Perturbative expansion of Chern–Simons theory

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An overview of the perturbative expansion of the Chern–Simons path integral is given. The main goal is to describe how trivalent graphs appear: as they already occur in the perturbative expansion of an analogous finite-dimensional integral, we discuss this case in detail.

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1 Introduction

While a standard technique used by physicists, many mathematicians are not familiar with the mechanisms and justifications behind perturbatively expanding Feynman path integrals. In this article we will describe how trivalent graphs arise when one expands a path integral whose Lagrangian contains a cubic term. We'll start with a finite-dimensional integral, and then indicate the necessary adjustments required to generalize to the infinite-dimensional integral arising in Chern–Simons theory.

This article is for the proceedings of the BIRS workshop "The interaction of finite type and Gromov–Witten invariants". The purpose of the workshop was to investigate 'large–N duality', which relates perturbative Chern–Simons theory with gauge group $\mathrm{SU}(N)$ to certain open and closed string theories. One of the more mysterious aspects of this correspondence, to this author at least, is that in string theory trivalent graphs (more precisely, fat graphs) have some geometric realization, whereas in perturbative Chern–Simons theory they enter purely as part of the technique of calculation. As we shall see, trivalent graphs arise merely as combinatorial objects which index terms in the series expansion.

The exact solution to Chern–Simons theory was described by Witten [25] in the late 1980s. The perturbative expansion of the path integral for flat $\mathbb{R}^3$ was then considered by Guadagnini, Martellini, and Mintchev [10; 11; 12; 13], and for general three-manifolds by Axelrod and Singer [3; 4]. Subsequently much attention was focused on understanding the perturbative series for knots and links in $\mathbb{R}^3$ (see Bar-Natan [5; 7], Bott and Taubes [8], Altschuler and Freidel [1], and Thurston [24]). More on perturbative expansion of path integrals may be found in Witten's lectures [27]. For

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a slightly different application, to the perturbative expansion of Hermitian matrix integrals, see Mulase [16], where Feynman diagram techniques are nicely explained.

While this article was being prepared, a preprint by Michael Polyak [19] appeared, covering very similar material. All the same, the author decided to complete this article for inclusion in the present volume. The interested reader may wish to consult Polyak's preprint, which in many cases provides additional details to what is covered here.

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2 Stationary-phase approximation

Fix a compact, connected, simply-connected gauge group G (for example, $\mathrm{SU}(n)$). These conditions imply that $\pi_1(G)=\pi_2(G)=1$ and $\pi_3(G)=\mathbb{Z}$, a fact that we shall use shortly. Let M be a three-manifold without boundary. The field in Chern–Simons theory is a connection ∇ on a principal G–bundle over M. The conditions on G imply that the bundle can be trivialized and hence we can write the connection as $\nabla = d + A$ where $A\in\Omega^1(M)\otimes\mathfrak{g}$ is a Lie algebra valued one-form. A change of trivialization is given by a gauge transformation, which is a smooth function $g\colon M\to G$. Under this transformation A changes to $g^{-1}Ag+g^{-1}dg$.

Definition 2.1 The Chern–Simons action is given by

$$S(A)=\frac{1}{4\pi}\int_M \mathrm{Tr}\big(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A\big).$$

Remark Let us explain the terms in the integrand. Write $A=\zeta^I E_I$ where ζ^I are one-forms and $\{E_I\}$ is a basis for $\mathfrak{g}$. On forms, $\wedge$ denotes the usual wedge product. On the Lie algebra part, the first term $\mathrm{Tr}(E_I E_J)$ is an invariant bilinear form B_{IJ} normalized so that $B(h_\alpha,h_\alpha)=2$, where h_α is a coroot corresponding to a long root α. The second term looks like

$$\begin{aligned}\mathrm{Tr}(A\wedge A\wedge A)&=\mathrm{Tr}(E_I E_J E_K)\zeta^I\wedge\zeta^J\wedge\zeta^K\\&=\tfrac{1}{2}\,\mathrm{Tr}([E_I,E_J]E_K)\zeta^I\wedge\zeta^J\wedge\zeta^K\\&=\tfrac{1}{2}c^L_{IJ}B_{LK}\zeta^I\wedge\zeta^J\wedge\zeta^K\end{aligned}$$

where c_{IJ}^K are the structure constants of $\mathfrak{g}$. We usually write $c_{IJ}^L B_{LK}$ simply as c_{IJK}. For a matrix group such as $\mathrm{SU}(n)$, these operations can be performed simply as they appear: by multiplying and taking traces of matrices.

Observe that c_{IJK} gives a three-form ω on $\mathfrak{g}$ defined by

$$\omega(X, Y, Z) := B([X, Y], Z).$$

where X, Y, $Z \in \mathfrak{g}$ and B is the invariant bilinear form as above. This can be made into a left-invariant three-form on G, and $\int_\gamma \omega = 8\pi^2$ where γ is a generator of $\mathrm{H}_3(G, \mathbb{Z}) = \mathbb{Z}$.

The action S gives a functional on the space of connections $\mathcal{A}$. Let us calculate its derivative.

Lemma 2.2 *The curvature F of A can be regarded as a one-form on $\mathcal{A}$, and the derivative of S equals $\frac{1}{2\pi}F$.*

Proof Let δA describe a variation of the connection A. We can regard δA as an element of $T_A\mathcal{A}$, and then the curvature F of A defines a one-form by

$$\delta A \mapsto \int_M \mathrm{Tr}(\delta A \wedge F).$$

Moreover,

$$\begin{aligned} S(A + \delta A) - S(A) &= \frac{1}{4\pi} \int_M \mathrm{Tr}(\delta A \wedge dA + A \wedge d\delta A + 2\delta A \wedge A \wedge A) \\ &= \frac{1}{2\pi} \int_M \mathrm{Tr}(\delta A \wedge (dA + A \wedge A)) \\ &= \frac{1}{2\pi} \int_M \mathrm{Tr}(\delta A \wedge F) \end{aligned}$$

where we have used integration by parts. □

Remark The gauge group $\mathcal{G}$ acts on $\mathcal{A}$, and F is an invariant one-form. On the other hand, a calculation shows that

$$S(g^{-1}Ag + g^{-1}dg) - S(A) = -2\pi \deg(g)$$

where $\deg(g)$ is the degree of g as a map from M to G, that is, g takes the fundamental class of M to $\deg(g)\gamma$, where γ is the generator of $\mathrm{H}_3(G, \mathbb{Z})$ mentioned earlier. Note that the degree gives an isomorphism $\pi_0(\mathcal{G}) \cong \mathbb{Z}$.

