

## The quadratic form $E_8$ and exotic homology manifolds

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An explicit  $(-1)^n$ -quadratic form over  $\mathbb{Z}[\mathbb{Z}^{2n}]$  representing the surgery problem  $E_8 \times T^{2n}$  is obtained, for use in the Bryant–Ferry–Mio–Weinberger construction of  $2n$ -dimensional exotic homology manifolds.

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*Dedicated to John Bryant on his 60th birthday*

### 1 Introduction

Exotic ENR homology  $n$ -manifolds,  $n \geq 6$ , were discovered in the early 1990s by Bryant, Ferry, Mio and Weinberger [1; 2]. In the 1970s, the existence of such spaces had become a widely debated problem among geometric topologists in connection with the works of Cannon [4], Edwards [9] and Daverman [8] on the characterization of topological manifolds. The *Resolution Conjecture*, formulated by Cannon in [3], implied the non-existence of exotic homology manifolds – compelling evidence supporting the conjecture was offered by the solution of the *Double Suspension Problem*. Quinn introduced methods of controlled  $K$ -theory and controlled surgery into the area. He associated with an ENR homology  $n$ -manifold  $X$ ,  $n \geq 5$ , a local index  $\iota(X) \in 8\mathbb{Z} + 1$  with the property that  $\iota(X) = 1$  if and only if  $X$  is resolvable. A *resolution* of  $X$  is a proper surjection  $f: M \rightarrow X$  from a topological manifold  $M$  such that, for each  $x \in X$ ,  $f^{-1}(x)$  is contractible in any of its neighborhoods in  $M$ . This led to the celebrated Edwards–Quinn characterization of topological  $n$ -manifolds,  $n \geq 5$ , as index-1 ENR homology manifolds satisfying the disjoint disks property (DDP) (see the articles by Quinn [17; 18] and Daverman [8]. More details and historical remarks on these developments can be found in the survey articles by Cannon [3], Edwards [9], Weinberger [26] and Mio [14], and in the book by Daverman [8].

In [1; 2], ENR homology manifolds with non-trivial local indexes are constructed as inverse limits of ever finer Poincaré duality spaces, which are obtained from topological manifolds using controlled cut-paste constructions. In the simply-connected case, for example, topological manifolds are cut along the boundaries of regular neighborhoods

of very fine 2–skeleta and pasted back together using  $\epsilon$ –homotopy equivalences that “carry non-trivial local indexes” in the form of obstructions to deform them to homeomorphisms in a controlled manner. The construction of these  $\epsilon$ –equivalences requires controlled surgery theory, the calculation of controlled surgery groups with trivial local fundamental group, and “Wall realization” of controlled surgery obstructions. The stability of controlled surgery groups is a key fact, whose proof was completed more recently by Pedersen, Quinn and Ranicki [15]; an elegant proof along similar lines was given by Pedersen and Yamasaki [16] at the 2003 Workshop on Exotic Homology Manifolds in Oberwolfach, employing methods of Yamasaki [27]. An alternative proof based on the  $\alpha$ –Approximation Theorem is due to Ferry [10].

The construction of exotic homology manifolds presented by Bryant, Ferry, Mio and Weinberger [2] is somewhat indirect. Along the years, many colleagues (notably Bob Edwards) voiced the desire to see – at least in one specific example – an explicit realization of the controlled quadratic form employed in the Wall realization of the local index. This became even clearer at the workshop in Oberwolfach. A detailed inspection of the construction of [2] reveals that it suffices to give this explicit description at the first (controlled) stage of the construction of the inverse limit, since fairly general arguments show that subsequent stages can be designed to inherit the local index.

The main goal of this paper is to provide explicit realizations of controlled quadratic forms that lead to the construction of compact exotic homology manifolds with fundamental group  $\mathbb{Z}^{2n}$ ,  $n \geq 3$ , which are not homotopy equivalent to any closed topological manifold. This construction was suggested in [2, Section 7], but details were not provided. Starting with the rank–8 quadratic form  $E_8$  of signature 8, which generates the Wall group  $L_0(\mathbb{Z}) \cong \mathbb{Z}$ , we explicitly realize its image in  $L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$  under the canonical embedding  $L_0(\mathbb{Z}) \rightarrow L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ .

Let

$$\psi_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be the  $8 \times 8$  matrix over  $\mathbb{Z}$  with symmetrization the unimodular  $8 \times 8$  matrix of the  $E_8$ -form:

$$\psi_0 + \psi_0^* = E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Write

$$\begin{aligned} \mathbb{Z}[\mathbb{Z}^{2n}] &= \mathbb{Z}[z_1, z_1^{-1}, \dots, z_{2n}, z_{2n}^{-1}] \\ &= \mathbb{Z}[z_1, z_1^{-1}] \otimes \mathbb{Z}[z_2, z_2^{-1}] \otimes \cdots \otimes \mathbb{Z}[z_{2n}, z_{2n}^{-1}]. \end{aligned}$$

For  $i = 1, 2, \dots, n$  define the  $2 \times 2$  matrix over  $\mathbb{Z}[z_{2i-1}, z_{2i-1}^{-1}, z_{2i}, z_{2i}^{-1}]$

$$\alpha_i = \begin{pmatrix} 1 - z_{2i-1} & z_{2i-1}z_{2i} - z_{2i-1} - z_{2i} \\ 1 & 1 - z_{2i} \end{pmatrix},$$

so that  $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$  is a  $2^n \times 2^n$  matrix over  $\mathbb{Z}[\mathbb{Z}^{2n}]$ . (See Section 6 for the geometric provenance of the matrices  $\alpha_i$ ).

**Theorem 8.1** *The surgery obstruction  $E_8 \times T^{2n} \in L_{2n}(\Lambda)$  ( $\Lambda = \mathbb{Z}[\mathbb{Z}^{2n}]$ ) is represented by the nonsingular  $(-1)^n$ -quadratic form  $(K, \lambda, \mu)$  over  $\Lambda$ , with*

$$K = \mathbb{Z}^8 \otimes \Lambda^{2^n} = \Lambda^{2^{n+3}}$$

the f.g. free  $\Lambda$ -module of rank  $8 \cdot 2^n = 2^{n+3}$  and

$$\lambda = \psi + (-1)^n \psi^*: K \rightarrow K^* = \text{Hom}_\Lambda(K, \Lambda),$$

$$\mu(x) = \psi(x)(x) \in \mathcal{Q}_{(-1)^n}(\Lambda) = \Lambda / \{g + (-1)^{n+1}g^{-1} \mid g \in \mathbb{Z}^{2n}\} \quad (x \in K)$$

with

$$\psi = \psi_0 \otimes \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n: K \rightarrow K^*.$$

Sections 2–8 contain background material on surgery theory and the arguments that lead to a proof of Theorem 8.1. Invariance of  $E_8 \times T^{2n}$  under transfers to finite covers is proven in Section 9. In Section 10, using a large finite cover  $T^{2n} \rightarrow T^{2n}$ , we describe how to pass from the non-simply-connected surgery obstruction  $E_8 \times T^{2n}$  to a controlled quadratic  $\mathbb{Z}$ -form over  $T^{2n}$ . Finally, in Section 11 we explain how the controlled version of  $E_8 \times T^{2n}$  is used in the construction of exotic homology  $2n$ -manifolds  $X$  with Quinn index  $\iota(X) = 9$ .

## 2 The Wall groups

We begin with some recollections of surgery obstruction theory – we only need the details in the even-dimensional oriented case.

Let  $\Lambda$  be a ring with an involution, that is a function

$$\bar{\phantom{x}}: \Lambda \rightarrow \Lambda; a \mapsto \bar{a}$$

satisfying

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{b}\bar{a}, \quad \overline{\bar{a}} = a, \quad \bar{1} = 1 \in \Lambda.$$

**Example 2.1** In the applications to topology  $\Lambda = \mathbb{Z}[\pi]$  is a group ring, with the involution

$$\bar{\phantom{x}}: \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi]; \quad \sum_{g \in \pi} a_g g \mapsto \sum_{g \in \pi} a_g g^{-1} \quad (a_g \in \mathbb{Z}).$$

The involution is used to define a left  $\Lambda$ -module structure on the dual of a left  $\Lambda$ -module  $K$

$$K^* := \text{Hom}_{\Lambda}(K, \Lambda),$$

with

$$\Lambda \times K^* \rightarrow K^*; (a, f) \mapsto (x \mapsto f(x)\bar{a}).$$

The  $2n$ -dimensional surgery obstruction group  $L_{2n}(\Lambda)$  is defined by Wall [25, Section 5] to be the Witt group of nonsingular  $(-1)^n$ -quadratic forms  $(K, \lambda, \mu)$  over  $\Lambda$ , with  $K$  a finitely generated free (left)  $\Lambda$ -module together with

- (i) a pairing  $\lambda: K \times K \rightarrow \Lambda$  such that

$$\begin{aligned} \lambda(x, ay) &= a\lambda(x, y), \\ \lambda(x, y+z) &= \lambda(x, y) + \lambda(x, z), \\ \lambda(y, x) &= (-1)^n \overline{\lambda(x, y)} \end{aligned}$$

and the adjoint  $\Lambda$ -module morphism

$$\lambda: K \rightarrow K^*; \quad x \mapsto (y \mapsto \lambda(x, y))$$

is an isomorphism,

- (ii) a  $(-1)^n$ -quadratic function  $\mu: K \rightarrow \mathcal{Q}_{(-1)^n}(\Lambda) = \Lambda / \{a + (-1)^{n+1}\bar{a} \mid a \in \Lambda\}$  with

$$\begin{aligned} \lambda(x, x) &= \mu(x) + (-1)^n \overline{\mu(x)}, \\ \mu(x+y) &= \mu(x) + \mu(y) + \lambda(x, y), \\ \mu(ax) &= a\mu(x)\bar{a}. \end{aligned}$$

For a f.g. free  $\Lambda$ -module  $K = \Lambda^r$  with basis  $\{e_1, e_2, \dots, e_r\}$  the pair  $(\lambda, \mu)$  can be regarded as an equivalence class of  $r \times r$  matrices over  $\Lambda$

$$\psi = (\psi_{ij})_{1 \leq i, j \leq r} \quad (\psi_{ij} \in \Lambda)$$

such that  $\psi + (-1)^n \psi^*$  is invertible, with  $\psi^* = (\overline{\psi_{ji}})$ , and

$$\psi \sim \psi' \text{ if } \psi' - \psi = \chi + (-1)^{n+1} \chi^* \text{ for some } r \times r \text{ matrix } \chi = (\chi_{ij}).$$

The relationship between  $(\lambda, \mu)$  and  $\psi$  is given by

$$\begin{aligned} \lambda(e_i, e_j) &= \psi_{ij} + (-1)^n \overline{\psi_{ji}} \in \Lambda, \\ \mu(e_i) &= \psi_{ii} \in Q_{(-1)^n}(\Lambda), \end{aligned}$$

and we shall write

$$(\lambda, \mu) = (\psi + (-1)^n \psi^*, \psi).$$

The detailed definitions of the odd-dimensional  $L$ -groups  $L_{2n+1}(\Lambda)$  are rather more complicated, and are not required here. The quadratic  $L$ -groups are 4-periodic

$$L_m(\Lambda) = L_{m+4}(\Lambda).$$

The simply-connected quadratic  $L$ -groups are given by

$$L_m(\mathbb{Z}) = P_m = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \pmod{4} \\ 0 & \text{if } m \equiv 1 \pmod{4} \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \pmod{4} \\ 0 & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

(Kervaire–Milnor). In particular, for  $m \equiv 0 \pmod{4}$  there is defined an isomorphism

$$L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; \quad (K, \lambda, \mu) \mapsto \frac{1}{8} \text{signature}(K, \lambda).$$

The *kernel form* of an  $n$ -connected normal map  $(f, b): M^{2n} \rightarrow X$  from a  $2n$ -dimensional manifold  $M$  to an oriented  $2n$ -dimensional geometric Poincaré complex  $X$  is the nonsingular  $(-1)^n$ -quadratic form over  $\mathbb{Z}[\pi_1(X)]$  defined by Wall [25, Section 5]

$$(K_n(M), \lambda, \mu)$$

with

$$K_n(M) = \ker(\tilde{f}_*: H_n(\tilde{M}) \rightarrow H_n(\tilde{X}))$$

the kernel (stably) f.g. free  $\mathbb{Z}[\pi_1(X)]$ -module,  $\tilde{X}$  the universal cover of  $X$ ,  $\tilde{M} = f^* \tilde{X}$  the pullback cover of  $M$  and  $(\lambda, \mu)$  given by geometric (intersection, self-intersection)

numbers. The *surgery obstruction* of Wall [25]

$$\sigma_*(f, b) = (K_n(M), \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

is such that  $\sigma_*(f, b) = 0$  if (and for  $n \geq 3$  only if)  $(f, b)$  is bordant to a homotopy equivalence.

