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Filling radius, quantitative K-theory and positive scalar curvature

JINMIN WANG, ZHIZHANG XIE, GUOLIANG YU AND BO ZHU

We prove a quantitative upper bound on the filling radius of complete, spin manifolds with uniformly positive scalar curvature using the quantitative operator K-theory and index theory.

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1 Introduction

In this paper, we study the following filling radius conjecture of Gromov.

Conjecture 1.1 [Gromov 2007] *If (M^n, g) is a complete Riemannian manifold with uniformly positive scalar curvature $\mathrm{Sc}_g \geq \sigma^2 > 0$, then there exists a universal constant c_n depending only on n such that*

$$\mathrm{Fill\,Rad}(M) \leq \frac{c_n}{\sigma}. \tag{1-1}$$

Here $\mathrm{Fill\,Rad}(M)$ stands for the filling radius of M, namely the smallest radius of the neighborhood of M inside any isometric embedding, in which M is homologous to zero. See Definition 2.2 for the precise definition of filling radius. A manifold M is said to be aspherical if $\pi_k(M) = 0$ for any $k \geq 2$. Since the universal cover of any closed aspherical manifold has infinite filling radius (see Example 2.6 for more details), Conjecture 1.1 implies the Gromov–Lawson conjecture [Gromov and Lawson 1983; Rosenberg 1986b] that any closed aspherical manifold admits no complete Riemannian metric with positive scalar curvature. Note that Conjecture 1.1 is weaker than Gromov's Uryson width conjecture [Gromov 2018a, Section 3]; also see Conjecture A.16 in the appendix.

Conjecture 1.1 has been proved for $n = 3$ [Gromov and Lawson 1983; Gromov 2023; Liokumovich and Wang 2026]. It remains open for $n \geq 4$ in general. The main purpose of this paper is to employ the quantitative operator K-theory and index theory to prove a large scale version of Conjecture 1.1 for manifolds with finite asymptotic dimension.

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Theorem 1.2 *Let (M^n, g) be a complete, noncompact, spin Riemannian manifold with finite asymptotic dimension m with control function $\mathcal{D}$. If (M, g) has uniformly positive scalar curvature $\mathrm{Sc}_g \geq \sigma^2 > 0$ and bounded geometry, then there exists a constant $c = c(\sigma, m, \mathcal{D})$ such that*

$$\mathrm{Fill}\,\mathrm{Rad}_{\mathbb{Q}}(M) \leq c. \tag{1-2}$$

From the proof of Theorem 1.2, the constant c is explicitly calculated as

$$c \leq \mathcal{D}\left(\frac{Ae^{Bm}}{\sigma}\right),$$

where A and B are some universal positive constants. Recall that a complete, noncompact Riemannian manifold (M^n, g) is said to have asymptotic dimension m with diameter control $\mathcal{D}$ if m is the minimal integer such that for any $r > 0$, M admits a uniformly bounded cover $\mathcal{C} = \{U_i\}_{i \in I}$ such that

(1) any ball $\mathcal{B}(p, r) = \{x \in M, \mathrm{dist}(x, p) < r\}$ intersects with at most $m+1$ members $\{U_{i_0}, U_{i_1}, \ldots, U_{i_m}\}$ in $\mathcal{C}$,

(2) $\sup_{i \in I} \mathrm{diam}(U_i) \leq \mathcal{D}(r)$. Here $\mathcal{D} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function with $\mathcal{D}(r) \geq r$.

The notion of asymptotic dimension was introduced by Gromov [1993]. The third author [Yu 1998] developed a quantitative operator K-theory and proved that any finitely generated group with finite asymptotic dimension satisfies the strong Novikov conjecture. The strong Novikov conjecture provides an algorithm for determining whether the higher index of an elliptic differential operator vanishes or not. The connection between the vanishing of higher index and positive scalar curvature was established by Rosenberg [1983; 1986a; 1986b]. We refer the reader to [Zhang 2017; Tang et al. 2018; Guo et al. 2019; Wang and Wang 2016; Wang et al. 2021; 2024a; 2024b; Wang and Xie 2022] for some of the further developments on the (higher) index theory, in particular for spaces with singularities, and its applications to scalar curvature problems.

In this paper, the estimate of the filling radius in Theorem 1.2 is obtained by studying the pairing of the Dirac operator and K-theory classes with a certain quantitative Lipschitz control. The K-theory with Lipschitz control was initially developed to study the scalar curvature decay in [Wang et al. 2024a]. Here is an outline of the proof of Theorem 1.2. If M has finite asymptotic dimension m, then for any $r > 0$, there exists an open cover with r-multiplicity bounded by $m + 1$. The natural map from M to the corresponding nerve complex $\mathcal{N}_r$ of this open cover has a small Lipschitz constant when r is large. Now if the filling radius of M is too large, then the K-homology class of the Dirac operator on M does not map to zero in the K-homology of the nerve complex $\mathcal{N}_r$ for some large r. This implies that the pairing between the Dirac operator D on M and the pullback of a certain K-theory class of $\mathcal{N}_r$ is not zero. However, by the Lipschitz controlled K-theory, the pullback of this K-theory class has a small Lipschitz constant when r is large. By a standard curvature estimate, the uniformly positive scalar curvature on M implies that the pairing of D with any vector bundle with small Lipschitz constant is zero. This leads to a contradiction, hence the filling radius cannot be too large under the uniformly positive scalar curvature condition. Our estimates in fact produce an explicit upper bound on the filling radius.

The paper is organized as follows. In Section 2, we review the notion of filling radius and some of its basic properties. In Section 3, we review the notion of asymptotic dimension and Lipschitz controlled K-theory. In Section 4, we prove the main theorem (Theorem 1.2). In the appendix, to put the main results of the paper in a broader context, we review a list of metric invariants of Riemannian manifolds and their connections with positive scalar curvature.

2 A basic introduction to filling radius

In this section, we review the concept of filling radius introduced by Gromov. We recall some of the basic properties of the filling radius and its connections to coarse geometry. Unless otherwise specified, all Riemannian manifolds in this article are assumed to be smooth, connected and orientable.

2.1 Kuratowski embedding

In order to define the filling radius, we first review the Kuratowski embedding, which isometrically embeds any given metric space into some Banach space.

Definition 2.1 Suppose that (X, d) is a metric space and x_0 is a point in X. Let $L^\infty(X)$ be the Banach space of all bounded functions on X with the supremum norm $\| \ \|_\infty$. We define the map $\Psi : (X, d) \to (L^\infty(X), \| \ \|_\infty)$ by

$$\Psi(x)(y) = d(x, y) - d(x_0, y) \quad \text{for all } x, y \in X.$$

The map Ψ is called a Kuratowski embedding of X.

Note that this embedding depends on the choice of the based point x_0 and is therefore not entirely canonical. It is not difficult to see that Ψ is an isometric embedding. Indeed, on the one hand, for any $x_1, x_2 \in X$, by the triangle inequality, we have

$$\|(\Psi(x_1) - \Psi(x_2)\|_\infty \leq d(x_1, x_2).$$

On the other hand,

$$\begin{aligned} d(x_1, x_2) &= d(x_1, x_2) - d(x_2, x_2) \\ &= \Psi(x_1)(x_2) - \Psi(x_2)(x_2) \\ &\leq \sup_{y \in X} |\Psi(x_1)(y) - \Psi(x_2)(y)| \\ &\leq \|\Psi(x_1) - \Psi(x_2)\|_\infty. \end{aligned}$$

Consequently, we have

$$d(x_1, x_2) = \|\Psi(x_1) - \Psi(x_2)\|_\infty \tag{2-1}$$

for all $x_1, x_2 \in X$. From now on, we shall identify X with $\Psi(X)$, and write X for both X and $\Psi(X)$, since no confusion is likely to arise. In particular, we will view X as a subspace of $L^\infty(X)$ and denote the inclusion by $i : X \hookrightarrow L^\infty(X)$.

If (X,d) is a compact metric space, then there is a more canonical way of defining the embedding Ψ. Indeed, in this case, we may define

$$\Psi : (X,d) \to L^\infty(X) \quad \text{by} \quad \Psi(x)(y) = d(y,x) \quad \text{for all } x, y \in X.$$

2.2 Filling radius

In this subsection, we review the locally finite singular homology theory, the notion of filling radius and some of its basic properties.

Let (X,d) be a complete metric space and F a ring. We denote by $C_k^{\mathrm{lf}}(X;F)$ the abelian group of formal sums (possibly infinite) of singular k-simplices

$$c = \sum_\sigma r_\sigma \sigma, \tag{2-2}$$

where σ runs over all continuous maps from the standard k-simplex to X and $r_\sigma \in F$, such that for any given bounded set $K \subset X$, there are only finitely many σ in c that have nonempty intersection with K. In this paper, we will mainly focus on the case where F equals the integers $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$. The usual definition of the boundary map ∂ for singular chains makes these abelian groups into a chain complex

$$\cdots \to C_k^{\mathrm{lf}}(X;F) \to \cdots \to C_2^{\mathrm{lf}}(X;F) \to C_1^{\mathrm{lf}}(X;F) \to C_0^{\mathrm{lf}}(X;F) \to 0. \tag{2-3}$$

The locally finite k-th homology group $H_k^{\mathrm{lf}}(X;F)$ of X is defined to be the k-th homology group of this complex chain, that is,

$$H_k^{\mathrm{lf}}(X;F) = \frac{\ker(\partial : C_k^{\mathrm{lf}}(X;F) \to C_{k-1}^{\mathrm{lf}}(X;F))}{\mathrm{Im}(\partial : C_{k+1}^{\mathrm{lf}}(X;F) \to C_k^{\mathrm{lf}}(X;F))}. \tag{2-4}$$

If (X,d) is compact, then the locally finite homology groups coincide with the usual singular homology groups. Note that for each oriented n-dimensional manifold M, the fundamental class $[M]$ of M generates $H_n^{\mathrm{lf}}(M;F) \cong F$.

Definition 2.2 [Gromov 1983] Suppose that (M^n, g) is a Riemannian manifold and $\Psi : M \to L^\infty(M)$ is a Kuratowski embedding as given in Definition 2.1. The filling radius of M (with respect to F-coefficients) is defined to be

$$\mathrm{Fill\,Rad}_F(M) := \inf_R \{R : \Psi_*([M]) = 0 \in H_n^{\mathrm{lf}}(N_R(\Psi(M)); F)\}, \tag{2-5}$$

where $N_R(\Psi(M)) = \{x \in L^\infty(M) : \mathrm{dist}(x, \Psi(M)) \le R\}$ is the R-neighborhood of $\Psi(M)$ in $L^\infty(M)$. In the case $F = \mathbb{Z}$, we shall simply write $\mathrm{Fill\,Rad}(M)$ in place of $\mathrm{Fill\,Rad}_{\mathbb{Z}}(M)$.

It is clear that $\mathrm{Fill\,Rad}_F(M)$ is independent of the choice of the base point $x_0 \in M$ in Definition 2.2. One can show that if (M^m, g) and (N^n, h) are two Riemannian manifolds that are coarsely equivalent, then $\mathrm{Fill\,Rad}_F(M) = \infty$ if and only if $\mathrm{Fill\,Rad}_F(N) = \infty$.

Lemma 2.3 *Let (M_1^n, g_1) and (M_2^n, g_2) be Riemannian manifolds. If $f : (M_1, g_1) \to (M_2, g_2)$ is a distance-decreasing map with $\deg(f) \neq 0$, then*

$$\operatorname{Fill\,Rad}_F(M_1) \geq \operatorname{Fill\,Rad}_F(M_2). \tag{2-6}$$

Proof We assume that both (M_1, g_1) and (M_2, g_2) are compact Riemannian manifolds, and the non-compact case can be proved similarly. We write $R_1 := \operatorname{Fill\,Rad}_F(M_1)$ and $R_2 := \operatorname{Fill\,Rad}_F(M_2)$. Let Ψ_i be the Kuratowski embedding of M_i for $i = 1, 2$ in Definition 2.1. Since $f : (M_1, g_1) \to (M_2, g_2)$ is a distance-decreasing map, we naturally obtain a distance-decreasing map $h = \Psi_2 \circ f : M_1 \to L^\infty(M_2)$.

