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XIAOCHUN RONG

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Gromov–Hausdorff limits of aspherical manifolds

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Let X be a compact Gromov–Hausdorff limit space of a sequence of compact n-manifolds, M_i, of Ricci curvature $\mathrm{Ric}_{M_i} \geq -(n-1)$, volume $\mathrm{vol}(M_i) \to 0$, and where all points in M_i are (δ, ρ)-local rewinding Reifenberg points, or the sectional curvature satisfies $\sec_{M_i} \geq -1$, respectively. We conjecture that if M_i is an aspherical manifold of fundamental group satisfying a certain condition (e.g., a nilpotent group), then X is a differentiable, or topological aspherical manifold, respectively. Our main result asserts that if M_i is diffeomorphic or homeomorphic to a nilmanifold, then X is diffeomorphic or homeomorphic to a nilmanifold, respectively.

0 Introduction

Consider a sequence of compact n-manifolds converging in the Gromov–Hausdorff distance (briefly, GH-distance) $M_i \xrightarrow{\mathrm{GH}} X$ such that

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad 0 < \mathrm{diam}(X) = d < \infty, \quad \mathrm{vol}(M_i) \to 0.$$

We study the following problem:

Problem 0.1 Under what (topological) constraints in M_i is a manifold X (referred to as a Ricci limit space) differentiable or topological?

Note that a Ricci limit space of a noncollapsing sequence, i.e, $\mathrm{vol}(M_i) \geq v > 0$, has been well studied: to have a diffeomorphism stability, a (optimal) local regularity condition is that there is $\rho > 0$ such that (any) $x_i \in M_i$ is a (δ, ρ)-Reifenberg point (briefly, (δ, ρ)-RP), i.e., for any $0 < s < \rho$, the GH-distance between s-balls in M_i centered at x_i and in $\mathbb{R}^n$ satisfies

$$d_{\mathrm{GH}}(B_s(x_i), \underline{B}^n_s(0)) < s \cdot \delta, \quad 0 < \delta \leq \delta(n).$$

According to Cheeger and Colding [1996] X is an n-manifold, and for i large there is a diffeomorphism, $f_i \colon M_i \to X$ such that distance functions $d_i = d_{M_i} \circ f_i^{-1} \colon X \to \mathbb{R}$ bi-Hölder converge. We will also use Perelman's stability result [Perelman 1991; Kapovitch 2007]: if $\sec M_i \geq -1$, then X is a topological n-manifold, and for i large there is a homeomorphism and ϵ_i-GH approximation $f_i \colon M_i \to X$.

Problem 0.1 is partially stimulated by the recent result of Bruè, Naber, and Semola below (see Conjecture 9 in [Zamora 2021]):

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Theorem 0.2 [Bruè et al. 2024] *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of n-manifolds satisfying*

$$\sec_{M_i} \geq -1, \quad 0 < \operatorname{diam}(X) = d < \infty, \quad \operatorname{vol}(M_i) \to 0.$$

If M_i is homeomorphic to an n-torus T^n, then X is homeomorphic to T^m $(m < n)$.

The proof of Theorem 0.2 in [Bruè et al. 2024] relies on a result in [Kloeckner and Sabourau 2020] that M_i has a finite covering space of diameter uniformly bounded above and volume bounded below, whose proof contains a gap recently resolved in [Zamora 2026].

Note that Theorem 0.2 will be false if one replaces "$\sec_{M_i} \geq -1$" with "$\mathrm{Ric}_{M_i} \geq -(n-1)$ and $\operatorname{vol}(M_i) \geq v > 0$" for $n \geq 4$ [Zhou 2023], or with "an Alexandrov metric of curvature ≥ -1" for $n \geq 5$ [Bruè et al. 2024].

In seeking a general answer to Problem 0.1, we recall the following related result by Fukaya:

Theorem 0.3 [Fukaya 1988] *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of aspherical n-manifolds (i.e., the universal covering space $\widetilde{M}_i$ is contractible) such that*

$$|\sec_{M_i}| \leq 1, \quad 0 < \operatorname{diam}(M_i) = d < \infty, \quad \operatorname{vol}(M_i) \to 0.$$

Then $X = X'/\Gamma$ is an aspherical orbifold, i.e., X' is an aspherical manifold on which a finite group of isometries Γ acts.

According to the collapsing theory of Cheeger, Fukaya, and Gromov [Cheeger and Gromov 1986; 1990; Fukaya 1987; 1989; Cheeger et al. 1992], for i large M_i admits a singular nilpotent fibration (see Theorem 1.13), $f_i\colon M_i \to X$, f_i is a continuous ϵ_i-GH approximation ($\epsilon_i \to 0$) such that each f_i-fiber, $f_i^{-1}(x)$ for $x \in X$ is an infranilmanifold of positive dimension, and M_i admits a C^1-close invariant metric of high regularities. Note that in Theorem 0.3, all fibers have the same dimension [Fukaya 1988].

Comparing Theorems 0.2 and 0.3 in dimension $n = 2$, consider an one-parameter family of flat metrics on a torus T^2, $(\epsilon S^1) \times S^1 \xrightarrow{\mathrm{GH}} S^1$ $(\epsilon \to 0)$, and its quotient metrics on a Klein bottle $K^2 = T^2/\mathbb{Z}_2$, $[(\epsilon S^1) \times S^1]/\mathbb{Z}_2 \xrightarrow{\mathrm{GH}} [0,1] \cong S^1/\mathbb{Z}_2$ (not a manifold). This example provides a motivation for the following notion.

Definition 0.4 Let Γ be a finitely generated group. We call Γ type $\mathcal{A}_s$ if, whenever $\Lambda < \Gamma$ is a normal virtually nilpotent subgroup, any finite subgroup of Γ/Λ is virtual normal, i.e, its normalizer has a finite index in Γ/Λ.

It is easy to see that if M is a compact nilmanifold, i.e., M is diffeomorphic to N/Γ, N is a simply connected nilpotent Lie group, and $\Gamma < N$ is a discrete subgroup, then $\pi_1(M)$ is type $\mathcal{A}_s$ and $\pi_1(M \times Y)$ is type $\mathcal{A}_s$ where Y is a compact hyperbolic manifold (because any normal nilpotent subgroup of $\pi_1(M \times Y)$ is a subgroup of $\pi_1(M)$), while $\pi_1(M')$ may not be type $\mathcal{A}_s$ for an infranilmanifold M' which is not a nilmanifold (M' may admit a sequence of metrics g_i of $|\sec_{g_i}| \leq 1$ collapsing to some X, not a topological manifold, e.g., $[(\epsilon S^1) \times S^1]/\mathbb{Z}_2 \xrightarrow{\mathrm{GH}} [0,1]$).

When $|\sec_{M_i}| \leq 1$, we have the following answer to Problem 0.1:

Theorem A (GH-limits of aspherical manifolds) *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of compact n-manifolds such that*

$$|\sec_{M_i}| \le 1, \quad 0 < \operatorname{diam}(X) = d < \infty, \quad \operatorname{vol}(M_i) \to 0.$$

If M_i is an aspherical manifold and $\pi_1(M_i)$ is type $\mathcal{A}_s$, then X is an aspherical manifold (with a C^α-metric tensor).

Let $\pi_i\colon (\widetilde{M}_i, \tilde{p}_i) \to (M_i, p_i)$ denote a Riemannian universal covering. Given a singular nilpotent fibration $f_i\colon M_i \to X$, f_i lifts to a singular nilpotent fibration on $\widetilde{M}_i$, $\tilde{f}_i\colon \widetilde{M}_i \to \widehat{X}$, where an $\tilde{f}_i$-fiber is a component of $\pi_i^{-1}(f_i^{-1}(x))$ for $x \in X$. It turns out that $\widetilde{M}_i$ admits a connected nilpotent Lie group G-action without fixed points whose orbits coincide with $\tilde{f}_i$-fibers (see Remark 1.14; cf. [Rong and Yao 2020]).

Note that M_i admits a C^1-close metric by which G and $\pi_1(M_i)$ generate a closed group of isometries on $\widetilde{M}_i$. The following result implies that X in Theorem A is an aspherical manifold:

Theorem 0.5 *Let $\widetilde{M}$ be a contractible Riemannian n-manifold, and let G be a closed group of isometries on $\widetilde{M}$ such that $\widetilde{M}/G$ is compact and not a point.*

(0.5.1) *If the identity component of G, G_0, is nilpotent, then G_0 contains no compact subgroup.*

(0.5.2) *If in* (0.5.1) *any finite subgroup of G/G_0 is virtually normal, then G/G_0 has no torsion element; thus $\widetilde{M}/G$ is an aspherical manifold.*

With the above preparation, let's now point out a relation and a difference between Theorem 0.2 and Theorem A: given a singular nilpotent fibration on a collapsed aspherical manifold M_i of $|\sec_{M_i}| \le 1$ and $\operatorname{diam}(M_i) \le d$, its lifting to $\widetilde{M}_i$ is an isometric connected nilpotent Lie group G_i-action on $\widetilde{M}_i$ with respect to a nearby metric; then Theorem 0.2 follows from (0.5.1), and Theorem A follows from both (0.5.1) and (0.5.2) (which relies on the fact that $\pi_1(M_i)$ is type $\mathcal{A}_s$).

In view of the above, the existence of a singular nilpotent fibration $f_i\colon M_i \to X$ is crucial in proving Theorem A. Indeed, a singular nilpotent fibration was conjectured on a collapsed manifold of curvature bounded below and a local bounded covering geometry [Rong 2018; Huang et al. 2020b].

For $\rho > 0$, let $\pi_i'\colon (\widetilde{B_\rho(x_i)}, \tilde{x}_i') \to (B_\rho(x_i), x_i)$ denote the Riemannian universal covering of the ρ-ball at $x_i \in M_i$.

Conjecture 0.6 (local bounded covering geometry, singular nilpotent fibration) *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of compact n-manifolds, $0 < \operatorname{diam}(X) = d < \infty$, and $\operatorname{vol}(M_i) \to 0$. For i large there is a singular nilpotent fibration $f_i\colon M_i \to X$ under either of the following local bounded covering geometry conditions*:

(0.6.1) $\operatorname{Ric}_{M_i} \ge -(n-1)$ *and every $x_i \in M_i$ is a (δ, ρ)-local rewinding Reifenberg point (briefly, (δ, ρ)-LRRP), i.e., $\tilde{x}_i'$ is a (δ, ρ)-RP.*

(0.6.2) $\sec_{M_i} \ge -1$ *and every $x_i \in M_i$ is a (ρ, v)-local rewinding volume point (briefly, (ρ, v)-LRVP), i.e.,* $\operatorname{vol}(B_\rho(\tilde{x}_i')) \ge v > 0$.

Note that Conjecture 0.6 will be false, if in (0.6.1) (resp. in (0.6.2)), one weakens "(δ,ρ)-LRRP" to "(ρ,v)-LRVP" (resp. one weakens "$\sec_{M_i}\geq -1$" to "$\mathrm{Ric}_{M_i}\geq -(n-1)$"); see Example 4.2.

Clearly, Theorem 0.5 and Conjecture 0.6 suggest the following conjectured answer to Problem 0.1, i.e., Theorem A should still hold under (0.6.1), or $\sec_{M_i}\geq -1$:

Conjecture 0.7 (GH-limit of aspherical manifolds) *Let $M_i\xrightarrow{\mathrm{GH}} X$ be a sequence of compact n-manifolds, $0<\mathrm{diam}(X)=d<\infty$, $\mathrm{vol}(M_i)\to 0$, and*

$$\begin{cases}\mathrm{Ric}_{M_i}\geq -(n-1),\\ \text{every } x_i\in M_i \text{ is a } (\delta,\rho)\text{-LRRP},\end{cases}\qquad \text{or}\quad \sec_{M_i}\geq -1,\ \text{respectively.}$$

If M_i is an aspherical manifold and $\pi_1(M_i)$ is type $\mathcal{A}_s$, then X is an aspherical manifold, it is differentiable, and its points are (δ,ρ')-RP for $0<\rho'\leq\rho$, or topological, respectively.

Proposition 0.8 *Conjecture 0.6 implies Conjecture 0.7.*

Note that "(ρ,v)-LRVP" in (0.6.2) is not assumed in Conjecture 0.7, which was implied by the following result of Gromov (see Theorem 4.5(D′) and E_4 (Appendix 1) in [Gromov 1983]):

Theorem 0.9 (Gromov) *Let M be a compact aspherical n-manifold of $\mathrm{Ric}_M\geq -(n-1)$ and where $\mathrm{diam}(M)\leq d$. Then for $\tilde{x}\in\widetilde{M}$, $\mathrm{vol}(B_1(\tilde{x}))\geq v(n,d)>0$, a constant depending on n and d.*

Theorem B (main result: Conjecture 0.7 in nilmanifolds) *Let $M_i\xrightarrow{\mathrm{GH}} X$ be a sequence of n-manifolds, $0<\mathrm{diam}(X)=d<\infty$, and $\mathrm{vol}(M_i)\to 0$.*

(B1) *If M_i is a nilmanifold of $\mathrm{Ric}_{M_i}\geq -(n-1)$ and every $x_i\in M_i$ is (ρ,δ)-LRRP, then X is diffeomorphic to a nilmanifold whose points are (δ,ρ')-RP for $0<\rho'\leq\rho$.*

(B2) *If M_i of $\sec_{M_i}\geq -1$ is homeomorphic to a nilmanifold, then X is homeomorphic to a nilmanifold.*

Note that (B1) will be false if one replaces (local bounded covering geometry) "$x_i\in M_i$ is a (δ,ρ)-LRRP" by "x_i is a (ρ,v)-LRVP" [Zhou 2023], (B2) generalizes Theorem 0.2 and also gives an affirmative answer to a problem of A. Petrunin,[1] and (B2) will be false if one replaces "$\sec_{M_i}\geq -1$" by an Alexandrov metric of curvature ≥ -1 [Bruè et al. 2024]. Moreover, our proof of (B2), when M_i is a torus, does not require the result in [Kloeckner and Sabourau 2020] (see [Zamora 2026]) as in the proof of Theorem 0.2.