Since S is well-defined on $\mathcal{A}/\mathcal{G}$ up to a multiple of 2π, the integrand in the next definition is invariant under gauge transformations.

Definition 2.3 The partition function of Chern–Simons theory is given by the path integral

$$Z_k(M) = \int_{\mathcal{A}} \mathcal{D}A e^{ikS(A)}$$

where the level k is a positive integer.

Remark The measure on the infinite-dimensional space $\mathcal{A}$ of connections is not well-defined, so the path integral is really a heuristic tool. It suggests that there should be a topological invariant $Z_k(M)$ of M. Moreover, the path integral suggests various ways of studying this invariant, for instance via a topological quantum field theory or via a perturbative series.

In the perturbative approach k should be thought of as being proportional to the inverse of Planck's constant $\hbar^{-1}$. In the classical limit $\hbar \to 0$, so we should expand in k^{-1}. In the $k \to \infty$ limit the path integral localizes around stationary points: this should be understood as follows. Assume that A is not a stationary point of the action S. The integrand $\exp(ikS(A))$ lies on the unit circle, and if we vary A slightly the value of the integrand rotates around the circle; in the large–k limit the values that the integrand takes are distributed evenly around the unit circle and so cancel out in the integral.

By Lemma 2.2 the derivative of S vanishes at flat connections, so these are the stationary points. To expand around a flat connection α, we let $A = \alpha + \beta$ where $\beta \in \Omega^1(M) \otimes \mathfrak{g}$ should be regarded as being the perturbation from α. A calculation shows that

$$S(A) = S(\alpha) + \frac{1}{4\pi} \int_M \mathrm{Tr}\big(\beta \wedge d_\alpha \beta + \tfrac{2}{3} \beta \wedge \beta \wedge \beta\big)$$

where d_α is the connection on one-forms induced by the connection α. The contribution to $Z_k(M)$ from the flat connection α is therefore

$$\int_{\beta \in \Omega^1(M) \otimes \mathfrak{g}} d\beta e^{ikS(\alpha)} e^{\frac{ik}{4\pi} \int_M \mathrm{Tr}(\beta \wedge d_\alpha \beta + \frac{2}{3} \beta \wedge \beta \wedge \beta)} \tag{1}$$

where the first factor $\exp(ikS(\alpha))$ in the integrand is a constant and the second factor involves both a quadratic and a cubic term in β. Note that the linear term in β vanishes because we are expanding around a stationary point.

This process of localization reduces the path integral to an integral over the moduli space of flat connections. If the flat connections are isolated then $Z_k(M)$ is simply a sum of integrals like the one above. However, the flat connections are never isolated because of the action of the gauge group - we will return to this point later. Note also that (1) is still an infinite-dimensional integral; our first step will be to calculate the integral in the simpler finite-dimensional case.

3 A finite-dimensional model

We will consider the following finite-dimensional integral

$$Z_k = \int_{\mathbb{R}^N} d^N x e^{ik\left(\frac{1}{2}Q(x,x)+\frac{1}{6}V(x,x,x)\right)}$$

where Q and V are respectively quadratic and cubic forms on $\mathbb{R}^N$ (linear in all entries and totally symmetric), and k is a positive real constant. One could drop the symmetry assumption on V, which would require some additional labeling of the half-edges at trivalent vertices of the graphs arising later, but in this article we will only consider symmetric V, which is why we have included the factor of $1/3!$.

The integral Z_k is not absolutely convergent, but it can be defined in some formal sense via analytic continuation. To begin with, the Gaussian integral on $\mathbb{R}$ can be evaluated

$$\int_{\mathbb{R}} dx e^{-\frac{1}{2}kx^2} = \sqrt{\frac{2\pi}{k}}$$

for k real and positive. Analytic continuation in k yields

$$\int_{\mathbb{R}} dx e^{\frac{1}{2}ikx^2} = \begin{cases} \sqrt{\frac{2\pi}{k}} e^{\frac{\pi i}{4}} & \text{if } k > 0, \\ \sqrt{\frac{2\pi}{|k|}} e^{-\frac{\pi i}{4}} & \text{if } k < 0. \end{cases}$$

In dimension greater than one, we can diagonalize the quadratic form to get

$$\int_{\mathbb{R}^N} d^N x e^{-k\frac{1}{2}Q(x,x)} = \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{1}{(\det Q)^{\frac{1}{2}}}$$

for Q positive definite, and

$$\int_{\mathbb{R}^N} d^N x e^{ik\frac{1}{2}Q(x,x)} = \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{e^{\pi i(\operatorname{sign} Q)/4}}{\left|\det Q\right|^{\frac{1}{2}}}$$

for Q indefinite (but still non-degenerate). We will refer to all of these integrals, which don't involve cubic terms, as *Gaussians*.

Returning to Z_k, we first deal with the dependence on k by a change of variable $x \mapsto x' := \sqrt{k}x$ (we immediately rewrite the dummy variable x' as x in our formula)

$$\begin{aligned} Z_k &= k^{-\frac{N}{2}} \int_{\mathbb{R}^N} d^N x e^{i\left(\frac{1}{2}Q(x,x)+\frac{1}{6}k^{-\frac{1}{2}}V(x,x,x)\right)} \\ &= k^{-\frac{N}{2}} \int_{\mathbb{R}^N} d^N x e^{i\frac{1}{2}Q(x,x)} \sum_{m=0}^{\infty} \frac{i^m}{m!6^m k^{\frac{m}{2}}} V(x,x,x)^m. \end{aligned}$$

We have used the power series for the exponential function involving the cubic term, but notice that the odd terms in the series give an integrand which is an odd function on $\mathbb{R}^N$, so those terms do not contribute to the integral. On the other hand, the $2m$th term gives the coefficient of $k^{-\frac{N}{2}-m}$ as

$$\int_{\mathbb{R}^N} d^N x e^{i\frac{1}{2}Q(x,x)} \frac{(-1)^m}{(2m)!6^{2m}} V(x,x,x)^{2m}.$$

To calculate this we need to understand how to integrate the Gaussian multiplied by a polynomial.