The Realization Theorem [25, Section 5] states that for a finitely presented group  $\pi$  and  $n \geq 3$  every nonsingular  $(-1)^n$ -quadratic form  $(K, \lambda, \mu)$  over  $\mathbb{Z}[\pi]$  is the kernel form of an  $n$ -connected  $2n$ -dimensional normal map  $f: M \rightarrow X$  with  $\pi_1(X) = \pi$ .

### 3 The instant surgery obstruction

Let  $(f, b): M^m \rightarrow X$  be an  $m$ -dimensional normal map with  $f_*: \pi_1(M) \rightarrow \pi_1(X)$  an isomorphism, and let  $\tilde{f}: \tilde{M} \rightarrow \tilde{X}$  be a  $\pi_1(X)$ -equivariant lift of  $f$  to the universal covers of  $M, X$ . The  $\mathbb{Z}[\pi_1(X)]$ -module morphisms  $\tilde{f}_*: H_r(\tilde{M}) \rightarrow H_r(\tilde{X})$  are split surjections, with the Umkehr maps

$$f^!: H_r(\tilde{X}) \cong H^{m-r}(\tilde{X}) \xrightarrow{\tilde{f}^*} H^{m-r}(\tilde{M}) \cong H_r(\tilde{M}),$$

such that

$$\tilde{f}_* f^! = 1: H_r(\tilde{X}) \rightarrow H_r(\tilde{X}).$$

The kernel  $\mathbb{Z}[\pi_1(X)]$ -modules

$$K_r(M) = \ker(\tilde{f}_*: H_r(\tilde{M}) \rightarrow H_r(\tilde{X}))$$

are such that

$$H_r(\tilde{M}) = K_r(M) \oplus H_r(\tilde{X}), \quad K_r(M) = H_{r+1}(\tilde{f}).$$

By the Hurewicz theorem,  $(f, b)$  is  $k$ -connected if and only if

$$K_r(M) = 0 \text{ for } r < k,$$

in which case  $K_k(M) = \pi_{k+1}(f)$ . If  $m = 2n$  or  $2n + 1$  then by Poincaré duality  $(f, b)$  is  $(n + 1)$ -connected if and only if it is a homotopy equivalence. In the even-dimensional case  $m = 2n$  the surgery obstruction of  $(f, b)$  is defined to be

$$\sigma_*(f, b) = \sigma_*(f', b') = (K_n(M'), \lambda', \mu') \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

with  $(f', b'): M' \rightarrow X$  any bordant  $n$ -connected normal map obtained from  $(f, b)$  by surgery below the middle dimension. The instant surgery obstruction of Ranicki [19] is an expression for such a form  $(K_n(M'), \lambda', \mu')$  in terms of the kernel  $2n$ -dimensional

quadratic Poincaré complex  $(C, \psi)$  such that  $H_*(C) = K_*(M)$ . In Section 8 we below we shall use a variant of the instant surgery obstruction to obtain an explicit  $(-1)^n$ -quadratic form over  $\mathbb{Z}[\mathbb{Z}^{2n}]$  representing  $E_8 \times T^{2n} \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ .

Given a ring with involution  $\Lambda$  and an  $m$ -dimensional f.g. free  $\Lambda$ -module chain complex

$$C: C_m \xrightarrow{d} C_{m-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d} C_0$$

let  $C^{m-*}$  be the dual  $m$ -dimensional f.g. free  $\Lambda$ -module chain complex, with

$$\begin{aligned} d_{C^{m-*}} &= (-1)^r d^*: \\ (C^{m-*})_r &= C^{m-r} = (C_{m-r})^* = \text{Hom}_\Lambda(C_{m-r}, \Lambda) \rightarrow C^{m-r+1}. \end{aligned}$$

Define a duality involution on  $\text{Hom}_\Lambda(C^{m-*}, C)$  by

$$T: \text{Hom}_\Lambda(C^p, C_q) \rightarrow \text{Hom}_\Lambda(C^q, C_p); \quad \phi \mapsto (-1)^{pq} \phi^*.$$

An  $m$ -dimensional quadratic Poincaré complex  $(C, \psi)$  over  $\Lambda$  is an  $m$ -dimensional f.g. free  $\Lambda$ -module chain complex  $C$  together with  $\Lambda$ -module morphisms

$$\psi_s: C^r \rightarrow C_{m-r-s} \quad (s \geq 0)$$

such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{m-s-1} (\psi_{s+1} + (-1)^{s+1} T\psi_{s+1}) = 0: C^{m-r-s-1} \rightarrow C_r$$

for  $s \geq 0$ , and such that  $(1+T)\psi_0: C^{m-*} \rightarrow C$  is a chain equivalence. The cobordism group of  $m$ -dimensional quadratic Poincaré complexes over  $\Lambda$  was identified in Ranicki [19] with the Wall surgery obstruction  $L_m(\Lambda)$ , and the surgery obstruction of an  $m$ -dimensional normal map  $(f, b): M \rightarrow X$  was identified with the cobordism class

$$\sigma_*(f, b) = (\mathcal{C}(f^\dagger), \psi_b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

of the kernel quadratic Poincaré complex  $(\mathcal{C}(f^\dagger), \psi_b)$ , with  $\mathcal{C}(f^\dagger)$  the algebraic mapping cone of the Umkehr  $\mathbb{Z}[\pi_1(X)]$ -module chain map

$$f^\dagger: C(\tilde{X}) \simeq C(\tilde{X})^{m-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{m-*} \simeq C(\tilde{M}).$$

The homology  $\mathbb{Z}[\pi_1(X)]$ -modules of  $\mathcal{C}(f^\dagger)$  are the kernel  $\mathbb{Z}[\pi_1(X)]$ -modules of  $f$

$$H_*(\mathcal{C}(f^\dagger)) = K_*(M) = \ker(\tilde{f}_*: H_*(\tilde{M}) \rightarrow H_*(\tilde{X})).$$

**Definition 3.1** The *instant form* of a  $2n$ -dimensional quadratic Poincaré complex  $(C, \psi)$  over  $\Lambda$  is the nonsingular  $(-1)^n$ -quadratic form over  $\Lambda$

$$(K, \lambda, \mu) = \left( \text{coker} \left( \begin{pmatrix} d^* & 0 \\ (-1)^{n+1}(1+T)\psi_0 & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \rightarrow C^n \oplus C_{n+1} \right), \begin{bmatrix} \psi_0 + (-1)^n \psi_0^* & d \\ (-1)^n d^* & 0 \end{bmatrix}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix} \right).$$

If  $C_r$  is f.g. free with  $\text{rank}_\Lambda C_r = c_r$  then  $K$  is (stably) f.g. free with

$$\text{rank}_\Lambda K = \sum_{r=0}^n (-1)^r (c_{n-r} + c_{n+r+1}) \in \mathbb{Z}.$$

If  $(1+T)\psi_0: C^{2n-*} \rightarrow C$  is an isomorphism then

$$c_{n+r+1} = c_{n-r-1}, \quad \text{rank}_\Lambda K = c_n,$$

with

$$(K, \lambda, \mu) = (C^n, \psi_0 + (-1)^n \psi_0^*, \psi_0).$$

**Proposition 3.2** (Instant surgery obstruction, Ranicki [19, Proposition I.4.3])

(i) The cobordism class of a  $2n$ -dimensional quadratic Poincaré complex  $(C, \psi)$  over  $\Lambda$  is the Witt class

$$(C, \psi) = (K, \lambda, \mu) \in L_{2n}(\Lambda)$$

of the instant nonsingular  $(-1)^n$ -quadratic form  $(K, \lambda, \mu)$  over  $\Lambda$ .

(ii) The surgery obstruction of a  $2n$ -dimensional normal map  $(f, b): M \rightarrow X$  is represented by the instant form  $(K, \lambda, \mu)$  of any quadratic Poincaré complex  $(C, \psi)$  which is chain equivalent to the kernel  $2n$ -dimensional quadratic Poincaré complex  $(C(f^!), \psi_b)$

$$\sigma_*(f, b) = (K, \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)]).$$

**Remark 3.3** (i) If  $(f, b)$  is  $n$ -connected then  $C$  is chain equivalent to the chain complex concentrated in dimension  $n$

$$C: 0 \rightarrow \cdots \rightarrow 0 \rightarrow K_n(M) \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

and the instant form is just the kernel form  $(K_n(M), \lambda, \mu)$  of Wall [25].

(ii) More generally, if  $(f, b)$  is  $k$ -connected for some  $k \leq n$  then  $C$  is chain equivalent to a chain complex concentrated in dimensions  $k, k+1, \dots, 2n-k$

$$C: 0 \rightarrow \cdots \rightarrow 0 \rightarrow C_{2n-k} \rightarrow \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_k \rightarrow 0 \rightarrow \cdots \rightarrow 0.$$



For  $n \geq 3$  the effect of surgeries killing the  $c_{2n-k}$  generators of  $H^{2n-k}(C) = K_k(M)$  represented by a basis of  $C^{2n-k}$  is a bordant  $(k+1)$ -connected normal map

$$(f', b'): M' \rightarrow X$$

with  $\mathfrak{C}(f': C(\tilde{X}) \rightarrow C(\tilde{M}'))$  chain equivalent to a chain complex of the type

$$C': 0 \rightarrow \cdots \rightarrow 0 \rightarrow C'_{2n-k-1} \rightarrow \cdots \rightarrow C'_n \rightarrow \cdots \rightarrow C'_{k+1} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

with

$$C'_r = \begin{cases} \ker((d(1+T)\psi_0): C_{k+1} \oplus C^{2n-k} \rightarrow C_k) & \text{if } r = k+1 \\ C_r & \text{if } k+2 \leq r \leq 2n-k-1. \end{cases}$$

Proceeding in this way, there is obtained a sequence of bordant  $j$ -connected normal maps

$$(f_j, b_j): M_j \rightarrow X \quad (j = k, k+1, \dots, n)$$

with

$$(f_k, b_k) = (f, b), \quad (f_{j+1}, b_{j+1}) = (f_j, b_j)'$$

The instant form of  $(C, \psi)$  is precisely the kernel  $(-1)^n$ -quadratic form

$$(K_n(M_n), \lambda_n, \mu_n)$$

of the  $n$ -connected normal map  $(f_n, b_n): M_n \rightarrow X$ , so that the surgery obstruction of  $(f, b)$  is given by

$$\begin{aligned} \sigma_*(f, b) &= \sigma_*(f_k, b_k) = \cdots = \sigma_*(f_n, b_n) \\ &= (K_n(M_n), \lambda_n, \mu_n) \in L_{2n}(\mathbb{Z}[\pi_1(X)]). \end{aligned}$$

## 4 The quadratic form $E_8$

For  $m \geq 2$  let  $M_0^{4m}$  be the  $(2m-1)$ -connected  $4m$ -dimensional  $PL$  manifold obtained from the Milnor  $E_8$ -plumbing of 8 copies of  $\tau_{S^{2m}}$  by coning off the (exotic)  $(4m-1)$ -sphere boundary, with intersection form  $E_8$  of signature 8. (For  $m=1$  we can take  $M_0$  to be the simply-connected 4-dimensional Freedman topological manifold with intersection form  $E_8$ ). The surgery obstruction of the corresponding  $2m$ -connected normal map  $(f_0, b_0): M_0^{4m} \rightarrow S^{4m}$  represents the generator

$$\sigma_*(f_0, b_0) = (K_{2m}(M_0), \lambda, \mu) = (\mathbb{Z}^8, E_8) = 1 \in L_{4m}(\mathbb{Z}) = L_0(\mathbb{Z}) = \mathbb{Z}$$

with

$$K_{2m}(M_0) = H_{2m}(M_0) = \mathbb{Z}^8$$

$$\lambda = E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\mu(0, \dots, 1, \dots, 0) = 1.$$

