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Formulae of one-partition and two-partition Hodge integrals

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Prompted by the duality between open string theory on noncompact Calabi–Yau threefolds and Chern–Simons theory on three-manifolds, M Mariño and C Vafa conjectured a formula of one-partition Hodge integrals in term of invariants of the unknot. Many Hodge integral identities, including the λ_g conjecture and the ELSV formula, can be obtained by taking limits of the Mariño–Vafa formula.

Motivated by the Mariño–Vafa formula and formula of Gromov–Witten invariants of local toric Calabi–Yau threefolds predicted by physicists, J Zhou conjectured a formula of two-partition Hodge integrals in terms of invariants of the Hopf link and used it to justify the physicists' predictions.

In this expository article, we describe proofs and applications of these two formulae of Hodge integrals based on joint works of K Liu, J Zhou and the author. This is an expansion of the author's talk of the same title at the BIRS workshop *The Interaction of Finite Type and Gromov–Witten Invariants*, November 15th–20th 2003.

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1 Introduction

In [46], Witten related topological string theory on the cotangent bundle T^*M of a three manifold M to the Chern–Simons gauge theory on M. Gopakumar and Vafa [11] related the topological string theory on the deformed conifold (the cotangent bundle T^*S^3 of the 3–sphere) to that on the resolved conifold (the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\to\mathbb{P}^1$). These works lead to a duality between the topological string theory on the resolved conifold and the Chern–Simons theory on S^3. A mathematical consequence of this duality is a surprising relationship between Gromov–Witten invariants, which arise in the topological string theory, and knot invariants, which arise in the Chern–Simons theory. Ooguri and Vafa [43] proposed that Chern–Simons knot invariants of a knot K in S^3 should be related to open Gromov–Witten invariants of (X, L_K), where X is the resolved conifold and L_K is a Lagrangian submanifold of X canonically associated to K.

Ooguri and Vafa's proposal for a general knot is not precise enough to be a mathematical conjecture for the following reasons. First of all, Ooguri and Vafa did not give an

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explicit description of L_K for a general knot K; later Taubes carried out a construction for a general knot [44] so we now have a candidate. Secondly, given a Lagrangian submanifold L of X, it is not known how to define open Gromov–Witten invariants of (X, L) in general, though these invariants are expected to count holomorphic curves in X with boundary in L.

When K is the unknot, Ooguri and Vafa's proposal can be made more precise. In this case, Ooguri and Vafa described L_K explicitly and conjectured a formula of open Gromov–Witten invariants of (X, L_K). Although these invariants were not defined, Katz–Liu [18] and Li–Song [26] carried out heuristic localization calculations which agreed with the Ooguri–Vafa formula. One-partition Hodge integrals arise in Katz and Liu's calculations with varying torus weight. Mariño and Vafa identified the weight dependence of one-partition Hodge integrals with the framing dependence of invariants of the unknot and conjectured a formula of one-partition Hodge integrals [35]. This formula is a precise mathematical statement, and has strong consequences on Hodge integrals.

Ooguri and Vafa's proposal can be generalized to links, Labastida–Mariño–Vafa [20]. In particular, the Hopf link corresponds to an explicit Lagrangian submanifold with two connected components, and the relevant open Gromov–Witten invariants can be transformed to two-partition Hodge integrals by heuristic localization calculations [4]. This leads to a formula of two-partition Hodge integrals. Zhou conjectured this formula in [47] and used it to verify the formula of Gromov–Witten invariants of any local toric Calabi–Yau threefold proposed by physicists, Aganagic–Mariño–Vafa [2], Iqbal [17], Zhou [50].

The discussion above is a brief description of the origin of the formulae of one-partition and two-partition Hodge integrals. See Mariño [34; 33] for much more complete survey on the duality between topological string theory on noncompact Calabi–Yau threefolds and Chern–Simons theory on three manifolds.

In this paper, we will describe proofs and applications of these two formulae of Hodge integrals based on Liu–Liu–Zhou [31; 30; 29; 28]. We now give an overview of the rest of this paper. In Section 2, we define one-partition Hodge integrals and state the formula of such integrals conjectured by Mariño and Vafa. In Section 3, we describe how to extract the values of all λ_g–integrals and the ELSV formula from the Mariño–Vafa formula, following [29]. In Section 4, we describe three approaches to the Mariño–Vafa formula: the first approach is based on cut-and-join equations and is used in the proof of the Mariño–Vafa formula given in [31; 30]; the second approach is based on convolution equations and can be generalized to prove the formula of two-partition Hodge integrals [28]; the third approach is based on bilinear localization equations and is used in Okounkov and Pandharipande's proof of the Mariño–Vafa formula [42]. In Section 5, we present a proof of the convolution equation for one-partition Hodge integrals. In Section 6, we generalize the previous discussion to the two-partition case.

Finally, this paper is based on the author's talk at the BIRS workshop *The Interaction of Finite Type and Gromov–Witten Invariants* and does not cover developments since the workshop took place. The two-partition Hodge integrals and their applications can be better understood in terms of the topological vertex Aganagic–Klemm–Mariño–Vafa [1]. The topological vertex is related to three-partition Hodge integrals Diaconescu–Florea [3], and a mathematical theory is developed in Li–Liu–Liu–Zhou [25]. Another exciting development is the correspondence between Gromov–Witten invariants and Donaldson–Thomas invariants Maulik–Nekrasov–Okounkov–Pandharipande [36; 37].

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2 The Mariño–Vafa formula of one-partition Hodge integrals

In this section, we state the formula of one-partition Hodge integrals conjectured by Mariño and Vafa. We first recall the definitions of partitions and Hodge integrals.

2.1 Partitions

Recall that a *partition* μ of a nonnegative integer d is a sequence of positive numbers

$$\mu = (\mu_1 \geq \cdots \geq \mu_h > 0)$$

such that $\mu_1 + \cdots + \mu_h = d$. We call $\ell(\mu) = h$ the *length* of μ and $|\mu| = d$ the *size* of μ. The automorphism group $\mathrm{Aut}(\mu)$ permutes μ_i and μ_j if $\mu_i = \mu_j$. For example, $\mathrm{Aut}(5,5,4,1,1,1) \cong S_2 \times S_3$, where S_n denotes the permutation group of n elements. In particular, $\mathrm{Aut}(\mu)$ is trivial if and only if all the components $\mu_1, \ldots, \mu_h$ of μ are distinct.

2.2 Hodge integrals

Let $\overline{\mathcal{M}}_{g,h}$ be the Deligne–Mumford compactification of moduli space of complex algebraic curves of genus g with h marked points. A point in $\overline{\mathcal{M}}_{g,h}$ is represented by $(C, x_1, \dots, x_h)$, where C is a complex algebraic curve of arithmetic genus g with at most nodal singularities, $x_1, \dots, x_h$ are distinct smooth points on C, and $(C, x_1, \dots, x_h)$ is *stable* in the sense that its automorphism group is finite.