Lemma 3.1 *Let $J(x) = J_j x^j$ be a linear functional on $\mathbb{R}^N$ (we are using the Einstein summation convention) and define*

$$Z(J) := \int_{\mathbb{R}^N} d^N x e^{i\left(\frac{1}{2}Q(x,x)+J(x)\right)}$$

so that $Z(0)$ is just the Gaussian integral. Then

$$Z(J) = Z(0) e^{-\frac{i}{2}Q^{-1}(J,J)}$$

where $Q^{-1}(J,J) = Q^{jk}J_j J_k$ and Q^{jk} is the matrix of the inverse of Q.

Proof This is a simple exercise in completing the square. □

The linear term J is known as a *source*, and the main trick is to differentiate $Z(J)$ with respect to J and then evaluate at $J=0$. We will illustrate this through some examples.

Example First observe that, using the definition of $Z(J)$,

$$\int_{\mathbb{R}^N} d^N x e^{i\frac{1}{2}Q(x,x)} x^j = \left(-i\frac{\partial}{\partial J_j}\right) Z(J)\big|_{J=0}.$$

Now using the expression for $Z(J)$ in Lemma 3.1, the right-hand side becomes

$$-iZ(0)\left(-iQ^{jk}J_k e^{-\frac{i}{2}Q^{-1}(J,J)}\right)\big|_{J=0}.$$

This vanishes when we evaluate at $J = 0$, as it should because it is the integral over $\mathbb{R}^N$ of an odd function.

Next observe that to integrate the product of the Gaussian and a degree two polynomial, we should differentiate $Z(J)$ twice

$$\int_{\mathbb{R}^N} d^N x e^{i\frac{1}{2}Q(x,x)} x^j x^k = \left(-i\frac{\partial}{\partial J_j}\right)\left(-i\frac{\partial}{\partial J_k}\right) Z(J)\big|_{J=0}.$$

Once again, we use Lemma 3.1 to evaluate this as

$$(-i)^2 Z(0)\left(-iQ^{jk}e^{-\frac{i}{2}Q^{-1}(J,J)} + J_k(\cdots)\right)\big|_{J=0} = iZ(0)Q^{jk}.$$

The precise expression $(\cdots)$ is irrelevant as it vanishes when we evaluate at $J = 0$.

In general, to integrate the product of the Gaussian and a degree d polynomial, we should differentiate $Z(J)$ d times.

Now let us apply this to the $m = 1$ term occurring in Z_k

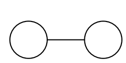
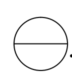
$$\int_{\mathbb{R}^N} d^N x e^{i\frac{1}{2}Q(x,x)} \left(\frac{-1}{2!6^2}\right) V(x,x,x)^2.$$

Writing $V(x,x,x)$ in coordinates as $V_{jkl}x^j x^k x^l$ or $V_{abc}x^a x^b x^c$, we see that the integrand is the product of a Gaussian and a degree six polynomial, and hence the integral equals

$$\begin{aligned}\left(\frac{-1}{2!6^2}\right) V_{jkl}V_{abc}\left(-i\frac{\partial}{\partial J_j}\right)\left(-i\frac{\partial}{\partial J_k}\right)\cdots\left(-i\frac{\partial}{\partial J_c}\right) Z(J)\Big|_{J=0} \\ = \left(\frac{-1}{2!6^2}\right) V_{jkl}V_{abc}i^3 Z(0)\left(Q^{jk}Q^{la}Q^{bc} + Q^{jk}Q^{lb}Q^{ac} + \cdots\right)\end{aligned}$$

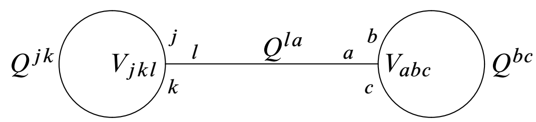
The last factor on the right consists of the sum over all ways of dividing the set of indices $\{j,k,l,a,b,c\}$ into pairs, and hence contains $\frac{1}{3!}\binom{6}{2,2,2} = 15$ terms. Any other terms which appear when we differentiate $Z(J)$ will vanish when we evaluate at $J = 0$.

Since we are using the summation convention, j, k, etc, are dummy indices. Up to a factor of $\left(\frac{i}{2!6^2}\right)Z(0)$, we get nine terms of the form $V_{jkl}V_{abc}Q^{jk}Q^{la}Q^{bc}$ and six of the form $V_{jkl}V_{abc}Q^{ja}Q^{kb}Q^{lc}$. These terms can be encoded using the trivalent graphs

$\Gamma_1 =$  and $\Gamma_2 =$ .

More precisely, to each trivalent graph Γ we associate a weight $W(\Gamma)$ given by labeling vertices with V and edges with Q^{-1}, and then contracting indices. For instance on

Γ_1 we first label the legs (ends of the edges) with indices. Then we label the vertices with V_{jkl} and V_{abc}, and the edges with Q^{jk}, Q^{la}, and Q^{bc}, in such a way that the indices agree with those on the legs.



Then we apply the summation convention and arrive at the weight

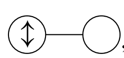
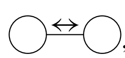
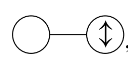
$$W(\Gamma_1) := V_{jkl}V_{abc}Q^{jk}Q^{la}Q^{bc}. \tag{2}$$

In this construction V and Q^{-1} are known as the *vertex* and *propagator*, respectively.

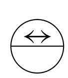
Summing up, the $m=1$ term becomes

$$\begin{aligned}\left(\frac{i}{2!6^2}\right)Z(0)\big(9W(\Gamma_1)+6W(\Gamma_2)\big) &= i\,Z(0)\big(\tfrac{1}{8}W(\Gamma_1)+\tfrac{1}{12}W(\Gamma_2)\big)\\ &= i\,Z(0)\left(\frac{1}{|\mathrm{Aut}\,\Gamma_1|}W(\Gamma_1)+\frac{1}{|\mathrm{Aut}\,\Gamma_2|}W(\Gamma_2)\right)\end{aligned}$$

where $\mathrm{Aut}\,\Gamma_1$ and $\mathrm{Aut}\,\Gamma_2$ are the automorphism groups of the graphs. We think of a trivalent graph as a collection of half-edges partitioned into two element subsets (edges) and three element subsets (vertices). Then in this context, an automorphism of the graph means a permutation of the half-edges which preserves both the partition into edges and the partition into vertices (this differs from the usual definition). For example, $\mathrm{Aut}\,\Gamma_1$ is generated by the involutions

, , and ,

whereas $\mathrm{Aut}\,\Gamma_2$ is the semi-direct product of the group generated by the involution



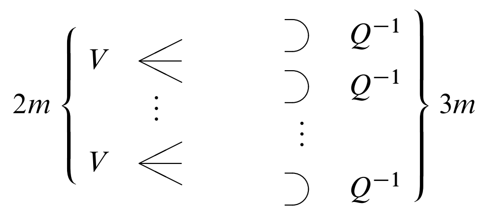
and the symmetric group S_3 permuting the three edges.

