjunction [Booth and Brown 1978]. Then we have a homotopy fibre diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{X}_b) & \longrightarrow & \mathcal{C}(\mathcal{X}/\mathcal{B}) \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & \mathcal{B} \end{array}$$

for each $b \in \mathcal{B}$, where $\mathcal{C}(\mathcal{X}_b) := \mathcal{C}(\mathcal{X}_b/\{b\})$. Note that

$$\pi_0(\mathcal{C}(\mathcal{X}_b)) = K^0_{\mathrm{top}}(\mathcal{X}_b)$$

is the topological K-theory of $\mathcal{X}_b$.

Let $\tilde{v}$ be a horizontal section of $\bigoplus_p \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Then $\tilde{v}^{\mathrm{top}}$ can be regarded as a section of the $\mathbb{C}$-local system $\bigoplus_p R^{2p} f_*^{\mathrm{top}}\mathbb{C}$ (Remark 3.31(1)). Hence we can consider an open subspace

$$\mathcal{C}(\mathcal{X}/\mathcal{B}, \tilde{v}) \subseteq \mathcal{C}(\mathcal{X}/\mathcal{B})$$

which fits into the fibre diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{X}_b, \tilde{v}_b) & \longrightarrow & \mathcal{C}(\mathcal{X}/\mathcal{B}, \tilde{v}) \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & \mathcal{B} \end{array}$$

for each $b \in \mathcal{B}$, where $\mathcal{C}(\mathcal{X}_b, \tilde{v}_b) = \mathcal{C}(\mathcal{X}_b/\{b\}, \tilde{v}_b) \subseteq \mathcal{C}(\mathcal{X}_b)$ is the union of connected components $\alpha \subseteq \mathcal{C}(\mathcal{X}_b)$ whose corresponding K-theory classes $[\alpha]$ map to $\tilde{v}_b \in H^*(\mathcal{X}_b, \mathbb{C})$ under the topological Chern character map

$$\mathrm{ch}^{\mathrm{top}} : K^0_{\mathrm{top}}(\mathcal{X}_b) \to H^*(\mathcal{X}_b, \mathbb{Q}).$$

Corollary C.2 *Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of smooth quasiprojective schemes. Assume that $\mathcal{B}$ is connected. Let $\tilde{v}$ be a horizontal section of $\bigoplus_p \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Let $O \to \mathcal{C}(\mathcal{X}/\mathcal{B}, \tilde{v})$ be a principal $\mathbb{Z}_2$-bundle. If the fibres $O|_{\mathcal{C}(\mathcal{X}_b, \tilde{v}_b)}$ are trivial for all $b \in \mathcal{B}$, then there exists a finite étale cover $u : \mathcal{B}' \to \mathcal{B}$ such that the pullback $O|_{\mathcal{C}(\mathcal{X}'/\mathcal{B}', \tilde{v}')}$ is trivial, where $\mathcal{X}' := \mathcal{X} \times_{\mathcal{B}} \mathcal{B}'$ and $\tilde{v}' := u^*(\tilde{v})$.*

Moreover, we can assume that $\deg(u) = |K^0_{\mathrm{top}}(\mathcal{X}_b)_{\mathrm{tor}}|$!, where $(-)_{\mathrm{tor}}$ denotes the torsion subgroup.

Proof By Ehresmann's theorem $\mathcal{C}(\mathcal{X}/\mathcal{B}, \tilde{v}) \to \mathcal{B}$ is a locally trivial fibration. To apply Proposition C.1, it suffices to show that

$$\pi_0(\mathcal{C}(\mathcal{X}_b, \tilde{v}_b)) = (\mathrm{ch}^{\mathrm{top}})^{-1}(\tilde{v}_b)$$

is finite. This set is bijective to $\ker(\mathrm{ch}^{\mathrm{top}}) = K^0_{\mathrm{top}}(\mathcal{X}_b)_{\mathrm{tor}}$. Since $K^0_{\mathrm{top}}(\mathcal{X}_b)$ is a finitely generated abelian group (by the Atiyah–Hirzebruch spectral sequence), $\pi_0(\mathcal{C}(\mathcal{X}_b, \tilde{v}_b)) \cong K^0_{\mathrm{top}}(\mathcal{X}_b)_{\mathrm{tor}}$ is also finite.

