Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc

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In this paper, we present foundational material towards the development of a rigorous enumerative theory of stable maps with Lagrangian boundary conditions, ie stable maps from bordered Riemann surfaces to a symplectic manifold, such that the boundary maps to a Lagrangian submanifold. Our main application is to a situation where our proposed theory leads to a well-defined algebro-geometric computation very similar to well-known localization techniques in Gromov–Witten theory. In particular, our computation of the invariants for multiple covers of a generic disc bounding a special Lagrangian submanifold in a Calabi–Yau threefold agrees completely with the original predictions of Ooguri and Vafa based on string duality. Our proposed invariants depend more generally on a discrete parameter which came to light in the work of Aganagic, Klemm, and Vafa which was also based on duality, and our more general calculations agree with theirs up to sign.

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1 Introduction

In recent years, we have witnessed the enormous impact that theoretical physics has had on the geometry of Calabi–Yau manifolds. One takes a version of string theory (or M–theory, or F–theory) and compactifies the theory on a Calabi–Yau manifold, obtaining an effective physical theory. One of the key tools is *duality*, whose assertion is that several a priori distinct theories are related in non-obvious ways. Perhaps the best known duality is mirror symmetry, which can be stated as the equality of the compactification of type IIA string theory on a Calabi–Yau manifold X with the compactification of type IIB string theory on a mirror Calabi–Yau manifold X°. This has deep mathematical consequences, which are still far from being completely understood. But there has certainly been remarkable progress, in particular in the realm of applications to enumerative geometry, as mirror symmetry (and considerations of the

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topological string) inspired the rigorous mathematical notions of the Gromov–Witten invariant and quantum cohomology. This has provided a precise context for the string-theoretic prediction of the number of rational curves of arbitrary degree on a quintic threefold (Candelas–de la Ossa–Green–Parkes [4]). Once the mathematical formulation was clarified, then a mirror theorem could be formulated and proven (Givental [8], Lian–Liu–Yau [16]). These aspects of the development of mirror symmetry are discussed in detail in Cox and Katz [5].

The enumerative consequences just discussed arose from considerations of closed strings. In closed string theory, closed strings sweep out compact Riemann surfaces as the string moves in time. Gromov–Witten theory assigns invariants to spaces of maps from these Riemann surfaces and their degenerations to collections of cohomology classes on X.

Recently, string theorists have obtained a better understanding of the open string sector of string theory and how it relates to dualities. Open strings sweep out compact Riemann surfaces with boundary, and string theory makes predictions about enumerative invariants of moduli spaces of maps from these and their degenerations to X, with boundary mapping to a special Lagrangian submanifold $L \subset X$. Enumerative predictions are now being produced by expected dualities, Ooguri–Vafa [20], Labastida–Mariño–Vafa [14], Aganagic–Vafa [2] and Aganagic–Klemm–Vafa [1].

Our goals in the present paper are twofold. First, we outline the mathematical framework for a theory of enumerative geometry of stable maps from bordered Riemann surfaces with boundary mapping to a Lagrangian submanifold (or more simply, with Lagrangian boundary conditions). In this direction, we give some foundational results and propose others which we believe to be true. Second, we support our hypotheses by carrying out a computation which under our assumptions completely agrees with the prediction of [20] for the contribution of multiple covers of a disc!

We were led to modify our conceptual framework after learning of the results of [1], where it is clarified that the enumerative invariants are not intrinsic to the geometry and necessarily depend on an additional discrete parameter. Happily, our modified calculations coincide up to sign with the results of [1], resulting in a doubly infinite collection of checks.

Our calculations carried out on prestable bordered Riemann surfaces, as well as on the prestable complex algebraic curves which arise from doubling the bordered surfaces.

Here is an overview of the paper. In Section 2, we give some background from Gromov–Witten theory and outline some desired properties for a similar theory of stable maps with Lagrangian boundary conditions. In Section 3 we describe bordered Riemann surfaces and doubling constructions that we will use. In Section 4 we discuss stable maps with Lagrangian boundary conditions and the idea of a virtual fundamental class on moduli spaces of stable maps. In Section 5 we describe $U(1)$ actions on these moduli spaces and perform calculations of the weights of $U(1)$ representations arising in deformations and obstructions of multiple covers of a disc. In Section 6 we give orientations on the bundles we need, so that an Euler class can be defined. The main results, Proposition 7.1 and Theorem 7.2, are formulated in Section 7. The proofs are completed in Section 8.

We hope that our techniques will prove useful in other situations including Floer homology and homological mirror symmetry.

Other approaches to defining enumerative invariants of stable maps with Lagrangian boundary conditions are possible. M. Thaddeus has an interesting approach using Gromov–Witten theory. This leads to a computation in agreement with the $g=0$ case of Proposition 7.1 Thaddeus [23]. J Li and Y S Song use relative stable morphisms in their approach. The result of their computation coincides with the result of Theorem 7.2 [15].

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2 Enumerative invariants and multiple covers

2.1 Gromov–Witten invariants and multiple covers of $\mathbf{P}^1$

We start by reviewing ideas from Gromov–Witten theory relating to multiple covers which will be helpful for fixing ideas.

Let X be a Calabi–Yau 3–fold and $C \subset X$ a smooth rational curve with normal bundle $N = N_{C/X} \cong \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Set $\beta = [C] \in H_2(X;\mathbb{Z})$. The virtual dimension of $\overline{M}_{g,0}(X, d\beta)$ is 0. Fixing an isomorphism $i\colon \mathbf{P}^1 \simeq C$, there is a natural embedding $j\colon \overline{M}_{g,0}(\mathbf{P}^1, d) \to \overline{M}_{g,0}(X, d\beta)$ given by $j(f) = i \circ f$, whose image is a connected component of $\overline{M}_{g,0}(X, d\beta)$, which we denote by $\overline{M}_{g,0}(X, d\beta)_C$. The virtual dimension of $\overline{M}_{g,0}(\mathbf{P}^1, d)$ is $2(d+g-1)$.

Let $\pi\colon \mathcal{U}\to \overline{M}_{g,0}(\mathbf{P}^1,d)$ be the universal family of stable maps, with $\mu\colon \mathcal{U}\to\mathbf{P}^1$ the evaluation map. Then

$$j_*\left(c(R^1\pi_*\mu^*N)\right)\cap[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{\mathrm{vir}})=[\overline{M}_{g,0}(X,d\beta)_C]^{\mathrm{vir}}$$

where $c(R^1\pi_*\mu^*N)$ is the top Chern operator of the rank $2(d+g-1)$ obstruction bundle $R^1\pi_*\mu^*N$ over $\overline{M}_{g,0}(\mathbf{P}^1,d)$ and $[\overline{M}_{g,0}(X,d\beta)_C]^{\mathrm{vir}}$ denotes the restriction of $[\overline{M}_{g,0}(X,d\beta)]^{\mathrm{vir}}$ to $[\overline{M}_{g,0}(X,d\beta)_C]$. The contribution $C(g,d)$ to the genus g Gromov–Witten invariant $N^g_{d[C]}$ of X from degree d multiple covers of C is defined to be the degree of $[\overline{M}_{g,0}(X,d\beta)_C]^{\mathrm{vir}}$. It follows that

$$C(g,d)=\int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{\mathrm{vir}}} c(R^1\pi_*\mu^*N).$$

Alternatively, $C(g,d)$ can be viewed as the Gromov–Witten invariants of the total space of $N_{C/X}$, which is a non-compact Calabi–Yau 3–fold. This follows because an analytic neighborhood of $C\subset X$ is isomorphic to an analytic neighborhood of the zero section in N, while stable maps to C cannot deform off C inside X.

The truncated genus g prepotential is

$$F_g(t)=\sum_{d=1}^{\infty}C(g,d)e^{-dt} \tag{1}$$

and the all genus truncated potential is

$$F(\lambda,t)=\sum_{g=0}^{\infty}\lambda^{-\chi(\Sigma_g)}F_g(t)=\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t), \tag{2}$$

where $\chi(\Sigma_g)=2-2g$ is the Euler characteristic of a smooth Riemann surface of genus g.[1] These potentials are "truncated" because they do not contain the contribution from constant maps. (2) is viewed as the contribution of C to the full potential

$$\sum_{g,\beta}\lambda^{2g-2}N^g_\beta q^\beta. \tag{3}$$

In (3), q^β is a formal symbol satisfying $q^\beta q^{\beta'}=q^{\beta+\beta'}$ for classes $\beta,\beta'\in H_2(X,\mathbb{Z})$. Alternatively, under suitable convergence hypotheses q^β can be identified with $\exp(-\int_\beta\omega)$, where ω is the Kähler form of X, as discussed in Cox–Katz [5].[2]

[1]The appearance of the Euler characteristic in the exponent is familiar from considerations of the topological string, where λ is the string coupling constant.

[2]A different constant is used here in the exponent since one of our goals is to compare to the result of Ooguri–Vafa [20].

Using the localization formula for the virtual fundamental classes proven in Graber–Pandharipande [10], it is shown in Faber–Pandharipande [6] that

$$C(g,d) = d^{2g-3} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} b_{g_1} b_{g_2}$$

where $d > 0$, and

$$b_g = \begin{cases} 1, & g=0 \\ \int_{\overline{M}_{g,1}} \psi_1^{2g-2}\lambda_g, & g>0 \end{cases}.$$

Therefore

$$\begin{aligned} F(\lambda,t) &= \sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{d=1}^{\infty} d^{2g-3} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} b_{g_1} b_{g_2} e^{-dt} \\ &= \sum_{d=1}^{\infty} \frac{e^{-dt}}{\lambda^2 d^3} \sum_{g=0}^{\infty} \left(\sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} b_{g_1} b_{g_2} \right) (\lambda d)^{2g} \\ &= \sum_{d=1}^{\infty} \frac{e^{-dt}}{\lambda^2 d^3} \left(\sum_{g=0}^{\infty} b_g (\lambda d)^{2g} \right)^2 . \end{aligned}$$

It is also shown in [6] that

$$\sum_{g=0}^{\infty} b_g u^{2g} = \frac{u/2}{\sin(u/2)}.$$

Therefore

$$F(\lambda,t) = \sum_{d=1}^{\infty} \frac{e^{-dt}}{d\,(2\sin(\lambda d/2))^2}$$

This is the multiple cover formula for the sphere predicted in Gopakumar–Vafa [9].

2.2 Vafa's enumerative invariants.

Several works of Vafa and collaborators deal with enumerative invariants arising from open string theory, and compute these invariants in several cases [20; 14; 2; 1]. It is of fundamental importance to formulate these invariants as precise mathematical objects. We do not do that here, but instead describe properties of the formulation, much in the same spirit as was done for Gromov–Witten theory in Kontsevich [13]. These properties will be sufficient to verify the multiple cover formula of [20], to be described at the end of this section. The proofs of these properties are left for later work.

Consider a Calabi–Yau 3–fold X, and let $L \subset X$ be a special Lagrangian submanifold. Fix non-negative integers g, h and a relative homology class $\beta \in H_2(X, L, \mathbb{Z})$. Choose classes $\gamma_1, \ldots, \gamma_h \in H_1(L, \mathbb{Z})$ such that $\sum_i \gamma_i = \partial\beta$, where

$$\partial\colon H_2(X, L, \mathbb{Z}) \to H_1(L, \mathbb{Z})$$

is the natural map. Then Vafa's enumerative invariants are rational numbers $N^{g,h}_{\beta;\gamma_1,\ldots,\gamma_h}$ which roughly speaking count continuous maps $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ where

- $(\Sigma, \partial\Sigma)$ is a bordered Riemann surface of genus g, whose boundary $\partial\Sigma$ consists of h oriented circles $R_1, \ldots, R_h$.[3]
- f is holomorphic in the interior of Σ.
- $f_*[\Sigma] = \beta$.
- $f_*[R_i] = \gamma_i$.

To define such an invariant, we will need several ingredients:

- A compact moduli space $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$ of stable maps which compactify the space of maps just described.
- An orientation on this moduli space.
- A virtual fundamental class $[\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)]^{\mathrm{vir}}$ of dimension 0.

Then we simply put

$$N^{g,h}_{\beta;\gamma_1,\ldots,\gamma_h} = \deg[\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)]^{\mathrm{vir}}.$$

In [1], it has come to light that these invariants are not completely intrinsic to the geometry but depend on an additional discrete parameter. In the context of the dual Chern–Simons theory, a framing is required to obtain a topological quantum field theory Witten [26], and the ambiguity we refer to arises precisely from this choice. In the context of enumerative geometry, this translates into additional structure required at

[3] More precisely, the bordered Riemann surfaces can be prestable, and the maps f stable. See Sections 3.6 and 4.

the boundary of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$, an extension in a sense of the additional data to be imposed on a complex bundle on a bordered Riemann surface in order to define its generalized Maslov index (see Section 3.3.3). In our application, this data can be deduced from the choice of a normal vector field to L.

We do not attempt to describe this circle of ideas here in detail, leaving this for future work. Instead, we content ourselves with showing how different choices lead to different computational results. We came to appreciate this point after the authors of [1] urged us to look for an ambiguity in the geometry, and T Graber suggested to us that our computations in an earlier draft might depend on choices of a certain torus action that we will describe later. An analogous point was also made in Li–Song [15]. We will explain the relationship between the choice of torus action and the additional geometric data in Section 8.1.

We introduce a symbol q^β for each $\beta \in H_2(X, L, \mathbb{Z})$ satisfying $q^\beta q^{\beta'} = q^{\beta+\beta'}$. For each $\gamma \in H_1(R_i, \mathbb{Z})$ we introduce a symbol q_i^γ satisfying $q_i^\gamma q_i^{\gamma'} = q_i^{\gamma+\gamma'}$.[4]

We put

$$F_{g;h}(q^\beta, q_1, \ldots, q_h) = \sum_\beta \sum_{\substack{\gamma_1 + \ldots + \gamma_h = \partial\beta \\ \gamma_1, \ldots, \gamma_h \neq 0}} N^{g,h}_{\beta;\gamma_1,\ldots,\gamma_h} q_1^{\gamma_1} \cdots q_h^{\gamma_h} q^\beta.$$

The potential can then be defined to be

$$\begin{aligned} F(\lambda, q^\beta, q_1, \ldots, q_h) &= \sum_{g=0}^\infty \sum_{h=1}^\infty \lambda^{-\chi(\Sigma_{g;h})} F_{g;h}(q^\beta, q_1, \ldots, q_h) \\ &= \sum_{g=0}^\infty \sum_{h=1}^\infty \lambda^{2g+h-2} F_{g;h}(q^\beta, q_1, \ldots, q_h), \end{aligned}$$

In the above, $\chi(\Sigma_{g;h}) = 2 - 2g - h$ is the Euler characteristic of a bordered Riemann surface of genus g with h discs removed.

Actually, the moduli space $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$ and virtual fundamental class should be defined for more general varieties (X, J, L) where X is a symplectic manifold, J is a compatible almost complex structure, and L is Lagrangian. In general however, the virtual dimension will not be 0, as will be discussed in Section 4.2,

[4]Choosing a generator $g_i \in H_1(R_i, \mathbb{Z})$ corresponding to the orientation of R_i, we can instead introduce ordinary variables $y_1, \ldots, y_h$ and write $y_i^{d_i}$ in place of $q_i^{d_i g_i}$.

illustrated for a disc. The orientation will be described in Section 6, and the virtual fundamental class will be discussed in Sections 4.2 and 8.

In many practical examples, $N^{g,h}_{\beta;\gamma_1,\ldots,\gamma_h}$ should be computable using a torus action and a modification of virtual localization introduced in [10]. This is in fact how we do our main computation, Proposition 7.1.

In our situation, the computation of the virtual fundamental class is simplified by exhibiting it as the Euler class of a computable obstruction bundle. The Euler class will be computed in Section 8.

3 Bordered Riemann surfaces and their moduli spaces

In this section, we describe bordered Riemann surfaces and their moduli spaces. We also describe the double of a bordered Riemann surface and related doubling constructions. These notions are connected by the idea of a symmetric Riemann surface, which is a Riemann surface together with an antiholomorphic involution. The considerations developed in this section play a fundamental role in later computations, as we will frequently alternate between the bordered Riemann surface and symmetric Riemann surface viewpoints. For the convenience of the reader, we have included proofs of well known results when an appropriate reference does not exist, as well as related foundational material.

3.1 Bordered Riemann surfaces

This section is a modification of Alling and Greenleaf [3, Chapter 1].

Definition 3.1.1 Let A and B be nonempty subsets of $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}$. A continuous function $f\colon A \to B$ is *holomorphic* on A if it extends to a holomorphic function $\tilde{f}\colon U \to \mathbb{C}$, where U is an open neighborhood of A in $\mathbb{C}$.