Using the same arguments, we deduce that the $2m^{\text{th}}$ term looks like

$$\frac{(-1)^m}{(2m)!6^{2m}}V_{jkl}\dots V_{abc}\,i^{3m}Z(0)\big(Q^{jk}\dots Q^{bc}+\cdots\big)$$

with $2m$ copies of V, and the last factor being the sum over all ways of dividing a set of $3\times 2m$ indices into pairs, thus a sum of $\frac{1}{(3m)!}\binom{6m}{2,\dots,2}$ terms. Each trivalent graph Γ

with $2m$ vertices gives us a weight $W(\Gamma)$ as before, which we claim occurs precisely $\frac{(2m)!6^{2m}}{|\operatorname{Aut}\Gamma|}$ times in the sum above. To understand why, it helps to separate the trivalent vertices from the edges of the graph.



Then Γ is given by gluing together the legs in some specific way. Note that if the legs are labeled with indices, then gluing corresponds to contracting indices. Now observe that the semi-direct product of groups $S_{2m} \ltimes (S_3)^{2m}$ acts on the trivalent vertices. Under this action, we still get Γ but the dummy indices change, unless the element of the group induces an automorphism of Γ. Thus the orbit has size $\frac{(2m)!6^{2m}}{|\operatorname{Aut}\Gamma|}$, corresponding to the number of times $W(\Gamma)$ occurs in the sum above.

Thus the $2m^{\text{th}}$ term is

$$i^m Z(0) \sum_{\substack{(m+1)-\text{loop}\\ \text{graphs}}} \frac{1}{|\operatorname{Aut}\Gamma|} W(\Gamma)$$

(trivalent graphs with $2m$ vertices are usually referred to as $(m+1)$–loop graphs), and

$$Z_k = \sum_{m=0}^{\infty} i^m Z(0) k^{-\frac{N}{2}-m} \sum_{\substack{(m+1)-\text{loop}\\ \text{graphs}}} \frac{1}{|\operatorname{Aut}\Gamma|} W(\Gamma).$$

This sum over graphs includes disconnected graphs. However, the multiplicativity of $W(\Gamma)$ and the simple behaviour of $\operatorname{Aut}\Gamma$ under disjoint union readily leads to the following formula:

$$\log\left(\frac{Z_k}{Z(0)}\right) = -\frac{N}{2}\log k + \sum_{m=0}^{\infty} i^m k^{-m} \sum_{\substack{\text{connected}\\ (m+1)-\text{loop}\\ \text{graphs}}} \frac{1}{|\operatorname{Aut}\Gamma|} W(\Gamma)$$

4 Degenerate quadratic forms

Before returning to the infinite-dimensional case, we need to consider how to modify our formulae in the case that the quadratic form is degenerate. Recall that the Gaussian integral for a non-degenerate form Q is given by

$$\int_{\mathbb{R}^N} d^N x e^{ik\frac{1}{2}Q(x,x)} = \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{e^{\pi i(\operatorname{sign} Q)/4}}{|\det Q|^{\frac{1}{2}}}.$$

Now suppose that a compact group K acts on $\mathbb{R}^N$ preserving Q. We can write

$$\int_{\mathbb{R}^N} d^N x e^{ik\frac{1}{2}Q(x,x)} = \int_{\mathbb{R}^N/K} d\mu e^{ik\frac{1}{2}Q(x,x)} \operatorname{Vol} \mathcal{O}_x$$

where $d\mu$ is the induced measure on the space of orbits R^N/K, and $\operatorname{Vol}\mathcal{O}_x$ is the volume of the orbit through $x \in \mathbb{R}^N$. More precisely, the integral over R^N/K can be thought of as an integral over a slice for the K–action, that is, a submanifold of R^N intersecting the generic orbit in exactly one point. The action induces a map

$$\rho_x\colon K \to \mathcal{O}_x \subset \mathbb{R}^N$$

taking $g \in K$ to $g(x) \in \mathcal{O}_x$. Denote the derivative of this map by

$$B := d\rho_x\colon \mathfrak{k} \to T_x\mathcal{O}_x \subset T_x\mathbb{R}^N = \mathbb{R}^N$$

where $\mathfrak{k}$ is the Lie algebra of K. Then

$$\begin{aligned}\operatorname{Vol}\mathcal{O}_x &= \int_{\mathcal{O}_x} d\eta' \\ &= \int_K d\eta |\det B| \\ &= \operatorname{Vol} K |\det B^* B|^{\frac{1}{2}}\end{aligned}$$

where $d\eta$ is the Haar measure on K, $d\eta'$ is the induced measure on $\mathcal{O}_x$, and B^* is the adjoint of B. Since ρ_x is K–equivariant, $\det B$ is constant on K, justifying the last equality. We usually refer to $|\det B^* B|^{\frac{1}{2}}$ as the Jacobian determinant. Summarizing, we have

$$\int_{\mathbb{R}^N} d^N x e^{ik\frac{1}{2}Q(x,x)} = \operatorname{Vol} K \int_{\mathbb{R}^N/K} d\mu e^{ik\frac{1}{2}Q(x,x)} |\det B^* B|^{\frac{1}{2}}. \tag{3}$$

Example The usual S^1–action on $\mathbb{R}^2$ preserves the standard quadratic form x^2+y^2. In this case, $B_{(x,y)}$ maps θ to $(-\theta y, \theta x)$ and the adjoint $B^*_{(x,y)}$ maps (X,Y) to $-Xy+Yx$, so

$$|\det B^* B|^{\frac{1}{2}} = (x^2+y^2)^{1/2} = r$$

and thus

$$\int_{\mathbb{R}^2} dx dy e^{-(x^2+y^2)} = 2\pi \int_{r=0}^{\infty} dr e^{-r^2} r = \pi.$$

In the example above, Q is still non-degenerate. However, if the action of the group K is affine, then this forces the form Q to be degenerate in the direction of the orbits. The author has been unable to find such an example with K compact, though in infinite-dimensions the action of the gauge group on the space of connections is affine. Now when K is non-compact, a K–invariant function will be non-integrable. However, the integral on the right-hand side of (3) can still be finite, and we will use this as a *definition* of the left-hand side (we will also ignore the infinite $\operatorname{Vol} K$ factor in this case).