We extend h to $H : L^\infty(M_1) \to L^\infty(M_2)$ by defining

$$\big(H(\varphi_1)\big)(m_2) = \inf_{m_1 \in M_1} \big((h(m_1))(m_2) + \|\varphi_1 - \Psi_1(m_1)\|_{L^\infty(M_1)}\big)$$

for any $\varphi_1 \in L^\infty(M_1)$ and $m_2 \in M_2$. By the construction, we have

$$\|H(\varphi_1) - H(\psi_1)\|_{L^\infty(M_2)} \leq \|\varphi_1 - \psi_1\|_{L^\infty(M_1)}$$

for any $\varphi_1, \psi_1 \in L^\infty(M_1)$, and $H(m_1) = h(m_1)$ for any $m_1 \in M$. It follows that for any $R > 0$, $H(N_R(M_1)) \subset N_R(M_2)$.

By the definition of $R_1 := \operatorname{Fill\,Rad}_F(M_1)$, for any $\varepsilon > 0$, there exists an $(n+1)$-chain c of $N_{R_1+\varepsilon}(M_1) \subset L^\infty(M_1)$ such that $\partial c = (\Psi_1)_*([M_1])$. It follows that

$$\partial(H_*(c)) = H_*(\partial c) = (\Psi_2)_* f_*([M_1]) = \deg f \cdot (\Psi_2)_*([M_2])$$

in the homology group of $N_{R_1+\varepsilon}(M_2)$. Consequently, $R_2 \leq R_1 + \varepsilon$. We conclude that $R_1 \geq R_2$ by letting $\varepsilon \to 0$. □

The following lemma will be useful in the proof of our main theorem. More precisely, it will allow us to reduce the proof for the odd-dimensional case of our main theorem to the even-dimensional case.

Lemma 2.4 *If (M^n, g) is a complete Riemannian manifold, then*

$$\operatorname{Fill\,Rad}_{\mathbb{Q}}(M \times \mathbb{R}) = \operatorname{Fill\,Rad}_{\mathbb{Q}}(M).$$

Proof For any $R > 0$, we set

$$X_1 = \bigcup_{k \in \mathbb{Z}} N_R(M \times [4R \cdot 2k, 4R(2k+2)]),$$
$$X_2 = \bigcup_{k \in \mathbb{Z}} N_R(M \times [4R(2k-1), 4R(2k+1)]).$$

Note that

$$X_1 \cap X_2 = \bigcup_{k \in \mathbb{Z}} \big(N_R(M \times [4R(k-2), 4Rk]) \cap N_R(M \times [4Rk, 4R(k+2)])\big).$$

Let

$$i_1 : X_1 \hookrightarrow N_R(M \times \mathbb{R}), \quad i_2 : X_2 \hookrightarrow N_R(M \times \mathbb{R}) \quad \text{and} \quad (j_1, j_2) : X_1 \cap X_2 \hookrightarrow X_1 \times X_2$$

be the inclusion maps. Note that each component of $X_1, X_2, X_1 \bigcap X_2$ is homotopic to $N_R(M)$ by a bounded homotopy. Thus the locally finite homology groups of X_1, X_2, and $X_1 \cap X_2$ are all isomorphic to $\prod_{\mathbb{Z}} H_*^{\mathrm{lf}}(N_R(M);\mathbb{Q})$. For notational simplicity, we omit the coefficient $\mathbb{Q}$ in the rest of the proof.

Consider the exact sequence

$$\cdots \to H_n^{\mathrm{lf}}(X_1 \cap X_2) \to H_n^{\mathrm{lf}}(X_1) \oplus H_n^{\mathrm{lf}}(X_2) \to H_n^{\mathrm{lf}}(N_R(M \times \mathbb{R})).$$

More explicitly, we have

$$\prod_{\mathbb{Z}} H_n^{\mathrm{lf}}(N_R(M)) \xrightarrow{(j_{1*}, j_{2*})} \prod_{\mathbb{Z}} H_n^{\mathrm{lf}}(N_R(M)) \oplus \prod_{\mathbb{Z}} H_n^{\mathrm{lf}}(N_R(M)) \xrightarrow{i_{1*} - i_{2*}} H_n^{\mathrm{lf}}(N_R(M \times \mathbb{R})).$$

Denote by $[M \times \mathbb{R}]$ the fundamental class on $M \times \mathbb{R}$, which is explicitly given by

$$[M \times \mathbb{R}] = \sum_{k \in \mathbb{Z}} [M] \times [4Rk, 4R(k+2)].$$

By the above exact sequence, the class $\Psi_{M \times \mathbb{R}}([M \times \mathbb{R}]) \in H_n^{\mathrm{lf}}(N_R(M \times \mathbb{R}))$ lies in the image of the map $i_{1*} - i_{2*}$. More explicitly, we have

$$\Psi_{M \times \mathbb{R}}([M \times \mathbb{R}]) = i_{1*}\left(\prod_{\mathbb{Z}} \Psi_M([M])\right) - i_{2*}\left(\prod_{\mathbb{Z}} (-\Psi_M([M]))\right).$$

Therefore, if $[M]$ vanishes in $H_*^{\mathrm{lf}}(N_R(M))$, then $\Psi_{M \times \mathbb{R}}([M \times \mathbb{R}])$ vanishes in $H_n^{\mathrm{lf}}(N_R(M \times \mathbb{R}))$ as well. Consequently, we have

$$\operatorname{Fill\,Rad}_{\mathbb{Q}}(M \times \mathbb{R}) \leq \operatorname{Fill\,Rad}_{\mathbb{Q}}(M).$$

We remark that so far this part of the proof works for the filling radius with arbitrary coefficients.

Conversely, if $\Psi_{M \times \mathbb{R}}([M \times \mathbb{R}]) = 0$, then

$$i_{1*}\left(\prod_{\mathbb{Z}} \Psi_M([M])\right) - i_{2*}\left(\prod_{\mathbb{Z}} (-\Psi_M([M]))\right) = \Psi_{M \times \mathbb{R}}([M \times \mathbb{R}]) = 0.$$

It follows from the above exact sequence that there exist $c_k \in H_*^{\mathrm{lf}}(N_R(M))$ for $k \in \mathbb{Z}$ such that

$$j_{1*}\left(\prod_{\mathbb{Z}} c_k\right) = \prod_{\mathbb{Z}} \Psi_M([M]) \quad \text{and} \quad j_{2*}\left(\prod_{\mathbb{Z}} c_k\right) = \prod_{\mathbb{Z}} (-\Psi_M([M])).$$

Note that each component of $X_1 \cap X_2$ only intersects with two components of X_1 and X_2, respectively. Therefore, for each $k \in \mathbb{Z}$, we have

$$\Psi_M([M]) = c_k = -\Psi_M([M]),$$

which implies that $2\Psi_M([M]) = 0$. It follows that $\Psi_M([M]) = 0$, since we are working with the coefficient $\mathbb{Q}$. To summarize, we have shown that if $\Psi_{M \times \mathbb{R}}([M \times \mathbb{R}])$ vanishes in $H_n^{\mathrm{lf}}(N_R(M \times \mathbb{R}))$,

then $[M]$ vanishes in $H_*^{\mathrm{lf}}(N_R(M))$. Consequently, we have

$$\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M\times\mathbb{R})\geq\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M).$$

This together with the inequality

$$\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M\times\mathbb{R})\leq\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M)$$

from the first part completes the proof. □

Remark 2.5 For the filling radius with $\mathbb{Z}_2$ coefficients, Gromov [1983] proved that

$$\operatorname{Fill}\operatorname{Rad}_{\mathbb{Z}_2}(M_1\times M_2)=\min\{\operatorname{Fill}\operatorname{Rad}_{\mathbb{Z}_2}(M_1),\operatorname{Fill}\operatorname{Rad}_{\mathbb{Z}_2}(M_2)\} \tag{2-7}$$

for any two Riemannian manifolds (M_1,g_1) and (M_2,g_2).

The following example shows that the filling radius of the universal cover of an aspherical manifolds is ∞.

Example 2.6 Let (M^n,g) be a closed aspherical manifold, then the universal cover $\widetilde{M}$ of M has $\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(\widetilde{M})=\infty$.

Proof We begin by proving $\widetilde{M}$ is uniformly contractible, that is, for any $r>0$, there exists $R=R(r)\geq r$ such that $B(y,r)$ is contractible in $B(y,R)$ for any $y\in M$. Fix $x_0\in\widetilde{M}$. Let $d=\operatorname{diam}(M)$. Then, for each $B(y,r)$, there exists $g\in\pi_1(M)$ such that $gB(y,r)\subset B(x_0,d+2r)$. Note that M admits a finite CW complex structure, which naturally lifts to a CW complex structure on $\widetilde{M}$. The ball $B(x_0,d+2r)$ is contained in some *finite* CW subcomplex C of $\widetilde{M}$. Since $\widetilde{M}$ is contractible, it follows by induction on the dimensions of cells that each cell in C contracts to a single point in $B(x_0,d+2r+s)$ for some $s>0$. Consequently, we have shown that $B(y,r)$ is contractible in $B(y,R)$ with $R(r)=d+2r+s$.

Now let us show that $\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(\widetilde{M})=\infty$. Assume to the contrary that $\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(\widetilde{M})<\infty$, then there exist $r>0$ and an $(n+1)$ dimension simplicial complex P such that $\partial P=\widetilde{M}$ and

$$\widetilde{M}\subset P\subset U_r(\widetilde{M})\subset L^\infty(\widetilde{M}).$$

However, since $\widetilde{M}$ is uniformly contractible, we see that the identity map $i:\widetilde{M}\to\widetilde{M}$ extends to a continuous map $\varphi:P\to\widetilde{M}$; see for example the proof of Lemma A.10 for more details. It follows that $[\widetilde{M}]=[\partial P]=0$ in $H_n^{\mathrm{lf}}(\widetilde{M})$. This contradicts the fact $[\widetilde{M}]$ is the fundamental class and does not vanish in $H_n^{\mathrm{lf}}(\widetilde{M})$. Therefore, we have shown that $\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(\widetilde{M})=\infty$. This finishes the proof. □

2.3 Filling radius and coarse geometry

In this subsection, we review some basic results in coarse geometry and their connections to filling radius. A discrete metric space Γ is said to be locally finite if every metric ball of finite radius in Γ contains only finitely many elements of Γ.

Definition 2.7 [Roe 1993] Suppose Γ is a locally finite discrete metric space. Given any $d > 0$, the Rips complex $P_d(\Gamma)$ is a simplicial complex given as follows: the set of vertices is Γ and a finite subset $\{x_1, \dots, x_n\} \subset \Gamma$ spans a simplex if and only if $d(x_i, x_j) \leq d$ for all $1 \leq i, j \leq n$. We define the k-th coarse homology group $HX_k(\Gamma; F)$ of Γ with coefficients in F to be

$$HX_k(\Gamma; F) = \lim_{d\to\infty} H_k^{\mathrm{lf}}(P_d(\Gamma); F). \tag{2-8}$$

Now suppose (M^n, g) is a complete Riemannian manifold with bounded geometry, where the latter condition means that (M^n, g) has uniformly bounded curvature and positive injectivity radius. A subspace $\Gamma \subset M$ is said to be a net in M if there exists $\delta > 0$ and $c > 0$ such that

- $d_g(x, y) \geq c$ for any distinct $x, y \in \Gamma$,
- $d_g(x, \Gamma) \leq \delta$ for any $x \in M$.

Since M has bounded geometry, there exists a triangulation of M such that for each $r > 0$, the number of simplices in the metric ball $B(x, r)$ of radius r is uniformly bounded for all $x \in M$. In particular, the set of vertices of this triangulation gives a net of M.

We define the coarse homology groups of Riemannian manifolds as follows.

Definition 2.8 Suppose that (M^n, g) is a complete Riemannian manifold with bounded geometry and Γ is a net in M. The k-th coarse homology group $HX_k(M; F)$ of M with coefficients in F is defined to be $HX_k(\Gamma; F)$ for each $k \in \mathbb{N}$.

With (M^n, g) and Γ as above, then there exists $d_0 > 0$ such that $M \subset P_{d_0}(\Gamma)$. Let $j : M \to P_{d_0}(\Gamma)$ be the inclusion map. Then j naturally induces a homomorphism

$$j_* : H_k^{\mathrm{lf}}(M; F) \to HX_k(M; F).$$

Definition 2.9 Suppose that (M^n, g) is a complete Riemannian manifold with bounded geometry. Let $[M] \in H_n^{\mathrm{lf}}(M; \mathbb{Q})$ be the fundamental class of M. We say M is rationally macroscopically large if $j_*([M]) \neq 0$ in $HX_n(M; \mathbb{Q})$. Otherwise, we say M is rationally macroscopically small. Similarly, by working with $\mathbb{Z}$ coefficients, we may define integral macroscopic largeness and integral macroscopic smallness.