## 5 The surgery product formula

Surgery product formulae were originally obtained in the simply-connected case, notably by Sullivan. We now recall the non-simply-connected surgery product formula of Ranicki [19] involving the Mishchenko symmetric  $L$ -groups. In Section 6 we shall recall the variant of the surgery product formula involving almost symmetric  $L$ -groups of Clauwens, which will be used in Theorem 8.1 below to write down an explicit nonsingular  $(-1)^n$ -quadratic form over  $\mathbb{Z}[\mathbb{Z}^{2n}]$  ( $n \geq 1$ ) representing the image of the generator

$$1 = E_8 \in L_{4m}(\mathbb{Z}) \cong \mathbb{Z} \ (m \geq 0)$$

under the canonical embedding

$$- \times T^{2n}: L_{4m}(\mathbb{Z}) \rightarrow L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}]);$$

$$\sigma_*((f_0, b_0): M_0 \rightarrow S^{4m}) = E_8 \mapsto \sigma_*((f_0, b_0) \times 1: M_0 \times T^{2n} \rightarrow S^{4m} \times T^{2n}).$$

An  $n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over a ring with involution  $\Lambda$  is an  $n$ -dimensional f.g. free  $\Lambda$ -module chain complex

$$C: C_n \xrightarrow{d} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d} C_0$$

together with  $\Lambda$ -module morphisms

$$\phi_s: C^r = \text{Hom}_\Lambda(C_r, \Lambda) \rightarrow C_{n-r+s} \ (s \geq 0)$$

such that

$$d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1} (\phi_{s-1} + (-1)^s T\phi_{s-1}) = 0:$$

$$C^{n-r+s-1} \rightarrow C_r \quad (s \geq 0, \phi_{-1} = 0)$$

and  $\phi_0: C^{n-*} \rightarrow C$  is a chain equivalence. The cobordism group of  $n$ -dimensional symmetric Poincaré complexes over  $\Lambda$  is denoted by  $L^n(\Lambda)$  – see Ranicki [19] for a detailed exposition of symmetric  $L$ -theory. Note that the symmetric  $L$ -groups  $L^*(\Lambda)$  are not 4-periodic in general

$$L^n(\Lambda) \neq L^{n+4}(\Lambda).$$

The symmetric  $L$ -groups of  $\mathbb{Z}$  are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{4} \\ 0 & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

For  $m \equiv 0 \pmod{4}$  there is defined an isomorphism

$$L^{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; \quad (C, \phi) \mapsto \text{signature}(H^{2k}(C), \phi_0).$$

A  $CW$  structure on an oriented  $n$ -dimensional manifold with  $\pi_1(N) = \rho$  and universal cover  $\tilde{N}$  and the Alexander–Whitney–Steenrod diagonal construction on the cellular complex  $C(\tilde{N})$  determine an  $n$ -dimensional symmetric Poincaré complex  $(C(\tilde{N}), \phi)$  over  $\mathbb{Z}[\rho]$  with

$$\phi_0 = [N] \cap -: C(\tilde{N})^{n-*} \rightarrow C(\tilde{N}).$$

The Mishchenko *symmetric signature* of  $N$  is the cobordism class

$$\sigma^*(N) = (C, \phi) \in L^n(\mathbb{Z}[\rho]).$$

For  $n = 4k$  the image of  $\sigma^*(N)$  in  $L^{4k}(\mathbb{Z}) = \mathbb{Z}$  is just the usual signature of  $N$ .

For any rings with involution  $\Lambda, \Lambda'$  there are defined products

$$L^n(\Lambda) \otimes L^{n'}(\Lambda') \rightarrow L^{n+n'}(\Lambda \otimes \Lambda'); \quad (C, \phi) \otimes (C', \phi') \mapsto (C \otimes C', \phi \otimes \phi'),$$

$$L_n(\Lambda) \otimes L_{n'}(\Lambda') \rightarrow L_{n+n'}(\Lambda \otimes \Lambda'); \quad (C, \psi) \otimes (C', \phi') \mapsto (C \otimes C', \psi \otimes \phi')$$

as in Ranicki [20]. The tensor product of group rings is given by

$$\mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi'] = \mathbb{Z}[\pi \times \pi'].$$

**Theorem 5.1** (Symmetric  $L$ -theory surgery product formula, Ranicki [19])

(i) The symmetric signature of a product  $N \times N'$  of an  $n$ -dimensional manifold  $N$  and an  $n'$ -dimensional manifold  $N'$  is the product of the symmetric signatures

$$\sigma^*(N \times N') = \sigma^*(N) \otimes \sigma^*(N') \in L^{n+n'}(\mathbb{Z}[\pi_1(N) \times \pi_1(N')]).$$

(ii) The product of an  $m$ -dimensional normal map  $(f, b): M \rightarrow X$  and an  $n$ -dimensional manifold  $N$  is an  $(m+n)$ -dimensional normal map

$$(g, c) = (f, b) \times 1: M \times N \rightarrow X \times N$$

with surgery obstruction

$$\sigma_*(g, c) = \sigma_*(f, b) \otimes \sigma^*(N) \in L_{m+n}(\mathbb{Z}[\pi_1(X) \times \pi_1(N)]).$$

**Proof** These formulae already hold on the chain homotopy level, and chain equivalent symmetric/quadratic Poincaré complexes are cobordant. In somewhat greater detail:

(i) The symmetric Poincaré complex of a product  $N'' = N \times N'$  is the product of the symmetric Poincaré complexes of  $N$  and  $N'$

$$(C(\tilde{N}''), \phi'') = (C(\tilde{N}) \otimes C(\tilde{N}'), \phi \otimes \phi').$$

(ii) The kernel quadratic Poincaré complex  $(C(g^\dagger), \psi_c)$  of the product normal map  $(g, c) = (f, b) \times 1: M \times N \rightarrow X \times N$  is the product of the kernel quadratic Poincaré complex  $(C(f^\dagger), \psi_b)$  of  $(f, b)$  and the symmetric Poincaré complex  $(C(\tilde{N}), \phi)$  of  $N$

$$(C(g^\dagger), \psi_c) = (C(f^\dagger) \otimes C(\tilde{N}), \psi_b \otimes \phi). \quad \square$$

**Theorem 5.2** (i) (Shaneson [24], Wall [25]) The quadratic  $L$ -groups of  $\mathbb{Z}[\mathbb{Z}^n]$  are given by

$$L_m(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{m-r}(\mathbb{Z}) \quad (m \geq 0),$$

interpreting  $L_{m-r}(\mathbb{Z})$  for  $m-r < 0$  as  $L_{m-r+4*}(\mathbb{Z})$ .

(ii) (Milgram and Ranicki [13], Ranicki [21, Section 19]) The symmetric  $L$ -groups of  $\mathbb{Z}[\mathbb{Z}^n]$  are given by

$$L^m(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L^{m-r}(\mathbb{Z}) \quad (m \geq 0)$$

interpreting  $L^{m-r}(\mathbb{Z})$  for  $m < r$  as

$$L^{m-r}(\mathbb{Z}) = \begin{cases} 0 & \text{if } m = r-1, r-2 \\ L_{m-r}(\mathbb{Z}) & \text{if } m < r-2. \end{cases}$$

The symmetric signature of  $T^n$

$$\sigma^*(T^n) = (C(\tilde{T}^n), \phi) = (0, \dots, 0, 1) \in L^n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L^{n-r}(\mathbb{Z}),$$

is the cobordism class of the  $n$ -dimensional symmetric Poincaré complex  $(C(\tilde{T}^n), \phi)$  over  $\mathbb{Z}[\mathbb{Z}^n]$  with

$$C(\tilde{T}^n) = \bigotimes_n C(\tilde{S}^1), \quad \text{rank}_{\mathbb{Z}[\mathbb{Z}^n]} C_r(\tilde{T}^n) = \binom{n}{r}.$$

The surgery obstruction

$$E_8 \times T^n = (0, \dots, 0, 1) \in L_n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{n-r}(\mathbb{Z})$$

is the cobordism class of the  $n$ -dimensional quadratic Poincaré complex over  $\mathbb{Z}[\mathbb{Z}^n]$

$$(C, \psi) = (\mathbb{Z}^8, E_8) \otimes (C(\tilde{T}^n), \phi)$$

with

$$\text{rank}_{\mathbb{Z}[\mathbb{Z}^n]} C_r = 8 \binom{n}{r}.$$

## 6 Almost $(-1)^n$ -symmetric forms

The surgery obstruction of the  $(4m + 2n)$ -dimensional normal map

$$(f, b) = (f_0, b_0) \times 1: M_0^{4m} \times T^{2n} \rightarrow S^{4m} \times T^{2n}$$

is given by the instant surgery obstruction of Section 3 and the surgery product formula of Section 5 to be the Witt class

$$\sigma_*(f, b) = (K, \lambda, \mu) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

of the instant form  $(K, \lambda, \mu)$  of the  $2n$ -dimensional quadratic Poincaré complex

$$(C, \psi) = (\mathbb{Z}^8, E_8) \otimes (C(\tilde{T}^{2n}), \phi),$$

with

$$\text{rank}_{\mathbb{Z}[\mathbb{Z}^{2n}]} K = 8 \text{rank}_{\mathbb{Z}[\mathbb{Z}^{2n}]} C_n(\tilde{T}^{2n}) = 8 \binom{2n}{n}.$$

In principle, it is possible to compute  $(\lambda, \mu)$  directly from the  $(4m + 2n)$ -dimensional symmetric Poincaré complex  $E_8 \otimes (C(\tilde{T}^n), \phi)$ . In practice, we shall use the almost symmetric form surgery product formula of Clauwens [7; 5; 6], which is the analogue

for symmetric Poincaré complexes of the instant surgery obstruction of Section 3. We establish a product formula for almost symmetric forms which will be used in Section 7 to obtain an almost  $(-1)^n$ -symmetric form for  $T^{2n}$  of rank  $2^n \leq \binom{2n}{n}$ , and hence a representative  $(-1)^n$ -quadratic form for  $\sigma_*(f, b) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$  of rank  $2^{n+3} \leq 8\binom{2n}{n}$ .

**Definition 6.1** Let  $R$  be a ring with involution.

(i) An *almost  $(-1)^n$ -symmetric form*  $(A, \alpha)$  over  $R$  is a f.g. free  $R$ -module  $A$  together with a nonsingular pairing  $\alpha: A \rightarrow A^*$  such that the endomorphism

$$\beta = 1 + (-1)^{n+1}\alpha^{-1}\alpha^*: A \rightarrow A$$

is nilpotent, i.e.  $\beta^N = 0$  for some  $N \geq 1$ .

(ii) A *sublagrangian* of an almost  $(-1)^n$ -symmetric form  $(A, \alpha)$  is a direct summand  $L \subset A$  such that  $L \subseteq L^\perp$ , where

$$L^\perp := \{b \in A \mid \alpha(b)(A) = \alpha(A)(b) = \{0\}\}.$$

A *lagrangian* is a sublagrangian  $L$  such that

$$L = L^\perp.$$

(iii) The *almost  $(-1)^n$ -symmetric Witt group*  $AL^{2n}(R)$  is the abelian group of isomorphism classes of almost  $(-1)^n$ -symmetric forms  $(A, \alpha)$  over  $R$  with relations

$$(A, \alpha) = 0 \text{ if } (A, \alpha) \text{ admits a lagrangian}$$

and addition by

$$(A, \alpha) + (A', \alpha') = (A \oplus A', \alpha \oplus \alpha').$$

**Example 6.2** A nonsingular  $(-1)^n$ -symmetric form  $(A, \alpha)$  is an almost  $(-1)^n$ -symmetric form such that

$$\alpha = (-1)^n\alpha^*: A \rightarrow A^*$$

so that  $1 + (-1)^{n+1}\alpha^{-1}\alpha^* = 0: A \rightarrow A$ .

