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On Donaldson's 4-6 question

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We prove that the examples of Smith (2000) and McMullen and Taubes (1999) provide infinitely many counterexamples to one direction of Donaldson's 4-6 question and the closely related stabilising conjecture. These are the first known counterexamples. In the other direction, we show that the Gromov–Witten invariants of two simply connected closed symplectic 4-manifolds agree if their products with $(S^2, \omega_{\text{std}})$ are deformation equivalent. In particular, when $b_2^+ \geq 2$, these 4-manifolds have the same Seiberg–Witten invariants. Furthermore, one can replace $(S^2, \omega_{\text{std}})$ by $(S^2, \omega_{\text{std}})^k$ for any $k \geq 1$ in both results.

1 Introduction

1.1 Main results

It is a classical fact of 4-manifold topology that two smooth closed simply connected 4-manifolds are homeomorphic if and only if they are diffeomorphic after taking the product with S^2, following [Smale 1962; Wall 1964; Freedman 1982]. The symplectic analogue of this result, attributed to Donaldson as Problem 5 of [McDuff and Salamon 2017, Section 14.1], has been an open problem for more than twenty-five years:

Donaldson's 4-6 question *Let (X_0, ω_0) and (X_1, ω_1) be two closed symplectic 4-manifolds such that X_0 and X_1 are homeomorphic. Then the product symplectic manifolds $(X_0 \times S^2, \omega_0 \oplus \omega_{\text{std}})$ and $(X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}})$ are deformation equivalent if and only if X_0 and X_1 are diffeomorphic.*

This problem appears in the seminal paper [Ruan and Tian 1997] for the first time as a conjecture, called the *stabilising conjecture*. In the same paper, Ruan and Tian show that the stabilising conjecture holds for simply connected elliptic surfaces. Additional supporting evidence for Donaldson's 4-6 question and the stabilising conjecture is given by [Ruan 1994; Ionel and Parker 1999; Cho 2014]. In contrast, we show the following:

Theorem 1 *There exist infinitely many pairwise nonhomeomorphic smooth closed simply connected 4-manifolds X admitting symplectic forms ω_0 and ω_1 such that the product forms $\omega_0 \oplus \omega_{\text{std}}^{\oplus k}$ and $\omega_1 \oplus \omega_{\text{std}}^{\oplus k}$ on $X \times (S^2)^k$ are deformation inequivalent for any $k \geq 1$.*

In particular, this gives the first known counterexamples to the "if" implication of Donaldson's 4-6 question, as well as the stabilising conjecture. In the "only if" implication, we significantly generalise previously known examples supporting the conjecture.

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Theorem 2 (Theorem 5.1) *Let (X_0, ω_0) and (X_1, ω_1) be two closed homeomorphic simply connected symplectic 4-manifolds so that $(X_0, \omega_0) \times (S^2, \omega_{\text{std}})$ and $(X_1, \omega_1) \times (S^2, \omega_{\text{std}})$ are deformation equivalent. Then the Gromov–Witten invariants of (X_0, ω_0) agree with those of (X_1, ω_1) up to a homeomorphism.*

The celebrated works of Taubes [2000b; 2000c] relating Seiberg–Witten invariants to pseudoholomorphic curves, combined with [Ionel and Parker 1997] and [Hirschi 2025, Theorem 6.3], give the following consequence.

Corollary 3 *If (X_0, ω_0) and (X_1, ω_1) satisfy the hypotheses of Theorem 2 and $b_2^+(X_j) \geq 2$ for $j = 0, 1$, then their Seiberg–Witten invariants agree up to homeomorphism.*

In a similar vein, we show that if two symplectic 4-manifolds have different Seiberg–Witten invariants, then their Gromov–Witten invariants remain different in arbitrary stabilisations; see Corollary 5.13. As a result, we generalise results of [Ruan and Tian 1997] and [Ionel and Parker 1999] on constructing infinitely many pairwise deformation-inequivalent[1] symplectic structures on a fixed smooth manifold in any given even dimension. Previously, this was done explicitly by considering the famous exotic elliptic surfaces constructed by [Fintushel and Stern 1998] and realising that a particular Gromov–Witten invariant in genus 1 does not vanish. Corollary 5.13 makes it easier to construct more examples, since we do not need to check for behaviours of specific moduli spaces under knot surgeries, but may rather conclude as soon as any two Seiberg–Witten invariants disagree. In particular, we can use the examples in the recent paper [Baykur and Hamada 2023] to show the following.

Corollary 4 *For any $k \geq 1$, $m \geq 6$ and $n \geq 5$, the smooth manifolds $\#_{2m+1}(S^2 \times S^2) \times (S^2)^k$ and $\#_{2n+1}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \times (S^2)^k$ admit infinitely many pairwise deformation inequivalent symplectic structures.*

The proofs of the two main results use fundamentally different strategies and techniques. Theorem 2 is one of the first applications of recent advances in symplectic Gromov–Witten theory, [Abouzaid et al. 2021; 2024; Hirschi and Swaminathan 2024; Hirschi 2025]. We will give a quick summary of Gromov–Witten invariants defined via global Kuranishi charts in Section 5.1. On the other hand, Theorem 1 is shown with classical techniques. The advantage of the classical approach lies in the fact that Gromov–Witten invariants, while being powerful symplectic deformation invariants, are very difficult to compute. In particular, there are few calculations of higher genus Gromov–Witten invariants in the symplectic category, which would be needed for at least one of our counterexamples as well as for minimal symplectic 4-manifolds by [Beyaz 2021, Theorem 3.5]. However, unexpectedly, we can use the first Chern class to show that the product symplectic forms are not deformation equivalent. Our proof analyses how diffeomorphisms act on the cohomology of product manifolds and offers a new strategy to distinguish symplectic structures on product manifolds.

[1] Some other related results are the following. In dimension 8, [McDuff 1987] constructed an example with infinitely many nonisotopic symplectic structures, where isotopy means a path of symplectic forms with a fixed cohomology class. Given a fixed arbitrary $k \geq 2$, [Smith 2000] also constructs k deformation inequivalent symplectic forms in any even dimensions.

1.2 Background

In general, spaces of symplectic structures are not well understood. Even in dimension 4, there are still many open questions discussed in, e.g., [Li 2008; Salamon 2013]. To simplify the problem of classifying symplectic structures, one can impose several equivalence relations on symplectic forms. We will work with the following one.

Definition 1.1 Two symplectic structures ω_0 on X_0 and ω_1 on X_1 are *deformation equivalent* if there exists a diffeomorphism $\varphi : X_0 \to X_1$ and a path of symplectic structures on X_0 from ω_0 to $\varphi^*\omega_1$.

A complete classification of symplectic structures up to deformation equivalence was achieved in the case of blowups of rational and ruled surfaces [McDuff 1990; Li and Liu 1995; Lalonde and McDuff 1996; Li 2008] building on work of Gromov [1985] and Taubes [1995]. See [McDuff 1997; Gompf and Stipsicz 1999; Fintushel and Stern 2009; McDuff and Salamon 2012, Section 9.4] for further results and references in the study of symplectic structures in dimension four.

In higher dimensions, smooth topology becomes much simpler since one can apply homotopy arguments; see [Smale 1962; Jupp 1973; Sullivan 1977]. However, many powerful tools of symplectic topology in dimension four are no longer readily available. In particular, we no longer have positivity of intersections and the adjunction inequality, which are used in almost all works on classifying closed symplectic 4-manifolds, e.g., [McDuff 1990; Li and Liu 1995; Lalonde and McDuff 1996]. Moreover, the fact that Seiberg–Witten invariants agree with certain Gromov–Witten invariants can be used to deduce the existence of pseudoholomorphic curves, which is a starting point for many arguments such as in [Taubes 1995; 2000a; Li and Liu 1995; Lalonde and McDuff 1996; Ohta and Ono 1996].

A natural way to go from four to higher dimensions is the so-called "stabilisation" procedure by taking products with $(S^2, \omega_{\text{std}})$. This is natural in the smooth category, since exotic smooth structures become diffeomorphic when stabilised, by the topological fact stated at the beginning of our article. In the symplectic category, since works of Taubes [2000b; 2000c] and Ionel and Parker [1997] identify Seiberg–Witten invariants with certain Gromov–Witten invariants, one could imagine that only the smooth information of the 4-manifolds is retained after stabilisation. More geometrically, a priori, one could expect for the additional factor of S^2 to provide enough space for deformation equivalence between the stabilised symplectic structures. In [Smith 2000], it was observed that S^2 cannot be replaced by $\mathbb{T}^2$ as the first Chern class of the latter vanishes, indicating that the case with S^2 requires additional considerations.

1.3 Strategy of proof

All manifolds are assumed to be connected in our paper. Let us elaborate on the counterexamples in Theorem 3.1, which are symplectic deformation inequivalent 4-manifolds. The first type of examples was constructed in [Smith 2000]. Starting with the standard torus $\mathbb{T}^4$, he perturbs the symplectic structure and then takes a symplectic fibre sum with $n+3$ copies of the rational elliptic surface $E(1)$. While the perturbation of the symplectic form does not affect the diffeomorphism type of the resulting manifold, denoted Z_n, it affects the resulting symplectic structures. Concretely, given $n \geq 2$, Smith obtains in

this way n symplectic structures on Z_n whose first Chern classes have coprime divisibilities. Since the signature of Z_n is $-8(n+3)$, they are pairwise nonhomeomorphic. This provides the examples of Theorem 1. The same arguments work for the manifolds constructed in [Vidussi 2001].

Our second type of examples is more topological in nature. McMullen and Taubes [1999] constructed a closed 3-manifold N that fibres in two different ways over S^1. They showed that $X := N \times S^1$ admits two symplectic forms ω_0 and ω_1 whose first Chern classes $c_1(X, \omega_0)$ and $c_1(X, \omega_1)$ lie in different orbits of the $\mathrm{Diff}(X)$-action on $H^2(X;\mathbb{Z})$. In fact, they lie on different orbits of the $\mathrm{Aut}(\pi_1(N))$-action as explained in Section 4. These examples show that the deformation equivalence classes remain distinct after stabilisation with any complex projective spaces; see Theorem 4.1. Note that the manifold X we use is not the main example of their paper.

While first Chern classes had been previously used to distinguish deformation classes on 4-manifolds, the key part of our proof was to realise that the action of the diffeomorphism group of the 6-manifold does, in certain examples, preserve the subring $H^*(X;\mathbb{Z}) \subset H^*(X \times S^2;\mathbb{Z})$. Together with the fact that the first Chern classes of the symplectic structures constructed by both Smith and McMullen–Taubes cannot even be related by homeomorphisms, we can deduce Theorem 1.