We assume that Q induces a non-degenerate form on the space of orbits. Thus we can reduce the original ill-defined integral involving a degenerate quadratic form to one over the space of orbits, with the inclusion of a factor proportional to the Jacobian determinant. This additional factor can be dealt with via the Faddeev–Popov method, which we will describe shortly, but first we will compute the one-loop term directly in the infinite-dimensional case.

5 The one-loop term

In this section we will calculate the one-loop term in perturbative Chern–Simons theory following Atiyah [2]. Recall that the contribution to the Chern–Simons path integral coming from connections $A = \alpha + \beta$ in a neighbourhood of the flat connection α is

$$\int_{\beta \in \Omega^1(M) \otimes \mathfrak{g}} d\beta e^{ikS(\alpha)} e^{\frac{ik}{4\pi} \int_M \operatorname{Tr}(\beta \wedge d_\alpha \beta + \frac{2}{3} \beta \wedge \beta \wedge \beta)}.$$

In this section we will ignore the cubic term in β and just focus on the infinite-dimensional Gaussian integral coming from the quadratic term, which is known as the one-loop term. In other words, we will evaluate

$$\int_{\beta \in \Omega^1(M) \otimes \mathfrak{g}} d\beta e^{\frac{ik}{4\pi} \int_M \operatorname{Tr}(\beta \wedge d_\alpha \beta)}.$$

First observe that after choosing a Riemannian metric on M, we can define a pairing on $\Omega^1(M) \otimes \mathfrak{g}$ by

$$\langle \beta_1, \beta_2 \rangle := -\frac{1}{2\pi} \int_M \operatorname{Tr}(\beta_1 \wedge *\beta_2)$$

where $*$ is the Hodge star operator acting on the one-form part of β_2 (which of course depends on the choice of metric). Recall that for X and $Y \in \mathfrak{su}(n)$, $-\operatorname{Tr}(XY)$ is a positive-definite invariant bilinear form, which explains why we have included a minus sign in the above pairing. The quadratic form Q is then given by the self-adjoint operator $-*d_\alpha$ on $\Omega^1(M)\otimes\mathfrak{g}$, as

$$\tfrac{1}{2}ik\langle\beta, Q(\beta)\rangle = \frac{ik}{4\pi}\int_M \mathrm{Tr}(\beta\wedge d_\alpha\beta).$$

This is completely analogous to the finite-dimensional case, where the quadratic form is given by a matrix combined with the standard pairing on $\mathbb{R}^N$.

Now the operator $Q=-*d_\alpha$ is degenerate on $\Omega^1(M)\otimes\mathfrak{g}$, and the analogue of $B\colon \mathfrak{k}\to T_x\mathcal{O}_x\subset\mathbb{R}^N$ is the map coming from infinitesimal gauge transformations

$$d_\alpha\colon \Omega^0(M)\otimes\mathfrak{g}\to\Omega^1(M)\otimes\mathfrak{g}.$$

In other words, Q is degenerate on the image $d_\alpha\Omega^0$ (we use Ω^j as an abbreviation for $\Omega^j(M)\otimes\mathfrak{g}$). If we assume that the flat connection α is irreducible and isolated, then the twisted de Rham complex

$$\Omega^0(M)\otimes\mathfrak{g}\xrightarrow{d_\alpha}\Omega^1(M)\otimes\mathfrak{g}\xrightarrow{d_\alpha}\Omega^2(M)\otimes\mathfrak{g}\xrightarrow{d_\alpha}\Omega^3(M)\otimes\mathfrak{g}$$

is exact. Specifically, a flat connection gives a representation of $\pi_1(M)$ which is irreducible if and only if $\mathrm{H}^0(M,\mathfrak{g})$ vanishes, and $\mathrm{H}^1(M,\mathfrak{g})$ is the space of first-order deformations of the connection, which vanishes if the connection is isolated. The remaining cohomology groups H^2 and H^3 are dual to H^1 and H^0 respectively.

According to the previous section, our Gaussian integral is defined as

$$\int_{\beta\in\Omega^1/d_\alpha\Omega^0} d\beta e^{ik\frac{1}{2}\langle\beta,Q'(\beta)\rangle}|\det B^*B|^{\frac{1}{2}}.$$

The quadratic form Q' is given by the reduction of Q on the quotient $\Omega^1/d_\alpha\Omega^0$; note that it is non-degenerate by exactness of the de Rham complex.

Define an operator P on

$$\Omega^0\oplus\Omega^1=\Omega^0\oplus d_\alpha\Omega^0\oplus(\Omega^1/d_\alpha\Omega^0)$$

by $(d_\alpha, d_\alpha^*, -*d_\alpha)$ where $d_\alpha^*:=-*d_\alpha*$ is the adjoint of d_α on one-forms. Writing this as a matrix (with respect to the three factors)

$$P=\begin{pmatrix}0 & B^* & 0\\ B & 0 & 0\\ 0 & 0 & Q'\end{pmatrix}$$

we see that

$$|\det P| = |\det B^*B||\det Q'|.$$

Moreover,

$$B^*B = \Delta^0_\alpha \qquad \text{(the Laplacian on } \Omega^0(M)\otimes\mathfrak{g})$$
$$P^2 = \Delta^0_\alpha \oplus \Delta^1_\alpha \qquad \text{(the Laplacian on } \Omega^0(M)\otimes\mathfrak{g}\oplus\Omega^1(M)\otimes\mathfrak{g}).$$

Note that although Q is degenerate on Ω^1, we can extend it to a non-degenerate quadratic form P on the larger space $\Omega^0 \oplus \Omega^1$. (This partly explains the addition of the ghost field $c \in \Omega^0$ in the Faddeev–Popov method – see the next section.)

The Jacobian determinant $|\det B^*B|^{\frac{1}{2}} = (\det \Delta^0_\alpha)^{\frac{1}{2}}$ is independent of β, while when we integrate the Gaussian we pick up a factor of $|\det Q'|^{-\frac{1}{2}}$. This combination can be rewritten

$$\frac{|\det B^*B|^{\frac{1}{2}}}{|\det Q'|^{\frac{1}{2}}} = \frac{|\det B^*B|}{|\det P|^{\frac{1}{2}}} = \frac{\left(\det \Delta^0_\alpha\right)^{\frac{3}{4}}}{\left(\det \Delta^1_\alpha\right)^{\frac{1}{4}}}.$$

For the determinants of the Laplacians, we need to calculate the product of their eigenvalues, which can be done using zeta-function regularization. This gives the one-loop term, up to a phase factor and an overall constant.