Theorem 3.1.2 (Schwartz reflection principle) *Let A and B be nonempty subsets of $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}$. A continuous function $f\colon A \to B$ is holomorphic if it is holomorphic on the interior of A and satisfies $f(A \cap \mathbb{R}) \subset B \cap \mathbb{R}$.*

Definition 3.1.3 A *surface* is a Hausdorff, connected, topological space Σ together with a family $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ such that $\{U_i \mid i \in I\}$ is an open covering of Σ and each map $\phi_i\colon U_i \to A_i$ is a homeomorphism onto an open subset A_i of $\mathbb{C}^+$. $\mathcal{A}$

is called a *topological atlas* on Σ, and each pair (U_i, ϕ_i) is called a *chart* of $\mathcal{A}$. The boundary of Σ is the set

$$\partial\Sigma = \{x \in \Sigma \mid \exists\, i \in I \text{ s.t. } x \in U_i, \phi_i(x) \in \mathbb{R}, \phi_i(U_i) \subseteq \mathbb{C}^+\}.$$

The mappings $\phi_{ij} \equiv \phi_i \circ \phi_j^{-1}\colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are surjective homeomorphisms, called the *transition functions* of $\mathcal{A}$. The atlas $\mathcal{A}$ is called a *holomorphic atlas* if all its transition functions are holomorphic.

Definition 3.1.4 A *bordered Riemann surface* is a compact surface with nonempty boundary equipped with the holomorphic structure induced by a holomorphic atlas on it.

Remark 3.1.5 A Riemann surface is canonically oriented by the holomorphic (complex) structure. In the rest of this paper, the boundary circles R_i of a bordered Riemann surface Σ with boundary $\partial\Sigma = R_1 \cup \ldots \cup R_h$ will always be given the orientation induced by the complex structure, which is a choice of tangent vector to R_i such that the basis (the tangent vector of R_i, inner normal) for the real tangent space is consistent with the orientation of Σ induced by the complex structure.

Definition 3.1.6 A *morphism* between bordered Riemann surfaces Σ and Σ' is a continuous map $f\colon (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ such that for any $x \in \Sigma$ there exist analytic charts (U, ϕ) and (V, ψ) about x and $f(x)$ respectively, and an analytic function $F\colon \phi(U) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \phi(U) & \xrightarrow{F} & \mathbb{C} \end{array}$$

A bordered Riemann surface is topologically a sphere with $g \geq 0$ handles and with $h > 0$ discs removed. Such a bordered Riemann surface is said to be of type $(g; h)$.

3.2 Symmetric Riemann surfaces

Definition 3.2.1 A *symmetric Riemann surface* is a Riemann surface Σ together with an antiholomorphic involution $\sigma\colon \Sigma \to \Sigma$. The involution σ is called the *symmetry* of Σ.

Definition 3.2.2 A *morphism* between symmetric Riemann surfaces (Σ, σ) and (Σ', σ') is a holomorphic map $f\colon \Sigma \to \Sigma'$ such that $f \circ \sigma = \sigma' \circ f$.

A compact symmetric Riemann surface is topologically a compact orientable surface without boundary Σ together with an orientation reversing involution σ, which is classified by the following three invariants:

(1) The genus $\widetilde{g}$ of Σ.

(2) The number $h = h(\sigma)$ of connected components of Σ^σ, the fixed locus of σ.

(3) The index of orientability, $k = k(\sigma) \equiv 2-$the number of connected components of $\Sigma \backslash \Sigma^\sigma$.

These invariants satisfy:

- $0 \leq h \leq \widetilde{g} + 1$.
- For $k = 0$, we have $h > 0$ and $h \equiv \widetilde{g} + 1 \pmod 2$.
- For $k = 1$, we have $0 \leq h \leq \widetilde{g}$.

The above classification was realized already by Felix Klein (see eg, Klein [12], Weichhold [25], Seppälä [21]). This classification is probably better understood in terms of the quotient $Q(\Sigma) = \Sigma/\langle\sigma\rangle$, where $\langle\sigma\rangle = \{id, \sigma\}$ is the group generated by σ. The quotient $Q(\Sigma)$ is orientable if $k = 0$ and nonorientable if $k = 1$, hence the name "index of orientability". Furthermore, h is the number of connected components of the boundary of $Q(\Sigma)$. If $Q(\Sigma)$ is orientable, then it is topologically a sphere with $g \geq 0$ handles and with $h > 0$ discs removed, and the invariants of (Σ, σ) are $(\widetilde{g}, h, k) = (2g + h - 1, h, 0)$. If $Q(\Sigma)$ is nonorientable, then it is topologically a sphere with $g > 0$ crosscaps and with $h \geq 0$ discs removed, and the invariants of Σ are $(\widetilde{g}, h, k) = (g + h - 1, h, 1)$.

From the above classification we see that symmetric Riemann surfaces of a given genus $\widetilde{g}$ fall into $[\frac{3\widetilde{g}+4}{2}]$ topological types.

3.3 Doubling constructions

3.3.1 The complex double of a bordered Riemann surface

Theorem 3.3.1 *Let Σ be a bordered Riemann surface. There exists a double cover $\pi\colon \Sigma_{\mathbb{C}} \to \Sigma$ of Σ by a compact Riemann surface $\Sigma_{\mathbb{C}}$ and an antiholomorphic involution $\sigma\colon \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ such that $\pi \circ \sigma = \pi$. There is a holomorphic embedding $i\colon \Sigma \to \Sigma_{\mathbb{C}}$ such that $\pi \circ i$ is the identity map. The triple $(\Sigma_{\mathbb{C}}, \pi, \sigma)$ is unique up to isomorphism.*

Proof See [3] for the construction of $\Sigma_{\mathbb{C}}$. The rest of Theorem 3.3.1 is clear from the construction. □

Definition 3.3.2 We call the triple $(\Sigma_{\mathbb{C}}, \pi, \sigma)$ in Theorem 3.3.1 the *complex double* of Σ, and $\overline{\Sigma} = \sigma(i(\Sigma))$ the *complex conjugate* of Σ.

If Σ is a bordered Riemann surface of type $(g; h)$, then $(\Sigma_{\mathbb{C}}, \sigma)$ is a compact symmetric Riemann surface of type $(2g+h-1, h, 0)$, and $\Sigma = Q(\Sigma_{\mathbb{C}})$. The two connected components of $\Sigma_{\mathbb{C}} \backslash (\Sigma_{\mathbb{C}})^{\sigma}$ are $i(\Sigma^\circ)$ and $\overline{\Sigma}^\circ = \sigma \circ i(\Sigma^\circ)$, where Σ° denotes the interior of Σ and $(\Sigma_{\mathbb{C}})^{\sigma}$ is the fixed locus of σ.

A bordered Riemann surface of type $(0; 1)$ is the disc, and its complex double is the projective line; a bordered Riemann surface of type $(0; 2)$ is an annulus, and its complex double is a torus. In the above two cases, Σ and $\overline{\Sigma}$ are isomorphic as bordered Riemann surfaces, which is in general not true for bordered Riemann surfaces of other topological types.

3.3.2 The complex double of a map

Theorem 3.3.3 *Let X be a complex manifold with an antiholomorphic involution $A\colon X \to X$, and $L = X^A$ be the fixed locus of A. Let Σ be a bordered Riemann surface, and $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ be a continuous map which is holomorphic in the interior of Σ. We identify Σ with its image under i in $\Sigma_{\mathbb{C}}$. Then f extends to a holomorphic map $f_{\mathbb{C}}\colon \Sigma_{\mathbb{C}} \to X$ such that $A \circ f_{\mathbb{C}} = f_{\mathbb{C}} \circ \sigma$.*

Proof For any $p \in \Sigma_{\mathbb{C}}$ define

$$f_{\mathbb{C}}(p) = \begin{cases} f(p) & \text{if } p \in \Sigma \\ A \circ f \circ \sigma(p) & \text{if } p \notin \Sigma \end{cases}$$

Then $f_{\mathbb{C}}$ is holomorphic by the Schwartz reflection principle, and the rest of Theorem 3.3.3 is clear. □

Definition 3.3.4 We call the holomorphic map $f_{\mathbb{C}}\colon \Sigma_{\mathbb{C}} \to X$ in Theorem 3.3.3 the *complex double* of $f\colon (\Sigma, \partial\Sigma) \to (X, L)$.

Remark 3.3.5 Let X be a complex manifold with an antiholomorphic involution $A\colon X \to X$, and let $L = X^A$ be the fixed locus of A. Let Σ be a bordered Riemann surface, and let $g\colon \Sigma_{\mathbb{C}} \to X$ be a holomorphic map such that $A \circ g = g \circ \sigma$. It is *not* always true that $g = f_{\mathbb{C}}$ for some continuous map $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ which is holomorphic on Σ°. For example, let $X = \mathbf{P}^1$ and $A(z) = 1/\overline{z}$, where z is the affine coordinate of $\mathbb{C} \subset \mathbf{P}^1$. Then $L = \{|z| = 1\} \cong S^1$. Let $\Sigma = D^2 = \{|z| \leq 1\} \subset \Sigma_{\mathbb{C}} = \mathbf{P}^1$ and put $\sigma(z) = 1/\overline{z}$. A degree 2 holomorphic map $g\colon \mathbf{P}^1 \to \mathbf{P}^1$ satisfies $A \circ g = g \circ \sigma$ if the set of two branch points is invariant under A, and it is the complex double of some continuous map $f\colon (D^2, \partial D^2) \to (\mathbf{P}^1, S^1)$ which is holomorphic on Σ° if the set of the two branch points of g is of the form $\{z_0, 1/\bar{z}_0\}$. In particular, if the branch points of g are two distinct points on S^1, then $A\circ g = g\circ\sigma$, but g is not the complex double of some continuous map $f\colon (D^2, \partial D^2) \to (\mathbf{P}^1, S^1)$ which is holomorphic on Σ°.

3.3.3 The topological complex double of a complex vector bundle We start by reviewing some facts about totally real subspaces of $\mathbb{C}^n$, following Oh [19].

A real subspace V of $\mathbb{C}^n$ is *totally real* (w.r.t the standard complex structure of $\mathbb{C}^n$) if $\dim_{\mathbb{R}} V = n$ and $V \cap iV = \{0\}$. Define

$$\mathcal{R}_n \equiv \{V \mid \text{V is a totally real subspace of } \mathbb{C}^n\}.$$

$GL(n,\mathbb{C})$ acts transitively on $\mathcal{R}_n$, and the isotropy group of $\mathbb{R}^n \subset \mathbb{C}^n$ is $GL(n,\mathbb{R})$, so $\mathcal{R}_n \cong GL(n,\mathbb{C})/GL(n,\mathbb{R})$. Concretely, this means that any totally real subspace $V \subset \mathbb{C}^n$ is of the form $A\cdot\mathbb{R}^n$ for some $A \in GL(n,\mathbb{C})$, and $A_1\cdot\mathbb{R}^n = A_2\cdot\mathbb{R}^n$ if and only if $A_2^{-1}A_1 \in GL(n,\mathbb{R})$, or equivalently, $A_1\bar{A}_1^{-1} = A_2\bar{A}_2^{-1}$. By [19, Proposition 4.4], the map

$$\begin{aligned} B\colon \mathcal{R}_n \cong GL(n,\mathbb{C})/GL(n,\mathbb{R}) &\longrightarrow \widetilde{\mathcal{R}}_n \equiv \{D \in GL(n,\mathbb{C}) \mid D\bar{D} = I_n\} \\ A\cdot\mathbb{R}^n &\mapsto A\bar{A}^{-1} \end{aligned}$$

is a diffeomorphism, where I_n is the $n\times n$ identity matrix. The *(generalized) Maslov index* $\mu(\gamma)$ of an oriented loop $\gamma\colon S^1 \to \mathcal{R}_n$ is defined to be the degree of the map $\phi = \det\circ B\circ\gamma\colon S^1 \to U(1)$, where $U(1) = \{e^{i\theta} \mid \theta\in\mathbb{R}\}$ is oriented by $\partial/\partial\theta$.

The (generalized) Maslov index gives an explicit way to detect homotopy of loops in $\mathcal{R}_n$; in fact $\pi_1(\mathcal{R}_n)\cong\mathbb{Z}$, and two loops $\gamma_1,\gamma_2\colon S^1\to\mathcal{R}_n$ are homotopic if and only if $\mu(\gamma_1)=\mu(\gamma_2)$.

A real subspace V of $\mathbb{C}^n$ is Lagrangian with respect to the standard symplectic structure on $\mathbb{C}^n$ if $\dim_{\mathbb{R}} V = n$ and V is orthogonal to iV with respect to the standard inner product on $\mathbb{R}^{2n}\cong\mathbb{C}^n$. The space $\mathcal{L}_n$ of Lagrangian subspaces of $\mathbb{C}^n$ is a submanifold of $\mathcal{R}_n$, and its image under B is the submanifold

$$\widetilde{\mathcal{L}}_n \equiv \{D \in GL(n,\mathbb{C}) \mid D\bar{D} = I_n,\ D = D^t\}$$

of $\widetilde{\mathcal{R}}_n$. The inclusion $\mathcal{L}_n \subset \mathcal{R}_n$ is a homotopy equivalence, and the restriction of the (generalized) Maslov index to loops in $\mathcal{L}_n$ is the usual Maslov index in symplectic geometry (McDuff–Salamon [17, Chapter 2]).

Let E be a complex vector bundle of rank n over a bordered Riemann surface Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$, so that

$$E|_{\partial\Sigma} \cong E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

We may assume that $h > 0$, so that E is a topologically trivial complex vector bundle. Fix a trivialization $\Phi\colon E \cong \Sigma \times \mathbb{C}^n$, and let $R_1, \ldots, R_h$ be the connected components of $\partial\Sigma$, with orientation induced by the orientation of Σ, as explained in Remark 3.1.5. Then $\Phi(E_{\mathbb{R}}|_{R_i})$ gives rise to a loop $\gamma_i\colon S^1 \to \mathcal{R}_n$. Setting $\mu(\Phi, R_i) = \mu(\gamma_i)$, we have

Proposition 3.3.6 *Let E be a complex vector bundle of rank n over a bordered Riemann surface Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. Let $R_1, \ldots, R_h$ be the connected components of $\partial\Sigma$, with orientation induced by the orientation of Σ. Then $\sum_{i=1}^{h} \mu(\Phi, R_i)$ is independent of the choice of trivialization $\Phi\colon E \cong \Sigma \times \mathbb{C}^n$.*

Proof Let Φ_1, Φ_2 be two trivializations of E. Then $\Phi_2 \circ \Phi_1^{-1}\colon \Sigma \times \mathbb{C}^n \to \Sigma \times \mathbb{C}^n$ is given by $(x, v) \mapsto (x, g(x)v)$ for an appropriate $g\colon \Sigma \to GL(n, \mathbb{C})$. Letting

$$\phi = \det(g\bar{g}^{-1})\colon \Sigma \to U(1),$$

it follows that

$$\mu(\Phi_2, R_i) - \mu(\Phi_1, R_i) = \deg(\phi|_{R_i}).$$

Furthermore, $\phi_*[R_i] = \deg(\phi|_{R_i})[U(1)]$, where $\phi_*\colon H_1(\Sigma; \mathbb{Z}) \to H_1(U(1); \mathbb{Z})$ and $[U(1)] \in H_1(U(1), \mathbb{Z})$ is the fundamental class of $U(1)$. Since $[R_1] + \cdots + [R_h] = [\partial\Sigma] = 0 \in H_1(\Sigma; \mathbb{Z})$, it follows that $\sum_{i=1}^{h} \deg(\phi|_{R_i}) = 0$, which implies $\sum_{i=1}^{h} \mu(\Phi_1, R_i) = \sum_{i=1}^{h} \mu(\Phi_2, R_i)$, as desired. □

Definition 3.3.7 The *Maslov index* of $(E, E_{\mathbb{R}})$ is defined by

$$\mu(E, E_{\mathbb{R}}) = \sum_{i=1}^{h} \mu(\Phi, R_i)$$

where $\Phi\colon E \to \Sigma \times \mathbb{C}^n$ is any trivialization.

Proposition 3.3.6 says that the Maslov index is well defined.

The following theorem is well-known. We include the proof for completeness.

Theorem 3.3.8 *Let E be a complex vector bundle over a bordered Riemann surface Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. Then there is a complex vector bundle $E_{\mathbb{C}}$ on $\Sigma_{\mathbb{C}}$ together with a conjugate linear involution $\tilde{\sigma}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ covering the* *antiholomorphic involution* $\sigma\colon \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ *such that* $E_{\mathbb{C}}|_{\Sigma} = E$ *(where* Σ *is identified with its image under* i *in* $\Sigma_{\mathbb{C}}$*) and the fixed locus of* $\tilde{\sigma}$ *is* $E_{\mathbb{R}} \to \partial\Sigma$*. Moreover, we have*

$$\deg E_{\mathbb{C}} = \mu(E, E_{\mathbb{R}}).$$

Here, $\deg E_{\mathbb{C}}$ means $\deg(c_1(E_{\mathbb{C}}) \cap [\Sigma_{\mathbb{C}}])$ as usual.