The coarse K-homology $KX_*(M)$ of M is defined to be

$$KX_*(M) = \lim_{d\to\infty} K_*(P_d(\Gamma)).$$

The Connes–Chern character map induces the following rational isomorphism[1] between (locally finite) K-homology and locally finite homology theory:

$$K_0(M) \otimes \mathbb{Q} \cong \bigoplus_{n=0}^{\infty} H_{2n}^{\mathrm{lf}}(M; \mathbb{Q}) \quad \text{and} \quad K_1(M) \otimes \mathbb{Q} \cong \bigoplus_{n=0}^{\infty} H_{2n+1}^{\mathrm{lf}}(M; \mathbb{Q}).$$

[1]This isomorphism can for example be proved by a Mayer–Vietoris sequence argument.

Similarly,

$$KX_0(M)\otimes\mathbb{Q}\cong\bigoplus_{n=0}^{\infty}HX_{2n}(M;\mathbb{Q})\quad\text{and}\quad KX_1(M)\otimes\mathbb{Q}\cong\bigoplus_{n=0}^{\infty}HX_{2n+1}(M;\mathbb{Q}).$$

In particular, if M is spin and the K-homology class of its Dirac operator vanishes in $KX_*(M)$, then M is rationally macroscopically small. This gives a useful K-homological way to decide when M is *rationally* macroscopically large or small via the techniques developed in the context of the coarse Novikov conjecture. We would like to remark that at the moment it is still unclear whether there is a similar K-homological approach to determining when M is *integrally* macroscopically large or small. In fact, there exist manifolds (M^n,g) with $\operatorname{Fill}\operatorname{Rad}(M)=\infty$, $\operatorname{width}_{n-1}(M)=\infty$ but $\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M)<\infty$ [Dranishnikov 2011]. Here $\operatorname{width}_{n-1}(M)$ is the $(n-1)$-th Uryson width of M. See Definition A.1 for the precise definition of Uryson width $\operatorname{width}_k(M)$ and its connection with the filling radius.

We conclude the section with the following theorem of Gong and the third author.

Theorem 2.10 [Gong and Yu 2000, Theorem 4.1] *Let (M^n,g) be a complete Riemannian manifold with uniformly positive scalar curvature, bounded geometry and subexponential volume growth. Then*

$$\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M,g)<\infty.$$

This result provides a qualitative characterization of the filling radii for a class of Riemannian manifolds with uniformly positive scalar curvature. The main purpose of the present paper is to prove a *quantitative* characterization of the filling radii for a large class of Riemannian manifolds with uniformly positive scalar curvature in the sense of Conjecture 1.1.

3 Lipschitz properties of spaces with finite asymptotic dimension

In this section, we recall the notion of asymptotic dimension. We also review the Lipschitz controlled K-theory from [Wang et al. 2024a], which is a key ingredient for our proof to the main theorem of the paper.

Definition 3.1 Suppose that (M,g) is a complete Riemannian manifold and $\mathcal{C}=\{U_i\}_{i\in I}$ is an open cover of M. The nerve $\mathcal{N}$ of M is a simplicial complex with one vertex v_i for each open set U_i with the property: given any k tuple of vertices $v_{i_0},\dots,v_{i_k}$, there exists a simplex in the nerve with these given vertices if and only if $\cap_{j=0}^{k}U_{i_j}$ is nonempty. Moreover, the cover $\mathcal{C}$ is said to have multiplicity $\le m+1$ if any point $p\in M$, it is contained in at most $(m+1)$ distinct members U_i in $\mathcal{C}$. The diameter of the cover $\mathcal{C}$ is defined as

$$\operatorname{diam}(\mathcal{C})=\sup_{i\in I}\operatorname{diam}(U_i).$$

A cover $\mathcal{C}$ of (M,g) is said to be uniformly bounded if $\operatorname{diam}(\mathcal{C})<\infty$.

We say an open cover $\mathcal{C}$ of (M,g) has Lebesgue number at least r if for all $p\in M$, the metric ball $B(p,r)$ of radius r centered at p is contained in some member of the cover $\mathcal{C}$. For each member U_i of $\mathcal{C}$, let $U_i^r:=N_r(U_i)=\{x\in M:\operatorname{dist}(x,U_i)<r\}$. Then $\mathcal{C}_r=\{U_i^r\}_{i\in I}$ is a new open cover of (M,g). Note

that $\mathcal{C}_r$ satisfies

$$\operatorname{diam}(\mathcal{C}_r) \leq \operatorname{diam}(\mathcal{C}) + 2r,$$

and has Lebesgue number $\geq r$. If we want to emphasize the dependence on the Lebesgue number, we will denote by $\mathcal{N}_r$ the nerve of (M, g) associated with the cover $\mathcal{C}_r$ of (M, g). Clearly, all the above definitions apply to any metric space (X, d). A metric space (X, d) is said to have Lebesgue dimension $\leq m$ if any open cover $\mathcal{C}$ of (X, d) has a refinement $\mathcal{C}'$ with multiplicity $m + 1$. The Lebesgue dimension of X is defined to be the minimum of such integer m. Now the notion of asymptotic dimension can be viewed as a large-scale version of the Lebesgue dimension.

Definition 3.2 A metric space (X, d) is said to have asymptotic dimension m if m is the minimal integer such that for any $R > 0$, X admits a uniformly bounded cover $\mathcal{C} = \{U_i\}_{i \in I}$ with R-multiplicity at most $m + 1$. Here R-multiplicity means that any ball $B(p, R) = \{x \in X : d(x, p) < R\}$ intersects with at most $m + 1$ members of $\mathcal{C}$.

Now we let $\mathcal{C} = \{U_i\}_{i \in I}$ be a uniformly bounded cover of (X, d) with $\operatorname{diam}(\mathcal{C}) = R$ and r-multiplicity at most $m + 1$ for any given $r > 0$. Let $\mathcal{N}_r$ be the nerve of (X, d) associated with the cover $\mathcal{C}_r = \{U_i^r\}_{i \in I}$ of (X, d) with Lebesgue number at least r. We equip $\mathcal{N}_r$ with the standard simplicial metric $d_{\mathcal{N}_r}$, which is equal to the standard l^1-metric d_{l^1} on each simplex and, for two points in different simplices, is the length of the shortest path connecting the two points. We define a map $f_r : X \to \mathcal{N}_r$ by

$$f_r(x) = \frac{\sum_{i \in I} d(x, X - U_i^r) U_i^r}{\sum_{i \in I} d(x, X - U_i^r)}. \tag{3-1}$$

Now suppose (X, d) has asymptotic dimension m. For each $r > 0$, let $\mathcal{N}_r$ be the nerve complex associated with the cover $\mathcal{C}_r$ from above. Then the map f_r from line (3-1) satisfies the following properties:

(1) f_r is $(m+1)^2/r$-Lipschitz;

(2) f_r is uniformly cobounded. More precisely, for any given bounded subset $K \subset \mathcal{N}_r$ with diameter $\operatorname{diam}(K) \leq d$, then $\operatorname{diam}(f^{-1}(K)) \leq Rd + 2(R + 2r)$, where $R = \operatorname{diam}(\mathcal{C})$.

Indeed, consider the function $f_{r,i} : X \to \mathbb{R}$ given by

$$f_{r,i}(x) = \frac{d(x, X - U_i^r)}{\sum_{i \in I} d(x, X - U_i^r)}.$$

Since the Lebesgue number of $\mathcal{C}_r$ is $\geq r$, we have

$$\sum_{i \in I} d(\cdot, X - U_i^r) \geq r, \quad \text{and the Lipschitz constant of } d(\cdot, X - U_i^r) \leq 1.$$

Hence, a direct computation shows that

$$\operatorname{Lip}(f_{r,i}) \leq \frac{m + 1}{r}.$$

Consequently,

$$\mathrm{Lip}(f_r) \le \frac{(m+1)^2}{r}.$$

Note that $\mathrm{diam}(f_r^{-1}(\sigma)) \le 2(R+2r)$ for any simplex $\sigma \in \mathcal{N}_r$. Hence, for any $K \subset \mathcal{N}_r$ with $\mathrm{diam}(K) \le d$, we have

$$\mathrm{diam}(f_r^{-1}(K)) \le Rd + 2(R+2r).$$

In particular, f_r is proper.

Remark 3.3 The above discussion shows that if (X,d) has asymptotic dimension m, then for any given $\varepsilon > 0$, there exists an m-dimensional locally finite simplicial complex P^m and a Lipschitz map $f : M \to P^m$ such that $\mathrm{Lip}(f) \le \varepsilon$. The latter in fact can be shown to be equivalent to the definition of finite asymptotic dimension. See the survey [Bell and Dranishnikov 2011] for more details.

Definition 3.4 Suppose that $\mathcal{D} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\mathcal{D}(s) \ge s$ and (X,d) is a metric space. We say X has asymptotic dimension $\le m$ with diameter control $\mathcal{D}$ if for any $r > 0$, X admits a uniformly bounded cover $\mathcal{C}_r = \{U_i\}_{i\in I}$ with r-multiplicity at most $m+1$ and $\mathrm{diam}(\mathcal{C}_r) \le \mathcal{D}(r)$.

Next, we will construct a left homotopy inverse g_r of the map $f_r : X \to \mathcal{N}_r$ in $L^\infty(X)$, where f_r is defined from line (3-1). More precisely, we prove:

Lemma 3.5 *Suppose that (X^n, g) is a complete, noncompact, Riemannian manifold with bounded geometry and finite asymptotic dimension. For each $r > 0$, let $f_r : X \to \mathcal{N}_r$ be the map from line* (3-1)*. Then there exists a continuous map $g_r : \mathcal{N}_r \to L^\infty(X)$ such that the following diagram commutes up to homotopy within $N_{\mathcal{D}(r)}(X)$:*

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & L^\infty(X) \\ & \searrow^{f_r} & \uparrow{\scriptstyle g_r} \\ & & \mathcal{N}_r \end{array}$$

Proof For any vertex U of $\mathcal{N}_r$, we associate U with a specific point $p_U \in U \subset X \subset L^\infty(X)$. This defines a map from the 0-skeleton of $\mathcal{N}_r$ to $L^\infty(X)$, which we denote by g_r^0. The map g_r^0 is proper since X has bounded geometry.

For any point $y \in \mathcal{N}_r$ that lies in the simplex generated by $U_0, U_1, \dots, U_m$, there are nonnegative numbers a_i with $\sum_{i=0}^m a_i = 1$ such that

$$y = \sum_{i=0}^m a_i U_i.$$

We define the map $g_r : \mathcal{N}_r \to L^\infty(X)$ by setting

$$g_r(y) = \sum_{i=0}^m a_i g_r^0(U_i).$$

In particular, g_r is proper.

Each point $p \in X$ is contained in finitely many open sets, say, $V_0, \dots, V_l$, with $l \le m$. Thus there exist nonnegative $b_0, \dots, b_l$ with $\sum_{i=0}^{l} b_i = 1$ such that

$$f_r(p) = \sum_{i=0}^{l} b_i V_i.$$

It follows that

$$\operatorname{dist}(g_r(f_r(p)), i(p)) = \left\| \sum_{i=0}^{l} b_i g_r(V_i) - i(p) \right\| \le \sum_{i=0}^{l} b_i \| g_r(V_i) - i(p) \| \le \mathcal{D}(r).$$

It follows that $\operatorname{Im}(g_r \circ f_r) \subset N_{\mathcal{D}(r)}(X) \subset L^\infty(X)$. Clearly, $g_r \circ f_r$ is homotopy equivalent to i by a linear homotopy within $N_{\mathcal{D}(r)}(X) \subset L^\infty(X)$. □

Finally, let us review the Lipschitz controlled quantitative K-groups from [Wang et al. 2024a]. Let (X, d) be a locally compact metric space, and $C_0(X)$ denote the algebra of continuous functions on X that vanish at infinity, equipped with the supremum norm. An element f in $M_k(C_0(X))$ is called L-Lipschitz if

$$\| f(x) - f(y) \| \le L \cdot d(x, y) \quad \text{for all } x, y \in X,$$

where the norm on the left is the operator norm of matrices. Let us denote by

- $C_0(X)_L$ the collection of L-Lipschitz functions in $C_0(X)$, and by
- $M_k(C_0(X))_L$ the collection of L-Lipschitz elements in $M_k(C_0(X))$.

