Problems on homology manifolds

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In these notes "homology manifold" means ENR (Euclidean neighborhood retract) $\mathbb{Z}$–coefficient homology manifold, unless otherwise specified, and "exotic" means not a manifold factor (ie local or "Quinn" index $\neq 1$. We use the multiplicative version of the local index, taking values in $1 + 8\mathbb{Z}$). In the last decade exotic homology manifolds have been shown to exist and quite a bit of structure theory has been developed. However they have not yet appeared in other areas of mathematics. The first groups of questions suggest ways this might happen. Later questions are more internal to the subject.

Section 1, Section 2, and Section 3 concern possible "natural" appearances of homology manifolds: as aspherical geometric objects; as Gromov–Hausdorff limits; and as boundaries of compactifications. Section 4 discusses group actions, where the use of homology manifold fixed sets may give simpler classification results. Section 5 and Section 6 consider possible generalizations to non-ANR and "approximate" homology manifolds. Section 7 concerns spaces with special metric structures. Section 8 describes still-open low dimensional cases of the current theory. Section 9 collects problems related to homeomorphisms and the "disjoint disk property" for exotic homology manifolds.

1   Aspherical homology manifolds

Geometric structures on aspherical spaces seem to be rigid. The "Borel conjecture" is that closed aspherical manifolds are determined up to homeomorphism by their fundamental groups, and this has been verified in many cases, see Farrell [21] for a survey. More generally it is expected that aspherical homology manifolds should be determined up to s-cobordism by their fundamental groups, so in particular the fundamental group should determine the local index. So far, however, there are no exotic examples.

Problem 1.1   *Is there a closed aspherical homology manifold with local index $\neq 1$?*

If so then exotic homology manifolds would be required for a full analysis of the aspherical question. See Section 6 for a "approximate" version of the problem.

2   Gromov–Hausdorff Limits

Differential geometers have investigated limits of Riemannian manifolds with various curvature and other constraints. As constraints are weakened one sees

(1) first, smooth manifold limits diffeomorphic to nearby manifolds (Anderson–Cheeger [2], Petersen [36]);

(2) next, topological manifold limits homeomorphic to nearby manifolds (Grove–Peterson–Wu [27]);

(3) then topologically stratified limits (Perel'man [33; 34], Perel'man–Petrunin [35]); and finally

(4) more-singular limits currently not good for much.

Limits in the homeomorphism case (2) were first only known to be homology manifolds, and nearby manifolds were analyzed using controlled topology. However Perel'man [33] later used the Alexandroff curvature structure to show the limits in (2) are in fact manifolds, and extended this to stratifications with manifold strata in some singular cases (3). This considerably simplified the analysis and removed the need for homology manifolds. An approach to singular cases using Ricci curvature is given by Cheeger–Colding [11], see also Zhu [48]. We might still hope for a role for the more sophisticated topology:

Problem 2.1   *Are there differential-geometric conditions or processes that give exotic homology manifold limits?*

Such conditions must involve something other than diameter, volume, and sectional curvature bounds. Exotic ENR homology manifolds cannot arise this way so the most interesting outcome would be to get infinite-dimensional limits. An analysis of manifolds near such limits has been announced by Dranishnikov and Ferry and apparently these can vary quite a lot, see also Section 5.

2.1   Stratified Gromov–Hausdorff limits

The most immediately promising problems about limits concern stratifications.

Problem 2.2 *Are there differential-geometric conditions on smooth stratified sets that imply Gromov–Hausdorff limits are homotopy stratified sets with homology-manifold strata? What is the structure of the nearby smooth stratified sets?*

There are two phenomena here: "collapse" in which new strata are generated, and convergence that is in some sense stratum-wise. The first case is typified by "volume collapse" of manifolds to stratified sets. The Cheeger–Fukaya–Gromov structure theory of collapsed manifolds suggests the nearby manifolds should be total spaces of stratified systems of fibrations with nilmanifold fibers. Note however this structure should only be topological: smooth structures on the limit and bundles are unlikely in general. In cases where curvature is bounded below Perel'man's analysis of the Alexandroff structure of the limit gives a topologically stratified space, solving the first part of the problem.