Proof Let $R_1, \ldots, R_h$ be the connected components of $\partial\Sigma$, and let $N_i \cong R_i \times [0, 1)$ be a neighborhood of R_i in Σ such that $N_1, \ldots, N_h$ are disjoint. Then $(N_i)_{\mathbb{C}} = N_i \cup \overline{N}_i$ is a tubular neighborhood of R_i in $\Sigma_{\mathbb{C}}$, and $N \equiv \cup_{i=1}^{h} (N_i)_{\mathbb{C}}$ is a tubular neighborhood of $\partial\Sigma$ in $\Sigma_{\mathbb{C}}$. Let $U_1 = \Sigma \cup N$, $U_2 = \overline{\Sigma} \cup N$, so that $U_1 \cup U_2 = \Sigma_{\mathbb{C}}$ and $U_1 \cap U_2 = N$.

Fix a trivialization $\Phi\colon E \cong \Sigma \times \mathbb{C}^n$, where n is the rank of E. Then $\Phi(E_{\mathbb{R}}|_{R_i})$ gives rise to a loop $B_i\colon R_i \to \widetilde{\mathcal{R}}_n \subset GL(n, \mathbb{C})$. To construct $E_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$, we glue trivial bundles $U_1 \times \mathbb{C}^n \to U_1$ and $U_2 \times \mathbb{C}^n \to U_2$ along N by identifying $(x, u) \in (N_i)_{\mathbb{C}} \times \mathbb{C}^n \subset U_1 \times \mathbb{C}^n$ with $(x, B_i^{-1} \circ p_i(x) u) \in (N_i)_{\mathbb{C}} \times \mathbb{C}^n \subset U_2 \times \mathbb{C}^n$, where $p_i\colon (N_i)_{\mathbb{C}} \cong R_i \times (-1, 1) \to R_i$ is the projection to the first factor and $B_i^{-1}\colon R_i \to \widetilde{\mathcal{R}}_n$ denotes the map $B_i^{-1}(x) = (B_i(x))^{-1}$. There is a conjugate linear involution $\tilde{\sigma}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ given by $(x, u) \in U_1 \times \mathbb{C}^n \mapsto (\sigma(x), \overline{u}) \in U_2 \times \mathbb{C}^n$ and $(y, v) \in U_2 \times \mathbb{C}^n \mapsto (\sigma(y), \overline{v}) \in U_1 \times \mathbb{C}^n$. It is clear from the above construction that $\tilde{\sigma}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ covers the antiholomorphic involution $\sigma\colon \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$, and the fixed locus of $\tilde{\sigma}$ is $E_{\mathbb{R}} \to \partial\Sigma$.

By considering the determinant line bundles, we only need to show $\deg E_{\mathbb{C}} = \mu(E, E_{\mathbb{R}})$ for a line bundle E. In this case, $B_i\colon R_i \to \widetilde{\mathcal{R}}_1 = U(1) \subset GL(1, \mathbb{C})$. Let $F \in \Omega^2(\Sigma_{\mathbb{C}}, i\mathbb{R})$ be the curvature of some $U(1)$ connection on E. The proof of [17, Theorem 2.70] shows that

$$\mu(E, E_{\mathbb{R}}) = \frac{i}{2\pi} \int_{\Sigma_{\mathbb{C}}} F = \deg(E_{\mathbb{C}}). \qquad \square$$

Definition 3.3.9 We call the bundle $E_{\mathbb{C}}$ as in Theorem 3.3.8 the *topological complex double* of $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$.

3.3.4 The holomorphic complex double of a Riemann–Hilbert bundle

Definition 3.3.10 Let A be a nonempty subset of $\mathbb{C}^+$. A continuous function $f\colon A \to \mathbb{C}^n$ is *holomorphic* on A if it extends to a holomorphic function $\tilde{f}\colon U \to \mathbb{C}^n$, where U is an open neighborhood of A in $\mathbb{C}$.

Definition 3.3.11 Let Σ be a bordered Riemann surface, and let X be a complex manifold. We say $f\colon \Sigma \to X$ is *holomorphic* if for any $x \in \Sigma$ there exist holomorphic charts (U,ϕ) and (V,ψ) about x and $f(x)$ respectively, and a holomorphic function $F\colon \phi(U)\to\mathbb{C}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \phi\downarrow & & \downarrow\psi \\ \phi(U) & \xrightarrow{F} & \mathbb{C}^n \end{array}$$

Definition 3.3.12 A complex vector bundle E over a bordered Riemann surface Σ is *holomorphic* if there is an open covering $\{U_i \mid i\in I\}$ of Σ together with trivializations $\Phi_i\colon E|_{U_i}\cong U_i\times\mathbb{C}^n$ such that if $U_i\cap U_j\neq\phi$ then

$$\begin{aligned} \Phi_{ij}\equiv\Phi_i\circ\Phi_j^{-1}\colon (U_i\cap U_j)\times\mathbb{C}^n &\longrightarrow (U_i\cap U_j)\times\mathbb{C}^n \\ (x,u) &\mapsto (x,g_{ij}(x)u) \end{aligned}$$

for some holomorphic map $g_{ij}\colon U_i\cap U_j\to GL(n,\mathbb{C})$.

Let E be a smooth complex vector bundle of rank n over a bordered Riemann surface Σ which is holomorphic on Σ°. More explicitly, there is an open covering $\{U_i \mid i\in I\}$ of Σ together with trivializations $\Phi_i\colon E|_{U_i}\cong U_i\times\mathbb{C}^n$ such that if $U_i\cap U_j\neq\phi$ then

$$\begin{aligned} \Phi_{ij}\equiv\Phi_i\circ\Phi_j^{-1}\colon (U_i\cap U_j)\times\mathbb{C}^n &\longrightarrow (U_i\cap U_j)\times\mathbb{C}^n \\ (x,u) &\mapsto (x,g_{ij}(x)u) \end{aligned}$$

for some smooth map $g_{ij}\colon U_i\cap U_j\to GL(n,\mathbb{C})$ which is holomorphic on $(U_i\cap U_j)\cap\Sigma^\circ$. We call $\{(U_i,\Phi_i)\mid i\in I\}$ a smooth trivialization of $(E,E_{\mathbb{R}})\to(\Sigma,\partial\Sigma)$ which is holomorphic on Σ°.

Theorem 3.3.13 *Let E be a smooth complex vector bundle of rank n over a bordered Riemann surface Σ which is holomorphic on Σ°, and let $E_{\mathbb{R}}$ be a smooth totally real subbundle of $E|_{\partial\Sigma}$. Then there exists a smooth trivialization $\{(U_i,\Phi_i)\mid i\in I\}$ of $(E,E_{\mathbb{R}})$ which is holomorphic on Σ° such that*

$$\Phi_i(E_{\mathbb{R}}|_{U_i\cap\partial\Sigma})=(U_i\cap\partial\Sigma)\times\mathbb{R}^n\subset U_i\times\mathbb{C}^n$$

whenever $U_i\cap\partial\Sigma\neq\phi$. In particular, $\{(U_i,\Phi_i)\mid i\in I\}$ determines a holomorphic structure on $E\to\Sigma$ such that $E_{\mathbb{R}}\to\partial\Sigma$ is a real analytic subbundle of $E|_{\partial\Sigma}\to\partial\Sigma$.

Proof Let $\{(U_i,\Phi'_i)\mid i\in I\}$ be a smooth trivialization of $(E,E_{\mathbb{R}})$ which is holomorphic on Σ°. If $U_i\cap\partial\Sigma$ is empty, set $\Phi_i=\Phi'_i$. If $U_i\cap\partial\Sigma$ is nonempty, our strategy is to find a smooth map $h\colon U_i\to GL(n,C)$ which is holomorphic on $U_i\cap\Sigma^\circ$

such that $H\circ\Phi_i(E_{\mathbb{R}}|_{U_i\cap\partial\Sigma})=(U_i\cap\partial\Sigma)\times\mathbb{R}^n$, where $H\colon U_i\times\mathbb{C}^n\to U_i\times\mathbb{C}^n$, $(x,u)\mapsto(x,h(x)u)$, and set $\Phi_i=H\circ\Phi_i'$.

By refining the open covering, we may assume that there is a holomorphic map $\phi_i\colon U_i\to\mathbb{C}^+$ such that (U_i,ϕ_i) is a holomorphic chart. Then $\phi_i(U_i\cap\partial\Sigma)\subset\mathbb{R}$. Let $\phi_i'\colon U_i\to D^2$ be the composition of ϕ_i with the isomorphism $\mathbb{C}^+\cong D^2$. We may assume that $U\equiv\phi_i'(U_i)\subset\{z\in D^2\mid \mathrm{Im} z>0\}\equiv T$. Then $E_{\mathbb{R}}$ gives rise to a smooth map $B\colon U\cap\partial D^2\to\widetilde{\mathcal{R}}_n$, and the proof will be completed if we can find a smooth map $A\colon U\to GL(n,\mathbb{C})$ which is holomorphic on the interior of U such that $A\bar{A}^{-1}|_{U\cap\partial D^2}=B$, since $h=A^{-1}\circ\phi_i'$ will have the desired property described in the previous paragraph.

We extend B to a smooth map $\widetilde{B}\colon\partial D^2\to\widetilde{\mathcal{R}}_n$. By [19, Lemma 4.6], there is a smooth map $\Theta\colon D^2\to GL(n,\mathbb{C})$ which is holomorphic on the interior of D^2 such that $B(z)=\Theta(z)\,\mathrm{diag}(z^{\kappa_1},\dots,z^{\kappa_n})\overline{\Theta(z)}^{-1}$, for some integers $\kappa_1,\dots,\kappa_n$. Let $r(z)$ be a holomorphic branch of $z^{1/2}$ on T. Set $A(z)=\Theta(z)\,\mathrm{diag}(r(z)^{\kappa_1}\dots r(z)^{\kappa_n})$ for $z\in U$. Then A is smooth and is holomorphic on the interior of U, and $A\bar{A}^{-1}|_{U\cap\partial D^2}=B$, as desired.

Finally, if $U_i\cap U_j\neq\phi$, then

$$\begin{aligned}\Phi_{ij}\equiv\Phi_i\circ\Phi_j^{-1}\colon (U_i\cap U_j)\times\mathbb{C}^n &\longrightarrow (U_i\cap U_j)\times\mathbb{C}^n\\ (x,u)&\mapsto (x,g_{ij}(x)u)\end{aligned}$$

for some smooth map $g_{ij}\colon U_i\cap U_j\to GL(n,\mathbb{C})$ which is holomorphic on $U_i\cap U_j\cap\Sigma^\circ$, and $g_{ij}(U_i\cap U_j\cap\partial\Sigma)\subset GL(n,\mathbb{R})$. Note that $U_i\cap U_j\cap\Sigma^\circ=U_i\cap U_j$ if and only if $U_i\cap U_j\cap\partial\Sigma$ is empty. By the Schwartz reflection principle, g_{ij} can be extended to a holomorphic map $(g_{ij})_{\mathbb{C}}\colon(U_i\cap U_j)_{\mathbb{C}}\to GL(n,\mathbb{C})$, and $g_{ij}|_{U_i\cap U_j\cap\partial\Sigma}$ is real analytic. Therefore, $\{(U_i,\Phi_i)\mid i\in I\}$ determines a holomorphic structure on $E\to\Sigma$ such that $E_{\mathbb{R}}\to\partial\Sigma$ is a real analytic subbundle of $E|_{\partial\Sigma}\to\partial\Sigma$. □

Definition 3.3.14 We call $(E,E_{\mathbb{R}})\to(\Sigma,\partial\Sigma)$ together with the holomorphic structure on $E\to\Sigma$ determined by $\{(U_i,\Phi_i)\mid i\in I\}$ as in Theorem 3.3.13 a *Riemann–Hilbert bundle* over Σ. The trivialization $\{(U_i,\Phi_i)\mid i\in I\}$ is called a *Riemann–Hilbert trivialization* of $(E,E_{\mathbb{R}})\to(\Sigma,\partial\Sigma)$.

The holomorphic maps $(g_{ij})_{\mathbb{C}}\colon(U_i\cap U_j)_{\mathbb{C}}\to GL(n,\mathbb{C})$ in the proof of Theorem 3.3.13 give a holomorphic vector bundle $E_{\mathbb{C}}\to\Sigma_{\mathbb{C}}$ together with a holomorphic trivialization $\{((U_i)_{\mathbb{C}},(\Phi_{ij})_{\mathbb{C}})\mid i\in I\}$. There is an antiholomorphic involution $\tilde{\sigma}\colon E_{\mathbb{C}}\to E_{\mathbb{C}}$ such that $\widetilde{\sigma}(E|_{(U_i)_{\mathbb{C}}}) = E|_{(U_i)_{\mathbb{C}}}$ and

$$\begin{aligned}\Phi_i \circ \widetilde{\sigma} \circ \Phi_i^{-1} \colon (U_i)_{\mathbb{C}} \times \mathbb{C}^n &\longrightarrow (U_i)_{\mathbb{C}} \times \mathbb{C}^n \\ (x, u) &\mapsto (\sigma(x), \overline{u})\end{aligned}$$

It is clear from the above construction that $E_{\mathbb{C}}|_{\Sigma} = E$ (where Σ is identified with its image under i in $\Sigma_{\mathbb{C}}$), $\widetilde{\sigma}$ covers the antiholomorphic involution $\sigma\colon \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$, and the fixed locus of $\widetilde{\sigma}$ is $E_{\mathbb{R}} \to \partial\Sigma$.

Definition 3.3.15 Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle over a bordered Riemann surface Σ, and construct $E_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ as above. We call $E_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ the *holomorphic complex double* of the Riemann–Hilbert bundle $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$.

Remark 3.3.16 Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle over a bordered Riemann surface Σ. The underlying topological complex vector bundle of the holomorphic complex double $E_{\mathbb{C}}$ of $(E, E_{\mathbb{R}})$ is isomorphic (as a topological complex vector bundle) to the topological complex double of the underlying topological bundles of $(E, E_{\mathbb{R}})$. In particular, $\deg E_{\mathbb{C}} = \mu(E, E_{\mathbb{R}})$.

3.4 Riemann–Roch theorem for bordered Riemann surfaces

Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle over a bordered Riemann surface Σ, and let $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$ denote the sheaf of local holomorphic sections of E with boundary values in $E_{\mathbb{R}}$. Let $\mathcal{E}_{\mathbb{C}}$ denote the sheaf of local holomorphic sections of $E_{\mathbb{C}}$, the holomorphic complex double of $(E, E_{\mathbb{R}})$. Let $\widetilde{U}$ be an open subset of $\Sigma_{\mathbb{C}}$. Define $\widetilde{\sigma}\colon \mathcal{E}_{\mathbb{C}}(\widetilde{U}) \to \mathcal{E}_{\mathbb{C}}(\sigma(\widetilde{U}))$ by $\widetilde{\sigma}(s)(x) = \widetilde{\sigma} \circ s \circ \sigma(x)$ for $x \in \sigma(\widetilde{U})$. In particular, if U is an open set in Σ, then $\widetilde{\sigma}\colon \mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}}) \to \mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}})$, and the fixed locus $\mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}})^{\widetilde{\sigma}} = (\mathcal{E}, \mathcal{E}_{\mathbb{R}})(U)$ is a totally real subspace of the complex vector space $\mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}})$.

We are interested in the sheaf cohomology groups $H^q(\Sigma, \partial\Sigma, E, E_{\mathbb{R}})$ of the sheaf $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$ on Σ. Here the sheaf cohomology functors are the right derived functors of the global section functor from the category of sheaves of $(\mathcal{O}, \mathcal{O}_{\mathbb{R}})$ modules on Σ to the category of $\mathbb{R}$ modules, where $(\mathcal{O}, \mathcal{O}_{\mathbb{R}})$ is the sheaf of local holomorphic functions on Σ with real boundary values. Let $\mathcal{A}^0(E, E_{\mathbb{R}})$ denote the sheaf of local C^∞ sections of E with boundary values in $E_{\mathbb{R}}$, and let $\mathcal{A}^{0,1}(E)$ denote the sheaf of local C^∞ E–valued $(0,1)$ forms. We claim that

$$0 \to (\mathcal{E}, \mathcal{E}_{\mathbb{R}}) \to \mathcal{A}^0(E, E_{\mathbb{R}}) \overset{\overline{\partial}}{\to} \mathcal{A}^{0,1}(E) \to 0 \tag{4}$$

is a fine resolution of $(\mathcal{E}, \mathcal{E}_{\boldsymbol{R}})$. The sheaves $\mathcal{A}^0(E, E_{\mathbb{R}})$ and $\mathcal{A}^{0,1}(E)$ are clearly fine. By definition, the kernel of $\overline{\partial}$ is $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$. To check the surjectivity of $\overline{\partial}$, it suffices to check the surjectivity of the stalk $\bar\partial_x$ at each point $x \in \Sigma$. For $x \in \Sigma^\circ$, this follows from the $\bar\partial$–Poincaré lemma. A neighborhood of $x \in \partial\Sigma$ in Σ can be identified with an open set $U \subset D^2$ such that $U \cap \partial D^2 \neq \phi$, and by shrinking U we may assume that $(E, E_{\mathbb{R}})|_U$ gets identified with the trivial bundle $(\mathbb{C}^n, \mathbb{R}^n) \to (U, U \cap \partial D^2)$. Let $F\colon U \to \mathbb{C}^n$ be a C^∞ function. We need to find a neighborhood V of x in U and a C^∞ function $\eta\colon V \to \mathbb{C}^n$ such that

$$\begin{cases} \frac{\partial\eta}{\partial\bar z} = F(z) & \text{on } V^\circ \\ \eta(z) \in \mathbb{R}^n & \text{on } V \cap \partial D^2. \end{cases}$$

Let ρ be a $\mathbb{C}^\infty$ cut-off function whose support is contained in U and satisfies $\rho \equiv 1$ on some neighborhood V of x. Then $G(z) \equiv \rho(z)F(z)$ defines a C^∞ function $G\colon D^2 \to \mathbb{C}^n$. It suffices to solve

$$\begin{cases} \frac{\partial\eta}{\partial\bar z} = G(z) & \text{on } (D^2)^\circ \\ \eta(z) \in \mathbb{R}^n & \text{on } \partial D^2. \end{cases}$$

The above system can be solved (see [24], [19]). We conclude that $\bar\partial_x$ is surjective for $x \in \partial\Sigma$.