An almost  $(-1)^n$ -symmetric form  $(R^q, \alpha)$  on a f.g. free  $R$ -module of rank  $q$  is represented by an invertible  $q \times q$  matrix  $\alpha = (\alpha_{rs})$  such that the  $q \times q$  matrix

$$1 + (-1)^{n+1}\alpha^{-1}\alpha^*$$

is nilpotent.

**Definition 6.3** The *instant form* of a  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over  $R$  is the almost  $(-1)^n$ -symmetric form over  $R$

$$(A, \alpha) = \left( \text{coker} \left( \begin{pmatrix} d^* & 0 \\ -\phi_0^* & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \rightarrow C^n \oplus C_{n+1} \right), \begin{bmatrix} \phi_0 + d\phi_1 & d \\ (-1)^n d^* & 0 \end{bmatrix} \right).$$

**Example 6.4** If  $\phi_0: C^{2n-*} \rightarrow C$  is an isomorphism the instant almost  $(-1)^n$ -symmetric form is

$$(A, \alpha) = (C^n, \phi_0 + d\phi_1).$$

Every  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over a ring with involution  $R$  is chain equivalent to a complex  $(C', \phi')$  such that  $\phi'_0: C'^{2n-*} \rightarrow C'$  is an isomorphism, with

$$C': C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{n-1} \rightarrow A^* \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0$$

and

$$\phi'_0 + d'\phi'_1 = \alpha: C'^n = A \rightarrow C'_n = A^*.$$

(We shall not actually need this chain equivalence, since  $\phi_0: C^{2n-*} \rightarrow C$  is an isomorphism for  $C = C(\widetilde{T}^{2n})$ , so Example 6.4 will apply). The instant form defines a forgetful map

$$L^{2n}(R) \rightarrow AL^{2n}(R); \quad (C, \phi) \mapsto (A, \alpha).$$

**Proposition 6.5** (Ranicki [22, 36.3]) *The almost  $(-1)^n$ -symmetric Witt group of  $\mathbb{Z}$  is given by*

$$AL^{2n}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{2} \\ 0 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

with  $L^{4k}(\mathbb{Z}) \rightarrow AL^{4k}(\mathbb{Z})$  an isomorphism. The Witt class of an almost symmetric form  $(A, \alpha)$  over  $\mathbb{Z}$  is

$$(A, \alpha) = \text{signature}(\mathbb{Q} \otimes A, (\alpha + \alpha^*)/2) \in AL^{4k}(\mathbb{Z}) = L^{4k}(\mathbb{Z}) = \mathbb{Z}.$$

The almost  $(-1)^n$ -symmetric  $L$ -group  $AL^{2n}(R)$  was denoted  $L\text{Asy}_{h, S_{(-1)^n}}^0(R)$  in [22].

**Definition 6.6** The *almost symmetric signature* of a  $2n$ -dimensional manifold  $N^{2n}$  with  $\pi_1(N) = \rho$  is the Witt class

$$\sigma^*(N) = (A, \alpha) \in AL^{2n}(\mathbb{Z}[\rho])$$

of the instant almost  $(-1)^n$ -symmetric form  $(A, \alpha)$  over  $\mathbb{Z}[\rho]$  of the  $2n$ -dimensional symmetric Poincaré complex  $(C(\widetilde{N}), \phi)$  over  $\mathbb{Z}[\rho]$ .

The forgetful map  $L^{2n}(\mathbb{Z}[\rho]) \rightarrow AL^{2n}(\mathbb{Z}[\rho])$  sends the symmetric signature  $\sigma^*(N) \in L^{2n}(\mathbb{Z}[\rho])$  to the almost symmetric signature  $\sigma^*(N) \in AL^{2n}(\mathbb{Z}[\rho])$ .

For any rings with involution  $R_1, R_2$  there is defined a product

$$AL^{2n_1}(R_1) \otimes AL^{2n_2}(R_2) \rightarrow AL^{2n_1+2n_2}(R_1 \otimes R_2);$$

$$(A_1, \alpha_1) \otimes (A_2, \alpha_2) \mapsto (A_1 \otimes A_2, \alpha_1 \otimes \alpha_2).$$

**Proposition 6.7** *The almost symmetric signature of a product  $N = N_1 \times N_2$  of  $2n_i$ -dimensional manifolds  $N_i$  with  $\pi_1(N_i) = \rho_i$  and almost  $(-1)^{n_i}$ -symmetric forms  $(\mathbb{Z}[\rho_i]^{q_i}, \alpha_i)$  ( $i = 1, 2$ ) is the product*

$$\sigma^*(N_1 \times N_2) = \sigma^*(N_1) \otimes \sigma^*(N_2)$$

$$\in \text{im}(AL^{2n_1}(\mathbb{Z}[\rho_1]) \otimes AL^{2n_2}(\mathbb{Z}[\rho_2]) \rightarrow AL^{2n_1+2n_2}(\mathbb{Z}[\rho_1 \times \rho_2])).$$

**Proof** The almost  $(-1)^{n_1+n_2}$ -symmetric form  $(A, \alpha)$  of  $N_1 \times N_2$  is defined on

$$A = C^{n_1+n_2}(\tilde{N}_1 \times \tilde{N}_2) = \bigoplus_{(p_1, p_2) \in S} C^{p_1}(\tilde{N}_1) \otimes C^{p_2}(\tilde{N}_2)$$

with

$$S = \{(p_1, p_2) \mid p_1 + p_2 = n_1 + n_2\}.$$

Define an involution

$$T: S \rightarrow S; \quad (p_1, p_2) \mapsto (2n_1 - p_1, 2n_2 - p_2),$$

and let  $U \subset S \setminus \{(n_1, n_2)\}$  be any subset such that  $S$  decomposes as a disjoint union

$$S = \{(n_1, n_2)\} \cup U \cup T(U).$$

The submodule

$$L = \bigoplus_{(p_1, p_2) \in U} C^{p_1}(\tilde{N}_1) \otimes C^{p_2}(\tilde{N}_2) \subseteq A$$

is a sublagrangian of  $(A, \alpha)$  such that

$$(L^\perp/L, [\alpha]) = (C^{n_1}(\tilde{N}_1), \alpha_1) \otimes (C^{n_2}(\tilde{N}_2), \alpha_2).$$

The submodule

$$\Delta_{L^\perp} = \{(b, [b]) \mid b \in L^\perp\} \subset A \oplus (L^\perp/L)$$

is a lagrangian of  $(A, \alpha) \oplus (L^\perp/L, -[\alpha])$ , and

$$(A, \alpha) = (L^\perp/L, [\alpha]) = (C^{n_1}(\tilde{N}_1), \alpha_1) \otimes (C^{n_2}(\tilde{N}_2), \alpha_2)$$

$$\in \text{im}(AL^{2n_1}(\mathbb{Z}[\rho_1]) \otimes AL^{2n_2}(\mathbb{Z}[\rho_2]) \rightarrow AL^{2(n_1+n_2)}(\mathbb{Z}[\rho_1 \times \rho_2])).$$

This completes the proof.  $\square$



The product of a nonsingular  $(-1)^m$ -quadratic form  $(K, \lambda, \mu)$  over  $\Lambda$  and a  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over  $R$  is a  $2(m+n)$ -dimensional quadratic Poincaré complex  $(K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$  over  $\Lambda' = \Lambda \otimes R$ , as in Ranicki [19], with  $K_{*-m}$  the  $2m$ -dimensional f.g. free  $\Lambda$ -module chain complex concentrated in degree  $m$

$$K_{*-m}: 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 0 \rightarrow \cdots \rightarrow 0.$$

The pairing

$$L_{2m}(\Lambda) \otimes L^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R); (K, \lambda, \mu) \otimes (C, \phi) \mapsto (K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$$

has the following generalization.

**Definition 6.8** The *product* of a nonsingular  $(-1)^m$ -quadratic form  $(K, \lambda, \mu)$  over  $\Lambda$  and an almost  $(-1)^n$ -symmetric form  $(A, \alpha)$  over  $R$  is the nonsingular  $(-1)^{m+n}$ -quadratic form over  $\Lambda' = \Lambda \otimes R$

$$(K', \lambda', \mu') = (K, \lambda, \mu) \otimes (A, \alpha)$$

with

$$K' = K \otimes A, (\lambda', \mu') = (\lambda, \mu) \otimes \alpha = (\psi' + (-1)^{m+n} \psi'^*, \psi')$$

determined by the  $\Lambda'$ -module morphism

$$\psi' = \psi \otimes \alpha: K' = K \otimes A \rightarrow K'^* = K^* \otimes A^*$$

with  $\psi: K \rightarrow K^*$  a  $\Lambda$ -module morphism such that

$$(\lambda, \mu) = (\psi + (-1)^m \psi^*, \psi).$$

In particular, if  $K = \Lambda^p$  then  $\psi$  is given by a  $p \times p$  matrix  $\psi = \{\psi_{ij}\}$  over  $\Lambda$ , and if  $A = R^q$  then  $\alpha = \{\alpha_{rs}\}$  is given by a  $q \times q$  matrix over  $R$ , so that

$$\psi' = \psi \otimes \alpha$$

is the  $pq \times pq$  matrix over  $\Lambda'$  with

$$\psi'_{tu} = \psi_{ij} \otimes \alpha_{rs} \text{ if } t = (i-1)p + r, u = (j-1)p + s.$$

If  $(A, \alpha)$  is an almost  $(-1)^n$ -symmetric form over  $R$  with a sublagrangian  $L \subset A$  the induced almost  $(-1)^n$ -symmetric form  $(L^\perp/L, [\alpha])$  over  $R$  is such that

$$\Delta_{L^\perp} = \{(b, [b]) \mid b \in L^\perp\} \subset A \oplus (L^\perp/L)$$

is a lagrangian of  $(A, \alpha) \oplus (L^\perp/L, -[\alpha])$ , and

$$(K, \lambda, \mu) \otimes (A, \alpha) = (K, \lambda, \mu) \otimes (L^\perp/L, [\alpha]) \in L_{2m+2n}(\Lambda').$$

In particular, if  $L$  is a lagrangian of  $(A, \alpha)$  then

$$(K, \lambda, \mu) \otimes (A, \alpha) = 0 \in L_{2m+2n}(\Lambda'),$$

so that the product

$$L_{2m}(\Lambda) \otimes AL^{2n}(R) \rightarrow L_{2m+2n}(\Lambda'); \quad (K, \lambda, \mu) \otimes (A, \alpha) \mapsto (K \otimes A, (\lambda, \mu) \otimes \alpha)$$

is well-defined.