In the second case (stratum-wise convergence) the given smooth stratifications need not converge, but some sort of "homotopy intrinsic" stratifications should converge. More detail and an elaborate proposal for the topological part of this question is given in Quinn [39]. In this case if strata in the limits are ENR homology manifolds then one expects nearby stratified sets to be stratified s-cobordant. Again cases where Perel'man's Alexandroff-space results apply should be much more accessible.

The motivation for this question is to study compactifications of collections of algebraic varieties or of stratifications arising in singularity theory. Therefore to be useful the "differential-geometric" hypotheses should have reasonable interpretations in these contexts. Other possibilities are to relate this to limits of special processes, eg the Ricci flow (Glickenstein [26]) or special limits defined by logical constraints (van den Dries [45]).

3   Compactifications

Negatively curved spaces and groups (in the sense of Gromov) have compactifications with "boundaries" defined by equivalence classes of geodesics. "Hyperbolization" procedures that mass-produce examples are described by Davis–Januszkiewicz [17], Davis–Januszkiewicz–Weinberger [18] and Charney–Davis [10]. "Visible boundaries" can be defined for nonpositively curved spaces using additional geometric information. Bestvina [3] has given axioms for compactifications and shown compactifications of Poincaré duality groups satisfying his axioms give homology manifolds.

In classical cases the space on which the group acts is homeomorphic to Euclidean space, the boundary is a sphere compactifying the space to a disk. Behavior of limits

in the sphere are more interesting than the sphere itself. More interesting examples arise with "Davis" manifolds: contractible nonpositively curved manifolds not simply connected at infinity. Fischer [23] shows that a class of these have boundaries that are

 (1) finite dimensional cohomology manifolds with the (Čech) homology of a sphere;

 (2) not locally 1–connected (so not ENR);

 (3) homogeneous, and

 (4) the double of the compactification along the boundary is a genuine sphere.

This connects nicely with the non-ENR questions raised in Section 5. However it seems unlikely that interesting ENR examples will arise this way: boundaries can be exotic only if the input space is exotic, for example the universal cover of an exotic closed aspherical manifold as in Section 1, and this is probably not compatible with nonpositive curvature assumptions, see Section 7.

To get more exotic behavior probably will require going outside the nonpositive curvature realm:

Problem 3.1 *Find non-curvature constructions for limits at infinity of Poincaré duality groups, and find (or verify) criteria for these to be homology manifolds.*

See Davis [16] for a survey of Poincaré duality groups. This question may provide an approach to closed aspherical exotic homology manifolds: first construct the "sphere at infinity" of the universal cover, then somehow fill in.

A variation on this idea is suggested by a proof of cases of the Novikov conjecture by Farrell–Hsiang [22] and many others since. They use a compactification of the universal cover to construct a fiberwise compactification of the tangent bundle. This suggests directly constructing completions of a bundle rather than a single fiber. The bundle might include parameters, for instance to resolve ambiguities arising in constructing limits without negative curvature. The context for this is discussed in Section 6.

Problem 3.2 *Construct "approximate" limits of duality groups, as "fibers" of the total space of an approximate fibration over a parameter space.*

4   Group actions and non–$\mathbb{Z}$ coefficients

This topic probably has the greatest potential for profound applications, but also has severe technical difficulty. Smith theory shows fixed sets of actions of p–groups on

homology manifolds must be $\mathbb{Z}/p\mathbb{Z}$ homology manifolds. In the PL case there is a remarkable near converse: Jones [31] shows PL $\mathbb{Z}/p\mathbb{Z}$ homology submanifolds satisfying the Smith conditions are frequently fixed sets of a $\mathbb{Z}/p\mathbb{Z}$ action. Better results are likely for topological actions.