The resolution (4) is fine, so the sheaf cohomology of $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$ is given by the cohomology of the two-term elliptic complex

$$0 \to A^0(E, E_{\mathbb{R}}) \xrightarrow{\bar\partial_{(E,E_{\mathbb{R}})}} A^{0,1}(E) \to 0, \tag{5}$$

where $A^0(E, E_{\mathbb{R}})$ is the space of global C^∞ sections of E with boundary values in $E_{\mathbb{R}}$, and $A^{0,1}(E)$ is the space of global C^∞ E–valued $(0,1)$ forms. In other words,

$$\begin{aligned} H^0(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) &= \operatorname{Ker} \bar\partial_{(E,E_{\mathbb{R}})} \\ H^1(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) &= \operatorname{Coker} \bar\partial_{(E,E_{\mathbb{R}})} \\ H^q(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) &= 0 \quad \text{for } q > 1 \end{aligned}$$

To summarize, the sheaf cohomology of $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$ can be identified with the *Dolbeault cohomology*, the cohomology of the *twisted Dolbeault complex* (5).

Let $\mathcal{A} = \{U_i \mid i \in I\}$ be an acyclic cover of Σ for the sheaf $(\mathcal{E}, \mathcal{E}_{\mathbb{R}})$ in the sense that

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, U_{i_1} \cap \cdots \cap U_{i_p} \cap \partial\Sigma, E, E_{\mathbb{R}}) = 0 \text{ for } q > 0, i_1, \ldots, i_p \in I.$$

Then the sheaf cohomology group $H^q(\Sigma, \partial\Sigma, E, E_{\mathbb{R}})$ is isomorphic to the Čech cohomology group $H^q(\mathcal{A}, \mathcal{E}, \mathcal{E}_{\mathbb{R}})$, for all q. We may further assume that $\mathcal{A} \equiv \{(U_i)_{\mathbb{C}} \mid$

$i \in I\}$ is an acyclic cover for the sheaf $\mathcal{E}_{\mathbb{C}}$ on $\Sigma_{\mathbb{C}}$ in the sense that the sheaf cohomology groups

$$H^q((U_{i_1})_{\mathbb{C}} \cap \cdots \cap (U_{i_p})_{\mathbb{C}}, E_{\mathbb{C}}) = 0 \text{ for } q > 0, i_1, \ldots, i_p \in I.$$

Then the sheaf cohomology group $H^q(\Sigma, E_{\mathbb{C}})$ of $E_{\mathbb{C}}$ is isomorphic to the Čech cohomology group $H^q(\mathcal{A}_{\mathbb{C}}, \mathcal{E})$, for all q. We have seen that for any open subset $U \subset \Sigma$, there is a conjugate linear involution $\tilde{\sigma}\colon \mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}}) \to \mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}})$ whose fixed locus $\mathcal{E}_{\mathbb{C}}(U_{\mathbb{C}})^{\tilde{\sigma}} = (\mathcal{E}, \mathcal{E}_{\mathbb{R}})(U)$. Note that $\tilde{\sigma}$ acts on the Čech complex $C^\bullet(\mathcal{A}_{\mathbb{C}}, \mathcal{E}_{\mathbb{C}})$, and the fixed locus $C^\bullet(\mathcal{A}_{\mathbb{C}}, \mathcal{E}_{\mathbb{C}})^{\tilde{\sigma}}$ is the Čech complex $C^\bullet(\mathcal{A}, \mathcal{E}, \mathcal{E}_{\mathbb{R}})$. So $\tilde{\sigma}$ acts on $H^q(\Sigma_{\mathbb{C}}, E_{\mathbb{C}}) \cong H^q(\mathcal{A}_{\mathbb{C}}, \mathcal{E}_{\mathbb{C}})$, and $H^q(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) \cong H^q(\Sigma_{\mathbb{C}}, E_{\mathbb{C}})^{\tilde{\sigma}}$. In particular,

$$\dim_{\mathbb{R}} H^q(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) = \dim_{\mathbb{C}} H^q(\Sigma_{\mathbb{C}}, E_{\mathbb{C}}).$$

Definition 3.4.1 Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle over a bordered Riemann surface Σ. The *index* of $(E, E_{\mathbb{R}})$ is the virtual real vector space

$$Ind(E, E_{\mathbb{R}}) \equiv H^0(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) - H^1(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}).$$

The *Euler characteristic* of $(E, E_{\mathbb{R}})$ is the virtual dimension

$$\chi(E, E_{\mathbb{R}}) \equiv \dim_{\mathbb{R}} H^0(\Sigma, \partial\Sigma, E, E_{\mathbb{R}}) - \dim_{\mathbb{R}} H^1(\Sigma, \partial\Sigma, E, E_{\mathbb{R}})$$

of the virtual vector space $Ind(E, E_{\mathbb{R}})$.

Note that $\chi(E, E_{\mathbb{R}})$ is the index of the Fredholm operator $\bar{\partial}_{(E, E_{\mathbb{R}})}$. Furthermore, $\chi(E, E_{\mathbb{R}}) = \chi(E_{\mathbb{C}})$. We have the following Riemann–Roch theorem for bordered Riemann surfaces.

Theorem 3.4.2 *Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle of rank n over a bordered Riemann surface Σ of type $(g; h)$. Then*

$$\chi(E, E_{\mathbb{R}}) = \mu(E, E_{\mathbb{R}}) + n\chi(\Sigma),$$

where $\chi(\Sigma) = 2 - 2g - h$ is the Euler characteristic of Σ.

Proof Let $\tilde{g} = 2g + h - 1$ be the genus of $\Sigma_{\mathbb{C}}$. Then

$$\begin{aligned} \chi(E, E_{\mathbb{R}}) &= \chi(E_{\mathbb{C}}) \\ &= \deg E_{\mathbb{C}} + n(1 - \tilde{g}) \\ &= \mu(E, E_{\mathbb{R}}) + n(2 - 2g - h) \\ &= \mu(E, E_{\mathbb{R}}) + n\chi(\Sigma), \end{aligned}$$

where the second equality follows from Riemann–Roch and the third equality comes from Theorem 3.3.8. □

The computations in the following examples will be useful in the sequel.

Example 3.4.3 Let m be an integer. Consider the line bundle $(L(m), L(m)_{\mathbb{R}})$ over (D^2, S^1), where $L(m)$ is the trivial bundle, and the fiber of $L(m)_{\mathbb{R}}$ over $z = e^{i\theta}$ is $ie^{(im\theta/2)}\mathbb{R} \subset \mathbb{C}$. Then $\mu(L(m), L(m)_{\mathbb{R}}) = m$. The complex double $D^2_{\mathbb{C}}$ can be identified with $\mathbf{P}^1$. The antiholomorphic involution is $\sigma(z) = 1/\overline{z}$ in an affine coordinate z on $\mathbf{P}^1$, and $L(m)_{\mathbb{C}}$ can be identified with $\mathcal{O}_{\mathbf{P}^1}(m)$.

Explicitly, let (z, u) and $(\widetilde{z}, \widetilde{u})$ be the two charts of the total space of $\mathcal{O}_{\mathbf{P}^1}(m)$, related by $(\widetilde{z}, \widetilde{u}) = (1/z, -z^{-m}u)$. There is an antiholomorphic involution

$$\begin{aligned} \widetilde{\sigma}\colon \mathcal{O}_{\mathbf{P}^1}(m) &\longrightarrow \mathcal{O}_{\mathbf{P}^1}(m) \\ (z, u) &\longmapsto \left(\frac{1}{\overline{z}}, -\overline{z}^{-m}\overline{u}\right) \end{aligned}$$

expressed in terms of the first chart (it clearly antiholomorphic over $z = 0$ as well, as $\widetilde{\sigma}(z, u) = (\overline{z}, \overline{u})$ when the image is expressed in the coordinates of the second chart).

Note that $\widetilde{\sigma}$ covers $\sigma\colon \mathbf{P}^1 \to \mathbf{P}^1$, and the fixed locus of $\widetilde{\sigma}$ is $L(m)_{\mathbb{R}}$. For any nonnegative integer m we have

$$\begin{aligned} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m)) &= \left\{ \sum_{j=0}^{m} a_j X_0^{m-j} X_1^j \;\middle|\; a_j \in \mathbb{C} \right\}, \\ &\quad \widetilde{\sigma}(a_0, a_1, \ldots, a_m) = (-\overline{a}_m, \ldots, -\overline{a}_1, -\overline{a}_0) \\ H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m)) &= 0 \\ H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-m-1)) &= 0 \\ H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-m-1)) &= \left\{ \sum_{j=1}^{m} \frac{a_j}{X_0^{m-j} X_1^j} \;\middle|\; a_j \in \mathbb{C} \right\}, \\ &\quad \widetilde{\sigma}(a_1, a_2, \ldots, a_m) = (-\overline{a}_m, \ldots, -\overline{a}_2, -\overline{a}_1), \end{aligned}$$

where (X_0, X_1) are homogeneous coordinates on $\mathbf{P}^1$, related to z by $z = \frac{X_1}{X_0}$, and the classes in H^1 are expressed as Čech cohomology classes using the standard open cover of $\mathbf{P}^1$. Therefore,

$$\begin{aligned} H^0(D^2, S^1, L(m), L(m)_{\mathbb{R}}) &= \left\{ f(z) = \sum_{j=0}^{m} a_j z^j \;\middle|\; a_{m-j} = -\overline{a}_j \right\} \\ H^1(D^2, S^1, L(m), L(m)_{\mathbb{R}}) &= 0 \\ H^0(D^2, S^1, L(-m-1), L(-m-1)_{\mathbb{R}}) &= 0 \end{aligned}$$

$$H^1(D^2,S^1,L(-m-1),L(-m-1)_{\mathbb{R}}) \cong \left\{ f(z)=\sum_{j=1}^{m}\frac{a_j}{z^j} \;\middle|\; a_{m+1-j}=-\overline{a}_j \right\}.$$

Example 3.4.4 Consider next the bundle $(N(d),N(d)_{\mathbb{R}})$ on (D^2,S^1), where $N(d)$ is the trivial rank 2 bundle, and the fiber of $N(d)_{\mathbb{R}}$ over $z\in S^1$ is

$$\left\{(u,v)\in\mathbb{C}^2 \mid v=\overline{z}^d\overline{u}\right\}.$$

Then $\mu(N(d),N(d)_{\mathbb{R}})=-2d$, and $N(d)_{\mathbb{C}}=\mathcal{O}_{\mathbf{P}^1}(-d)\oplus\mathcal{O}_{\mathbf{P}^1}(-d)$. Let (z,u,v) and $(\widetilde{z},\widetilde{u},\widetilde{v})$ be the two charts of $\mathcal{O}_{\mathbf{P}^1}(-d)\oplus\mathcal{O}_{\mathbf{P}^1}(-d)$, related by

$$(\widetilde{z},\widetilde{u},\widetilde{v})=(1/z,z^du,z^dv).$$

There is an antiholomorphic involution

$$\begin{aligned}\widetilde{\sigma}\colon \mathcal{O}_{\mathbf{P}^1}(-d)\oplus\mathcal{O}_{\mathbf{P}^1}(-d) &\longrightarrow \mathcal{O}_{\mathbf{P}^1}(-d)\oplus\mathcal{O}_{\mathbf{P}^1}(-d)\\ (z,u,v)&\longrightarrow (\frac{1}{\overline{z}},\overline{z}^d\overline{v},\overline{z}^d\overline{u})\end{aligned}$$

in terms of the first chart. The involution $\widetilde{\sigma}$ covers $\sigma\colon\mathbf{P}^1\to\mathbf{P}^1$, and the fixed locus of $\widetilde{\sigma}$ is $N(d)_{\mathbb{R}}$. Note that $H^0(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}(-d)\oplus\mathcal{O}_{\mathbf{P}^1}(-d))=0$, and

$$H^1(\mathbf{P}^1,\mathcal{O}(-d)\oplus\mathcal{O}(-d))=\left\{\left(\sum_{j=1}^{d-1}\frac{a_{-j}}{X_0^jX_1^{d-j}},\sum_{j=1}^{d-1}\frac{a_j}{X_0^{d-j}X_1^j}\right)\;\middle|\; a_{-j},a_j\in\mathbb{C}\right\},$$

while

$$\widetilde{\sigma}(a_{-(d-1)},\dots,a_{-1},a_1,\dots,a_{d-1})=(\overline{a}_{d-1},\dots,\overline{a}_1,\overline{a}_{-1},\dots,\overline{a}_{-(d-1)}).$$

Therefore, $H^0(D^2,S^1,N(d),N(d)_{\mathbb{R}})=0$, and

$$H^1(D^2,S^1,N(d),N(d)_{\mathbb{R}})\cong\left\{\left(\sum_{j=1}^{d-1}\frac{\overline{a}_j}{z^{d-j}},\sum_{j=1}^{d-1}\frac{a_j}{z^j}\right)\;\middle|\; a_j\in\mathbb{C}\right\}.$$

3.5 Moduli spaces of bordered Riemann surfaces

Let $M_{g;h}$ denote the moduli space of isomorphism classes of bordered Riemann surfaces of type $(g;h)$, where $2g+h>2$. It is a double cover of $M(2g+h-1,h,0)$, the moduli space of symmetric Riemann surface of type $(2g+h-1,h,0)$, which is a semialgebraic space of real dimension $-3\chi(\Sigma_{g;h})=6g+3h-6$ Seppälä–Silhol [22], where $\chi(\Sigma_{g;h})=2-2g-h$ is the Euler characteristic of a bordered Riemann surface of type $(g;h)$. The covering map is given by $[\Sigma]\mapsto[\Sigma_{\mathbb{C}}]$, and the preimage of $[\Sigma_{\mathbb{C}}]$

is $\{[\Sigma], [\overline{\Sigma}]\}$. Note that Σ and $\overline{\Sigma}$ are not isomorphic bordered Riemann surfaces for generic Σ, so this cover is generically 2 to 1.

3.6 Stable bordered Riemann surfaces

Definition 3.6.1 A *node* on a singular bordered Riemann surface Σ is either a singularity on Σ° isomorphic to $(0,0) \in \{xy = 0\}$ or a singularity on $\partial\Sigma$ isomorphic to $(0,0) \in \{xy = 0\}/A$, where (x, y) are coordinates on $\mathbb{C}^2$, $A(x,y) = (\bar{x}, \bar{y})$ is the complex conjugation. A *nodal bordered Riemann surface* is a singular bordered Riemann surface whose singularities are nodes.

The notion of morphisms and complex doubles can be easily extended to nodal bordered Riemann surfaces. The complex double of a nodal bordered Riemann surface is a nodal compact symmetric Riemann surface. The boundary of a nodal Riemann surface is a union of circles, where the intersection of any two distinct circles is a finite set.

Definition 3.6.2 Let Σ be a nodal bordered Riemann surface. The antiholomorphic involution σ on its complex double $\Sigma_{\mathbb{C}}$ can be lifted to $\hat{\sigma}\colon \widehat{\Sigma_{\mathbb{C}}} \to \widehat{\Sigma_{\mathbb{C}}}$, where $\widehat{\Sigma_{\mathbb{C}}}$ is the normalization of $\Sigma_{\mathbb{C}}$ (viewed as a complex algebraic curve). The *normalization* of $\Sigma = \Sigma_{\mathbb{C}}/\langle\sigma\rangle$ is defined to be $\hat{\Sigma} = \widehat{\Sigma_{\mathbb{C}}}/\langle\hat{\sigma}\rangle$.

From the above definition, the complex double of the normalization is the normalization of the complex double, ie, $\hat{\Sigma}_{\mathbb{C}} = \widehat{\Sigma_{\mathbb{C}}}$.

Definition 3.6.3 A *prestable* bordered Riemann surface is either a smooth bordered Riemann surface or a nodal bordered Riemann surface. A *stable* bordered Riemann surface is a prestable bordered Riemann surface whose automorphism group is finite.

The complex double of a stable bordered Riemann surface is a stable complex algebraic curve. A smooth bordered Riemann surface of type $(g; h)$ is stable if and only if $2g + h > 2$.