It is clear that $\{C_0(X)_L\}_{L \ge 0}$ gives a filtration of $C_0(X)$. If X is noncompact, let $C_0(X)^+$ be the unitization of $C_0(X)$, that is, $C_0(X)^+$ is the algebra of continuous functions on the one-point compactification $X \cup \{*\}$ of X. And we define

$$\pi : C_0(X)^+ \to \mathbb{C} \tag{3-2}$$

to be the evaluation map at the point $*$.

For any given $L \ge 0$, we define $P_k^L(C_0(X)^+)$ to be the set of projections in $M_k(C_0(X)^+)_L$ with the inclusion

$$P_k^L(C_0(X)^+) \to P_{k+1}^L(C_0(X)^+), \quad p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}. \tag{3-3}$$

Similarly, we define $U_k^L(C_0(X)^+)$ to be the set of unitaries in $M_k(C_0(X)^+)_L$ with the inclusion

$$U_k^L(C_0(X)^+) \to U_{k+1}^L(C_0(X)^+), \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}. \tag{3-4}$$

We set

$$P^L(C_0(X)^+) = \lim_{k \to \infty} P_k^L(C_0(X)^+) \quad \text{and} \quad U^L(C_0(X)^+) = \lim_{k \to \infty} U_k^L(C_0(X)^+) \tag{3-5}$$

under the natural inclusions from above.

Definition 3.6 The Lipschitz-controlled quantitative K-groups of X are defined as follows.

(1) Let $(p,s),(q,t)\in P^L(C_0(X)^+)\times\mathbb{N}$. We say $(p,s)\sim(q,t)$ if $p\oplus I_{j+t}$ and $q\oplus I_{j+s}$ are homotopic in $P^{2L}(C_0(X)^+)$ for some $j\in\mathbb{N}$, where I_{j+t} is the $(j+t)\times(j+t)$ identity matrix. We define

$$K_0^L(C_0(X)) := \{(p,\ell)\in P^L(C_0(X)^+) \mid \operatorname{rank}(\pi(p))=\ell\}/\sim,$$

where π is the evaluation map from line (3-2).

(2) Let u and $v\in U^L(C_0(X)^+)$. We say $u\sim v$ if u and v are homotopic in $U^{2L}(C_0(X)^+)$. We define

$$K_1^L(C_0(X)) := U^L(C_0(X)^+)/\sim.$$

It is not difficult to see that $K_*^L(C_0(X)^+)$ are abelian groups for any $L\geq 0$. In fact, the quantitative K-theory applies to a much broader class of C^*-algebras. We refer the reader to [Oyono-Oyono and Yu 2015] for a systematic approach to the quantitative K-theory for more general C^*-algebras.

Recall that we used l^1-metric on the nerve space $\mathcal{N}_r$ above. In fact, there is quite a bit of flexibility when it comes to the choice of such a metric. It turns out that, in order to improve the various constants that will appear in the proof of our main theorem, let us consider a l^2-version of simplicial metric on $\mathcal{N}_r$ as follows. For each $n>0$, let $\overline{\Delta}_n$ be the regular n-simplex in $\mathbb{R}^n$ centered at the origin. More precisely, the vertices $\{v_n^j\}_{0\leq j\leq n}$ of $\overline{\Delta}_n$ are defined inductively by

$$\begin{cases} v_1^0=1,\ v_1^1=-1, \\ v_n^n=0\oplus e_n,\ v_n^j=\sqrt{1-\dfrac{1}{n^2}}\,v_{n-1}^j\oplus -\dfrac{1}{n}e_n, \end{cases}$$

with e_n the unit vector of the n-th direction of $\mathbb{R}^n$.

Let $\widetilde{\Delta}_n$ be the simplex in the unit ball of $\mathbb{R}^n$, whose vertices are $\{v_n^j\}_{0\leq j\leq n}$ and faces are the spherical simplices generated by the vertices. Equip $\widetilde{\Delta}_n$ with the Euclidean metric $\widetilde{d}$. Now for the simplicial complex $\mathcal{N}_r$, we equip each top-dimensional simplex with the metric $\widetilde{d}$ and define the distance between any two points in two different simplices to the distance of the short path connecting the two points. We denote the resulting metric on $\mathcal{N}_r$ by $\widetilde{d}$, since no confusion is likely to arise. Note that, with respect to the new metric $\widetilde{d}$ on $\widetilde{\Delta}_n$, every Lipschitz function $f:\partial\widetilde{\Delta}_n\to M_k(\mathbb{C})$ with Lipschitz function L extends to a Lipschitz function on $\widetilde{\Delta}_n$ by

$$\widetilde{f}(x)=\begin{cases} f\Big(\dfrac{x}{|x|}\Big)(2|x|-1) & \text{if } |x|\geq\frac{1}{2}, \\ 0 & \text{if } |x|\leq\frac{1}{2}, \end{cases}$$

where $|x|$ is the distance of x to the origin. A key observation is that the Lipschitz constant of $\widetilde{f}$ is at most $2L+2\|f\|$, which is *independent* of the dimension of $\widetilde{\Delta}_n$.

Lemma 3.7 *Let $\pi_n:(\Delta_n,d_{l^1})\to(\widetilde{\Delta}_n,\widetilde{d})$ be the radial projection, defined by $\pi_n(ty)=t\pi_n(y)$ for all $t>0$, and $\pi_n(x)=x/|x|$ for $x\in\partial\Delta_n$. Then*

$$\operatorname{Lip}(\pi_n)\leq n\cdot\frac{2}{\sqrt{3}}.$$

Proof By the triangle inequality, we have

$$\|x_1 - x_2\|_{l^2} \leq \|x_1 - x_2\|_{d_{l^1}} \quad \text{for all } x_1, x_2 \in \Delta_n.$$

Thus, it suffices to estimate the Lipschitz constant L of π_n with respect to the Euclidean norm $\|\cdot\|_{l^2}$. Since π_n is homogeneous along radial directions, the derivative of π_n at tx equals the derivative at x,

$$d\pi_n|_{tx} = d\pi_n|_x,$$

for $x \in \Delta_n$ and $t > 0$. Therefore, it suffices to consider points on $\partial\Delta_n$.

Let x lie in a flat face Σ with inward unit normal $\boldsymbol{v}$. Let $r = \|x\|$ and $\hat{x} = x/r$. For y in the cone of Σ, the radial projection is explicitly given by $\pi_n(y) = n\langle y, \boldsymbol{v}\rangle y/\|y\|$. Differentiating at x (where $n\langle x, \boldsymbol{v}\rangle = 1$) gives

$$A(h) := d\pi_n|_x(h) = \frac{1}{r}h + \langle w, h\rangle \hat{x},$$

where

$$w = n\boldsymbol{v} - \frac{1}{r}\hat{x}, \quad \text{with } \|w\|^2 = n^2 + \frac{1}{r^2} - \frac{2}{r^2} = n^2 - \frac{1}{r^2}.$$

We need to compute the largest singular value of A. We have

$$r^2 A^T A = I + xw^T + wx^T + r^2 ww^T.$$

Since $\langle x, \boldsymbol{v}\rangle = 1/n$, we have

$$\langle x, w\rangle = \left\langle x, n\boldsymbol{v} - \frac{1}{r}\hat{x}\right\rangle = n \cdot \frac{1}{n} - 1 = 0.$$

Notice that for any vector h orthogonal to both x and w, $r^2 A^T A$ acts as the identity, namely $r^2 A^T Ah = h$, and $r^2 A^T A$ maps the subspace spanned by $\{x, w\}$ to itself. It suffices to compute the maximum eigenvalue of $r^2 A^T A$ on the subspace spanned by $\{x, w\}$.

The matrix representation of $r^2 A^T A$ with respect to the orthonormal basis $\{x/r, w/\|w\|\}$ is given by

$$G = \begin{pmatrix} 1 & r\|w\| \\ r\|w\| & 1 + r^2\|w\|^2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{n^2r^2 - 1} \\ \sqrt{n^2r^2 - 1} & n^2r^2 \end{pmatrix}.$$

Let $\beta = nr$, which is ≥ 1 (since $r \geq 1/n$). Therefore $\|A\|^2$ is given by scaling the maximum eigenvalue of G:

$$\|A\|^2 = \frac{\lambda_{\max}(G)}{r^2} = \frac{n^2}{2}\left(1 + \frac{1}{\beta^2} + \sqrt{1 + \frac{2}{\beta^2} - \frac{3}{\beta^4}}\right),$$

which reaches its maximum at $\beta^2 = \frac{3}{2}$. It follows that

$$\|A\|^2 \leq \frac{n^2}{2}\left(\frac{8}{3}\right) = \frac{4n^2}{3}.$$

Therefore, we conclude that $\mathrm{Lip}(\pi_n) \leq 2n/\sqrt{3}$. □

Now let f_r be the map from line (3-1). It follows that with respect to the new metric $\widetilde{d}$ on $\mathcal{N}_r$, we have

$$\mathrm{Lip}(f_r) \leq \frac{2}{\sqrt{3}} \frac{(m+1)^3}{r}. \tag{3-6}$$

A key result of the Lipschitz controlled K-theory is the following theorem from [Wang et al. 2024a], which states that K-theory classes of finite-dimensional simplicial complexes have Lipschitz controlled representatives.

Theorem 3.8 [Wang et al. 2024a] *Let $(\mathcal{N}, \widetilde{d})$ be a locally compact m-dimensional simplicial complex equipped with the metric $\widetilde{d}$ as above. Then there exists a constant L_m such that every class in $K_*(C_0(\mathcal{N}))$ admits an L_m-Lipschitz representative in $K_*^{L_m}(C_0(\mathcal{N}))$. Here $L_m \leq C_1 e^{C_2 m}$ for some universal positive constants C_1 and C_2 independent of $\mathcal{N}$. Moreover, if $\alpha \in K_*(C_0(\mathcal{N}))$ can be represented by an element that is constant outside a compact $K \subset \mathcal{N}$, then one can choose an L_m-Lipschitz representative of α that is constant outside the 1-neighborhood of K.*

For the convenience of the reader, let us briefly sketch the proof of Theorem 3.8. The complete details can be found in [Wang et al. 2024a]. For simplicity, let us assume that $\mathcal{N}$ is compact. Let $\mathcal{N}_1$ be a small neighborhood of the $(m-1)$-skeleton of $\mathcal{N}$, and $\mathcal{N}_2$ be the interior of all m-simplices of $\mathcal{N}$. Then $\mathcal{N}_1$ is Lipschitz homotopic to the $(m-1)$-skeleton of $\mathcal{N}$, $\mathcal{N}_2$ is Lipschitz homotopic to a disjoint union of points, and $\mathcal{N}_1 \cap \mathcal{N}_2$ is Lipschitz homotopic to a disjoint union of $(m-1)$-spheres.

Theorem 3.8 is proved by induction. First, Theorem 3.8 obviously holds for $\mathcal{N}_2$ with zero Lipschitz constant. Note that $\mathcal{N}_1 \cap \mathcal{N}_2$ and $\mathcal{N}_1$ are essentially of dimension $m-1$. By induction, assume that Theorem 3.8 holds for $\mathcal{N}_1 \cap \mathcal{N}_2$ and $\mathcal{N}_1$ with Lipschitz constants L_{m-1}. For $i = 1, 2$, let

$$\pi_i : C(\mathcal{N}_i) \to C(\mathcal{N}_1 \cap \mathcal{N}_2)$$

be the restriction map between the algebras of continuous functions, which is surjective for $i = 1, 2$. Furthermore, π_i is also surjective with respect to the Lipschitz filtration, in the sense that every Lipschitz function $f \in M_k(C(\mathcal{N}_1 \cap \mathcal{N}_2))_L$ extends to a Lipschitz function in $M_k(C(\mathcal{N}_i))$ with Lipschitz constant at most $2L + 2\|f\|$. Now the partition $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ induces a Mayer–Vietoris six-term exact sequence of K-theory groups $K^*(\mathcal{N})$, $K^*(\mathcal{N}_1) \oplus K^*(\mathcal{N}_2)$ and $K^*(\mathcal{N}_1 \cap \mathcal{N}_2)$, which is asymptotically exact if we additionally keep track of the Lipschitz filtration; see [Wang et al. 2024a, Definition 4.5 and Proposition 4.10]. Since the Lipschitz constant of the extension (of Lipschitz functions) is independent of m, there exist universal constants $A_1, A_2 > 0$ such that the desired constant L_m is at most

$$L_m \leq A_1 \cdot L_{m-1} + A_2. \tag{3-7}$$

See [Wang et al. 2024a, Appendix]. Now Theorem 3.8 follows by the induction on the dimension of the skeleton, which ends in m steps. This shows that

$$L_m \leq C_1 e^{C_2 m},$$

for some universal constants $C_1, C_2 > 0$.

