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Genus two Heegaard splittings: an omission

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In Rubinstein and Scharlemann [4], the given list of ways in which a closed orientable 3–manifold could have distinct genus two Heegaard splittings misses a significant case. A brief description of the case (discovered by John Berge) is given here, and the proof that the list is complete is corrected, now incorporating the missed case. Full details and further discussion appear elsewhere.

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1 Preliminaries

The argument in Rubinstein and Scharlemann [4, Theorem 9.4] contains a significant gap[1], discovered in 2008 by John Berge. The details and significance of the omitted case are discussed elsewhere (see Berge [1; 2], Berge and Scharlemann [3], and Scharlemann [5]); the present short note simply indicates how the proof of [4, Theorem 9.4] should be corrected.

Definition 1.1 *A simple closed curve λ in the boundary of a handlebody H is primitive if there is a properly embedded disk $D \subset H$ whose boundary intersects λ in a single point.*

For such a primitive curve, let α be the curve obtained by pushing λ slightly into the interior of H. We can view H as the boundary connect sum of a handlebody H' and a solid torus W, with the curve λ a longitude of W. Then α is a core curve of W, so Dehn surgery on $\alpha \subset W$ still gives a solid torus. Hence any Dehn surgery on $\alpha \subset H$ leaves H still a handlebody.

Definition 1.2 *Suppose $M_0 = H_a \cup_S H_b$ is a Heegaard splitting of a closed 3–manifold M_0. A simple closed curve $\lambda \subset S$ is* doubly primitive *if λ is a primitive curve in both handlebodies H_a and H_b.*

[1] The error is on page 533: The last sentence of the first paragraph of Case 2 should have read, "The *same curves* cannot then be twisted in X since M is hyperbolike." This leaves open an additional possibility for P_X, P_Y, which appears as Subcase B in Section 2 below.

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Suppose M_0 is a closed orientable 3–manifold and that $M_0 = H_a \cup_S H_b$ is a genus 2 Heegaard splitting of M_0. Suppose further that $\lambda_1, \lambda_2 \subset S$ are two disjoint doubly primitive curves in S.

Proposition 1.3 *Suppose M is a manifold obtained by some specified Dehn surgeries on λ_1 and λ_2. For $i = 1, 2$, let A_i (respectively B_i) be the manifold obtained from the handlebody H_a (respectively H_b) by pushing the curve λ_i into $\text{int}(H_a)$ (respectively $\text{int}(H_b)$) and performing the specified Dehn surgery on the curve.*

Then $A_1 \cup_S B_2$ and $A_2 \cup_S B_1$ are two (possibly different) genus 2 Heegaard splittings of M.

Proof A_i (respectively B_i) is obtained from H_a (resp H_b) by Dehn surgery on a pushed in copy α_i of a single primitive curve in S. It was just observed that this makes each A_i (respectively B_i) a handlebody. □

Definition 1.4 *Two genus 2 Heegaard splittings $X \cup_Q Y$ and $A \cup_P B$ of a closed 3–manifold M are called* Dehn derived *(from the splitting $M_0 = H_a \cup_S H_b$ via $\lambda_1 \cup \lambda_2 \subset S$) if the two splittings are created as in Proposition 1.3.*

Berge has a nice classification of pairs of disjoint primitive curves in a genus 2 handlebody. The classification leads to concrete descriptions of how Dehn-derived pairs of splittings can arise. See [1; 3].

2 Filling the gap in [4]

The gap in [4] arises because of a faulty sentence in the midst of a long and technical argument which would be difficult to summarize. Here we jump in on p. 533, in the midst of trying to prove that all cases of multiple genus 2 Heegaard splittings have been covered in the earlier examples listed in that paper. M is a closed hyperbolike 3–manifold with Heegaard splittings $M = A \cup_P B = X \cup_Q Y$. The two splitting surfaces P and Q have been made to coincide on sub-surfaces $P_0 \subset P$ and $Q_0 \subset Q$. Then $P - P_0$ consists of annular components P_X and P_Y properly embedded in the handlebodies X and Y respectively, and $Q - Q_0$ consists of annular components Q_A, Q_B properly embedded in the handlebodies A, B respectively. With this as background, we now re-enter the proof of [4, Theorem 9.4, Case 2] on page 533, at first echoing what is written there as the proof of Subcase A below. In filling the gap in the argument we also broaden the possible outcome, as expressed in Proposition 2.1.

Case 2 P_X and P_Y are both non-empty and the end of each curve in $\partial P_X \cup \partial P_Y$ is parallel to one of c_1 or c_2.

Proposition 2.1 *In this case, either*

(I) *the splittings* $M = A \cup_P B = X \cup_Q Y$ *are related as in [4, Example 4.4] or*

(II) *the two splittings* $M = A \cup_P B = X \cup_Q Y$ *are Dehn-derived from a single genus* 2 *Heegaard splitting of another manifold* M_0.

Proof If at least one annulus in each of P_X or P_Y is non-separating, then together they would give a non-separating, hence essential, torus in M. This contradicts our assumption that M is hyperbolike. So we may as well assume that each annulus in P_Y is separating. Hence the ends of P_Y are twisted in Y (see [4, Definition 5.4]). No end of P_Y can also be twisted in X, for the union along the curve of the solid tori (one in X, one in Y) on which the curve is a torus knot would be a Seifert submanifold of M, contradicting the assumption that M is hyperbolike.