The stable compactification $\overline{M}_{g;h}$ of $M_{g;h}$ is a double cover of the stable compactification $\overline{M}(2g+h-1, h, 0)$ of $M(2g+h-1, h, 0)$, which is compact and Hausdorff [21]. The covering map described in Section 3.5 has an obvious extension to $\overline{M}_{g;h} \to \overline{M}(2g+h-1, h, 0)$.

3.7 Riemann–Roch theorem for prestable bordered Riemann surfaces

Let Σ be a nodal bordered Riemann surface such that $H_2(\Sigma;\mathbb{Z})=0$. Let E be a complex vector bundle over Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. In this situation, E is a topologically trivial complex vector bundle. One can trivialize E and define the *Maslov index* $\mu(E,E_{\mathbb{R}})$ which is independent of the choice of the trivialization as before, since the proof of Proposition 3.3.6 is valid for nodal Σ as well.

Definition 3.7.1 A prestable bordered Riemann surface is *irreducible* if its complex double is an irreducible complex algebraic curve.

Let Σ be a nodal bordered Riemann surface, let $C_1,\dots,C_\nu$ be the irreducible components of Σ which are (possibly nodal) Riemann surfaces, and let $\Sigma_1,\dots,\Sigma_{\nu'}$ be the remaining irreducible components of Σ, which are (possibly nodal) bordered Riemann surfaces. Then the irreducible components of $\Sigma_{\mathbb{C}}$ are

$$C_1,\dots,C_\nu,\ \overline{C}_1,\dots,\overline{C}_\nu,\ (\Sigma_1)_{\mathbb{C}},\dots,(\Sigma_{\nu'})_{\mathbb{C}}. \tag{6}$$

Let E be a complex vector bundle over Σ and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. Observe that $H_2(\Sigma_{i'};\mathbb{Z})=0$, so $\mu(E|_{\Sigma_{i'}},E_{\mathbb{R}}|_{\partial\Sigma_{i'}})$ is defined for $i'=1,\dots,\nu'$.

Definition 3.7.2 Let Σ, E, $E_{\mathbb{R}}$ be as above. The *Maslov index* of (E,E_{R}) is defined by

$$\mu(E,E_{\mathbb{R}})=2\sum_{i=1}^{\nu}\deg(E|_{C_i})+\sum_{i'=1}^{\nu'}\mu(E|_{\Sigma_{i'}},E_{\mathbb{R}}|_{\partial\Sigma_{i'}}).$$

Let Σ, C_i, $\Sigma_{i'}$ be as above, $i=1,\dots,\nu$, $i'=1,\dots,\nu'$. Then

$$H_2(\Sigma_{\mathbb{C}},\mathbb{Z})=\bigoplus_{i=1}^{\nu}\mathbb{Z}[C_i]\oplus\bigoplus_{i=1}^{\nu}\mathbb{Z}[\overline{C}_i]\oplus\bigoplus_{i'=1}^{\nu'}\mathbb{Z}[(\Sigma_i)_{\mathbb{C}}].$$

Set

$$[\Sigma_{\mathbb{C}}]=\sum_{i=1}^{\nu}[C_i]+\sum_{i=1}^{\nu}[\overline{C}_i]+\sum_{i'=1}^{\nu'}[(\Sigma_i)_{\mathbb{C}}]\in H_2(\Sigma_{\mathbb{C}},\mathbb{Z}).$$

With this notation, we have the following generalization of Theorem 3.3.8:

Theorem 3.7.3 *Let E be a complex vector bundle over a prestable bordered Riemann surface Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. Then there is a complex vector bundle $E_{\mathbb{C}}$ on $\Sigma_{\mathbb{C}}$ together with a conjugate linear involution $\tilde{\sigma}\colon E_{\mathbb{C}}\to E_{\mathbb{C}}$*

covering the antiholomorphic involution $\sigma\colon \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ *such that* $E_{\mathbb{C}}|_{\Sigma} = E$ *and the fixed locus of* $\tilde{\sigma}$ *is* $E_{\mathbb{R}} \to \partial\Sigma$*. Moreover, we have*

$$\deg E_{\mathbb{C}} = \mu(E, E_{\mathbb{R}}).$$

Proof The construction of $E_{\mathbb{C}}$ is elementary, as in the smooth case. The reasoning in the proof of Theorem 3.3.8 shows that

$$\deg(E|_{\Sigma_{i'}})_{\mathbb{C}} = \mu(E|_{\Sigma_{i'}}, E_{\mathbb{R}}|_{\partial\Sigma_{i'}}),$$

so that

$$\begin{aligned}
\deg E_{\mathbb{C}} &= \sum_{i=1}^{\nu} \deg(E_{\mathbb{C}}|_{C_i}) + \sum_{i=1}^{\nu} \deg(E_{\mathbb{C}}|_{\overline{C}_i}) + \sum_{i'=1}^{\nu'} \deg(E_{\mathbb{C}}|_{\Sigma_{i'}}) \\
&= 2\sum_{i=1}^{\nu} \deg(E|_{C_i}) + \sum_{i'=1}^{\nu'} \deg(E|_{\Sigma_{i'}})_{\mathbb{C}} \\
&= 2\sum_{i=1}^{\nu} \deg(E|_{C_i}) + \sum_{i'=1}^{\nu'} \mu(E|_{\Sigma_{i'}}, E_{\mathbb{R}}|_{\partial\Sigma_{i'}}) \\
&= \mu(E, E_{\mathbb{R}}).
\end{aligned}$$

□

Definition 3.7.4 We call the bundle $E_{\mathbb{C}}$ as in Theorem 3.7.3 the *topological complex double* of $(E, E_{\mathbb{R}})$.

Definition 3.7.5 Let E be a complex vector bundle of rank n over a prestable bordered Riemann surface Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ is called a *Riemann–Hilbert bundle* over Σ if there is an open cover $\{U_i \mid i \in I\}$ of Σ together with trivializations $\Phi_i\colon E|_{U_i} \cong U_i \times \mathbb{C}^n$ such that

$$\begin{aligned}
\Phi_{ij} \equiv \Phi_i \circ \Phi_j^{-1}\colon (U_i \cap U_j) \times \mathbb{C}^n &\longrightarrow (U_i \cap U_j) \times \mathbb{C}^n \\
(x, u) &\mapsto (x, g_{ij}(x)u)
\end{aligned}$$

where $g_{ij}\colon U_i \cap U_j \to GL(n, \mathbb{C})$ is holomorphic, and $g_{ij}(U_i \cap U_j \cap \partial\Sigma) \subset GL(n, \mathbb{R})$. The trivialization $\{(U_i, \Phi_i) \mid i \in I\}$ is called a *Riemann–Hilbert trivialization* of $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$.

The construction of the *holomorphic complex double* of a Riemann–Hilbert bundle can be easily extended to Riemann–Hilbert bundles over nodal bordered Riemann surfaces. Let $(E, E_{\mathbb{R}}) \to (\Sigma, \partial\Sigma)$ be a Riemann–Hilbert bundle of rank n over a prestable bordered Riemann surface Σ, and let $E_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ be its complex double. We define $Ind(E, E_{\mathbb{R}})$ and $\chi(E, E_{\mathbb{R}})$ as in Definition 3.4.1. Letting $\widetilde{g}$ denote the arithmetic genus of $\Sigma_{\mathbb{C}}$, we have

$$\begin{aligned}\chi(E, E_{\mathbb{R}}) &= \chi(E_{\mathbb{C}}) \\ &= \deg(E_{\mathbb{C}}) + n(1-\widetilde{g}) \\ &= \mu(E, E_{\mathbb{R}}) + n(1-\widetilde{g}),\end{aligned}$$

where the second identity follows from Riemann–Roch and the third comes from Theorem 3.7.3. Therefore, we have the following Riemann–Roch theorem for prestable bordered Riemann surfaces:

Theorem 3.7.6 *Let E be a Riemann–Hilbert bundle of rank n over a prestable bordered Riemann surface Σ. Then*

$$\chi(E, E_{\mathbb{R}}) = \mu(E, E_{\mathbb{R}}) + n(1-\widetilde{g}),$$

where $\widetilde{g}$ is the arithmetic genus of $\Sigma_{\mathbb{C}}$, the complex double of Σ.

4 Stable maps and their moduli spaces

4.1 Stable maps

Definition 4.1.1 Let Σ be a prestable bordered Riemann surface, let (X, J, ω) be a symplectic manifold together with an almost complex structure J compatible with the symplectic form ω, and let $L \subset X$ be a Lagrangian submanifold of X with respect to ω. A *prestable map* $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ is a continuous map which is J–holomorphic on Σ°.

Definition 4.1.2 A *morphism* between two prestable maps $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ and $f'\colon (\Sigma', \partial\Sigma') \to (X, L)$ is a morphism $g\colon (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ such that $f = f' \circ g$. If $f = f'$ and g is an isomorphism, then g is called an automorphism of f. A *stable map* is a prestable map whose automorphism group is finite.

Let (X, J, ω) and $L \subset X$ be as in Definition 4.1.1. Let

$$\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$$

denote the moduli space of isomorphism classes of stable maps $f\colon (\Sigma, \partial\Sigma) \to (X, L)$ such that $f_*[\Sigma] = \beta \in H_2(X, L; \mathbb{Z})$ and $f_*[R_i] = \gamma_i \in H_1(L)$, where Σ is a prestable bordered Riemann surface of type $(g; h)$, and R_1, ..., R_h are the connected components of $\partial\Sigma$ with the standard orientations on the R_i. Here an isomorphism between stable maps is an isomorphism in the sense of Definition 4.1.2 which preserves the ordering of the boundary components. A necessary condition for the moduli space to be non-empty is $\partial\beta = \sum \gamma_i$.

The moduli space $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$ has extra structure, which will be described elsewhere. For our present purposes, we content ourselves with remarking that $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$ should be thought of as an orbi-space due to the presence of nontrivial automorphisms. The only additional structure we will use arises from deformations and obstructions, to be described in Section 4.2. We will frequently be in the situation where X is a complex algebraic manifold, J is the complex structure, and ω is a Kähler form. In this situation, we will frequently use algebro-geometric language and related constructions in describing $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$.

4.2 Virtual dimension of the moduli space

Let (X, J, ω) be as in Definition 4.1.1, and let $L \subset X$ be Lagrangian with respect to ω. We want to define and compute the virtual dimension of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$, where $\beta \in H_2(X, L; \mathbb{Z})$ and $\gamma_1, ..., \gamma_h \in H_1(L)$. We propose that there is a tangent-obstruction exact sequence of sheaves on $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$ much as in ordinary Gromov–Witten theory[5]

$$
\begin{aligned}
0 \;\to\;\; & H^0(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma}) \to H^0(\Sigma, \partial\Sigma, f^*T_X, (f|_{\partial\Sigma})^*T_L) \to \mathcal{T}^1 \\
\to\;\; & H^1(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma}) \to H^1(\Sigma, \partial\Sigma, f^*T_X, (f|_{\partial\Sigma})^*T_L) \to \mathcal{T}^2 \to 0
\end{aligned} \tag{7}
$$

The 4 terms other than the $\mathcal{T}^i$ are labeled by their fibers. We will continue to use this description, analogous to that described in [5, Section 7.1] in Gromov–Witten theory, leaving a more formal treatment for future work.

The virtual (real) dimension of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \dots, \gamma_h)$ at f is defined to be

$$
\operatorname{rank}\left(\mathcal{T}^1\right) - \operatorname{rank}\left(\mathcal{T}^2\right). \tag{8}
$$

The $\mathcal{T}^i$ need not be bundles, but the rank can be computed fiberwise. Using (7), the virtual dimension is just

$$
\chi\left(f^*T_X, (f|_{\partial\Sigma})^*T_L\right) - \chi(T_\Sigma, T_{\partial\Sigma}). \tag{9}
$$

For ease of exposition, assume that $(\Sigma, \partial\Sigma)$ is smooth. By Theorem 3.4.2, we get for the virtual dimension

$$
\mu\left(f^*T_X, (f|_{\partial\Sigma})^*T_L\right) - \mu\left(\chi(T_\Sigma, T_{\partial\Sigma})\right) + (n-1)\chi(\Sigma),
$$

[5]The term $H^0(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma})$ is correct only for smooth Σ. In general, we need to consider instead vector fields which vanish at the nodes, together with a contribution for the smoothing of the nodes [13].

where $n = \dim X$. Noting that

$$\mu\left(\chi(T_\Sigma, T_{\partial\Sigma})\right) = 2\chi(\Sigma)$$

(as can be seen for example by doubling), the virtual dimension (9) becomes

$$\mu\left(f^*T_X, (f|_{\partial\Sigma})^*T_L\right) + (n-3)\chi(\Sigma). \tag{10}$$

The above discussion leading up to (10) can be extended to general Σ using Theorem 3.7.6 in place of Theorem 3.4.2.

Suppose that L is the fixed locus of an antiholomorphic involution $A\colon X \to X$. Then in this situation, $f_{\mathbb{C}}^*T_X$ is the complex double of $(f^*T_X, (f|_{\partial\Sigma})^*T_L)$ and $T_{\Sigma_{\mathbb{C}}}$ is the complex double of $(T_\Sigma, T_{\partial\Sigma})$ (we only need the complex double of a bundle, but in this case it happens that the double comes from the complex double of a map). This implies that

$$\mu\left(f^*T_X, (f|_{\partial\Sigma})^*T_L\right) = \deg\left(f_{\mathbb{C}}^*T_X\right).$$

This gives for the virtual dimension

$$\deg\left(f_{\mathbb{C}}^*T_X\right) + (n-3)\chi(\Sigma). \tag{11}$$

We could have obtained the same result by Riemann–Roch on $\Sigma_{\mathbb{C}}$, noting that the genus $\widetilde{g} = 2g + h - 1$ of $\Sigma_{\mathbb{C}}$ satisfies $1 - \widetilde{g} = \chi(\Sigma)$. It is clear from (11) that in this situation, the virtual dimension (8) is independent of the chosen map f in the moduli space.

The formula (11) coincides with the well known formula for the virtual dimension of $\overline{M}_{\widetilde{g},0}(X, (f_{\mathbb{C}})_*[\Sigma_{\mathbb{C}}])$ in ordinary Gromov–Witten theory [5, Section 7.1.4]. Each of the terms in (7) is in fact the fixed locus of the induced antiholomorphic involution on the terms of the corresponding sequence in Gromov–Witten theory.

Remark 4.2.1 The stack $\overline{M}_{\widetilde{g},0}(X, (f_{\mathbb{C}})_*[\Sigma_{\mathbb{C}}])$ has a natural antiholomorphic involution induced from that of X, and it is natural to try to compare the fixed locus of this involution with $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$. To address this, we could have expressed our theory in terms of *symmetric stable maps*, ie, stable maps from symmetric Riemann surfaces to X which are compatible with the symmetry of the Riemann surface and the antiholomorphic involution on X in the natural sense. But there are subtleties: non-isomorphic symmetric stable maps can give rise to isomorphic stable maps, and Remark 3.3.5 says in this context that not all symmetric stable maps arise as doubles of stable maps of bordered Riemann surfaces.

Remark 4.2.2 Note that there are subtleties in the notion of the virtual dimension in the general case, since the Maslov index is a homotopy invariant rather than a homology invariant in general.

We will be considering two situations in this paper. For the first, consider $(\mathbf{P}^1, J, \omega)$, where J is the standard complex structure, and ω is the Kähler form of the Fubini–Study metric. Set $\beta = [D^2] \in H_2(\mathbf{P}^1, S^1; \mathbb{Z})$, and $\gamma = [S^1] \in H_1(S^1)$. We denote $\overline{M}_{g;h}(\mathbf{P}^1, S^1 | d\beta; n_1\gamma, ..., n_h\gamma)$ by $\overline{M}_{g;h}(D^2, S^1 \mid d; n_1, \ldots, n_h)$.

If $f \in \overline{M}_{g;h}(D^2, S^1 \mid d; n_1, \ldots, n_h)$, then $\deg(f_{\mathbb{C}}^* T_X) = 2d$. So (11) gives $2(d + 2g + h - 2)$ as the virtual dimension of $\overline{M}_{g;h}(D^2, S^1 \mid d; n_1, \ldots, n_h)$.

The other situation is (X, ω, Ω), where X is a Calabi–Yau n–fold, ω is the Kähler form of a Ricci flat metric on X, Ω is a holomorphic n–form, and

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega},$$

where Ω is normalized so that it is a calibration Harvey–Lawson [11, Definition 4.1]. Let $L \subset X$ be a submanifold which is the fixed locus of an antiholomorphic isometric involution $A\colon X \to X$ so that L is a special Lagrangian submanifold of (X, ω, Ω).

In this situation, $\deg(f_{\mathbb{C}}^* T_X) = 0$ because $c_1(T_X) = 0$. So the virtual real dimension of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$ is

$$(n-3)\chi(\Sigma) = (n-3)(2-2g-h).$$

Observe that the virtual dimension does not depend on the classes $\beta, \gamma_1, ..., \gamma_h$. In particular, the virtual dimension of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$ is 0 if $n = 3$.