The Hodge bundle $\mathbb{E}$ is a rank–g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over the moduli point

$$[(C, x_1, \dots, x_h)] \in \overline{\mathcal{M}}_{g,h}$$

is $H^0(C, \omega_C)$. When C is smooth, it can be viewed as a compact Riemann surface and $H^0(C, \omega_C)$ is the space of holomorphic one forms on C. The λ–classes are defined by

$$\lambda_j = c_j(\mathbb{E}) \in H^{2i}\big(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}\big).$$

The cotangent line $T^*_{x_i} C$ of C at the i th marked point x_i gives rise to a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ–classes are defined by

$$\psi_i = c_1(\mathbb{L}_i) \in H^2\big(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}\big).$$

The λ–classes and ψ–classes lie in $H^*(\overline{\mathcal{M}}_{g,h}; \mathbb{Q})$ instead of $H^*(\overline{\mathcal{M}}_{g,h}; \mathbb{Z})$ because $\mathbb{E}$ and ψ are orbibundles on the compact orbifold $\overline{\mathcal{M}}_{g,h}$.

Hodge integrals are intersection numbers of λ–classes and ψ–classes:

$$\int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \in \mathbb{Q}$$

The ψ–integrals (also known as *descendent integrals*)

$$\int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{j_1} \cdots \psi_h^{j_h}$$

can be computed recursively by Witten's conjecture [45] proved by Kontsevich [19]. Okounkov and Pandharipande gave another proof of Witten's conjecture in [40], and M Mirzakhani recently gave a third proof [38].

Using Mumford's Grothendieck–Riemann–Roch calculations [39], Faber and Pandharipande showed [7] that general Hodge integrals can be uniquely reconstructed from descendent integrals. See [10, Section 4] for Givental's reformulation of this result.

2.3 One-partition Hodge integrals

Given a triple (g,μ,τ), where g is a nonnegative integer, μ is a partition, and $\tau\in\mathbb{Z}$, define a *one-partition Hodge integral* $G_{g,\mu}(\tau)$ as

$$\frac{-\sqrt{-1}^{|\mu|+\ell(\mu)}(\tau(\tau+1))^{\ell(\mu)-1}}{|\mathrm{Aut}(\mu)|}\left(\prod_{i=1}^{\ell(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}\right)\int_{\overline{\mathcal{M}}_{g,\ell(\mu)}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}$$

where $$\Lambda_g^\vee(u)=u^g-\lambda_1u^{g-1}+\ldots+(-1)^g\lambda_g.$$

One-partition Hodge integrals can be simplified in special cases. If $g=0$, then $\Lambda_0^\vee(u)=1$ and

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{0,h}}\frac{1}{\prod_{i=1}^h(1-\mu_i\psi_i)}&=\sum_{k_1+\cdots+k_h=h-3}\mu_1^{k_1}\cdots\mu_h^{k_h}\int_{\overline{\mathcal{M}}_{0,h}}\psi_1^{k_1}\cdots\psi_h^{k_h}\\&=\sum_{k_1+\cdots+k_h=h-3}\mu_1^{k_1}\cdots\mu_h^{k_h}\frac{h!}{k_1!\cdots k_h!}\\&=|\mu|^{h-3}.\end{aligned}$$

If $\tau=0$, then $G_{g,\mu}(0)=0$ if $\ell(\mu)>1$, and

$$G_{g,(d)}(0)=-\sqrt{-1}^{d+1}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\lambda_g}{1-d\psi}=-\sqrt{-1}^{d+1}d^{2g-2}b_g,$$

where $$b_g=\begin{cases}1, & g=0,\\ \int_{\overline{\mathcal{M}}_{g,1}}\lambda_g\psi^{2g-2}, & g>0.\end{cases}$$

2.4 Formula of one-partition Hodge integrals

To state the formula of one-partition Hodge integrals, we need to introduce some generating functions.

We first define generating functions of one-partition Hodge integrals. Introduce variables λ and $p=(p_1,p_2,\ldots)$. Given a partition μ, let

$$p_\mu=p_{\mu_1}\ldots p_{\mu_{\ell(\mu)}}.$$

Define generating functions

$$
\begin{aligned}
G_\mu(\lambda;\tau) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\mu)} G_{g,\mu}(\tau),\\
G(\lambda;\tau;p) &= \sum_{\mu\neq\varnothing} G_\mu(\lambda;\tau) p_\mu,\\
G^\bullet(\lambda;\tau;p) &= \exp\left(G(\lambda;\tau;p)\right) = \sum_{\mu} G^\bullet_\mu(\lambda;\tau) p_\mu = 1 + \sum_{\mu\neq\varnothing} G^\bullet_\mu(\lambda;\tau)p_\mu.
\end{aligned}
$$

Here $\varnothing$ is the empty partition, the unique partition such that $|\varnothing| = \ell(\varnothing) = 0$, and $G^\bullet_\mu(\lambda;\tau)$ is the disconnected version of $G_\mu(\lambda;\tau)$.

We next define generating functions of symmetric group representations. Introduce

$$
\mathcal{W}_\mu(q) = q^{\kappa_\mu/4} \prod_{1\leq i<j\leq \ell(\mu)} \frac{[\mu_i-\mu_j+j-i]}{[j-i]} \prod_{i=1}^{\ell(\mu)} \frac{1}{\prod_{v=1}^{\mu_i}[v-i+\ell(\mu)]} \tag{1}
$$

where

$$
\kappa_\mu = |\mu| + \sum_i (\mu_i^2 - 2i\mu_i), \qquad [m] = q^{m/2} - q^{-m/2}, \qquad q = e^{\sqrt{-1}\lambda}.
$$

The expression $\mathcal{W}_\mu(q)$ is related to the HOMFLY polynomial of the unknot. Let χ_μ denote the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ, and let C_μ denote the conjugacy class of $S_{|\mu|}$ indexed by μ. Define

$$
R^\bullet_\mu(\lambda;\tau) = \sum_{|\nu|=|\mu|} \frac{\chi_\nu(C_\mu)}{z_\mu} e^{\sqrt{-1}\tau\kappa_\nu\lambda/2} \sqrt{-1}^{|\mu|} \mathcal{W}_\nu(q), \tag{2}
$$

where $z_\mu = |\mathrm{Aut}(\mu)|\mu_1\cdots\mu_{\ell(\mu)}$. Finally, define

$$
R^\bullet(\lambda;\tau;p) = \sum_\mu R^\bullet_\mu(\lambda;\tau)p_\mu
$$

and its connected version

$$
R(\lambda;\tau;p) = \log R^\bullet(\lambda;\tau;p) = \sum_{\mu\neq\varnothing} R_\mu(\lambda;\tau)p_\mu.
$$

Conjecture 1 (Mariño–Vafa [35])

$$
G(\lambda;\tau;p) = R(\lambda;\tau;p). \tag{3}
$$

The Mariño–Vafa formula (3) provides a highly nontrivial link between geometry (Hodge integrals) and combinatorics (representations of symmetric groups). Note that for each fixed partition μ, the Mariño–Vafa formula gives a closed and finite formula of $G_\mu(\lambda;\tau)$, a generating function of all genera.