Because Conjecture 0.6 has been open for X a singular space, let's consider the commutative diagram on an equivariant GH-convergence (passing to a subsequence)

$$\begin{array}{ccc}(\widetilde{M}_i,\tilde{p}_i,\Gamma_i) & \xrightarrow{\mathrm{eqGH}} & (\widetilde{X},\tilde{p},G)\\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle /G}\\ (M_i,p_i) & \xrightarrow{\mathrm{GH}} & (X=\widetilde{X}/G,p)\end{array}\tag{0.10}$$

where $\Gamma_i=\pi_1(M_i,p_i)$, and G is the limit group of isometries on $\widetilde{X}$ (a Lie group [Colding and Naber 2012]). Obviously X is a manifold if $\widetilde{X}$ is an n-manifold and the G-action is free.

[1] In a private communication with Zamora about Theorem B, the author was told that Petrunin asked (B2) as a problem concerning [Bruè et al. 2024].

By Theorem 0.9 and stability results mentioned earlier [Cheeger and Colding 1996; Perelman 1991], one concludes that in Conjecture 0.7, $\widetilde{X}$ (in (0.10)) is an n-manifold, and is differentiable or topological, respectively. In the proof of Theorem B, a main technical result is the following:

Theorem 0.11 (contractibility) *Let the assumptions be as in Theorem B. Then $\widetilde{X}$ in* (0.10) *is a contractible n-manifold, and is differentiable or topological, respectively.*

A consequence of Theorem 0.11 and a version of Theorem 0.5 for GH-limit spaces (see Lemma 3.5) is that X is a topological aspherical manifold. Because G is nilpotent, $\pi_1(X)$ is a nilpotent group; thus X is homotopy equivalent to a nilmanifold. Now (B2) follows from a result of Farrell and Hsiang [1983]: a homotopy nilmanifold is homeomorphic to a nilmanifold (see Theorem 3.6).

In (B1), to show that X is diffeomorphic to a nilmanifold, the following partial verification of (0.6.1) plays an essential role:

Theorem 0.12 (nilpotent fibration [Rong 2026]; see Remark 1.17) *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a collapsing sequence of compact n-manifolds satisfying*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad 0 < \mathrm{diam}(X) = d < \infty, \quad \text{every } x_i \in M_i \text{ is a } (\rho, v)\text{-LRVP}.$$

If all points in X are (δ, ρ)-RP, then for i large there is a smooth fiber bundle $F_i \to M_i \xrightarrow{f_i} X$ with F_i an infranilmanifold and affine structural group, and f_i is an ϵ_i-Gromov–Hausdorff approximation, $\epsilon_i \to 0$.

First, because in (B1) points $x_i \in M_i$ are (δ, ρ)-LRRP and $\mathrm{vol}(B_1(\tilde{x}_i')) \geq v > 0$ (Theorem 0.9), it is easy to check that points in $\widetilde{M}_i$ are (δ, ρ')-RP for $0 < \rho' = \rho'(n, d, \rho, v)$ (see (1.11.2)), and thus points in $\widetilde{X}$ are (δ, ρ')-RP [Cheeger and Colding 1996]. Because the G action on $\widetilde{X}$ is free, points in X are $(\delta, \hat{\rho})$-RP for $0 < \hat{\rho} < \rho'$. Applying Theorem 0.12, for i large one gets a fiber bundle $F_i \to M_i \xrightarrow{f_i} X$ with F_i a nilmanifold (because M_i is nilmanifold) and affine structural group. Consequently, the lifting fiber bundle on $\widetilde{M}_i$ is a principal $\widetilde{F}_i$-bundle $\widetilde{F}_i \to \widetilde{M}_i \xrightarrow{\tilde{f}_i} \widehat{X}$, with $\widetilde{F}_i$ a simply connected nilpotent Lie group and $\widehat{X}$ the universal covering space of X. Because X is aspherical and $\widehat{X}$ is contractible, the principal $\widetilde{F}_i$-bundle is a trivial bundle. Because the inclusion $F_i \hookrightarrow M_i$ induces an injection $\pi_1(F_i) \hookrightarrow \pi_1(M_i)$, by Maltsev rigidity we identify $\widetilde{F}_i$ with a normal subgroup of $\widetilde{M}_i$; thus X is diffeomorphic to the quotient of a simply connected nilpotent Lie group $\widetilde{M}_i/\widetilde{F}_i$ by a cocompact discrete subgroup $\pi_1(M_i)/\pi_1(F_i)$.

Our proof of Theorem 0.11 relies on the property that a simply connected nilpotent Lie group N is exponential, i.e., the exponential map from the Lie algebra Exp: $T_eN \to N$ is a diffeomorphism (see Theorem 3.1).

Nevertheless, the following is possible (see Remark 3.4):

Conjecture 0.13 (contractibility) *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of compact aspherical n-manifolds and*

$$\begin{cases} \mathrm{Ric}_{M_i} \geq -(n-1), \\ \text{every } x_i \in M_i \text{ is a } (\delta, \rho)\text{-LRRP}, \end{cases} \quad \text{or} \quad \mathrm{sec}_{M_i} \geq -1, \text{ respectively.}$$

Then $\widetilde{X}$ in (0.10) *is a contractible n-manifold, it is differentiable, and its points are (δ, ρ')-RP, or topological, respectively.*

Proposition 0.14 *Conjecture 0.13 implies Conjecture 0.7.*

In dimension 3, we have the following result, which implies Conjecture 0.7 in dimension 3:

Theorem C *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of aspherical 3-manifolds such that*

$$\mathrm{Ric}_{M_i} \geq -2, \quad 0 < \mathrm{diam}(X) = d < \infty, \quad \mathrm{vol}(M_i) \to 0.$$

If $\pi_1(M_i)$ is type $\mathcal{A}_s$, then X is an aspherical manifold.

Theorem C generalizes a result in [Bruè et al. 2024]: Theorem 0.2 holds in dimension 3 under $\mathrm{Ric}_{M_i} \geq -2$; Theorem C applies to aspherical 3-manifolds, e.g., $M \in \{S^1 \times \Sigma_g, g \geq 1\}$, where Σ_g denotes a closed surface of genus g in $\mathbb{R}^3$, and $(M, g_\epsilon) = (\epsilon S^1) \times \Sigma_g \xrightarrow{\mathrm{GH}} \Sigma_g$ $(\epsilon \to 0)$ with bounded sectional curvature.

The rest of the paper is organized as follows:

- In Section 1, we will review notions and results that will be used throughout this paper.
- In Section 2, we will prove Theorems 0.5 and A, and Proposition 0.8
- In Section 3, we will prove Theorems 0.11 and B, and Proposition 0.14.
- In Section 4, we will prove Theorem C.

1 Preliminaries

For convenience of readers, we will give a brief review of notions and results that are used in the rest of the paper.

1.1 Equivariant Gromov–Hausdorff convergence

Let X and Y be compact metric spaces. For $\epsilon > 0$, a map $f: X \to Y$ is called an ϵ-Gromov–Hausdorff approximation (briefly, GHA) if f is an ϵ-isometry, i.e., $|d_Y(f(x), f(x')) - d_X(x, x')| < \epsilon$ and $f(X) \subseteq Y$ is ϵ-dense.

We say that a sequence of compact metric spaces X_i Gromov–Hausdorff (briefly, GH) converges to a compact metric space, denoted by $X_i \xrightarrow{\mathrm{GH}} X$, if there is a sequence of ϵ_i-GHA, $f_i: X_i \to X$, $\epsilon_i \to 0$.

A sequence of compact metric spaces X_i contains a GH-convergent subsequence if the following Ascoli-type precompactness conditions are satisfied:

(1.1.1) **Uniform bound** $\mathrm{diam}(X_i) \leq d$.

(1.1.2) **Equicontinuous** For $\epsilon > 0$, the number of points in an ϵ-net of X_i satisfies $|X_i(\epsilon)| \leq c(\epsilon)$, a constant depending on ϵ (an ϵ-net of X is a subset of points that are ϵ-separated and ϵ-dense).

There is a natural metric on the isometry group of X: for $\alpha, \beta \in \mathrm{Isom}(X)$,

$$d(\alpha, \beta) = \max\{d(\alpha(x) \text{ for } \beta(x)), x \in X\},$$

and it is easy to check that $(\mathrm{Isom}(X), d)$ is a compact metric space.

For a sequence of pairs (X_i, Γ_i) and a pair (X, Γ), where Γ_i and Γ are closed subgroups of $\mathrm{Isom}(X_i)$, we say that (X_i, Γ_i) is equivariant GH-convergent to (X, Γ), denoted by $(X_i, \Gamma_i) \xrightarrow{\text{eqGH}} (X, \Gamma)$, if there is a sequence of pairs, $(f_i, \phi_i)\colon (X_i, \Gamma_i) \to (X, \Gamma)$, $f_i\colon X_i \to X$ and $\phi_i\colon \Gamma_i \to \Gamma$ are ϵ_i-GHAs such that

$$d_X\big(f_i(\gamma_i(x_i)), \phi_i(\gamma_i)(f_i(x_i))\big) < \epsilon_i \quad \text{for } x_i \in X_i \text{ and } \gamma_i \in \Gamma_i.$$

Lemma 1.2 *Assume that $X_i \xrightarrow{\text{GH}} X$ is a sequence of compact metric spaces. Let $\Gamma_i < \mathrm{Isom}(X_i)$ be a closed subgroup. Then passing to a subsequence, $(X_i, \Gamma_i) \xrightarrow{\text{eqGH}} (X, \Gamma)$, where Γ is a closed subgroup of $\mathrm{Isom}(X)$. Moreover, for the sequence of compact orbit spaces equipped with quotient metrics, $X_i/\Gamma_i \xrightarrow{\text{GH}} X/\Gamma$.*

Next, we review notions that extend the above GH-convergence and equivariant GH-convergence to a sequence of complete locally compact metric spaces (which may not be compact).

For $\epsilon > 0$, a map between two pointed metric spaces $f\colon (X, p) \to (Y, q)$, $q = f(p)$ is called an ϵ-GHA if $f\colon B_{\epsilon^{-1}}(p) \to B_{\epsilon^{-1}+\epsilon}(q)$ is an ϵ-GHA.

A sequence of pointed complete locally compact metric spaces (X_i, p_i) GH-converges to a pointed metric space (X, p), denoted by $(X_i, p_i) \xrightarrow{\text{GH}} (X, p)$, if there is a sequence of ϵ_i-GHA, $f_i\colon (X_i, p_i) \to (X, p)$, $f_i(p_i) = p$, $\epsilon_i \to 0$. If Γ_i is a closed subgroup of $\mathrm{Isom}(X_i)$, and Γ a closed subgroup of $\mathrm{Isom}(X)$, a pair of maps $(f_i, \phi_i)\colon (X_i, p_i, \Gamma_i) \to (X, p, \Gamma)$ is called an ϵ_i-eqGHA, if $f_i\colon (X_i, p_i) \to (X, p)$ is an ϵ_i-GHA, and $\phi_i\colon \Gamma_i(\epsilon_i) \to \Gamma(\epsilon_i)$ such that

$$d_X\big(f_i\big(\gamma_i(x_i), \phi_i(\gamma_i)(f_i(x_i))\big)\big) < \epsilon_i \quad \text{for } x_i \in B_{\epsilon_i^{-1}}(p_i),\ \gamma_i \in \Gamma_i, \text{ and } \gamma_i(x_i) \in B_{\epsilon_i^{-1}}(p_i),$$

where $\Gamma_i(\epsilon) = \{\alpha_i \in \Gamma_i : d_i(\tilde{p}_i, \alpha_i(p_i)) < \frac{1}{2}\epsilon^{-1}\}$.

A metric space X is called a geodesic length space if, for $p, q \in X$, $d(p, q)$ is realized by the length of a path from p to q.

Lemma 1.3 *Assume that (X_i, p_i) are complete locally compact geodesic length metric spaces.*

(1.3.1) *Assume that $(X_i, p_i) \xrightarrow{\text{GH}} (X, p)$ if and only if for any $r > 0$, for the closed balls $\overline{B}_r$,*

$$(\overline{B}_r(p_i), p_i) \xrightarrow{\text{GH}} (\overline{B}_r(p), p).$$

(1.3.2) *Assume that $(X_i, p_i) \xrightarrow{\text{GH}} (X, p)$, and that we have a closed subgroup $\Gamma_i < \mathrm{Isom}(X_i)$. Then passing to a subsequence, $(X_i, p_i, \Gamma_i) \xrightarrow{\text{eqGH}} (X, p, \Gamma)$, where Γ is a closed subgroup of $\mathrm{Isom}(X)$. Moreover, $(X_i/\Gamma_i, \bar{p}_i) \xrightarrow{\text{GH}} (X/\Gamma, \bar{p})$.*

(1.3.3) *(See the appendix.) Let $\Lambda_i < \Gamma_i$ denote a normal closed subgroup of Γ_i such that $\Lambda_i \to G_1$. Then the following diagram commutes:*

$$\begin{array}{ccc} (X_i, p_i, \Lambda_i < \Gamma_i) & \xrightarrow{\text{eqGH}} & (X, p, G_1 < G) \\ \big\downarrow /\Lambda_i & & \big\downarrow /G_1 \\ (X/\Lambda_i, \bar{p}_i, \Gamma_i/\Lambda_i) & \xrightarrow{\text{eqGH}} & (X/G_1, \bar{p}, G/G_1) \end{array}$$

For a reference for this subsection, see [Gromov 2007; Fukaya and Yamaguchi 1992; Rong 2010].

1.2 Cheeger–Colding–Naber theory on Ricci limit spaces

Let M be a complete n-manifold of $\mathrm{Ric}_M \geq (n-1)H$, where H is a constant. A basic geometric property is the Bishop–Gromov relative volume comparison, where Gromov extended the Bishop volume comparison, aiming to obtain the following precompactness, which has been a fundamental platform in metric Riemannian geometry:

Theorem 1.4 (precompactness, Gromov) *Any sequence of complete n-manifolds M_i of $\mathrm{Ric}_{M_i} \geq -(n-1)$ contains a GH-convergent subsequence $(M_i, p_i) \xrightarrow{\mathrm{GH}} (X, p)$.*

In literature, X in Theorem 1.4 is referred as a Ricci limit space, which is a length metric space. An application of Theorem 1.4 is that for $x \in X$, and any sequence of reals $r_i \to \infty$, there is a subsequence such that $(r_{i_k} X, x) \xrightarrow{\mathrm{GH}} (C_x X, x)$, a tangent cone at x, which may depend on a choice of a subsequence.