The proof of Theorem 2 uses the product formula of symplectic Gromov–Witten invariants recently developed by [Hirschi and Swaminathan 2024]. Additionally, we use the Kontsevich–Manin axioms of [Kontsevich and Manin 1994], proven in [Hirschi 2025] for the Gromov–Witten invariants that we use in this paper, to show that a certain Gromov–Witten invariant of S^2 evaluated on the point class does not vanish. This allows us to access the Gromov–Witten invariants of the 4-manifolds through the Gromov–Witten invariants of the 6-manifolds.

2 From four to higher dimensions

We call a continuous map a *cohomology equivalence* if it induces an isomorphism on singular cohomology with integral coefficients.

Definition 2.1 Given two spaces X and Y, define $G_{X,Y}$ to be the set of cohomology equivalences ψ of $X \times Y$ satisfying

(a) ψ^* maps $H^2(X;\mathbb{Z})$ to itself setwise,

(b) $\mathrm{pr}_1\, \psi(\cdot, y)$ is a cohomology equivalence on X for each $y \in Y$.

Subsequently, Y will always be path-connected, so we can consider $H^*(X;\mathbb{Z})$ as a subspace of $H^*(X \times Y;\mathbb{Z})$. We will omit the projection maps and inclusion to keep the notation uncluttered.

Let $\widetilde{G}_{X,Y} \subset G_{X,Y}$ be the H-group of homotopy equivalences ψ such that $\mathrm{pr}_1\, \psi(\cdot, y)$ is a homotopy equivalence on X for some (and hence any) $y \in Y$. By slight abuse of language, we call the map from the set of cohomology equivalences to $\mathrm{Aut}(H^*(X;\mathbb{Z}))$ the *action of cohomology equivalences*, and similarly for $\widetilde{G}_{X,Y}$.

Proposition 2.2 *Let X be a closed smooth manifold with two symplectic forms ω_0 and ω_1. Suppose $c_1(\omega_0)$ and $c_1(\omega_1)$ lie in different orbits of the action of cohomology equivalences (or homotopy equivalences) of X on $H^2(X;\mathbb{Z})$. Then $c_1(\omega_0 \oplus \omega_{\text{std}})$ and $c_1(\omega_1 \oplus \omega_{\text{std}})$ lie in different orbits of the action of $G_{X,\mathbb{C}P^k}$ (respectively $\widetilde{G}_{X,\mathbb{C}P^k}$) for any $k \geq 1$.*

Proof Denote by $h \in H^2(\mathbb{C}P^k;\mathbb{Z}) \subset H^2(X \times \mathbb{C}P^k;\mathbb{Z})$ the algebraic dual of $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^k;\mathbb{Z})$. Suppose

$$\psi^* c_1(\omega_1 \oplus \omega_{\text{std}}) = c_1(\omega_0 \oplus \omega_{\text{std}}) \tag{2.0.1}$$

for some $\psi \in G_{X,\mathbb{C}P^k}$ or $\widetilde{G}_{X,\mathbb{C}P^k}$, and define $\hat{\psi} := \text{pr}_1(\psi(\cdot, z))$ for some $z \in \mathbb{C}P^k$.

Then $\psi^* h = h + \alpha$ for some $\alpha \in H^2(X;\mathbb{Z})$ by the definition of $G_{X,\mathbb{C}P^k}$ and (2.0.1). Therefore,

$$0 = \psi^*(h^{k+1}) = (h+\alpha)^{k+1} = \sum_{j=1}^{k+1} \binom{k+1}{j} \alpha^j h^{k+1-j}.$$

By the Künneth isomorphism,

$$H^*(X \times \mathbb{C}P^k;\mathbb{Z}) \cong H^*(X)[h]/(h^{k+1}),$$

so $(k+1)\alpha = 0$. As $c_1(\mathbb{C}P^k, \omega_{\text{std}}) = (k+1)h$, (2.0.1) implies

$$c_1(\omega_0) + (k+1)h = \psi^* c_1(\omega_1) + (k+1)\psi^* h = \psi^* c_1(\omega_1) + (k+1)h.$$

Since ψ^* preserves $H^2(X;\mathbb{Z})$, this shows that $\hat{\psi}^* c_1(\omega_1) = \psi^* c_1(\omega_1) = c_1(\omega_0)$.[2] As $\hat{\psi}$ is a cohomology equivalence (respectively homotopy equivalence), we may conclude by contraposition. □

Proposition 2.3 *Let X, ω_0 and ω_1 be as in Proposition 2.2 and let $(Y, \omega_Y) = (\Sigma, \sigma)^k$ for some closed Riemann surface Σ with a symplectic form σ and $k \geq 1$. If Σ has positive genus, suppose additionally that X is simply connected. Then the first Chern classes $c_1(\omega_0 \oplus \omega_Y)$ and $c_1(\omega_1 \oplus \omega_Y)$ lie in different orbits of the action of $G_{X,Y}$ (respectively $\widetilde{G}_{X,Y}$).*

Proof We give the proof for the case $k = 2$. The same line of reasoning applies to higher products of Σ. Let $h_0, h_1 \in H^2(Y;\mathbb{Z})$ be the algebraic duals of the fundamental classes of the factors and abbreviate $\chi := \chi(\Sigma)$. Given $\psi \in G_{X,Y}$, or $\widetilde{G}_{X,Y}$, there exist $a_0, a_1, b_0, b_1 \in \mathbb{Z}$ and $\alpha, \beta \in H^2(X;\mathbb{Z})$ such that

$$\psi^* h_0 = a_0 h_0 + a_1 h_1 + \alpha, \quad \psi^* h_1 = b_0 h_0 + b_1 h_1 + \beta.$$

Then $a_0 b_1 - a_1 b_0 = \pm 1$ since this is the determinant of the composition

$$H^2(Y;\mathbb{Z}) \xrightarrow{\psi^*} H^2(X \times Y;\mathbb{Z}) \to H^2(Y;\mathbb{Z}),$$

which is an isomorphism by Definition 2.1(a) and the Künneth theorem. As $\psi^*(h_0)^2 = 0$, we have that

$$2\alpha(a_0 h_0 + a_1 h_1) = 0.$$

[2] To be precise, the equality is up to the canonical identification of $H^2(X;\mathbb{Z})$ with $H^2(X;\mathbb{Z}) \otimes H^0(\mathbb{C}P^k;\mathbb{Z})$.

Hence $2a_0\alpha = 2a_1\alpha = 0$, so

$$\pm 2\alpha = 2(a_0 b_1 - a_1 b_0)\alpha = 0.$$

Similarly, $\psi^*(h_1)^2 = 0$ and we see that $2\beta = 0$. Using the splitting of $H^2(X \times Y; \mathbb{Z})$ given by the Künneth theorem and Definition 2.1(a), the equality $\psi^* c_1(\omega_1 \oplus \omega_Y) = c_1(\omega_0 \oplus \omega_Y)$ implies that

$$c_1(\omega_0) = \psi^* c_1(\omega_1) + \chi\alpha + \chi\beta = \psi^* c_1(\omega_1). \tag{2.0.2}$$

However, (2.0.2) contradicts the choice of ω_0 and ω_1 by Definition 2.1(b). □

3 Simply connected examples

Using Section 2 we will first give a general theorem about the behaviour of deformation equivalence classes of symplectic structures under taking the product with $(S^2, \omega_{\text{std}})$. Then we explain the examples of [Smith 2000] and show that they satisfy our assumptions, completing the proof of Theorem 1. In fact, we can replace the sphere in said theorem by an arbitrary closed Riemann surface.

Theorem 3.1 *Let Σ be a closed surface with a symplectic form σ and X a smooth simply connected 4-manifold with nonvanishing signature. If ω_0 and ω_1 are two symplectic forms on X with $c_1(\omega_0)$ and $c_1(\omega_1)$ in different orbits of action of cohomology equivalences on $H^2(X;\mathbb{Z})$, then $\omega_0 \oplus \sigma^{\oplus k}$ and $\omega_1 \oplus \sigma^{\oplus k}$ on $X \times \Sigma^k$ are deformation inequivalent for any $k \geq 1$.*

Proof Since Σ is a surface, its first Pontryagin class $p_1(\Sigma)$ vanishes. Since $H^*(X \times \Sigma^k; \mathbb{Z})$ is torsion-free and Pontryagin classes satisfy the Whitney formula up to 2-torsion, we have

$$p_1(X \times \Sigma^k) = p_1(X) = 3\sigma(X)\,\mathrm{PD}([\Sigma^k]),$$

where the second equality follows from the Hirzebruch signature theorem and $\sigma(X)$ denotes the signature of X. As diffeomorphisms preserve the Pontryagin classes up to sign and $H_*(X \times \Sigma^k; \mathbb{Z})$ is free, any $\psi \in \mathrm{Diff}(X \times \Sigma^k)$ satisfies $\psi_*[\Sigma^k] = \pm[\Sigma^k]$. Given $\alpha \in H^2(X;\mathbb{Z})$ we have $\psi^*\alpha = \alpha' + a_1 h_1 + \cdots + a_k h_k$, for some $\alpha' \in H^2(X;\mathbb{Z})$ and $a_i \in \mathbb{Z}$, where $h_i \in H^2(\Sigma^k;\mathbb{Z})$ is the algebraic dual of the fundamental class of the i-th factor. Then

$$0 = \pm\alpha \cap [\Sigma^k] = \psi_*(\psi^*\alpha \cap [\Sigma^k]) = \psi_* \sum_{i=1}^{k} a_i [\Sigma \times \cdots \times \hat{\Sigma} \times \cdots \times \Sigma],$$

so $a_1 = \cdots = a_k = 0$. Thus ψ^* preserves $H^2(X;\mathbb{Z})$. Define $\hat{\psi} : X \to X$ by $\hat{\psi}(x) = \mathrm{pr}_1\, \psi(x, y)$ for some $y \in \Sigma^k$. As ψ preserves $[\Sigma^k]$ up to sign, $\hat{\psi}$ has degree ± 1. Then

$$\hat{\psi}^*\alpha \cdot \hat{\psi}^*\beta = \hat{\psi}^*(\alpha \cdot \beta) = \pm\alpha \cdot \beta$$

for any $\alpha, \beta \in H^2(X;\mathbb{Z})$, so $\hat{\psi}$ is a cohomology equivalence. This shows that $\psi \in G_{X,\Sigma^k}$, so the claim follows from Proposition 2.3. □

Remark 3.2 During a revision of this paper, we found a similar argument in the proof of the main theorem of [Ruan 1994], which shows that a minimal complex surface and a nonminimal one never become deformation inequivalent when stabilised with an arbitrary product of Riemann surfaces. However, our final conclusions and the overall strategy are quite different from his and rely essentially on the first Chern classes.