Remark 4.2.3 This computation can be done without A and the doubling construction using the vanishing of the Maslov index for special Lagrangians.

Continuing to assume the existence of an antiholomorphic involution, we propose the existence of a virtual fundamental class $[\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)]^{\mathrm{vir}}$ on $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$ of dimension equal to the virtual dimension (10) or (11). The only assumption we will need to make on this class is that it can be computed using (7) and a torus action in our situation, as will be explained in more detail later.

More precisely, we are not proposing the existence of an intrinsic notion of the virtual fundamental class, but rather a family of virtual fundamental classes depending on additional choices made at the boundary of $\overline{M}_{g;h}(X, L \mid \beta; \gamma_1, \ldots, \gamma_h)$. We will see later that different choices lead to different computations of the desired enumerative invariants in examples.

5 Torus action

In this section, we define $U(1)$ actions on moduli spaces and compute the weights of certain $U(1)$ representations that we will need later.

5.1 Torus action on moduli spaces

Let z be an affine coordinate on $\mathbf{P}^1$, and put $D^2 = \{z : |z| \leq 1\} \subset \mathbf{P}^1$, with its boundary circle denoted by S^1 with the standard orientation induced by the complex structure.

Consider the action of $U(1)$ on $(\mathbf{P}^1, S^1)$:

$$z \mapsto e^{-i\theta} z, \qquad e^{i\theta} \in U(1). \tag{12}$$

This action has been chosen for consistency with [5] and [6]. The action (12) also preserves D^2 and therefore induces an action of $U(1)$ on $\overline{M}_{g;h}(D^2, S^1 \,|\, d; n_1, \dots, n_h)$ by composing a stable map with the action on the image.

More generally, if (X, L) is a pair with a $U(1)$ action, then $U(1)$ acts naturally on any space of stable maps to (X, L).

The fixed locus $F_{0;1|d;d}$ of the $U(1)$ action on $\overline{M}_{0;1}(D^2, S^1|d; d)$ consists of a single point corresponding the map $f\colon D^2 \to D^2$, $x \mapsto x^d$, which has an automorphism group of order d. The fixed locus $F_{0;2|d;n_1,n_2}$ of the $U(1)$ action on $\overline{M}_{0;2}(D^2, S^1|d; n_1, n_2)$ consists of a single point corresponding the map $f\colon D_1 \cup D_2 \to D^2$, where D_1 and D_2 are two discs glued at the origin to form a node. The restrictions of f to D_1 and to D_2 are $x_1 \mapsto x_1^{n_1}$ and $x_2 \mapsto x_2^{n_2}$, where x_1 and x_2 are coordinates on D_1 and D_2, respectively. For $(g; h) \neq (0; 1), (0; 2)$, the $U(1)$ action on $\overline{M}_{g;h}(D^2, S^1 \,|\, d; n_1, \dots, n_h)$ has a single fixed component $F_{g;h|n_1,\dots,n_h}$, consisting of the following stable maps $f\colon (\Sigma, \partial\Sigma) \to (D, S^1)$:

- The source curve Σ is a union $\Sigma_0 \cup D_1 \dots \cup D_h$, where $\Sigma_0 = (\Sigma_0, p_1, \dots, p_h)$ is an h–pointed stable curve of genus g, and $D_1, \dots, D_h$ are discs. The origin of D_i is glued to Σ_0 at p_i to form a node.
- f collapses Σ_0 to 0.
- $f|_{D_i}\colon D_i \to D$ is the cover $f(x) = x^{n_i}$.

The fixed locus $F_{g;h|n_1,\dots,n_h}$ in $\overline{M}_{g;h}(D^2, S^1 \,|\, d; n_1, \dots, n_h)$ is an algebraic stack contained in $\overline{M}_{g;h}(D^2, S^1 \,|\, d; n_1, \dots, n_h)$. It is isomorphic to a quotient of $\overline{M}_{g;h}$ by two types of automorphisms: the automorphisms induced by the covering transformations of the $f|_{D_i}$, and the automorphisms induced by permutations of the p_i which leave the n_i unchanged.

5.2 Torus action on holomorphic vector bundles

Let $E_{\mathbb{C}}$ be a holomorphic vector bundle on $\mathbf{P}^1$, and $\widetilde{\sigma}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ an antiholomorphic involution which covers $\sigma\colon \mathbf{P}^1 \to \mathbf{P}^1$, given by $\sigma(z) = 1/\overline{z}$ in an affine coordinate.

The fixed locus of $\widetilde{\sigma}$ is the total space of a real vector bundle $E_{\mathbb{R}} \to S^1$, and $E_{\mathbb{C}}|_{S^1} = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The group $\mathbf{C}^*$ acts on $\mathbf{P}^1$ by $z \to \lambda z$ for $\lambda \in \mathbf{C}^*$. We want to lift the $\mathbf{C}^*$ action on $\mathbf{P}^1$ to E in such a way that $U(1) \subset \mathbf{C}^*$ acts on $E_{\mathbb{R}}$.

Definition 5.2.1 A lifting of the $\mathbf{C}^*$ action on $\mathbf{P}^1$ to E is *compatible with* $\widetilde{\sigma}$ if $\widetilde{\sigma}(\lambda \cdot v) = \overline{\lambda}^{-1} \cdot \widetilde{\sigma}$ for all $\lambda \in \mathbf{C}^*$ and $v \in E$.

If the $\mathbf{C}^*$ action is compatible with $\widetilde{\sigma}$, then $U(1)$ automatically acts on $E_{\mathbb{R}}$ as desired.

We now establish some terminology. We say that a lifting of a $\mathbf{C}^*$ action from $\mathbf{P}^1$ to a line bundle L has weight $[a,b]$ if the induced $\mathbf{C}^*$ actions on the fibers over the fixed points on $\mathbf{P}^1$ have respective weights a and b. We say that a lifting to a sum $L_1 \oplus L_2$ of line bundles has weight $[a,b]$, $[c,d]$ if the lifting to L_1 has weight $[a,b]$ and the lifting to L_2 has weight $[c,d]$.

Example 5.2.2 Let $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ be given in affine coordinates by $z = f(x) = x^d$. The antiholomorphic involution σ on $\mathbf{P}^1$ has a canonical lifting to $T_{\mathbf{P}^1}$, thus a canonical lifting $\widetilde{\sigma}$ to $f^*T_{\mathbf{P}^1}$. The canonical lifting of the torus action on $\mathbf{P}^1$ to $f^*T_{\mathbf{P}^1}$ has weights $[1,-1]$ at the respective fixed points [10, Section 2] and is compatible with $\widetilde{\sigma}$.

The corresponding weights of the $\mathbf{C}^*$ action on $H^0(\mathbf{P}^1, f^*T_{\mathbf{P}^1})$ are

$$\left\{\frac{d}{d}, \frac{d-1}{d}, \cdots, \frac{1}{d}, 0, -\frac{1}{d}, \cdots, -\frac{d}{d}\right\}$$

with respect to the ordered basis

$$\mathcal{B} = \left(x^{2d}\frac{\partial}{\partial z}, x^{2d-1}\frac{\partial}{\partial z}, \cdots, \frac{\partial}{\partial z}\right)$$

expressed in affine coordinates.

We compute that

$$\begin{aligned} H^0(\mathbf{P}^1, f^*T_{\mathbf{P}^1})^{\widetilde{\sigma}} &\simeq H^0(D^2, S^1, (f|_{D^2})^*T_{\mathbf{P}^1}, (f|_{S^1})^*T_{S^1}) \\ &= H^0(D^2, S^1, L(2d), L(2d)_{\mathbb{R}}) \end{aligned} \tag{13}$$

(see Example 3.4.3 for the definition of $(L(m), L(m)_{\mathbb{R}})$). For later use, note that (13) can be identified with the kernel V of

$$H^0(\mathbf{P}^1, \mathcal{O}(d)) \oplus H^0(D^2, S^1, \mathbb{C}, \mathbb{R}) \to \mathbb{C}$$

$$(\sum_{j=0}^{d} a_j X_0^{d-i} X_1^i, b) \mapsto a_d - b.$$

Explicitly, the identification is given by

$$\begin{array}{rcl} H^0(D^2, S^1, L(2d), L(2d)_{\mathbb{R}}) & \to & V \\ \sum_{j=0}^{d-1}(a_j x^j - \overline{a}_j x^{2d-j}) + ibx^d & \mapsto & (\sum_{j=0}^{d-1} a_j X_0^{d-j} X_1^j + bX_1^d, b) \end{array} \tag{14}$$

where $a_j \in \mathbb{C}$, $b \in \mathbb{R}$.

We have a $U(1)$ action on V defined by giving X_0 weight $1/d$ and X_1 weight 0. Then (14) preserves the $U(1)$ actions.

We now introduce notation for real representations of $U(1)$. We let $(0)_{\mathbb{R}}$ denote the trivial representation on $\mathbb{R}^1$, and (w) denote the representation

$$e^{i\theta} \mapsto \begin{pmatrix} \cos w\theta & -\sin w\theta \\ \sin w\theta & \cos w\theta \end{pmatrix}$$

on $\mathbb{R}^2$ (or $e^{i\theta} \mapsto e^{iw\theta}$ on $\mathbb{C}$ if we identify $\mathbb{R}^2$ with $\mathbb{C}$).

With this notation, the corresponding representation of $U(1) \subset \mathbb{C}^*$ on V is

$$\left(\frac{d}{d}\right) \oplus \left(\frac{d-1}{d}\right) \oplus \cdots \left(\frac{1}{d}\right) \oplus (0)_{\mathbb{R}}.$$

Example 5.2.3 Let $N = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, and consider the standard antiholomorphic involution on N

$$\widetilde{\sigma}\colon N \to N, \qquad \widetilde{\sigma}(z,u,v) = (\frac{1}{\overline{z}}, \overline{zv}, \overline{zu}).$$

A lifting of the $\mathbf{C}^*$ action to $\mathcal{O}_{\mathbf{P}^1}(-1)$ must have weights $[a-1, a]$ at the respective fixed points for some integer a.

The lifting $([a-1,a],[b-1,b])$ is compatible with $\widetilde{\sigma}$ if $a+b=1$, ie, if it is of the form $([a-1,a],[-a,1-a])$.

We now consider such a compatible lifting with weights $([a-1,a],[-a,1-a])$. Let $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ be given by $f(x) = x^d$ as in Example 5.2.2. The antiholomorphic involution on N induces an antiholomorphic involution on $f^*N \cong \mathcal{O}_{\mathbf{P}^1}(-d) \oplus \mathcal{O}_{\mathbf{P}^1}(-d)$. The corresponding representation of $\mathbf{C}^*$ on $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-d) \oplus \mathcal{O}_{\mathbf{P}^1}(-d))$ has weights

$$\left\{a - \frac{d-1}{d}, a - \frac{d-2}{d}, \cdots, a - \frac{1}{d}, \frac{1}{d} - a, \cdots, \frac{d-1}{d} - a\right\}$$

with respect to the ordered basis

$$\left(\frac{1}{X_0^{d-1}X_1},0\right),\left(\frac{1}{X_0^{d-2}X_1^2},0\right),\ldots,\left(\frac{1}{X_0X_1^{d-1}},0\right),$$
$$\left(0,\frac{1}{X_0^{d-1}X_1}\right),\ldots,\left(0,\frac{1}{X_0X_1^{d-1}}\right),$$

where Čech cohomology has been used (see [5, page 292] for the $a=0$ case). The orientation on

$$\begin{aligned} H^1(\mathbf{P}^1,f^*N)^{\tilde{\sigma}} &\simeq H^1(D^2,S^1,(f|_{D^2})^*N,(f|_{S^1})^*N_{\mathbb{R}}) \\ &= H^1(D^2,S^1,N(d),N(d)_{\mathbb{R}}) \end{aligned}$$

is given by identifying it with the complex vector space $H^1(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-d))$:

$$H^1(D^2,S^1,N(d),N(d)_{\mathbb{R}})\to H^1(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-d))$$
$$\left(\sum_{j=1}^{d-1}\frac{\bar{a}_j}{z^{d-j}},\sum_{j=1}^{d-1}\frac{a_j}{z^j}\right)\mapsto\left(0,\sum_{j=1}^{d-1}\frac{a_j}{X_0^{d-j}X_1^j}\right).$$

Here, $(N(d),N(d)_{\mathbb{R}})$ is as in Section 3.3.2. The corresponding representation of $U(1)\subset\mathbb{C}^*$ on $H^1(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-d))$ is

$$\left(\frac{1}{d}-a\right)\oplus\left(\frac{2}{d}-a\right)\oplus\cdots\left(\frac{d-1}{d}-a\right).$$

6 Orientation and the Euler class

Our computations in Section 8 will use the Euler class of oriented bundles on moduli spaces of stable maps. Rather than attempt to define orientations of the bundles directly, it suffices for computation to define orientations on their restrictions to the fixed loci under the $U(1)$ actions. We presume that the eventual careful formulation, left for future work, will coincide with the natural choices which we make here.

The essential point is to understand orientations on bundles on $BU(1)$, the classifying space for $U(1)$. We consider these in turn.

The $U(1)$ representation (w) gives rise to a rank 2 bundle on $BU(1)$. We abuse notation by denoting this bundle by (w) as well. Since the $U(1)$ action preserves the standard orientation of $\mathbb{R}^2$, the bundle (w) inherits a standard orientation.

With this choice, the Euler class of (w) is $w\lambda\in H^2(BU(1),\mathbb{Z})$, where λ is a generator of $H^2(BU(1),\mathbb{Z})$.

Similarly, the trivial representation $(0)_{\mathbb{R}}$ gives rise to the trivial $\mathbb{R}$–bundle on $BU(1)$, which inherits the standard orientation of $\mathbb{R}$. The Euler class is 0 in this case.

In the rest of this paper, we will always use these orientations on $BU(1)$ bundles. We will also have occasion to consider bundles B on algebraic stacks F with a trivial $U(1)$ action, arising as fixed loci of a $U(1)$ action on a larger space. In this case, we have $F_{U(1)} = F \times BU(1)$, and correspondingly the bundles $B_{U(1)}$ decompose into sums of bundles obtained by tensoring pullbacks of the above $BU(1)$ bundles with pullbacks of holomorphic bundles on F. Since holomorphic bundles have canonical orientations, we are able to orient all of our bundles in the sequel. These orientations will be used without further comment.

We now show that our choices are compatible with the orientations defined for $g = 0$ in Fukaya–Oh–Ohta–Ono [7]. Consider the fixed point component $F_{g;h|n_1,\ldots,n_h}$, isomorphic to a quotient of $\overline{M}_{g,h}$. Consider a map f in this component described as in Section 5.1. Let $f_i = f|_{D_i}\colon D_i \to D^2$ be the multiple cover map. The tangent space to moduli space of maps at this point can be calculated via

$$T_{\Sigma,p_1,\ldots,p_h}\overline{M}_{g,h} \oplus_i T_{p_i}\Sigma_0 \otimes T_{p_i}D_i \oplus H^0(\Sigma_i, f_i^*T_D, f_i^*T_{\partial D}).$$

All spaces except the last have canonical complex structures, hence canonical orientations. The last space can be oriented as in [7]. By pinching the disc along a circle centered at the origin, a union of a sphere and a disc are obtained. A standard gluing argument identifies

$$H^0(\Sigma_i, f_i^*T_{D^2}, f_i^*T_{\partial D^2}) \simeq \mathrm{Ker}\left(H^0(\mathbf{P}^1, \mathcal{O}(n_i)) \oplus H^0(\mathrm{Disc}, \mathbb{C}, \mathbb{R}) \to \mathbb{C}\right).$$

The key point is that $H^0(\mathrm{Disc}, \mathbb{C}, \mathbb{R})$ is identified with $\mathbb{R}$ by the orientation on S^1 so has a canonical orientation. Everything else is complex, so has a canonical orientation.

Using (14), this orientation coincides with the orientation induced by the $U(1)$ action. This orientation is used in the proof of Proposition 7.1.

The infinitesimal automorphisms are oriented similarly, so the tangent space to moduli is the quotient space, which is therefore oriented.

7 Main results

Let C be a smooth rational curve in a Calabi–Yau threefold X with normal bundle $N = N_{C/X} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, where we identify C with $\mathbf{P}^1$.

Suppose X admits an antiholomorphic involution A with the special Lagrangian L as its fixed locus. Suppose in addition that A preserves C and $C \cap L = S^1$. Then A

induces an antiholomorphic involution $A_*\colon N\to N$ which covers $A|_C\colon C\to C$, and the fixed locus of A_* is $N_{\mathbb{R}}=N_{S^1/L}$, the normal bundle of S^1 in L, a real subbundle of rank two in $N_{C/X}|_{S^1}$.