Since the 1990s, there has been intensive study on the interplay between geometric and topological structures on M_i and on X, and the Cheeger–Colding–Naber theory on Ricci limit spaces provides basic tools [Cheeger and Colding 1996; 1997; 2000; Colding 1996; 1997; Cheeger and Naber 2015; Colding and Naber 2012; Cheeger et al. 2021], and a fairly good understanding has been obtained for a noncollapsing sequence, i.e., $\mathrm{vol}(B_1(p_i)) \geq v > 0$. For instance, X has an open dense subset that is an n-manifold, whose complement has Hausdorff codimension at least 2, every tangent cone is an n-dimensional metric cone, etc.

For a noncollapsing sequence, a fundamental result is the following stability theorem:

Theorem 1.5 (stability [Cheeger and Colding 1996; Cheeger et al. 2021]) *Let $(M_i, p_i) \xrightarrow{\mathrm{GH}} (X, p)$ be a sequence of complete n-manifolds such that $\mathrm{Ric}_{M_i} \geq -(n-1)$ and every $x_i \in M_i$ is (δ, ρ)-RP for $0 < \delta \leq \delta(n)$.*

(1.5.1) *If $0 < \mathrm{diam}(X) = d < \infty$, then X is an n-manifold, and for i large, there is a diffeomorphism $f_i\colon X \to M_i$ such that the pullback distance functions bi-Hölder converge.*

(1.5.2) *If X is not compact, then fixing $r > 0$ and $\epsilon > 0$, for i large there is a diffeomorphic embedding $f_i\colon B_r(p_i) \to B_{r+\epsilon}(p)$ (open balls) such that the pullback distance functions bi-Hölder converge.*

We also use a stability result of Perelman; for simplicity, we state it for Riemannian manifolds.

Theorem 1.6 (stability [Perelman 1991; Kapovitch 2007]) *Let $(M_i, p_i) \xrightarrow{\mathrm{GH}} (X, p)$ be a sequence of complete n-manifolds such that $\mathrm{sec}_{M_i} \geq -1$ and $\mathrm{vol}(B_1(p_i)) \geq v > 0$.*

(1.6.1) *If $0 < \mathrm{diam}(X) = d < \infty$, then X is a topological n-manifold, and for i large, there is a homeomorphism $f_i\colon M_i \to X$ which is also ϵ_i-GHA, $\epsilon_i \to 0$.*

(1.6.2) *If X is not compact, then fixing $r > 0$ and $\epsilon > 0$, for i large there is a homeomorphism $f_i\colon B_r(p_i) \to B_{r+\epsilon}(p)$ which is also ϵ_i-GHA, $\epsilon_i \to 0$.*

In contrast with our understanding for a noncollapsing sequence, very limited knowledge has been gained for a collapsing sequence (see Theorems 1.7–1.9 below). On the other hand, recent examples have revealed properties of X as "bad" as one could imagine; the Hausdorff dimension $\dim_H(X)$ may not be

an integer [Pan and Wei 2022; Pan 2023], and X may not contain a single topological manifold point [Hupp et al. 2025; Zhou 2024].

Theorem 1.7 [Colding and Naber 2012] *The isometry group of a Ricci limit space X is a Lie group. In particular, any closed group of isometries on X is a Lie group.*

Theorem 1.7 is required in Kapovitch and Wilking's resolution on the following conjecture of Gromov:

Theorem 1.8 (generalized Margulis lemma [Kapovitch and Wilking 2011]) *For $n \geq 2$, there are constants $\epsilon(n), c(n) > 0$ such that for any complete n-manifold M of $\mathrm{Ric}_M \geq -(n-1)$, and $x \in M$, $\Lambda_x = \mathrm{Im}[\pi_1(B_{\epsilon(n)}(x)) \to \pi_1(B_1(x))]$ is $c(n)$-nilpotent, i.e., Λ_x contains a nilpotent subgroup of index $\leq c(n)$.*

Theorem 1.7 is also used in Wang's result on a topological property of a Ricci limit space:

Theorem 1.9 (semilocally simply connectedness [Wang 2024]) *A Ricci limit space X is semilocally simply connected. In particular, the universal covering space of X is simply connected.*

We now summarize a list of properties which will be used in the rest of the paper, and which are either from the above results, or can be found in a given reference.

Proposition 1.10 *Let $M_i \xrightarrow{\mathrm{GH}} X$ (X is compact) be a sequence of compact n-manifolds of $\mathrm{Ric}_{M_i} \geq -(n-1)$, and let $\pi_i\colon (\widetilde{M}_i, \tilde{p}_i) \to (M_i, p_i)$ denote the Riemannian universal covering space. Then passing to a subsequence, the equivariant GH-convergence diagram (see Lemma 1.3):*

$$\begin{array}{ccc} (\widetilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{\mathrm{eqGH}} & (\widetilde{X}, \tilde{p}, G) \\ \big\downarrow \pi_i & & \big\downarrow /G \\ (M_i, p_i) & \xrightarrow{\mathrm{GH}} & (X = \widetilde{X}/G, p) \end{array}$$

commutes, where $\Gamma_i = \pi_1(M_i, p_i)$.

(1.10.1) Theorem 1.7 *The isometry group of $\widetilde{X}$ is a Lie group, and thus G is a Lie group.*

(1.10.2) [Fukaya and Yamaguchi 1992] *There is $\epsilon = \epsilon(X) > 0$ such that the subgroup $\Gamma_i(\epsilon) < \Gamma_i$ generated by elements of maximal displacement in $B_1(\tilde{p}_i)$ is less than ϵ, is normal, $\Gamma_i(\epsilon) \to G_0$ (the identity component of G), and for i large, $\Gamma_i / \Gamma_i(\epsilon) \cong G/G_0$.*

(1.10.3) Theorem 1.8 *$\Gamma_i(\epsilon)$ is $c(n)$-nilpotent, and thus G_0 is a nilpotent Lie group.*

(1.10.4) Theorem 1.9 *The universal covering of X, $\widehat{X}$, is simply connected, and $\widehat{X} = (\widetilde{X}/G_0)/H$, where $H < G/G_0$ is the subgroup generated by isotropy groups of the G/G_0-action on $\widetilde{X}/G_0$* [Sormani and Wei 2001].

Finally, we give a result mentioned in the introduction following Theorem 0.12. Note that Theorem 0.12 was stated in [Rong 2026] for the case that X is a Riemannian manifold, and the construction of a nilpotent fiber bundle map $F_i \to M_i \xrightarrow{f_i} X$ requires the local regularity that every $x_i \in M_i$ is a (ρ, δ)-LRRP, which is derived from the fact that x_i is a (ρ, v)-LRVP and X is a Riemannian manifold; see Lemma 2.1 in [Huang et al. 2020a].

Our next result will show that the above still holds when X is a metric space whose points are (ρ,δ)-RP (see (1.11.1) below).

Proposition 1.11 (improving regularity) *Let $M_i \xrightarrow{\text{GH}} X$ be a sequence of compact n-manifolds satisfying*

$$\begin{array}{ccc}(\widetilde{M}_i,\tilde{p}_i,\Gamma_i) & \xrightarrow{\text{eqGH}} & (\widetilde{X},\tilde{p},G)\\ \big\downarrow{\scriptstyle \pi_i} & & \big\downarrow{\scriptstyle /G}\\ (M_i,p_i) & \xrightarrow{\text{GH}} & (X=\widetilde{X}/G,p)\end{array} \qquad \mathrm{Ric}_{M_i}\ge -(n-1),\quad 0<\mathrm{diam}(X)=d<\infty,\quad \mathrm{vol}(M_i)\to 0.$$

Let $x_i\in M_i, x_i\to x\in X$.

(1.11.1) *Assume that x is a (δ,ρ)-RP. Then that $x_i\in M_i$ is a (ρ,v)-LRVP implies that for i large x_i is a (δ,ρ')-LRRP, where $0<\rho'=\rho'(n,d,v)<\rho$.*

(1.11.2) *Assume $\mathrm{vol}(B_1(\tilde{p}_i))\ge v>0$. Then that $x_i\in B_{1/2}(p_i)$ is a (δ,ρ)-LRRP implies that $\tilde{x}_i\in \pi_i^{-1}(x_i)\subset\widetilde{M}_i$ is a (δ,ρ')-RP, where $0<\rho'=\rho'(n,d,v)<\rho$.*

Proof (1.11.1) The improvement of local regularity in (1.11.1) is based on the following observation: Let $\pi_i'\colon (\widetilde{B_\rho(x_i)},\tilde{x}_i')\to (B_\rho(x_i),x_i)$ denote the Riemannian universal covering space and $\Lambda_i=\pi_1(B_\rho(x_i),x_i)$. Passing to a subsequence, we may assume the following commutative diagram (where points in consideration are away from $(\pi_i')^{-1}(\partial B_\rho(x_i))$ [Xu 2025]):

$$\begin{array}{ccc}(\widetilde{B_\rho(x_i)},\tilde{x}_i',\Lambda_i) & \xrightarrow{\text{eqGH}} & (\widetilde{Y},\tilde{x}',Q)\\ \big\downarrow{\scriptstyle \pi_i'} & & \big\downarrow{\scriptstyle \text{proj}}\\ (B_\rho(x_i),x_i) & \xrightarrow{\text{GH}} & (B_\rho(x)=\widetilde{Y}/Q\subset X,x)\end{array}$$

In (1.1.1), that x_i is a (ρ,v)-LRVP, i.e., $\mathrm{vol}(B_\rho(\tilde{x}_i'))\ge v>0$, implies that $\dim(\widetilde{Y})=n$. Because x is a (ρ,δ)-PR, $B_{\rho/2}(x)$ is diffeomorphic and bi-Hölder to an Euclidean $\frac{1}{2}\rho$-ball [Cheeger and Colding 1996]. Consequently, the region of $\widetilde{Y}$ consisting of points away from the "boundary" is a manifold on which $Q=Q_0$ acts freely (Q is a Lie group by Theorem 1.7).

It remains to show that $\tilde{x}'$ is a (ρ',δ)-PR ($0<\rho'<\rho$), and thus for i large $\tilde{x}_i'$ is a (ρ',δ)-RP [Cheeger and Colding 1996]. By definition, whether $\tilde{x}'$ is a (ρ',δ)-RP is scaling invariant, and thus one may blow up proj: $(\widetilde{Y},\tilde{x}',Q)\to(X=\widetilde{Y}/Q,x)$ at $\tilde{x}'$: assuming $r_j\to\infty$ is such that $(r_j\widetilde{Y},\tilde{x}')\to(C_{\tilde{x}'}\widetilde{Y},\tilde{x}')$, $C_{\tilde{x}'}\widetilde{Y}=\mathbb{R}^k\times\widetilde{Z}$, where $\mathbb{R}^k$ is tangent to $Q_0(\tilde{x}')$ (a homogeneous k-manifold for $k=\dim(Q_0)\ge 1$), because any line in $C_{\tilde{x}'}\widetilde{Y}$ splits off an $\mathbb{R}$-factor [Cheeger and Colding 1996].

We claim that passing to a subsequence, we obtain the commutative diagram

$$\begin{array}{ccc}(r_j\widetilde{Y},\tilde{x}',Q_0) & \xrightarrow{\text{eqGH}} & (C_{\tilde{x}'}\widetilde{Y}\cong\mathbb{R}^k\times\widetilde{Z},\tilde{x}',\mathbb{R}^k)\\ \big\downarrow{\scriptstyle \text{proj}} & & \big\downarrow{\scriptstyle \text{proj}}\\ (r_jX,x) & \xrightarrow{\text{GH}} & (C_xX=\widetilde{Z},x)\end{array}$$

where, restricting to $\widetilde{Z}$, proj is identity map. Because $x \in \widetilde{Z}$ is a (δ, ρ)-RP, for r_j large or ρ' suitably small, $\tilde{x}'$ is a (δ, ρ')-RP.

Let's now verify that for each r_j, proj: $(r_j \widetilde{Y}, \tilde{x}', G) \to (r_j X, x)$ is a submetry. Passing to a subsequence, we obtain a limiting submetry on the right-hand side of the claimed diagram. Passing to a subsequence, we may assume that $(r_{j_k} X, x) \to (C_x X = \widetilde{Z}, x)$; passing to a subsequence of r_{j_k}, still denoted by r_{j_k}, one obtains the top eqGH-convergence and thus completes the commutative diagram.

(1.11.2) Let $\pi_i \colon (\widetilde{M}_i, \tilde{x}_i) \to (M_i, x_i)$ denote the Riemannian universal covering map. Then

$$\widehat{\pi}_i \colon (\widetilde{B_\rho(x_i)}, \tilde{x}_i') \to (\pi_i^{-1}(B_\rho(x_i))_0, \tilde{x}_i)$$

is a covering map, where $\pi_i' = \pi_i \circ \widehat{\pi}_i$, and $\pi_i^{-1}(B_\rho(x_i))_0$ denotes the component of $\pi_i^{-1}(B_\rho(x_i))$ at $\tilde{x}_i$. By the Bishop–Gromov relative volume comparison and that $\mathrm{vol}(B_1(\tilde{p}_i)) \geq v > 0$, we derive

$$|\widehat{\pi}_i^{-1}(\tilde{x}_i) \cap B_{\rho/4}(\tilde{x}_i')| \leq \frac{\mathrm{vol}(B_\rho(\tilde{x}_i'))}{\mathrm{vol}(B_{\rho/4}(\tilde{x}_i))} \leq \frac{\mathrm{vol}(\underline{B}_\rho(\mathbb{H}^n))}{c(n, \rho, v)} \leq c'(n, d, \rho, v),$$

where $\mathbb{H}^n$ denotes the simply connected n-dimensional hyperbolic space. We claim that for any $\gamma_i \in \pi_1(B_\rho(\tilde{x}_i))$, the orbit of the cyclic subgroup generated by γ_i, $\langle \gamma_i \rangle(\tilde{x}_i')$, is not contained in $B_{\rho/4}(\tilde{x}_i')$. Consequently, the minimal displacement of γ_i at $\tilde{x}'$ is $\geq \rho' = \rho'(n, d, \rho, v) > 0$; thus $\tilde{x}_i$ is a (δ, ρ')-RP.