3 Applications of the Mariño–Vafa formula

Many Hodge integral identities can be obtained by taking limits of the Mariño–Vafa formula. We illustrate this by some examples, following [29].

We have

$$G_{g,\mu}(\tau) = \sum_{k=\ell(\mu)-1}^{2g-2+|\mu|+\ell(\mu)} G_{g,\mu}^k \tau^k,$$

where

$$G_{g,\mu}^{\ell(\mu)-1} = \frac{-\sqrt{-1}^{|\mu|+\ell(\mu)}}{|\mathrm{Aut}(\mu)|} \frac{\lambda_g}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)},$$

$$G_{g,\mu}^{2g-2+|\mu|+\ell(\mu)} = \frac{-\sqrt{-1}^{|\mu|+\ell(\mu)}}{|\mathrm{Aut}(\mu)|}\left(\prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}\right)\int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}.$$

3.1 λ_g–integrals

Extracting the part corresponding to $G_{g,\mu}^{\ell(\mu)-1}$ from $R(\lambda;\tau;p)$, we obtain

$$\sum_{g=0}^{\infty} \lambda^{2g}\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = |\mu|^{n-3}\frac{|\mu|\lambda/2}{\sin(|\mu|\lambda/2)}. \tag{4}$$

The identity (4) is true for any partition of length n, so we may view it as an identity in $\mathbb{Q}[\mu_1,\ldots,\mu_n][[\lambda]]$. This gives us the values of all λ_g–integrals:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\cdots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!} \tag{5}$$

where B_{2g} are Bernoulli numbers.

Equation (5) also follows from the following two identities:

Fact 2 (A formula for b_g)

$$\sum_{g=0}^{\infty} b_g t^{2g} = \frac{t/2}{\sin(t/2)} \tag{6}$$

Fact 3 (The λ_g conjecture)

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \cdots, k_n} b_g \tag{7}$$

Recall that b_g are defined in Section 2.3. Formula (6) gives values of b_g which are λ_g–integrals on $\overline{\mathcal{M}}_{g,1}$, while the λ_g conjecture tells us how general λ_g–integrals are determined by b_g. The formula (6) was proved by Faber and Pandharipande in [7]. The λ_g conjecture was found by Getzler and Pandharipande [9] as a consequence of the degree–0 Virasoro conjecture of $\mathbb{P}^1$ and was first proved by Faber and Pandharipande [8]. Later the Virasoro conjecture was proved for projective spaces [10] and curves [41]; both cases include $\mathbb{P}^1$ as a special case.

3.2 ELSV formula

The part corresponding to $G_{g,\mu}^{2g-2+|\mu|+\ell(\mu)}$ in $R(\lambda;\tau;p)$ reduces to the Burnside formula of Hurwitz numbers. Recall that the Hurwitz number $H_{g,\mu}$ is the weighted count of genus–g, degree–$|\mu|$ ramified covers of $\mathbb{P}^1$ with fixed ramification type μ over a point $q^1 \in \mathbb{P}^1$. We obtain the ELSV formula:

$$\frac{1}{|\mathrm{Aut}(\mu)|}\left(\prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}\right) \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)} = \frac{H_{g,\mu}}{(2g-2+|\mu|+\ell(\mu))!}. \tag{8}$$

This identity was first proved by Ekedahl, Lando, Shapiro, and Vainshtein [5; 6]. In [14], T Graber and R Vakil gave a proof by virtual localization on moduli spaces $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$ of stable maps to $\mathbb{P}^1$, and described a simplified proof by virtual localization on moduli spaces $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ of relative stable maps to $(\mathbb{P}^1, q^1)$ (see Section 5.1 for precise definition). Actually, virtual localization on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ with suitable choices of weights gives both the ELSV formula and the cut-and-join equation of Hurwitz numbers [30, Section 7]. The latter was first proved using combinatorics in [12], and later using the symplectic sum formula in [22] and [15, Section 15.2].

3.3 Other identities

The following identities, proved in [7], are also consequences the Mariño–Vafa formula:

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g \sum_{i=1}^{2g-1}\frac{1}{i} - \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1}b_{g_2}$$

See [29] for details.

4 Three approaches to the Mariño–Vafa formula

4.1 The cut-and-join equation and functorial localization

In this subsection, we described the strategy of the proof of the Mariño–Vafa formula given in [30]. We have seen in Section 2.3 that the left hand side of the Marño–Vafa formula can be greatly simplified at $\tau = 0$:

$$G(\lambda; 0; p) = -\sum_{d=1}^{\infty} \frac{\sqrt{-1}^{d+1} p_d}{\lambda d^2} \sum_{g=0}^{\infty} b_g (\lambda d)^{2g} \tag{9}$$

It turns out that the right hand side of the Mariño–Vafa formula can also be greatly simplified at $\tau = 0$ (see [49] or [30, Section 2] for details):

$$R(\lambda; 0; p) = -\sum_{d=1}^{\infty} \frac{\sqrt{-1}^{d+1} p_d}{2d\,\sin(\lambda d/2)} \tag{10}$$

Expressions (9) and (10) are equal by Fact 2, so the Mariño–Vafa formula (3) holds at $\tau = 0$:

$$G(\lambda; 0; p) = R(\lambda; 0; p) \tag{11}$$

Note that both sides of the Mariño–Vafa formula are valid for $\tau \in \mathbb{C}$. From (2) and the orthogonality of characters

$$\sum_{\sigma} \frac{\chi_\mu(C_\sigma)\chi_\nu(C_\sigma)}{z_\sigma} = \delta_{\mu\nu} \tag{12}$$

it follows immediately that

$$R^\bullet_\mu(\lambda;\tau)=\sum_{|\nu|=|\mu|}R^\bullet_\nu(\lambda;0)z_\nu\Phi^\bullet_{\nu,\mu}(\sqrt{-1}\lambda\tau) \tag{13}$$

where

$$\Phi^\bullet_{\nu,\mu}(\lambda)=\sum_\eta \frac{\chi_\eta(C_\nu)}{z_\nu}\frac{\chi_\eta(C_\mu)}{z_\mu}e^{\kappa_\eta\lambda/2}. \tag{14}$$

The convolution equation (13) is equivalent to the following *cut-and-join equation*:

$$\frac{\partial R}{\partial\tau}=\frac{\sqrt{-1}\lambda}{2}\sum_{i,j=1}^\infty\left((i+j)p_ip_j\frac{\partial R}{\partial p_{i+j}}+ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j}+\frac{\partial^2 R}{\partial p_i\partial p_j}\right)\right) \tag{15}$$

See [12], [49], and [30, Section 3] for details.

