

## On the rigidity of stable maps to Calabi–Yau threefolds

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If  $X \subset Y$  is a nonsingular curve in a Calabi–Yau threefold whose normal bundle  $N_{X/Y}$  is a generic semistable bundle, are the local Gromov–Witten invariants of  $X$  well defined? For  $X$  of genus two or higher, the issues are subtle. We will formulate a precise line of inquiry and present some results, some positive and some negative.

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### 1 Introduction

In 1998, Gopakumar and Vafa [5] proposed a duality between  $SU(N)$  Chern–Simons theory on the 3–sphere and topological string theory on the resolved conifold. As evidence, Gopakumar and Vafa showed the large– $N$  free energy in Chern–Simons theory exactly matches (after a change of variables) the topological string partition function on the resolved conifold.

Mathematically, the *topological string partition function* is just the natural generating function for the Gromov–Witten invariants. The *resolved conifold* is the total space of the bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , considered as a Calabi–Yau threefold. The Gromov–Witten theory of the noncompact total space  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  is well-defined: all (nonconstant) stable maps have image contained in the zero section and thus their moduli spaces are compact.

The Gromov–Witten invariants of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  are often regarded as the *local Gromov–Witten invariants* of  $\mathbb{P}^1$ . Indeed, if  $X \subset Y$  is any smoothly embedded rational curve in a Calabi–Yau threefold  $Y$  with normal bundle isomorphic to  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ , then the contribution of  $X$  to the Gromov–Witten invariants of  $Y$  is well-defined and is given by the corresponding invariants of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ .

We consider here the local theory of higher genus curves. If  $X \subset Y$  is a nonsingular curve in a Calabi–Yau threefold whose normal bundle  $N_{X/Y}$  is a generic semistable bundle, are the local Gromov–Witten invariants of  $X$  well defined? For  $X$  of genus two or higher, the issues are subtle. We will formulate a precise line of inquiry and present some results, some positive and some negative.

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## 2 Definitions and results

Let  $X \subset Y$  be a nonsingular genus- $g$  curve in a threefold  $Y$  with normal bundle  $N_{X/Y}$  of degree  $2g-2$ . If  $Y$  is Calabi–Yau, the condition on the normal bundle is always satisfied. We define the following notions of rigidity:

### Definition 2.1

- (i) A curve  $X \subset Y$  is  $(d, h)$ -rigid if for every degree- $d$ , genus- $h$  stable map  $f: C \rightarrow X$ , we have  $H^0(C, f^* N_{X/Y}) = 0$ .
- (ii) A curve  $X \subset Y$  is  $d$ -rigid if  $X \subset Y$  is  $(d, h)$ -rigid for all genera  $h$ .
- (iii) A curve  $X \subset Y$  is super-rigid if  $X \subset Y$  is  $d$ -rigid for all  $d > 0$ .

For example, a nonsingular rational curve with normal bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  is super-rigid. An elliptic curve  $E \subset Y$  is  $d$ -rigid if and only if  $N_{E/Y} \cong L \oplus L^{-1}$  where  $L \rightarrow E$  is a flat line bundle which is not  $d$ -torsion (see Pandharipande [7]).

For a  $(d, h)$ -rigid curve  $X \subset Y$ , the contribution of  $X$  to  $N_{d[X]}^h(Y)$ , the genus- $h$  Gromov–Witten invariant of  $Y$  in the class  $d[X]$ , is well-defined and given by

$$(1) \quad \int_{[\overline{M}_h(X, d[X])]^{\text{vir}}} c_{\text{top}}(R^1 \pi_* f^* N),$$

where  $\overline{M}_h(X, d)$  is the moduli space of degree- $d$ , genus- $h$  stable maps to  $X$ ,

$$\pi: U \rightarrow \overline{M}_h(X, d)$$

is the universal curve,

$$f: U \rightarrow X$$

is the universal map, and  $[\ ]^{\text{vir}}$  denotes the virtual fundamental class. The  $(d, h)$ -rigidity of  $X$  guarantees that  $R^1 \pi_* f^* N$  is a bundle. See Bryan–Pandharipande [2] for an expanded discussion.

By definition,  $(d, h)$ -rigidity is a condition on the normal bundle  $N_{X/Y}$ . Assuming  $N_{X/Y}$  is generic, we may ask for which pairs  $(d, h)$  does  $(d, h)$ -rigidity hold. The 1-rigidity of a generic normal bundle is straightforward and was used in [7]. We prove the following positive result.

