Some root invariants at the prime 2

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The first part of this paper consists of lecture notes which summarize the machinery of filtered root invariants. A conceptual notion of “homotopy Greek letter element” is also introduced, and evidence is presented that it may be related to the root invariant. In the second part we compute some low dimensional root invariants of v_1–periodic elements at the prime 2.

55Q45; 55Q51, 55T15

1 Introduction

This paper consists of two parts. The first part consists of the lecture notes of a series of talks on the root invariant given by the author at a workshop held at the Nagoya Institute of Technology. The second part is a detailed computation, using the methods of Part I, of some low dimensional root invariants at the prime 2. More detailed descriptions of the contents of these parts are given at the beginning of each part.

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ideas of [2] without all of the technical details. Haynes Miller provided some comments on the homotopy Greek letter construction, and Mike Hopkins clued the author into the existence of Mahowald's useful paper [17]. W-H Lin explained to the author how to show that a certain element of the Adams spectral sequence for P_{-23} was a permanent cycle. The author is grateful to the referee for discovering a mistake in a previous version of Corollary 6.2, and to R R Bruner for pointing out some typographical errors. The computations of Part II began as part of the author's thesis, which was completed under the guidance of Peter May at the University of Chicago. The author is, here and elsewhere, heavily influenced by the work of (and discussions with) Mark Mahowald. Finally, the author would like to extend his heartfelt gratitude to Norihiko Minami, both for organizing a very engaging workshop at the Nagoya Institute of Technology, and for his role as a mentor, introducing the author to the field of homotopy theory as an undergraduate at the University of Alabama. The author is supported by the NSF.

Part I Lectures on root invariants

Throughout this paper we are always working in the p–local stable homotopy category for p a fixed prime number. In this first section we will summarize the chromatic filtration and the machinery of filtered root invariants. A very detailed treatment of this theory appeared in [2]. Our intention here is to ignore many of the subtleties, sometimes to the point of omitting or simplifying hypotheses and ignoring indeterminacy, to communicate to the reader the underlying ideas. Our hope is that the reader will be able to use this overview as a motivation to drudge through the more precise treatment of [2].

In Section 2 we review the chromatic filtration of the stable stems. In Section 3 we review the Greek letter construction of Miller, Ravenel, and Wilson. The Greek letter elements are distinguished chromatic families of elements in the E_2–term $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ of the Adams–Novikov spectral sequence (ANSS). These elements need not be nontrivial permanent cycles in the ANSS. We introduce the notion of a homotopy Greek letter element to remedy this. In Section 4 we define the root invariant and recall some computational examples that occur throughout the literature. The interesting thing is that, at least for the limited number of root invariants we know, it seems to be the case that the root invariant has a tendency to take v_n–periodic homotopy Greek letter elements to v_{n+1}–periodic homotopy Greek letter elements. In Section 5 we define filtered root invariants. In Section 6 we summarize the main theorems that make the filtered root invariants compute root invariants.

Conventions We shall be using the following abbreviations.

ASS: The classical Adams spectral sequence based on $H\mathbb{F}_p$.

ANSS: The Adams–Novikov spectral sequence based on BP.

E–ASS: The generalized Adams spectral sequence based on E.

AHSS: The Atiyah–Hirzebruch spectral sequence.

$\mathrm{Ext}(E_*X)$: For nice spectra E, the comodule Ext group $\mathrm{Ext}_{E_*E}(E_*, E_*X)$.

AAHSS: The algebraic Atiyah–Hirzebruch spectral sequence that computes the group $\mathrm{Ext}(E_*X)$ using the cellular filtration on X.

Modified AAHSS: The AAHSS for $\mathrm{Ext}(BP_*P^\infty)$ arising from the filtration of P^∞ by Moore spectra.

Often we shall place a bar over the name of a permanent cycle in an Adams spectral sequence to denote an element of homotopy that it detects. We shall place dots over binary relations to indicate that they only hold up to multiplication by a unit in $\mathbb{Z}_{(p)}$. For instance, we shall write $x \doteq y$ if $x = \alpha \cdot y$ for some $\alpha \in \mathbb{Z}_{(p)}^\times$. We shall use $\overset{\cap}{=}$ to indicate that two quantities are equal modulo some indeterminacy group. We shall always use the Hazewinkel generators of BP_*.

2 The chromatic filtration

We shall first describe the chromatic filtration on the stable homotopy groups of spheres. What we are describing is referred to as the "geometric chromatic filtration" by Ravenel in [28]. We first need to discuss type–n complexes and v_n–self maps.

Let $K(n)$ be n^{th} Morava K–theory, with coefficient ring

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}].$$

Here v_n has degree $2(p^n - 1)$. By $K(0)$ we shall mean the rational Eilenberg–MacLane spectrum $H\mathbb{Q}$, and by v_0 we shall mean p.

If X is a finite complex, it is said to be *type–n* if $K(n-1)_*X = 0$ and $K(n)_*X \neq 0$. It is a consequence of the Landweber filtration theorem (see Landweber [14] and Ravenel [28]) that the condition $K(n-1)_*X = 0$ implies that $K(m)_*X = 0$ for $m \leq n-1$.

A self map $v\colon \Sigma^N X \to X$ is said to be a v_n–self map if it induces an isomorphism on $K(n)$–homology

$$v_*\colon K(n)_*\Sigma^N X \to K(n)_*X.$$

If v_* induces, up to an element in $\mathbb{F}_p^\times$, multiplication by v_n^k for some k, we shall say that X *has v_n^k multiplication*. By [28, Lemma 6.1.1], if v is a v_n–self map, there is some power of v which induces v_n^k multiplication. If v induces v_n^k–multiplication, we shall often denote the map v_n^k. This practice is slightly objectionable because a complex can have many different v so that v_* is v_n^k, but there is some consolation in that the Devinatz–Hopkins–Smith Nilpotence Theorem may be used to show that any two such maps are equal after a finite number of iterates.

The following important theorem was conjectured by Ravenel [26] and proved by Hopkins and Smith [8] using the nilpotence theorem [12].

Theorem 2.1 (Hopkins–Smith Periodicity Theorem) *If X is type–n, then X possesses a v_n–self map.*

We will now define the chromatic filtration of an element $\alpha \in \pi_n(S)$. We shall refer to the following diagram.

$$
\begin{tikzcd}
S^n \arrow[r, "p^{k_0}"'] & S^n \arrow[r, "\alpha"] \arrow[d] & S^0 \\
\Sigma^{N_1} M(p^{k_0}) \arrow[r, "v_1^{k_1}"'] & M(p^{k_0}) \arrow[ur, dotted, "\alpha_1"] \arrow[d] & \\
\Sigma^{N_2} M(p^{k_0}, v_1^{k_1}) \arrow[r, "v_2^{k_2}"'] & M(p^{k_0}, v_1^{k_1}) \arrow[uur, dotted, "\alpha_2"'] \arrow[d] & \\
 & \vdots &
\end{tikzcd}
$$

v_0–periodic: If $\alpha \circ p^k$ is non-zero for every k, then α is said to be *v_0–periodic*.