If the claim is false, because $B_\rho(\tilde{x}_i')$ is diffeomorphic and bi-Hölder close to $\underline{B}_\rho^n(0)$, using the Euclidean metric the energy function $e(\cdot) = \sum_{j=1}^{o(\gamma_i)} \frac{1}{2} d_0^2(\gamma_i^j(\tilde{x}_i'), \cdot)$ achieves the minimum at a unique point $\tilde{z}_i \in B_{\rho/2}(\tilde{x}_i')$, where the order $o(\gamma_i)$ is at most $c'(n, d, \rho, v)$. Consequently, $\langle \gamma_i \rangle(\tilde{z}_i)$ concentrates around $\tilde{z}_i$, a contradiction. □

1.3 Collapsed manifolds of local Ricci bounded covering geometry

In this section, we will briefly describe the background of Theorem 0.12, and an outline of the proof.

An n-manifold M with a (normalized) bound on curvature (to prevent a possible scaling) is called ϵ-collapsed if for any $x \in M$, $\mathrm{vol}(B_1(x)) < \epsilon$. Examples of collapsed n-manifolds of various curvature bounds

$$|\mathrm{sec}_M| \leq 1, \quad \mathrm{sec}_M \geq -1, \quad |\mathrm{Ric}_M| \leq n-1, \quad \mathrm{Ric}_M \geq -(n-1),$$

indicate that the complexity in the underlying geometric and topological structure of M is consistent with the regularity strength of a curvature bound; roughly speaking, geometric and topological structures on a collapsed manifold M of $|\mathrm{sec}_M| \leq 1$ is well understood, while collapsed manifolds under other curvature bounds are less (or far) from understood.

According to the collapsing theory of Cheeger, Fukaya, and Gromov [Gromov 1978; Cheeger and Gromov 1986; 1990; Fukaya 1987; 1989, Cheeger et al. 1992], an ϵ-collapsed n-manifold M of $|\mathrm{sec}_M| \leq 1$ admits a mixed nilpotent structure [Cheeger et al. 1992] and a C^1-close invariant metric of high regularities. Conversely, if a manifold M admits a topological mixed nilpotent structure, then M admits a one-parameter family of ϵ-collapsed metrics g_ϵ of $|\mathrm{sec}_{g_\epsilon}| \leq 1$, $\epsilon \to 0$ [Cai and Rong 2009].

In the Cheeger–Fukaya–Gromov theory, a cornerstone is the following:

Theorem 1.12 (almost flat manifolds [Gromov 1978; Ruh 1982]) *For $n \geq 2$, there are constants $\epsilon(n), c(n) > 0$ such that if an n-manifold M of $|\sec_M| \leq 1$ satisfies that $\operatorname{diam}(M) < \epsilon(n)$, then M is diffeomorphic to an infranilmanifold N/Γ, where N is a simply connected nilpotent Lie group, $\Gamma < N \rtimes \operatorname{Aut}(N)$ is a discrete subgroup, and $[\Gamma : \Gamma \cap N] \leq c(n)$.*

The following can be treated as a parametrized Theorem 1.12:

Theorem 1.13 (singular nilpotent fibrations [Fukaya 1987; 1989; Cheeger et al. 1992]) *Assume $M_i \xrightarrow{\mathrm{GH}} X$ is a sequence of compact n-manifolds converging in the Gromov–Hausdorff topology such that*

$$|\sec_{M_i}| \leq 1, \quad 0 < \operatorname{diam}(X) = d, \quad \dim(X) < n.$$

Then for i large, there is an $O(n)$-invariant fiber bundle on the frame bundle $F(M_i)$ equipped with a canonical metric such that the diagram

$$\begin{array}{ccc} \widehat{F}_i \to (F(M_i), O(n)) & \xrightarrow{\hat{f}} & (Y, \mathrm{O(n)}) \\ \big\downarrow{\scriptstyle \phi_i} & & \big\downarrow{\scriptstyle /O(n)} \\ M_i & \xrightarrow{f_i} & X = Y/O(n) \end{array} \tag{1.13.1}$$

commutes, where $\widehat{F}_i$ is a nilmanifold and the structural group is affine, and $\phi_i(\widehat{F}_i)$ is an infranilmanifold of a positive dimension. Here Y is a $C^{1,\alpha}$-manifold and $\hat{f}_i$ satisfies the following regularities:

(1.13.2) *$\hat{f}_i$ is an ϵ_i-Gromov–Hausdorff approximation (briefly, GHA), $\epsilon_i \to 0$.*

(1.13.3) *$\hat{f}_i$ is an ϵ_i-Riemannian submersion, i.e., for a vector ξ orthogonal to $\widehat{F}_i$,*

$$e^{-\epsilon_i} \leq |d\hat{f}_i(\xi_i)|/|\xi_i| \leq e^{\epsilon_i}.$$

(1.13.4) *The second fundamental form of $\widehat{F}_i$ satisfies $|\mathrm{II}(\widehat{F}_i)| \leq c(n)$.*

Here a singular nilpotent fibration refers to a continuous map $f_i : M_i \to X$ satisfying the commutative diagram (1.13.1), and $\tilde{f}_i$ may be a continuous map satisfying (only) (1.13.2).

For applications of Theorem 1.13, see [Rong 2010].

Remark 1.14 Let $f : M \to X$ be a singular nilpotent fibration, and let $\pi : (\widetilde{M}, \tilde{p}) \to (M, p)$ denote the Riemannian universal covering map. The lifting singular fibration $\tilde{f} : \widetilde{M} \to \widehat{X}$ is defined by requiring that an $\tilde{f}$-fiber is a component of $\pi^{-1}(f^{-1}(x))$, $x \in X$, and the following diagram commutes:

$$\begin{array}{ccc} \widetilde{F} \to (F(\widetilde{M}), O(n)) & \xrightarrow{\hat{\tilde{f}}} & (\widehat{Y}, O(n)) \\ \big\downarrow{\scriptstyle /O(n)} & & \big\downarrow{\scriptstyle /O(n)} \\ (\widetilde{M}, \tilde{p}) & \xrightarrow{\tilde{f}} & (\widehat{X} = Y/O(n), \hat{p}) \end{array}$$

From the homotopy exact sequence $\widetilde{F} \to F(\widetilde{M}) \to \widehat{Y}$, we conclude that $\pi_1(\widetilde{F})$ is abelian; thus $\widetilde{F}$ is a nilpotent Lie group. Because $\widehat{Y}$ is simply connected, the $\widetilde{F}$-bundle is a principal $\widetilde{F}$-bundle. Consequently, the lifting singular fibration on $\widetilde{M}$ coincides with an $\widetilde{F}$-action on $\widetilde{M}$ in the sense that an $\widetilde{F}$-orbit coincides with an $\tilde{f}$-fiber [Rong and Yao 2020].

Let's now consider a sequence of collapsed n-manifolds of $\mathrm{Ric}_{M_i} \geq -(n-1)$, and where every $x_i \in M_i$ is a (ρ, v)-LRVP (resp. a (δ, ρ)-LRRP).

A first step in resolving (0.6.1) is to generalize Theorem 1.12 to the following:

Theorem 1.15 (maximal collapsed manifolds with Ricci bounded covering geometry [Huang et al. 2020a; Rong 2026]) *Given $n, v > 0$, there exists a constant $\epsilon(n, v) > 0$ such that if a compact n-manifold satisfies*

$$\mathrm{Ric}_M \geq -(n-1), \quad \text{there is } \tilde{p} \in \widetilde{M} \text{ with } \mathrm{vol}(B_1(\tilde{p})) \geq v, \quad \mathrm{diam}(M) < \epsilon(n, v),$$

then M is diffeomorphic to an infranilmanifold (in Theorem 1.12).

Note that "$\mathrm{diam}(M) < \rho$ implies that $x \in M$ is a (ρ, v)-LRVP" is equivalent to "there is $p \in \widetilde{M}$ such that $\mathrm{vol}(B_1(\tilde{p})) \geq v$".

The proof of Theorem 1.15 in [Huang et al. 2020a] relies on a smoothing method via Ricci flows [Perelman 2002]: one first shows that points on $\widetilde{M}$ are (δ, ρ)-RP (Proposition 1.11), by which one applies Perelman's pseudolocality [2002] to conclude that Ricci flows exists for a definite time, which yields a metric satisfying Theorem 1.12. So Theorem 1.15 follows from Theorem 1.12.

A second step is to prove Conjecture 0.6; Theorem 0.12 verifies (0.5.1) in the case that X is a manifold all whose points are (δ, ρ')-RP (see Remark 1.17). Roughly speaking, the main part of Theorem 0.12 is a new proof of Theorem 1.15 without involving any smoothing technicalities (and thus independent of Theorem 1.12) [Rong 2026].

For convenience of readers, we will briefly outline the proof of Theorem 0.12 (for details, see [Rong 2026]). Recall that Huang [2020] constructed a fiber bundle map, $F_i \to M_i \xrightarrow{f_i} X$, obtained by gluing locally defined δ-splitting maps over a locally finite cover for X, $\{B_{\rho/2}(x_\alpha)\}$ [Cheeger and Colding 1996]. First, one obtains a locally finite open cover for M_i, $\{U_{i,\alpha}\}$, such that the local splitting map over $B_{\rho/2}(x_\alpha)$, $F_i \to U_{i,\alpha} \xrightarrow{f_{i,\alpha}} B_{\rho/2}(x_\alpha)$, is nondegenerate based on the fact that $\tilde{x}_\alpha$ is a (δ, ρ)-RP [Cheeger et al. 2021]. Second, the gluing of local δ-splitting maps $\{f_{i,\alpha}\}$ via an associate partition of unity for $\{B_{\rho/2}(x_\alpha)\}$ yields a smooth map $F_i \to M_i \to X$, which is also nondegenerate. By now the proof of Theorem 0.12 is to show that a fiber F_i is an infranilmanifold, and the structural group can be reduced to an affine group; these two properties play an essential role in the proof of Theorem B. Unfortunately, the techniques in [Huang 2020] are unlikely to prove the two desired properties because of a lack of regularity on f_i (note that one may try a local smoothing technique [Huang and Wang 2022], which likely fails because local smoothed metrics are not well-related from one chart to another).

Overcoming the above problem requires a direct proof of Theorem 1.15 without a smoothing. Let's recall the following criterion for a compact manifold to be diffeomorphic to a nilmanifold:

Theorem 1.16 (a nilmanifold: a sequence of bundles [Rong 2026]) *A compact manifold M_0 is diffeomorphic to a nilmanifold if and only if M_0 carries a sequence of bundles*

$$M_1 \to M_0 \to B_1, \quad M_2 \to M_0 \to B_2, \quad \dots \quad M_{s+1} \to M_0 \to B_{s+1}$$

such that:

(1.16.1) *Each M_i-fiber is a union of M_j-fibers for all $j \geq i$, and $M_{s+1} = \{\mathrm{pt}\}$.*

(1.16.2) *For each $1 \leq i \leq s$, the natural projection $B_{i+1} \to B_i$ is a principal T^{k_i}-bundle $(k_1 + \cdots + k_s = n)$.*

Note that Theorem 1.16 is an alternative formulation of the criterion in [Nakayama 2014] (see [Belegradek 2020]) that M is diffeomorphic to a nilmanifold if and only if M admits iterated principal circle bundles:

$$S^1 \to M \to M_1, \quad S^1 \to M_1 \to M_2, \quad \ldots \quad S^1 \to M_n \to \mathrm{pt}\,.$$

By a standard compactness argument (Theorem 1.4), we may consider a sequence in Theorem 1.15, $M_i \xrightarrow{\mathrm{GH}} \mathrm{pt}$, and show that for i large, $\widetilde{M}_i$ is diffeomorphic to an infranilmanifold. Furthermore, with suitable rescaling and passing to a subsequence, we may assume that $\mathrm{Ric}_{M_i} \geq -\epsilon_i(n-1) \to 0$, and show that $(\widetilde{M}_i, \tilde{p}_i, \Gamma_i) \xrightarrow{\mathrm{eqGH}} (\widetilde{X}, \tilde{p}, G)$, so that $\widetilde{X}$ is isometric to $\mathbb{R}^n$. Consequently, points in $\widetilde{M}_i$ are (δ_i, ρ)-RP as $\delta_i \to 0$.

First, by Theorem 1.8 M_i has a bounded normal covering space $M_{i,0}$, with $\pi_1(M_{i,0})$ a nilpotent group. The rest of proof consists of two steps:

(1) Main work: $M_{i,0}$ satisfies the criterion of Theorem 1.16.

(2) M_i is diffeomorphic to an infranilmanifold.

We point out that in each step, the condition that $\tilde{x}_i \in \widetilde{M}_i$ is a (δ_i, ρ)-RP is essential.

To obtain a sequence of bundle structures on $M_{i,0}$, we blow up $M_{i,0}$ at every point $x_i \in M_{i,0}$ with an almost canonically determined rate, and show that the geometry and topology of a GH-limit is somewhat rigid, i.e., independent of x_i and a choice of a convergent subsequence. For instance, consider a blowup at x_i by $\ell_{i,1}^{-1}$, where $\ell_{i,1}$ is the length of the first short generator for $\pi_1(M_i, x_i)$; then any GH-limit is a product of a flat torus T^{k_1} with $\mathbb{R}^{n-k_1}$, where k is independent of x_i, $\mathrm{diam}(T^{k_1}) \leq d$, and $\mathrm{injrad}(T^{k_1}) \geq \epsilon > 0$ (ϵ may depend on $\{M_i\}$). We then show that for i large, we obtain a locally finite open cover on $M_{i,0}$ by elementary T-structures in the sense of Cheeger and Gromov [1986; 1990]. Employing the gluing method in [Cheeger and Gromov 1990], we construct a T^{k_1}-bundle on $M_{i,0}$, and show that, based on $\pi_1(M_{i,0})$ a torsion free nilpotent group, this is a principal T^{k_1}-bundle.