The Mariño–Vafa formula will follow from the initial values (11), the cut-and-join equation (15) of $R(\lambda;\tau;p)$, and the following cut-and-join equation of $G(\lambda;\tau;p)$.

Theorem 4 (Liu–Liu–Zhou [30])

$$\frac{\partial G}{\partial\tau}=\frac{\sqrt{-1}\lambda}{2}\sum_{i,j=1}^\infty\left((i+j)p_ip_j\frac{\partial G}{\partial p_{i+j}}+ijp_{i+j}\left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j}+\frac{\partial^2 G}{\partial p_i\partial p_j}\right)\right) \tag{16}$$

In [30], Theorem 4 was proved by applying (virtual) functorial localization [27] to the branch morphism

$$\mathrm{Br}\colon \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)\to \mathrm{Sym}^r\mathbb{P}^1\cong\mathbb{P}^r,$$

where $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ is the moduli space of relative stable maps from a genus–g curve to $\mathbb{P}^1$ with fixed ramification type $\mu=(\mu_1,\dots,\mu_h)$ over $q^1\in\mathbb{P}^1$, and

$$r=2g-2+|\mu|+\ell(\mu)$$

is the virtual dimension of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$. The precise definition of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ is given in Section 5.1. Note that the $\mathbb{C}^*$–action on $\mathbb{P}^1$ induce $\mathbb{C}^*$–actions on the domain and the target of Br, and Br is $\mathbb{C}^*$–equivariant. This proof was outlined in [31] and presented in detail in [30].

4.2 Convolution equation and double Hurwitz numbers

We now describe a variant of the above approach, which is even more direct and can be generalized to prove the formula of two-partition Hodge integrals [28]. This alternative proof of the Mariño–Vafa formula was discovered by the authors of [30] while working on [28].

The Mariño–Vafa formula will follow from the initial values (11), the convolution equation (13) of $R^\bullet_\mu(\lambda;\tau)$, and the following convolution equation (17) of $G^\bullet_\mu(\lambda;\tau)$:

Theorem 5 (Liu–Liu–Zhou)

$$G^\bullet_\mu(\lambda;\tau) = \sum_{|\nu|=|\mu|} G^\bullet_\nu(\lambda;0) z_\nu \Phi^\bullet_{\nu,\mu}(\sqrt{-1}\lambda\tau). \tag{17}$$

Recall that the λ_g conjecture (7) tells us how b_g (λ_g–integrals on $\overline{\mathcal{M}}_{g,1}$) determine general λ_g–integrals. The convolution equation (17) tells us how the b_g determine all one-partition Hodge integrals (which contain all λ_g–integrals and more), so it can be viewed as a generalization of the λ_g conjecture.

By the Burnside formula of Hurwitz numbers, $\Phi^\bullet_{\nu,\mu}(\lambda)$ is the generating function of disconnected double Hurwitz numbers $H^\bullet_{\chi,\nu,\mu}$:

$$\Phi^\bullet_{\nu,\mu}(\lambda) = \sum_\chi \lambda^{-\chi+\ell(\nu)+\ell(\mu)} \frac{H^\bullet_{\chi,\nu,\mu}}{(-\chi+\ell(\nu)+\ell(\mu))!} \tag{18}$$

where $|\nu| = |\mu| = d$, and $H^\bullet_{\chi,\nu,\mu}$ is the weighted count of degree–d covers of $\mathbb{P}^1$ with prescribed ramification types ν,μ over two points $q^0, q^1 \in \mathbb{P}^1$ by possibly disconnected Riemann surfaces of Euler characteristic χ. From (14) it is clear that

$$\sum_{|\sigma|=d} \Phi^\bullet_{\nu,\sigma}(\lambda_1) z_\sigma \Phi^\bullet_{\sigma,\mu}(\lambda_2) = \Phi^\bullet_{\nu,\mu}(\lambda_1+\lambda_2) \quad \text{and} \quad \Phi^\bullet_{\nu,\mu}(0) = \frac{\delta_{\nu,\mu}}{z_\nu}. \tag{19}$$

In Section 5, we will define a generating function $K^\bullet_\mu(\lambda)$ of relative Gromov–Witten invariants of $(\mathbb{P}^1,\infty)$. By virtual localization, $K^\bullet_\mu(\lambda)$ can be expressed in terms of one-partition Hodge integrals and double Hurwitz numbers:

Proposition 6

$$K^\bullet_\mu(\lambda) = \sum_{|\nu|=|\mu|} G^\bullet_\nu(\lambda;\tau) z_\nu \Phi^\bullet_{\nu,\mu}\big(-\sqrt{-1}\lambda\tau\big). \tag{20}$$

Proposition 6 is a special case of [28, Proposition 7.1] and, by (19), is equivalent to

$$G^\bullet_\mu(\lambda;\tau) = \sum_{|\nu|=|\mu|} K^\bullet_\nu(\lambda) z_\nu \Phi^\bullet_{\nu,\mu}\big(\sqrt{-1}\lambda\tau\big), \qquad G^\bullet_\mu(\lambda;0) = K^\bullet_\mu(\lambda). \tag{21}$$

This gives Theorem 5.

4.3 Bilinear localization equations

This approach is due to Okounkov and Pandharipande [42] and was motivated by Faber and Pandharipande's proof of the λ_g conjecture [8].

By virtual localization on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$, Okounkov and Pandharipande derived homogeneous bilinear equations of the form

$$\sum(\text{linear Hodge integrals}) \cdot (\text{special cubic Hodge integrals}) = 0 \tag{22}$$

and inhomogeneous bilinear equations of the form

$$\sum(\text{linear Hodge integrals}) \cdot (\text{special cubic Hodge integrals}) = (\lambda_g \text{ integrals}) \tag{23}$$

where "linear Hodge integrals" are those in the ELSV formula, and "special cubic Hodge integrals" are those in the Mariño–Vafa formula. The values of linear Hodge integrals are given by the ELSV formula (8) and the Burnside formula of Hurwitz numbers; the values of λ_g–integrals are given by (5). Therefore, (22) and (23) can be viewed as linear equations satisfied by special cubic Hodge integrals. It was shown in [41] that there is a unique solution to this system of linear equations, and the Mariño–Vafa formula gives a solution.

5 Proof of Proposition 6

We fix a degree d, and a partition $\mu = (\mu_1, \dots, \mu_h)$ of d.

5.1 Moduli spaces

Let $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$ be the moduli space of ramified covers

$$f\colon (C, x_1, \dots, x_h) \to (\mathbb{P}^1, q^1)$$

of degree–d from a smooth curve C of genus g to $\mathbb{P}^1$ such that the ramification type over a distinguished point $q^1 \in \mathbb{P}^1$ is specified by the partition μ, that is,

$$f^{-1}(q^1) = \mu_1 x_1 + \cdots + \mu_h x_h$$

as Cartier divisors. The moduli space $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$ is not compact. To compactify it, we consider the moduli space

$$\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

of relative stable maps to $(\mathbb{P}^1, q^1)$. The moduli spaces of relative stable maps were constructed by A Li–Ruan [21] and Ionel–Parker [16; 15] in symplectic geometry. Later J Li carried out the construction in algebraic geometry [23; 24]. We need to use J

Li's algebraic version. Such moduli spaces are defined for a general pair (X, D) where X is a smooth projective variety and D is a smooth divisor. Here we content ourselves with the definition for this special case.