v_1–periodic: Otherwise, α is v_0–torsion, and there is some k_0 such that $\alpha \circ p^{k_0} = 0$. Let $M(p^{k_0})$ denote the cofiber of p^{k_0}. Then there exists a lift α_1 of α to $M(p^{k_0})$. The complex $M(p^{k_0})$ is type–1, and thus has v_1^m multiplication for some m. If $\alpha_1 \circ v_1^{mk}$ is non-zero for every k, then α is said to be *v_1–periodic*.

v_2–periodic: Otherwise, α is v_1–torsion, and there is some k_1 with $\alpha_1 \circ v_1^{k_1} = 0$. Let $M(p^{k_0}, v_1^{k_1})$ denote the cofiber of $v_1^{k_1}$. Then there exists a lift α_2 of α_1 to $M(p^{k_0}, v_1^{k_1})$. The complex $M(p^{k_0}, v_1^{k_1})$ is type–1, and thus has v_2^m multiplication for some m. If $\alpha_2 \circ v_2^{mk}$ is non-zero for every k, then α is said to be *v_2–periodic*.

v_3–periodic: Otherwise, α is said to be v_1–torsion, and there is some k_2 so that $\alpha_2 \circ v_2^{k_2} = 0$. The process continues.

In this way, we have defined a decreasing filtration

$$\pi_*(S) \supseteq \{v_0 - torsion\} \supseteq \{v_1 - torsion\} \supseteq \{v_2 - torsion\} \supseteq \cdots$$

which is the chromatic filtration.

It is not clear that the chromatic filtration is independent of the sequence of lifts. The (geometric) chromatic filtration may be more succinctly described by means of finite localization (see Mahowald–Sadofsky [23]), and from this perspective it is clear that the chromatic filtration is well defined. The *finite localization* functor

$$L^f_{E(n)}\colon Spectra \to Spectra$$

is initial amongst endofunctors that kill finite $E(n)$–acyclic spectra. The finitely localized sphere $L^f_{E(n)}S$ may be explicitly described as a colimit of finite spectra, and in this manner one finds that the v_n–torsion elements of $\pi_*(S)$ are precisely those that make up the kernel of the map

$$\pi_*(S) \to \pi_*(L^f_{E(n)}S).$$

There are fiber sequences

$$M^f_n S = v_n^{-1} M(p^\infty, \dots, v_{n-1}^\infty) \to L^f_n S \to L^f_{n-1} S.$$

Remark 2.2 In [28], Ravenel discusses a different filtration which he calls the "algebraic chromatic filtration," which is what is more commonly meant by the chromatic filtration these days. The n^{th} filtration is the kernel of the localization map

$$\pi_*(S) \to \pi_*(L_{E(n)}S)$$

where $E(n)$ is the Johnson–Wilson spectrum with coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}].$$

If the telescope conjecture is true, than the "geometric" and "algebraic" chromatic filtrations agree. However, it has been the case in the past decade that the more people have thought about the telescope conjecture, the more they have believed it to be false (see Mahowald–Ravenel–Shick [22]).

3 Greek letter elements

We shall now outline a standard method of constructing v_n–periodic elements of the stable stems called the *Greek letter construction*. Suppose that the generalized Moore spectrum $M(I)$ exists, where I is the ideal $(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \subset BP_*$, and assume that $M(I)$ has v_n^k–multiplication. The spectrum $M(I)$ is a finite complex of dimension

$$d = n + i_1|v_1| + \cdots + i_{n-1}|v_{n-1}|.$$

Then we can form the composite

$$\alpha^{(n)}_{lk/i_{n-1},\dots,i_0} \colon S^{lk|v_n|-d} \xrightarrow{\iota} \Sigma^{lk|v_n|-d} M(I) \xrightarrow{v_n^{lk}} \Sigma^{-d} M(I) \xrightarrow{\nu} S^0$$

where ι is the inclusion of the bottom cell, ν is the projection onto the top cell, and $\alpha^{(n)}$ is the n^{th} letter in the Greek alphabet $\alpha, \beta, \gamma, \delta, \dots$. Miller, Ravenel, and Wilson in [25] described Greek letter elements in $\mathrm{Ext}(BP_*)$, and the Greek letter elements of $\pi_*(S)$ that we have described are detected by their elements in the Adams–Novikov spectral sequence (ANSS). We shall refer to the Greek letter elements in $\mathrm{Ext}(BP_*)$ as "algebraic Greek letter elements."

We give a different interpretation of the Greek letter construction with an eye towards generalization. The existence of v_n^k multiplication on $M(I)$ gives homotopy elements

$$v_n^{lk} \in \pi_{lk|v_n|}(M(I))$$

detected by v_n^{lk} in BP–homology. Fix a (minimal) cellular decomposition of $M(I)$. Consider the Atiyah–Hirzebruch spectral sequence (AHSS)

$$E_1^{n,i} = \bigoplus_{n\text{–cells in } M(I)} \pi_i(S^n) \Rightarrow \pi_i(M(I)).$$

Suppose that the element $\alpha^{(n)}_{lk/i_{n-1},\dots,i_0}$ is non-trivial. Then in the AHSS, $v_n^{lk} \in \pi_*(M(I))$ is detected on the top cell by $\alpha^{(n)}_{lk/i_{n-1},\dots,i_0}$.

But how does one define $\alpha^{(n)}_{k/i_{n-1},\dots,i_0}$ if the appropriate $M(I)$ does not exist? Or if $M(I)$ does not have v_n^k multiplication? What do we do if the homotopy element $\alpha^{(n)}_{k/i_{n-1},\dots,i_0}$ turns out to be trivial? We give a "definition" of *homotopy Greek letter elements* to be the homotopy replacement of the Greek letter element when any of the above calamities befalls us. The author does not believe this is the right way to define these elements, but has no better ideas.

Definition 3.1 (Homotopy Greek letter elements) Suppose X is a type–n p–local finite complex for which BP_*X is free module over BP_*/I, for $I = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$. Suppose that X has v_n^k–multiplication. Then we define the homotopy Greek letter element $(\alpha^{(n)})^h_{k/i_{n-1},\dots,i_0}$ to be the element of $\pi_*(S)$ which detects $v_n^k \in \pi_*(X)$ in the E_1–term of the AHSS.

Remark 3.2 This definition is full of flaws. Different choices of X, or even different choices of detecting element in the AHSS, could yield some ambiguity in the definition of $(\alpha^{(n)})^h_{k/i_{n-1},\dots,i_0}$. There is no reason to believe that these homotopy Greek letter elements are of chromatic filtration n. In fact, the stem in which the element appears is even ambiguous. We do point out that there is already some ambiguity in the standard definition of the Greek letter elements — there can exist many complexes with the same BP–homology as $M(I)$, and the choice of v_n–self map is not unique. The v_n–self map is, however, unique after a finite number of iterations (see Hopkins–Smith [12]).

Remark 3.3 Mahowald and Ravenel [21] propose a different definition for "homotopy Greek letter elements" using iterated root invariants. Their definition suggests that we should be defining $(\alpha^{(n)})^h_i$ as

$$(\alpha^{(n)})^h_i = R^n(p^i)$$

This notion suffers the same sorts of indeterminacy issues that our notion of homotopy Greek letter element suffers.