Next, choosing the blowup rate at x_i by $\rho_{i,2}^{-1}$, $\ell_{i,2}$ is the length of the first short generator that is not contained in T^{k_1} at x_i. Then the limit can be viewed as a blowup at a point in the orbit space of the principle T^{k_1}-bundle B_1. Repeating the above operation, we conclude that B_1 admits a principal T^{k_2}-structure, and thus $M_{i,0} \to B_1 \to B_2$ is a fiber bundle and $B_1 \to B_2$ is a principal T^{k_2}-bundle. Iterating the above operations, after at most n steps, one finishes step 1.

In the proof of Theorem 0.12, by performing the same successive blowups at a point as in the above independent proof of Theorem 1.15, we obtain a locally finite open cover on M with elementary N-structure $f_\alpha : U_\alpha \to B_\alpha$, where f_α is a smooth trivial bundle map and $\Psi(\epsilon|n, v)$-GHA (a function in ϵ such that $\Psi(\epsilon|n) \to 0$ as $\epsilon \to 0$), with fiber an infranilmanifold, such that on overlaps, elementary N-structures satisfy the patching criterion in [Cheeger et al. 1992]. Hence we are able glue elementary N-structures into a nilpotent fiber bundle with an affine structural group.

Remark 1.17 In [Rong 2026], Theorem 0.12 was stated for X a Riemannian manifold and $x_i \in M_i$ a (ρ, v)-LRVP; see (1.11.1). In the proof, that X is a Riemannian manifold is used only in two places: the construction of a local (m, δ)-splitting map on M_i, and the gluing of local (m, δ)-splitting map via center of mass technique using a positive convexity radius of X. It is easy to see that in both places, the condition that X is a differentiable manifold whose points are (δ, ρ)-RP is enough in carrying out the above two operations.

2 Proofs of Theorems 0.5 and A, and Proposition 0.8

2.1 Proof of Theorem 0.5

First, for completeness, we present a proof of the following lemma, which is likely known to experts (cf. [Wright 1992]):

Lemma 2.1 *Let $\widetilde{M}$ be an open contractible manifold. Let $G = T^k$ or $\mathbb{Z}_p^k$. If G effectively acts on $\widetilde{M}$, then the G-fixed-point set is not empty and connected.*

In the proof, we will use the following Smith fixed-point theorem:

Theorem 2.2 *Let M be a compact manifold, let $G = T^k$ or $\mathbb{Z}_p^k$ (for p a prime) act effectively on M, and let $\ell = \mathbb{Q}$ for $G = T^k$ and $\ell = \mathbb{Z}_p$ for $G = \mathbb{Z}_p^k$. If M is a homology n-sphere of coefficient ℓ, then the G-fixed-point set $F(M; G)$ is a homology sphere of coefficient ℓ.*

Proof of Lemma 2.1 Let $\widetilde{M}^*$ denote the one-point compactification of $\widetilde{M}$, and suppose the G-action extends to $\widetilde{M}^*$ and G fixes $\widetilde{M}^* \setminus \widetilde{M}$. Because $\widetilde{M}$ is contractible, $\widetilde{M}^*$ is a homology n-sphere with a coefficient field ℓ in Theorem 2.2. Thus the G-action has a nonempty fixed-point set in $\widetilde{M}$; otherwise the G-fixed-point set in $\widetilde{M}^*$ is a point, not a homology sphere, a contradiction to Theorem 2.2.

If the G-action has at least two fixed-point components F_1 and F_2, then either the G-action on $\widetilde{M}^*$ has two fixed-point components, or the G-fixed-point set in $\widetilde{M}^*$ is not a homology sphere (e.g., the one-point union of at least two homology spheres), again a contradiction to Theorem 2.2. □

Lemma 2.3 *Let the assumptions be as in Theorem 0.5. Then $\widetilde{M}/G_0$ is not compact.*

Proof Let T^k denote the maximal compact subgroup of G_0. If $k = 0$, then G_0 is a simply connected nilpotent Lie group acting freely on $\widetilde{M}$; thus $\widetilde{M}/G_0$ is contractible (not compact). Assume that $k \geq 1$. Because the G_0-action is not free (Lemma 2.1), we consider the free G_0-action on the principal $\mathrm{Spin}(n)$-bundle over $\widetilde{M}$, which is a double covering space of the oriented orthogonal frame bundle over $\widetilde{M}$, with the following commutative diagram:

$$\begin{array}{ccc} G_0 \to (\mathrm{Sp}(\widetilde{M}), \mathrm{Spin}(n)) & \longrightarrow & (\mathrm{Sp}(\widetilde{M})/G_0, \mathrm{Spin}(n)) \\ \downarrow /\mathrm{Spin}(n) & & \downarrow /\mathrm{Spin}(n) \\ \widetilde{M} & \longrightarrow & \widetilde{M}/G_0 \end{array}$$

Clearly $\widetilde{M}/G_0$ is not compact if and only if $\mathrm{Sp}(\widetilde{M})/G_0$ is not compact.

Arguing by contradiction, assume that $\mathrm{Sp}(\widetilde{M})/G_0$ is compact. Because $\widetilde{M}$ is contractible, $\mathrm{Sp}(\widetilde{M})$ is diffeomorphic to $\mathrm{Spin}(n)\times\widetilde{M}$. Thus $\mathrm{Sp}(\widetilde{M})$ homotopically retracts to $\mathrm{Spin}(n)$. We claim that $\mathrm{Sp}(\widetilde{M})$ also homotopically retracts to a compact submanifold $\mathrm{Sp}(\widetilde{M})/(G_0\sim T^k)$, which is a T^k-bundle over $\mathrm{Sp}(\widetilde{M})/G_0$. Consequently, $\dim(\mathrm{Spin}(n))=\dim(\mathrm{Sp}(\widetilde{M})/(G_0\sim T^k))$, i.e., $n+k=\dim(G_0)$, a contradiction.

To verify the claim, observe that by Theorem 3.1, $\mathrm{Exp}\colon T_{\bar e}(G_0/T^k)\to G_0/T^k$ is a diffeomorphism. Thus the principal bundle $G_0/T^k\to \mathrm{Sp}(\widetilde{M})/T^k\to \mathrm{Sp}(\widetilde{M})/G_0$ is equivalent to a vector bundle $T_{\bar e}(G_0/T^k)\to \mathrm{Sp}(\widetilde{M})/T^k\to \mathrm{Sp}(\widetilde{M})/G_0$. Consequently, we may assume a homotopy retraction $H_s\colon[0,1]\times\mathrm{Sp}(\widetilde{M})/T^k\to\mathrm{Sp}(\widetilde{M})/T^k$ such that $H_0=\mathrm{id}_{\mathrm{Sp}(\widetilde{M})/T^k}$ and $H_1\colon\mathrm{Sp}(\widetilde{M})/T^k\to\mathrm{Sp}(\widetilde{M})/G_0\subset \mathrm{Sp}(\widetilde{M})/T^k$. Consider the one-parameter family of pullback T^k-bundles by H_s

$$\begin{array}{ccc} H_s^*(\mathrm{Sp}(\widetilde{M})) & \xrightarrow{H_s^*} & \mathrm{Sp}(\widetilde{M}) \\ \big\downarrow{\scriptstyle /T^k} & & \big\downarrow{\scriptstyle /T^k} \\ [0,1]\times\mathrm{Sp}(\widetilde{M})/T^k & \xrightarrow{H_s} & \mathrm{Sp}(\widetilde{M})/T^k \end{array}$$

Observe that each pullback T^k-bundle homotopy retracts to $T^k\to H_s^*(\mathrm{Sp}(\widetilde{M}))'\to\mathrm{Sp}(\widetilde{M})/T^k$ as $[0,1]$ homotopy retracts to 0, and $H_s^*(\mathrm{Sp}(\widetilde{M}))'$ is one-parameter family of T^k-bundles (isomorphic to $T^k\to\mathrm{Sp}(\widetilde{M})\to\mathrm{Sp}(\widetilde{M})/T^k$). Because H_s homotopy retracts $\mathrm{Sp}(\widetilde{M})/T^k$ to $\mathrm{Sp}(\widetilde{M})/G_0$ (which is the "0" cross section), $H_1^*(\mathrm{Sp}(\widetilde{M}))'$ homotopy retracts to the pullback T^k-bundle $\mathrm{Sp}(\widetilde{M})/G_0\to\mathrm{Sp}(\widetilde{M})/T^k$ denoted by $\mathrm{Sp}(\widetilde{M})/(G_0\sim T^k)$ as in the above. □

Proof of Theorem 0.5 (0.5.1) Because the identity component G_0 is a connected nilpotent Lie group, its maximal compact subgroup is contained in the center, i.e., a torus T^k. We claim that G_0 has no compact subgroup, i.e., $k=0$. Consequently, G_0 acts freely on $\widetilde{M}$.

Arguing by contradiction, assume that G_0 has a maximal compact subgroup, a torus T^k for $k\geq 1$. Because $\widetilde{M}$ is contractible, by Lemma 2.1 the T^k-fixed-point set $F_0=F(T^k,\widetilde{M})\neq\varnothing$ is connected. Because $G_0<G$ is normal and $T^k<G_0$ is the unique maximal compact subgroup, T^k is normal in G. Consequently, G preserves F_0, i.e., $G(F_0)=F_0$. Thus the discrete group G/G_0 preserves the connected and closed subset of codimension ≥ 2, $F_0/G_0\subset\widetilde{M}/G_0$. Because $\widetilde{M}$ is contractible, $\widetilde{M}/G_0$ is not compact (Lemma 2.3). Let $\hat c(t)$ denote a ray at $\bar x_0\in F_0/G_0\subset\widetilde{M}/G_0$ which is also normal to F_0/G_0. Because $\widetilde{M}\to\widetilde{M}/G_0$ is a submetry, there is a horizontal ray $c(t)$ at $\tilde x_0\in F_0$ such that $\hat\pi(c(t))=\hat c(t)$. Observe that for any $\hat\gamma\in G/G_0$, $\hat\gamma(\hat c(t))$ is also a ray normal to F_0/G_0, which implies that $X=\hat X/(G/G_0)$ is not compact, a contradiction.

(0.5.2) Because G_0 has no compact subgroup, G_0 is diffeomorphic to $\mathbb{R}^s$. Because $\widetilde{M}$ is contractible, $\hat X=\widetilde{M}/G_0$ is contractible. We claim that G/G_0 acts freely on $\hat X$; thus $\hat X$ is a universal covering space of $X=\hat X/(G/G_0)$, an aspherical manifold.

Arguing by contradiction, assume a finite isotropy subgroup $\hat H<G/G_0$ that fixes $\hat x$. Without loss of generality, we may assume that $\hat H\cong\mathbb{Z}_p$, where p is a prime. Then $\hat F_0=F(\hat X,\mathbb{Z}_p)\neq\varnothing$ is connected

(Lemma 2.1). Because any subgroup group of G/G_0 is virtually normal, the normalizer of $\widehat{H}$, $N_{G/G_0}(\widehat{H})$, is a subgroup of G/G_0 with a finite index. Thus $N_{G/G_0}(\widehat{H})(\widehat{F}_0) = \widehat{F}_0$. By a similar argument as in the above, we conclude that $\widehat{X}/N_{G/G_0}(\widehat{H})$ is not compact, a contradiction to $X = \widetilde{X}/G$ being compact. □

Remark 2.4 It was pointed out by a referee that Theorem 1.2 in [Farb and Weinberger 2008] asserts that if a compact aspherical manifold M satisfies that $\mathrm{Isom}(\widetilde{M})$ is not discrete, then the identity component of $\mathrm{Isom}(\widetilde{M})$ contains no compact subgroup (see Claim II in [Farb and Weinberger 2008]), from which one also gets Lemma 2.3.

2.2 Proof of Theorem A

By Theorem 1.13, we may assume that, for i large, there is a singular nilpotent fibration $f_i\colon M_i \to X$, and a C^1-close left invariant metric. Then the lifting singular nilpotent fibration on $\widetilde{M}_i$, $\tilde{f}_i\colon \widetilde{M}_i \to \widehat{X}$, coincides with the isometric action of a connected nilpotent Lie group $\widetilde{F}_i$-action, i.e., an $\tilde{f}_i$-fiber is an $\widetilde{F}_i$-orbit; thus the Γ_i-action preserves the $\widetilde{F}_i$-orbits fibers (see Remark 1.14).

By Theorem 0.5, $\widetilde{F}_i$ contains no compact subgroup; thus $\widetilde{F}_i$ is diffeomorphic to $\mathbb{R}^k$. Let Λ_i denote the subgroup that preserves a $\widetilde{F}_i$-orbit. Because the Γ_i-action preserves fibers of the principal $\widetilde{F}_i$-bundle, $\Lambda_i < \Gamma_i$ is normal. Because Γ_i is type $\mathcal{A}_s$, any finite group of Γ_i/Λ_i is virtually normal. Again by Theorem 0.5, the singular nilpotent fibration on M_i is a fiber bundle $F_i \to M_i \xrightarrow{f_i} X$; thus X is an aspherical differentiable manifold.

By Theorem 1.13, $X = Y/O(n)$, where Y is $C^{1,\alpha}$-manifold on which $O(n)$ acts freely and isometrically. Consequently, the quotient metric on $X = Y/O(n)$ is defined by a C^α-metric tensor. □

2.3 Proof of Proposition 0.8

Proof Let $M_i \xrightarrow{\mathrm{GH}} X$ be as in Conjecture 0.7. Assume that, for i large, there is a singular nilpotent fibration $f_i\colon M_i \to X$ such that the induced metric on an f_i-fiber is almost left invariant. According to [Cheeger et al. 1992], there is a nearby left-invariant metric whose sectional curvature depends on $\max|\mathrm{sec}_{M_i}|$ (obtained by averaging the metric by (consistent) infinitesimal actions of a nilpotent Lie group). The rest of the proof follows the proof of Theorem A, and we conclude that X is an aspherical manifold, differentiable or homeomorphic, respectively. It remains to show that in (0.6.1), all $x \in X$ are (δ, ρ')-PR for $0 < \rho' \le \rho$.