We first introduce some notation. Let $\mathbb{P}^1(m) = \mathbb{P}^1_1 \cup \cdots \cup \mathbb{P}^1_m$ be a chain of m copies of $\mathbb{P}^1$. For $l = 1, \ldots, m-1$, let q^1_l be the node at which $\mathbb{P}^1_l$ and $\mathbb{P}^1_{l+1}$ intersect. Let $q^1_0 \in \mathbb{P}^1_1$ and $q^1_m \in \mathbb{P}^1_m$ be smooth points.

A point in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is represented by a morphism

$$f\colon (C, x_1, \ldots, x_h) \to (\mathbb{P}^1[m], q^1_m)$$

where C has at most nodal singularities, and $\mathbb{P}^1[m]$ is obtained by identifying $q^1 \in \mathbb{P}^1$ with $q^1_0 \in \mathbb{P}^1(m)$. In particular, $\mathbb{P}^1[0] = \mathbb{P}^1$. We call the original $\mathbb{P}^1 = \mathbb{P}^1_0$ the root component and $\mathbb{P}^1_1, \ldots, \mathbb{P}^1_m$ bubble components. We have $C = C_0 \cup C_1 \cup \cdots \cup C_m$, where C_l is the preimage of $\mathbb{P}^1_l$. Let $f_l\colon C_l \to \mathbb{P}^1_l$ be the restriction of f. Then:

(1) (*degree*) $\deg f_l = d$, for $l = 0, \ldots, h$.

(2) (*ramification*) $f^{-1}(q^1_m) = \sum_{j=1}^h \mu_j x_j$.

(3) (*predeformability*) The preimage of each node of the target consists of nodes, at each of which two branches have the same contact order. This is the predeformable condition: so that one can smooth both the target and the domain to a morphism to $\mathbb{P}^1$.

(4) (*stability*) The automorphism group of f is finite.

Two morphisms satisfying (1)–(3) are equivalent if they have the same target $\mathbb{P}^1[m]$ for some nonnegative integer m and they differ by an isomorphism of the domain and an element of $\mathrm{Aut}(\mathbb{P}^1[m], q^1_0, q^1_m) \cong (\mathbb{C}^*)^m$. In particular, this defines the automorphism group in (4). For fixed g, μ, the stability condition (4) gives an upper bound of the number m of bubble components of the target.

By results in [23; 24], $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is a proper, separated Deligne–Mumford stack with a perfect obstruction theory of virtual dimension $2g-2+d+h$. Roughly speaking, this means that it is a compact, Hausdorff singular orbifold with a "virtual tangent bundle" of rank $2g-2+d+h$. For later convenience, we will also consider the disconnected version $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu)$, where the domain C is allowed to be disconnected with $2\big(h^0(\mathcal{O}_C)-h^1(\mathcal{O}_C)\big) = \chi$. Note that when C is smooth, $2\big(h^0(\mathcal{O}_C)-h^1(\mathcal{O}_C)\big)$ is also the Euler characteristic of C as a compact surface. $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu)$ is a proper, separated Deligne–Mumford stack with a perfect obstruction theory of virtual dimension $-\chi + d + h$.

Similarly, we may specify ramification types ν, μ over two points $q^0, q^1 \in \mathbb{P}^1$ and define the corresponding moduli spaces $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \nu, \mu)$ and $\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)$ of relative stable maps. We will also consider the quotient

$$\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)//\mathbb{C}^* \equiv \big(\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu) \setminus \overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)^{\mathbb{C}^*}\big)/\mathbb{C}^*$$

by the automorphism group $\mathbb{C}^*$ of the target $(\mathbb{P}^1, q^0, q^1)$.

5.2 Double Hurwitz numbers as relative Gromov–Witten invariants

The moduli space $\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)$ parametrizes morphisms with targets of the form $\mathbb{P}^1[m_0, m_1]$, where $\mathbb{P}^1[m_0, m_1]$ is obtained by attaching $\mathbb{P}^1(m_0)$ and $\mathbb{P}^1(m_1)$ to $\mathbb{P}^1$ at q^0 and q^1 respectively. The distinguished points on $\mathbb{P}^1[m_0, m_1]$ are $q^0_{m_0}$ and $q^1_{m_1}$. Let $\pi_{m_0,m_1}\colon \mathbb{P}^1[m_0, m_1] \to \mathbb{P}^1$ be the contraction to the root component.

There is a branch morphism

$$\mathrm{Br}\colon \overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu) \to \mathrm{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$$

sending $\big[f\colon C \to \mathbb{P}^1[m_0, m_1]\big]$ to

$$\mathrm{div}\big(\widetilde{f}\,\big) - (d - \ell(\nu))q^0 - (d - \ell(\mu))q^1$$

where $\mathrm{div}\big(\widetilde{f}\,\big)$ is the branch divisor of $\widetilde{f} = \pi_{m_0,m_1} \circ f\colon C \to \mathbb{P}^1$, and

$$r = -\chi + \ell(\nu) + \ell(\mu)$$

is the virtual dimension of $\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)$.

The structures on $\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)$ allow one to construct a *virtual fundamental class*

$$\big[\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu)\big]^{\mathrm{vir}} \in H_{2r}(\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1, \nu, \mu); \mathbb{Q})$$

which plays the role of the fundamental class of a compact oriented manifold. Let $H \in H^2(\mathbb{P}^r; \mathbb{Z})$ be the hyperplane class. The disconnected double Hurwitz number is equal to

$$H^{\bullet}_{\chi,\nu,\mu} = \frac{1}{|\mathrm{Aut}(\nu)||\mathrm{Aut}(\mu)|} \int_{[\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1,\nu,\mu)]^{\mathrm{vir}}} \mathrm{Br}^* H^r$$

which is equal to

$$\frac{r!}{|\mathrm{Aut}(\nu)||\mathrm{Aut}(\mu)|} \int_{[\overline{\mathcal{M}}^{\bullet}_{\chi}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*]^{\mathrm{vir}}} \psi_0^{r-1}$$

by virtual localization, where ψ_0 is the *target* ψ*–class*, the first Chern class of the line bundle $\mathbb{L}_0$ over $\overline{\mathcal{M}}_g(\mathbb{P}^1,\nu,\mu)$ whose fiber at

$$\big[f\colon C\to\mathbb{P}^1[m_0,m_1]\big]$$

is the cotangent line $T^*_{q^0_{m_0}}\mathbb{P}^1[m_0,m_1]$.