We give some examples to illustrate the sorts of phenomena that the reader should expect at bad primes.

Let $p = 2$, and consider chromatic level $n = 1$. The complex $M(2)$ only has v_1^4 multiplication (see Adams [1]), giving us the Greek letter elements $\alpha_{4k} \in \pi_{8k-1}(S)$ (these are elements of order 2 in the image of J). However the complex $X = M(2) \wedge C(\eta)$, where $C(\eta)$ is the cofiber of η, has v_1–multiplication (see Mahowald [19]). Using this complex, we get the following homotopy Greek letter elements. These are precisely the elements on the edge of the ASS vanishing line, and, we believe, quite worthy of the designation "Greek letter element."

n (mod 4)	α_n^h (ANSS name)
0	α_n
1	α_n
2	$\alpha_{n-1}\alpha_1$
3	$\alpha_{n-2}\alpha_1^2$

Likewise, we list some low dimensional homotopy Greek letters for $p = 3$ and chromatic level $n = 2$. The complex $M(3, v_1) = V(1)$ exists, but only has v_2^9 multiplication (see Behrens–Pemmaraju [3]). Thus we are only able to define β_{9t} using the conventional methods. However the complex

$$X = V(1) \wedge (S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8)$$

can be shown to have v_2–multiplication, and using this complex we find the following homotopy Greek letter elements.

Greek name	Adams–Novikov name
β^h_1	β_1
β^h_2	$\beta_1^2\alpha_1$
β^h_3	β_3
β^h_4	β_1^5
β^h_5	β_5 (*)
β^h_6	β_6 (*)

(*) Tentative calculation

The reader will note that although the algebraic Greek letter element β_2 exists in $\mathrm{Ext}(BP_*)$ and is a non-trivial permanent cycle in the ANSS, it does not agree with the homotopy Greek letter element β^h_2.

We shall present evidence that these homotopy Greek letter elements may behave nicely with respect to root invariants.

4 The root invariant

Mahowald defined an invariant called the *root invariant* that takes an element α in the stable stems and outputs another element $R(\alpha)$ in the stable stems.

$$R\colon \pi_*(S) \rightsquigarrow \pi_*(S)$$

Our reason for using the wavy arrow "$\rightsquigarrow$" is that R is not a well defined map, but has indeterminacy, much in the way that Toda brackets do. $R(\alpha)$ is actually a coset, but in this first part, we shall often ignore this indeterminacy to clarify the exposition. In the literature this invariant is sometimes called the "Mahowald invariant," with good reason.

In this section we shall define the root invariant. We shall then summarize some of the computations of root invariants that appear in the literature.

We first need to define stunted projective spectra. We first assume that we are working at the prime $p=2$. Let ξ be the canonical line bundle over $\mathbb{R}P^n$. Then the Thom space may be identified (see Bruner–May–McClure–Steinberger [6, V.2.14]) as

$$(\mathbb{R}P^n)^{s\xi} \cong \mathbb{R}P^{n+s}/\mathbb{R}P^{s-1}.$$

We may allow s to be negative in the above definition if we use Thom *spectra* instead of Thom spaces. This motivates the definition of the spectrum

$$P_s^{n+s} = (\mathbb{R}P^n)^{s\xi}$$

for any integer s and any non-negative integer n. This spectrum has one cell in each degree in the interval $[s, n+s]$.

At an odd prime we can replace $\mathbb{R}P^n = B\Sigma_2$ with the classifying space $B\Sigma_p$. This complex only has cells in degrees congruent to $0, -1 \pmod{2(p-1)}$. We shall also refer to the resulting spectra as P_s^{n+s}.

We can take the colimit over n to obtain the spectrum P_s^∞. Taking the homotopy inverse limit of these spectra over s yields a spectrum $P_{-\infty}^\infty$. The inclusion of the -1–cell extends to a map

$$S^{-1} \xrightarrow{l} P_{-\infty}^\infty.$$

For $p = 2$, Lin [15] proved the following remarkable theorem. The theorem was conjectured by Mahowald, and is equivalent to the Segal conjecture for the group $\mathbb{Z}/2$. The odd primary version was proved by Gunawardena [9].

Theorem 4.1 *The map $l\colon S^{-1} \to P_{-\infty}^\infty$ is equivalent to the p–completion of S^{-1}.*

This theorem makes the following definition possible.

Definition 4.2 (Root invariant) Let α be an element of $\pi_t(S^0)$. The *root invariant* of α is the coset of all dotted arrows making the following diagram commute.

$$\begin{tikzcd} S^{t-1} \arrow[r, dotted] \arrow[d, "\alpha"'] & S^{-N} \arrow[dd] \\ S^{-1} \arrow[d, "l"'] & \\ P_{-\infty}^\infty \arrow[r] & P_{-N} \end{tikzcd}$$

This coset is denoted $R(\alpha)$. Here N is chosen to be minimal such that the composite $S^{t-1} \to P_{-N}$ is non-trivial.

One way to think of the root invariant is that it is the coset $R(\alpha)$ of elements in the E_1–term of the AHSS

$$E_1^{k,n} = \pi_k(S^n) \Rightarrow \pi_k(P_{-\infty}^\infty) = \pi_k(S_2^{-1})$$

that detects α.

The root invariant is interesting for two reasons:

(1) Elements which are root invariants behave quite differently in the EHP sequence as opposed to elements which are not root invariants (see Mahowald–Ravenel [21]).
(2) The root invariant appears to generically take things in chromatic filtration n to things in chromatic filtration $n+1$.

Our purpose in this paper is to concentrate on the latter. For instance, we give the following sampler of results:

- For $p \geq 3$ we have $R(p^i) = \alpha_i$ [21].
- For $p \geq 5$ we have $R(\alpha_i) = \beta_i$ and $R(\alpha_{p/2}) = \beta_{p/2}$ (see [21] and Sadofsky [29]).
- For $p = 2$ we have $R(2^i) = \alpha_i^h$ (see [21], Johnson [13]).
- For $p = 3$ we have $R(\alpha_i) = \beta_i^h$ for $i \leq 6$, and we have $R(\alpha_i) = \beta_i$ for for $i \equiv 0, 1, 5 \pmod 9$ (see Behrens [2]).

A conjecture that the root invariant increases chromatic filtration appears in Mahowald–Ravenel [20]. However, we warn the reader that the conjecture takes some time to begin working. For instance, at $p = 2$, $R(\eta) = \nu$, and $R(\nu) = \sigma$, and η, ν, and σ are all v_1–periodic elements.

5 Filtered root invariants

Let E be a ring-spectrum for which the E–Adams spectral sequence converges. In [2], the author investigated a series of approximations to the root invariant which live in the E_1–term of the E–Adams spectral sequence called *filtered root invariants*.

$$R_E^{[k]} \colon \pi_*(S) \rightsquigarrow E_1^{k,*}$$

We shall give a brief outline of their definition, but refer the reader to [2] for a completely detailed treatment.