**Theorem 6.9** (Almost symmetric  $L$ -theory surgery product formula, Clauwens [7])

(i) *The product*

$$L_{2m}(\Lambda) \otimes L^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R); \quad (K, \lambda, \mu) \otimes (C, \phi) \mapsto (K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$$

factors through the product

$$L_{2m}(\Lambda) \otimes AL^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R); \quad (K, \lambda, \mu) \otimes (A, \alpha) \mapsto (K \otimes A, (\lambda, \mu) \otimes \alpha).$$

(ii) Let  $(f, b): M \rightarrow X$  be a  $2m$ -dimensional normal map with surgery obstruction

$$\sigma_*(f, b) = (\mathbb{Z}[\pi]^p, \lambda, \mu) \in L_{2m}(\mathbb{Z}[\pi]) \quad (\pi = \pi_1(X)),$$

and let  $N$  be a  $2n$ -dimensional manifold with almost  $(-1)^n$ -symmetric signature

$$\sigma^*(N) = (\mathbb{Z}[\rho]^q, \alpha) \in AL^{2n}(\mathbb{Z}[\rho]) \quad (\rho = \pi_1(N)).$$

The surgery obstruction of the  $(2m + 2n)$ -dimensional normal map

$$(g, c) = (f, b) \times 1: M \times N \rightarrow X \times N$$

is given by

$$\begin{aligned} \sigma_*(g, c) &= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha) \\ &\in \text{im}(L_{2m}(\mathbb{Z}[\pi]) \otimes AL^{2n}(\mathbb{Z}[\rho]) \rightarrow L_{2m+2n}(\mathbb{Z}[\pi \times \rho])). \end{aligned}$$

(iii) *The surgery obstruction of the product  $2(m + n_1 + n_2)$ -dimensional normal map*

$$(g, c) = (f, b) \times 1: M \times N_1 \times N_2 \rightarrow X \times N_1 \times N_2$$

is given by

$$\sigma_*(g, c) = (\mathbb{Z}[\pi \times \rho_1 \times \rho_2]^{pq_1q_2}, (\lambda, \mu) \otimes \alpha_1 \otimes \alpha_2) \in L_{2(m+n_1+n_2)}(\mathbb{Z}[\pi \times \rho_1 \times \rho_2]).$$

**Proof** (i) By construction.

(ii) It may be assumed that  $(f, b): M \rightarrow X$  is an  $m$ -connected  $2m$ -dimensional normal map, with kernel  $(-1)^m$ -quadratic form over  $\mathbb{Z}[\pi]$

$$(K_m(M), \lambda, \mu) = (\mathbb{Z}[\pi]^p, \lambda, \mu).$$

The product  $(g, c) = (f, b) \times 1: M \times N \rightarrow X \times N$  is  $m$ -connected, with quadratic Poincaré complex

$$(C, \psi) = (K_m(M), \lambda, \mu) \otimes (C(\tilde{N}), \phi)$$

and kernel  $\mathbb{Z}[\pi \times \rho]$ -modules

$$K_*(M \times N) = K_m(M) \otimes H_{*-m}(\tilde{N}).$$

Let  $(f', b'): M' \rightarrow X \times N$  be the bordant  $(m+n)$ -connected normal map obtained from  $(g, c)$  by surgery below the middle dimension, using  $(C, \psi)$  as in Remark 3.3 (ii). The kernel  $(-1)^{m+n}$ -quadratic form over  $\mathbb{Z}[\pi \times \rho]$  of  $(f', b')$  is the instant form of  $(C, \psi)$ , which is just the product of  $(K_m(M), \lambda, \mu)$  and the almost  $(-1)^n$ -symmetric form  $(\mathbb{Z}[\rho]^q, \alpha)$

$$(K_{m+n}(M'), \lambda', \mu') =$$

$$\left( \text{coker} \left( \begin{pmatrix} d^* & 0 \\ (-1)^{m+n+1}(1+T)\psi_0 & d \end{pmatrix} : C^{m+n-1} \oplus C_{m+n+2} \rightarrow C^{m+n} \oplus C_{m+n+1} \right), \right.$$

$$\left. \begin{matrix} [\psi_0 + (-1)^{m+n}\psi_0^* & d] \\ (-1)^{m+n}d^* & 0 \end{matrix}, \begin{matrix} [\psi_0 & d] \\ 0 & 0 \end{matrix} \right)$$

$$= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha).$$

The surgery obstruction of  $(g, c)$  is thus given by

$$\begin{aligned} \sigma_*(g, c) &= \sigma_*(f', b') = (K_{m+n}(M'), \lambda', \mu') \\ &= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha) \in L_{2m+2n}(\mathbb{Z}[\pi \times \rho]). \end{aligned}$$

(iii) Combine (i) and (ii) with Proposition 6.7. □

## 7 The almost $(-1)^n$ -symmetric form of $T^{2n}$

Geometrically,  $-- \times T^{2n}$  sends the surgery obstruction  $\sigma_*(f_0, b_0) = E_8 \in L_{4m}(\mathbb{Z})$  to the surgery obstruction

$$E_8 \times T^{2n} = \sigma_*(f_n, b_n) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

of the  $(4m+2n)$ -dimensional normal map

$$(f_n, b_n) = (f_0, b_0) \times 1: M_0^{4m} \times T^{2n} \rightarrow S^{4m} \times T^{2n}$$

given by product with the almost symmetric signature of

$$\begin{aligned} T^{2n} &= S^1 \times S^1 \times \cdots \times S^1 \quad (2n \text{ factors}) \\ &= T^2 \times T^2 \times \cdots \times T^2 \quad (n \text{ factors}). \end{aligned}$$

In order to apply the almost symmetric surgery product formula (see Theorem 6.9) for  $N^{2n} = T^{2n}$  it therefore suffices to work out the almost  $(-1)$ -symmetric form  $(C^1(\tilde{T}^2), \alpha)$  of  $T^2$ .

The symmetric Poincaré structure  $\phi = \{\phi_s | s \geq 0\}$  of the universal cover  $\tilde{S}^1 = \mathbb{R}$  of  $S^1$  is given by

$$\begin{aligned} d = 1 - z: C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] &\rightarrow C_0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}], \\ \phi_0 &= \begin{cases} 1: C^0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \\ z: C^1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}], \end{cases} \\ \phi_1 = -1: C^1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] &\rightarrow C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}]. \end{aligned}$$

Write

$$\Lambda = \mathbb{Z}[\pi_1(T^2)] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}].$$

The Poincaré duality of  $\tilde{T}^2 = \mathbb{R}^2$  is the  $\Lambda$ -module chain isomorphism given by the chain-level Künneth formula to be

$$\begin{array}{ccccc} C(\tilde{T}^2)^{2-*} \phi_0: C^0 = \Lambda & \xrightarrow{d^* = \begin{pmatrix} z_2 - 1 \\ 1 - z_1^{-1} \end{pmatrix}} & C^1 = \Lambda \oplus \Lambda & \xrightarrow{d^* = \begin{pmatrix} 1 - z_1^{-1} & 1 - z_2 \end{pmatrix}} & C^2 = \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ C(\tilde{T}^2): C_2 = \Lambda & \xrightarrow{d = \begin{pmatrix} 1 - z_1 \\ 1 - z_2^{-1} \end{pmatrix}} & C_1 = \Lambda \oplus \Lambda & \xrightarrow{d = \begin{pmatrix} z_2^{-1} - 1 & 1 - z_1 \end{pmatrix}} & C_0 = \Lambda. \end{array}$$

$\begin{matrix} \downarrow & & \downarrow & & \downarrow \\ & 1 & & \begin{pmatrix} 0 & -z_1 \\ z_2^{-1} & 0 \end{pmatrix} & & -z_1 z_2^{-1} \end{matrix}$

The chain homotopy

$$\phi_1: \phi_0 \simeq T\phi_0: C(\tilde{T}^2)^{2-*} \rightarrow C(\tilde{T}^2)$$

is given by

$$\phi_1 = \begin{cases} \begin{pmatrix} 1 & -z_2 \end{pmatrix}: C^1 = \Lambda \oplus \Lambda \rightarrow C_2 = \Lambda \\ \begin{pmatrix} -z_1 \\ 1 \end{pmatrix}: C^2 = \Lambda \rightarrow C_1 = \Lambda \oplus \Lambda. \end{cases}$$

**Proposition 7.1** The almost  $(-1)$ -symmetric form of  $T^2$  is given by  $(C^1, \alpha)$  with

$$\alpha = \phi_0 - \phi_1 d^* = \begin{pmatrix} 1 - z_1 & z_1 z_2 - z_1 - z_2 \\ 1 & 1 - z_2 \end{pmatrix}: C^1 = \Lambda \oplus \Lambda \rightarrow C_1 = \Lambda \oplus \Lambda.$$

**Proof** By construction, noting that

$$1 + \alpha^{-1} \alpha^* = \begin{pmatrix} -(1 - z_1)(1 - z_2^{-1}) & z_1(1 - z_2)(1 - z_2^{-1}) \\ -z_2^{-1}(1 - z_1)(1 - z_1^{-1}) & (1 - z_1)(1 - z_2^{-1}) \end{pmatrix}: \\ C^1 = \Lambda \oplus \Lambda \rightarrow C^1 = \Lambda \oplus \Lambda$$

is nilpotent, with

$$(1 + \alpha^{-1} \alpha^*)^2 = 0: C^1 = \Lambda \oplus \Lambda \rightarrow C^1 = \Lambda \oplus \Lambda. \quad \square$$

**Remark 7.2** An almost  $(-1)^n$ -symmetric form  $(R^q, \alpha)$  over  $R$  determines a nonsingular  $(-1)^n$ -quadratic form  $(R[1/2]^q, \lambda, \mu)$  over  $R[1/2]$ , with

$$\lambda(x, y) = (\alpha(x, y) + (-1)^n \overline{\alpha(y, x)})/2, \quad \mu(x) = \alpha(x)(x)/2.$$

In particular, the almost  $(-1)$ -symmetric form  $(\Lambda \oplus \Lambda, \alpha)$  of  $T^2$  determines the nonsingular  $(-1)$ -quadratic form  $(\Lambda[1/2] \oplus \Lambda[1/2], \lambda, \mu)$  over  $\Lambda[1/2] = \mathbb{Z}[\mathbb{Z}^2][1/2]$ , with

$$\lambda = (\alpha - \alpha^*)/2 \\ = \begin{pmatrix} ((z_1)^{-1} - z_1)/2 & (1 - z_1 z_2 - z_1 - z_2)/2 \\ (-1 + (z_1)^{-1}(z_2)^{-1} + (z_1)^{-1} + (z_2)^{-1})/2 & ((z_2)^{-1} - z_2)/2 \end{pmatrix}$$

the invertible skew-symmetric  $2 \times 2$  matrix exhibited in [11, Example, p120].

## 8 An explicit form representing $E_8 \times T^{2n} \in L_{4*+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$

Write the generators of the free abelian group  $\pi_1(T^{2n}) = \mathbb{Z}^{2n}$  as

$$z_1, z_2, \dots, z_{2n-1}, z_{2n},$$

so that

$$\mathbb{Z}[\mathbb{Z}^{2n}] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_{2n}, z_{2n}^{-1}].$$

The expression of  $T^{2n}$  as an  $n$ -fold cartesian product of  $T^2$ 's

$$T^{2n} = T^2 \times T^2 \times \dots \times T^2$$

gives

$$\mathbb{Z}[\mathbb{Z}^{2n}] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}] \otimes \mathbb{Z}[z_3, z_3^{-1}, z_4, z_4^{-1}] \otimes \cdots \otimes \mathbb{Z}[z_{2n-1}, z_{2n-1}^{-1}, z_{2n}, z_{2n}^{-1}].$$

For  $i = 1, 2, \dots, n$  define the invertible  $2 \times 2$  matrix over  $\mathbb{Z}[z_{2i-1}, z_{2i-1}^{-1}, z_{2i}, z_{2i}^{-1}]$

$$\alpha_i = \begin{pmatrix} 1 - z_{2i-1} & z_{2i-1}z_{2i} - z_{2i-1} - z_{2i} \\ 1 & 1 - z_{2i} \end{pmatrix}.$$

The generator  $1 = E_8 \in L_0(\mathbb{Z}) = \mathbb{Z}$  is represented by the nonsingular quadratic form  $(\mathbb{Z}^8, \psi_0)$  over  $\mathbb{Z}$  with:

$$\psi_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Theorem 8.1** *The  $2^{n+3} \times 2^{n+3}$  matrix over  $\mathbb{Z}[\mathbb{Z}^{2n}]$*

$$\psi_n = \psi_0 \otimes \alpha_1 \otimes \alpha_2 \cdots \otimes \alpha_n$$

is such that

$$E_8 \times T^{2n} = (\mathbb{Z}[\mathbb{Z}^{2n}]^{2^{n+3}}, \psi_n) \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]).$$

**Proof** A direct application of the almost symmetric surgery product formula (Theorem 6.9), noting that  $\alpha_1, \alpha_2, \dots, \alpha_n$  are copies of the almost  $(-1)$ -symmetric form of  $T^2$  obtained in Proposition 7.1.  $\square$