We now describe a construction of Smith [2000], which provides infinitely many examples of smooth 4-manifolds X satisfying the assumption of Theorem 3.1.

Proof of Theorem 1 By [Smith 2000, Theorem 1.2], for any $n \geq 2$ there exists a simply connected smooth 4-manifold Z with symplectic forms $\omega_1, \dots, \omega_n$ whose first Chern classes have pairwise different divisibilities. Specifically, given $n \geq 2$, the manifold Z is the fibre sum of $\mathbb{T}^4$, with the coordinates x, y, z and w, and $n+3$ copies of $E(1)$ along

$$T_x = \langle x, t \rangle, \quad T_y = \langle y, t \rangle, \quad T_z = \langle z, t \rangle$$

and n parallel copies of $T_w = \langle x = y = z, t \rangle$.

As $E(1)$ has signature -8, its first Pontryagin number is $p_1(E(1)) = -24$ by the Hirzebruch signature theorem [Milnor and Stasheff 1974], while $p_1(\mathbb{T}^4) = 0$. Thus, by [Gompf 1995, page 535], we have

$$\langle p_1(Z), [Z] \rangle = -24(n+3) \neq 0.$$

Now, applying Theorem 3.1 finishes the proof. □

Remark 3.3 The homotopy K3's constructed in [Vidussi 2001] are also simply connected and have nonvanishing first Pontryagin number. Thus they could also be used to prove Theorem 1.

4 McMullen and Taubes' example

This section expands on the discussion of the "further example" in [McMullen and Taubes 1999] and then proves the following variant of Theorem 1:

Theorem 4.1 *There exists a smooth 4-manifold X with symplectic forms ω_0 and ω_1 such that for any $k \geq 1$ the symplectic forms $\omega_0 \oplus \omega_{\mathrm{FS}}$ and $\omega_1 \oplus \omega_{\mathrm{FS}}$ on $X \times \mathbb{C}P^k$ are not deformation equivalent.*

As in [McMullen and Taubes 1999], by using the 0-surgery along the Borromean rings, define

$$M := \mathbb{T}^3 \setminus \mathcal{N}(L) \cong S^3 \setminus \mathcal{N}(K),$$

where $L = L_1 \sqcup L_2 \sqcup L_3 \sqcup L_4$ is the 4-component link consisting of four closed geodesics representing the three S^1-factors of $\mathbb{T}^3$ (and thus forming a basis of $H_1(\mathbb{T}^3; \mathbb{Z})$) and a fourth component satisfying $[L_4] = [L_1] + [L_2] + [L_3]$, and $\mathcal{N}(L)$ denotes a tubular neighbourhood of the link L. The link K has four components K_i given by the Borromean link together with an axis through them. Let $m_i \subset S^3$ be the meridian linking K_i positively. Then $m_1, \dots, m_4$ forms a basis of $H_1(M; \mathbb{Z})$.

We let $\pi : N \to \mathbb{T}^3$ be the double branched cover over L associated to the homomorphism

$$\xi : H_1(M;\mathbb{Z}) \to \{\pm 1\}$$

with $\xi(m_j) = 1$ for $1 \le j \le 3$ and $\xi(m_4) = -1$. By the discussion of [McMullen and Taubes 1999, page 11], the pullback $H^1(\mathbb{T}^3;\mathbb{R}) \to H^1(N;\mathbb{R})$ is an isomorphism. Recall that the *Euler class*[3] of a fibration $\rho : N \to S^1$ and its associated one-form $\alpha := d\rho$ is given by

$$e(\alpha) = [s^{-1}(0)] \in H_1(N;\mathbb{Z})/\text{torsion}$$

where $s : N \to \ker(\alpha)$ is a section transverse to the zero section.

We elaborate the discussion of the "further example" in [McMullen and Taubes 1999] in the following three lemmas.

Lemma 4.2 *There exist two closed one-forms, $\alpha_0, \alpha_1 \in H^1(N;\mathbb{Z})$, induced by submersions*

$$\rho_0, \rho_1 : N \to S^1$$

such that the associated Euler classes have different orbits under the $\mathrm{Aut}(\pi_1(N))$*-action on* $H_1(N;\mathbb{Z})$.

Proof The proof expands on the "further example" of [McMullen and Taubes 1999]. By the local model of a cover branched over a link, e.g., [Hempel 1992, page 121], a submersion $\rho' : \mathbb{T}^3 \to S^1$ with fibres transverse to L pulls back to a submersion $\rho : N \to S^1$. Since intersecting L transversely is a generic property, $\mathbb{Q}\langle\{[d\rho] \mid \rho : N \to S^1 \text{ a submersion}\}\rangle$ is dense in $H^1(N;\mathbb{R})$.

By [McMullen and Taubes 1999, Theorem 2.6], the pullback π^* identifies the unit ball, which is a polytope by [Thurston 1986, Theorem 2], in the Thurston norm

$$B_T(N) := \{\alpha \in H^1(N;\mathbb{R}) \mid \|\alpha\|_T \le 1\}$$

with (a scaling of) $B_L(\mathbb{T}^3) := \big\{\beta \in H^1(\mathbb{T}^3;\mathbb{R}) \mid \sum_{i=1}^4 \big|\int_{L_i} \beta\big| \le 1\big\}$.[4] For the definition of the Thurston norm, see [Thurston 1986] and expositions in [McMullen and Taubes 1999, Section 2].

Fix $\phi_0, \phi_1 \in H^1(\mathbb{T}^3;\mathbb{Z})$, which are represented by fibrations transverse to L, that live in the positive cones over polygonally distinct top-dimensional faces F_0 and F_1 of $B_L(\mathbb{T}^3)$. Set $\alpha_i := \pi^*\phi_i$. By [McMullen and Taubes 1999, Theorem 2.1], the corresponding face F_i' of $B_T(N)$ is contained in the affine hyperplane

$$H_i := \{\beta \in H^1(N;\mathbb{R}) \mid \langle \beta, e(\alpha_i)\rangle = -1\}.$$

By [McMullen and Taubes 1999, Theorems 2.1 and 2.2], we have $\|\alpha\|_A = 1$ for any α in a top-dimensional face of the polytope $B_T(N)$, where $\|\cdot\|_A$ denotes the Alexander norm on N, defined in [McMullen 2002]. Therefore, $B_T(N) = B_A(N)$. Since the Alexander norm is invariant under the action of $\mathrm{Aut}(\pi_1(N))$ on $H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ by definition, the action restricts to an action on $B_T(N)$. By assumption,

[3]This Euler class, as used in [McMullen and Taubes 1999], is Poincaré dual modulo torsion of the usual Euler class $e(d\rho) \in H^2(N;\mathbb{Z})$.

[4]$B_L(\mathbb{T}^3)$ was completely determined by McMullen and Taubes, and is pictured at the bottom of [McMullen and Taubes 1999, Figure 3].

F_0' and F_1' live in different orbits of this action, hence so do the affine hyperplanes H_0 and H_1. This shows that $e(\alpha_1) \notin \mathrm{Aut}(\pi_1(N)) \cdot e(\alpha_0)$. □

Define $X := N \times S^1_t$, where t denotes the coordinate on S^1.

Lemma 4.3 *To a closed one-form α represented by the differential of a fibre bundle $\rho : N \to S^1$ we can associate a symplectic form ω on X such that*

$$c_1(X, \omega) = \mathrm{PD}_X(e(\alpha) \times [S^1])$$

and ω is invariant under the canonical free S^1-action on X that leaves N invariant.

Proof This is a straightforward adaptation of the proof of [McMullen and Taubes 1999, Theorem 3.4] to the given example. As ρ is a submersion, N is identified with the mapping torus of a diffeomorphism ϕ on a fibre S of ρ. Let σ be an area form on S so that $[\sigma] \in H^2(S; \mathbb{Z})$. As the mapping class group of S is generated by Dehn twists, each element has a representative which is an area-preserving diffeomorphism. Hence, there exists a smooth homotopy $\{\phi_r\}_{r\in[0,1]}$ from $\phi_0 \in \mathrm{Symp}(S, \sigma)$ to $\phi = \phi_1$ that is constant near the ends. Set $\tilde{\phi} : S \times I \to S$, $(x, r) \mapsto \phi_r(x)$, and define $\tilde{\beta} := \tilde{\phi}^*\sigma$. Then $\tilde{\beta}$ descends to a 2-form β on N, so that $[\beta]$ admits an integral lift and β is nondegenerate on the fibres of ρ. Given $k \gg 0$, the form

$$\omega := k\alpha \wedge dt + \beta$$

is symplectic and invariant under the canonical S^1-action acting by rotation on the S^1_t-factor of X. In particular, we obtain a symplectically orthogonal splitting $TX = \ker(d\rho) \oplus \mathrm{pr}_2^* TS^1_t \oplus H$ where $H \cong \rho^* TS^1_\theta \cong \underline{\mathbb{R}}$, where θ is the coordinate on the base of the fibration $N \to S^1$. As $\alpha = \rho^* d\theta$, we may thus fix a β-tame complex structure on $\ker(d\rho)$ and extend it in the obvious way to $\underline{\mathbb{C}} = \mathrm{pr}_2^* TS^1 \oplus H$. This shows that

$$c_1(\omega) = \mathrm{pr}_1^* c_1(\ker(d\rho), \beta) = \mathrm{pr}_1^* \mathrm{PD}_N(e(\alpha)) = \mathrm{PD}_X(e(\alpha) \times [S^1]),$$

by definition of $e(\alpha)$. □

Recall that a space is *aspherical* if its universal cover is contractible. In particular, all homotopy groups, except possibly the fundamental group, are trivial.

Lemma 4.4 *The 4-manifold X equipped with the symplectic form of Lemma 4.3 is aspherical.*

Proof By Lemma 4.3 and the discussion at the beginning of [McCarthy 2001], X is either diffeomorphic to $S^2 \times \mathbb{T}^2$ or aspherical. As $H^1(X; \mathbb{R}) \cong H^1(\mathbb{T}^4; \mathbb{R})$, the 4-manifold is not isomorphic to $S^2 \times \mathbb{T}^2$. □

Remark 4.5 In particular, Lemma 4.4 implies that Gromov–Witten theory of X equipped with the Thurston symplectic form ω given in Lemma 4.3 is trivial in genus 0. However, it has nontrivial Gromov–Witten invariants of genus 1. This follows from [Taubes 2000c, Theorem 0.2(5)] and [Ionel and Parker 1997, Theorem 4.5].

Lemma 4.6 *Let α_0 and α_1 be the closed one-forms given in Lemma 4.2. Then for any homotopy equivalence ϕ on X, we have that*

$$\phi_*(e(\alpha_0)\times[S^1]) \neq e(\alpha_1)\times[S^1].$$

In particular, $\phi^* c_1(\omega_1) \neq c_1(\omega_0)$.