5.3 Obstruction bundle

We will define a vector bundle $V^\bullet_{\chi,\mu}$ over the moduli space $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$.

For $f\colon (C,x_1,\dots,x_h)\to\mathbb{P}^1[m]$, let $\widetilde{f}$ be the composition $\pi_m\circ f\colon C\to\mathbb{P}^1$, where $\pi_m\colon \mathbb{P}^1[m]\to\mathbb{P}^1$ is the contraction to the root component. The fiber of $V^\bullet_{\chi,\mu}$ at

$$\big[f\colon (C,x_1,\dots,x_h)\to\mathbb{P}^1[m]\big]\in\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$$

is

$$H^1(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))\oplus H^1(C,\mathcal{O}_C(-x_1-\cdots-x_h)). \tag{24}$$

Note that

$$H^0(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))=H^0(C,\mathcal{O}_C(-x_1-\cdots-x_h))=0,$$

so (24) forms a vector bundle over $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$ which, by Riemann–Roch, has rank $-\chi+d+h$, which is equal to the virtual dimension of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$. So

$$K^\bullet_{\chi,\mu}=\frac{1}{|\mathrm{Aut}(\mu)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} e\big(V^\bullet_{\chi,\mu}\big)$$

is a topological invariant, where $e\big(V^\bullet_{\chi,\mu}\big)$ is the Euler class (top Chern class) of $V^\bullet_{\chi,\mu}$.

The generating function in Proposition 6 is given by

$$K^\bullet_\mu(\lambda)=\sqrt{-1}^{\,h-d}\sum_\chi \lambda^{-\chi+h}K^\bullet_{\chi,\mu}. \tag{25}$$

5.4 Virtual localization

Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $t\cdot[X,Y]=[tX,Y]$ for $t\in\mathbb{C}^*$ and $[X,Y]\in\mathbb{P}^1$. The two fixed points are $q^0=[0,1]$ and $q^1=[1,0]$. This induces a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$. We would like to lift the $\mathbb{C}^*$–action to the vector bundle $V^\bullet_{\chi,\mu}$ so that we can calculate the integral of the equivariant Euler class $e_{\mathbb{C}^*}(V^\bullet_{\chi,\mu})$ by virtual localization. It suffices to lift the $\mathbb{C}^*$–action on $\mathbb{P}^1$ to the line bundles $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$, which will induce actions on the cohomology groups (24). We only need to specify the weights of $\mathbb{C}^*$–actions on the fibers of these line bundles over q^0 and q^1:

	q^0	q^1
$\mathcal{O}_{\mathbb{P}^1}(-1)$	$t^{-\tau-1}$	$t^{-\tau}$
$\mathcal{O}_{\mathbb{P}^1}$	t^{τ}	t^{τ}

where $\tau \in \mathbb{Z}$. We have

$$\begin{aligned} K^\bullet_{\chi,\mu} &= \frac{1}{|\mathrm{Aut}(\mu)|}\int_{\left[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)\right]^{\mathrm{vir}}} e\big(V^\bullet_{\chi,\mu}\big) \\ &= \frac{1}{|\mathrm{Aut}(\mu)|}\int_{\left[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)\right]^{\mathrm{vir}}} e_{\mathbb{C}^*}\big(V^\bullet_{\chi,\mu}\big) \\ &= \frac{1}{|\mathrm{Aut}(\mu)|}\sum_F \int_{[F]^{\mathrm{vir}}} \frac{e_{\mathbb{C}^*}\big(V^\bullet_{\chi,\mu}\big)}{e_{\mathbb{C}^*}\big(N_F^{\mathrm{vir}}\big)}. \end{aligned}$$

If $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$ is a compact complex manifold and each fixed locus F is a compact complex submanifold, then $[F]^{\mathrm{vir}}$ is the fundamental class of F, N_F^{vir} is the normal bundle of F in $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)$, and the last equality is the Atiyah–Bott localization formula. Here we need to apply virtual localization formula proved by Graber and Pandharipande in [13], where fundamental classes and normal bundles are replaced by *virtual* fundamental classes and *virtual* normal bundles, respectively. The equivariant Euler class $e_{\mathbb{C}^*}\big(V^\bullet_{\chi,\mu}\big)$ and the contribution from each fixed locus F depend on τ.

Let $f\colon (C,x_1,\dots,x_h)\to \mathbb{P}^1[m]$ be a point in the fixed points set $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu)^{\mathbb{C}^*}$, let $\widetilde{f}=\pi_m\circ f\colon C\to\mathbb{P}^1$, and let $C^i=\widetilde{f}^{-1}(q^i)$, for $i=0,1$. Then the complement of $C^0\cup C^1$ in C is a disjoint union of twice-punctured spheres $L_1,\dots,L_k$, and $f|_{L_j}\colon L_j\to\mathbb{P}^1\setminus\{q^0,q^1\}$ is an honest covering map of some degree ν_j. This gives a partition $\nu=(\nu_1,\dots,\nu_k)$ of $d=|\mu|$. The restriction $f|_{C^0}$ is a constant map to q^0, and $f|_{C^1}\colon C^1\to\mathbb{P}^1(m)$ represents a point in $\overline{\mathcal{M}}^\bullet_{\chi^1}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*$, so the fixed locus is a finite quotient of

$$\overline{\mathcal{M}}^\bullet_{\chi^0,\ell(\nu)}\times\big(\overline{\mathcal{M}}^\bullet_{\chi^1}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*\big) \tag{26}$$

where $\chi^0+\chi^1-2\ell(\nu)=\chi$, and $\overline{\mathcal{M}}^\bullet_{\chi,n}$ is the moduli stack of possibly disconnected stable curves C with n marked points and $2(h^0(\mathcal{O}_C)-h^1(\mathcal{O}_C))=\chi$. The contribution from the above fixed locus is of the form

$$\int_{\overline{\mathcal{M}}^\bullet_{\chi^0,\ell(\nu)}}(\cdots)\cdot A(\tau)\cdot\int_{\left[\overline{\mathcal{M}}^\bullet_{\chi^1}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*\right]^{\mathrm{vir}}}(\cdots)$$

where $A(\tau)$ is some combinatorial factor. Calculations in [31; 30] show that the integral over $\overline{\mathcal{M}}^{\bullet}_{\chi^0,\ell(\nu)}$ is exactly the one-partition Hodge integral $G^{\bullet}_{\chi^0,\nu}(\tau)$ and the integrand of the integral over $\overline{\mathcal{M}}^{\bullet}_{\chi^1}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*$ is a power of the target ψ class. We have

$$K^{\bullet}_{\chi,\mu} = \sum_{\chi^0+\chi^1-2\ell(\nu)=\chi} G^{\bullet}_{\chi^0,\nu}(\tau)\cdot A_{\chi^0,\nu,\chi^1}(\tau)\cdot H^{\bullet}_{\chi^1,\nu,\mu} \tag{27}$$

where $A_{\chi^0,\nu,\chi^1}(\tau)$ is some τ–dependent combinatorial factor. The expression (27) can be neatly packaged in terms of generating functions:

$$K^{\bullet}_{\chi}(\lambda) = \sum_{|\nu|=d} G^{\bullet}_{\nu}(\lambda;\tau) z_\nu \Phi^{\bullet}_{\nu,\mu}\big(-\sqrt{-1}\lambda\tau\big).$$

This completes the proof of Proposition 6.