Remark 3.9 By tracing through the proofs of the five lemma [Wang et al. 2024a, Lemma A.19] and the asymptotic exact sequences [Wang et al. 2024a, Proposition A.26 and Theorem A.35], one sees that the constants A_1 and A_2 in the above discussion are at most 10^{20}. As a result, we have $C_1 \leq 10^{20}$ and $C_2 \leq 50$, where C_1 and C_2 are the constants in Theorem 3.8. The bounds 10^{20} and 50 are not optimal. It is an interesting question what are the optimal constants.

4 Proof of Theorem 1.2

In this section, we prove the main theorem (Theorem 1.2).

4.1 Index pairing

To prepare the proof of Theorem 1.2, let us first review some basics of the index pairing. We shall mainly focus on the even-dimensional case. The odd-dimensional case is similar. Let M be a complete spin manifold of even dimension. The Dirac operator on M defines a K-homology class $[D]$ in $K_0(M)$, and there is a pairing map

$$K^0(M) = K_0(C_0(M)) \to K_0(\mathcal{K}) \cong \mathbb{Z}, \quad \alpha \mapsto \langle [D], \alpha \rangle, \tag{4-1}$$

where $C_0(M)$ is the algebra of continuous functions on M vanishing at infinity, and $\mathcal{K}$ is the algebra of compact operators on a Hilbert space. Let us describe an explicit formula for $\langle [D], \alpha \rangle$; cf. [Willett and Yu 2020].

Let us first review the difference construction of K-theory elements from [Kasparov and Yu 2006].

Lemma 4.1 *If A is a C^*-algebra and I is an ideal of A, and we define*

$$D_A(I) = \{(a, a') \in A \oplus A \mid a - a' \in I\},$$

then there is a natural surjective homomorphism from $K_*(D_A(I))$ to $K_*(I)$. In other words, each K-theory class of the ideal I can be expressed by $[\alpha]-[\beta]$ for some $\alpha, \beta \in M_k(A^+)$ with $\alpha - \beta \in M_k(I)$.

Proof Consider the short exact sequence

$$0 \to I \to D_A(I) \to A \to 0,$$

where the map $I \to D_A(I)$ is given by $a \mapsto (a, 0)$. This exact sequence splits since there exists a section $A \to D_A(I)$ given by $a \mapsto (a, a)$. Thus, $K_*(D_A(I)) \cong K_*(I) \oplus K_*(A)$, and the required surjective homomorphism from $K_*(D_A(I))$ to $K_*(I)$ is given by the projection from $K_*(D_A(I))$ to the first component. □

Now suppose $[\alpha]-[\beta]$ is a K-theory class in $K_0(I)$ from Lemma 4.1. We shall review the difference construction [Kasparov and Yu 2006], which provides an explicit way to construct idempotents of the

unitization I^+ that represent the same class as $[\alpha]-[\beta]$ in $K_0(I)$. Without loss of generality, we may assume α, β are idempotents in A^+ such that $\alpha-\beta \in I$. Set

$$Z(\beta)=\begin{pmatrix} \beta & 0 & 1-\beta & 0 \\ 1-\beta & 0 & 0 & \beta \\ 0 & 0 & \beta & 1-\beta \\ 0 & 1 & 0 & 0 \end{pmatrix} \in M_4(A^+).$$

It is invertible with inverse

$$Z(\beta)^{-1}=\begin{pmatrix} \beta & 1-\beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-\beta & 0 & \beta & 0 \\ 0 & \beta & 1-\beta & 0 \end{pmatrix} \in M_4(A^+).$$

We define

$$d(\alpha,\beta) := Z(\beta)^{-1}\begin{pmatrix} \alpha & & & \\ & 1-\beta & & \\ & & 0 & \\ & & & 0 \end{pmatrix} Z(\beta) \tag{4-2}$$

$$= \begin{pmatrix} 1+\beta(\alpha-\beta)\beta & 0 & \beta(\alpha-\beta) & 0 \\ 0 & 0 & 0 & 0 \\ (\alpha-\beta)\alpha\beta & 0 & (1-\beta)(\alpha-\beta)(1-\beta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(I^+),$$

$$e_{1,3} := \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Note that both $d(\alpha,\beta)$ and $e_{1,3}$ are idempotents in $M_4(I^+)$, and

$$[\alpha]-[\beta]=[d(\alpha,\beta)]-[e_{1,3}] \in K_0(I).$$

Now we return to the construction of the index pairing. Let M be an even-dimensional complete spin manifold and D the Dirac operator acting on its spinor bundle S. The spinor bundle is naturally $\mathbb{Z}_2$-graded and the Dirac operator is an odd operator of the form

$$D=\begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}.$$

Recall that a continuous function $\chi:\mathbb{R}\to[-1,1]$ is called *normalizing function* if χ is a nondecreasing odd function such that $\lim_{x\to\pm\infty}\chi(x)=\pm 1$. To be specific, we consider the the smooth function

$$\chi(x)=\frac{2}{\pi}\int_0^x \frac{1-\cos y}{y^2}\,dy.$$

It is easy to verify that χ is a normalizing function. Moreover, the distributional Fourier transform of $\widehat{\chi}$ is given by

$$\widehat{\chi}(\xi) = \frac{i}{\xi}\max\{2(1-|\xi|,0)\},$$

which has compact support. Note that, for any $t>0$, the operator $\chi(t^{-1}D)$ is a self-adjoint odd operator of the form

$$\chi(t^{-1}D) = \begin{pmatrix} 0 & U_{t,D} \\ V_{t,D} & 0 \end{pmatrix}, \tag{4-3}$$

where $U_{t,D}$ and $V_{t,D}$ are bounded operators acting on the Hilbert space $L^2(M,S)$ of L^2-sections of the spinor bundle S. Since $\widehat{\chi}$ has compact support, the operator $\chi(t^{-1}D)$ has finite propagation, and its propagation goes to zero as $t\to\infty$. It follows that

$$\|[\chi(t^{-1}D), f]\| \to 0 \quad \text{as } t\to\infty \tag{4-4}$$

for any $f\in C_0(M)$, where $[\chi(t^{-1}D), f] = \chi(t^{-1}D)\cdot f - f\cdot\chi(t^{-1}D)$ is the commutator of $\chi(t^{-1}D)$ and f. Moreover, since $\chi^2-1\in C_0(\mathbb{R})$, the operators

$$f\cdot\big(\chi(t^{-1}D)^2-1\big) = \begin{pmatrix} f\cdot(U_{t,D}V_{t,D}-1) & 0 \\ 0 & f\cdot(V_{t,D}U_{t,D}-1) \end{pmatrix}, \tag{4-5}$$

$$\big(\chi(t^{-1}D)^2-1\big)\cdot f = \begin{pmatrix} (U_{t,D}V_{t,D}-1)\cdot f & 0 \\ 0 & (V_{t,D}U_{t,D}-1)\cdot f \end{pmatrix}, \tag{4-6}$$

are compact operators for any $f\in C_0(M)$.

Set

$$\begin{aligned} W_{t,D} &= \begin{pmatrix} 1 & U_{t,D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -V_{t,D} & 1 \end{pmatrix}\begin{pmatrix} 1 & U_{t,D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ e_{1,1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ P_{t,D} &= W_{t,D}e_{1,1}W_{t,D}^{-1} \\ &= \begin{pmatrix} 1-(1-U_{t,D}V_{t,D})^2 & (2-U_{t,D}V_{t,D})U_{t,D}(1-V_{t,D}U_{t,D}) \\ V_{t,D}(1-U_{t,D}V_{t,D}) & (1-V_{t,D}U_{t,D})^2 \end{pmatrix}. \end{aligned} \tag{4-7}$$

The formal difference of $P_{t,D}$ and $e_{1,1}$ gives an explicit representative of the K-homology class $[D]$; see [Willett and Yu 2020, Remark 8.3.15].

Now let $[p]-[q]\in K_0(C_0(M))$, where p,q are projections in $M_k(C_0(M)^+)$ such that $p-q\in M_k(C_0(M))$ for some $k\in\mathbb{N}$. To simplify notation, let $P_t = P_{t,D_k}$, where $D_k = D\otimes I_k$ for some $k\in\mathbb{N}$. It follows from line (4-4) that

$$\|[P_t,p]\|\to 0 \quad \text{and} \quad \|[P_t,q]\|\to 0 \quad \text{as } t\to\infty.$$

Consequently,

$$\|(P_t \cdot p)^2 - P_t \cdot p\| \to 0 \quad \text{and} \quad \|(P_t \cdot q)^2 - P_t \cdot q\| \to 0 \quad \text{as } t \to \infty. \tag{4-8}$$

Now we apply the difference construction in line (4-2) to define

$$a_{t,p,q} = d(P_t \cdot p, P_t \cdot q) \quad \text{and} \quad b_{p,q} = d(e_{1,1} \otimes p, e_{1,1} \otimes q),$$

where $e_{1,1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We see that $b_{p,q}^2 = b_{p,q}$, and $a_{t,p,q}$ is an almost idempotent in the sense that

$$\|a_{t,p,q}^2 - a_{t,p,q}\| \to 0 \quad \text{as } t \to \infty. \tag{4-9}$$

We further consider the difference construction of $a_{t,p,q}$ and $b_{p,q}$ by setting

$$d_{t,p,q} := d(a_{t,p,q}, b_{p,q}) \quad \text{and} \quad e := e_{1,3} \otimes I_k \otimes I_4,$$

where I_k denotes the $k \times k$ identity matrix and

$$e_{1,3} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

By the construction in (4-2) and (4-7), each entry of the matrix $d_{t,p,q} - e$ is a (noncommutative) polynomial of $U_{t,D}$, $V_{t,D}$, p and q. Each monomial term of the polynomials contains the factors $1 - U_{t,D}V_{t,D}$ (or $1 - V_{t,D}U_{t,D}$) and $p - q$. It follows from (4-5) and (4-6) that

$$d_{t,p,q} \in M_{32k}(\mathcal{K}^+) \quad \text{for all } t > 0,$$
$$d_{t,p,q} - e = d(a_{t,p,q}, b_{p,q}) - e_{1,3} \otimes I_{2k} \otimes I_4 \in M_{32k}(\mathcal{K}) \quad \text{for all } t > 0.$$

By line (4-8) and (4-9), $d_{t,p,q}$ is an almost idempotent in the sense that

$$\|d_{t,p,q}^2 - d_{t,p,q}\| \to 0 \quad \text{as } t \to \infty. \tag{4-10}$$

More precisely, we derive the following quantitative estimate of $\|d_{t,p,q}^2 - d_{t,p,q}\|$.

Lemma 4.2 [Wang et al. 2024a, Lemma 3.4] *With the same notation as above, if $p, q \in M_k(C_0(M)^+)$ be Lipschitz functions with Lipschitz constant L, then there exists a universal positive constant c_1 such that*

$$\|d_{t,p,q}^2 - d_{t,p,q}\| \leq \frac{c_1 L}{t}.$$

In particular, if $t > 4c_1 L$, Lemma 4.2 implies that

$$\|d_{t,p,q}^2 - d_{t,p,q}\| < \tfrac{1}{4}.$$

In particular, the spectrum of $d_{t,p,q}$ is contained in $\left\{z \in \mathbb{C} \mid \mathrm{Re}(z) \neq \frac{1}{2}\right\}$. Now we apply the standard holomorphic functional calculus to obtain a genuine idempotent from $d_{t,p,q}$. More precisely, since

$$\Theta(z) = \begin{cases} 0 & \text{if } \mathrm{Re}(z) < \frac{1}{2}, \\ 1 & \text{if } \mathrm{Re}(z) > \frac{1}{2}, \end{cases}$$

is holomorphic on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \neq \frac{1}{2}\}$, which contains the spectrum of $d_{t,p,q}$, the functional calculus $\Theta(d_{t,p,q})$ is well-defined by

$$\Theta(d_{t,p,q}) = \frac{1}{2\pi i}\int_\gamma (d_{t,p,q}-\xi)^{-1}d\xi, \tag{4-11}$$

where $\gamma = \{z \in \mathbb{C} \mid |z-1| = \frac{1}{2}\}$, and $\Theta(d_{t,p,q})$ is an idempotent.