Proof Let $\phi : X \to X$ be a homotopy equivalence and define $\hat{\phi} := \mathrm{pr}_1\,\phi(\cdot, z)$ for some $z \in S^1$. By [McMullen and Taubes 1999, page 18], $\pi_1(S^1)$ is the centre of $\pi_1(X)$ and thus preserved by ϕ_* (up to outer automorphism). In particular, $\phi_*[S^1] = \pm[S^1]$. It follows that $\hat{\phi}_* : \pi_1(N) \to \pi_1(N)$ corresponds to the quotient morphism induced by ϕ_* on $\pi_1(N) \cong \pi_1(X)/\pi_1(S^1)$. In particular, it is an isomorphism and this construction induces an action of $\mathrm{Diff}(X)$ on $H_1(N;\mathbb{Z})$. As N is aspherical, $\hat{\phi}$ is thus a (weak) homotopy equivalence. Moreover, this shows that the action of $\mathrm{Diff}(X)$ (or the H-group of homotopy equivalences) on $H_1(N;\mathbb{Z})$ is given by $(\phi,\sigma) \mapsto \hat{\phi}_*\sigma$. Let $\mu \in H^1(S^1;\mathbb{Z})$ be the positive generator and write $\phi^*\mu = \pm\mu + \beta$ for some $\beta \in H^1(N;\mathbb{Z})$. The multiplicativity of the cap product shows that

$$\mu \cap \phi_*(\sigma\times[S^1]) = \pm\phi_*\sigma + \beta(\sigma)[S^1] = \pm\hat{\phi}_*\sigma + a[S^1] + \beta(\sigma)[S^1]$$

for some $a \in \mathbb{Z}$ depending on σ. Thus the equality $\phi_*(e(\alpha_0)\times[S^1]) = e(\alpha_1)\times[S^1]$ would imply that

$$e(\alpha_1) = \mu \cap (e(\alpha_1)\times[S^1]) = \mu \cap \phi_*(e(\alpha_0)\times[S^1]) = \pm\hat{\phi}_* e(\alpha_0). \tag{4.0.1}$$

This contradicts the choice of α_0 and α_1 by Lemma 4.2 on the nose when the sign in (4.0.1) is positive. For the negative case, observe that in the proof of Lemma 4.2, the fibered faces were distinguished by their polygonal information, implying that they also cannot be related by multiplication by -1. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1 We will show that the H-group of homotopy equivalences of $X\times\mathbb{C}P^k$ agrees with $\widetilde{G}_{X,\mathbb{C}P^k}$. Let ψ be any homotopy equivalence of $X\times\mathbb{C}P^k$. As X is aspherical by Lemma 4.4,

$$\pi_2(X\times\mathbb{C}P^k) \cong \pi_2(\mathbb{C}P^k) \cong \mathbb{Z}$$

on which ψ induces an isomorphism. Since the Hurewicz homomorphism is functorial, $\psi_* L = \pm L$, where L is the homology class of $\mathbb{C}P^1 \subset \mathbb{C}P^k$. Thus ψ^* preserves the annihilator of L, which is $H^2(X;\mathbb{Z})$. Define $\hat{\psi} : X \to X$ by $\hat{\psi}(x) = \mathrm{pr}_1\,\psi(x,z)$ for some $z \in \mathbb{C}P^k$. As $\pi_1(X) \cong \pi_1(X\times\mathbb{C}P^k)$, $\hat{\psi}$ is a weak homotopy equivalence and thus a homotopy equivalence. By Lemma 4.6 and Proposition 2.2, we may conclude. □

As in Section 3, these symplectic forms remain deformation inequivalent after arbitrarily many stabilisations.

Corollary 4.7 *The symplectic forms $\omega_0 \oplus \omega_{\mathrm{std}}^{\oplus k}$ and $\omega_1 \oplus \omega_{\mathrm{std}}^{\oplus k}$ on $X\times(S^2)^k$ are deformation inequivalent for any $k \geq 1$.*

Proof Let ψ be any homotopy equivalence of $X \times (S^2)^k$. As $\pi_2(X \times (S^2)^k) = \pi_2((S^2)^k)$, we obtain via the Hurewicz homomorphism that ψ_* preserves $H_2((S^2)^k; \mathbb{Z})$ and restricts to an isomorphism on it. Hence ψ^* preserves the annihilator of $H_2((S^2)^k; \mathbb{Z})$, which is exactly $H^2(X; \mathbb{Z})$. As above, $\hat{\psi} = \mathrm{pr}_1 \psi(\cdot, z)$ is a homotopy equivalence for any $z \in (S^2)^k$. Thus the claim follows from Lemma 4.6 and Proposition 2.3. □

Remark 4.8 This line of reasoning fails for the main example in [McMullen and Taubes 1999]. This is a simply connected 4-manifold Y with two symplectic forms ω_0' and ω_1' where $c_1(\omega_0')$ and $c_1(\omega_1')$ lie in different orbits of the $\mathrm{Diff}(Y)$-action. However, they can be related by a homeomorphism ϑ of Y. Combining [Wall 1964, Theorem 2] with [Smale 1962], one can show that $\vartheta \times \mathrm{id}_{S^2}$ is homotopic to a diffeomorphism ψ of $Y \times S^2$. Thus $\psi^* c_1(\omega_1' \oplus \omega_{\mathrm{std}}) = c_1(\omega_0' \oplus \omega_{\mathrm{std}})$. Nonetheless, they could be deformation inequivalent: Ruan [1994] showed that there exist 6-manifolds with inequivalent symplectic forms that have the same first Chern class.

5 From higher to four dimensions

This section is devoted to the proof of Theorem 2, which we first state more precisely.

Theorem 5.1 *Suppose (X_0, ω_0) and (X_1, ω_1) are closed homeomorphic simply connected symplectic 4-manifolds. If $(X_0, \omega_0) \times (S^2, \omega_{\mathrm{std}})$ and $(X_1, \omega_1) \times (S^2, \omega_{\mathrm{std}})$ are deformation equivalent, then there exists a homeomorphism $\phi : X_0 \to X_1$ such that $\phi^* c_1(\omega_1) = c_1(\omega_0)$, and for any $g, n \geq 0$ and $A \in H_2(X_0; \mathbb{Z})$ we have the equality of Gromov–Witten invariants*

$$\mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\phi^*\alpha_1, \dots, \phi^*\alpha_n)(\mathrm{c}) = \mathrm{GW}^{X_1,\omega_1}_{g,n,\phi_* A}(\alpha_1, \dots, \alpha_n)(\mathrm{c}) \tag{5.0.1}$$

for any $\alpha_1, \dots, \alpha_n \in H^*(X_1; \mathbb{Q})$ and $\mathrm{c} \in H_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$.

To introduce ingredients in the above theorem, we start with a short digression into Gromov–Witten theory. We will finish the proof of Theorem 5.1 in Section 5.2. At its very end, in Proposition 5.12, we state and prove the analogue of Theorem 5.1 with $(S^2, \omega_{\mathrm{std}})^k$ for $k \geq 1$.

5.1 Gromov–Witten invariants

Given a symplectic manifold (Y, σ) and an almost complex structure J tamed by σ, the *Gromov–Witten invariants* are symplectic invariants based on an intersection theory on the *moduli space of stable J-holomorphic maps* $\overline{\mathcal{M}}_{g,n}(Y, B, J)$, where $g, n \geq 0$ and $B \in H_2(Y; \mathbb{Z})$. These moduli spaces were first defined in [Kontsevich and Manin 1994]; we refer the reader to [McDuff and Salamon 2012, Chapters 5–7] for a detailed exposition in genus 0. Unfortunately, these moduli spaces are often highly singular, complicating the construction of the invariants, in particular in higher genus.

While there are many frameworks available, we will use the Gromov–Witten invariants constructed in [Hirschi and Swaminathan 2024] and their properties, shown in [Hirschi 2025]. In particular, in dimensions 4 and 6, they agree with the Gromov–Witten invariants constructed in [Ruan and Tian 1997].

The main result of [Hirschi and Swaminathan 2024, Theorem 1.1] asserts that we have the following model for $\overline{\mathcal{M}}_{g,n}(Y, B; J)$, called a *derived orbifold chart*.[5] It consists of a finite-rank orbibundle $\overline{\mathcal{E}}_n$ over an orbifold $\overline{\mathcal{T}}_n$, equipped with a section $\mathfrak{s}_n : \overline{\mathcal{T}}_n \to \overline{\mathcal{E}}_n$ such that

$$\mathfrak{s}_n^{-1}(0) \cong \overline{\mathcal{M}}_{g,n}(Y, B; J).$$

The *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(Y, B; J)]^{\mathrm{vir}} \in H_*(\overline{\mathcal{T}}_n; \mathbb{Q})$ is

$$[\overline{\mathcal{M}}_{g,n}(Y, B; J)]^{\mathrm{vir}} = \mathfrak{s}_n^* \tau(\overline{\mathcal{E}}_n) \cap [\overline{\mathcal{T}}_n],$$

where $\tau(\overline{\mathcal{E}}_n)$ is the (rational) Thom class of $\overline{\mathcal{E}}_n$.[6] The moduli space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ admits natural evaluation maps

$$\mathrm{ev}_j : \overline{\mathcal{M}}_{g,n}(Y, B; J) \to Y$$

evaluating a stable map at its j-th marked point, as well as a stabilisation map

$$\mathrm{st} : \overline{\mathcal{M}}_{g,n}(Y, B; J) \to \overline{\mathcal{M}}_{g,n}$$

which forgets the map u to Y and stabilises the domain. Both extend over the thickening $\overline{\mathcal{T}}_n$. The *Gromov–Witten homomorphisms* of (Y, σ) are the maps

$$\mathrm{GW}^{Y,\sigma}_{g,n,B} : H^*(Y; \mathbb{Q})^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

given by

$$\mathrm{GW}^{Y,\sigma}_{g,n,B}(\beta_1, \dots, \beta_n) := \mathrm{PD}\left(\mathrm{st}_*\left(\prod_{i=1}^{n} \mathrm{ev}_i^* \beta_i \cap [\overline{\mathcal{M}}_{g,n}(Y, B; J)]^{\mathrm{vir}}\right)\right)$$

for $\beta_1, \dots, \beta_n \in H^*(Y; \mathbb{Q})$. Its *Gromov–Witten invariants* are the (a priori rational) numbers

$$\mathrm{GW}^{Y,\sigma}_{g,n,B}(\beta_1, \dots, \beta_n)(\mathrm{c})$$

where $\mathrm{c} \in H_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$. The *full Gromov–Witten invariants* are given by evaluating at $\mathrm{c} = [\overline{\mathcal{M}}_{g,n}]$.