Let $\overline{E}$ be the fiber of the unit $S \to E$. The E–Adams resolution of the sphere is given by

$$\begin{tikzcd}
S \arrow[r, equal] & W_0 \arrow[d] & W_1 \arrow[l] \arrow[d] & W_2 \arrow[l] \arrow[d] & W_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
& Y_0 & Y_1 & Y_2 & Y_3 &
\end{tikzcd}$$

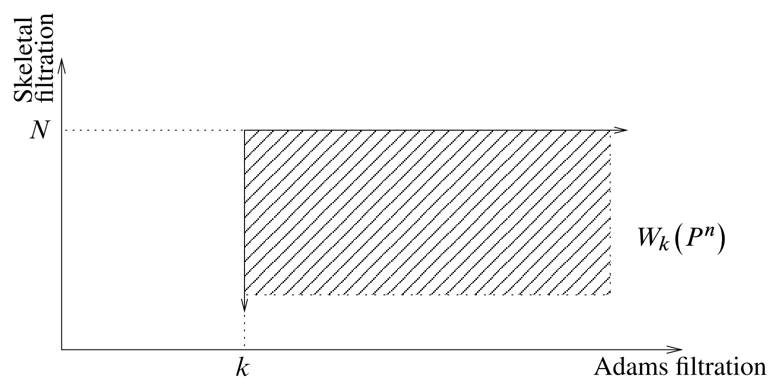
where $W_k = \overline{E}^{(k)}$ and $Y_k = E \wedge \overline{E}^{(k)}$. The skeletal filtration of $P_{-\infty}^{\infty}$ is given by

$$\begin{tikzcd} \cdots \arrow[r] & P_{-\infty}^{N} \arrow[r] \arrow[d] & P_{-\infty}^{N+1} \arrow[r] \arrow[d] & P_{-\infty}^{N+2} \arrow[r] \arrow[d] & \cdots \\ & S^N & S^{N+1} & S^{N+2} & \end{tikzcd}$$

We wish to mix the two filtrations. We may regard $P_{-\infty}^{\infty}$ as being a bifiltered object, with (k, N)–bifiltration given by

$$W_k(P^N) = (W_k \wedge P^N)_{-\infty}$$

where we take the homotopy limit after smashing with W_k. We may pictorially represent this bifiltration by a region of the Cartesian plane where we let the x–axis represent the Adams filtration and the y–axis represent the skeletal filtration.



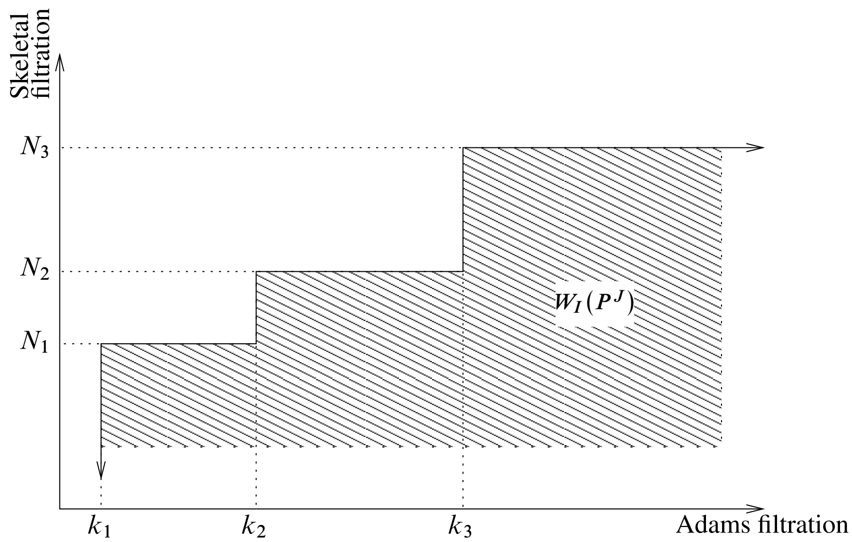
The spectra W_k may be replaced by weakly equivalent approximations so that for every k the maps $W_{k+1} \to W_k$ are inclusions of subcomplexes. We then have that for $k_1 \geq k_2$ and $N_1 \leq N_2$, the bifiltration $W_{k_1}(P^{N_1})$ is a subcomplex of $W_{k_2}(P^{N_2})$. We shall consider spectra which are unions of these bifiltrations, which appeared in [2] as "filtered Tate spectra." Given sequences

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{N_1 < N_2 < \cdots < N_l\}$$

with $k_i \geq 0$, we define the filtered Tate spectrum as the union

$$W_I(P^J) = \bigcup_i W_{k_i}(P^{N_i}).$$

A picture of the bifiltrations that compose this spectrum is given below.



For $1 \leq i \leq l$, there are natural projection maps

$$p_i\colon W_I(P^J) \to Y_{k_i} \wedge S^{N_i}.$$

We shall now define the filtered root invariants.

Definition 5.1 Let α be an element of $\pi_t(S)$, with image $l(\alpha) \in \pi_{t-1}(P_{-\infty}^{\infty})$. Choose a multi-index (I, J) where $I = (k_1, k_2, \ldots)$ and $J = (N_1, N_2, \ldots)$ so that the filtered Tate spectrum $W_I(P^J)$ is initial amongst the Tate spectra $W_K(P^L)$ so that $l(\alpha)$ is in the image of the map

$$\pi_{t-1}(W_K(P^L)) \to \pi_{t-1}(P_{-\infty}^{\infty}).$$

(This initial multi-index is not unique with this property, but in [2] we give a convention for choosing a unique preferred initial multi-index.) Let $\widetilde{\alpha}$ be a lift of $l(\alpha)$ to $\pi_{t-1}(W_I(P^J))$. Then the k_i^{th} E–filtered root invariant is given by

$$R_E^{[k_i]}(\alpha) = p_i(\widetilde{\alpha}) \in \pi_{t-1}(Y_{k_i} \wedge S^{N_i}).$$

We shall refer to (I, J) as the E–bifiltration of α.

The k_i^{th} filtered root invariant thus lives in the E_1–term of the E–ASS for the sphere. It should be regarded as an approximation to the root invariant in E–Adams filtration k_i. There is indeterminacy in this invariant given by the various choices of lifts $\widetilde{\alpha}$.

6 Some theorems

We shall now outline the manner in which filtered root invariants may be used to compute homotopy root invariants. The statements of these theorems appeared in [2, Section 5], with proofs appearing in Section 6. The theorems as stated in [2] are rather difficult to conceptualize due to the complicated hypotheses and the nature of the indeterminacy. The statements we give below are imprecise, but easier to read and understand. Throughout this section, let α be an element of $\pi_t(S)$ of E–bifiltration (I, J), where $I = (k_i)$ and $J = (-N_i)$. Our first theorem tells us how to determine if a filtered root invariant detects the homotopy root invariant in the E–ASS.

Theorem 6.1 [2, Theorem 5.1] *Suppose that $R_E^{[k_i]}(\alpha)$ contains a permanent cycle β. Then there exists an element $\overline{\beta} \in \pi_*(S)$ which β detects in the E–ASS such that the following diagram commutes up to elements of E–Adams filtration greater than or equal to $k_i + 1$.*

$$\begin{tikzcd}
S^{t-1} \arrow[r, "\overline{\beta}"'] \arrow[d, "\alpha"'] & S^{-N_i} \arrow[dd] \\
S^{-1} \arrow[d, "l"'] & \\
P_{-\infty}^{\infty} \arrow[r] & P_{-N_i}
\end{tikzcd}$$

We present a practical reinterpretation of this theorem. This essentially appears in [2] as Procedure 9.1.