**Theorem 2.2** *If  $X \subset Y$  is a genus- $g$  curve in a threefold  $Y$  and  $N_{X/Y}$  is a generic stable bundle of degree  $2g-2$ , then  $X$  is 2-rigid.*

However, 3-rigidity is *not* satisfied for genus-3 curves.

**Theorem 2.3** *If  $X \subset Y$  is a genus-3 curve in a threefold  $Y$  with*

$$\deg(N_{X/Y}) = 4,$$

*then  $X$  is not 3-rigid.*

Let  $N \rightarrow X$  be a generic stable bundle of degree  $2g-2$ . By [Theorem 2.2](#), the degree-2 Gromov–Witten theory of the total space of  $N$  considered as a noncompact threefold is well-defined by the integral (1).

In the case when  $X$  embeds in a threefold  $Y$  with normal bundle  $N$ , we may regard the above theory as the degree-2 local Gromov–Witten theory of  $X \subset Y$ . Such embeddings of  $X$  can always be found. For example, let  $Y$  be the threefold  $\mathbb{P}(\mathcal{O}_X \oplus N)$  with the embedding  $X \subset Y$  determined by the trivial factor. It would be interesting to construct a curve in a Calabi–Yau threefold with a 2-rigid normal bundle.

The degree-2 *local* theory of  $X$  is a *global* theory for  $N$ . Strong global integrality constraints, obtained from the Gopakumar–Vafa conjecture [4] and more recently from the conjectural Gromov–Witten/Donaldson–Thomas correspondence of Maulik–Nekrasov–Okounkov–Pandharipande [6], should therefore hold for the degree-2 local theory of  $X$ .

In [3; 2], a formal local theory is defined in all degrees for  $X \subset Y$ . By [Theorem 2.2](#), the above local theory coincides with the degree-2 formal local theory defined in [3; 2]. However by [Theorem 2.3](#), the formal local theory of [3; 2] does *not* correspond exactly to a well-defined global theory of  $N$  in degree 3.

Using degeneration techniques, we have recently obtained the complete computation of the formal local invariants of curves in all degrees [1]. The expected integrality holds for the degree-2 invariants. Integrality for the degree-3 formal local invariants fails, in fact the breakdown occurs for precisely the *same* domain genus- $h$  for which rigidity fails.

### 3 Proof of [Theorem 2.2](#)

Let  $X$  be a nonsingular curve of genus  $g$ . Let  $N$  be a bundle of rank 2 and degree  $2g-2$  on  $X$ . The bundle  $N$  is 2-rigid if

$$(2) \quad H^0(C, f^*(N)) = 0$$

for all stable maps  $f: C \rightarrow X$  of degree 2. If  $g = 0$ , then

$$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

is 2-rigid. For  $g > 0$ , we will prove 2-rigidity holds on an open set of the irreducible moduli space of semistable bundles on  $X$ . As the 2-rigidity statement was already proven for  $g = 1$  in [7], we will assume  $g > 1$ .

Let  $\Lambda$  be a fixed line bundle of degree  $2g-2$ . Let  $\overline{M}_X(2, \Lambda)$  be the moduli space of rank-2, semistable bundles with determinant  $\Lambda$ . Since 2-rigidity is a generic condition, we need only prove the existence of a 2-rigid bundle  $[N] \in \overline{M}_X(2, \Lambda)$ .

We first prove  $H^0(X, N) = 0$  for an open set  $V \subset \overline{M}_X(2, \Lambda)$ . Since the vanishing of sections is an open condition and the moduli space is irreducible, we need only find an example. If  $L \in \text{Pic}^{g-1}(X)$  then the bundle

$$N = L \oplus L^{-1}\Lambda$$

is semistable. Since the locus of  $\text{Pic}^{g-1}$  determined by bundles with nontrivial sections is a divisor, neither  $L$  nor  $L^{-1}\Lambda$  have sections for generic  $L$ . Thus

$$H^0(X, L \oplus L^{-1}\Lambda) = 0.$$

If there exists a nonzero section  $s \in H^0(C, f^*(N))$  for a stable map  $f$ , then  $s$  must be nonzero on some dominant, irreducible component of  $C' \subset C$ . Also,  $s$  must be nonzero when pulled-back to the normalization of  $C'$ . Therefore, to check 2-rigidity for  $N$ , we need only prove

$$H^0(C, f^*(N)) = 0$$

for maps  $f: C \rightarrow X$  where the domain is nonsingular and irreducible.