Remark 5.2 There is a slight subtlety due to the fact that $\overline{\mathcal{M}}_{g,n}(Y, B; J)$ may be nonempty even though $2g - 2 + n \leq 0$. If this is the case, one can still define the map $\mathrm{GW}^{Y,\sigma}_{g,n,B} : H^*(X^n; \mathbb{Q}) \to \mathbb{Q}$ by

$$\mathrm{GW}^{Y,\sigma}_{g,n,B}(\beta_1, \dots, \beta_n) = \left\langle \prod_{i=1}^{n} \mathrm{ev}_i^* \beta_i, [\overline{\mathcal{M}}_{g,n}(Y, B; J)]^{\mathrm{vir}} \right\rangle.$$

[5] To be precise, [Hirschi and Swaminathan 2024] uses the equivalent notion of a global Kuranishi chart where instead of an orbi-bundle orbifold $\overline{\mathcal{E}} \to \overline{\mathcal{T}}$, one has a G-vector bundle $\mathcal{E} \to \mathcal{T}$ over an almost free G-manifold.

[6] This is the definition of the virtual fundamental class we need for our applications; in [Hirschi 2025], a slightly different one is used, which becomes the same as soon as we integrate cohomology classes, which extend over the thickening $\overline{\mathcal{T}}$, over it. This will always be the case in our paper.

By [Hirschi 2025, Proposition 3.7], a mild generalisation of the divisor axiom stated below, we have for $n \geq 1$ that

$$\mathrm{GW}^{Y,\sigma}_{g,n,B}(\beta_1,\dots,\beta_n) = \frac{1}{\langle \beta, B\rangle^2}\,\mathrm{GW}^{Y,\sigma}_{g,n+2,B}(\beta_1,\dots,\beta_n,\beta,\beta)([\overline{\mathcal{M}}_{g,n+2}])$$

if $2g-2+n \leq 0$, where $\beta \in H^2(Y;\mathbb{Z})$ is any element such that $\langle \beta, B\rangle \neq 0$. Thus, in the proof of Theorem 5.1, it suffices to consider the case of $2g-2+n>0$, which we will do throughout Section 5.2.

We will never need the explicit form of the thickening, so we refer the interested reader to [Hirschi and Swaminathan 2024] for more details. The only properties we use are that the evaluation and stabilisation maps on $\overline{\mathcal{M}}_{g,n}(Y,B;J)$ lift to maps $\overline{\mathcal{T}}_n \to X^n$ and $\overline{\mathcal{T}}_n \to \overline{\mathcal{M}}_{g,n}$. Moreover, the map

$$\overline{\mathcal{M}}_{g,n+1}(Y,B;J) \to \overline{\mathcal{M}}_{g,n}(Y,B;J),$$

which forgets the last marked point, lifts to a map $\pi_{n+1}:\overline{\mathcal{T}}_{n+1}\to\overline{\mathcal{T}}_n$.

The key ingredient for Theorem 5.1 is the following product formula, shown in [Hirschi and Swaminathan 2024] except for $(g,n)\in\{(1,1),(2,0)\}$ and extended to these cases in [Hirschi 2023, Corollary 3.2.2].

Theorem 5.3 (product formula) *Suppose (Y_0,σ_0) and (Y_1,σ_1) are symplectic manifolds. Denoting the projection $Y_0\times Y_1\to Y_i$ by pr_i, we have*

$$\sum_{(\mathrm{pr}_i)_*B=B_i} \mathrm{GW}^{Y_0\times Y_1,\sigma_0\oplus\sigma_1}_{g,n,B}(\alpha_1\times\beta_1,\dots,\alpha_n\times\beta_n) = \mathrm{GW}^{Y_0,\sigma_0}_{g,n,B_0}(\alpha_1,\dots,\alpha_n)\cdot\mathrm{GW}^{Y_1,\sigma_1}_{g,n,B_1}(\beta_1,\dots,\beta_n) \tag{5.1.1}$$

in $H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ for any

$$B_0\in H_2(Y_0;\mathbb{Z}),\quad B_1\in H_2(Y_1;\mathbb{Z}),\quad \alpha_1,\dots,\alpha_n\in H^*(Y_0;\mathbb{Q}),\quad \beta_1,\dots,\beta_n\in H^*(Y_1;\mathbb{Q}).$$

In particular, if Y_1 is simply connected, we have

$$H_2(Y_0\times Y_1;\mathbb{Z})\cong H_2(Y_0;\mathbb{Z})\oplus H_2(Y_1;\mathbb{Z})$$

and (5.1.1) reduces to

$$\mathrm{GW}^{Y_0\times Y_1,\sigma_0\oplus\sigma_1}_{g,n,B_0+B_1}(\alpha_1\times\beta_1,\dots,\alpha_n\times\beta_n) = \mathrm{GW}^{Y_0,\sigma_0}_{g,n,B_0}(\alpha_1,\dots,\alpha_n)\cdot\mathrm{GW}^{Y_1,\sigma_1}_{g,n,B_1}(\beta_1,\dots,\beta_n).$$

Therefore, if the degree of the cohomology class $\mathrm{GW}^{Y_1,\sigma_1}_{g,n,B_1}(\beta_1,\dots,\beta_n)$ is 0, then

$$\mathrm{GW}^{Y_0\times Y_1,\sigma_0\oplus\sigma_1}_{g,n,(B_0,B_1)}(\alpha_1\times\beta_1,\dots,\alpha_n\times\beta_n) = \mathrm{GW}^{Y_1,\sigma_1}_{g,n,B_1}(\beta_1,\dots,\beta_n)([\mathrm{pt}])\,\mathrm{GW}^{Y_0,\sigma_0}_{g,n,B_0}(\alpha_1,\dots,\alpha_n). \tag{5.1.2}$$

The GW invariants satisfy a famous set of recursion relations, called the Kontsevich–Manin axioms [Kontsevich and Manin 1994]. These axioms were shown for the construction of GW invariants we use here in [Hirschi 2025, Section 2] and are used throughout our proof. We review the relevant axioms here for easy reference.

Effective If $\langle[\omega],B\rangle<0$, then $\mathrm{GW}^{Y,\sigma}_{g,n,B}=0$.

Fundamental class If 1_Y denotes the unit of $H^*(Y;\mathbb{Q})$ and π_{n+1} the map that forgets the $(n+1)$-st marked point, then

$$\mathrm{GW}^{Y,\sigma}_{g,n+1,B}(\alpha_1,\dots,\alpha_n,1_Y)(\mathrm{c}) = \mathrm{GW}^{Y,\sigma}_{g,n,B}(\alpha_1,\dots,\alpha_n)(\pi_{n+1*}\mathrm{c}).$$

for any $\alpha_1,\dots,\alpha_n \in H^*(Y;\mathbb{Q})$ and $\mathrm{c} \in H_*(\overline{\mathcal{M}}_{g,n+1};\mathbb{Q})$.

Mapping to a point in genus zero In the case $B = 0$, we have

$$\mathrm{GW}^{Y,\sigma}_{0,n,0}(\alpha_1,\dots,\alpha_n)([\mathrm{pt}]) = \langle \alpha_1 \cdots \alpha_n, [Y]\rangle. \tag{5.1.3}$$

Genus reduction Let $\psi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$ denote the map that creates a nonseparating node by gluing the last two marked points. Then

$$\mathrm{GW}^{Y,\sigma}_{g+1,n-2,B}(\alpha_1,\dots,\alpha_{n-2})(\psi_*\mathrm{c}) = \mathrm{GW}^{Y,\sigma}_{g,n,B}(\alpha_1,\dots,\alpha_{n-2},\mathrm{PD}(\Delta_Y))(\mathrm{c})$$

for any $\alpha_i \in H^*(Y;\mathbb{Q})$ and $\mathrm{c} \in H_*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$.

Divisor If $|\alpha_{n+1}| = 2$, then

$$\pi_{n+1!}\,\mathrm{GW}^{X,\sigma}_{g,n+1,B}(\alpha_1,\dots,\alpha_n) = \langle \alpha_{n+1}, B\rangle\,\mathrm{GW}^{X,\sigma}_{g,n,B}(\alpha_1,\dots,\alpha_n).$$

for $\alpha_1,\dots,\alpha_n \in H^*(Y;\mathbb{Q})$.

Splitting Write $\mathrm{PD}(\Delta_Y) = \sum_k \gamma_k \times \gamma_k'$ with $\gamma_k, \gamma_k' \in H^*(Y;\mathbb{Q})$. Let

$$\varphi : \overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1} \to \overline{\mathcal{M}}_{g,n}$$

be the associated map that glues two curves together at their (n_0+1)-st and first marked point respectively, and renumbers the remaining n_1 marked points on the second curve. Then

$$\begin{aligned}\mathrm{GW}^{Y,\sigma}_{g,n,B}&(\alpha_1,\dots,\alpha_n)(\varphi_*(\mathrm{c}_0\otimes\mathrm{c}_1))\\ &= \sum_{\substack{B_0+B_1=B\\ k}} \mathrm{GW}^{Y,\sigma}_{g_0,n_0+1,B_0}(\alpha_1,\dots,\alpha_{n_0},\gamma_k)(\mathrm{c}_0)\cdot \mathrm{GW}^{Y,\sigma}_{g_1,n_1+1,B_1}(\gamma_k',\alpha_{n_0+1},\dots,\alpha_n)(\mathrm{c}_1).\end{aligned}$$

for any $\mathrm{c}_0 \in H_*(\overline{\mathcal{M}}_{g_0,n_0+1};\mathbb{Q})$ and $\mathrm{c}_1 \in H_*(\overline{\mathcal{M}}_{g_1,n_1+1};\mathbb{Q})$.

We will apply (5.1.2) to $X_0 \times S^2$ and $X_1 \times S^2$. However, we need to first prove a nonvanishing result for the GW invariants of S^2 for any genus.

Lemma 5.4 *Given $g \geq 0$, $n \geq 1$ with $2g-2+n > 0$ and $d = \lceil \frac{1}{2}(g-1) \rceil$ we have*

$$\mathrm{GW}^{S^2}_{g,n,d}(1,\dots,1)([\mathrm{pt}]) = 2^g$$

if g is odd, while

$$\mathrm{GW}^{S^2}_{g,n,d}(1,\dots,1,h)([\mathrm{pt}]) = 2^g$$

if g is even.