Problem 4.1 *Extend the Jones analysis to determine when 1–LC embedded $\mathbb{Z}/p\mathbb{Z}$ homology submanifolds are fixed sets of $\mathbb{Z}/p\mathbb{Z}$ actions.*

If the submanifold is an ENR then there are tools available (eg mapping cylinder neighborhoods) that should bring this within reach. Unfortunately the non-ENR case is likely to be the one with powerful applications. As a test case we formulate a stable version of Problem 4.1 in which some difficulties should be avoided:

Problem 4.2 *Suppose $X \subset R^n$ is an even-codimension properly embedded possibly non-ENR $\mathbb{Z}/p\mathbb{Z}$ homology manifold and is $\mathbb{Z}/p\mathbb{Z}$ acyclic. Is there a $\mathbb{Z}/p\mathbb{Z}$ action on R^{n+2k} for some k with fixed set $X \times \{0\}$?*

Extending Problem 4.1 to a systematic classification theory for topological group actions will require a good understanding of the corresponding homology manifolds:

Problem 4.3 *Are there "surgery theories" for $\mathbb{Z}/n\mathbb{Z}$ and rational homology manifolds?*

Surgery for PL manifolds up to $\mathbb{Z}/p\mathbb{Z}$ homology equivalence was developed in the 1970s by Quinn [37], Anderson [1], Taylor–Williams [43], and a speculative sketch for PL $\mathbb{Z}/p\mathbb{Z}$ homology manifolds is given in Quinn [37]. There are two serious difficulties for a topological version. The first problem is that the local and "normal" structures do not decouple. The boundary of a regular neighborhood in Euclidean space is the appropriate model for the Spanier–Whitehead dual of a space. $\mathbb{Z}$–Poincaré spaces are characterized by this neighborhood being equivalent to a spherical fibration over the space. When manipulating a Poincaré space within its homotopy type (eg while constructing homology manifolds) the bundle gives easy and controlled access to the Spanier–Whitehead dual. $\mathbb{Z}/p\mathbb{Z}$ Poincaré spaces have regular neighborhoods that are $\mathbb{Z}/n\mathbb{Z}$ spherical fibrations, but this only specifies the $\mathbb{Z}/p\mathbb{Z}$ homotopy type. Local structure at other primes can vary from place to place, and the normal structure must conform to this. Some additional structure is probably needed, but this is unclear.

The second problem is that constructions of $\mathbb{Z}/p\mathbb{Z}$ homology manifolds are unlikely to give ENRs. In the $\mathbb{Z}$ case ENRs are obtained as limits of sequences of controlled homotopy equivalences. Homotopy equivalences are obtained because obstructions to constructing these can be identified with global data (essentially the topological

structure on the normal bundle). In the $\mathbb{Z}/p\mathbb{Z}$ case there will probably only be enough data to get controlled $\mathbb{Z}/p\mathbb{Z}$ homology equivalences. It seems likely that n–dimensional homology manifolds can be arranged to have covering dimension n and have nice $\left[\frac{n-1}{2}\right]$ skeleta, but above the middle dimension infinitely generated homology prime to p is likely to be common.

$\mathbb{Z}/p\mathbb{Z}$ homology manifolds might geometrically implement some of the remarkable but formal "p–complete" manifold theory proposed in Sullivan [42].

Dranishnikov [20] gives constructions of rational homology 5–manifolds with large but still finite covering and cohomological dimension. If this complicates the development it may be appropriate to consider only homology manifolds with covering dimension equal to the duality dimension.

4.1 Circle actions

A group action is "semifree" if points are either fixed by the whole group or moved freely. In this case the fixed set is also the fixed set of the $\mathbb{Z}/p\mathbb{Z}$ subgroups, all p, so it follows from Smith theory that it is a $\mathbb{Z}$ homology manifold. Problems 4.1 and 4.2 therefore have analogs for semifree S^1 actions and $\mathbb{Z}$–coefficient homology manifolds:

Problem 4.4 *Determine when $\mathbb{Z}$ coefficient homology submanifolds satisfying Smith conditions are fixed sets of semifree S^1 actions.*

This setting has the advantages that there are fewer obstructions, and in the ENR case the analog of Problem 4.3 is already available. Again the significance of non-ENR case depends on how many non-ENR homology manifolds there are (see Section 5). If they all occur as boundaries with ENR interior then it seems likely a general action will be concordant to one with ENR fixed set. At the other extreme if there are fixed sets with non-integer local index then a full treatment of group actions will probably need non-ENRs.