Definition 4.3 [Willett and Yu 2020, Section 9.1] The index pairing (4-1) of the Dirac operator D and $[p]-[q] \in K_0(M)$ is explicitly given by

$$\langle [D], [p]-[q]\rangle = [\Theta(d_{t,p,q})]-[e] \in K_0(\mathcal{K}) = \mathbb{Z} \quad \text{for any } t > 4c_1L.$$

Now suppose that Riemannian manifold (M^n, g) has uniformly positive scalar curvature $\mathrm{Sc}_g \geq \sigma^2 > 0$. It follows from the Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{\mathrm{Sc}_g}{4}$$

that the Dirac operator D of M has a spectral gap at zero. In this case, we see that

$$\|\chi(t^{-1}D)^2 - 1\| \to 0 \quad \text{as } t \to 0.$$

Moreover, we have the following lemma from [Wang et al. 2024a].

Lemma 4.4 [Wang et al. 2024a, Lemma 3.4] *Let (M^n, g) be a complete, noncompact Riemannian manifold. If $p-q$ is supported on a compact set in M and $\mathrm{Sc}_g \geq \sigma^2 > 0$ on M, then there exists a universal constant c_2 such that*

$$\|d_{t,p,q} - e\| \leq \frac{c_2 t}{\sigma} \quad \text{for all } t > 0.$$

With the above preparation, we have the following key estimate, which shows that if a complete Riemannian manifold (M^n, g) has uniformly positive scalar curvature, then the pairing of its Dirac operator with a bundle with sufficiently small Lipschitz constant has zero index.

Proposition 4.5 *Suppose that (M^n, g) is a complete, noncompact, spin Riemannian manifold with uniformly positive scalar curvature $\mathrm{Sc}_g \geq \sigma^2 > 0$. Let D be the Dirac operator on M and $[p]-[q] \in K_0^L(M)$ with p, q projections in $M_k(C_0(M)^+)$. If*

$$L < \frac{\sigma}{16c_1c_2}, \tag{4-12}$$

then the index pairing satisfies $\langle [D], [p]-[q]\rangle = 0$.

Proof Let us retain the same notation from Definition 4.3. We have

$$\Theta(d_{t,p,q}) = \frac{1}{2\pi i}\int_\gamma (d_{t,p,q}-\xi)^{-1}d\xi,$$

with $\gamma = \{z \in \mathbb{C} \mid |z-1| = \frac{1}{2}\}$ for $t > 4c_1L$.

Let $t_0 = 4c_1L$. We have

$$\|d_{t_0,p,q} - e\| \leq \frac{c_2 t_0}{\sigma} \leq \frac{c_2 \cdot 4c_1 L}{\sigma} < \tfrac{1}{4}. \tag{4-13}$$

Moreover, we have

$$\begin{aligned} \Theta(d_{t_0,p,q}) - e &= \frac{1}{2\pi i}\int_\gamma ((d_{t_0,p,q}-\xi)^{-1} - (e-\xi)^{-1})\, d\xi \\ &= \frac{1}{2\pi i}\int_\gamma (d_{t_0,p,q}-\xi)^{-1}(e - d_{t_0,p,q})(e-\xi)^{-1}\, d\xi. \end{aligned} \tag{4-14}$$

Note that

$$(d_{t_0,p,q}-\xi)^{-1} = (e-\xi)^{-1}(1 + (d_{t_0,p,q}-e)(e-\xi)^{-1})^{-1}.$$

For any $\xi \in \gamma = \{z \mid |z-1| = \frac{1}{2}\}$, as e is a diagonal matrix with entries 0 or 1, we have

$$\|(e-\xi)^{-1}\| = \max\left\{\frac{1}{|\xi|}, \frac{1}{|1-\xi|}\right\} \leq 2.$$

Thus,

$$\|(d_{t_0,p,q}-\xi)^{-1}\| \leq 2 \cdot \|(1 + (d_{t_0,p,q}-e)(e-\xi)^{-1})^{-1}\| < 4 \tag{4-15}$$

by (4-13). Applying the inequalities (4-13) and (4-15) to the integral (4-14), we have

$$\|\Theta(d_{t_0,p,q}) - e\| < 1. \tag{4-16}$$

Recall that if two idempotents f_1 and f_2 in a C^*-algebra satisfy the inequality

$$\|f_1 - f_2\| < \frac{1}{\|2f_1 - 1\|},$$

then f_1 is equivalent to f_2; see [Blackadar 1998, Proposition 4.3.2]. In our case, since e is a projection,

$$\|2e - 1\| = 1.$$

In particular, the inequality in line (4-16) implies that

$$\|\Theta(d_{t_0,p,q}) - e\| < \frac{1}{\|2e-1\|}.$$

Hence $\Theta(d_{t_0,p,q})$ is equivalent to e. Thus $[\Theta(d_{t_0,p,q})] - [e] = 0 \in K_0(\mathcal{K})$, implying $\langle [D], [p]-[q]\rangle = 0$. □

Remark 4.6 Equivalently, we may state Proposition 4.5 as follows: if the index pairing $\langle [D], [p]-[q]\rangle$ is nonzero, then

$$L \geq \frac{\sigma}{16 c_1 c_2}. \tag{4-17}$$

4.2 Proof of the main theorem

Proof of Theorem 1.2 We first prove the even-dimensional case. Let (M, g) be a complete, noncompact spin Riemannian manifold of even dimension. For a given $r_0 > 0$, suppose that $\mathcal{D}(r_0) < \operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M)$. By the definition of filling radius in Definition 2.2, $[M]$ is nonzero in $H_n^{\mathrm{lf}}(N_{\mathcal{D}(r_0)}(M;\mathbb{Q}))$. Let f_{r_0} be the map constructed in line (3-1) and g_{r_0} the corresponding map constructed in Lemma 3.5. In particular,

$$g_{r_0} f_{r_0} \quad \text{is homotopic to} \quad i : M \to N_{\mathcal{D}(r_0)}(M).$$

Consequently,

$$[M] = i_*([M]) = (g_{r_0})_*\big((f_{r_0})_*([M])\big).$$

Since $[M]$ is nonzero in $H_n^{\mathrm{lf}}(N_{\mathcal{D}(r_0)}(M;\mathbb{Q}))$, it follows that $(f_{r_0})_*([M])$ is nonzero in $H_n^{\mathrm{lf}}(\mathcal{N}_{r_0};\mathbb{Q})$.

The Dirac operator D of M defines a K-homology class $[D]$ in $K_0(M)$. Under the natural isomorphism[2] $K_0(M)\otimes\mathbb{Q} \cong \bigoplus_{\text{even}} H_*^{\mathrm{lf}}(M,\mathbb{Q})$, the top-degree component of $[D]$ in $H_n^{\mathrm{lf}}(M,\mathbb{Q})$ coincides with the fundamental class $[M] \in H_n^{\mathrm{lf}}(M,\mathbb{Q})$. For example, see the Poincaré duality theorem in [Willett and Yu 2020, Section 9.6]. In particular, we conclude that $(f_{r_0})_*([D])$ is a nonzero element in $K_0(\mathcal{N}_{r_0})$, since its top-degree component in $H_n^{\mathrm{lf}}(\mathcal{N}_r,\mathbb{Q})$ is equal to the nonzero element $(f_{r_0})_*([M])$ in $H_n^{\mathrm{lf}}(\mathcal{N}_r,\mathbb{Q})$. By [Willett and Yu 2020, Theorem 9.2.8], the index pairing

$$K_0(\mathcal{N}_{r_0}) \otimes K^0(\mathcal{N}_{r_0}) \to \mathbb{Z}$$

is rationally nondegenerate as rationally the Ext-group vanishes. Thus there exists a K-theory class $\beta \in K^0(\mathcal{N}_{r_0})$ such that

$$\langle (f_{r_0})_*([D]), \beta\rangle \neq 0.$$

By Theorem 3.8, there exist L_m-Lipschitz projections $p, q \in M_k(C_0(\mathcal{N}_{r_0})^+)$ for some $k \in \mathbb{N}$ such that

$$\beta = [p] - [q] \in K^0(\mathcal{N}_{r_0}).$$

It follows that

$$0 \neq \langle (f_{r_0})_*([D]), \beta\rangle = \langle [D], f_{r_0}^*([p]-[q])\rangle.$$

By Lemma 3.7 and the inequality (3-6), we have

$$\operatorname{Lip}(f_{r_0}) \leq \frac{2}{\sqrt{3}}\frac{(m+1)^3}{r_0}. \tag{4-18}$$

It follows that

$$\operatorname{Lip}(f_{r_0}^*(p)) \leq \frac{2}{\sqrt{3}}\frac{(m+1)^3 L_m}{r_0} \quad \text{and} \quad \operatorname{Lip}(f_{r_0}^*(q)) \leq \frac{2}{\sqrt{3}}\frac{(m+1)^3 L_m}{r_0}.$$

Now by applying Proposition 4.5,

$$\frac{2}{\sqrt{3}}\frac{(m+1)^3 L_m}{r_0} \geq \frac{\sigma}{16 c_1 c_2}.$$

[2] This isomorphism is induced by the Chern character map.

Equivalently,

$$r_0 \leq \frac{32}{\sqrt{3}} \frac{c_1 c_2 (m+1)^3 L_m}{\sigma}.$$

To summarize, we have shown that

$$\text{if } r_0 > \frac{32}{\sqrt{3}} \frac{c_1 c_2 (m+1)^3 L_m}{\sigma}, \quad \text{then } \mathcal{D}(r_0) \geq \operatorname{Fill\,Rad}_{\mathbb{Q}}(M).$$

In other words,

$$\operatorname{Fill\,Rad}_{\mathbb{Q}}(M) \leq \mathcal{D}\left(\frac{32}{\sqrt{3}} \frac{c_1 c_2 (m+1)^3 L_m}{\sigma} \right).$$

This proves the theorem for the even-dimensional case.

Now assume that M is odd-dimensional, and consider the direct product manifold $(M \times \mathbb{R}, g + dt^2)$. Since the scalar curvature of g is uniformly bounded below by σ^2, it follows that the scalar curvature of $g + dt^2$ is also bounded below by σ^2. Recall that $f_r : M \to \mathcal{N}_r$ and $g_r : \mathcal{N}_r \to L^\infty(X)$ are given in line (3-1) and Lemma 3.5. We consider $\mathcal{N}_r \times \mathbb{R}$ equipped with the ℓ^1-metric $d_{\mathcal{N}_r} + d_{\mathbb{R}}$, where $d_{\mathcal{N}_r}$ is the standard ℓ^1-simplicial metric on $\mathcal{N}_r$. Then we obtain

$$\widetilde{f}_r : \ (M \times \mathbb{R}, g + dt^2) \ \to (\mathcal{N}_r \times \mathbb{R}, d_{\mathcal{N}_r} + d_{\mathbb{R}}), \qquad (x,t) \mapsto \left(f_r(x), \frac{t}{r} \right),$$

$$\widetilde{g}_r : (\mathcal{N}_r \times \mathbb{R}, d_{\mathcal{N}_r} + d_{\mathbb{R}}) \to (L^\infty(X) \times \mathbb{R}, d_{L^\infty(X)} + d_{\mathbb{R}}) \subset L^\infty(X \times \mathbb{R}), \quad (y,s) \mapsto (g_r(y), rs),$$

such that $\widetilde{f}_r$ is $2(m+1)^2/r$-Lipschitz and proper, and $\widetilde{g}_r \widetilde{f}_r$ is homotopic to the inclusion map within $N_{\mathcal{D}(r)}(X \times \mathbb{R})$. Now we equip $\mathcal{N}_r \times \mathbb{R}$ with a simplicial structure. Equip the vertex set of $\mathcal{N}_r$ with any total order. Then for any simplex $\Delta = \{v_{i_1}, \dots, v_{i_k}\}$ of $\mathcal{N}_r$ with increasing indices, we define corresponding simplices

$$\Delta^{r,j} = \{(v_{i_1}, r), \dots, (v_{i_j}, r), (v_{i_j}, r+1), \dots, (v_{i_k}, r+1)\}$$

for any $1 \leq j \leq k$ and $r \in \mathbb{Z}$. Let $d_{\mathcal{N}_r \times \mathbb{R}}$ be the standard simplicial ℓ^1-metric on the above simplicial complex $\mathcal{N}_r \times \mathbb{R}$. The map

$$T : (\mathcal{N}_r \times \mathbb{R}, d_{\mathcal{N}_r} + d_{\mathbb{R}}) \to (\mathcal{N}_r \times \mathbb{R}, d_{\mathcal{N}_r \times \mathbb{R}})$$

is given as follows. For any point $(x, r+t) = \left(\sum \lambda_\ell v_{e_\ell}, r+t \right) \in \Delta \times \mathbb{R}$ with $t \in [0,1]$ and $\sum \lambda_\ell = 1$, there exists an index j such that $s_{j-1} \leq 1 - t \leq s_j$, where $s_j = \sum_{\ell \leq j} \lambda_\ell$ denotes the partial sum. Then we map (x,t) into $\Delta^{r,j}$ by

$$T(x,t) = \sum_{\ell < j} \lambda_j \cdot (v_{i_\ell}, r) + (1 - t - s_{j-1}) \cdot (v_{i_j}, r) + (s_j - 1 + t) \cdot (v_{i_j}, r+1) + \sum_{\ell > j} \lambda_j \cdot (v_{i_\ell}, r+1).$$