Proof We prove the claim by induction on the genus g. In the case of $g = 0$, we have by the mapping to a point axiom that

$$\mathrm{GW}^{S^2}_{0,n,0}(1,\dots,1,h)([\mathrm{pt}]) = \langle h, [S^2]\rangle = 1.$$

This shows the base case. As we will use it later in the proof, let us also explain why

$$\mathrm{GW}^{S^2}_{0,3,d}(h,h,h)([\mathrm{pt}]) = \delta_{1,d}. \tag{5.1.4}$$

If $d > 1$, $\mathrm{GW}^{S^2}_{0,3,d}(h,h,h) = 0$ for degree reasons, while $\mathrm{GW}^{S^2}_{0,3,0}(h,h,h)([\mathrm{pt}]) = \langle h^3, [S^2]\rangle = 0$. To see the case $d = 1$, we observe that $\overline{\mathcal{M}}_{0,n}(S^2, [S^2])$ is regular by [McDuff and Salamon 2012, Proposition 7.4.3]. Thus, $\mathrm{GW}^{S^2}_{0,3,1}(h,h,h)([\mathrm{pt}])$ is the number of holomorphic spheres passing through three fixed general points on S^2. Since there exists a unique such curve up to automorphism, we obtain the case $d = 1$ and thus (5.1.4).

For the inductive step, i.e., the claim in higher genus, we need some of the Kontsevich–Manin axioms. The fundamental class axiom with $\mathrm{c} = [\mathrm{pt}]$ is

$$\mathrm{GW}^{S^2,\omega}_{g,n+1,B}(\alpha_1,\dots,\alpha_n,1_{S^2})([\mathrm{pt}]) = \mathrm{GW}^{S^2,\omega}_{g,n,B}(\alpha_1,\dots,\alpha_n)([\mathrm{pt}]). \tag{5.1.5}$$

For the sphere S^2, the splitting axiom takes the form

$$\begin{aligned}\mathrm{GW}^{S^2,\omega}_{g,n,d[S^2]}&(\alpha_1,\dots,\alpha_n)([\mathrm{pt}]) \\ &= \sum_{d_1+d_2=d} \mathrm{GW}^{S^2,\omega}_{0,3,d_1[S^2]}(\alpha_1,\alpha_2,h)([\mathrm{pt}])\cdot \mathrm{GW}^{S^2,\omega}_{g,n-1,d_2[S^2]}(1,\alpha_3,\dots,\alpha_n)([\mathrm{pt}]) \\ &\quad + \sum_{d_1+d_2=d} \mathrm{GW}^{S^2,\omega}_{0,3,d_1[S^2]}(\alpha_1,\alpha_2,1)([\mathrm{pt}])\cdot \mathrm{GW}^{S^2,\omega}_{g,n-1,d_2[S^2]}(h,\alpha_3,\dots,\alpha_n)([\mathrm{pt}])\end{aligned} \tag{5.1.6}$$

Since

$$\mathrm{vdim}(\overline{\mathcal{M}}(S^2,d)) = 2(1-3) + 4d + 6 = 2 + 4d,$$

we have

$$\mathrm{GW}^{S^2,\omega}_{g,3,d_i[S^2]}(\alpha_1,\alpha_2,\gamma)([\mathrm{pt}]) = 0$$

unless $|\alpha_1| + |\alpha_2| + |\gamma| = 2 + 4d_i$, where $\gamma = 1$ or h. As

$$\mathrm{GW}^{S^2,\omega}_{0,3,0}(h,h,1)([\mathrm{pt}]) = 0$$

by the mapping to a point axiom and

$$\mathrm{GW}^{S^2,\omega}_{0,3,[S^2]}(h,h,h)([\mathrm{pt}]) = 1$$

by (5.1.4), equation (5.1.6) simplifies to

$$\begin{aligned}\mathrm{GW}^{S^2,\omega}_{g,n,d[S^2]}(h,h,\alpha_3,\dots,\alpha_n)([\mathrm{pt}]) &= \mathrm{GW}^{S^2,\omega}_{0,3,0}(h,h,1)([\mathrm{pt}])\,\mathrm{GW}^{S^2,\omega}_{g,n-1,d[S^2]}(h,\alpha_3,\dots,\alpha_n)([\mathrm{pt}]) \\ &\quad + \mathrm{GW}^{S^2,\omega}_{0,3,[S^2]}(h,h,h)([\mathrm{pt}])\,\mathrm{GW}^{S^2,\omega}_{g,n-1,(d-1)[S^2]}(1,\alpha_3,\dots,\alpha_n)([\mathrm{pt}]) \\ &= \mathrm{GW}^{S^2,\omega}_{g,n-1,(d-1)[S^2]}(\alpha_3,\dots,\alpha_n)([\mathrm{pt}]),\end{aligned} \tag{5.1.7}$$

where the second equality is by (5.1.5). Meanwhile, the genus reduction axiom becomes

$$\begin{aligned}(5.1.8)\quad &\mathrm{GW}^{S^2,\omega}_{g,n,d[S^2]}(\alpha_1,\dots,\alpha_n)([\mathrm{pt}])\\ &\quad=\mathrm{GW}^{S^2,\omega}_{g-1,n+2,d[S^2]}(\alpha_1,\dots,\alpha_n,h,1)([\mathrm{pt}])+\mathrm{GW}^{S^2,\omega}_{g-1,n+2,d[S^2]}(\alpha_1,\dots,\alpha_n,1,h)([\mathrm{pt}])\\ &\quad=2\,\mathrm{GW}^{S^2,\omega}_{g-1,n+1,d[S^2]}(\alpha_1,\dots,\alpha_n,h)([\mathrm{pt}]),\end{aligned}$$

where we use that $\mathrm{PD}(\Delta_{S^2})=h\times 1+1\times h$ and apply (5.1.5).

We can now prove the inductive step. Suppose $g\geq 1$ and that the claim holds for $g'<g$. If g is odd, apply (5.1.8) once, to obtain that

$$(5.1.9)\qquad \mathrm{GW}^{S^2,\omega}_{g,n,d[S^2]}(1,\dots,1)([\mathrm{pt}])=2\,\mathrm{GW}^{S^2,\omega}_{g-1,n+1,d[S^2]}(1,\dots,1,h)([\mathrm{pt}])=2^g.$$

If g is even, we compute

$$\begin{aligned}&\mathrm{GW}^{S^2,\omega}_{g,n,d[S^2]}(1,\dots,1,h)([\mathrm{pt}])\\ &\quad=2\,\mathrm{GW}^{S^2,\omega}_{g-1,n+2,d[S^2]}(1,\dots,1,h,h)([\mathrm{pt}])=2\,\mathrm{GW}^{S^2,\omega}_{g-1,n+2,d[S^2]}(h,h,1,\dots,1)([\mathrm{pt}])\\ &\quad=2\,\mathrm{GW}^{S^2,\omega}_{0,3,[S^2]}(h,h,h)([\mathrm{pt}])\,\mathrm{GW}^{S^2,\omega}_{g-1,n+1,(d-1)[S^2]}(1,\dots,1)([\mathrm{pt}])\quad\text{(by (5.1.7))}\\ &\quad=2^{1+g-1},\end{aligned}$$

using that $\lceil\frac{1}{2}(g-1)\rceil=\lceil\frac{1}{2}((g-1)-1)\rceil+1$ for even g. This proves the inductive step and thus the claim. □

5.2 Proof of Theorem 5.1

Recall that (X_0,ω_0) and (X_1,ω_1) are simply connected 4-manifolds and we are given a diffeomorphism $\tilde{\phi}:X_0\times S^2\to X_1\times S^2$ such that $\tilde{\phi}^*(\omega_1\oplus\omega_{\mathrm{std}})$ is homotopic to $\omega_0\oplus\omega_{\mathrm{std}}$. In particular,

$$\tilde{\phi}^*c_1(\omega_1\oplus\omega_{\mathrm{std}})=c_1(\omega_0\oplus\omega_{\mathrm{std}}).$$

Remark 5.5 Throughout the proof, we will identify elements of $H^*(X_j;\mathbb{Z})$ and $H^*(S^2;\mathbb{Z})$ with their image in $H^*(X_j\times S^2;\mathbb{Z})$. In particular, the notation does not distinguish between the cross and the cup product.

Recall from the beginning of Section 1.1 that if $X_0\times S^2$ and $X_1\times S^2$ are diffeomorphic and simply connected, then X_0 and X_1 are homeomorphic. We distinguish three cases to prove the existence of the homeomorphism ϕ in the statement of Theorem 5.1:

- when $\sigma(X_0)\neq 0$ (Lemma 5.6);
- when X_0 is homeomorphic to $\#_{2n+1}(\mathbb{C}P^2\,\#\,\overline{\mathbb{C}P}^2)$ for some $n\geq 0$ (Lemma 5.8);
- when X_0 is homeomorphic to $\#_{2m+1}(S^2\times S^2)$ for some $m\geq 0$ (Lemma 5.9).

In the first case, we can rely on an algebraic argument similar to the one used in the proof of Theorem 3.1.

Lemma 5.6 *Suppose that $(X_0,\omega_0)\times(S^2,\omega_{\mathrm{std}})$ and $(X_1,\omega_1)\times(S^2,\omega_{\mathrm{std}})$ are deformation equivalent via a diffeomorphism $\tilde\phi$. If X_0 and X_1 are 4-manifolds with nonvanishing signatures, then there exists a homeomorphism $\phi: X_0\to X_1$ such that $\tilde\phi^*: H^*(X_1\times S^2;\mathbb{Z})\to H^*(X_0\times S^2;\mathbb{Z})$ agrees with the pullback by $\phi\times \mathrm{id}_{S^2}$ and $\phi^*c_1(\omega_1)=c_1(\omega_0)$.*

Proof The beginning of the proof is similar to the proof of Theorem 3.1. As the first Pontryagin class is a smooth invariant, $\tilde\phi^* p_1(X_1\times S^2)=p_1(X_0\times S^2)$. We have

$$p_1(X_j\times S^2)=3\sigma(X_j)\,\mathrm{PD}([S^2]),$$

because S^2 is a Riemann surface and $H^*(X_j\times S^2;\mathbb{Z})$ is torsion-free. Hence, $\tilde\phi_*[S^2]=\pm[S^2]$, so

$$\tilde\phi^*(H^2(X_1;\mathbb{Z}))\subset H^2(X_0;\mathbb{Z}). \tag{5.2.1}$$

Thus, $\tilde\phi^*h=ah+\beta$ for some $\beta\in H^2(X_0;\mathbb{Z})$ and some $a\neq 0$. Since $h^2=0$ we have $2\beta=0$ and since $H^2(X_0\times S^2;\mathbb{Z})$ is torsion-free, this shows that $\tilde\phi^*h=ah$. As $\tilde\phi$ maps $c_1(\omega_1)+2h$ to $c_1(\omega_0)+2h$, we must have $a=1$ as well as $\tilde\phi^*c_1(\omega_1)=c_1(\omega_0)$. The induced map $H^*(X_1;\mathbb{Z})\to H^*(X_0;\mathbb{Z})$ is an isomorphism preserving the intersection form (up to sign) and therefore is induced by a homeomorphism $\phi: X_0\to X_1$ by [Freedman 1982]. □

Suppose now X_0 (and thus also X_1) has vanishing signature. We need the following observation.