Corollary 6.2 *Let β be as in Theorem 6.1. Then in order for β to detect the homotopy root invariant in the E–ASS, it is sufficient to check the following two things.*

(1) *No element $\gamma \in \pi_{t-1}(P_{-N_i})$ of E–Adams filtration greater that k_i can detect the root invariant of α in P_{-N_i+1}.*

(2) *The image of the element $\overline{\beta}$ under the inclusion of the bottom cell*

$$\pi_{t-1}(S^{-N_i}) \to \pi_{t-1}(P_{-N_i})$$

is non-trivial.

For our next set of theorems we shall need to introduce a variant of the Toda bracket construction. Let K be a finite CW complex with a single cell in top dimension n and bottom dimension 0. There is an inclusion map

$$\iota\colon S^0 \to K$$

and the n–cell is attached to the $n-1$ skeleton $K^{[n-1]}$ by an attaching map

$$a\colon S^{n-1} \to K^{[n-1]}.$$

The following definition was the subject of [2, Section 4].

Definition 6.3 Let β be an element of $\pi_j(S)$. Then the K–Toda bracket $\langle K\rangle(\beta)$ is a lift

$$\begin{tikzcd} S^{j+n-1} \arrow[r, "\beta"] \arrow[drr, dotted, "\langle K\rangle(\beta)"'] & S^{n-1} \arrow[r, "a"] & K^{[n-1]} \\ & & S^0 \arrow[u, "\iota"'] \end{tikzcd}$$

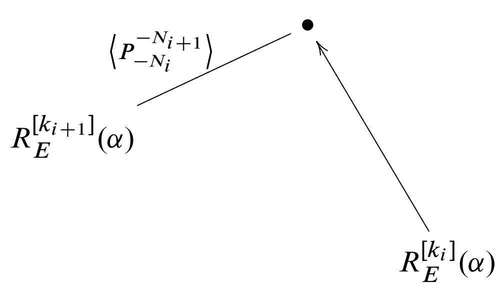
The K–Toda bracket may not exist, or may not be well defined.

Given a k_i^{th} filtered root invariant, the k_{i+1}^{th} filtered root invariant may be revealed by the presence of an Adams differential.

Theorem 6.4 [2, Theorem 5.3] *There is a (possibly trivial) E–Adams differential*

$$d_r(R_E^{[k_i]}(\alpha)) = \left\langle P_{-N_i}^{-N_i+1}\right\rangle\left(R_E^{[k_{i+1}]}(\alpha)\right).$$

Theorem 6.4 is saying the following differential happens in the E–ASS chart.



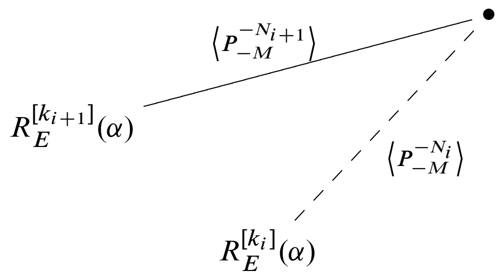
If this differential is zero, there may still be a hidden extension that reveals the k_{i+1}^{th} filtered root invariant. In the next theorem, we use the notation $\overline{\beta}$ for an element that $\beta \in E_1^{*,*}$ detects in the E–ASS.

Theorem 6.5 [2, Theorem 5.4] *There is an equality of (possibly trivial) elements of $\pi_*(S)$*

$$\left\langle P_{-M}^{-N_i}\right\rangle\overline{\left(R_E^{[k_i]}(\alpha)\right)} = \overline{\left\langle P_{-M}^{-N_i+1}\right\rangle\left(R_E^{[k_{i+1}]}(\alpha)\right)}.$$

Here M is the largest integer for which $\left\langle P_{-M}^{-N_i}\right\rangle\overline{\left(R_E^{[k_i]}(\alpha)\right)}$ exists and is non-trivial, and the second Toda bracket is taken in the E–ASS.

Theorem 6.5 says that the following hidden extension happens on the E–ASS chart.



We have given tools to move from one filtered root invariant to the next, but we need a place to start this process. For $E = BP$ in [2] we used BP–root invariants and $BP \wedge \overline{BP}$–root invariants. For $E = H\mathbb{F}_p$ the first filtered root invariant is given by the algebraic root invariant R_{alg} (see, for example Mahowald–Ravenel [21]). The nice thing about R_{alg} is that it is very computable, especially with the help of a computer (see Bruner [5]).

Theorem 6.6 [2, Theorem 5.10] *Let α be of Adams filtration s, detected in the ASS by $\widetilde{\alpha}$ in* Ext*. Then the first $H\mathbb{F}_p$–filtered root invariant is given by*

$$R^{[s]}_{H\mathbb{F}_p}(\alpha) = R_{\mathrm{alg}}(\alpha).$$

Part II 2–primary calculations

In this part we are always implicitly working 2–locally. Our goal is to explain how the theory of Part I play out in low dimensions in the ANSS and the ASS at the prime 2. Unlike in Part I, we intend to be completely precise about these calculations. This part is really an extension of [2] to the prime 2.

Our main result is to compute the homotopy root invariants of all of the v_1–periodic elements through the 12–stem (Theorem 11.1). These root invariants turn out fit into the primary v_2–family investigated by Mahowald [18] (but beware! v_2^8 does not exist, v_2^{32} does exist ; see Hopkins–Mahowald [10]).

Our plan of attack is the following. In [2], we computed the $BP \wedge \widetilde{BP}$–root invariants of the elements $\alpha_{i/j}$. These give the first filtered root invariant modulo an indeterminacy group as described in Proposition 7.4. Once we know the indeterminacy group, we can identify $R^{[1]}_{BP}(\alpha_{i/j})$, and then get $R^{[2]}_{BP}(\alpha_{i/j})$. The higher filtered root invariants are deduced from differentials and hidden extensions in the ANSS. We then check to see that these top filtered root invariants must detect the homotopy root invariants.

In Section 7, we compute this indeterminacy group completely. This essentially involves understanding how the v_1–periodic elements act in the algebraic Atiyah–Hirzebruch spectral sequence for $\mathrm{Ext}(BP_*P^\infty)$. Unfortunately, there is no J–homomorphism in Ext to produce these differentials, so we must resort to explicit computation of the differentials using the BP_*BP–coaction on BP_*P^∞. We find generators for this BP–homology group that make a complete determination of the AAHSS differentials possible. This method may be interesting in its own right, in the sense that it gives a particularly clean and pleasant description of the coaction.

In Section 8 we describe how the theory of BP–filtered root invariants reproduces the expected root invariants of the elements 2^i. The differentials and hidden extensions amongst the elements $\alpha_{i/j}\alpha_1^k$ get a rather natural interpretation: the ANSS must deal with the fact that the homotopy Greek letter elements α_i^h are different from the algebraic Greek letter elements α_i.