Let  $C$  be nonsingular and irreducible, and let  $f: C \rightarrow X$  be a (possibly ramified) double cover. Let  $B \subset X$  be the branch divisor of  $f$ . The map  $f$  is well-known to determine a square root of  $\mathcal{O}(-B)$  in the Picard group of  $X$  by the following construction. Let  $E = f_*(\mathcal{O}_C)$ .  $E$  is a rank-2 bundle on  $X$  with a  $\mathbb{Z}/2\mathbb{Z}$ -action induced by the Galois group of  $f$ , and decomposes into a direct sum

$$E \cong \mathcal{O} \oplus Q$$

of +1 and -1 eigenbundles for the action. The +1 eigenbundle is the trivial line bundle  $\mathcal{O}$  while the -1 eigenbundle is a line bundle  $Q$  with  $Q^2 \cong \mathcal{O}(-B)$ .

If  $[N] \in V$ , a sequence of isomorphisms

$$H^0(C, f^*(N)) \cong H^0(X, f_*f^*(N)) \cong H^0(X, (\mathcal{O} \oplus Q) \otimes N) \cong H^0(X, Q \otimes N)$$

is obtained from the geometry of the double cover  $f$ . To prove the 2–rigidity of  $N$ , it therefore suffices to prove the vanishing of  $H^0(X, Q \otimes N)$  for all double covers  $f$ .

Suppose  $H^0(X, Q \otimes N) \neq 0$  for a double cover  $f$ . Then there is a nonzero sheaf map

$$\iota: Q^{-1} \rightarrow N.$$

Let  $S$  be the saturation of the image of  $\iota$ . Then  $S$  has the following properties:

- (i)  $S$  is a subbundle of  $N$ ,
- (ii)  $S^2$  has a section.

Property (ii) is proven as follows. The saturation short exact sequence of sheaves on  $X$  is

$$0 \rightarrow Q^{-1} \rightarrow S \rightarrow \mathcal{O}_D \rightarrow 0$$

for some effective divisor  $D$  on  $X$ . Hence  $S \cong Q^{-1}(D)$ , and

$$S^2 \cong Q^{-2}(2D) \cong \mathcal{O}(B + 2D).$$

The line bundle clearly has a section since both  $B$  and  $D$  are effective.

For  $d \geq 0$ , define  $\Delta(d) \subset \overline{M}_X(2, \Lambda)$  to be the following locus in the moduli space:

$$\{[N] \mid \text{there exists a subbundle } S \subset N \text{ with } \deg(S) = d \text{ and } H^0(X, S^2) \neq 0\}$$

If  $[N] \in V$  and  $N$  is not 2–rigid, then we have proven  $[N]$  must lie in  $\Delta(d)$  for some  $d$ . By Lemma 3.1 below, the proof of Theorem 2.2 is complete.  $\square$

**Lemma 3.1** For all  $d$ ,  $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$ .

**Proof** By stability,  $\Delta(d)$  is empty if  $d \geq g - 1$ . We may assume

$$0 \leq d < g - 1.$$

If  $[N] \in \Delta(d)$ , then  $N$  is given by an extension

$$0 \rightarrow S \rightarrow N \rightarrow S^{-1} \otimes \Lambda \rightarrow 0.$$

The number of parameters for  $S$  is  $g$ , the dimension of  $\text{Pic}^d(X)$ . The number of parameters of the space of extensions is

$$h^1(X, S^2 \otimes \Lambda^{-1}) - 1 = 3g - 3 - 2d - 1,$$

by Riemann–Roch. Therefore

$$\dim \Delta(d) \leq g + 3g - 3 - 2d - 1 = 4g - 4 - 2d.$$

If  $d > (g - 1)/2$ , then  $\dim \Delta(d) < \dim M_X(2, \Lambda) = 3g - 3$ .