## 9 Transfer invariance

A covering map  $p: T^n \rightarrow T^n$  induces an injection of the fundamental group in itself

$$p_*: \pi_1(T^n) = \mathbb{Z}^n \rightarrow \pi_1(T^n) = \mathbb{Z}^n$$

as a subgroup of finite index, say  $q = [\mathbb{Z}^n: p_*(\mathbb{Z}^n)]$ . Given a  $\mathbb{Z}[\mathbb{Z}^n]$ -module  $K$  let  $p^!K$  be the  $\mathbb{Z}[\mathbb{Z}^n]$ -module defined by the additive group of  $K$  with

$$\mathbb{Z}[\mathbb{Z}^n] \times p^!K \rightarrow p^!K; \quad (a, b) \mapsto p_*(a)b.$$

In particular

$$p^!\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[\mathbb{Z}^n]^q.$$

The restriction functor

$$p^!: \{\mathbb{Z}[\mathbb{Z}^n]\text{-modules}\} \rightarrow \{\mathbb{Z}[\mathbb{Z}^n]\text{-modules}\}; \quad K \mapsto p^!K$$

induces transfer maps in the quadratic  $L$ -groups

$$p^!: L_m(\mathbb{Z}[\mathbb{Z}^n]) \rightarrow L_m(\mathbb{Z}[\mathbb{Z}^n]); \quad (C, \psi) \mapsto p^!(C, \psi).$$

**Proposition 9.1** *The image of the (split) injection*

$$L_0(\mathbb{Z}) \rightarrow L_n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{n-r}(\mathbb{Z}); \quad E_8 \mapsto E_8 \times T^n$$

is the subgroup of the transfer-invariant elements

$$L_n(\mathbb{Z}[\mathbb{Z}^n])^{INV} = \{x \in L_n(\mathbb{Z}[\mathbb{Z}^n]) \mid p^!x = x \text{ for all } p: T^n \rightarrow T^n\}.$$

**Proof** See Ranicki [21, Chapter 18]. □

**Example 9.2** (i) Write

$$\Lambda = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}].$$

Here is an explicit verification that

$$p^!(E_8 \times T^2) = E_8 \times T^2 \in L_2(\Lambda)$$

for the double cover

$$p: T^2 = S^1 \times S^1 \rightarrow T^2; \quad (w_1, w_2) \mapsto ((w_1)^2, w_2)$$

with

$$p_*: \pi_1(T^2) = \mathbb{Z}^2 \rightarrow \mathbb{Z}^2; \quad z_1 \mapsto (z_1)^2, \quad z_2 \mapsto z_2$$

the inclusion of a subgroup of index 2. For any  $j_1, j_2 \in \mathbb{Z}$  the transfer of the  $\Lambda$ -module morphism  $z_1^{j_1} z_2^{j_2}: \Lambda \rightarrow \Lambda$  is given by the  $\Lambda$ -module morphism

$$p^!(z_1^{j_1} z_2^{j_2}) = \begin{cases} \begin{pmatrix} (z_1)^{j_1/2} z_2^{j_2} & 0 \\ 0 & (z_1)^{j_1/2} z_2^{j_2} \end{pmatrix}: \\ p^!\Lambda = \Lambda \oplus \Lambda \rightarrow p^!\Lambda = \Lambda \oplus \Lambda & \text{if } j_1 \text{ is even} \\ \begin{pmatrix} 0 & (z_1)^{(j_1+1)/2} z_2^{j_2} \\ (z_1)^{(j_1-1)/2} z_2^{j_2} & 0 \end{pmatrix}: \\ p^!\Lambda = \Lambda \oplus \Lambda \rightarrow p^!\Lambda = \Lambda \oplus \Lambda & \text{if } j_1 \text{ is odd.} \end{cases}$$

The transfer of the almost  $(-1)$ -symmetric form of  $T^2$  over  $\Lambda$

$$(C^1(\widetilde{T}^2), \alpha) = \left( \Lambda \oplus \Lambda, \begin{pmatrix} 1-z_1 & z_1z_2 - z_1 - z_2 \\ 1 & 1-z_2 \end{pmatrix} \right)$$

is the almost  $(-1)$ -symmetric form over  $\Lambda$

$$p^1(C^1(\widetilde{T}^2), \alpha) = \left( \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda, \begin{pmatrix} 1 & -z_1 & -z_2 & z_1z_2 - z_1 \\ -1 & 1 & z_2 - 1 & -z_2 \\ 1 & 0 & 1 - z_2 & 0 \\ 0 & 1 & 0 & 1 - z_2 \end{pmatrix} \right)$$

The  $\Lambda$ -module morphisms

$$i = \begin{pmatrix} z_1 - z_1z_2 \\ 0 \\ -z_1 \\ 1 \end{pmatrix} : \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda,$$

$$j = \begin{pmatrix} 1 & 0 & z_1 - z_1z_2 \\ z_1^{-1} & 0 & 0 \\ 0 & 1 & -z_1 \\ 0 & 0 & 1 \end{pmatrix} : \Lambda \oplus \Lambda \oplus \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$$

are such that  $i = j|_{0 \oplus 0 \oplus \Lambda}$  and there is defined a (split) exact sequence

$$0 \longrightarrow \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{j} \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{i^* p^1 \alpha} \Lambda \longrightarrow 0$$

with

$$j^*(p^1 \alpha)j = \begin{pmatrix} 1-z_1 & z_1z_2 - z_1 - z_2 & 0 \\ 1 & 1-z_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \Lambda \oplus \Lambda \oplus \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda.$$

The submodule

$$L = i(\Lambda) \subset p^1(\Lambda \oplus \Lambda) = \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$$

is thus a sublagrangian of the almost  $(-1)$ -symmetric form  $p^1(C^1(\widetilde{T}^2), \alpha)$  over  $\mathbb{Z}[\mathbb{Z}^2]$  such that

$$(L^\perp/L, [p^1 \alpha]) = (C^1(\widetilde{T}^2), \alpha)$$

and

$$\begin{aligned} p^1(E_8 \times T^2) &= E_8 \otimes p^1(C^1(\widetilde{T}^2), \alpha) \\ &= E_8 \otimes (L^\perp/L, [p^1 \alpha]) \\ &= E_8 \otimes (C^1(\widetilde{T}^2), \alpha) = E_8 \times T^2 \in L_2(\Lambda). \end{aligned}$$



(ii) For any  $n \geq 1$  replace  $p$  by

$$p_n = p \times 1: T^{2n} = T^2 \times T^{2n-2} \rightarrow T^{2n} = T^2 \times T^{2n-2}$$

to likewise obtain an explicit verification that

$$p_n^!(E_8 \times T^{2n}) = E_8 \times T^{2n} \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]).$$

## 10 Controlled surgery groups

A *geometric  $\mathbb{Z}[\pi]$ -module* over a metric space  $B$  is a pair  $(K, \varphi)$ , where  $K = \mathbb{Z}[\pi]^r$  is a free  $\mathbb{Z}[\pi]$ -module with basis  $S = \{e_1, \dots, e_r\}$  and  $\varphi: S \rightarrow B$  is a map. The  $(\epsilon, \delta)$ -*controlled surgery group*  $L_n(B; \mathbb{Z}, \epsilon, \delta)$  (with trivial local fundamental group) is defined as the group of  $n$ -dimensional quadratic  $\mathbb{Z}$ -Poincaré complexes (see [19]) over  $B$  of radius  $< \delta$ , modulo  $(n+1)$ -dimensional quadratic  $\mathbb{Z}$ -Poincaré bordisms of radius  $< \epsilon$ . Elements of  $L_{2n}(B; \mathbb{Z}, \epsilon, \delta)$  are represented by non-singular  $(-1)^n$ -quadratic forms  $(K, \lambda, \mu)$ , where  $K = \mathbb{Z}^r$  is a geometric  $\mathbb{Z}$ -module over  $B$ , and  $\lambda$  has radius  $< \delta$ , i.e.,  $\lambda(e_i, e_j) = 0$  if  $d(\varphi(e_i), \varphi(e_j)) \geq \delta$ . In matrix representation  $(K, \psi)$ , this is equivalent to  $\psi_{ij} = 0$  if  $d(\varphi(e_i), \varphi(e_j)) \geq \delta$ . The radius of a bordism is defined similarly.

In effect, Yamasaki [27] defined an assembly map  $H_n(B; \mathbb{L}) \rightarrow L_n(B; \mathbb{Z}, \epsilon, \delta)$ , where  $H_*(B; \mathbb{L})$  denotes homology with coefficients in the 4-periodic simply-connected surgery spectrum  $\mathbb{L}$  of Ranicki [20, Chapter 25].

The following Stability Theorem is a key ingredient in the construction of exotic ENR homology manifolds.

**Theorem 10.1** (Stability; Pedersen, Quinn and Ranicki [15]; Ferry [10]; Pedersen and Yamasaki [16])

Let  $n \geq 0$  and suppose  $B$  is a compact metric ENR. Then there exist constants  $\epsilon_0 > 0$  and  $\kappa > 1$ , which depend on  $n$  and  $B$ , such that the assembly map  $H_n(B; \mathbb{L}) \rightarrow L_n(B; \mathbb{Z}, \epsilon, \delta)$  is an isomorphism if  $\epsilon_0 \geq \epsilon \geq \kappa\delta$ , so that

$$\lim_{\epsilon \leftarrow} \lim_{\delta \leftarrow} L_n(B; \mathbb{Z}, \epsilon, \delta) = H_n(B; \mathbb{L}).$$

We are interested in controlled surgery over the torus  $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$  equipped with the usual geodesic metric. Let  $(K, \psi)$  represent an element of  $L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ , where  $K = \mathbb{Z}[\mathbb{Z}^{2n}]^r$ . Our next goal is to show that passing to a sufficiently large covering

space  $p: T^{2n} \rightarrow T^{2n}$ ,  $(K, \psi)$  defines an element of  $L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta)$ . For simplicity, we assume that

$$p_*: \pi_1(T^{2n}) \cong \mathbb{Z}^{2n} \rightarrow \pi_1(T^{2n}) \cong \mathbb{Z}^{2n}$$

is given by multiplication by  $k > 0$ , so that  $p$  is a  $k^{2n}$ -sheeted covering space.

Let  $(\bar{K}, \bar{\psi}) = \mathbb{Z}[\mathbb{Z}_k^{2n}] \otimes_{\mathbb{Z}[\mathbb{Z}^{2n}]} (K, \psi)$ , where the (right)  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module structure on  $\mathbb{Z}[\mathbb{Z}_k^{2n}]$  is induced by reduction modulo  $k$ . The  $\mathbb{Z}$ -module  $\tilde{K}$  underlying  $\bar{K}$  has basis  $\mathbb{Z}_k^{2n} \times S$ ; if  $g \in \mathbb{Z}_k^{2n}$  and  $e_i \in S$ , we write  $(g, e_i) = g e_i$ . Pick a point  $x_0$  in the covering torus  $T^{2n}$  viewed as a  $\mathbb{Z}_k^{2n}$ -space under the action of the group of deck transformations. Let  $\varphi(e_i) = x_0$ , for every  $e_i \in S$ , and extend it  $\mathbb{Z}_k^{2n}$ -equivariantly to obtain  $\varphi: \mathbb{Z}_k^{2n} \times S \rightarrow T^{2n}$ . Then, the pair  $(\tilde{K}, \varphi)$  is a geometric  $\mathbb{Z}$ -module over  $T^{2n}$  of dimension  $rk^{2n}$ .

We now describe the quadratic  $\mathbb{Z}$ -module  $(\tilde{K}, \tilde{\psi})$  induced by  $(K, \psi)$  and the covering  $p$ . Write

$$\bar{\psi} = \sum_{g \in \mathbb{Z}_k^{2n}} g \bar{\psi}_g,$$

where each  $\bar{\psi}_g$  is a matrix with integer entries. For basis elements  $g e_i, f e_j \in \mathbb{Z}_k^{2n} \times S$ , let  $\tilde{\psi}(g e_i, f e_j) = \bar{\psi}_{g^{-1}f}(e_i, e_j)$ ; this defines a bilinear  $\mathbb{Z}$ -form on the geometric  $\mathbb{Z}$ -module  $\tilde{K}$ . For a given quadratic  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module  $(K, \psi)$ , we show that  $(\tilde{K}, \tilde{\psi})$  has diameter  $< \delta$  over the (covering) torus  $T^{2n}$ , if  $k$  is sufficiently large.