Finally, there is a natural equivalence of categories

$$\{\text{finite étale covers of } \mathcal{B}\} \cong \{\text{finitely sheeted topological coverings of } \mathcal{B}^{\mathrm{top}}\}$$

by [SGA 1 1971, corollaire 5.2]. Hence the finite covering in Proposition C.1 comes from a finite étale map. □

Remark C.3 Let $f : \mathcal{X} \to \mathcal{B}$ be a smooth projective morphism of smooth quasiprojective schemes of relative dimension 4 with connected fibres. Assume that the relative canonical bundle $\omega_{\mathcal{X}/\mathcal{B}}$ is trivial and $\mathcal{B}$ is connected. Let $\tilde{v}$ be a horizontal section of $\bigoplus_p \mathcal{H}^{2p}_{\mathrm{DR}}(\mathcal{X}/\mathcal{B})$. Let $\mathscr{P}erf(\mathcal{X}/\mathcal{B}, \tilde{v})$ be the moduli stack of all perfect complexes on the fibres of f (which exists as a higher Artin stack) whose Chern characters are pullbacks of $\tilde{v}$. Then there exists a canonical $\mathbb{Z}_2$-bundle $O \to \mathscr{P}erf(\mathcal{X}/\mathcal{B}, \tilde{v})$ given by the determinant line bundle $\det(\mathbb{E})$ of the restriction $\mathbb{E}$ of the cotangent complex of the derived enhancement of $\mathscr{P}erf(\mathcal{X}/\mathcal{B}, \tilde{v})$. We want to know whether the $\mathbb{Z}_2$-bundle O is trivial.

If the canonical map

$$p^{\mathrm{top}} : \mathscr{P}erf(\mathcal{X}/\mathcal{B}, \tilde{v})^{\mathrm{top}} \to \mathcal{B}^{\mathrm{top}}$$

(given by the topological realization functor [Simpson 1996; Blanc 2016]) were a locally trivial fibration, then this would follow from Proposition C.1. However, p^{top} is *not* locally trivial in general.

What we can use instead is the topological mapping space. There is a canonical map

$$\mathscr{P}erf(\mathcal{X}/\mathcal{B},\tilde{v})^{\mathrm{top}} \to \mathcal{C}(\mathcal{X}/\mathcal{B},\tilde{v})$$

and also the map

$$c : \mathcal{C}(\mathcal{X}/\mathcal{B},\tilde{v}) \to \mathcal{B}^{\mathrm{top}},$$

which is a locally trivial fibration.

In view of Corollary C.2, the remaining ingredients required to prove Conjecture 5.7 are to

(1) construct a natural $\mathbb{Z}_2$-bundle on $\mathcal{C}(\mathcal{X}/\mathcal{B})$ whose fibres on $\mathcal{C}(\mathcal{X}_b)$ are the ones defined by Joyce, Tanaka and Upmeier [Joyce et al. 2020], and

(2) compare the $\mathbb{Z}_2$-bundle in (1) with the natural orientation on $\mathscr{P}erf(\mathcal{X}/\mathcal{B})$.

We note that this strategy is exactly the same as the fibrewise result of [Cao et al. 2020], which showed the orientability on $\mathscr{P}erf(\mathcal{X}_b)$ by

(1) constructing an orientation bundle on $\mathcal{C}(\mathcal{X}_b)$,

(2) showing the compatibility of orientations of $\mathscr{P}erf(\mathcal{X}_b)$ and $\mathcal{C}(\mathcal{X}_b)$, and

(3) showing the orientability of $\mathcal{C}(\mathcal{X}_b)$.

In their setting, (3) was the most difficult part using topological facts on 5-manifolds. In our situation, we only need to generalize parts (1) and (2) to the relative setting.

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YOUNGHAN BAE younghan@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI, United States

MARTIJN KOOL m.kool1@uu.nl
Mathematical Institute, Utrecht University, Utrecht, Netherlands

HYEONJUN PARK hyeonjunpark@kias.re.kr
June E Huh Center for Mathematical Challenges, Korea Institute for Advanced Study, Seoul, South Korea

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