By Theorem 0.9, for all $\tilde{x}_i \in \widetilde{M}_i$, $\mathrm{vol}(B_1(\tilde{x}_i)) \ge v(n,d) > 0$. By (1.11.2), all points $\tilde{x}_i \in \widetilde{M}_i$ are $(\delta, \tilde{\rho})$-RP $(0 < \tilde{\rho} < \rho)$; thus all points in $\widetilde{X}$ are $(\delta, \tilde{\rho})$-RP for Theorem 1.5.

Because G_0 acts freely and isometrically on $\widetilde{X}$ and $\widetilde{X} \to \widetilde{X}/G_0$ is a submetry, it is clear that points in $\widetilde{X}/G_0$ are $(\delta, \tilde{\rho})$-RP. Because the discrete group G/G_0 acts freely and discontinuously on $\widetilde{X}/G_0$ with a compact quotient manifold X, $\pi_1(X) \cong G/G_0$ and $\rho' = \min\{d_{\widetilde{X}/G_0}(\tilde{z}, \gamma(\tilde{z})) : \tilde{z} \in D, \gamma \in G/G_0\} > 0$, where D denotes a fundamental domain in $\widetilde{X}/G_0$ (the universal covering of X). Without loss of generality we may assume that $\rho' < \rho$. Then points in $X = \widetilde{X}/G$ are (δ, ρ')-RP. □

Note that in the above proofs of Theorem A and Proposition 0.8, Theorem 0.9 is not needed.

3 Proofs of Theorems 0.11 and B, and Proposition 0.14

3.1 Proof of Theorem 0.11

Our strategy is to introduce a left-invariant metric $\hat{g}_i$ on $\widetilde{M}_i$ (identified with a simply connected nilpotent Lie group with identity $\tilde{p}_i$) and show that there is a constant $c(n,d)>0$ such that for any $R>0$,

$$B_R(\tilde{p}_i) \subset \bigcup_{\gamma_i \in \Gamma_i(R)} B_{c(n,d)}(\gamma_i(D(\tilde{p}_i, \hat{g}_i))) \subset B_{R+c(n,d)}(\tilde{p}_i),$$

where D_i denotes a Dirichlet fundamental domain at $\tilde{p}_i \in \widetilde{M}_i$ and $\Gamma_i(R)=\{\gamma_i \in \Gamma_i : \gamma_i(D_i)\cap B_R(\tilde{p}_i)\neq \varnothing\}$.

Given a higher homotopy class of $\widetilde{X}$ represented by a continuous map $\phi\colon S^k \to \widetilde{X}$, where $\phi(S^k)\subset B_R(\tilde{p})$ for some $R>0$, we shall show that there is homotopy H'_s deforming $f_i^{-1}(\phi(S^k_1))$ to $\tilde{p}_i$ inside $B_{R+c(n,d)}(\tilde{p}_i)$ for $0\le s\le 1$, where for i large $f_i\colon B_{R+c(n,d)}(\tilde{p}_i)\to B_{R+c(n,d)}(\tilde{p})$ is a diffeomorphism (Theorem 1.5) or homeomorphism (Theorem 1.6). Then $f_i\circ H'_s\circ f_i^{-1}$ deforms $\phi(S^k_1)$ to $\tilde{p}$; thus, i.e., $\widetilde{X}$ is contractible.

Our construction of H'_s relies on the following result:

Theorem 3.1 [Le Donne 2025, Theorem 9.4.6] *A simply connected nilpotent Lie group is exponential (i.e., Exp: $T_eG\to G$ is a diffeomorphism).*

We now define a left-invariant metric on $\widetilde{M}_i$ based on information from a set of (Gromov) short generators for $\Gamma_i=\pi_1(M_i,p_i)$ as follows.

Recall that for a compact n-manifold M of $\operatorname{diam}(M)=d$ and $p\in M$, $\Gamma=\pi_1(M,p)$ can be generated by a finite number of elements whose displacement at $\tilde{p}$ is $\le 2d+\epsilon\le 3d$ (for any $0<\epsilon<\operatorname{injrad}(M)$, the injectivity radius). A set of short generators for $\pi_1(M,p)$ is picked up in order as follows: let $\gamma_1\in\Gamma$ such that $|\gamma_1|=d(\tilde{p},\gamma(\tilde{p}))$ achieves the minimum among all $\gamma\in\Gamma$. Let $\gamma_2\in\Gamma\setminus\langle\gamma_1\rangle$ (the subgroup generated by γ_1) such that $|\gamma_2|$ is the minimum among all $\gamma\in\Gamma\setminus\langle\gamma_1\rangle$. Iterating this process, one gets a set of short generators, denoted by $S_{\tilde{p}}(\Gamma)$.

For a finitely generated torsion free nilpotent group Λ, we call the minimal number of generators the rank of Λ.

Lemma 3.2 *Let $M_i \xrightarrow{\mathrm{GH}} X$ be a sequence of compact n-manifolds satisfying*

$$\begin{array}{ccc} (\widetilde{M}_i,\tilde{p}_i,\Gamma_i) & \xrightarrow{\text{eqGH}} & (\widetilde{X},\tilde{p},G) \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle /GV} \\ (M_i,p_i) & \xrightarrow{\mathrm{GH}} & (X=\widetilde{X}/G,p) \end{array} \qquad \mathrm{Ric}_{M_i}\ge -(n-1),\quad 0<\operatorname{diam}(X)=d<\infty.$$

If $\Gamma_i(\epsilon)$ is a torsion free nilpotent group of rank s, then G_0 is diffeomorphic to $\mathbb{R}^a\times T^b$ for $a\ge s, b\ge 0$.

Proof We proceed by induction on s, starting with $s=1$. Because $\Gamma_i(\epsilon)$ is a torsion free nilpotent group of rank one, $\Gamma_i(\epsilon)\cong\mathbb{Z}$. Because $\Gamma_i(\epsilon)(\tilde{p}_i)$ is not bounded, which converges $G_0(\tilde{p})$ which is not bounded, i.e., $G_0(\tilde{p})$ is an open submanifold of dimension ≥ 1. Consequently, G_0 is a noncompact nilpotent Lie group; thus $G_0\cong\mathbb{R}^a\times T^k$, $a\ge 1$, and $k\ge 0$, where T^k is a maximal compact subgroup.

Consider torsion free nilpotent $\Gamma_i(\epsilon)$ of rank $s+1$. Then $\Gamma_i(\epsilon)$ has a nontrivial center. Let $\gamma_i \in C(\Gamma_i(\epsilon))$ such that $|\gamma_i| \to 0$. Let $\hat{M}_i = \tilde{M}_i/\langle\gamma_i\rangle$ and $\hat{\Gamma}_i(\epsilon) = \Gamma_i(\epsilon)/\langle\gamma_i\rangle$, $\langle\gamma_i\rangle \to G_1$, which is contained in the center of G_0. Consider the following commutative diagram (see (1.3.3)):

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i(\epsilon)) & \xrightarrow{\text{eqGH}} & (\tilde{X}, \tilde{p}, G_0) \\ \big\downarrow /\langle\gamma_i\rangle & & \big\downarrow /G_1 \\ (\hat{M}_i, \hat{p}_i, \hat{\Gamma}_i(\epsilon)) & \xrightarrow{\text{GH}} & (\hat{X} = \tilde{X}/G_1, \hat{p}, G_0/G_1) \end{array}$$

Because $\hat{\Gamma}_i(\epsilon)$ is a torsion free nilpotent group of rank s and $\hat{\Gamma}_i(\epsilon) \to G_0/G_1$, by induction we have that $G_0/G_1 \cong \mathbb{R}^a \times T^{k'}$ for $a \geq s, b \geq 0$ (note that in the induction, that $\tilde{M}_i$ is simply connected is not required). Because $G_1 = \mathbb{R}^{a'} \times T^{b'}$ for $a' \geq 1, b' \geq 0$, we conclude that $G_0 \cong \mathbb{R}^{a+a'} \times T^b$ for $a + a' \geq s+1, b \geq 0$. □

Identifying $\tilde{M}_i$ with a simply connected nilpotent Lie group with identity $e = \tilde{p}_i$, one may define an inner product on $T_{\tilde{p}}\tilde{M}_i$, by assigning an inner product for any linearly independent n vectors in $T_{\tilde{p}_i}\tilde{M}_i$. Note that because $\Gamma_i < \tilde{M}_i$ acts on $\tilde{M}_i$ isometrically with respect to the left-invariant metric, the left-invariant metric descends to a metric on M_i, denoted by $\hat{g}_i^*$.

First, each short generator $\gamma_{i\alpha} \in S_{\tilde{p}_i}(\Gamma_i)$ determines a vector in $T_{\tilde{p}_i}\tilde{M}_i$

$$u_{i\alpha} = d_i(\tilde{p}_i, \gamma_{i\alpha}(\tilde{p}_i)) \uparrow_{\tilde{p}_i}^{\gamma_{i\alpha}(\tilde{p}_i)},$$

where $\uparrow_{\tilde{p}_i}^{\gamma_{i\alpha}(\tilde{p}_i)}$ denotes the direction of a minimal geodesic from $\tilde{p}_i$ to $\gamma_{i\alpha}(\tilde{p}_i)$ (which may not be unique). We claim that the set of vectors $\{u_{i\alpha}, \gamma_{i\alpha} \in S_{\tilde{p}_i}(\Gamma_i)\}$ spans $T_{\tilde{p}_i}\tilde{M}_i$.

To see the claim, recall that the vector n-space $T_{\tilde{p}_i}\tilde{M}_i$ is solely determined by the differential structure of $\tilde{M}_i$. Because $\tilde{M}_i$ is also a Lie group, $T_{\tilde{p}_i}\tilde{M}_i$ is identified with the Lie algebra. Because $\tilde{M}_i$ is a simply connected nilpotent Lie group, the exponential map Exp: $T_{\tilde{p}_i}\tilde{M}_i \to \tilde{M}_i$ is a diffeomorphism (see Theorem 3.1). Consequently, if $\Gamma_i < \tilde{M}_i$ is a cocompact lattice, then $\text{Exp}^{-1}(\Gamma_i)$ contains a basis for $T_{\tilde{p}_i}\tilde{M}_i$. On the other hand, the selection of short generators for Γ_i at $\tilde{p}_i$ naturally divide into groups in terms of lengths (elements in each group have comparable lengths, e.g., the first group consisting of short generators of almost minimal lengths). Clearly, the nilpotency implies that selecting short generators in groups corresponds to a process of extending the sub-Lie algebra spanned by vectors from the first group to the sub-Lie algebra spanned by vectors from first and second groups and so on, until the vectors of short generators span the whole Lie algebra $T_{\tilde{p}_i}\tilde{M}_i$.

Let's now pick up a basis from $\{u_{i\alpha}, \gamma_{i\alpha} \in S_{\tilde{p}_i}(\Gamma_i)\}$: we start with $u_{ik} = u_{in}$ (the last one) and extend it linearly independently, in the reverse order, to a basis for $T_{\tilde{p}_i}\tilde{M}_i$, denoted by $\{u_{i1}, \dots, u_{in}\}$. Note that because $\langle\gamma_{i1}, \dots, \gamma_{in}\rangle < \Gamma_i$ is a torsion free nilpotent group, γ_{i1} is from the center of Γ_i.

We now assign an inner product for the basis, $\{u_{i1}, \dots, u_{in}\}$:

$$\hat{g}_i(u_{i\alpha}, u_{i\beta}) = d_i(\tilde{p}_i, \gamma_{i\alpha}(\tilde{p}_i)) \cdot d_i(\tilde{p}_i, \gamma_{i\beta}(\tilde{p}_i)) \cos \measuredangle(\uparrow_{\tilde{p}_i}^{\gamma_{i\alpha}(\tilde{p}_i)} \uparrow_{\tilde{p}_i}^{\gamma_{i\beta}(\tilde{p}_i)}).$$

Let $d_i = \text{diam}(M_i)$ and $\hat{d}_i = \text{diam}(M_i, \hat{g}_i^*)$.

Lemma 3.3 (ratio of diameters estimate) *Let the assumptions be as in Proposition 0.8, and let $\hat{g}_i^*$ be the left-invariant metric on M_i defined in the above. Then there exist constants $a(n), b(n,d) > 0$ such that for i large,*

$$a(n)\hat{d}_i \le d_i \le b(n,d)\hat{d}_i.$$

Proof We first claim that there is a constant $c(n)$ such that the following inequality holds:

$$\ell_i < \hat{d}_i \le c(n)\ell_i, \quad \ell_i = \max\{|\gamma_{i,j}| : \gamma_{i,j} \in S_{\tilde{p}_i}(\Gamma_i)\}. \tag{3.3.1}$$

We proceed by induction on n, starting with $n = 2$: $\hat{g}_i$ is flat, and $\hat{d}_i \le \sqrt{2}\ell_i$. Assume the claim holds for $\dim(M_i) = n-1$.

In general, by the choice of γ_{i1} we may assume that $\langle\gamma_{i1}\rangle$ is contained in the center of Γ_i. Then there is an $\mathbb{R}^1$-subgroup of $\widetilde{M}_i$ that contains $\langle\gamma_{i1}\rangle$. Let $\widehat{M}_i = (\widetilde{M}_i/\mathbb{R}^1)/(\Gamma_i/\langle\gamma_{i1}\rangle)$, which is a nilmanifold of dimension $n-1$. Applying the inductive assumption on $((\widetilde{M}_i, \hat{g}_i)/\mathbb{R}^1)/(\Gamma_i/\langle\gamma_{i1}\rangle)$, we have that $\hat{d}_i \le c(n-1)\ell_i^* \le c(n-1)\ell_i$ and that $S^1 \to M_i \to \widehat{M}_i$ is a Riemannian submersion ($S^1 = \mathbb{R}^1/\langle\gamma_{i1}\rangle$); it is clear that $\hat{d}_i \le c(n-1)\ell_i + \ell_i = (c(n-1)+1)\ell_i = c(n)\ell_i$.

We then claim that there is a constant $b(n,d) > 0$ such that the following inequality holds:

$$\tfrac{1}{3}\ell_i \le d_i \le b(n,d)\ell_i. \tag{3.3.2}$$

Note that the desired inequality follows from a combination of (3.3.1) and (3.3.2), with $a(n) = 1/(3c(n))$.