Elements of  $\mathbb{Z}^{2n}$  can be expressed uniquely as monomials

$$z^i = z_1^{i_1} \dots z_{2n}^{i_{2n}}$$

where  $i = (i_1, \dots, i_{2n}) \in \mathbb{Z}^{2n}$  is a multi-index. We use the notation

$$|i| = \max \{|i_1|, \dots, |i_{2n}|\}.$$

Any  $z \in \mathbb{Z}[\mathbb{Z}^{2n}]$  can be expressed uniquely as

$$z = \sum_{i \in \mathbb{Z}^{2n}} \alpha_i z^i,$$

where  $\alpha_i \in \mathbb{Z}$  is zero for all but finitely many values of  $i$ . We define the *order* of  $z$  to be

$$o(z) = \max \{|i| : \alpha_i \neq 0\}$$

and let

$$|\psi| = \max \{o(\psi_{ij}), 1 \leq i, j \leq r\}.$$

Then,  $(\tilde{K}, \tilde{\psi})$  is a quadratic  $\mathbb{Z}$ -module over  $T^{2n}$  of radius  $< \delta$ , provided that  $k > 2|\psi|/\delta$ . Similarly, quadratic  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -Poincaré bordisms induce quadratic  $\mathbb{Z}$ -Poincaré  $\epsilon$ -bordisms for  $k$  large.

### 10.1 The forgetful map

We give an algebraic description of the forget-control map

$$\mathfrak{F}: L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta) \rightarrow L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]),$$

for  $\epsilon$  and  $\delta$  small. Let  $\sigma \in L(T^{2n}; \mathbb{Z}, \epsilon, \delta)$  be represented by the  $(-1)^n$ -quadratic  $\mathbb{Z}$ -module  $(K, \psi)$  over  $T^{2n}$  of radius  $< \delta$ , where  $K$  has basis  $S = \{e_1, \dots, e_r\}$  and projection  $\varphi: S \rightarrow T^{2n}$ . Consider the free  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module  $\tilde{K}$  of rank  $r$  generated by  $\tilde{S} = \{\tilde{e}_1, \dots, \tilde{e}_r\}$  and let  $\tilde{\varphi}: \tilde{S} \rightarrow \mathbb{R}^{2n}$  be a map satisfying  $q \circ \tilde{\varphi}(\tilde{e}_i) = \varphi(e_i)$ ,  $1 \leq i \leq r$ , where  $q: \mathbb{R}^{2n} \rightarrow T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$  is the universal cover. If  $\psi_{ij} \neq 0$  and  $\delta$  is small, there is a unique element  $g_{ij}$  of  $\mathbb{Z}^{2n}$  such that  $d(\tilde{\varphi}(\tilde{e}_j) + g_{ij}, \tilde{\varphi}(\tilde{e}_i)) < \delta$ , where  $d$  denotes Euclidean distance. Let  $\tilde{\psi} = (\tilde{\psi}_{ij})$ ,  $1 \leq i, j \leq r$  be the matrix whose entries in  $\mathbb{Z}[\mathbb{Z}^{2n}]$  are

$$(1) \quad \tilde{\psi}_{ij} = \begin{cases} 0, & \text{if } \psi_{ij} = 0; \\ \psi_{ij} g_{ij}, & \text{if } \psi_{ij} \neq 0. \end{cases}$$

The quadratic  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module  $(\tilde{K}, \tilde{\psi})$  represents  $\mathfrak{F}(\sigma) \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ . Likewise, quadratic  $\mathbb{Z}$ -Poincaré  $\epsilon$ -bordisms over  $T^{2n}$  induce quadratic  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -Poincaré bordisms.

### 10.2 Controlled $E_8$ over $T^{2n}$

Starting with the  $(-1)^n$ -quadratic  $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module  $E_8 \times T^{2n}$ , pass to a large covering space  $p: T^{2n} \rightarrow T^{2n}$  to obtain a  $\delta$ -controlled quadratic  $\mathbb{Z}$ -module  $\tilde{E}_8$  over  $T^{2n}$  representing an element of  $L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta)$ . It is simple to verify that  $\mathfrak{F}(\tilde{E}_8) = p^!(E_8 \times T^{2n})$ , where  $p^!$  is the  $L$ -theory transfer. The transfer invariance results discussed in Section 9 imply that  $\mathfrak{F}(\tilde{E}_8) = E_8 \times T^{2n}$ . Thus,  $\tilde{E}_8$  gives a  $\delta$ -controlled realization of the form  $E_8$  over  $T^{2n}$ .

### 10.3 Controlled surgery obstructions

**Definition 10.2** Let  $p: X \rightarrow B$  be a map to a metric space  $B$  and  $\epsilon > 0$ . A map  $f: Y \rightarrow X$  is an  $\epsilon$ -homotopy equivalence over  $B$ , if there exist a map  $g: X \rightarrow Y$  and homotopies  $H_t$  from  $g \circ f$  to  $1_Y$  and  $K_t$  from  $f \circ g$  to  $1_X$  such that  $\text{diam}(p \circ f \circ H_t(y)) < \epsilon$  for every  $y \in Y$ , and  $\text{diam}(p \circ K_t(x)) < \epsilon$ , for every  $x \in X$ . This means that the tracks of  $H$  and  $K$  are  $\epsilon$ -small as viewed from  $B$ .

Controlled surgery theory addresses the question of the existence and uniqueness of controlled manifold structures on a space. Polyhedra homotopy equivalent to compact topological manifolds satisfy the Poincaré duality isomorphism. Likewise, there is a notion of  $\epsilon$ -Poincaré duality satisfied by polyhedra finely equivalent to a manifold. Poincaré duality can be estimated by the diameter of cap product with a fundamental class as a chain homotopy equivalence.

**Definition 10.3** Let  $p: X \rightarrow B$  be a map, where  $X$  is a polyhedron and  $B$  is a metric space.  $X$  is an  $\epsilon$ -Poincaré complex of formal dimension  $n$  over  $B$  if there exist a subdivision of  $X$  such that simplices have diameter  $\ll \epsilon$  in  $B$  and an  $n$ -cycle  $y$  in the simplicial chains of  $X$  so that  $\cap y: C^\#(X) \rightarrow C_{n-\#}(X)$  is an  $\epsilon$ -chain homotopy equivalence in the sense that  $\cap y$  and the chain homotopies have the property that the image of each generator  $\sigma$  only involves generators whose images under  $p$  are within an  $\epsilon$ -neighborhood of  $p(\sigma)$  in  $B$ .

To formulate simply-connected controlled surgery problems, the notion of locally trivial fundamental group from the viewpoint of the control space is needed. This can be formalized using the notion of  $UV^1$  maps as follows.

**Definition 10.4** Given  $\delta > 0$ , a map  $p: X \rightarrow B$  is called  $\delta$ - $UV^1$  if for any polyhedral pair  $(P, Q)$ , with  $\dim(P) \leq 2$ , and maps  $\alpha_0: Q \rightarrow X$  and  $\beta: P \rightarrow B$  such that  $p \circ \alpha_0 = \beta|_Q$ ,

$$\begin{array}{ccc} Q_i & \xrightarrow{\alpha_0} & X \\ \downarrow & \nearrow \alpha & \downarrow p \\ P_\beta & \longrightarrow & B \end{array}$$

there is a map  $\alpha: P \rightarrow X$  extending  $\alpha_0$  so that  $p \circ \alpha$  is  $\delta$ -homotopic to  $\beta$  over  $B$ . The map  $p$  is  $UV^1$  if it is  $\delta$ - $UV^1$ , for every  $\delta > 0$ .

Let  $B$  be a compact metric ENR and  $n \geq 5$ . Given  $\epsilon > 0$ , there is a  $\delta > 0$  such that if  $p: X \rightarrow B$  is a  $\delta$ -Poincaré duality space over  $B$  of formal dimension  $n$ ,  $(f, b): M^n \rightarrow X$  is a surgery problem, and  $p$  is  $\delta$ - $UV^1$ . By the Stability Theorem 10.1 there is a well-defined surgery obstruction

$$\sigma_*(f, b) \in \lim_{\leftarrow \epsilon} \lim_{\leftarrow \delta} L_n(B; \mathbb{Z}, \epsilon, \delta) = H_n(B; \mathbb{L})$$

such that  $(f, b)$  is normally cobordant to an  $\epsilon$ -homotopy equivalence for any  $\epsilon > 0$  if and only if  $\sigma_*(f, b) = 0$ . See Ranicki and Yamasaki [23] for an exposition of controlled  $L$ -theory.

The main theorem of Pedersen–Quinn–Ranicki [15] is the following controlled surgery exact sequence (see also Ferry [10] and Ranicki–Yamasaki [23]).

**Theorem 10.5** *Suppose  $B$  is a compact metric ENR and  $n \geq 4$ . There is a stability threshold  $\epsilon_0 > 0$  such that for any  $0 < \epsilon < \epsilon_0$ , there is  $\delta > 0$  with the property that if  $p: N \rightarrow B$  is a  $\delta$ - $UV^1$  map, with  $N$  is a compact  $n$ -manifold, there is an exact sequence*

$$H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\epsilon, \delta}(N, f) \rightarrow [N, \partial N; G/TOP, *] \rightarrow H_n(B; \mathbb{L}).$$

Here,  $\mathcal{S}_{\epsilon, \delta}$  is the controlled structure set defined as the set of equivalence classes of pairs  $(M, g)$ , where  $M$  is a topological manifold and  $g: (M, \partial M) \rightarrow (N, \partial N)$  restricts to a homeomorphism on  $\partial N$  and is a  $\delta$ -homotopy equivalence relative to the boundary. The pairs  $(M_1, g_1)$  and  $(M_2, g_2)$  are equivalent if there is a homeomorphism  $h: M_1 \rightarrow M_2$  such that  $g_1$  and  $h \circ g_2$  are  $\epsilon$ -homotopic rel boundary. As in classical surgery, the map  $H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\epsilon, \delta}(N, f)$  is defined using controlled Wall realization.

## 11 Exotic homology manifolds

In [1], exotic ENR homology manifolds of dimensions greater than 5 are constructed as limits of sequences of controlled Poincaré complexes  $\{X_i, i \geq 0\}$ . These complexes are related by maps  $p_i: X_{i+1} \rightarrow X_i$  such that  $X_{i+1}$  is  $\epsilon_{i+1}$ -Poincaré over  $X_i$ ,  $i \geq 0$ , and  $p_i$  is an  $\epsilon_i$ -homotopy equivalence over  $X_{i-1}$ ,  $i \geq 1$ , where  $\sum \epsilon_i < \infty$ . Beginning, say, with a closed manifold  $X_0$ , the sequence  $\{X_i\}$  is constructed iteratively using cut-paste constructions on closed manifolds. The gluing maps are obtained using the Wall realization of controlled surgery obstructions, which emerge as a non-trivial local index in the limiting ENR homology manifold. As pointed out in the Introduction, our main goal is to give an explicit construction of the first controlled stage  $X_1$  of this construction using the quadratic form  $E_8$ , beginning with the  $2n$ -dimensional torus  $X_0 = T^{2n}$ ,  $n \geq 3$ . The construction of subsequent stages follows from fairly general arguments presented in [1] and leads to an index-9 ENR homology manifold not homotopy equivalent to any closed topological manifold. Since an explicit algebraic description of the controlled quadratic module  $\tilde{E}_8$  over  $T^{2n}$  has already been given in Section 10.2, we conclude the paper with a review of how this quadratic module can be used to construct  $X_1$ .

Let  $P$  be the 2-skeleton of a fine triangulation of  $T^{2n}$ , and  $C$  a regular neighborhood of  $P$  in  $T^{2n}$ . The closure of the complement of  $C$  in  $T^{2n}$  will be denoted  $D$ , and the common boundary  $N = \partial C = \partial D$  (see Figure 1). Given  $\delta > 0$ , we may assume

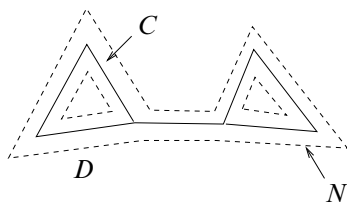


Figure 1

that the inclusions of  $C$ ,  $D$  and  $N$  into  $T^{2n}$  are all  $\delta$ - $UV^1$  by taking a fine enough triangulation.