Now it follows directly by the triangle inequality that under the map T,

$$\tfrac{1}{2}(d_{\mathcal{N}_r} + d_{\mathbb{R}}) \leq d_{\mathcal{N}_r \times \mathbb{R}} \leq 2(d_{\mathcal{N}_r} + d_{\mathbb{R}}).$$

Therefore,

$$\widetilde{f}_r : (M \times \mathbb{R}, g + dt^2) \to (\mathcal{N}_r \times \mathbb{R}, d_{\mathcal{N}_r \times \mathbb{R}}), \quad (x,t) \mapsto \left(f_r(x), \frac{t}{r} \right),$$

is $4(m+1)^2/r$-Lipschitz. Furthermore, if we denote by $\widetilde{d}$ the ℓ^2-Euclidean metric on $\mathcal{N}_r \times \mathbb{R}$ introduced in Section 3, then the map $\widetilde{f}_r$ is now $(8/\sqrt{3})(m+2)(m+1)^2/r$-Lipschitz with respect to $(\mathcal{N}_r \times \mathbb{R}, \widetilde{d})$. Applying the same argument as above, we obtain

$$\operatorname{Fill}\operatorname{Rad}_{\mathbb{Q}}(M \times \mathbb{R}) \leq \mathcal{D}\left(16 c_1 c_2 \cdot \frac{8}{\sqrt{3}} \frac{(m+2)(m+1)^2 L_{m+1}}{\sigma} \right).$$

This finishes the proof by Lemma 2.4. □

While the proof of Theorem 1.2 does not rely on the (coarse) Baum–Connes conjecture, the techniques used in the proof are closely related to those developed for the (coarse) Baum–Connes conjecture [Connes 1985; 1994; Kasparov 1988; Baum and Connes 2000; Guentner et al. 2024; Willett and Yu 2024]. In fact, a counterexample to Conjecture 1.1 could potentially lead to counterexamples to the Baum–Connes conjecture.

Appendix Metric invariants of Riemannian manifolds

In this appendix, we review a list of metric invariants of Riemannian manifolds and their relations with each other. We also briefly explain how these metric invariants are connected with some of the recent developments on metric invariants and positive scalar curvature problems. We refer the readers to [Gromov 1983; 2007; 2018a; 2018b; 2025; Guth 2007; 2010; 2011; 2017] for more systematic studies of these metric invariants.

A.1 Metric invariants

We first recall the definition of Uryson width due to Gromov [2007].

Definition A.1 Suppose that (M^n, g) is a Riemannian manifold. The k-th Uryson width of (M, g) is defined as

$$\operatorname{width}_k(M) = \inf_{f, P^k} \sup_{p \in P^k} \{\operatorname{diam}(f^{-1}(p)) \mid f : M \to P^k\}, \tag{A-1}$$

where P^k is a locally finite k-dimensional polyhedron and f is a continuous map from M to P^k. We then define the macroscopic dimension of M to be

$$\dim_{\mathrm{mac}}(M) = \inf_k \{k \mid \operatorname{width}_k(M) < \infty\}.$$

The Uryson width can be viewed as a generalization of the notion of diameter of Riemannian manifolds. Intuitively speaking, the k-th Uryson width of M quantifies the shortest "distance" from the n-dimensional Riemannian manifold (M^n, g) to a locally finite simplicial complex P^k of dimension k. In particular, $\operatorname{width}_0(M) = \operatorname{diam}(M)$, where $\operatorname{diam}(M) = \sup_{p,q \in M} d_g(p,q)$ is the diameter of (M, g).

Lemma A.2 *If (M^n, g) is a complete Riemannian manifold, then*

(1) $\mathrm{width}_{k+1}(M) \le \mathrm{width}_k(M)$ *for any* $k \ge 0$,

(2) $\mathrm{width}_0(M) = \mathrm{diam}(M)$ *and* $\mathrm{width}_k(M) = 0$ *for* $k \ge n$.

Proof First, it follows from definition that $\mathrm{width}_{k+1}(M) \le \mathrm{width}_k(M)$ for any k. For $\mathrm{width}_n(M)$, let us fix a locally finite triangulation of M and denote the resulting simplicial complex by P. Then it is clear that the identity map from M to P implies that $\mathrm{width}_n(M) = 0$. It follows by part (1) that $\mathrm{width}_k(M) = 0$ for all $k \ge n$. For $\mathrm{width}_0(M)$, we consider the constant map from M to a single point, which implies that $\mathrm{width}_0(M) = \mathrm{diam}(M)$. □

Lemma A.3 *Suppose that (M^n, g) is a complete Riemannian manifold. For any given $W > 0$, the following statements are equivalent*:

(1) $\mathrm{width}_k(M) < W$.

(2) *There exists an open cover $\mathcal{C}$ of M such that* $\mathrm{diam}(\mathcal{C}) < W$ *and the multiplicity of $\mathcal{C}$ is at most $k+1$.*

Proof We first prove (1) $\Longrightarrow$ (2). Since $\mathrm{width}_k(M) < W$, it follows from the definition of $\mathrm{width}_k(M)$ that there exists a k-dimensional locally finite simplicial complex P^k and a continuous map $f : M \to P^k$ such that

$$\mathrm{diam}(f^{-1}(p)) < W - 2\varepsilon$$

for any $p \in P$, where ε is a sufficiently small positive constant. We see that for each $p \in P$, there exists $r_p > 0$ such that

$$\mathrm{diam}(f^{-1}(B(p, r_p))) < W - \varepsilon,$$

where $B(p, r_p) = \{q \in P \mid \mathrm{dist}(p, q) < r_p\}$. Note that $\{B(p, r_p)\}_{p \in M}$ forms an open cover of P. Since $\dim(P) = k$, the cover $\{B(p, r_p)\}_{p \in M}$ admits a refinement open cover $\mathcal{C} = \{C_i\}_{i \in I}$ of P such that the multiplicity of $\mathcal{C}$ is no greater than $k+1$. Since $\mathrm{diam}(f^{-1}(B(p, r_p))) < W - \varepsilon$ for any $p \in P$ and any member $C_i \in \mathcal{C}$ is contained in a member of $\{B(p, r_p)\}_{p \in M}$, the collection $\mathcal{C}_M = \{f^{-1}(C_i)\}_{i \in I}$ forms an open cover of M with

$$\mathrm{diam}(\mathcal{C}_M) \le W - \varepsilon < W,$$

which is the desired open cover in (2).

Now we prove (2) $\Longrightarrow$ (1). Suppose that $\mathcal{C} = \{C\}_{i \in I}$ is an open cover of M such that the $\mathrm{diam}(\mathcal{C}) < W$ and the multiplicity of $\mathcal{C}$ is no greater than $k+1$. Let $\mathcal{N}$ be the nerve associated with the cover $\mathcal{C}$ and $\varphi : M \to \mathcal{N}$ the map constructed as in line (3-1). Then for each $p \in \mathcal{N}$, $\mathrm{diam}(\varphi^{-1}(p)) < W$ since $\varphi^{-1}(p) \subset C_i \subset \mathcal{C}$ for some $i \in I$. Moreover, $\mathcal{N}$ is a simplicial complex of dimension k. It follows that $\mathrm{width}_k(M) < W$. □

Remark A.4 By comparing Lemma A.3 with Remark 3.3, we see that

$$\dim_{\mathrm{mac}}(M) \le \dim_{\mathrm{asy}}(M), \tag{A-2}$$

where $\dim_{\mathrm{asy}}(M)$ is the asymptotic dimension of (M, g).

Lemma A.5 [Gromov 2020, 4.C] *If (M^n, g) is complete Riemannian manifold, then*

$$\operatorname{Fill}\operatorname{Rad}(M) \leq \tfrac{1}{2}\operatorname{width}_{n-1}(M). \tag{A-3}$$

Proof Let $f : M \to P^{n-1}$ be a continuous map such that

$$\operatorname{diam}(f^{-1}(p)) \leq \operatorname{width}_{n-1}(M) + 2\varepsilon.$$

Choose $\omega = \frac{1}{2}(\operatorname{width}_{n-1}(M) + 2\varepsilon)$. Let

$$C_f = ([0, \omega] \times M) \cup P /\sim$$

be the mapping cylinder, where (ω, x) is identified with $f(x)$ for each $x \in M$. We can extend the metric on M to a metric on C_f such that the length of each path $\{x\} \times [0, \omega]$ is ω. Since $\dim(P) = n-1$, we conclude that $\operatorname{Fill}\operatorname{Rad}(M) \leq \omega$. Since ε is arbitrary, it follows that $\operatorname{Fill}\operatorname{Rad}(M) \leq \frac{1}{2}\operatorname{width}_{n-1}(M)$. □

Conversely, we propose the following conjecture.

Conjecture A.6 *There exists a universal constant c_n such that*

$$\operatorname{width}_{n-1}(M) \leq c_n \operatorname{Fill}\operatorname{Rad}(M) \tag{A-4}$$

for all complete Riemannian manifolds (M^n, g).

Moreover, motivated by Llarull's theorem [1998], we introduce a metric invariant as follows.

Definition A.7 Given a complete Riemannian manifold (M^n, g), we define

$$\operatorname{Rad}_{\mathbb{S}^n}(M) = \inf_{f,R}\{R \mid f : M \to \mathbb{S}^n(R) \text{ with } \operatorname{Lip}(f) \leq 1 \text{ and } \deg(f) \neq 0\}, \tag{A-5}$$

where f is assumed to be locally constant near infinity; in particular, $\deg(f)$ is well-defined.

Lemma A.8 *If (M^n, g) is a complete Riemannian manifold, then*

$$\operatorname{Rad}_{\mathbb{S}^n}(M) \leq \frac{4}{\pi}\operatorname{Fill}\operatorname{Rad}(M). \tag{A-6}$$

Proof For any small $\varepsilon > 0$, we set $\boldsymbol{r} = \operatorname{Rad}_{\mathbb{S}^n}(M) - \varepsilon$. By the definition of $\operatorname{Rad}_{\mathbb{S}^n}(M)$, there exists a map $f : M \to \mathbb{S}^n(R)$ such that $\operatorname{Lip}(f) \leq 1$ and $\deg(f) \neq 0$. Note that Lemma 2.3 implies that

$$\operatorname{Fill}\operatorname{Rad}(M) \geq \operatorname{Fill}\operatorname{Rad}(\mathbb{S}^n(R)).$$

It is known that $\operatorname{Fill}\operatorname{Rad}(\mathbb{S}^n(R)) = \frac{1}{2}\boldsymbol{r}\arccos(-1/(n+1))$. See for example [Gromov 1983, page 8]. In particular, we have

$$\operatorname{Fill}\operatorname{Rad}(M) \geq \tfrac{1}{2}\boldsymbol{r}\arccos\left(-\frac{1}{n+1}\right).$$

Since $\varepsilon > 0$ can be arbitrarily small, we have

$$\operatorname{Fill}\operatorname{Rad}(M) \geq \tfrac{1}{2}\operatorname{Rad}_{\mathbb{S}^n}(M)\arccos\left(-\frac{1}{n+1}\right).$$

Note that $\frac{1}{2}\arccos(-1/(n+1)) \geq \frac{1}{4}\pi$ for all $n \geq 1$. It follows that

$$\operatorname{Fill}\operatorname{Rad}(M) \geq \tfrac{1}{4}\pi \operatorname{Rad}_{\mathbb{S}^n}(M). \qquad \square$$

Now we recall the definition of injectivity radius.