6 Generalization to the two-partition case

6.1 Gromov–Witten invariants of local toric Calabi–Yau threefolds

Let S be a Fano surface, and let X be the total space of the canonical line bundle K_S of S (for example, $\mathcal{O}(-3)\to\mathbb{P}^2$). Then X is a noncompact Calabi–Yau threefold. We call such a noncompact Calabi–Yau threefold a *local Calabi–Yau threefold*. The image of any nonconstant morphism from a complex algebraic curve to X is contained in S, so for any nontrivial $d\in H_2(S;\mathbb{Z})\cong H_2(X;\mathbb{Z})$ we have

$$\overline{\mathcal{M}}_{g,0}(X,d)=\overline{\mathcal{M}}_{g,0}(S,d)$$

as Deligne–Mumford stacks. However, they have different perfect obstruction theories: the virtual dimension of the perfect obstruction theory of $\overline{\mathcal{M}}_{g,0}(X,d)$ is zero, while that of $\overline{\mathcal{M}}_{g,0}(S,d)$ is

$$g-1-\int_d c_1(K_S).$$

The genus–g, degree–d (for $d\neq 0$) Gromov–Witten invariant of X is defined by

$$N^X_{g,d}=\int_{\left[\overline{\mathcal{M}}_{g,0}(X,d)\right]^{\mathrm{vir}}} 1\in\mathbb{Q}.$$

Let $V_{g,d}$ be the vector bundle over $\overline{\mathcal{M}}_{g,0}(S,d)$ whose fiber over a point represented by $f\colon C\to S$ is given by

$$H^1(C,f^*K_S). \tag{28}$$

Note that $H^0(C, f^*K_S)=0$, so (28) forms a vector bundle over $\overline{\mathcal{M}}_{g,0}(S,d)$. By Riemann–Roch, its rank is $g-1-\int_d c_1(K_S)$ which is equal to the virtual dimension of $\overline{\mathcal{M}}_{g,0}(S,d)$. We have

$$N^X_{g,d} = \int_{[\overline{\mathcal{M}}_{g,0}(X,d)]^{\mathrm{vir}}} 1 = \int_{[\overline{\mathcal{M}}_{g,0}(S,d)]^{\mathrm{vir}}} e(V_{g,d}). \tag{29}$$

When X is a local *toric* Calabi–Yau threefold, that is, the total space of the canonical line bundle of a *toric* Fano surface, the integral in (29) can be reduced to *two-partition Hodge integrals* by virtual localization.

6.2 Two-partition Hodge integrals

Given μ^+,μ^- partitions, let $\ell^\pm=\ell(\mu^\pm)$ and define

$$\begin{aligned}
G_{g,\mu^+,\mu^-}(\tau) = &\frac{-\sqrt{-1}^{\ell^++\ell^-}}{|\mathrm{Aut}(\mu^+)||\mathrm{Aut}(\mu^-)|}(\tau(\tau+1))^{\ell^++\ell^--1}\\
&\cdot\prod_{i=1}^{\ell^+}\frac{\prod_{a=1}^{\mu_i^+-1}(\mu_i^+\tau+a)}{(\mu_i^+-1)!}\prod_{j=1}^{\ell^-}\frac{\prod_{a=1}^{\mu_j^--1}\left(\frac{\mu_j^-}{\tau}+a\right)}{(\mu_j^--1)!}\\
&\cdot\int_{\overline{\mathcal{M}}_{g,\ell^++\ell^-}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{\ell^+}\left(1-\mu_i^+\psi_i\right)\prod_{j=1}^{\ell^-}\tau\left(\tau-\mu_j^-\psi_{\ell^++j}\right)}
\end{aligned}$$

In particular,

$$\sqrt{-1}^{|\mu|}G_{g,\mu,\varnothing}(\tau) = G_{g,\mu}(\tau) \tag{30}$$

where $G_{g,\mu}(\tau)$ is the one-partition Hodge integral defined in Section 2.3.

6.3 Formula of two-partition Hodge integrals

To state the formula of two-partition Hodge integrals, we introduce some generating functions.

We first define generating function of two-partition Hodge integrals. Introduce variables

$$p^+ = (p_1^+, p_2^+, \ldots) \quad \text{and} \quad p^- = (p_1^-, p_2^-, \ldots).$$

Given a partition μ, define

$$p_\mu^\pm = p_1^\pm \ldots p_{\ell(\mu)}^\pm$$

Define generating functions

$$\begin{aligned}
G_{\mu^+,\mu^-}(\lambda;\tau) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell^++\ell^-} G_{g,\mu^+,\mu^-}(\tau) \\
G(\lambda;p^+,p^-;\tau) &= \sum_{(\mu^+,\mu^-)\neq(\varnothing,\varnothing)} G_{\mu^+,\mu^-}(\lambda;\tau)p^+_{\mu^+}p^-_{\mu^-} \\
G^\bullet(\lambda;p^+,p^-;\tau) &= \exp\big(G\big(\lambda;p^+,p^-;\tau\big)\big) \\
&= \sum_{\mu^+,\mu^-} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)p^+_{\mu^+}p^-_{\mu^-}
\end{aligned}$$

$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)$ is the disconnected version of $G_{\mu^+,\mu^-}(\lambda;\tau)$. By (30),

$$\sqrt{-1}^{|\mu|}G^\bullet_{\mu,\varnothing}(\lambda;\tau) = G^\bullet_{\mu}(\lambda;\tau). \tag{31}$$

We next define generating functions of symmetric group representations. Let

$$s_\mu(x_1,x_2,\dots)$$

be *Schur functions* (see [32] for definitions). Recall that

$$s_\nu s_\rho = \sum_\mu c^\mu_{\nu\rho}s_\mu \quad \text{and} \quad s_{\mu/\rho} = \sum_\rho c^\mu_{\nu\rho}s_\rho$$

where $c^\mu_{\nu\rho}$ are known as *Littlewood–Richardson coefficients* and $s_{\mu/\rho}$ are known as *skew Schur functions*. Let $q = e^{\sqrt{-1}\lambda}$ and write

$$q^{-\rho} = \big(q^{\frac{1}{2}}, q^{\frac{3}{2}}, \dots\big)$$

Introduce

$$\mathcal{W}_{\mu,\nu}(q) = (-1)^{|\mu|+|\nu|}q^{\frac{\kappa_\mu+\kappa_\nu}{2}}\sum_\eta s_{\mu/\eta}(q^{-\rho})s_{\nu/\eta}(q^{-\rho})$$

which is related to the HOMFLY polynomial of the Hopf link. In particular,

$$\mathcal{W}_{\mu,\varnothing}(q) = (-1)^{|\mu|}q^{\kappa_\mu/2}s_\mu(q^{-\rho}) = s_\mu(q^\rho) = \mathcal{W}_\mu(q)$$

where $\mathcal{W}_\mu(q)$ is defined by (1) (see [48] for details). Define

$$R^\bullet_{\mu^+,\mu^-}(\lambda;\tau) = \sum_{|\nu^\pm|=|\mu^\pm|} \frac{\chi_{\nu^+}(C_{\mu^+})}{z_{\mu^+}}\frac{\chi_{\nu^-}(C_{\mu^-})}{z_{\mu^-}}e^{\sqrt{-1}(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1})\lambda/2}\mathcal{W}_{\nu^+,\nu^-}(q). \tag{32}$$