In Section 9, we use our computation of the indeterminacy group to compute the the filtered root invariants $R_{BP}^{[k]}(\alpha_{i/j})$ for $k = 1, 2$. We find that the indeterminacy is essential to allow for the root invariants $R(\eta) = \nu$ and $R(\nu) = \sigma$. For the higher dimensional v_1–periodic elements, we find that for all other i and j,

$$R_{BP}^{[2]}(\alpha_{i/j}) \doteq \beta_{i/j} + \text{something}$$

where the "something" is rather innocuous. One could take this calculation as further evidence that the root invariants wants to take v_n–periodic families of Greek letter elements to v_{n+1}–periodic families of Greek letter elements.

In Section 10 we compute all of the higher BP–filtered root invariants of the elements $\alpha_{i/j}$ which lie within the 12–stem. These higher filtered root invariants are deduced from differentials and hidden extensions in the ANSS, and, amusingly enough, actually account for most of the differentials and hidden extensions in this range. We saw this sort of behavior at the prime 3 in [2]. The reason the range is so limited is the author's limited knowledge of the ANSS at the prime 2.

In Section 11 we show that the filtered root invariants of Section 10 actually detect homotopy root invariants. This is done by brute force. We show that there are no elements of $\pi_*(P_{-N}^\infty)$ that could survive to the difference of the filtered root invariant and the homotopy root invariant. We use the BP–filtered root invariants for the elements in BP–Adams filtration 1, and the $H\mathbb{F}_2$–filtered root invariants for the rest.

7 The indeterminacy spectral sequence

Recall that

$$BP_*(P_{2l-1}^{2k}) = BP_*\{e_{2m-1} \,:\, l \le m \le k\}/\Big(\sum_{i\ge 0} c_i e_{2(m-i)-1}\Big)$$

where the universal 2–typical 2–series is given by

$$[2]_F(x) = \sum_{i\ge 0} c_i x^{i+1} \in BP_*[\![x]\!].$$

In particular, the first couple of values of c_i are

$$c_0 = 2, \qquad c_1 = -v_1.$$

We will first define a v_1–self map of BP_*BP–comodules. Define a map

$$\widetilde{v}_1 \colon BP_*P_{2l+1}^{2k} \to BP_*P_{2l-1}^{2(k-1)}$$

by

$$\widetilde{v}_1(e_{2m-1}) = \sum_{i\ge 1} -c_i e_{2(m-i)-1}$$

This is a map of comodules since in $BP_*P_{2l-1}^{2k}$, we have

$$\widetilde{v}_1(e_{2m-1}) = 2e_{2m-1}.$$

Thus $\widetilde{v}_1$ is just a certain factorization of multiplication by 2.

The short exact sequences

$$0 \to BP_*P^{2(k-1)} \to BP_*P^{2k} \to \Sigma^{2k-1}BP_*/(2) \to 0$$

gives rise to long exact sequences of Ext groups, which piece together to give a *modified AAHSS*

$$E_1^{k,m,s} = \mathrm{Ext}^{s,s+k}(\Sigma^{2m-1}BP_*/(2)) \Rightarrow \mathrm{Ext}^{s,s+k}(BP_*P^\infty).$$

We shall refer to elements of $E_1^{k,m,s}$ of the modified AAHSS by $x[2m-1]$, where x is an element of $\mathrm{Ext}^{s,k+s}(\Sigma^{2m-1}BP_*/(2))$. The existence of the map $\widetilde{v}_1$ gives the following propagation result in the modified AAHSS.

Proposition 7.1 *Suppose $x[2m-1]$ is an element of the modified AAHSS, and that there is a differential*

$$d_r(x[2m-1]) = y[2(m-r)-1].$$

Then we have

$$d_r(v_1 x[2(m-1)-1]) = v_1 y[2(m-1-r)-1].$$

Proof The map $\widetilde{v}_1$ induces a map of modified AAHSS's:

$$\begin{array}{ccc}
\bigoplus_{l+1\le m\le k} \mathrm{Ext}^{s,s+k}(\Sigma^{2m-1}BP_*/(2)) & \Longrightarrow & \mathrm{Ext}^{s,s+k}\big(BP_* P^{2k}_{2l+1}\big) \\
{\scriptstyle v_1}\big\downarrow & & \big\downarrow{\scriptstyle (\widetilde{v}_1)_*} \\
\bigoplus_{l+1\le m\le k} \mathrm{Ext}^{s,s+k}(\Sigma^{2m-3}BP_*/(2)) & \Longrightarrow & \mathrm{Ext}^{s,s+k}\big(BP_* P^{2(k-1)}_{2l-1}\big)
\end{array}$$

This proves the proposition. □

Proposition 7.2 *The differentials on the elements* $1[2m-1]$ *in the modified AAHSS are given as follows:*

$$\begin{array}{ll}
d_1(1[2m-1]) = \alpha_1[2(m-1)-1] & m \text{ odd} \\
d_2(1[2m-1]) = \widetilde{\beta}_1[2(m-2)-1] & \nu_2(m) = 1 \\
d_r(1[2m-1]) = v_1^{k-2}(x_7 + \widetilde{\beta}_{2/2})[2(m-k-2)-1] & \nu_2(m) = k, k = 2,3 \\
d_r(1[2m-1]) = v_1^{k-2} x_7[2(m-k-2)-1] & \nu_2(m) = k, k \ge 4
\end{array}$$

Proof The formulas for d_1 and d_2 follow immediately from the well known attaching map structure of P^∞. We shall prove the formulas for the higher differentials by working with the negative cells of P^∞_{-N}, and then by using James periodicity. It suffices to consider $m = -2^k$. There is the following equivalence to the Spanier–Whitehead dual (see Bruner–May–McClure–Steinberger [6]).

$$\Sigma P^{-2\cdot 2^k}_{-2(2^k+l)-1} \simeq DP^{2(2^k+l)}_{2\cdot 2^k-1}$$

It follows that we have an isomorphism of BP_*BP–comodules

$$BP_*(\Sigma P^{-2\cdot 2^k}_{-2(2^k+l)-1}) \cong BP^{-*} P^{2(2^k+l)}_{2\cdot 2^k-1}.$$

Here, for finite X, the cohomology group $BP^{-*}X$ is viewed as a BP_*BP–comodule by the coaction given by the composite

$$\begin{aligned}
BP^{-*}X = \pi_*(F(X,BP)) \xrightarrow{(\eta_R)_*} {} & \pi_*(F(X, BP\wedge BP)) \\
\cong {} & \pi_*(BP\wedge BP\wedge DX) \\
\cong {} & BP_*BP \otimes_{BP_*} BP^{-*}X.
\end{aligned}$$

We recall from [2] that there are short exact sequences

$$0 \to BP^*\mathbb{C}P_a^b \xrightarrow{\cdot[2]_F(x)/x} BP^*\mathbb{C}P_a^b \to BP^*P_{2a-1}^{2b} \to 0.$$

Here we have

$$BP^{-*}\mathbb{C}P^\infty \cong BP_*[\![x]\!]$$

where x has (homological degree) -2, and $BP^{-*}\mathbb{C}P_a^b$ is given by the ideal

$$BP^{-*}\mathbb{C}P_a^b \cong (x^a) \subseteq BP_*[x]/(x^{b+1}) \cong BP^{-*}\mathbb{C}P^b.$$

We recall from [2] that the coaction of BP_*BP on $h(x) \in BP^{-*}\mathbb{C}P_a^b$ is given by

$$\psi(h(x)) = (f_*h)(f(x))$$

where f is the universal isomorphism of 2–typical formal groups, whose inverse is given by

$$f^{-1}(x) = \sum_{i \geq 0}^F t_i x^{2^i}.$$

The polynomial $f_*h(x)$ is the polynomial obtained by applying the right unit to all of the coefficients of $h(x)$. The surjection of $BP^*\mathbb{C}P_a^b$ onto $BP^*P_{2a-1}^{2b}$ completely determines the latter as a BP_*BP–comodule. In what follows we shall refer to elements of $BP^*P_{2a-1}^{2b}$ by the names of elements in $BP^*\mathbb{C}P_a^b$ which project onto them.