We now let (z,u,v) and $(\widetilde{z},\widetilde{u},\widetilde{v})$ be the two charts of $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$, related by $(\widetilde{z},\widetilde{u},\widetilde{v})=(\frac{1}{z},zu,zv)$. Let $\widetilde{X}$ denote the total space of $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$. We assume that under a suitable identification $N_{C/X}\cong\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$, the map A_* takes the form

$$\begin{aligned} A_*\colon \widetilde{X} &\longrightarrow \widetilde{X} \\ (z,u,v) &\longmapsto (\frac{1}{\overline{z}},\overline{zv},\overline{zu}) \end{aligned}$$

in terms of the first chart (see Example 3.4.4). The fixed locus $\widetilde{L}$ of A_* gets identified with the total space of $N_{C\cap L/L}$, and $\widetilde{L}$ is a special Lagrangian submanifold of the noncompact Calabi–Yau 3–fold $\widetilde{X}$. Then $(\widetilde{X},\widetilde{L})$ can be thought of as a local model for (X,L) near C.

Set $\beta=f_*[D^2]\in H_2(X,L;\mathbb{Z})$ and $\gamma=f_*[S^1]\in H_1(L)$. We consider $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)$ with $n_1+\dots+n_h=d$. The virtual dimension of $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)$ is 0.

Given a stable map f in $\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$, $i\circ f$ is an element of $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)$, so there is an embedding

$$j\colon \overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)\to\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma),$$

whose image is a connected component of $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)$ which we denote by $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)_D$. We will sometimes refer to maps in $\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$ or $\overline{M}_{g;h}(X,L\mid d\beta;n_1\gamma,\dots,n_h\gamma)_D$ as type $(g;h)$ multiple covers of D^2 of degree $(n_1,\dots,n_h)$. We have

$$\dim[\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)]^{\mathrm{vir}}=2(d+2g+h-2)$$

by (11). Let

$$\pi\colon\mathcal{U}\to\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$$

be the universal family of stable maps, $\mu\colon\mathcal{U}\to D^2$ be the evaluation map, and let $\mathcal{N}$ be the sheaf of local holomorphic sections of $N|_{D^2}$ with boundary values in $N_{\mathbb{R}}$.

Now let $f\colon(\Sigma,\partial\Sigma)\to(D^2,S^1)$ be a stable map with

$$f_*[\Sigma]=d[D^2]\in H^2(D^2,S^1,\mathbb{Z}).$$

Then

$$\begin{aligned} H^0(\Sigma, f^*N, f^*N_{\mathbb{R}}) &= 0 \\ \dim H^1(\Sigma, f^*N, f^*N_{\mathbb{R}}) &= 2(d+2g-2+h). \end{aligned}$$

Thus the sheaf $R^1\pi_*\mu^*\mathcal{N}$ is a rank $2(d+2g+h-2)$ real vector bundle over $\overline{M}_{g;h}(D^2, S^1 \mid d; n_1, \ldots, n_h)$, which we have called the obstruction bundle.

Note the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ H^i(T_\Sigma, T_{\partial\Sigma}) & \to & H^i(f^*T_{D^2}, f^*T_{S^1}) \\ \downarrow & & \downarrow \\ H^i(T_\Sigma, T_{\partial\Sigma}) & \to & H^i(f^*T_X, f^*T_L)) \\ & & \downarrow \\ & & H^i(f^*N_{\mathbb{C}}, f^*N_{\mathbb{R}}) \\ & & \downarrow \\ & & 0 \end{array} \tag{15}$$

Combining (7) (for (X, L) and for (D^2, S^1)) and (15), we are led to the following claim:

The virtual fundamental classes are related by

$$j_*\left(e(R^1\pi_*\mu^*\mathcal{N}) \cap [\overline{M}_{g,h}(D^2, S^1 \mid d; n_1, \ldots, n_h)]^{\text{vir}}\right) = [\overline{M}_{g;h}(X, L \mid d\beta; n_1\gamma, \ldots, n_h\gamma)_D]^{\text{vir}}. \tag{16}$$

We leave a more precise formulation of this claim and its proof for future work.

Therefore,[6]

$$C(g;h|d;n_1,...,n_h) = \int_{[\overline{M}_{g;h}(D^2,S^1|d;n_1,...,n_h)]^{\text{vir}}} e(R^1\pi_*\mu^*\mathcal{N})$$

is the contribution to the type $(g;h)$ enumerative invariants of (X, L) from multiple covers of D^2 of degree $(n_1, ..., n_h)$.

Alternatively, $C(g;h|d;n_1,...,n_h)$ can be viewed as enumerative invariants of $(\widetilde{X}, \widetilde{L})$.

[6] Recall that we mentioned at the end of Section 4 that the virtual fundamental class depends on choices. In the following formula and the rest of this section, we are actually fixing a particularly simple choice, corresponding to a specific torus action, in order to compare with [20]. The general case and its relationship to this section will be discussed in Section 8.1.

Since for a Kähler class ω on X we have

$$\int_{D^2} f^*\omega = \frac{1}{2}\int_{\mathbf{P}^1} f^*\omega$$

(extending f to $\mathbf{P}^1$ as its complex double, see Section 3.3.2), the truncated type $(g;h)$ prepotential takes the form

$$F_{g;h}(t, y_1, \ldots, y_h) = \sum_{\substack{n_1+\cdots+n_h=d \\ n_1,\ldots,n_h\geq 0}} C(g;h|d;n_1,\ldots,n_h)\, y_1^{n_1}\cdots y_h^{n_h} e^{-\frac{dt}{2}}$$

where we have put $d = n_1 + \ldots + n_h$. The truncated all-genus potential is

$$\begin{aligned} F(\lambda, t, y_1, y_2, \ldots) &= \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} \lambda^{-\chi(\Sigma_{g;h})} F_{g;h}(t, y_1, \ldots, y_h) \\ &= \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} \lambda^{2g+h-2} F_{g;h}(t, y_1, \ldots, y_h) \end{aligned}$$

Our final assumption is that a natural extension of the virtual localization formula of [10] holds in this context.[7] Rather than formulate this assumption in generality, we will illustrate it in our situation. The generalization is straightforward.

Proposition 7.1 *Under our assumptions, we have*

$$C(g;1|d;d) = d^{2g-2} b_g$$
$$C(g;h|n_1 + \cdots n_h, n_1, \ldots, n_h) = 0 \quad \text{for } h > 1.$$

We will prove Proposition 7.1 in Section 8. The next result follows immediately.

Theorem 7.2

$$F(\lambda, t, y) = \sum_{d=1}^{\infty} \frac{e^{-\frac{dt}{2}} y^d}{2d \sin\frac{\lambda d}{2}}.$$

This is the multiple cover formula for the disc, first obtained obtained by string duality in [20].

[7] In particular, this extension requires that the torus action is chosen to preserve the additional structure, as will be described in Section 8.1.

Proof We compute

$$F_{g;1}(t,y) = \sum_{d=1}^{\infty} C(g;1|d;d) y^d e^{-\frac{dt}{2}} = b_g \sum_{d=1}^{\infty} d^{2g-2} y^d e^{-\frac{dt}{2}}$$
$$F_{g;h}(t,y) = 0 \quad \text{for } h > 1$$

Then

$$\begin{aligned} F(\lambda, t, y_1, y_2, \ldots) &= \sum_{g=0}^{\infty} \lambda^{2g-1} F_{g;1}(t,y) \\ &= \sum_{g=0}^{\infty} \lambda^{2g-1} b_g \sum_{d=1}^{\infty} d^{2g-2} y^d e^{-\frac{dt}{2}} \\ &= \sum_{d=1}^{\infty} \frac{y^d e^{-\frac{dt}{2}}}{\lambda d^2} \sum_{g=0}^{\infty} b_g (\lambda d)^{2g} \\ &= \sum_{d=1}^{\infty} \frac{y^d e^{-\frac{dt}{2}}}{\lambda d^2} \frac{\frac{\lambda d}{2}}{\sin \frac{\lambda d}{2}} \\ &= \sum_{d=1}^{\infty} \frac{e^{-\frac{dt}{2}} y^d}{2d \sin \frac{\lambda d}{2}}. \end{aligned}$$

□

8 Final calculations

8.1 Outline

In this section, we perform our main calculation and use the result to prove Proposition 7.1. We want to calculate

(17) $$\int_{[\overline{M}_{g;h}(D^2, S^1|d;n_1,\ldots,n_h)]^{\mathrm{vir}}} e(R^1\pi_*\mu^*\mathcal{N})$$

by localization. As before, let $\mathbf{C}^*$ act on $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ with weights $([a-1,a],[-a,1-a])$. From Section 5.1 we know that there is only one fixed component. This lifting is compatible with the antiholomorphic involution on $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, as explained in Example 5.2.3.

Recall that over $z = e^{i\theta} \in \partial D^2$, we have $(N_{\mathbb{R}})_z = \{(u,v) \in \mathbb{C}^2 | v = \overline{z u}\}$. The additional data needed to define our invariants turns out to be completely determined by the topological class of a real 1 dimensional subbundle of $N_{\mathbb{R}}$, just like the topological class of a real 1 dimensional subbundle of a complex line bundle on a bordered Riemann surface is needed to define its generalized Maslov index. In order for localization to be valid, we propose that the torus action is required to preserve this subbundle.

We define the subbundle $N_a \subset N_{\mathbb{R}}$ as the subbundle characterized by

$$(N_a)_z = \{(e^{i\theta(a-1)}r, e^{-i\theta a}r) | r \in \mathbb{R}\}.$$

Note that the $U(1)$ action on N with weights $([a-1,a],[-a,1-a])$ is the only compatible lifting of the torus action on $\mathbf{P}^1$ which preserves N_a. Note also that the $\{N_a\}$ represent all of the topological classes of rank 1 subbundles of $N_{\mathbb{R}}$.

We denote by $C(g,h|d;n_1,\ldots,n_h|a)$ the invariant (17) to emphasize its dependence on a. The invariant $C(g,h|d;n_1,\ldots,n_h)$ of Section 7 is the special case $a=0$ of (17).

8.2 The obstruction bundle

Let $f\colon \Sigma \to \mathbf{P}^1$ be a generic stable map in $F_{g;h|d;n_1,\ldots,n_h}$, where $(g;h) \neq (0;1),(0;2)$. Write $\Sigma = \Sigma_0 \cup D_1 \cup \cdots \cup D_h$, where Σ_0 is a curve of genus g, and D_i is a disc for $i=1,\ldots,h$. Let z_i be the coordinate on D_i, identifying D_i with $D^2 \subset \mathbb{C}$. The point $z_i = 0$ is identified with $p_i \in \Sigma_0$, where $p_1,\ldots,p_h$ are distinct points on Σ_0, forming nodes on the nodal bordered Riemann surface Σ. Note that $(\Sigma_0, p_1,\ldots,p_h)$ represents an element in $\overline{M}_{g,h}$. The restriction of f on D_i is $z_i \mapsto z_i^{n_i}$. Conversely, all $f\colon \Sigma \to \mathbf{P}^1$ as described above are elements of $F_{g;h|d;n_1,\ldots,n_h}$.

Let $\pi\colon \mathcal{U} \to \overline{M}_{g;h}(D^2,S^1 \mid d;n_1,\ldots,n_h)$ be the universal family of stable maps, let $\mu\colon \mathcal{U} \to D^2$ be the evaluation map, and let $\mathcal{N}$ be the sheaf of local holomorphic sections of $N|_{D^2}$ with boundary values in $N_{\mathbb{R}}$, as in Section 7. The fiber of the obstruction bundle $R^1\pi_*\mu^*\mathcal{N}$ at $[f]$ is $H^1(\Sigma, f^*\mathcal{N}) = H^1(\Sigma, \partial\Sigma, f^*N, f^*N_{\mathbb{R}})$. Consider the normalization sequence

$$0 \to f^*\mathcal{N} \to \bigoplus_{i=1}^h (f|_{D_i})^*\mathcal{N} \oplus (f|_{\Sigma_0})^*\mathcal{N} \to \bigoplus_{i=1}^h (f^*\mathcal{N})_{p_i} \to 0.$$

The corresponding long exact sequence reads

$$\begin{aligned} 0 \;\to\; & H^0(\Sigma_0,(f|_{\Sigma_0})^*\mathcal{N}) \to \textstyle\bigoplus_{i=1}^h H^0((f^*\mathcal{N})_{p_i}) \to H^1(\Sigma, f^*\mathcal{N}) \\ \to\; & \textstyle\bigoplus_{i=1}^h H^1(D_i,(f|_{D_i})^*\mathcal{N}) \oplus H^1(\Sigma_0,(f|_{\Sigma_0})^*\mathcal{N}) \to 0 \end{aligned} \tag{18}$$

The vector spaces $H^0(\Sigma_0,(f|_{\Sigma_0})^*\mathcal{N})$, $H^0((f^*\mathcal{N})_{p_i})$, and $H^1(D_i,(f|_{D_i})^*\mathcal{N})$ appearing in (18) fit together to form trivial complex vector bundles over the component $F_{g;h|d;n_1,\ldots,n_h}$ which are not necessarily trivial as equivariant vector bundles. Let $\mathbf{C}^*$

act on $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$ with weights $([a-1,a],[-a,1-a])$. Then the weights of $U(1)$ in (18) are (see Example 5.2.3)

$$\begin{aligned}0\to(a-1)\oplus(-a)\to\bigoplus_{i=1}^h((a-1)\oplus(-a))\to H^1(\Sigma,f^*\mathcal{N})\\ \to H^1(\Sigma_0,\mathcal{O}_{\Sigma_0})\otimes((a-1)\oplus(-a))\oplus\\ \bigoplus_{i=1}^h\left(\left(\tfrac{1}{n_i}-a\right)\oplus\left(\tfrac{2}{n_i}-a\right)\oplus\cdots\oplus\left(\tfrac{n_i-1}{n_i}-a\right)\right)\to 0.\end{aligned}$$

The map

$$i_{g;h|d;n_1,\dots,n_h}\colon \overline{M}_{g,h}\to\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$$

has degree $n_1\cdots n_h$ onto its image $F_{g;h|d;n_1,\dots,n_h}$, and

$$\begin{aligned}&i^*_{g;h|d;n_1,\dots,n_h}R^1\pi_*\mu^*\mathcal{N}=\\&\left(\mathbb{E}^\vee\otimes((a-1)\oplus(-a))\right)\oplus\bigoplus_{i=1}^h\prod_{j=1}^{n_i-1}\left(\tfrac{j}{n_i}-a\right)\oplus\bigoplus_{i=1}^{h-1}((a-1)\oplus(-a))\end{aligned}$$

where $\mathbb{E}$ is the Hodge bundle over $\overline{M}_{g,h}$. Therefore,

$$\begin{aligned}&e_{U(1)}(i^*_{g;h|d;n_1,\dots,n_h}R^1\pi_*\mu^*\mathcal{N})\\ =\ &c_g(\mathbb{E}^\vee((a-1)\lambda))c_g(\mathbb{E}^\vee(-a\lambda))\lambda^{d+h-2}(a(1-a))^{h-1}\prod_{i=1}^h\frac{\prod_{j=1}^{n_i-1}(j-n_ia)}{n_i^{n_i-1}}\end{aligned}\tag{19}$$

where $\lambda_g=c_g(\mathbb{E})$, and $\lambda\in H^2(BU(1),\mathbb{Z})$ is the generator discussed in Section 6.

We now consider the case $(g;h)=(0;1)$. Then $F_{0;1|d;d}$ consists of a single point corresponding to the map $f\colon D^2\to D^2$, $x\mapsto x^d$. In this case,

$$H^1(D^2,f^*\mathcal{N})=\left(\frac{1}{d}-a\right)\oplus\left(\frac{2}{d}-a\right)\oplus\cdots\oplus\left(\frac{d-1}{d}-a\right).$$

This gives

$$e_{U(1)}(i^*_{0,d}R^1\pi_*\mu^*\mathcal{N})=\lambda^{d-1}\frac{\prod_{j=1}^{d-1}(j-da)}{d^{d-1}}.\tag{20}$$

We finally consider the case $(g;h)=(0;2)$. Then $F_{0;2|d;n_1,n_2}$ consists of a single point corresponding to the map $f\colon D_1\cup D_2\to D^2$. Here D_1 and D_2 are identified at the origin, and the restrictions of f to D_1 and to D_2 are $x_1\mapsto x_1^{n_1}$ and $x_2\mapsto x_2^{n_2}$, where x_1 and x_2 are coordinates on D_1 and D_2, respectively. In this case,

$$H^1(D^2,f^*\mathcal{N})=(a-1)\oplus(-a)\bigoplus_{i=1}^{2}\bigoplus_{j=1}^{n_i-1}\left(\frac{j}{d}-a\right).$$

This gives

$$e_{U(1)}(i_{0,d}^* R^1\pi_*\mu^*\mathcal{N}) = \lambda^d a(1-a)\prod_{i=1}^{2}\frac{\prod_{j=1}^{n_i-1}(j-n_i a)}{n_i^{n_i-1}}. \tag{21}$$

We conclude that (19) is valid for $(g;h)=(0;1),(0;2)$ as well.

8.3 The virtual normal bundle

We next compute the equivariant Euler class of the virtual normal bundle of the fixed point component $F_{g;h|d;n_1,\dots,n_h}$ in $\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$. The computation is very similar to that in [10, Section 4].