Definition A.9 Let (M^n, g) be a complete Riemannian manifold.

(1) We define the injectivity radius of M to be

$$\operatorname{Inj}(M) = \inf_{p\in M} \sup_r\{r \mid \exp_p \text{ is a diffeomorphism on } B(p,r) \subset T_pM\}.$$

(2) We define the conjugate radius of M to be

$$\operatorname{Conj}(M) = \inf_{p\in M} \sup_r\{r \mid \exp_p \text{ is an immersion on } B(p,r) \subset T_pM\}.$$

(3) We define the convexity radius of M to be

$$\operatorname{Conv}(M) = \inf_{p\in M} \sup_r\{r \mid \text{the metric ball } \mathcal{B}(p,r) \subset M \text{ is convex}\}.$$

Here $\exp_p$ is the exponential map from the tangent space T_pM to the manifold M and

$$\mathcal{B}(p,r) = \{x \in M \mid d_g(p,x) < r\}$$

is the metric ball of radius r centered at p.

It clear that

$$2\operatorname{Conv}(M) \leq \operatorname{Inj}(M) \leq \operatorname{Conj}(M).$$

Lemma A.10 *If (M^n, g) is a complete Riemannian manifold, then*

$$\operatorname{Inj}(M) \leq (n+2)\operatorname{Fill}\operatorname{Rad}(M). \tag{A-7}$$

Proof Let

$$R \leq \frac{\operatorname{Inj}(M)}{n+2} - \delta$$

for some small $\delta > 0$. If P is an $(n+1)$-dimensional locally finite simplicial complex such that $M \subset P \subset N_R(M) \subset L^\infty(M)$, then we can subdivide P into sufficiently small simplices such that any simplex $\triangle^{n+1} \subset P$, whose vertices are denoted by $p_0, p_1, \ldots, p_n$, satisfy $d(p_i, p_j) \leq 2\delta$ for all $0 \leq i, j \leq n+1$. In other words, we have extended the identity map $i : M \to M$ to a continuous map $f_0 : M \cup P^{(0)} \to M$, where $P^{(0)}$ is the 0-th skeleton of P. From now on, we denote the k-th skeleton of P by $P^{(k)}$. Now we shall inductively extend the map $i : M \to M$ to a continuous map $f : P \to M$.

Let $m_i \in M$ be a point that realizes the shortest distance between p_i and M. We connect m_i and m_j by a geodesic segment $\triangle^1(m_i, m_j)$. By mapping the 1-simplex $\triangle^1(p_i, p_j)$ to $\triangle^1(m_i, m_j)$, we obtain the desired extension $f_1 : M \cup P^{(1)} \to M$.

Now we need to fill in each triangle formed by three edges, say, $\triangle^1(m_0, m_1)$, $\triangle^1(m_1, m_2)$ and $\triangle^1(m_0, m_2)$. First, in the previous step, it follows from the triangle inequality that $d_g(m_i, m_j) < 2\delta + 2R$.

By applying the triangle inequality again, we see that each point on $\triangle_1(m_0, m_1)$ lies within distance $\frac{3}{2}(2R + 2\delta)$ of m_2. It follows that all three edges lie in the metric ball $\mathcal{B}\big(m_2, \frac{3}{2}(2R + 2\delta)\big)$ centered at m_2. Since $\frac{3}{2}(2R + 2\delta) < (n + 2)(R + \delta)$, the exponential map $\exp_{m_2}$ is a diffeomorphism from $B\big(0, \frac{3}{2}(2R + 2\delta)\big) \subset T_{m_2}M$ to $\mathcal{B}\big(m_2, \frac{3}{2}(2R + 2\delta)\big)$. By the contractibility of the ball, it follows that the map f_1 defined on the three edges $\triangle^1(p_0, p_1)$, $\triangle^1(p_1, p_2)$ and $\triangle^1(p_0, p_2)$ extends to the corresponding 2-simplex $\Delta^2(p_0, p_1, p_2)$ in P. Denote the image of $\Delta^2(p_0, p_1, p_2)$ in M by $\Delta^2(m_0, m_1, m_2)$. Repeating this process for each triangle, we obtain the desired extension $f_2 : M \cup P^{(2)} \to M$.

Now we want to fill in the nonsolid tetrahedron formed by, say, $\Delta^2(m_0, m_1, m_2)$, $\Delta^2(m_1, m_2, m_3)$, $\Delta^2(m_0, m_1, m_3)$ and $\Delta^2(m_0, m_2, m_3)$. Using the triangle inequality, a similar argument as above shows that every point in $\Delta^2(m_0, m_1, m_2)$ lies within distance $2(2R+2\delta)$ of m_3. It follows all $\Delta^2(m_0, m_1, m_2)$, $\Delta^2(m_1, m_2, m_3)$, $\Delta^2(m_0, m_1, m_3)$ and $\Delta^2(m_0, m_2, m_3)$ lie in the metric ball $\mathcal{B}(m_3, 2(2R + 2\delta))$ centered at m_3. Since $2(2R + 2\delta) < (n + 2)(R + \delta)$, the same argument in the previous inductive step shows that the map $f_2 : M \cup P^{(2)} \to M$ extends to a map $f_3 : M \cup P^{(3)} \to M$.

By repeating the above steps inductively, we eventually obtain an extension of the identity map $i : M \to M$ to a continuous map $f : P \to M$, as long as

$$R \leq \frac{\mathrm{Inj}(M)}{n+2} - \delta.$$

This implies M cannot bound any $(n+1)$-dimensional locally finite simplicial complex P in $N_R(M)$. It follows that

$$\mathrm{Fill\,Rad}(M) \geq \frac{\mathrm{Inj}(M)}{n+2}. \qquad \square$$

To summarize, we have the following inequalities of the above metric invariants:

$$\frac{\mathrm{Inj}(M)}{n+2} \leq \mathrm{Fill\,Rad}(M) \leq \mathrm{width}_k(M) \leq \mathrm{diam}(M) \quad \text{for } 0 \leq k \leq n-1, \tag{A-8}$$

$$\mathrm{Rad}_{\mathbb{S}^n}(M) \leq \frac{4}{\pi}\,\mathrm{Fill\,Rad}(M). \tag{A-9}$$

Motivated by the above inequalities, we propose the following conjecture.

Conjecture A.11 *There exists a universal constant c_n such that*

$$\mathrm{Inj}(M) \leq c_n\,\mathrm{Rad}_{\mathbb{S}^n}(M) \tag{A-10}$$

for all complete Riemannian manifolds (M^n, g).

A.2 Metric invariants and uniformly positive scalar curvature

In this subsection, we review some classical results and conjectures on the connections between (uniformly) positive scalar curvature and the various metric invariants that we considered in the previous subsection.

Let us start with the maximal diameter theorem of Myers and Cheng, which states that strictly positive Ricci curvature can control the diameter of the space. More precisely:

Theorem A.12 [Cheng 1975; Myers 1941] *If (M^n, g) is a complete Riemannian manifold with Ricci curvature $\mathrm{Ric}(g) \geq n-1$, then $\mathrm{diam}(M) \leq \pi$, and equality holds if and only if M is the unit round sphere $\mathbb{S}^n(1)$.*

The above maximal diameter theorem is a rigidity type result in the context of Ricci curvature bounded from below. Intuitively speaking, it states that uniformly positive Ricci curvature controls the diameter of the underlying space. Moreover, by applying the theorem to the universal cover, one sees that if a manifold admits a complete Riemannian metric with uniformly positive Ricci curvature, then its fundamental group is finite. Generally speaking, scalar curvature, being the trace of the Ricci tensor, imposes far fewer constraints on the topology and geometry of the underlying space. However, in some cases surprisingly, rigidity-type results persist in the context of scalar curvature bounded from below. See [Gromov 2018a; 2023] for more systematic surveys on this subject.

For example, the following theorem of Green shows that positive scalar curvature imposes constraints on the conjugate radius, hence the injectivity radius.

Theorem A.13 [Green 1963] *Suppose that (M^n, g) is a closed Riemannian manifold. If the scalar curvature is uniformly positive $\mathrm{Sc}(g) \geq n(n-1)$, then $\mathrm{Conj}(M) \leq \pi$, and equality holds if and only if M is isometric to the unit round sphere $\mathbb{S}^n(1)$.*

Since the injectivity radius satisfies $\mathrm{Inj}(M) \leq \mathrm{conj}(M)$, Theorem A.13 implies that for any closed Riemannian manifold with positive scalar curvature $\mathrm{Sc}_g \geq n(n-1)$, we have $\mathrm{Inj}(M) \leq \pi$, and equality holds if and only if M is isometric to the unit round sphere $\mathbb{S}^n(1)$. Note that the manifold M is assumed to be closed in Green's theorem. If one assumes that (M^n, g) is a complete, noncompact Riemannian manifold with uniformly positive scalar curvature $\mathrm{Sc}_g \geq n(n-1)$, then it is still an open question whether $\mathrm{Inj}(M) \leq \pi$. In fact, Gromov proposed the following stronger conjecture.

Conjecture A.14 [Gromov 2025] *Suppose that (M^n, g) is a complete, noncompact Riemannian manifold with uniformly positive scalar curvature $\mathrm{Sc}(g) \geq (n-1)(n-2)$. Then*

$$\mathrm{Inj}(M) \leq \pi,$$

and equality holds if and only if M is isometric to $(\mathbb{S}^{n-1}(1) \times \mathbb{R}, g_{\mathbb{S}^{n-1}} + dt^2)$.

The following theorem is essentially a formulation of Llarull's theorem on scalar rigidity of round spheres.

Theorem A.15 [Llarull 1998] *Suppose that (M^n, g) is a complete, spin, noncompact manifold with uniformly positive scalar curvature $\mathrm{Sc}_g \geq n(n-1)$. Then*

$$\mathrm{Rad}_{\mathbb{S}^n}(M) \leq 1,$$

and equality holds if and only if (M, g) is isometric to the unit round sphere $\mathbb{S}^n(1)$.

In particular, if we assume Conjecture A.11, then the above theorem of Llarull will give a universal (but possibly nonoptimal) upper bound on the injectivity radius, which would lead to a positive solution of a nonoptimal version of Conjecture A.14.

Moreover, in view of the inequalities from (A-8) and (A-9), the following Uryson width conjecture of Gromov is closely related to Conjecture A.14.

Conjecture A.16 (Gromov 2007; 2018a) *There exists a universal constant c_n such that*

$$\operatorname{width}_{n-2}(M) \leq c_n \tag{A-11}$$

for all complete Riemannian manifolds (M^n, g) with uniformly positive scalar curvature $\mathrm{Sc}_g \geq n(n-1)$.

An immediate consequence of Conjecture A.16 is the following conjecture.

Conjecture A.17 (Gromov–Lawson, Schoen–Yau) *If M^n is a closed aspherical manifold, then M admits no Riemannian metric with positive scalar curvature.*

Let us briefly comment on the current status of the various conjectures that have appeared in the present paper, and the relations among them. It follows from the metric inequalities (A-8) that Conjecture A.16 implies Conjecture 1.1, where the latter together with Example 2.6 implies Conjecture A.17. For three-dimensional manifolds, Conjecture A.16 has been confirmed in [Gromov and Lawson 1983; Gromov 2023; Liokumovich and Wang 2026]. Moreover, Conjecture A.17 has been proved for dimensions $n = 3, 4, 5$ [Gromov and Lawson 1983; Gromov 2020; Chodosh and Li 2024]. In dimension higher than 5, Conjecture A.17 remains open in general. On the other hand, by applying noncommutative geometric methods, Conjecture A.17 has been verified in all dimensions for a large class of aspherical manifolds whose fundamental group satisfies some geometric conditions. For example, the third author [Yu 1997; 1998; 2000] proved Conjecture A.17 for all n-dimensional aspherical manifolds whose fundamental group has finite asymptotic dimension or more generally whose fundamental group coarsely embeds into a Hilbert space. Moreover, in conjunction with Conjecture A.6, a weaker version of Conjecture A.16 that asserts $\operatorname{width}_{n-1}(M) \leq c_n$ instead of $\operatorname{width}_{n-2}(M) \leq c_n$ is equivalent to Conjecture 1.1. In this sense, the main theorem of the present paper could also be viewed as a solution to this weaker version of Conjecture A.16 for the class of manifolds with finite asymptotic dimension.

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