We shall need the following formulas. (We are using Hazewinkel generators.)

$$\begin{aligned}
f(x) &= x - t_1x^2 + (2t_1^2 + v_1t_1)x^3 + \cdots \\
[2]_F(x) &= 2x - v_1x^2 + 2v_1^2x^3 + \cdots \\
\eta_R(v_1) &= v_1 + 2t_1 \\
\eta_R(v_2) &= v_2 + 2t_2 - 4t_1^3 - 5v_1t_1^2 - 3v_1^2t_1
\end{aligned}$$

We shall now use our very specific knowledge of the BP_*BP coaction to determine the differential $d(1[-2\cdot 2^k - 1])$ in the modified AAHSS for

$$\mathrm{Ext}\big(BP_*\Sigma P_{-2(2^k-l)-1}^{-2\cdot 2^k}\big) = \mathrm{Ext}\big(BP^{-*}P_{2\cdot 2^k-1}^{2(2^k+l)}\big).$$

We do this for $l = k+2$, so in what follows we work modulo x^{2^k+k+3}. The desired differential is governed by the coaction on $x^{2^k} \in BP^{-*}P_{2\cdot 2^k-1}^{2(2^k+l)}$. Actually, the case $k = 2$ must be handled separately, because in the computations that follow we are implicitly using the fact that $2^k > k+2$. However, the method, and conclusion, for

$k=2$ are completely identical.

$$\begin{aligned}
\psi\big(x^{2^k}\big) &= \big(f(x)\big)^{2^k} \\
&= \big(x - t_1x^2 + \big(2t_1^2 + v_1t_1\big)x^3 + \cdots\big)^{2^k} \\
&= x^{2^k} - \tbinom{2^k}{1}t_1x^{2^k+1} + \tbinom{2^k}{1}\big(2t_1^2+v_1t_1\big)x^{2^k+2} + \\
&\quad \tbinom{2^k}{2}t_1^2x^{2^k+2} + \tbinom{2^k}{4}t_1^4x^{2^k+4} + \cdots \\
&= x^{2^k} - 2^kt_1x^{2^k+1} + 2^{k+1}t_1^2x^{2^k+2} + 2^kv_1t_1x^{2^k+2} + \\
&\quad \big(2^k-1\big)2^{k-1}t_1^2x^{2^k+2} + 2^{k-2}\cdot\tfrac{1}{3}\big(2^k-1\big)\big(2^{k-1}-1\big)\big(2^k-3\big)t_1^4x^{2^k+4} + \cdots \\
&= x^{2^k} - v_1^kt_1x^{2^k+k+1} - v_1^{k-1}t_1^2x^{2^k+k+1} + v_1^{k-2}t_1^4x^{2^k+k+2} + \\
&\quad v_1^{k+1}t_1x^{2^k+k+2} + \cdots
\end{aligned}$$

We conclude that in the cobar complex for $\mathrm{Ext}\big(BP^*P_{2\cdot 2^k-1}^{2(2^k+k+2)}\big)$, we have:

$$d\big(x^{2^k}\big) = v_1^kt_1x^{2^k+k+1} + v_1^{k-1}t_1^2x^{2^k+k+1} + v_1^{k-2}t_1^4x^{2^k+k+2} + v_1^{k+1}t_1x^{2^k+k+2} + \cdots$$

We compute the differential in the modified AAHSS by adding a coboundary supported on an element of lower cellular filtration. Namely, we compute the coaction on $v_1^{k-2}v_2x^{2^k+k+1}$ as:

$$\begin{aligned}
\psi\big(v_1^{k-2}v_2x^{2^k+k+1}\big) &= \eta_R\big(v_1^{k-2}v_2\big)\big(f(x)\big)^{2^k+k+1} \\
&= (v_1+2t_1)^{k-2}\big(v_2+2t_2-4t_1^3-5v_1t_1^2-3v_1^2t_1\big)\big(x-t_1x^2+\cdots\big)^{2^k+k+1} \\
&= v_1^{k-2}\big(v_2+2t_2-4t_1^3-5v_1t_1^2-3v_1^2t_1\big)x^{2^k+k+1} + \\
&\quad 2(k-2)v_1^{k-3}t_1\big(v_2+2t_2-4t_1^3-5v_1t_1^2-3v_1^2t_1\big)x^{2^k+k+1} - \\
&\quad (2^k+k+1)v_1^{k-2}t_1\big(v_2+2t_2-4t_1^3-5v_1t_1^2-3v_1^2t_1\big)x^{2^k+k+2} + \cdots \\
&= \big(v_1^{k-2}v_2 + v_1^{k-1}t_1^2 + v_1^kt_1\big)x^{2^k+k+1} + v_1^{k-1}\big(t_2+v_1t_1^2\big)x^{2^k+k+2} + \\
&\quad \big(2^k+2k-1\big)v_1^{k-2}t_1\big(v_2+2t_2-4t_1^3-5v_1t_1^2-3v_1^2t_1\big)x^{2^k+k+2} + \cdots \\
&= \big(v_1^{k-2}v_2 + v_1^{k-1}t_1^2 + v_1^kt_1\big)x^{2^k+k+1} + \\
&\quad \big(v_1^{k-1}t_2 + v_1^{k-2}v_2t_1 + v_1^{k-1}t_1^3\big)x^{2^k+k+2} + \cdots
\end{aligned}$$

We conclude that in the cobar complex for $\mathrm{Ext}\big(BP^*P_{2\cdot 2^k-1}^{2(2^k+k+2)}\big)$ we have:

$$\begin{aligned}
d\big(v_1^{k-2}v_2x^{2^k+k+1}\big) &= \\
&\big(v_1^{k-1}t_1^2 + v_1^kt_1\big)x^{2^k+k+1} + \big(v_1^{k-1}t_2 + v_1^{k-2}v_2t_1 + v_1^{k-1}t_1^3\big)x^{2^k+k+2} + \cdots
\end{aligned}$$

We therefore have:

$$d\big(x^{2^k} - v_1^{k-2}v_2x^{2^k+k+1}\big) = \\ \big(v_1^{k-2}t_1^4 + v_1^{k+1}t_1 + v_1^{k-1}t_2 + v_1^{k-2}v_2t_1 + v_1^{k-1}t_1^3\big)x^{2^k+k+2}$$