Bearing in mind that the Chern–Simons partition function is supposed to represent a topological invariant of M, one might wonder about the appearance of a Riemannian metric in our calculations above. While it is true that the Laplacians (and their determinants) depend on the choice of metric, Ray and Singer showed that the ratio

$$\frac{\left(\det \Delta^0_\alpha\right)^{\frac{3}{2}}}{\left(\det \Delta^1_\alpha\right)^{\frac{1}{2}}}$$

is independent of this choice [20]. They showed this by explicitly calculating the variation under a change of metric, and this quantity became known as the Ray–Singer analytic torsion. It was later shown that it equals the Reidemeister torsion, which is defined combinatorially. In particular, our factor above is the square root of the Ray–Singer torsion, and is therefore a topological invariant.

6 The Faddeev–Popov method

There is another way of dealing with the Jacobian determinant, which is necessary to extend our results to the higher order terms in the perturbative expansion. This is the method used by Axelrod and Singer [3] and Bar-Natan [5; 7], and involves introducing additional fields and integrating over them in the path integral, as we now explain.

To begin with, we want to integrate over the space of orbits of the gauge group acting on $\Omega^1(M)\otimes\mathfrak{g}$. One way to achieve this is to find a function F whose level sets meet each orbit transversely in a single point, and then integrate over $F^{-1}(0)$. Of course, we must also include the Jacobian determinant, which represents the volume of the orbit. Equivalently, we could include a δ–function term $\delta(F(\beta))$ in the path integral. Up to a factor, this can be written as

$$\delta(F(\beta)) \propto \int d\phi e^{i\langle F(\beta),\phi\rangle}$$

by using the Fourier representation of the δ–function. The term $F(\beta)\phi$ can then be added to the Lagrangian, and ϕ becomes an additional field to integrate over. In the Chern–Simons case, F comes from the gauge-fixing condition $d^*_\alpha\beta=0$, and ϕ should be a $\mathfrak{g}$–valued function.

The Jacobian determinant can also be written in terms of additional fields. For this we must introduce ghost fields c and $\bar{c}$, which lie in $\Omega^0(M)\otimes\mathfrak{g}$ but are Fermionic, that is, anti-commutative (whereas β and ϕ are Bosonic). Then, up to a factor,

$$|\det B^*B|^{\frac{1}{2}} = \left(\det\Delta^0_\alpha\right)^{\frac{1}{2}} \propto \int dc\, d\bar{c}\, e^{ik\frac{1}{2}\langle\bar{c},\Delta^0_\alpha c\rangle},$$

which is the analogue of the Gaussian in the theory of anti-commutative integration.

So our new path integral looks like

$$\int d\beta\, d\phi\, dc\, d\bar{c}\, e^{ikS(\beta,\phi,c,\bar{c})}$$

where after the addition of gauge-fixing and ghost terms the action is given by

$$S(\beta,\phi,c,\bar{c}) = \frac{1}{4\pi}\int_M \mathrm{Tr}\big(\beta\wedge d_\alpha\beta + \tfrac{2}{3}\beta\wedge\beta\wedge\beta + d^*_\alpha\beta\wedge *\phi + \bar{c}\wedge *\Delta^0_\alpha c\big).$$

One might be concerned about the appearance of metric dependent terms in the new Lagrangian, such as the $*$–operator and Laplacian. To show that the path integral is independent of the choice of metric we introduce the BRST operator. In this subsection only, Q will denote the BRST operator, which should not be confused with the quadratic form Q defined earlier. It is an odd (interchanges Bosonic and Fermionic fields) derivation on the space of functionals in β, ϕ, c, and $\bar{c}$ defined by

$$\begin{aligned} Q\beta &= -(d_\alpha + \beta)c \\ Q\phi &= 0 \\ Q\bar{c} &= \phi \\ Qc &= \tfrac{1}{2}[c, c]. \end{aligned}$$

When we deform the metric, only the gauge-fixing and ghost terms vary. However, one can show that this variation is Q–exact. Moreover, Q is like a vector field with zero divergence, so the integral of a Q–exact term must vanish. See Axelrod and Singer [3] or Bar-Natan [7] for the details.

The introduction of the BRST operator can perhaps be understood as an algebraic method of forming a quotient, though as far as the author knows, this idea has not yet been mathematically formalized[1]. In our case, we are interested in the quotient of the space of connections by gauge transformations. Note that Q extends the infinitesimal gauge action on β to the other fields ϕ, c, and $\bar{c}$. It is straightforward to check that $Q^2 = 0$; essentially what we are doing above is taking Q–cohomology. Only the metric invariant part

$$\frac{1}{4\pi}\int_M \mathrm{Tr}\big(\beta \wedge d_\alpha \beta + \tfrac{2}{3}\beta \wedge \beta \wedge \beta\big)$$

survives to give an element of $\mathrm{H}^0(Q)$, which is our algebraic representation of the space of functionals on the quotient. The BRST approach to gauge theory is described in Witten's third lecture in [26]; Chapter 7 of Bar-Natan's thesis [5] is another useful reference.

It should also be noted that a more natural way to view the above path integral is in its superspace formulation. However, since we wish to write the terms of the perturbative expansion as configuration space integrals we must stick to the language of differential forms. Both of these approaches are described by Axelrod and Singer [3].

7 Higher order terms

We will now give a rough outline of how one constructs the higher order terms in the perturbative expansion of the Chern–Simons partition function. These terms will be indexed by trivalent graphs, and formally the series will look the same as the finite-dimensional case, once we have properly interpreted the quadratic and cubic forms on

[1] Note added in proof: for a homological algebraic interpretation due to Jim Stasheff see [21; 22; 23].

$\Omega^1(M)\otimes\mathfrak{g}$. Actually, the series has only been fully computed in the case that α is the trivial connection, in which case the twisted de Rham complex decouples

$$\Omega^0(M)\otimes\mathfrak{g}\xrightarrow{d\otimes\mathrm{Id}}\Omega^1(M)\otimes\mathfrak{g}\xrightarrow{d\otimes\mathrm{Id}}\Omega^2(M)\otimes\mathfrak{g}\xrightarrow{d\otimes\mathrm{Id}}\Omega^3(M)\otimes\mathfrak{g}$$

where d is the usual exterior derivative on forms on M, and Id is the identity on $\mathfrak{g}$. We will assume α is trivial for the remainder of this article; of course if M is Euclidean space then the only flat connection is the trivial connection.