By the above analysis, we may assume  $0 \leq d \leq (g - 1)/2$ . We will now redo the dimension count for  $\Delta(d)$  using the condition  $H^0(S^2) \neq 0$ . By Brill–Noether theory, the dimension of the space of line bundles of degree  $2d < g$  having nonzero sections is  $2d$ . Recomputing, we find

$$\dim \Delta(d) \leq 2d + 3g - 3 - 2d - 1 = 3g - 4.$$

Therefore  $\dim \Delta(d) < \dim \overline{M}_X(2, \Lambda)$  for all  $d$ .  $\square$

## 4 Proof of Theorem 2.3

Let  $X$  be a nonsingular genus–3 curve. For any rank–2 bundle  $N$  on  $X$  of degree 4, we will show there exists a degree–3 stable map

$$f: C \rightarrow X$$

for which  $H^0(C, f^*(N)) \neq 0$ .

If  $N$  contains a rank–1 subbundle  $L \subset N$  of positive degree, then  $H^0(X, L^3) \neq 0$  by Riemann–Roch. Let  $s$  be a nonzero section of  $L^3$  and let  $f: C \rightarrow X$  be the cube root of  $s$  in  $L$ . Since  $H^0(C, f^*(L))$  has a canonical section, we have shown  $H^0(C, f^*(N)) \neq 0$ . If  $N$  is unstable, then it has a destabilizing rank–1 subbundle of positive degree, and is not 3–rigid.

Let  $\Lambda$  be a line bundle on  $X$  of degree 4. Let  $\overline{M}_X(2, \Lambda)$  be the moduli space of rank–2, semistable bundles with determinant  $\Lambda$ . We will prove, for general

$$[N] \in \overline{M}_X(2, \Lambda),$$

that  $N$  contains a rank–1 subbundle of degree 1. Hence the general semistable bundle  $N$  with determinant  $\Lambda$  is not 3–rigid. By semicontinuity and the irreducibility of the moduli of semistable bundles, we conclude no semistable bundle on  $X$  is 3–rigid.

The dimension of the moduli space  $\overline{M}_X(2, \Lambda)$  is 6. The extensions

$$(3) \quad 0 \rightarrow L \rightarrow N \rightarrow L^{-1}\Lambda \rightarrow 0,$$

where  $L$  has rank 1 and degree 1, are parametrized (up to scale) by a projective bundle  $B$  with fiber  $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$  over  $\text{Pic}^1(X)$ . Since the dimensions of  $\text{Pic}^1(X)$  and  $\mathbb{P}(H^1(X, L^2\Lambda^{-1}))$  are both 3, the dimension of  $B$  is 6. An elementary argument shows the map to moduli,

$$\epsilon: B \rightarrow \overline{M}_X(2, \Lambda),$$

is well-defined on a (nonempty) open set of  $B$ . If  $\epsilon$  is dominant, the theorem is proven.

It suffices to prove that the tangent space to  $B$  generically surjects onto the tangent space of  $\overline{M}_X(2, \Lambda)$ . The following argument was provided by Michael Thaddeus.

Let  $\text{End}_0 N$  be the bundle of traceless endomorphisms of  $N$ . The tangent space to  $\overline{M}_X(2, \Lambda)$  at  $N$  is given by  $H^1(X, \text{End}_0 N)$ . The exact sequence (3) induces a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = \text{End}_0 N$$

where

$$E_1/E_0 = L^2\Lambda^{-1}, \quad E_2/E_1 = \mathcal{O}, \quad \text{and } E_3/E_2 = L^{-2}\Lambda.$$

For generic  $L$ ,

$$H^0(X, L^{-2}\Lambda) = H^1(X, L^{-2}\Lambda) = 0.$$

Hence, the inclusion  $E_2 \subset E_3$  induces isomorphisms

$$H^1(X, E_2) \cong H^1(X, \text{End}_0 N) \quad \text{and} \quad H^0(X, E_2) \cong H^0(X, \text{End}_0 N) = 0.$$

Consequently, the exact sequence for the inclusion  $E_1 \subset E_2$  reduces to

$$0 \rightarrow H^1(X, L^2\Lambda^{-1})/H^0(X, \mathcal{O}) \rightarrow H^1(X, \text{End}_0 N) \rightarrow H^1(X, \mathcal{O}) \rightarrow 0.$$

The tangent space of  $\overline{M}_X(2, \Lambda)$  at  $N$  is therefore identified with the tangent space of  $B$  at  $N$ . Indeed,  $H^1(X, L^2\Lambda^{-1})/H^0(X, \mathcal{O})$  is the space of deformations of the extension class and  $H^1(X, \mathcal{O})$  is the space of deformations of  $L$ .  $\square$

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