Let  $(K, \varphi)$  be a geometric  $\mathbb{Z}$ -module over  $T^{2n}$  representing the controlled quadratic form  $\widetilde{E}_8$ , where  $K \cong \mathbb{Z}^r$  is a free  $\mathbb{Z}$ -module with basis  $S = \{e_1, \dots, e_r\}$  and  $\varphi: S \rightarrow T^{2n}$  is a map. If  $Q \subset T^{2n}$  is the dual complex of  $P$ , after a small perturbation, we can assume that  $\varphi(S) \cap (P \cup Q) = \emptyset$ . Composing this deformation with a retraction  $T^{2n} \setminus (P \cup Q) \rightarrow N$ , we can assume that  $\varphi$  factors through  $N$ , that is, the geometric module is actually realized over  $N$ .

Using a controlled analogue of the Wall Realization Theorem [25, Theorem 5.8] applied to the identity map of  $N$ , realize this quadratic module over  $N \subset T^{2n}$  to obtain a degree-one normal map  $F: (V, N, N') \rightarrow (N \times I, N \times \{0\}, N \times \{1\})$  satisfying:

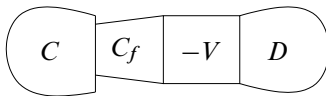
- (a)  $F|_N = 1_N$ .
- (b)  $f = F|_{N'}: N' \rightarrow N$  is a fine homotopy equivalence over  $T^{2n}$ .
- (c) The controlled surgery obstruction of  $F$  rel  $\partial$  over  $T^{2n}$  is  $\widetilde{E}_8 \in H_{2n}(T^{2n}; \mathbb{L})$ .

The map  $F$  can be assumed to be  $\delta$ - $UV^1$  using controlled analogues of  $UV^1$  deformation results of Bestvina and Walsh [12].

Let  $C_f$  be the mapping cylinder of  $f$ . Form a Poincaré complex  $X_1$  by pasting  $C_f \cup_{N'} (-V)$  into  $T^{2n}$  along  $N$ , that is,

$$X_1 = C \cup_N C_f \cup_{N'} (-V) \cup_N D,$$

as shown in Figure 2. Our next goal is to define the map  $p_1: X_1 \rightarrow X_0 = T^{2n}$ .

Figure 2: The Poincaré complex  $X_1$

Let  $g: N \rightarrow N'$  be a controlled homotopy inverse of  $f$ . Composing  $f$  and  $g$ , and using an estimated version of the Homotopy Extension Theorem (see e.g. [1]) and the controlled Bestvina–Walsh Theorem, one can modify  $F$  to a  $\delta$ - $UV^1$  map  $G: V \rightarrow C_g$ , so that  $G|_{N'} = 1_{N'}$  and  $G|_N = 1_N$ .

Let  $X'_1 = C \cup_N C_f \cup_{N'} C_g \cup_N D$  and  $p_1^*: X_1 \rightarrow X'_1$  be as indicated in Figure 3. Crushing  $C_f \cup_{N'} C_g$  to  $N = \partial C$ , we obtain the desired map  $p_1: X_1 \rightarrow T^{2n} = C \cup_N D$ .

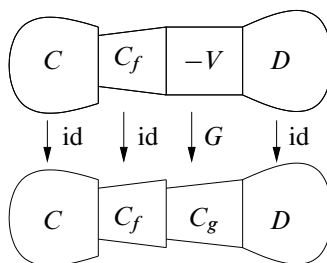


Figure 3: The map  $p_1^*: X_1 \rightarrow X'_1$

To conclude, as in [2], we argue that  $X_1$  is not homotopy equivalent to any closed topological manifold. To see this, consider the closed manifold

$$M = C \cup_N V \cup_{N'} N' \times I \cup_{N'} (-V) \cup_N D$$

and the degree-one normal map  $\phi: M \rightarrow X_1$  depicted in Figure 4, where  $\pi: N' \times I \rightarrow C_f$  is induced by  $f: N' \rightarrow N$ . The controlled surgery obstruction of  $\phi$  over  $T^{2n}$  is the generator

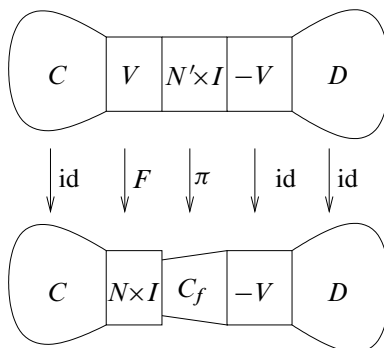
$$\begin{aligned} \sigma_*(\phi) &= E_8 \times T^{2n} = (0, \dots, 0, 1) \in L_0(\mathbb{Z}) = L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])^{INV} \\ &\subset H_{2n}(T^{2n}; \mathbb{L}) = L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = \sum_{r=0}^{2n} \binom{2n}{r} L_{2n-r}(\mathbb{Z}) \end{aligned}$$

of the subgroup of the transfer invariant elements (Proposition 9.1). Let  $\mathbb{L}\langle 1 \rangle$  be the 1–connective cover of  $\mathbb{L}$ , the simply-connected surgery spectrum with 0th space (homotopy equivalent to)  $G/TOP$ . Now

$$L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = H_{2n}(T^{2n}; \mathbb{L}) = H_{2n}(T^{2n}; \mathbb{L}\langle 1 \rangle) \oplus L_0(\mathbb{Z})$$

with

$$H_{2n}(T^{2n}; \mathbb{L}\langle 1 \rangle) = [T^{2n}, G/TOP] = \sum_{r=1}^{2n} \binom{2n}{r} L_{2n-r}(\mathbb{Z}) \subset L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

Figure 4: The map  $\phi: M \rightarrow X_1$ 

the subgroup of the surgery obstructions of normal maps  $M_1 \rightarrow T^{2n}$ . The surgery obstruction of any normal map  $\phi_1: M_1 \rightarrow X_1$  is of the type

$$\sigma_*(\phi_1) = (\tau, 1) \neq 0 \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = [T^{2n}, G/TOP] \oplus L_0(\mathbb{Z})$$

for some  $\tau \in [T^{2n}, G/TOP]$ , since the variation of normal invariant only changes the component of the surgery obstruction in  $[T^{2n}, G/TOP] \subset L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ . Thus,  $X_1$  is not homotopy equivalent to any topological manifold. In the terminology of Ranicki [20, Chapter 17] the total surgery obstruction  $s(X_1) \in \mathfrak{S}_{2n}(X_1)$  has image

$$(p_1)_*s(X_1) = 1 \in \mathfrak{S}_{2n}(T^{2n}) = L_0(\mathbb{Z}).$$

The Bryant–Ferry–Mio–Weinberger procedure for constructing an ENR homology manifold starting with  $p_1: X_1 \rightarrow T^{2n}$  leads to a homology manifold homotopy equivalent to  $X_1$ . Thus, from the quadratic form  $E_8$ , we obtained a compact index–9 ENR homology  $2n$ –manifold  $\mathfrak{X}_8$  which is not homotopy equivalent to any closed topological manifold.

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### References

- [1] **J Bryant, S Ferry, W Mio, S Weinberger**, *Topology of homology manifolds*, Bull. Amer. Math. Soc. (N.S.) 28 (1993) 324–328 MR1183997
- [2] **J Bryant, S Ferry, W Mio, S Weinberger**, *Topology of homology manifolds*, Ann. of Math. (2) 143 (1996) 435–467 MR1394965



- [3] **J W Cannon**, *The recognition problem: what is a topological manifold?*, Bull. Amer. Math. Soc. 84 (1978) 832–866 MR0494113
- [4] **J W Cannon**, *Shrinking cell-like decompositions of manifolds. Codimension three*, Ann. of Math. (2) 110 (1979) 83–112 MR541330
- [5] **F Clauwens**, *The  $K$ -theory of almost symmetric forms*, from: “Topological structures, II (Proc. Sympos. Topology and Geom., Amsterdam, 1978), Part 1”, Math. Centre Tracts 115, Math. Centrum, Amsterdam (1979) 41–49 MR565824
- [6] **F Clauwens**, *Product formulae for surgery obstructions*, from: “Algebraic topology, Aarhus 1978”, Lecture Notes in Math. 763, Springer, Berlin (1979) 198–211 MR561223
- [7] **F Clauwens**, *Surgery on products. I, II*, Nederl. Akad. Wetensch. Indag. Math. 41 (1979) 121–132, 133–144 MR535561
- [8] **R J Daverman**, *Decompositions of manifolds*, Pure and Applied Mathematics 124, Academic Press Inc., Orlando, FL (1986) MR872468
- [9] **R D Edwards**, *The topology of manifolds and cell-like maps*, from: “Proceedings of the International Congress of Mathematicians (Helsinki, 1978)”, Acad. Sci. Fennica, Helsinki (1980) 111–127 MR562601
- [10] **S C Ferry**, *Epsilon-Delta surgery over  $\mathbb{Z}$* , preprint, Rutgers University (2003)
- [11] **M Gromov**, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, from: “Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)”, Progr. Math. 132, Birkhäuser, Boston (1996) 1–213 MR1389019
- [12] **K Kawamura**, *An inverse system approach to Menger manifolds*, Topology Appl. 61 (1995) 281–292 MR1317082
- [13] **R J Milgram, A A Ranicki**, *The  $L$ -theory of Laurent extensions and genus 0 function fields*, J. Reine Angew. Math. 406 (1990) 121–166 MR1048238
- [14] **W Mio**, *Homology manifolds*, from: “Surveys on surgery theory, Vol. 1”, Ann. of Math. Stud. 145, Princeton Univ. Press, Princeton, NJ (2000) 323–343 MR1747540
- [15] **E K Pedersen, F Quinn, A Ranicki**, *Controlled surgery with trivial local fundamental groups*, from: “High-dimensional manifold topology”, World Sci. Publishing, River Edge, NJ (2003) 421–426 MR2048731
- [16] **E Pedersen, M Yamasaki**, *Stability in controlled  $L$ -theory*, from: “Exotic homology manifolds (Oberwolfach 2003)”, Geom. Topol. Monogr. 9 (2006) 69–88 arXiv: math.GT/0402218
- [17] **F Quinn**, *Resolutions of homology manifolds, and the topological characterization of manifolds*, Invent. Math. 72 (1983) 267–284 MR700771
- [18] **F Quinn**, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. 34 (1987) 285–291 MR894878

- [19] **A Ranicki**, *The algebraic theory of surgery. I. Foundations*, Proc. London Math. Soc. (3) 40 (1980) 87–192 MR560997
- [20] **A A Ranicki**, *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics 102, Cambridge University Press, Cambridge (1992) MR1211640
- [21] **A Ranicki**, *Lower K- and L-theory*, London Mathematical Society Lecture Note Series 178, Cambridge University Press, Cambridge (1992) MR1208729
- [22] **A Ranicki**, *High-dimensional knot theory*, Springer Monographs in Mathematics, Springer, New York (1998) MR1713074
- [23] **A Ranicki, M Yamasaki**, *Controlled L-theory*, from: “Exotic homology manifolds (Oberwolfach 2003)”, Geom. Topol. Monogr. 9 (2006) 107–156 arXiv: math.GT/0402217
- [24] **J L Shaneson**, *Wall’s surgery obstruction groups for  $G \times Z$* , Ann. of Math. (2) 90 (1969) 296–334 MR0246310
- [25] **C T C Wall**, *Surgery on compact manifolds*, Mathematical Surveys and Monographs 69, American Mathematical Society, Providence, RI (1999) MR1687388
- [26] **S Weinberger**, *Nonlocally linear manifolds and orbifolds*, from: “Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)”, Birkhäuser, Basel (1995) 637–647 MR1403964
- [27] **M Yamasaki**, *L-groups of crystallographic groups*, Invent. Math. 88 (1987) 571–602 MR884801

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