Because Γ_i is generated by elements in $S_{\tilde{p}_i}(\Gamma_i)$, whose distortions at $\tilde{p}_i$ is less than $3d_i$, $\frac{1}{3}\ell_i \le d_i \to d > 0$. We now prove the other inequality by a contradiction: assuming, passing to a subsequence, that $\ell_i \to 0$ as $i \to \infty$. By Lemma 3.2, $G_0 \cong \mathbb{R}^n \times T^k$ for $k \ge 0$, and thus G_0 acts transitively on $\widetilde{X}$; so $X = \widetilde{X}/G_0$ is a point, a contradiction to $\operatorname{diam}(X) = d > 0$. □

Proof of Theorem 0.11 Let $M_i \xrightarrow{\mathrm{GH}} X$ be as in Theorem B, and let $\widetilde{X}$ be as in (0.10). Given a higher homotopy class of $\widetilde{X}$ represented by a continuous map $\phi\colon S^k \to \widetilde{X}$, $\phi(S^k) \subset B_R(\tilde{p})$ for some $R > 0$.

Let $D(\tilde{p}_i)$ denote a Dirichlet fundamental domain at $\tilde{p}_i$ (with respect to $\tilde{g}_i$). Let

$$\Gamma_i(R) = \{\gamma_i \in \Gamma_i : \gamma_i(D_i(\tilde{p}_i)) \cap B_R(\tilde{p}_i) \ne \varnothing\}$$

(a finite set). Then

$$B_R(\tilde{p}_i) \subset \bigcup_{\gamma_i \in \Gamma_i(R)} \gamma_i(D(\tilde{p}_i)).$$

By Lemma 3.3, we get that $\gamma_i(D(\tilde{p}_i)) \subset B_{c(n,d)}(\gamma_i(\tilde{p}_i))$ and $c_1(n,d) = b(n,d) + 3c(n)$. Because $d(\tilde{p}_i, \gamma_i(\tilde{p}_i)) \le R + d$, we derive

$$\bigcup_{\gamma_i \in \Gamma_i(R)} \gamma_i(D(\tilde{p}_i)) \subset \bigcup_{\gamma_i \in \Gamma_i(R)} B_{c(n,d)}(\gamma_i(D(\tilde{p}_i, \hat{g}_i))) \subset B_{R+c(n,d)}(\tilde{p}_i),$$

where $c(n,d) = d + c_1(n,d)$. For i large, we may assume a diffeomorphism (Theorem 1.5) or a homeomorphism (Theorem 1.6)

$$f_i\colon B_{R+c(n,d)}(\tilde{p}_i) \to B_{R+c(n,d)}(\tilde{p}).$$

Let H_s denote the radial contraction from $B_{R+c(n,d)}(0_i) \subset T_{p_i}\widetilde{M}_i$ to 0_i. Identify $(\widetilde{M}_i, \tilde{p})$ with a simply connected nilpotent Lie group (diffeomorphically or homeomorphically). Then because Exp: $T_{\tilde{p}_i}\widetilde{M}_i \to \widetilde{M}_i$ is a diffeomorphism or a homeomorphism (if M_i is homeomorphic to a nilmanifold),

$$\exp_{\tilde{p}_i}(H_s(\mathrm{Exp}^{-1}(f_i^{-1}(\phi(S_1^k))))) \subset B_{R+c(n,d)}(p_i), \quad 0 < s \leq 1$$

defines a deformation contraction $f_i^{-1}(\phi(S_1^k))$ to $\tilde{p}_i$ inside $B_{R+c(n,d)}(\tilde{p}_i)$, and thus

$$f_i \circ \exp_{\tilde{p}_i} \circ H_s \circ \mathrm{Exp}^{-1} \circ f_i^{-1} \colon [0,1] \times \phi(S_1^k) \to \widetilde{X}$$

defines a deformation contraction of $\phi(S_1^k)$ to $\tilde{p}$, i.e., $\widetilde{X}$ is contractible. □

Remark 3.4 Inspecting the above proof, it is clear that Theorem 0.11 is valid when replacing $\widetilde{M}$ (identified as a simply connected nilpotent Lie group) by an exponential solvable Lie group.

3.2 Proofs of Theorem B and Proposition 0.14

Let $M_i \xrightarrow{\mathrm{GH}} X$ be as in Theorem B, and let $(\widetilde{X}, G)$ be as in (0.10). We first extend Theorem 0.5 to $(\widetilde{X}, G)$, where $\widetilde{X}$ is a contractible differentiable, or a topological, n-manifold, respectively (Theorem 0.11).

Lemma 3.5 *Let $M_i \xrightarrow{\mathrm{GH}} X$ be as in Theorem B, and let $(\widetilde{X}, G)$ be as in* (0.10)*. Then G acts freely on $\widetilde{X}$; thus $\widetilde{X}/G$ is a differentiable, or topological, manifold, respectively.*

Note that in (B2), $\widetilde{X}$ is a smoothable Alexandrov space of curvature ≥ -1, i.e., iterated spaces of directions at a point are spheres [Kapovitch 2002]. According to Proposition 2.7 in [Bruè et al. 2024], if a closed group of isometries G freely acts on a smoothable Alexandrov space X, then X/G is a topological manifold.

Proof of Lemma 3.5 Let $\mathrm{Sp}(\widetilde{M}_i)$ denote the $\mathrm{Spin}(n)$ bundle over $\widetilde{M}_i$ equipped with a canonical metric. Passing to a subsequence, we may assume the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{Sp}(\widetilde{M}_i), (\tilde{p}_i, u_i), \mathrm{Spin}(n), \Gamma_i(\epsilon), \Gamma_i) & \xrightarrow{\mathrm{eqGH}} & (\mathrm{Sp}(\widetilde{X}), (\tilde{p}, u), \mathrm{Spin}(n), G_0 < G) \\ \big\downarrow /\mathrm{Spin}(n) & & \big\downarrow /\mathrm{Spin}(n) \\ (\widetilde{M}_i, \tilde{p}_i, \Gamma_i(\epsilon), \Gamma_i) & \xrightarrow{\mathrm{eqGH}} & (\widetilde{X}, \tilde{p}, G_0 < G) \end{array}$$

By Theorem 0.11, $\widetilde{X}$ is a contractible n-manifold, and because Γ_i is nilpotent, G is nilpotent. Note that the Ricci curvature of the canonical metric on $\mathrm{Sp}(\widetilde{M}_i)$ may not have a uniform lower bound; so the equivariant GH-convergence is from that for any $R > 0$. There is a uniform upper bound on the number of points in an ϵ-net in $B_R(\tilde{p}_i, u_i)$, $s(\epsilon) \cdot \ell_i(R, \epsilon)$, where $s(\epsilon)$ denotes an ϵ-net in $\mathrm{Spin}(n)$ with a bi-invariant metric, and $\ell_i(R, \epsilon)$ is the size of an ϵ-net in $B_R(\tilde{p}_i, \widetilde{M}_i)$.

Note that one may view the free isometric G-action on $\mathrm{Sp}(\widetilde{X})$ (equipped with a length metric) as the lifting of isometric G-action on $\widetilde{X}$. Consequently, $\widetilde{X}/G_0$ is not compact. By now the rest of proof follows from the proof of Theorem 0.5. □

Proof of (B1) By Theorems 0.9, 1.5, and 0.11, in (0.10) $\widetilde{X}$ is a contractible n-manifold. Because M_i is a nilmanifold, G is nilpotent. Applying Lemma 3.5 to $(\widetilde{X}, G)$, we conclude that X is an aspherical manifold, and by Proposition 0.8, points in X are (δ, ρ')-RP.

We now apply Theorem 0.12 to $M_i \xrightarrow{\text{GH}} X$, and conclude that for i large there is a smooth fiber bundle map $F_i \to M_i \xrightarrow{f_i} X$ with F_i an infranilmanifold and affine structural group.

Let $\pi_i\colon (\widetilde{M}_i, \tilde{p}_i) \to (M_i, p_i)$ denote the Riemannian universal covering map, and consider the lifting bundle $\widetilde{F}_i \to \widetilde{M}_i \to \widehat{X}$, where $\widetilde{F}_i$ is a component of $\pi_i^{-1}(F_i)$. Because $\widetilde{M}_i$ is simply connected, $\widehat{X}$ is simply connected, and $\hat{\pi}\colon (\widehat{X}, \hat{p}) \to (X, p)$ is a universal covering map. Because $\widehat{X}$ is contractible, $\widetilde{F}_i \to \widetilde{M}_i \to \widehat{X}$ is a trivial bundle $\widetilde{M}_i \cong \widetilde{F}_i \times \widehat{X}$; thus $\widetilde{F}_i$ is simply connected, i.e., $\widetilde{F}_i$ is a simply connected nilpotent Lie group.

We now identify $(\widetilde{M}_i, \tilde{p}_i)$ with a simply connected nilpotent Lie group of $e = \tilde{p}_i$, and Γ_i a cocompact lattice, and identify $(\widetilde{F}_i, \tilde{p}_i)$ with a simply connected nilpotent Lie group of $e = \tilde{p}_i$, and a cocompact lattice $\Gamma_i(\epsilon) = \pi_1(F_i)$; see (1.10.2).

Observe that the inclusion map $F_i \hookrightarrow M_i$ induces a homomorphic embedding, $\Gamma_i(\epsilon) \hookrightarrow \Gamma_i$, which, by Maltsev rigidity, induces an isomorphic embedding on simply connected nilpotent Lie groups $\widetilde{F}_i \hookrightarrow \widetilde{M}_i$. By now the principal $\widetilde{F}_i$-bundle on $\widetilde{M}_i$ is identified as a normal subgroup Lie group $\widetilde{F}_i$-action of the nilpotent Lie group $\widetilde{M}_i$. Thus $\widehat{X}$ is diffeomorphic to the simply connected nilpotent Lie group $\widetilde{M}_i/\widetilde{F}_i$ with a cocompact lattice $\Gamma_i/\Gamma_i(\epsilon)$; see (1.10.2). We conclude that X is diffeomorphic to $\widehat{X}/(\Gamma_i/\Gamma_i(\epsilon))$, a nilmanifold. □

In the proof of (B2), we shall apply the following result, which confirms the Borel conjecture for nilmanifolds (the Borel conjecture asserts that two homotopy equivalent compact aspherical manifolds are homeomorphic).

Theorem 3.6 [Farrell and Hsiang 1983] *Homotopy equivalent compact nilmanifolds are homeomorphic.*

Proof of (B2) We start with (0.10): By Theorems 0.9, 1.6, and 0.11 $\widetilde{X}$ is a contractible topological n-manifold. Because M_i is a nilmanifold, G is nilpotent. By Lemma 3.5, X is a topological aspherical manifold of $\pi_1(X) \cong G/G_0$, a nilpotent group. Because X is an Eilenberg–Mac Lane space (a $K(\pi, 1)$-space), a classifying space of the group G/G_0, X is unique up to homotopy equivalence. On the other hand, a finitely generated torsion free nilpotent group G/G_0 can be realized as a compact lattice in a simply connected nilpotent Lie group N (see Theorem 2.18 in [Raghunathan 1972]). Consequently, X is homotopically equivalent to the nilmanifold $N/(G/G_0)$.

By now (B2) follows from Theorem 3.6, i.e., X is homeomorphic to a nilmanifold. □

Remark 3.7 The above proof of (B2), restricting to the case that M_i is homeomorphic to a torus, does not require that a finite covering space of M_i is not collapsed and has bounded diameter [Kloeckner and Sabourau 2020; Zamora 2026], on which the proof of Theorem 0.2 in [Bruè et al. 2024] relies.

Proof of Proposition 0.14 Consider a sequence of aspherical n-manifolds $M_i \xrightarrow{\text{GH}} X$ satisfying the conditions in Conjecture 0.7. By Theorems 0.9, 1.5 and 1.6, $\widetilde{X}$ in (0.10) is an n-dimensional manifold,

differentiable or homeomorphic, respectively. Assume that $\widetilde{X}$ is contractible (Conjecture 0.13). By (1.10.1) G_0 is nilpotent, and by (1.10.2), for i large, G/G_0 is isomorphic to $\Gamma_i/\Gamma_i(\epsilon)$. Because Γ_i is type $\mathcal{A}_s$, any finite subgroup of G/G_0 is virtually normal. We now apply Lemma 3.5 to conclude that $X = \widetilde{X}/G$ is an aspherical manifold, differentiable or topological, respectively, i.e., Conjecture 0.7 is proved. □

4 Proof of Theorem C

Proof Let $M_i \xrightarrow{\text{GH}} X$ be a sequence of compact aspherical 3-manifolds satisfying

$$\text{Ric}_{M_i} \geq -1, \quad 0 < \text{diam}(X) = d < \infty, \quad \text{vol}(M_i) \to 0,$$

and the commutative diagram (0.10), where $X = \widetilde{X}/G$.

By Theorem 0.9, $\text{Vol}(B_1(\tilde{p}_i)) \geq v > 0$, and by Theorem 1.7 in [Simon 2012], $\widetilde{X}$ is an open 3-manifold. Because $0 < \text{diam}(X) = d < \infty$, $1 \leq \dim(G) \leq 2$.

Case 1 Assume $\dim(G) = 1$. By Lemma 3.2, $G_0 = \mathbb{R}^1$. Because $\mathbb{R}^1$ acts freely on $\widetilde{X}$, $\widetilde{X}/G_0$ is an open surface. We claim that $\widetilde{X}/G_0$ is simply connected, and thus $\widetilde{X}/G_0$ is homeomorphic to $\mathbb{R}^2$. By Lemma 3.5, G/G_0 acts freely on $\widetilde{X}/G_0$; thus X is an aspherical manifold.

If $\widetilde{X}/G_0$ is not simply connected, by (1.10.4) $(\widetilde{X}/G_0)/H$ is a simply connected open surface, where H is generated by finite isotropy groups of the G/G_0-action on $\widetilde{X}/G_0$. Because the fixed-point set of each isotropy group is isolated, that $\widetilde{X}/G_0$ is not simply connected implies that $(\widetilde{X}/G_0)/H$ is not simply connected, a contradiction.