Lemma 5.7 *Let (X,ω) be a closed simply connected symplectic 4-manifold with signature zero. Then X is either homeomorphic to $\#_{2m+1}(S^2\times S^2)$ or homeomorphic to $\#_{2n+1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ for some $m,n\geq 0$.*

Proof As X is symplectic, it admits an almost complex structure and $b_2^+(X)$ is odd; see e.g., [Gompf and Stipsicz 1999, Exercise 1.4.16(b)]. Theorem 1.5 of [Freedman 1982], combined with the classification of indefinite unimodular symmetric bilinear forms [Serre 1973, Section V.2.2], implies that X is homeomorphic to $\#_{2m+1}(S^2\times S^2)$ if X has an even intersection form, and to $\#_{2n+1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ otherwise. □

Lemma 5.8 *Suppose that $(X_0,\omega_0)\times(S^2,\omega_{\mathrm{std}})$ and $(X_1,\omega_1)\times(S^2,\omega_{\mathrm{std}})$ are deformation equivalent via a diffeomorphism $\tilde\phi$. If X_0 has vanishing signature and is not homeomorphic to $\#_{2m+1}(S^2\times S^2)$, then there exists a homeomorphism $\phi: X_0\to X_1$ with $\phi^*c_1(\omega_1)=c_1(\omega_0)$ such that $\tilde\phi^*=\phi^*\otimes \mathrm{id}^*_{S^2}$.*

Proof By Lemma 5.7, X_0 and X_1 are homeomorphic to $\#_{2n+1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ for some $n\geq 0$. Let $\tilde\phi: X_0\times S^2\to X_1\times S^2$ be a diffeomorphism such that $\tilde\phi^*(\omega_1\oplus\omega_{\mathrm{std}})$ is homotopic to $\omega_0\oplus\omega_{\mathrm{std}}$. Then $\tilde\phi^*h=ah+\alpha$ for some $a\in\mathbb{Z}$ and $\alpha\in H^2(X_0;\mathbb{Z})$ with $2a\alpha=0$. Similarly, $(\tilde\phi^{-1})^*h=bh+\beta$ for some $b\in\mathbb{Z}$ and $\beta\in H^2(X_1;\mathbb{Z})$ with $2b\beta=0$. As $H^2(X_0;\mathbb{Z})$ and $H^2(X_1;\mathbb{Z})$ are torsion free, $a\neq 0$ implies $\alpha=0$. Thus, if $a\neq 0$, then

$$h=(\tilde\phi^{-1})^*(ah)=bah+a\beta,$$

so $a=b\in\{\pm1\}$ and $\beta=0$ as well. The same holds if $b\neq 0$.

Suppose thus that $a=b=0$. Then $\tilde\phi_*^{-1}[S^2]=A$ for some $A\in H_2(X_0;\mathbb{Z})$ and $\tilde\phi^*h=\alpha$. Let $\gamma\in H^2(X_1;\mathbb{Z})$ be such that $\langle\gamma^2,[X_1]\rangle=1$. Write $\tilde\phi^*\gamma=\tilde\gamma+ch\in H^2(X_0\times S^2;\mathbb{Z})$. To unclutter the

notation, we omit the symplectic forms from the notation of the GW invariant whenever it is clear. As $\overline{\mathcal{M}}_{0,3} = *$, we implicitly identify $H_*(\overline{\mathcal{M}}_{0,3};\mathbb{Q})$ with $\mathbb{Q}$ below.

Then, together with the computation in Lemma 5.4, we have that

$$\begin{aligned}
1 &= \mathrm{GW}^{X_1}_{0,3,0}(\gamma,\gamma,1)\cdot \mathrm{GW}^{S^2}_{0,3,[S^2]}(h,h,h) && \text{(by (5.1.3))}\\
&= \mathrm{GW}^{X_1\times S^2}_{0,3,(0,[S^2])}(\gamma\times h,\gamma\times h,1\times h) && \text{(by (5.1.1))}\\
&= \mathrm{GW}^{X_0\times S^2}_{0,3,(A,0)}(\tilde\phi^*(\gamma\times h),\tilde\phi^*(\gamma\times h),\tilde\phi^* h)\\
&= \mathrm{GW}^{X_0\times S^2}_{0,3,(A,0)}((\tilde\gamma+ch)\cdot\alpha,(\tilde\gamma+ch)\cdot\alpha,\alpha)\\
&= \mathrm{GW}^{X_0}_{0,3,A}(\tilde\gamma\alpha,\tilde\gamma\alpha,\alpha)\cdot \mathrm{GW}^{S^2}_{0,3,0}(1,1,1) + c^2\,\mathrm{GW}^{X_0}_{0,3,A}(\alpha,\alpha,\alpha)\cdot \mathrm{GW}^{S^2}_{0,3,0}(h,h,1)\\
&\qquad + 2c\,\mathrm{GW}^{X_0}_{0,3,A}(\tilde\gamma\alpha,\alpha,\alpha)\cdot \mathrm{GW}^{S^2}_{0,3,0}(1,h,1) && \text{(by (5.1.1))}\\
&= 2c\,\mathrm{GW}^{X_0}_{0,3,A}(\tilde\gamma\alpha,\alpha,\alpha) && \text{(by (5.1.3)).}
\end{aligned}$$

By the comparison [Hirschi 2025, Theorem 6.3] with the genus-zero Ruan–Tian invariants, we know that $\mathrm{GW}^{X_0}_{0,3,A}(\tilde\gamma\alpha,\alpha,\alpha)$ is an integer. Hence, we obtain the desired contradiction. In particular, $\tilde\phi_*[S^2] = a[S^2] + A$ for some $A \in H_2(X_1;\mathbb{Z})$. To see that $a = 1$ and $A = 0$, we compute

$$1 = \mathrm{GW}^{X_0}_{0,3,0}(\tilde\phi^*\gamma,\tilde\phi^*\gamma,1)\cdot \mathrm{GW}^{S^2}_{0,3,[S^2]}(h,h,h) = \mathrm{GW}^{X_1}_{0,3,A}(\gamma,\gamma,1)\cdot \mathrm{GW}^{S^2}_{0,3,a[S^2]}(ah,ah,ah).$$

By the fundamental class axiom, $\mathrm{GW}^{X_1}_{0,3,A}(\gamma,\gamma,1) = 0$ unless $A = 0$ and for degree reasons, the invariant $\mathrm{GW}^{S^2}_{0,3,a[S^2]}(h,h,h)$ vanishes unless $a = 1$. It follows that $\tilde\phi^*$ preserves the splitting

$$H^2(X_j\times S^2;\mathbb{Z}) \cong H^2(X_j;\mathbb{Z})\oplus H^2(S^2;\mathbb{Z})$$

and $\tilde\phi^* h = h$. The map $H^*(X_1;\mathbb{Z}) \to H^*(X_0;\mathbb{Z})$ induced by $\tilde\phi^*$ is an isomorphism intertwining the intersection forms. By Freedman's theorem [Freedman 1982], it is realised by a homeomorphism $\phi : X_0 \to X_1$. As $\tilde\phi^* h = h$ and $\tilde\phi^* c_1(\omega_1 \oplus \omega_{\mathrm{std}}) = c_1(\omega_1 \oplus \omega_{\mathrm{std}})$, we obtain $\phi^* c_1(\omega_1) = c_1(\omega_0)$. □

Lemma 5.9 *Theorem 5.1 holds if X_0 and X_1 are not homeomorphic to a connected sum of copies of $S^2\times S^2$.*

Proof Let $\phi : X_0 \to X_1$ be a homeomorphism as given by Lemmas 5.6 and 5.8 respectively, for the cases of manifolds that have nonzero signatures and manifolds homeomorphic to $\#_{2n+1}(\mathbb{C}P^2\,\#\,\overline{\mathbb{C}P^2})$. Suppose $g \geq 1$. Let $d := \lceil \frac{1}{2}(g-1)\rceil$ and let $\delta(g) = 0$ if g is odd and $\delta(g) = 1$ if g is even. Given $J_j \in \mathcal{J}_\tau(X_j,\omega_j)$, we have for any $\alpha_1,\dots,\alpha_n \in H^*(X_1;\mathbb{Q})$ that

$$\begin{aligned}
&\mathrm{GW}^{X_1,\omega_1}_{g,n,\phi_* A}(\alpha_1,\dots,\alpha_n)\cdot \mathrm{GW}^{S^2}_{g,n,d[S^2]}(1_{S^2},\dots,1_{S^2},h^{\delta(g)})\\
&\qquad = \mathrm{GW}^{X_1\times S^2,\tilde\omega_1}_{g,n,\tilde\phi_*(A,d)}(\alpha_1\times 1_{S^2},\dots,\alpha_n\times h^{\delta(g)})\\
&\qquad = \mathrm{GW}^{X_0\times S^2,\tilde\omega_0}_{g,n,(A,d)}(\phi^*\alpha_1\times 1_{S^2},\dots,\phi^*\alpha_n\times h^{\delta(g)})\\
&\qquad = \mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\phi^*\alpha_1,\dots,\phi^*\alpha_n)\cdot \mathrm{GW}^{S^2}_{g,n,d[S^2]}(1_{S^2},\dots,1_{S^2},h^{\delta(g)}).
\end{aligned}$$

By Lemma 5.4, we thus have

$$\mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\phi^*\alpha_1,\dots,\phi^*\alpha_n) = \mathrm{GW}^{X_1,\omega_1}_{g,n,\phi_*A}(\alpha_1,\dots,\alpha_n). \qquad \square$$

We now treat the case where the underlying 4-manifolds are homeomorphic to a connected sum of $S^2\times S^2$. We first consider the case of $S^2\times S^2$ itself.

Proposition 5.10 *The assertion of Theorem 5.1 holds when X_0 and X_1 are homeomorphic to $S^2\times S^2$.*

Proof Let $\psi : X_0 \to S^2\times S^2$ by any homeomorphism such that

$$A := \psi_*^{-1}([S^2]\otimes[\mathrm{pt}]) \quad \text{and} \quad B := \psi_*^{-1}([\mathrm{pt}]\otimes[S^2])$$

have positive ω_0-energy. Writing u_1 and u_2 for the algebraic duals of A and B respectively, we have $c_1(\omega_0) = au_1 + bu_2$ where a and b satisfy

$$0 = \sigma(X_0) = 2c_2(\omega_0) - c_1(\omega_0)^2 = 2\chi(S^2\times S^2) - 2ab.$$

Hence $a = b = \pm 2$ or $\{a,b\} = \{1,4\}$ up to sign. As X_0 has even intersection form, it is spin. Thus, $w_2(X_0) = c_1(X_0) \bmod 2$ vanishes and we are in the first case. This also implies that X_0 is minimal: if $S\subset X_0$ is a sphere, then $[S] = nA + mB$ for some $n,m\in\mathbb{Z}$. Thus, $S\cdot S = 2nm \neq -1$.