We saw that the quadratic form Q is given by the self-adjoint operator $-*d_\alpha$, when combined with the pairing

$$\langle\beta_1,\beta_2\rangle := -\frac{1}{2\pi}\int_M \mathrm{Tr}(\beta_1\wedge *\beta_2)$$

on $\Omega^1(M)\otimes\mathfrak{g}$. When α is the trivial connection this decouples into $-*d$ (combined with the induced inner product) on $\Omega^1(M)$ and the invariant quadratic form B on $\mathfrak{g}$. Recall that to calculate weights on trivalent graphs we needed the inverse Q^{-1}. As far as $\mathfrak{g}$ is concerned, this is just the inverse B^{-1} of the invariant quadratic form.

On $\Omega^1(M)$ we need to calculate the inverse of the elliptic operator $-*d$. Strictly speaking, this inverse does not exist as $-*d$ has non-trivial kernel. However, the kernel is the image of the infinitesimal gauge action, and is taken care of by the Faddeev–Popov method described in the last section. After adding gauge-fixing terms, we find that the new Lagrangian has a quadratic term which looks like

$$ik\tfrac{1}{2}\langle\beta+\phi, L(\beta+\phi)\rangle$$

where $L := -(*d + d*)$. Moreover, L^2 is the Laplacian (on functions and one-forms), whose inverse is given in terms of a Green's function. For example, in Euclidean space $\mathbb{R}^3$ the solution to $\Delta u = w$ is given by

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} dy\,\frac{w(y)}{|x-y|}.$$

The Green's function is

$$G(x,y) = \frac{1}{4\pi|x-y|}$$

and the inverse of L itself is given by

$$L\circ G = \frac{\epsilon^{ijk}(x_i-y_i)dx_j\wedge dx_k}{4\pi|x-y|^3}$$

(using Einstein summation), meaning that this is the kernel of an integral operator representing L^{-1}.

The cubic term V is much simpler. Recall that if we write $\beta = \zeta^I E_I$ where ζ^I are one-forms and $\{E_I\}$ is a basis for $\mathfrak{g}$, then

$$\mathrm{Tr}(\beta \wedge \beta \wedge \beta) = \tfrac{1}{2} c_{IJK} \zeta^I \wedge \zeta^J \wedge \zeta^K$$

where c_{IJK} comes from the structure constants of $\mathfrak{g}$. So the cubic term decouples into simply wedging the three one-forms together (and then integrating over M) on $\Omega^1(M)$ and c_{IJK} on $\mathfrak{g}$.

The Faddeev–Popov method also introduces ghost fields, so there is an additional propagator and interaction coming from ghosts. These are usually drawn with dashed lines in our Feynman diagrams, whereas the propagator and interaction described above are drawn with solid lines. See Bar-Natan [5; 7] for more details.

Let Γ be a trivalent graph. Since both the quadratic and cubic terms decouple, the weight $W(\Gamma)$ will become a product

$$W(\Gamma) = I_\Gamma(M) \times b_{\mathfrak{g}}(\Gamma) \tag{4}$$

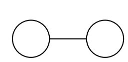
where $I_\Gamma(M)$ does not depend on the Lie algebra $\mathfrak{g}$ and $b_{\mathfrak{g}}(\Gamma)$ does not depend on the three-manifold M. The fact that α is the trivial connection is crucial for this decoupling, as otherwise the quadratic form would not decouple.

Remark For each finite-dimensional Lie algebra $\mathfrak{g}$, $b_{\mathfrak{g}}$ is a weight system on trivalent graphs. It is the same as the weight system arising in the theory of Vassiliev invariants [6]; for example

$$b_{\mathfrak{g}}(\Gamma_2) := c_{IJK} c_{ABC} B^{IA} B^{JB} B^{KC}.$$

Notice that the cubic terms on $\Omega^1(M)$ and $\mathfrak{g}$ are both skew-symmetric. Consequently we really need to introduce the notion of an orientation on Γ (an equivalence class of cyclic orderings of the legs at each trivalent vertex), so that $I_\Gamma(M)$ and $b_{\mathfrak{g}}(\Gamma)$ make sense. The skew-symmetry cancels out in $W(\Gamma)$: if we reverse the orientation on Γ, both $I_\Gamma(M)$ and $b_{\mathfrak{g}}(\Gamma)$ will change sign but $W(\Gamma)$ will remain the same.

One consequence of this skew-symmetry of vertices is that many weights automatically vanish. For example, write Γ_1 and $\widetilde{\Gamma}_1$ for the graph



with its two orientations. Then Γ_1 and $\widetilde{\Gamma}_1$ are actually isomorphic as oriented graphs, so

$$b_{\mathfrak{g}}(\Gamma_1) = b_{\mathfrak{g}}(\widetilde{\Gamma}_1) = -b_{\mathfrak{g}}(\Gamma_1)$$

must vanish.

8 Configuration space integrals

We will describe in more detail the terms $I_\Gamma(M)$ arising from (4). In our finite-dimensional model we summed over indices when evaluating $W(\Gamma)$, as in (2), and this still works fine for the Lie algebra factor $b_{\mathfrak{g}}(\Gamma)$ since $\mathfrak{g}$ is finite-dimensional. However, on the infinite-dimensional part summation is replaced by integration: heuristically, one can regard the δ–functions $\{\delta(x-y)\}_{y\in M}$ as a basis for functions on M, so that summation over elements in the basis becomes $\int_{y\in M}$ (and something similar happens for one-forms). The term $I_{\Gamma_2}(M)$, for example, therefore looks like

$$\int_M dx \int_M dy K(x,y)\wedge K(x,y)\wedge K(x,y).$$

In this formula, $K(x,y)$ is the Green's function for L^{-1} on M. We can only write this out explicitly when M is Euclidean space, since we have an explicit formula for the Green's function on $\mathbb{R}^3$.

For a graph Γ with m vertices, we get an integral over M^m. Now the Green's function $K(x,y)$ will have a singularity at $x=y$, so the integrand only makes sense on the configuration space

$$C_m^0(M) := \{(x^{(1)},\ldots,x^{(m)})\in M^m \mid x^{(i)}\neq x^{(j)} \text{ for } i\neq j\}.$$

However, by analyzing the severity of the singularity Axelrod and Singer [3] showed that the integrals on M^m are actually finite. One approach here is to extend the integrand to the compactification of $C_m^0(M)$ due to Fulton and MacPherson [9], which is a manifold with boundary and corners, and then show that this extension is finite.