There is a tangent-obstruction exact sequence of sheaves on $F_{g;h|d;n_1,\dots,n_h}$:

$$\begin{aligned} 0 \;\to\;\; & H^0(\Sigma,\partial\Sigma,T_\Sigma,T_{\partial\Sigma}) \to H^0(\Sigma,\partial\Sigma,f^*T_{\mathbf{P}^1},(f|_{\partial\Sigma})^*T_{S^1}) \to \mathcal{T}^1 \\ \to\;\; & H^1(\Sigma,\partial\Sigma,T_\Sigma,T_{\partial\Sigma}) \to H^0(\Sigma,\partial\Sigma,f^*T_{\mathbf{P}^1},(f|_{\partial\Sigma})^*T_{S^1}) \to \mathcal{T}^2 \to 0 \end{aligned} \tag{22}$$

The 4 terms in (22) other than the sheaves $\mathcal{T}^i$ are vector bundles and are labeled by fibers. We use [10] and the notation there, so that $\mathcal{T}^i=\mathcal{T}^{i,f}\oplus\mathcal{T}^{i,m}$, where $\mathcal{T}^{i,f}$ is the fixed part and $\mathcal{T}^{i,m}$ is the moving part of $\mathcal{T}^i$ under the $U(1)$ action. Then the $\mathcal{T}^{i,m}$ will determine the virtual normal bundle of $F_{g;h|d;n_1,\dots,n_h}$ in $\overline{M}_{g;h}(D^2,S^1\mid d;n_1,\dots,n_h)$:

$$e_{U(1)}(N^{\mathrm{vir}}) = \frac{e_{U(1)}(B_2^m)e_{U(1)}(B_4^m)}{e_{U(1)}(B_1^m)e_{U(1)}(B_5^m)},$$

where the B_i^m denote the moving part of the ith term in the tangent-obstruction exact sequence (22).

We first consider the cases $(g;h)\neq(0;1),(0;2)$. We have $B_1=B_1^f=\oplus_{i=1}^h(0)_{\mathbb{R}}$, which corresponds to the rotations of the h disc components of the domain curve, so $e_{U(1)}(B_1^m)=1$. Next, $B_4^m=\oplus_{i=1}^h T_{p_i}\Sigma_0\otimes T_0D_i$ corresponds to deformations of the h nodes of the domain. To compute this, note that $T_{p_i}^*\Sigma_0$ is the fiber of $\mathbb{L}_i=s_i^*\omega_\pi\to\overline{M}_{g,h}$ at $[(\Sigma_0,p_1,\dots,p_h)]$, where ω_π is the relative dualizing sheaf of the universal curve over $\overline{M}_{g,h}$, and s_i is the section corresponding to the ith marked point. Since $T_0D_i=(1/n_i)$, we get

$$e_{U(1)}(B_4^m)=\prod_{i=1}^h\left(c_1(\mathbb{L}_i^\vee)+\frac{\lambda}{n_i}\right)=\prod_{i=1}^h\left(\frac{\lambda}{n_i}-\psi_i\right),$$

where $\psi_i=c_1(\mathbb{L}_i)$.

For B_2^m and B_5^m, consider the normalization exact sequence

$$0 \to (f^* T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^* T_{S^1}) \to (f|_{\Sigma_0})^* T_{\mathbf{P}^1} \oplus \bigoplus_{i=1}^{h} \big((f|_{D_i})^* T_{\mathbf{P}^1}, (f|_{\partial D_i})^* T_{S^1}\big) \to \bigoplus_{i=1}^{h} (T\mathbf{P}^1)_0 \to 0.$$

The corresponding long exact sequence reads

$$\begin{aligned} 0 &\to H^0(\Sigma, \partial\Sigma, f^* T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^* T_{S^1}) \\ &\to H^0(\Sigma_0, (f|_{\Sigma_0})^* T_{\mathbf{P}^1}) \oplus \bigoplus_{i=1}^{h} H^0(D_i, \partial D_i, (f|_{D_i})^* T_{\mathbf{P}^1}, (f|_{\partial D_i})^* T_{S^1}) \\ &\to \bigoplus_{i=1}^{h} (T\mathbf{P}^1)_0 \to H^1(\Sigma, \partial\Sigma, f^* T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^* T_{S^1}) \to H^1(\Sigma_0, (f|_{\Sigma_0})^* T_{\mathbf{P}^1}) \\ &\to 0. \end{aligned}$$

The representations of $U(1)$ are (see Example 5.2.2)

(23)
$$\begin{aligned} 0 &\to H^0(\Sigma, \partial\Sigma, f^* T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^* T_{S^1}) \\ &\to H^0(\Sigma_0, \mathcal{O}_{\Sigma_0}) \otimes (1) \oplus \textstyle\bigoplus_{i=1}^{h} \Big(\bigoplus_{j=1}^{n_i} \Big(\frac{j}{n_i}\Big) \oplus (0)_{\mathbb{R}} \Big) \\ &\to \textstyle\bigoplus_{i=1}^{h} (1) \to H^1(\Sigma, \partial\Sigma, f^* T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^* T_{S^1}) \to H^1(\Sigma_0, \mathcal{O}_{\Sigma_0}) \otimes (1) \to 0. \end{aligned}$$

The trivial representation lies in the fixed part, so (23) gives

$$\frac{e_{U(1)}(B_2^m)}{e_{U(1)}(B_5^m)} = \frac{\lambda^{d+1-h}}{c_g(\mathbb{E}^\vee(\lambda))} \prod_{i=1}^{h} \frac{n_i!}{n_i^{n_i}}.$$

Therefore

$$e_{U(1)}(N^{\mathrm{vir}}) = \frac{\lambda^{d+1-h}}{c_g(\mathbb{E}^\vee(\lambda))} \prod_{i=1}^{h} \left(\frac{n_i!}{n_i^{n_i}} \left(\frac{\lambda}{n_i} - \psi_i \right) \right). \tag{24}$$

We now consider the case $(g; h) = (0; 1)$. The representations of $U(1)$ in (22) are

$$0 \to \left(\frac{1}{d}\right) \oplus (0)_{\mathbb{R}} \to \bigoplus_{i=1}^{d} \left(\left(\frac{i}{d}\right) \oplus (0)_{\mathbb{R}} \right) \to \mathcal{T}^1 \to 0 \to 0 \to \mathcal{T}^2 \to 0.$$

Therefore

$$e_{U(1)}(N^{\mathrm{vir}}) = \lambda^{d-1}\frac{d!}{d^{d-1}}. \tag{25}$$

We finally consider the case $(g;h)=(0;2)$. We have $B_1 = B_1^f = (0)_{\mathbb{R}} \oplus (0)_{\mathbb{R}}$, which corresponds to the rotations of the two disc components, so $e_{U(1)}(B_1^m)=1$. Next, $B_4^m = T_0 D_1 \otimes T_0 D_2$ corresponds to deformation of the node of the domain. Since $T_0 D_i = \left(\frac{1}{n_i}\right)$, we get $e_{U(1)}(B_4^m) = \left(\frac{1}{n_1}+\frac{1}{n_2}\right)\lambda$.

For B_2^m and B_5^m, consider the normalization exact sequence

$$0 \to (f^*T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^*T_{S^1}) \to \bigoplus_{i=1}^{2}\big((f|_{D_i})^*T_{\mathbf{P}^1}, (f|_{\partial D_i})^*T_{S^1}\big) \to (T\mathbf{P}^1)_0 \to 0.$$

The corresponding long exact sequence reads

$$\begin{aligned}
&0 \to H^0(\Sigma,\partial\Sigma, f^*T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^*T_{S^1})\\
&\to \bigoplus_{i=1}^{2} H^0(D_i,\partial D_i, (f|_{D_i})^*T_{\mathbf{P}^1}, (f|_{\partial D_i})^*T_{S^1})\\
&\to (T\mathbf{P}^1)_0 \to H^1(\Sigma,\partial\Sigma, f^*T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^*T_{S^1}) \to 0.
\end{aligned}$$

The representations of $U(1)$ are (see Example 5.2.2)

$$\begin{aligned}
0 \;&\to\; H^0(\Sigma,\partial\Sigma, f^*T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^*T_{S^1}) \to \textstyle\bigoplus_{i=1}^{2}\left(\bigoplus_{j=1}^{n_i}\left(\frac{j}{n_i}\right)\oplus(0)_{\mathbb{R}}\right)\\
&\to\; (1) \to H^1(\Sigma,\partial\Sigma, f^*T_{\mathbf{P}^1}, (f|_{\partial\Sigma})^*T_{S^1}) \to 0.
\end{aligned} \tag{26}$$

The trivial representation lies in the fixed part, so (26) gives

$$\frac{e_{U(1)}(B_2^m)}{e_{U(1)}(B_5^m)} = \lambda^{d-1}\frac{n_1!n_2!}{n_1^{n_1}n_2^{n_2}}.$$

Therefore

$$e_{U(1)}(N^{\mathrm{vir}}) = \lambda^{d}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{n_1!n_2!}{n_1^{n_1}n_2^{n_2}}. \tag{27}$$

8.4 Conclusion

From our calculations in Sections 8.2 and 8.3, we can now compute the sought-after invariants.

$$C(0;1|d;d|a) = \frac{1}{d}\int_{\mathrm{pt}_{U(1)}}\frac{i^*_{0;1|d;d}e_{U(1)}(R^1\pi_*\mu^*\mathcal{N})}{e_{U(1)}(N^{\mathrm{vir}})}$$

$$= \left(\frac{\prod_{j=1}^{d-1}(j - da)}{(d-1)!} \right) \frac{1}{d^2},$$

$$C(0;2|d;n_1,n_2|a) = \frac{1}{n_1 n_2} \int_{\mathrm{pt}_{U(1)}} \frac{i^*_{0;2|d;n_1,n_2} e_{U(1)}(R^1\pi_*\mu^*\mathcal{N})}{e_{U(1)}(N^{\mathrm{vir}})}$$

$$= a(1-a) \left(\prod_{i=1}^{2} \frac{\prod_{j=1}^{n_i-1}(j - n_i a)}{(n_i - 1)!} \right) \frac{1}{d}.$$

Here $\mathrm{pt}_{U(1)}$ denotes the "equivariant point", which is isomorphic to $BU(1)$.

For $(g;h) \neq (0;1),(0;2)$, we get

$$\begin{aligned} &C(g;h|d;n_1,\dots,n_h|a) \\ &= \frac{1}{n_1 \cdots n_h} \int_{\overline{M}_{g,h\,U(1)}} \frac{i^*_{g;h|d;n_1,\dots,n_h} e_{U(1)}(R^1\pi_*\mu^*\mathcal{N})}{e_{U(1)}(N^{\mathrm{vir}})} \\ &= (a(1-a))^{h-1} \left(\prod_{i=1}^{h} \frac{\prod_{j=1}^{n_i-1}(j - n_i a)}{(n_i-1)!} \right) \cdot \\ &\quad \int_{\overline{M}_{g,h}} \frac{c_g(\mathbb{E}^\vee(\lambda)) c_g(\mathbb{E}^\vee((a-1)\lambda)) c_g(\mathbb{E}^\vee(-a\lambda)) \lambda^{2h-3}}{\prod_{i=1}^{h}(\lambda - n_i\psi_i)}, \end{aligned}$$

where $\overline{M}_{g,h\,U(1)}$ denotes $\overline{M}_{g,h} \times BU(1)$. In particular, we get for $h \geq 3$,

$$\begin{aligned} &C(0;h|d;n_1,\dots,n_h|a) \\ &= (a(1-a))^{h-1} \left(\prod_{i=1}^{h} \frac{\prod_{j=1}^{n_i-1}(j - n_i a)}{(n_i-1)!} \right) \int_{\overline{M}_{0,h\,U(1)}} \frac{\lambda^{2h-3}}{\prod_{i=1}^{h}(\lambda - n_i\psi_i)} \\ &= (a(1-a))^{h-1} \left(\prod_{i=1}^{h} \frac{\prod_{j=1}^{n_i-1}(j - n_i a)}{(n_i-1)!} \right) \cdot \\ &\quad \sum_{\substack{k_1 + \dots k_h = h-3 \\ k_1,\dots,k_h \geq 0}} n_1^{k_1} \cdots n_h^{k_h} \int_{\overline{M}_{0,h}} \psi_1^{k_1} \cdots \psi_h^{k_h}. \end{aligned}$$

But

$$\int_{\overline{M}_{0,h}} \psi_1^{k_1} \cdots \psi_h^{k_h} = \binom{h-3}{k_1,\dots,k_h},$$

so

$$\begin{aligned}
&C(0;h|d;n_1,\ldots,n_h|a)\\
&=(a(1-a))^{h-1}\left(\prod_{i=1}^{h}\frac{\prod_{j=1}^{n_i-1}(j-n_ia)}{(n_i-1)!}\right)\cdot\\
&\qquad\sum_{\substack{k_1+\ldots k_h=h-3\\ k_1,\ldots k_h\geq 0}} n_1^{k_1}\cdots n_h^{k_h}\binom{h-3}{k_1,\ldots,k_h}\\
&=(a(1-a))^{h-1}\left(\prod_{i=1}^{h}\frac{\prod_{j=1}^{n_i-1}(j-n_ia)}{(n_i-1)!}\right)(n_1+\ldots+n_h)^{h-3}\\
&=(a(1-a))^{h-1}\left(\prod_{i=1}^{h}\frac{\prod_{j=1}^{n_i-1}(j-n_ia)}{(n_i-1)!}\right)d^{h-3}.
\end{aligned}$$

Observe the symmetry

$$C(g;h|d;n_1,\ldots,n_h|1-a)=(-1)^{d-h}C(g;h|d;n_1,\ldots,n_h|a),$$

which is a consequence of the symmetry of $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$ obtained from interchanging the two factors, together with our orientation choices (or it can be checked directly). It is also of interest to observe that for $a>0$, we have the identity

$$\frac{\prod_{j=1}^{n_i-1}(j-n_ia)}{(n_i-1)!}=(-1)^{n_i-1}\binom{n_ia-1}{n_i-1}$$

which is an integer. There is a similar formula for $a\leq 0$.

We may summarize our results as follows:

Proposition 8.4.1 *Let a be a positive integer. Then*

$$\begin{aligned}
&(-1)^{d-h}C(0;h|d;n_1,\ldots,n_h|a)=C(0;h|d;n_1,\ldots,n_h|1-a)\\
&=(a(1-a))^{h-1}\prod_{i=1}^{h}\binom{n_ia-1}{n_i-1}d^{h-3}.
\end{aligned}$$

For $g>0$,

$$(-1)^{d-h}C(g;h|d;n_1,\ldots,n_h|a)=C(g;h|d;n_1,\ldots,n_h|1-a)$$

$$= (a(1-a))^{h-1} \prod_{i=1}^{h} \binom{n_i a - 1}{n_i - 1} \cdot$$
$$\int_{\overline{M}_{g,h\,U(1)}} \frac{c_g(\mathbb{E}^\vee(\lambda)) c_g(\mathbb{E}^\vee((a-1)\lambda)) c_g(\mathbb{E}^\vee(-a\lambda))\lambda^{2h-3}}{\prod_{i=1}^{h}(\lambda - n_i\psi_i)}.$$

Our results for $g = 0$ agree with results of [1] up to a sign (eg, up to a choice of orientation). Our conventions differ from theirs by a sign of $(-1)^{ad}$ (our "a" is their "p"). They also note integer invariants N_d depending on a, yielding the identity

$$(-1)^{ad} C(0, 1|d, d|a) = \sum_{k|d} \frac{N_{d/k}}{k^2}. \tag{28}$$

In the case $a = 0$, we just have $N_1 = 1$ and $N_d = 0$ for all $d \geq 1$, in which case (28) is just the multiple cover formula for the disc. In the general case, (28) can be viewed as multiple cover formula with different integer invariants, in the same sense as the formula of [9] for stable maps of closed Riemann surfaces.

Proof of Proposition 7.1 Setting

$$C(g; h|d; n_1, ..., n_h) = C(g; h|d; n_1, ..., n_h|0) = (-1)^{d-h} C(g; h|d; n_1, ..., n_h|1),$$

we have

$$C(g; h|d; n_1, n_2, ..., n_h) = 0 \quad \text{if } h > 1.$$

For $g = 0$, we have

$$C(0; 1|d; d) = \frac{1}{d^2}.$$

For $g > 0$, we get

$$C(g; 1|d; d) = \int_{\overline{M}_{g,1\,U(1)}} \frac{(-1)^g \lambda_g}{\lambda(\lambda - d\psi_1)} c_g(\mathbb{E}^\vee(\lambda)) c_g(\mathbb{E}^\vee(-\lambda)),$$

where $\lambda_g = c_g(\mathbb{E})$. But $c(\mathbb{E})c(\mathbb{E}^\vee) = 1$ Mumford [18, 5.4]. It is straightforward to check that this implies

$$c_g(\mathbb{E}^\vee(\lambda)) c_g(\mathbb{E}^\vee(-\lambda)) = (-1)^g \lambda^{2g}.$$

It follows that

$$C(g; 1|d; d) = d^{2g-2} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = d^{2g-2} b_g. \qquad \square$$

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