Subcase A Some end of P_Y is parallel to an end of P_X. [The gap in [4] was to view this as the only possibility.]

In this case, by [4, Lemma 5.6] all of P_X is a collection of parallel non-separating longitudinal annuli in X. If P_Y has ends at both c_1 and c_2 then neither curve can be twisted in X. In this case each annulus in P_X is non-separating and so has ends that are non-parallel in Q. This implies that each annulus in P_X has one end at c_1 and one at c_2. Attach such an annulus in X to the tori in Y on which the c_i are twisted. The boundary of the thickened result would exhibit a Seifert manifold in M, again contradicting the assumption that M is hyperbolike. We conclude that P_Y has ends only at c_2, say.

If there were three or more annuli in P_Y (hence six or more ends of ∂P_Y at c_2) then there would be at least four ends of P_X at c_2. No annulus in P_X could have both ends at c_2 (since c_2 is not twisted in X) so there would also be at least four ends of P_X at c_1. This would contradict [4, Lemma 9.5]. So we conclude that P_Y is made up of one or two annuli. If it's two annuli, necessarily separating and parallel in Y, then, again by [4, Lemma 9.5] some annulus in P_X has an end at c_2. It cannot have both ends at c_2 and must be non-separating and longitudinal in X, since c_2 is not twisted in X. In this case the relation between P and Q can be seen as follows (See [4, Figure 32]): In [4, Example 4.4, Variation 2], let P be the splitting given there with Dehn surgery curve in μ_{a+} and Q be the same splitting given there but with Dehn surgery curve in μ_{a-}. To view these simultaneously as splittings of the same manifold M, of course, the Dehn surgery curve has to be moved from μ_{a+} to μ_{a-}, dragging some annuli along, until the splitting surfaces P and Q intersect as described.

Suppose then that P_Y is a single annulus. It may have both ends on P_0 or it may have one end on P_0 and one end on an end of P_X. (If both ends of P_Y were also ends of

P_X then these, together with ends of P_X at ∂P_0 parallel to c_2 would exhibit more than two annuli in P_X, hence more than two ends of P_X at c_1, contradicting [4, Lemma 9.5].) If P_Y has one end on P_0 and one end on an end of P_X, the initial splitting by Q is as in [4, Example 4.4, Variation 1] ($X = A_- \cup \sigma$), with a Dehn surgery curve lying in μ_{b_+}, say. If the splitting is altered by first putting the Dehn surgery curve in μ_{a_+} (yielding the same manifold M), then altering as in [4, Example 4.4] (i. e. considering $A \cup_P B$ where $B = B_- \cup \sigma$) and then dragging the Dehn surgery curve from μ_{a_+} to μ_{b_+}, pushing before it an annulus from the 4-punctured sphere along which A_- and B_- are identified, we get the splitting surface P, intersecting Q as required. (See [4, Figure 33].) This completes the proof that Proposition 2.1, (I) holds in Subcase A.

Subcase B No end of P_Y is parallel to an end of P_X.

In view of [4, Lemma 9.5], in this subcase P_Y and P_X each consist of exactly one separating annulus, P_Y twisted in Y with boundary curves parallel to c_2 (say) and P_X twisted in X with boundary curves parallel to c_1. This case is symmetric: the annulus in Q lying between the ends of P_Y is Q_A (say) and the annulus in Q lying between the ends of P_X is exactly Q_B. The annulus P_Y cannot be parallel to the annulus Q_A (else P_0 and Q_0 could be extended to include both) but rather the region between them is a solid torus $W_2 = A \cap Y$ on whose boundary the cores of the annuli are torus knots. Similarly $B \cap X$ is a solid torus W_1 on whose boundary the cores of the annuli Q_B and P_X are torus knots. The annulus P_Y ∂-compresses in Y to become a separating disk; it follows that $Y - W_2 = B \cap Y$ is a genus 2 handlebody H_{BY} on which the core a_2 of the annulus P_Y is primitive. Symmetrically, the curve c_1 (viewed as the core of the annulus Q_B) is primitive in H_{BY}, the curve c_2 is primitive in the genus 2 handlebody $H_{AX} = X - W_1 = X \cap A$, as is the core curve a_1 of P_X.

Here is another way to describe the manifold M above: begin with the two genus 2 handlebodies H_{AX} (which contains disjoint primitive simple closed curves a_1, c_2 on its boundary) and H_{BY} (which contains disjoint primitive simple closed curves a_2, c_1 on its boundary). Construct a closed 3–manifold M_0 by identifying ∂H_{AX} to ∂H_{BY} by a homeomorphism that identifies a_i with c_i, $i = 1, 2$. Call the resulting curves α_1, α_2. Now recover M from M_0 by removing a tubular neighborhood of each α_i and replacing with the solid torus W_i; equivalently, do an appropriate surgery on each α_i in M_0. The two Heegaard splittings of M are then seen as follows: if α_1 is pushed into H_{AX} and α_2 into H_{BY} before the surgery on the curves, then the resulting Heegaard splitting is $M = X \cup_Q Y$; if α_1 is pushed into H_{BY} and α_2 into H_{AX} before the surgery then the resulting splitting is $M = A \cup_P B$. That is, the splittings of M are both Dehn-derived from the Heegaard splitting $M_0 = H_{AX} \cup H_{BY}$. Thus Proposition 2.1, (II) holds in Subcase B. □

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References

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