5 Non-ENR homology manifolds

It is a folk theorem that a homology manifold that is finite dimensional and locally 1–connected is an ENR. The proof goes as follows: duality shows homology manifolds are homologically locally n–connected, all n, and a local Whitehead theorem shows local 1–connected and homological local n–connected implies local n–connected in the usual homotopic sense. Finally finite-dimensional and locally n–connected for large n implies ENR. The point here is that the ENR condition can fail in two ways: failure of finite dimensionality or failure of local 1–connectedness. These are discussed separately.

5.1 Infinite dimensional homology manifolds

Here we consider locally compact metric spaces that are locally contractible (or at least locally 1–connected) and homology manifolds in the usual finite-dimensional sense, but with infinite covering dimension. This is not related to manifolds modeled on infinite dimensional spaces.

Problem 5.1 *Is there a "surgery theory" of infinite dimensional homology manifolds?*

These were shown to arise as cell-like images of manifolds by Dranishnikov [19], following a proposal of Edwards. Recently Dranishnikov and Ferry have announced that there are examples with arbitrarily close (in the Gromov–Hausdorff sense) topological manifolds with different homotopy types. This contrasts with the ENR case where sufficiently close manifolds are all homeomorphic, and suggests this is a way to loosen the strait-jacket constraints of homotopy type in standard surgery. In particular the "surgery theory" should *not* follow the usual pattern of fixing a homotopy type, and "structures" should include manifolds of different homotopy type. This might be done by following Dranishnikov–Ferry in assuming existence of metrics that are sufficiently Gromov–Hausdorff close. See Problem 3.1.

The source dimension for infinite-dimensionality is not quite settled:

Problem 5.2 *Are there infinite-dimensional $\mathbb{Z}$–homology 4–manifolds? Are there infinite-dimensional homology 4– or 5–manifolds with nearby manifolds of different homotopy type?*

Walsh [46] has shown homology manifolds of homological dimension ≤ 3 are finite dimensional. Dydak–Walsh [15] produced infinite-dimensional examples of homology 5–manifolds but these do not connect with the Dranishnikov–Ferry analysis. It may be that an interesting "surgery theory" does not start until dimension 6.

5.2 Non locally–1–connected homology manifolds

Now consider finite dimensional metric homology manifolds that may fail to be locally 1–connected. These arise as "spheres at infinity" for certain groups, see Section 3. The first question seeks to locate these spaces relative to ENR and "virtual" homology manifolds. This is important for applications to group actions, see Problem 4.4.

Problem 5.3 *Extend the definition of local index to finite dimensional non-ENR homology manifolds*

If the extended definition still takes values in $1 + 8\mathbb{Z}$ then the next question (motivated by Section 3) would be:

Problem 5.4 *Does every finite dimensional homology manifold arise as the "weakly tame" boundary of one with ENR interior? Is the union of two such extensions along their boundary an ENR?*

Here "weakly tame" should be as close to "locally 1–connected complement" as possible. An affirmative answer to this question would suggest thinking of non-ENR homology manifolds as "puffed up" versions not much different from ENRs.

At another extreme "approximate" homology manifolds are defined in section 6 in terms of approximate fibrations with homology manifold base and total space. These behave as though they have "fibers" that are homology manifolds with local indices in $1 + 8\mathbb{Z}_{(2)}$. If an extension of the local index to non ENRs can take non-integer values then the ENR boundary question above is wrong and we should ask:

Problem 5.5 *Does every finite dimensional homology manifold occur as a fiber of an approximate fibration with ENR homology manifold base and total space? Conversely is any such approximate fibration concordant to one with such a fiber?*

An appropriate relative version of this would show approximate homology manifolds are equivalent to finite dimensional non-ENR homology manifolds.