Case 2 Assume that $\dim(G) = 2$. Then $G_0 = \mathbb{R}^1 \times S^1$ or $\mathbb{R}^2$. Because $\widetilde{X}/G_0$ is not compact, $\widetilde{X}/G_0 = \mathbb{R}$ or $\mathbb{R}_+$. Note that $\mathbb{R}_+$ has a trivial isometry group, and thus $G/G_0 = e$, contradicting that X is compact. If $\widetilde{X}/G_0 = \mathbb{R}$, then that $\pi_1(M_i)$ is type $\mathcal{A}_s$ implies that G/G_0 has no torsion element (otherwise, $G/G_0 = \mathbb{Z}_2 \rtimes \mathbb{Z}$, a contradiction). Consequently, $X = \widetilde{X}/G$ is a circle, an aspherical 1-manifold. □

Recall that if a compact simply connected M admits a torus T^k-action without fixed points, then M admits an one-parameter family of T^k-invariant metrics g_t of $|\text{sec}_{g_t}| \leq 1$ and $(M, g_t) \xrightarrow{\text{GH}} M/T^k$, a compact metric space [Rong 2018]. Consequently, if a compact manifold satisfies that any metrics g_i of $|\text{sec}_{g_i}| \leq 1$, $(M, g_i) \xrightarrow{\text{GH}} X$ and X is a topological manifold, then any torus action on M has to be free.

Next, we supply a justification of the following assertion in discussion before Definition 0.4:

Lemma 4.1 *Let M be compact simply connected n-manifold M for $n \leq 4$. If M admits a torus action, then M admits a nonfree T^1-action.*

Proof For $n = 2$, M is a sphere and any T^1-action on M has a fixed point.

For $n = 3$, if M admits a free T^1-action, then M/T^1 is a compact simply connected surface; thus M/T^1 is homeomorphic to S^2, and therefore M is homeomorphic to S^3 (one may conclude this directly from the Poincaré conjecture). But S^3 also admits not free T^1- and T^2-actions.

For $n = 4$, the Euler characteristic of M is ≥ 2. Thus M admits no free T^1-action. □

The following example was mentioned in the introduction following Conjecture 0.6.

Example 4.2 Let $M = S^k \times S^{n-k}$ for $k \geq 2$ and $n \geq 5$. Otsu [1991] constructed a family of metrics g_i on M such that $\mathrm{Ric}_{g_i} \geq n-1$, $\mathrm{vol}(M, g_i) \geq v > 0$, and $\mathrm{diam}(g_i) \to \pi$. The following property on $S^3 \times S^3$ was pointed out to the author by J. Pan: g_i has a free S^1-symmetry, $(S^3 \times S^3, g_i) \xrightarrow{\mathrm{GH}} \widetilde{X}$, where $\widetilde{X}$ is a spherical suspension over $S^3 \times S^2$ equipped with a metric of diameter π. Let $\mathbb{Z}_i < S^1$. Then

$$\begin{array}{ccc} (S^3 \times S^3, g_i, \mathbb{Z}_i) & \xrightarrow{\mathrm{eqGH}} & (\widetilde{X}, G = S^1) \\ \big\downarrow /\mathbb{Z}_i & & \big\downarrow /S^1 \\ (S^3 \times S^3)/\mathbb{Z}_i & \xrightarrow{\mathrm{GH}} & X = \widetilde{X}/S^1 \end{array}$$

Because the S^1-action fixes a vertex of $\widetilde{X}$, for all i large, $(S^3 \times S^3)/\mathbb{Z}_i$ admits no singular nilpotent fibration, because otherwise the lifting of the singular nilpotent fibration on $S^3 \times S^3$ is a S^1-action without fixed points. This converges to the S^1-action on $\widetilde{X}$, which must have no fixed point, a contradiction.

In Example 4.2, $(S^3 \times S^3)/\mathbb{Z}_i$ admits an isometric S^1-action with isolated fixed points, which may be treated as a generalized singular fibration if one drops the condition that each fiber is an infranilmanifold of positive dimension.

To partially justify the existence of a generalized singular nilpotent fiber bundle on a collapsed manifold M of $\mathrm{Ric}_M \geq -(n-1)$ and where $x \in M$ is a (ρ, v)-LRVP, we first recall the following property on a Ricci limit space of a collapsing sequence of n-manifolds of (ρ, v)-LRVP [Rong 2018]:

Lemma 4.3 (a Ricci limit space of a collapsing sequence with LRVP) *Let $(M_i, p_i) \xrightarrow{\mathrm{GH}} (X, p)$ be a sequence of complete n-manifolds satisfying*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(p_i)) \to 0, \quad \textit{every } x_i \in M_i \textit{ is a } (\rho, v)\textit{-LRVP.}$$

(4.3.1) *For $x \in X$, any tangent cone at x is a metric cone of dimension $m \in \mathbb{N}$. Thus the Hausdorff dimension $\dim_H(X)$ is m.*

(4.3.2) *The set of ϵ-regular points in X, X_ϵ, is an open m-manifold which is dense in X.*

(4.3.3) *For small $0 < \rho < 1$, points in $X_0 = \{x \in X_\epsilon : d(x, X \setminus X_\epsilon) \geq \rho\}$ are $\big(\delta, \frac{1}{4}\rho\big)$-RP for $0 < \delta \leq \delta(n)$.*

Recall that if X is a Ricci limit space of a collapsing sequence of n-manifolds M_i of $\mathrm{Ric}_{M_i} \geq -(n-1)$, then the local structure on X could be very complicated; a tangent cone at $x \in X$ may not be a metric cone, different tangent cones at x are not related (e.g., in dimension, topology, or geometry), the Hausdorff dimension of X may not be an integer [Pan and Wei 2022], X may contain no topological point [Hupp et al. 2025], etc.

In contrast, Lemma 4.3 says that if M_i satisfies that any $x_i \in M_i$ is a (ρ, v)-LRVP, then X shares the local properties on a Ricci limit space of a noncollapsing sequence; see (4.3.1)–(4.3.3).

Because [Rong 2018] is in Chinese, for convenience of readers we present the following proof:

Proof of Lemma 4.3 (4.3.1) Let $x \in X$ and let $r_j \to \infty$ be such that $(r_j X, x) \xrightarrow{\text{GH}} (C_x X, x)$. We may assume $x_i \in M_i$ such that $x_i \to x$. We start with the commutative equivariant convergence [Xu 2025]

$$\begin{array}{ccc} (\widetilde{B_\rho(x_i)}, \tilde{x}_i, \Gamma_i) & \xrightarrow{\text{eqGH}} & (\widetilde{X}, \tilde{x}, G) \\ \big\downarrow \pi_i & & \big\downarrow /G \\ (B_\rho(x_i), x_i) & \xrightarrow{\text{GH}} & (B_\rho(x) = \widetilde{X}/G, x) \end{array}$$

where $\pi_i \colon (\widetilde{B_\rho(x_{i_j})}, \tilde{x}_{i_j}) \to (B_\rho(x_{i_j}), x_{i_j})$ denotes the Riemannian universal covering map. Then passing to a subsequence, $(r_j B_\rho(x_{i_j}), x_{i_j}) \xrightarrow{\text{GH}} (C_x X, x)$, and we have the commutative diagram

$$\begin{array}{ccc} (r_j B_\rho(\tilde{x}_{i_j}), \tilde{x}_{i_j}, \Gamma_{i_j}) & \xrightarrow{\text{eqGH}} & (C_{\tilde{x}} \widetilde{X}, \tilde{x}, dG) \\ \big\downarrow \pi_i & & \big\downarrow /G \\ (r_j B_\rho(x_{i_j}), x_{i_j}) & \xrightarrow{\text{GH}} & (C_x X = C_{\tilde{x}} \widetilde{X}/dG, x) \end{array}$$

where dG denotes the "differential" of the G-action at $\tilde{x}$, i.e., the limit of the subgroup of G generated by G_0 (the identity component) and isotropy group at $\tilde{x}$. Because $\operatorname{vol}(B_\rho(x_{i_j})) \geq v > 0$, $C_{\tilde{x}} \widetilde{X} = \mathbb{R}^k \times C(\Sigma_{\tilde{x}})$ is a metric cone of dimension n, and $\operatorname{diam}(\Sigma_{\tilde{x}}) < \pi$ [Cheeger and Colding 1996]. We may assume that $dG = \mathbb{R}^l \times G_1$ for $l \leq k$, $\mathbb{R}^l$ acts by translation on the $\mathbb{R}^k$-factor, and G_1 is a compact subgroup acting on the cross section of $C_{\tilde{x}} X/\mathbb{R}^l = C(\Sigma_{\tilde{x}}(\widetilde{X}/\mathbb{R}^l))$. It is clear that $C_x X \cong C(\Sigma_{\bar{x}}(C_{\tilde{x}}(\widetilde{X}/\mathbb{R}^l)/G_1))$, a metric cone.

(4.3.2) This is a consequence of a result in [Cheeger and Colding 2000]: if a Ricci limit space satisfies that every tangent cone is a pole space, then $\dim_H(X) \in \mathbb{N}$.

(4.3.3) Let $x \in X_0$; we may assume $x_i \in M_i$ such that $x_i \to x$. Because $B_{\rho/2}(x) \subset X_\epsilon$, consider the following commutative equivariant GH-convergence:

$$\begin{array}{ccc} (\widetilde{B_{\rho/2}(x_i)}, \tilde{x}_i, \Gamma_i) & \xrightarrow{\text{eqGH}} & (\widetilde{X}, \tilde{x}, G) \\ \big\downarrow \pi_i & & \big\downarrow \text{proj} \\ (B_{\rho/2}(x_i), x_i) & \xrightarrow{\text{GH}} & (B_{\rho/2}(x) = \widetilde{X}/G, x) \end{array}$$

Since $\widetilde{X}$ is an n-manifold and $B_{\rho/2}(x)$ is a manifold, the isometric (Lie group) G-action on $\text{proj}^{-1}(B_{\rho/2}(x))$ is free. Because $\tilde{x}$ is $\big(\delta, \frac{1}{2}\rho\big)$-RP (1.11.2), x is a (δ, ρ')-RP for $0 < \rho' < \frac{1}{2}\rho$. □

Combining Lemma 4.3 and Theorem 0.12, in the case that $0 < \operatorname{diam}(X) = d < \infty$ we get a fiber bundle $F_i \to f_i^{-1}(X_0) \to X_0$ with F_i an infranilmanifold and affine structural group.

Open Problem 4.4 (generalized singular nilpotent fibration [Huang et al. 2020b]) *Let $M_i \xrightarrow{\text{GH}} X$ be a sequence of compact n-manifolds such that*

$$\operatorname{Ric}_{M_i} \geq -(n-1), \quad 0 < \operatorname{diam}(X) = d < \infty, \quad x_i \in M_i \text{ is a } (\rho, v)\text{-LRVP}.$$

Then for i large, there is continuous ϵ_i-GHA, $f_i \colon M_i \to X$, such that $F_i \to f_i^{-1}(X_0) \to X_0$ is a smooth fiber bundle.

Note that a positive answer to Open Problem 4.4 would likely provide information on a singular f_i-fiber; a singular fiber could be a point (Example 4.2), or may not be a topological infranilmanifold (e.g., an orbifold).

Appendix

In the appendix, we will supply a proof of (1.3.3) due to Pan.

Proof Let $(X, p, \Lambda, <\Gamma) := (X_i, p_i, \Lambda_i < \Gamma_i)$, let $\pi\colon \Gamma \to \Gamma/\Lambda$, and let $\overline{X} = X/\Lambda$ denote the quotient space with quotient metric $d_{\overline{X}}$. Following a standard argument, one obtains (1.3.3) if for $R > 0$, we have[2]

$$\frac{\Gamma}{\Lambda}(R) = \frac{\Gamma(R)\Lambda}{\Lambda},$$

where $\Gamma(R) = \{\gamma \in \Gamma : d_X(p, \gamma(p)) \le R\}$.

First, we show that $(\Gamma/\lambda)(R) \subseteq (\Gamma(R)\Lambda)/\Lambda$. For any $\bar\gamma \in (\Gamma/\Lambda)(R)$ with $d_{\overline{X}}(\bar p, \bar\gamma(\bar p)) \le R$, let $\gamma \in \Gamma$ be such that $\pi(\gamma) = \bar\gamma$. We shall show that there is $h_\gamma \in \Lambda$ such that $\gamma h_\gamma \in \Gamma(R)$, and thus $\pi(\gamma h_\gamma) = \bar\gamma \in (\Gamma(R)\Lambda)/\Lambda$. First, we may assume $h_1, h_2 \in \Lambda$ such that $d_{\overline{X}}(\bar p, \bar\gamma\,\bar p) = d_X(h_1 p, h_2\gamma(p))$. Because Λ is a normal subgroup of Γ,

$$\begin{aligned} R \ge d_{\overline{X}}(\bar p, \bar\gamma(\bar p)) &= d_X(h_1 p, h_2\gamma(p)) = d_X(p, h_1^{-1}h_2\gamma(p)) \\ &= d_X(p, \gamma(\gamma^{-1}h^{-1}h_2\gamma)(p)) = d_X(p, \gamma h_\gamma(p)). \end{aligned}$$

Second, we show that $(\Gamma(R)\Lambda)/\Lambda \subseteq (\Gamma/\Lambda)(R)$. Let $\bar\gamma \in (\Gamma(R)\Lambda)/\Lambda$, $\bar\gamma = \pi(\gamma)$, and $\gamma \in \Gamma(R)$. Then

$$d_{\overline{X}}(\bar\gamma(\bar p), \bar p) = d_{\overline{X}}(\Lambda(\gamma(p), \Lambda(p))) = d_X(\gamma(p), \Lambda(p)) \le d_X(\gamma(p), p) \le R.$$

Thus $\bar\gamma \in (\Gamma/\Lambda)(R)$. □

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XIAOCHUN RONG rong@math.rutgers.edu
School of Mathematics Sciences, Capital Normal University, Beijing, China
Current address: *Department of Mathematics, Rutgers University, Piscataway, NJ, United States*

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