It is important to note that for a fixed trivalent graph Γ, I_Γ is *not* a topological invariant of three-manifolds: it varies as we deform the metric on M. The variation can be understood as an integral over the boundary of the Fulton–MacPherson compactification of $C_m^0(M)$. However, when we combine the terms I_Γ in the sum

$$\sum_{\substack{(m+1)-\text{loop}\\ \text{graphs}}} \frac{1}{|\operatorname{Aut}\Gamma|} I_\Gamma(M)\times b_{\mathfrak{g}}(\Gamma) \tag{5}$$

the variations should cancel out. In fact there are two kinds of variation, and variations of the first kind cancel due to a special property of the weights $b_{\mathfrak{g}}(\Gamma)$, namely that they satisfy the IHX relations. Variations of the second kind are known as "anomalous", and are more difficult to deal with. If we choose a framing of M, then we can correct the anomaly with a counter-term and thereby construct a genuine invariant of M (see Sections 5 and 6 of Axelrod and Singer [3]).

9 Invariants of knots and links

Finally let us say a few words about the generalization to knot and link invariants. Given an oriented knot $\mathcal{K}$ in M, we can include an additional term in the Chern–Simons path integral given by the monodromy of the connection A around $\mathcal{K}$. The monodromy would take values in $\mathfrak{g}$, so we evaluate its trace in a representation V of G. This gives the Wilson loop

$$\mathcal{W}_V(\mathcal{K}) := \mathrm{Tr}_V\,\mathrm{Pexp}\int_{\mathcal{K}\subset M} A.$$

In this expression, the path ordered exponential Pexp can be represented by a sum of iterated integrals

$$\sum_{k=0}^{\infty} \int\cdots\int_{0\le t_1\le t_2\le\ldots\le t_k\le 1} A(\mathcal{K}(t_1))\otimes\cdots\otimes A(\mathcal{K}(t_k)) \tag{6}$$

where we have used $\mathcal{K}\colon S^1\to M$ to denote a parametrization of the knot. We then represent the $\mathfrak{g}$–valued $A(\mathcal{K}(t_i))$ in $\mathrm{End}(V)$, multiply them, and take the trace. The expectation value of the Wilson loop

$$Z_k(M;\mathcal{K}) = \int_{\mathcal{A}} \mathcal{D}A e^{ikS(A)}\mathcal{W}_V(\mathcal{K})$$

should be an invariant of the knot. For links one includes several Wilson loops, one for each component of the link.

The expectation value $Z_k(M;\mathcal{K})$ admits a perturbative expansion just like $Z_k(M)$, and once again each term splits into the product of a weight system (coming from the Lie algebra $\mathfrak{g}$ and representation V) and a configuration space integral. The Feynman graphs consist of unitrivalent graphs, whose univalent vertices lie on a circle S^1 (or collection of l circles in the case of an l–component link). The interactions at the univalent vertices come from the terms $A(\mathcal{K}(t_i))$ in (6). The configuration space looks like

$$C^0_{a,b}(M;\mathcal{K}) := \left\{(x^{(1)},\ldots,x^{(a)},x^{(a+1)},\ldots,x^{(a+b)})\in M^a\times\mathcal{K}^b \mid x^{(i)}\neq x^{(j)} \text{ for } i\neq j\right\}$$

and the integral can be replaced by an integral over a suitable compactification of $C^0_{a,b}(M;\mathcal{K})$ (see Thurston [24]).

A great deal of work has gone into understanding these configuration space integrals when M is Euclidean space, since we then have an explicit formula for the Green's function of L^{-1} (see Bar-Natan [5; 7], Bott and Taubes [8], Altschuler and Freidel [1], and Thurston [24]). The simplest example comes from the graph given by a single chord connecting two circles, in the case of a two component link $\mathcal{L}$ (it looks like Γ_1, but with the circles corresponding to the two components of the link). Then

$$C^0_{0,\{1,1\}}(M;\mathcal{L}) \cong \mathcal{L}_1 \times \mathcal{L}_2 \cong S^1 \times S^1$$

is already compact (we write $\{1,1\}$ to indicate that there is a point on each component of the link). The integral is given by

$$\int_{\mathcal{L}_1} dx_j \int_{\mathcal{L}_2} dy_k \frac{\epsilon^{ijk}(x_i - y_i)}{4\pi|x-y|^3}$$

and it computes the degree of the map from $\mathcal{L}_1 \times \mathcal{L}_2 \cong S^1 \times S^1$ to S^2 which takes (x,y) to $\frac{x-y}{|x-y|}$. It is known as the Gauss linking number of $\mathcal{L}_1$ and $\mathcal{L}_2$.

The Gauss linking number is a genuine invariant, but for a general configuration space integral there are two kinds of variation when we isotope the knot or link. It is easy to show that, when combined in a sum like (5), the variations of the first kind cancel out. However, the second kind of variation is more difficult to deal with, and once again leads to an "anomaly".

For example, the first-order term $I_\Theta(\mathbb{R}^3;\mathcal{K})$ for a knot $\mathcal{K}$ is known as the writhe. It corresponds to the graph given by a circle with a single chord (it looks like Γ_2, but with the circle corresponding to the knot), and can be expressed as a Gauss integral similar to the one above. This integral varies as we isotope the knot, and can take all real values, so there is an anomaly. However, if the knot is framed, that is, a trivialization of its normal bundle is given, then we get a genuine invariant by adding the total torsion τ (see Pohl [17]). In fact, up to a factor,

$$I_\Theta(\mathbb{R}^3;\mathcal{K}) + \tau$$

is the linking number of the knot with its parallel (the knot obtained by using the framing to displace $\mathcal{K}$ slightly).

It was shown independently by Altschuler and Freidel [1] and by Thurston [24] that, up to a correction by the anomaly, the perturbative series gives an invariant of knots and links in $\mathbb{R}^3$. This extended earlier work by Bar-Natan [5; 7] and Bott and Taubes [8], who had investigated the series up to the degree two term. Moreover, the perturbative series was shown to be a universal Vassiliev invariant (c.f. Bar-Natan [6]). It is expected that it should agree with the universal Vassiliev invariant of Kontsevich [14], which would follow from the vanishing of the anomaly in degree greater than one. This is still an open conjecture, though recently Poirier [18] showed that the anomaly does indeed vanish in degrees two to six (see also Lescop [15]).

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