6 Approximate homology manifolds

The intent is that approximate homology manifolds should be fibers of approximate fibrations with base and total space ENR homology manifolds. Actual point-inverses are not topologically well-defined and do not encode the interesting information, so we use a germ approach. For simplicity we restrict to the compact case (fibers of proper maps).

A *compact approximate homology manifold* is a pair $(f\colon E \to B, b)$ where f is a proper approximate fibration with homology manifold base and total space, and b is a point in B. "Concordance" is the equivalence relation generated by

(1) changing the basepoint by an arc in the base;

(2) restricting to a neighborhood of the basepoint; and

(3) product with identity maps of homology manifolds.

6.1 Basic structure

Suppose F is a compact virtual homology manifold defined by an proper approximate fibration $f\colon E \to B$ with E, B connected homology manifolds, and $b \in B$.

(1) F has a well-defined homotopy type (the homotopy fiber of the map) that is a Poincaré space (with universal coefficients);

(2) this Poincaré space has a canonical topological reduction of the normal fibration, given by restriction of the difference of the canonical reductions of E and B; and

(3) there is a local index defined by $i(F) = i(E)/i(B)$.

If f is a locally trivial bundle then the fiber is a ENR homology manifold. Multiplicativity of the local index shows the formula in (3) does give the local index of the fiber in this case. In general the quotients in (3) lie in $1 + 8\mathbb{Z}_{(2)}$, where $\mathbb{Z}_{(2)}$ is the rationals with odd denominator.

6.2 Example

Suppose X is a homology manifold, and choose a 1–LC embedding in a manifold of dimension at least 5. This has a mapping cylinder neighborhood; let $f\colon \partial N \to X$ be the map. Duality shows this is an approximate fibration with fiber the homotopy type of a sphere. As an approximate homology manifold the index is $1/i(X)$, which is not an integer unless $i(X) = 1$.

Products of these examples with genuine homology manifolds show that all elements of $1 + 8\mathbb{Z}_{(2)}$ are realized as indices of approximate homology manifolds.

6.3 Example

A tame end of a manifold has an "approximate collar" in the sense of a neighborhood of the end that approximately fibers over R. In the controlled case the local fundamental group is required to be stratified; see Hughes [28] for a special case and Quinn [40] in general.

There is a finiteness obstruction to finding a genuine collar. In some cases it follows that the approximate homology manifold appearing as the "fiber" of the approximate collar does not have the homotopy type of a finite complex.

Problem 6.1 *Show that the exotic behavior of the examples are the only differences: if an approximate homology manifold has integral local index and is homotopy equivalent to a finite complex then it is concordant to an ENR homology manifold.*

Problem 6.2 *Define "approximate transversality" of homology manifolds, and deter-mine when a map from one homology manifold to another can be made approximate transverse to a submanifold.*

The examples give maps approximately transverse to a point, but for which more geometric forms of transversality are obstructed. Transversality theories restricted to situations where indices must be integers have been developed by Johnston [29], Johnston–Ranicki [30] and Bryant–Mio [8], and a finiteness-obstruction case has been investigated by Bryant–Kirby [7]. The hope is that a more complete approximate transversality theory is possible. There still will be restrictions however: a degree–1 map of homology manifolds of different index cannot be made geometrically transverse to a point in any useful sense.

Problem 6.3 *Develop a surgery theory for approximate homology manifolds.*

The obstructions should lie in the $L^{-\infty}$ groups introduced by Yamasaki [47].

7   Special metric spaces

Several special classes of metric spaces have been developed, particularly by the Russian school, as general settings for some of the results of differential geometry. It is natural to ask how these hypotheses relate to manifolds and homology manifolds, but for the question to have much real significance it is necessary to have sources of examples not a priori known to be manifolds. Gromov–Hausdorff convergence gives Alexandroff spaces, see Section 1.