Let v_1, v_2 be a basis of $H^2(X_1;\mathbb{Z})$ defined in the same way as u_1, u_2. Recall that we are given a deformation equivalence $\tilde\phi : X_0\times S^2 \to X_1\times S^2$. Identifying $H^2(X_0\times S^2;\mathbb{Z}) = \mathbb{Z}\langle u_1,u_2,h\rangle$ and $H^2(X_1\times S^2;\mathbb{Z}) = \mathbb{Z}\langle v_1,v_2,h\rangle$ with $\mathbb{Z}^3$ via these bases, we see that $\tilde\phi^*$ is a permutation matrix. If $\tilde\phi^* h = h$, then we may use the argument of Lemma 5.9 to conclude. Suppose, thus, without loss of generality, that $\tilde\phi^* h = u_1$. Then, the equality

$$\tilde\phi^*(c_1(\omega_1) + 2h) = c_1(\omega_0) + 2h$$

implies that $a = b = 2$. Moreover, we have

$$\begin{aligned} 1 = \mathrm{GW}^{X_1}_{0,3,0}(v_1,v_2,1_{X_1}) &= \mathrm{GW}^{X_1\times S^2}_{0,3,(0,0,1)}(v_1h, v_2h, h) \\ &= \mathrm{GW}^{X_0\times S^2}_{0,3,(1,0,0)}(u_1h, u_2u_1, u_1) = \mathrm{GW}^{X_0}_{0,3,(1,0)}(u_1, u_1u_2, u_1) \end{aligned}$$

since $\mathrm{GW}^{S^2}_{0,3,1}(h,h,h) = 1 = \mathrm{GW}^{S^2}_{0,3,0}(h, 1_{S^2}, 1_{S^2})$. Hence, A can be represented by a possibly nodal J-holomorphic curve of genus 0 for (any) $J\in\mathcal{J}_\tau(X_0,\omega_0)$. By [McDuff 1990, Corollary 1.5(ii)], (X_0,ω_0) is a symplectic S^2-bundle over S^2 as soon as we have shown that A or B is represented by a symplectically embedded sphere. Due to the adjunction inequality and the fact that A is primitive, it suffices to prove that A is represented by a J-holomorphic sphere for some $J\in\mathcal{J}_\tau(X,\omega)$. Fix such an almost complex structure.

By the computation of GW invariants above, we know that there exists homology classes $A_1,\dots,A_d$, not necessarily distinct, such that $A = \sum_{i=1}^d A_i$, where each A_i is represented by a simple J-holomorphic map $u_i : S^2 \to X_0$.

Write $A_i = n_i A + m_i B$. Then $n_1 + \cdots + n_d = 1$, $m_1 + \cdots + m_d = 0$, $n_i\omega_0(A) + m_i\omega_0(B) > 0$ and, by the adjunction inequality,

$$2n_i m_i \geq 2(n_i + m_i - 1) \iff (n_i - 1)(m_i - 1) \geq 0. \tag{5.2.2}$$

Suppose first that $\omega_0(A) \leq \omega_0(B)$. This, together with (5.2.2), shows that if $n_i \geq 1$, then $m_i \geq 0$. However, if $n_i < 1$, then due to energy consideration we also must have $m_i \geq 1$. Thus, $m_i \geq 0$ regardless of the value of n_i and this inequality is strict if $n_i \neq 1$. Together with $m_1 + \cdots + m_d = 0$, we must have $n_i = 1$ for any i. Furthermore, since $n_1 + \cdots + n_d = 1$, we have that $d = 1$. Now $A_1 = A$, whence we may conclude.

If $\omega_0(A) > \omega_0(B)$, we could have $n_i = 1$ and $m_i < 0$ for some i, say $i = 1$. Then $A' := A - A_1 = |m_1|B$ is represented by a (possibly nodal) J-holomorphic sphere. Decompose $A' = A'_1 + \cdots + A'_{d'}$ into classes represented by simple J-holomorphic spheres. By the argument in the previous paragraph, we must have $A'_j = B$ for each j. Thus, applying [McDuff 1990, Corollary 1.5(ii)] to (X_0, ω_0) and B, we see that (X_0, ω_0) is a ruled surface over S^2 with spherical fibres. By [Smale 1959, Theorem A], we have

$$\pi_1(\mathrm{Ham}(S^2)) \cong \pi_1(\mathrm{Diff}^+(S^2)) \cong \mathbb{Z}/2\mathbb{Z}.$$

Thus, X_0 is either deformation equivalent to the standard $S^2 \times S^2$ or to the unique nontrivial ruled surface over S^2 and the same is true for X_1. Since X_0 and X_1 are homeomorphic to $S^2 \times S^2$, we have that both X_0 and X_1 are deformation equivalent to $(S^2 \times S^2, \omega_{\mathrm{std}})$. □

Lemma 5.11 *The assertion of Theorem 5.1 holds when X_0 and X_1 are homeomorphic to $\#_{2m+1}(S^2 \times S^2)$ for some $m > 0$.*

Proof Amusingly enough, this case resembles again the proof of Lemma 5.6: we will show that given a diffeomorphism

$$\tilde{\phi} : X_0 \times S^2 \to X_1 \times S^2$$

such that $\tilde{\phi}^*(\omega_1 \oplus \omega_{\mathrm{std}})$ is homotopic to $\omega_0 \oplus \omega_{\mathrm{std}}$, we can find a homeomorphism ϕ such that $\tilde{\phi}^* = (\phi \times \mathrm{id}_{S^2})^*$ as maps $H^*(X_1 \times S^2; \mathbb{Z}) \to H^*(X_0 \times S^2; \mathbb{Z})$. To this end, let $\ell := 2m + 1$ and write

$$H^*(X_0; \mathbb{Z}) \cong \mathbb{Z}[u_1, u'_1, \ldots, u_\ell, u'_\ell]/(u_i u_j, (1 - \delta_{ij})u_i u'_j, u'_i u'_j, u_i u'_i - u_j u'_j)$$

where $|u_i| = |u'_i| = 2$. Write $\tilde{\phi}^* h = ah + \alpha$ for some $\alpha \in H^*(X_0; \mathbb{Z})$. As used throughout the paper, this implies $2a\alpha = 0$, so either $\alpha = 0$, in which case we are done, or $a = 0$. Suppose thus that $a = 0$. Given any $1 \leq i \leq \ell$, we may write

$$(\tilde{\phi}^{-1})^* u_i = bh + \gamma, \quad (\tilde{\phi}^{-1})^* u'_i = b'h + \gamma'.$$

As $u_i^2 = {u'_i}^2 = 0$, we see that

$$2b\gamma = 2b'\gamma' = 0.$$

On the other hand,

$$0 \neq (\tilde{\phi}^{-1})^* u_i (\tilde{\phi}^{-1})^* u'_i (\tilde{\phi}^{-1})^* h = (bh + \gamma)(b'h + \gamma')\beta = (b\gamma' + b'\gamma)\beta h,$$

where $\beta := (\tilde{\phi}^{-1})^* h \in H^*(X_1;\mathbb{Z})$. This shows that $b \neq 0$ or $b' \neq 0$. Without loss of generality, assume the former. Then we must have $\gamma = 0$, so $(\tilde{\phi}^{-1})^* u_i = bh$. As u_i is primitive, $b \in \{\pm 1\}$. Hence $\tilde{\phi}^* h = \pm u_i$. But $\ell \geq 2$ and i was arbitrary, so this contradicts the well-definedness of $\tilde{\phi}$. Now we may use the argument of Lemma 5.9 to conclude. □

Lemma 5.9, Proposition 5.10 and Lemma 5.11 together complete the proof of Theorem 5.1. We conclude with the observation that Theorem 5.1 can be strengthened to the case of $(S^2, \omega_{\text{std}})^k$ for any $k \geq 1$.

Proposition 5.12 *If (X_0, ω_0) and (X_1, ω_1) are simply connected symplectic 4-manifolds with different Gromov–Witten invariants, then $(X_0, \omega_0) \times (S^2, \omega_{\text{std}})^k$ and $(X_1, \omega_1) \times (S^2, \omega_{\text{std}})^k$ are deformation inequivalent for any $k \geq 1$.*

Proof This follows from the observation that in all arguments above, S^2 could have been replaced by $(S^2)^k$. The extension of Lemma 5.6 is essentially contained in the proof of Theorem 3.1 and the proofs of Lemmas 5.8 and 5.11 carry over verbatim. In the proof of Lemma 5.9 and Proposition 5.10, one has to use that

$$\mathrm{GW}^{(S^2)^k, \omega_{\text{std}}^{\oplus k}}_{g,n,(d[S^2],\dots,d[S^2])}(1^{\times k}, \dots, 1^{\times k}, (h^{\delta(g)})^{\times k})([\mathrm{pt}]) = \big(\mathrm{GW}^{S^2,\omega_{\text{std}}}_{g,n,d[S^2]}(1,\dots,1,h^{\delta(g)})([\mathrm{pt}])\big)^k \neq 0,$$

where $\gamma^{\times k}$ the denotes the k-th cross product of γ with itself and $\delta(g) = 0$ if g is odd and $\delta(g) = 1$ if g is even. Similarly, the proof of Proposition 5.10 extends to the case of $k \geq 1$ since we only used the additional factor of $(S^2, \omega_{\text{std}})$ to conclude the existence of a nonvanishing GW invariant, which can just as well be done using $(S^2, \omega_{\text{std}})^k$. □

Corollary 5.13 *Suppose (X_0, ω_0) and (X_1, ω_1) are two simply connected closed symplectic 4-manifolds with $b_2^+(X_i) \geq 2$. If the Seiberg–Witten invariants of X_0 and X_1 disagree, then $(X_0, \omega_0) \times (S^2, \omega_{\text{std}})^k$ and $(X_1, \omega_1) \times (S^2, \omega_{\text{std}})^k$ are deformation inequivalent for any $k \geq 1$.*

Proof Taubes [2000a; 2000b] related the Seiberg–Witten invariants of a closed simply connected symplectic 4-manifold (X, ω) to a pseudoholomorphic curve invariant Gr. By [Ionel and Parker 1997], this invariant can be computed from the Ruan–Tian Gromov–Witten invariants of (X, ω), which in turn agree with the Gromov–Witten invariants used in this paper by [Hirschi 2025, Theorem 6.3]. Using these comparisons, the claim now follows from Proposition 5.12. □

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