In particular,

$$\sqrt{-1}^{|\mu|} R^{\bullet}_{\mu,\varnothing}(\lambda;\tau) = \sqrt{-1}^{|\mu|} \sum_{\nu} \frac{\chi_\nu(C_\mu)}{z_\mu} e^{\sqrt{-1}\kappa_\nu \tau\lambda/2} \mathcal{W}_\nu(q) = R^{\bullet}_{\mu}(\lambda;\tau) \tag{33}$$

where $R^{\bullet}_{\mu}(\lambda;\tau)$ is defined by (2).

Conjecture 7 (Zhou [47])

$$G^{\bullet}_{\mu^+,\mu^-}(\lambda;\tau) = R^{\bullet}_{\mu^+,\mu^-}(\lambda;\tau). \tag{34}$$

By (31) and (33), Conjecture 7 reduces to the Mariño–Vafa formula when $\mu^- = \varnothing$.

6.4 Application

Let X be a local toric Calabi–Yau threefold. The *Gromov–Witten potential* of X is defined by

$$F^X(\lambda,t) = \sum_{g=0}^{\infty} \sum_{\substack{d \in H_2(X,\mathbb{Z}) \\ d \neq 0}} \lambda^{2g-2} N^X_{g,d} e^{-d\cdot t}$$

where $t=(t_1,t_2,\dots)$ are coordinates on $H^{1,1}(X)$. Note that $H^2(X,\mathbb{C}) = H^{1,1}(X)$ because X is toric. The *partition function* is defined by

$$Z^X(\lambda,t) = \exp\big(F^X(\lambda,t)\big)$$

By virtual localization, $Z^X(\lambda,t)$ can be expressed in terms of $G^{\bullet}_{\mu^+,\mu^-}(\lambda;\tau)$, so (34) gives an explicit formula of $Z^X(\lambda,t)$ for any local toric Calabi–Yau threefold X in terms of $\mathcal{W}_{\mu\nu}(q)$.

For example, let X be the total space of $\mathcal{O}_{\mathbb{P}^1}(-3) \to \mathbb{P}^2$. Then

$$Z^X(\lambda,t) = \sum_{\mu^i \neq \varnothing} \big((-1)^{\sum_{i=1}^3 |\mu^i|}\big)\big(e^{-\sum_{i=1}^3 |\mu^i| t}\big)\big(q^{\sum_{i=1}^3 \kappa_{\nu^i}/2}\big) \mathcal{W}_{\mu^1\mu^2}(q)\mathcal{W}_{\mu^2\mu^3}(q)\mathcal{W}_{\mu^3\mu^1}(q).$$

The algorithm for computing $Z^X(\lambda,t)$ for local toric Calabi–Yau threefolds in terms of $\mathcal{W}_{\mu\nu}(q)$ was described by Aganagic–Mariño–Vafa [2], and an explicit formula was given by Iqbal in [17]. Motivated by the Mariño–Vafa and Iqbal formulæ, Zhou conjectured a formula (34) for two-partition Hodge integrals and used it, together with virtual localization, to give a mathematical derivation of Iqbal's formula. The formula for two-partition Hodge integrals was proved in [28].

6.5 Outline of proof

In this subsection, we outline the proof of the formula of two-partition Hodge integrals given in [28]. The strategy is similar to the second approach to the Mariño–Vafa formula described in Section 4.2.

It follows from (32) that

(35)
$$R^\bullet_{\mu^+,\mu^-}(\lambda;\tau) = \sum_{|\nu^\pm|=|\mu^\pm|} R^\bullet_{\mu^+,\mu^-}(\lambda;\tau_0)\cdot z_{\nu^+}\Phi^\bullet_{\nu^+,\mu^+}\big(\sqrt{-1}\lambda(\tau-\tau_0)\big)\cdot z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}\big(\sqrt{-1}\lambda\big(\tfrac{1}{\tau}-\tfrac{1}{\tau_0}\big)\big)$$

for any $\tau\in\mathbb{C}^*$. Here we cannot specialize at $\tau=0$ because both sides of (34) have a pole at $\tau=0$. At $\tau=-1$, the two partition integrals vanish unless $\ell(\mu^+)+\ell(\mu^-)=1$ and we are left with b_g (the λ_g–integrals on $\overline{\mathcal{M}}_{g,1}$).

Theorem 8 (Zhou [47])

$$G^\bullet_{\mu^+,\mu^-}(\lambda;-1) = R^\bullet_{\mu^+,\mu^-}(\lambda;-1) \tag{36}$$

The authors of [28] defined generating functions $K^\bullet_{\mu^+,\mu^-}(\lambda)$ for relative Gromov–Witten invariants of $\mathbb{P}^1\times\mathbb{P}^1$ blown up at a point, and used virtual localization to derive the following expression.

Proposition 9 (Liu–Liu–Zhou [28])

(37)
$$K^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu^\pm|=|\mu^\pm|} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)z_{\nu^+}\Phi^\bullet_{\nu^+,\mu^+}\big(-\sqrt{-1}\lambda\tau\big)z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}\big(-\sqrt{-1}\lambda/\tau\big)$$

This proposition implies the following convolution equation of $G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)$:

Theorem 10

(38)
$$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) = \sum_{|\nu^\pm|=|\mu^\pm|} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau_0)\cdot z_{\nu^+}\Phi^\bullet_{\nu^+,\mu^+}(\sqrt{-1}\lambda\big(\tau-\tau_0\big)\big)\cdot z_{\nu^-}\Phi^\bullet_{\nu^-,\mu^-}\big(\sqrt{-1}\lambda\big(\tfrac{1}{\tau}-\tfrac{1}{\tau_0}\big)\lambda\big)$$

for any $\tau_0\in\mathbb{C}^*$.

The formula (34) for two-partition Hodge integrals follows from the convolution equation (35) for $R^\bullet_{\mu^+,\mu^-}(\lambda;\tau)$, the convolution equation (38) for $G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)$, and the initial values (36).

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