A Busemann space is a metric space in which geodesic (locally length-minimizing) segments can be extended, and small metric balls are cones parameterized by geodesics starting at the center point. The standard question is:

Problem 7.1 *Must a Busemann space be a manifold?*

This is true in dimensions ≤ 4; the 4–dimensional case was done by Thurston [44] and is not elementary.

An Alexandroff space is a metric space in which geodesics and curvature constraints make sense, but with less structure than Busemann spaces. These need not be homology manifolds, so the appropriate question seems to be:

Problem 7.2 *Is an Alexandroff space that is a homology manifold in fact a manifold in the complement of a discrete set?*

The problematic discrete set should be detectable by local fundamental groups of complements, as with the cone on a non-simply-connected homology sphere. The answer is "yes" when there is a lower curvature bound, because the analysis in Perel'man [33; 34] and Perel'man–Petrunin [35] shows it is a topological stratified set and topological stratified sets have this property (Quinn [38]).

8 Low dimensions

1– and 2–dimensional ENR homology manifolds are manifolds. In dimensions ≥ 5 exotic homology manifolds of arbitrary local index exist, and there are "many" of them in the senses that

- there is a "full surgery theory" given by Bryant–Ferry–Mio–Weinberger [6] for dimensions ≥ 6 and announced by Ferry–Johnston for dimension 5; and
- in dimensions ≥ 6 Bryant–Ferry–Mio–Weinberger have announced a proof that an arbitrary homology manifold is the cell-like image of one with the DDP.

The 5–dimensional case of (2) is still open:

Problem 8.1 *Can 5–dimensional exotic homology manifolds be resolved by ones with the DDP?*

In dimension 4:

Problem 8.2 *Are there exotic 4–dimensional homology manifolds?*

The expected answer is "yes." In a little more detail the possibilities seem to be:

(1) exotic homology manifolds don't exist; or

(2) sporadic examples exist; or

(3) there is a "full surgery theory".

Even in higher dimensions there are currently no methods for getting isolated examples: to get anything one essentially has to go through the full surgery theory. More-direct examples in higher dimensions would be useful in approaching (2) as well as interesting in their own right. In (3) note there is currently a fundamental group restriction in the manifold case Freedman–Quinn [24], Freedman–Teichner [25], Krushkal–Quinn [32]. Homology-manifold surgery would imply manifold surgery so "full surgery theory" should be interpreted to mean "as full as the manifold case."

Finally in dimension 3:

Problem 8.3 *Are there exotic 3–dimensional homology manifolds?*

The expected answer is "no". See Repovš [41] for special conclusions in the resolvable case.

9 Homeomorphisms and the DDP

The basic question is: do the key homeomorphism theorems for manifolds extend to homology manifolds? The question should include a nondegeneracy condition that gives manifolds in the index $= 1$ case. Here we use the DDP (Disjoint Disk Property): any two maps $i, j\colon D^2 \to X$ can be arbitrarily closely approximated by maps with disjoint images. However see Problem 9.6.

There is a feeling that the first three problems should be roughly equivalent in the sense that one good idea could resolve them all.

Problem 9.1 *Is the "α approximation theorem" of Chapman–Ferry [9] true for DDP homology manifolds?*

This expected answer is "yes", and current techniques suggest a proof might break into two sub-problems:

- A compact metric homology manifold X has $\epsilon > 0$ so if X' is ϵ homotopy equivalent to X then there is a DDP Y with cell-like maps onto both X and X'; and

- if X, X', or both, have the DDP then the corresponding cell-like maps can be chosen to be homeomorphisms.

As a testbed for technique for the second part it would be valuable to have a surgery-type proof of Edwards' theorem: a cell-like map from a (genuine) manifold to a homology manifold with DDP can be arbitrarily closely approximated by homeomorphisms.

Problem 9.2 *Is the h-cobordism theorem true for DDP homology manifolds?*

h-cobordisms appear in a natural way in the definition of "homology manifold structure sets", among other places, and can be produced by surgery.

Problem 9.3 *Is a homology manifold with the DDP arc-homogeneous?*

"Arc-homogeneous" means if x, y are in the same component of M then there is a homeomorphism $M \times I \to M \times I$ that is the identity on one end and on the other takes x to y. "Isotopy-homogeneous" is the sharper version in which the homeomorphism is required to preserve the I coordinate, so gives an ambient isotopy taking x to y. The "arc" in the terminology refers to the track of the point under the homeomorphism or isotopy.

An affirmative answer to 9.3 would show DDP homology manifolds have coordinate charts homeomorphic to subsets of standard models in the same way manifolds have Euclidean charts. Note this is consistent with a number of different models in each index: only one model could occur in a *connected* homology manifold but different components might have different models.

The "Bing–Borsuk conjecture" is that a homogeneous ENR is a manifold, where "homogeneous" is used in the traditional sense that any two points have homeomorphic neighborhoods. A version more in line with current expectations, and avoiding low dimensional problems, is that the homogeneous ENRs of dimension at least 5 are exactly the DDP homology manifolds. For applications and philosophical reasons we prefer arc versions of homogeneity, and split the question into Problem 9.3 and a homological question:

Problem 9.4 *Is a locally 1–connected homologically arc-homogeneous space a homology manifold (possibly infinite-dimensional)?*

A space is *homologically* arc-homogeneous if for any arc $f \colon I \to X$ the induced maps

$$H_i(X \times \{0\}, X \times \{0\} - (f(0), 0); \mathbb{Z}) \to H_i(X \times I, X \times I - \mathrm{graph}(f); \mathbb{Z})$$

is an isomorphism. This is clearly an analog of "arc-homogeneous" as defined above. A homology manifold satisfies this by Alexander duality. In fact it holds for $(I, 0)$ replaced by a n–disk and a point in the boundary.

It was shown by Bredon that homogeneous (in the traditional point sense) ENRs are homology manifolds *provided* the local homology groups are finitely generated, see Bryant [4], Dydak–Walsh [15]. The problem is to show the local homology groups form a locally constant sheaf. The arc version of homogeneity gives local isomorphisms so the problem becomes showing these are locally well-defined. This would follow immediately from a "homologically 2–disk-homogeneous" hypothesis, so is equivalent to this condition. The question is whether this follows from arc-homogeneity and local 1–connectedness. Bryant [5] has recently proved Problem 9.4 under the assumption that the space is an ENR. Note that a finite dimensional locally 1–connected homology

manifold is an ENR, so the question remaining is whether "ENR" can be shifted from hypothesis to conclusion.

In a somewhat different direction the following is still unknown even in the manifold case:

Problem 9.5 *Is the product of a homology manifold and R homogeneous?*

This can be disengaged from the homogeneity questions by asking "does $X \times R$ have DDP?", but see the discussion of the DDP in Problem 9.6. $X \times R^2$ does have DDP (Daverman [12] and there are quite a number of properties of X that imply $X \times R$ has DDP, see Halverson [28], Daverman–Halverson [14]. However there are ghastly (in the technical sense) examples of homology manifolds that show none of these properties holds in general, see Daverman–Walsh [15], Halverson [28].

The final question is vague but potentially important:

Problem 9.6 *Is there a weaker condition than DDP that implies index $= 1$ homology manifolds are manifolds?*

If so then this condition should be substituted for the DDP in the other problems in this section. This could make some of them significantly easier, and may also help with understanding dimension 4. A good way to approach this would be to find a surgery-based proof of Edwards' approximation theorem (see 9.1), then inspect it closely to find the minimum needed to make it work. Edwards' proof (see Daverman [13]) uses unobstructed cases of engulfing and approximation theorems. Surgery by contrast proceeds by showing an obstruction vanishes. Potentially-obstructed proofs (when they work) are often more flexible and have led to sharper results.

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