We recall from Ravenel [27] that the generators of $\mathrm{Ext}^{1,8}(BP_*/(2))$ are x_7 and $\widetilde{\beta}_{2/2}$, and they are represented in the cobar complex by the elements

$$x_7 = v_1t_2 + v_2t_1 + v_1t_1^3$$
$$\widetilde{\beta}_{2/2} = t_1^4 + v_1^3t_1$$

We conclude that for $k = 2, 3$ we have the modified AAHSS differentials

$$d_{k+2}\big(1[-2\cdot 2^k - 1]\big) = v_1^{k-2}\big(x_7 + \widetilde{\beta}_{2/2}\big)\big[-2\big(2^k + k + 2\big) - 1\big].$$

If $k \geq 4$, then we may add an additional coboundary to obtain the cobar formula

$$d\big(x^{2^k} + v_1^{k-2}v_2x^{2^k+k+1} + v_1^{k-4}v_2^2x^{2^k+k+2}\big) = x_7x^{2^k+k+2}$$

from which it follows that for $k \geq 4$, we have the modified AAHSS differential

$$d_{k+2}\big(1\big[-2\cdot 2^k - 1\big]\big) = v_1^{k-2}x_7\big[-2\big(2^k + k + 2\big) - 1\big]. \qquad \square$$

Combining Proposition 7.2 with Proposition 7.1, we get the following differentials.

Proposition 7.3 *The differentials on the elements $v_1^i[2m-1]$ for $i \geq 1$ in the modified AAHSS are given as follows.*

$$\begin{array}{ll}
d_1\big(v_1^i[2m-1]\big) = v_1^i\alpha_1[2(m-1)-1] & m+i \text{ odd} \\
d_3\big(v_1^i[2m-1]\big) = v_1^{i-1}x_7[2(m-3)-1] & \nu_2(m+i) = 1 \\
d_4\big(v_1^i[2m-1]\big) = v_1^i(x_7+\beta_{2/2})[2(m-4)-1] & \nu_2(m+i) = 2 \\
d_r\big(v_1^i[2m-1]\big) = v_1^{k+i-2}x_7[2(m-k-2)-1] & \nu_2(m+i) = k, k \geq 3
\end{array}$$

Proof The differentials follow from applying v_1–propagation as described in Proposition 7.1 to the differentials of Proposition 7.2. However, the element $v_1\widetilde{\beta}_1$ is null in $\mathrm{Ext}(BP_*/(2))$. An explicit computation similar to that in the proof of Proposition 7.2 yields the modified AAHSS differential

$$d_3(v_1[2m-1]) = x_7[2(m-3)-1]$$

for $m+1 \equiv 2 \pmod 4$. The rest of the d_3's then follow by v_1–propagation. $\square$

Recall that we have the following computation, which is given by combining Corollary 5.9 and Proposition 10.2 of [2].

Proposition 7.4 *There is an indeterminacy group $A_{i/j} \subseteq BP_* \widetilde{BP}$ such that*

$$R^{[1]}_{BP}(\alpha_{i/j}) \subseteq \widetilde{\beta}_{i/j} + 2BP_* \widetilde{BP} + A_{i/j}.$$

In [2] (spectral sequence (10.8) and the discussion which follows it), a method of computing this indeterminacy group was described in terms of the differentials of an *indeterminacy spectral sequence*. This spectral sequence is a truncated version of the AAHSS. The differentials given in Proposition 7.2 and Proposition 7.3 give differentials in the indeterminacy spectral sequence, which translate to the following result.

Proposition 7.5 *The indeterminacy group $A_{i/j}$ for $R^{[1]}_{BP}(\alpha_{i/j})$ is contained in the $\mathbb{Z}_{(2)}$ module spanned by $2BP_*BP$ and the generator given in the table below, where $a = 3i - j$.*

a	Generator	Condition
2	$-$	$\nu_2(i) \geq 2$
	$\widetilde{\beta}_1$	$\nu_2(i) = 1$
	$v_1\alpha_1$	$\nu_2(i) = 0$
3	$-$	$\nu_2(i) \geq 1$
	$v_1^2\alpha_1$	$\nu_2(i) = 0$
4	$-$	$\nu_2(i) \geq 3$
	$x_7 + \widetilde{\beta}_{2/2}$	$\nu_2(i) = 2$
	x_7	$\nu_2(i) = 1$
	$v_1^3\alpha_1$	$\nu_2(i) = 0$
5	$-$	$\nu_2(i) \geq 4$
	$v_1(x_7 + \widetilde{\beta}_{2/2})$	$2 \leq \nu_2(i) \leq 3$
	$v_1 x_7$	$\nu_2(i) = 1$
	$v_1^4\alpha_1$	$\nu_2(i) = 0$
$a \geq 6$	$-$	$\nu_2(i) \geq a-1$
	$v_1^{a-4}x_7$	$1 \leq \nu_2(i) \leq a-2$
	$v_1^{a-1}\alpha_1$	$\nu_2(i) = 0$

Remark 7.6 Not all of the entries of the table in Proposition 7.5 actually occur with the allowable values of i and j for $\alpha_{i/j}$.

8 *BP* filtered root invariants of 2^k

The root invariants of the elements 2^k were determined by Mahowald and Ravenel [21] and by Johnson [13]. In this section we will explain how the root invariants of the elements 2^k are formed from the perspective of the ANSS. This analysis provides an explanation for the pattern of differentials and hidden extensions in the v_1–periodic part of the ANSS. We only compute filtered root invariants — our treatment is not an independent verification of the know values of the homotopy root invariants $R(2^k)$ because we do not eliminate the higher Adams–Novikov filtration obstructions required by Theorem 6.1. The proper thing to do would be to combine the use of the ASS and the ANSS, in a manner employed in the infinite family computations of [2].

Throughout this section and the rest of the paper, the reader should find it helpful to refer to Figure 1, which depicts the ANSS chart for the sphere at the prime 2 through the 29–stem.

[2, Proposition 10.1] states that

$$\alpha_k \in R^{[1]}_{BP}(2^k).$$

Our description of $R(2^k)$ depends on the value of k modulo 4.

$k \equiv 1 \pmod 4$: The element α_k is a permanent cycle which detects the homotopy root invariant $R(2^k)$.

$k \equiv 2 \pmod 4$: While α_k is a permanent cycle, it does not detect an element of $R(2^k)$. This is an instance where Theorem 6.5 comes into play. The $-2k$–cell attaches to the $(-2k-1)$–cell of P^{-2k}_{-2k-1} by the degree 2 map, and the $(-2k+1)$–cell attaches to the $(-2k-1)$–cell in P^{-2k+1}_{-2k-1} by the map α_1. There is a hidden extension $\overline{\alpha}_1 \cdot \overline{\alpha}_{k-1}\overline{\alpha}_1 = 2 \cdot \overline{\alpha}_k$, so we may use Theorem 6.5 to deduce that

$$\alpha_{k-1}\alpha_1 \in R^{[2]}_{BP}\big(2^k\big).$$

The element $\alpha_{k-1}\alpha_1$ detects the homotopy root invariant $R\big(2^k\big)$.

$k \equiv 3 \pmod 4$: The first filtered root invariant α_k is not a permanent cycle, so we turn to Theorem 6.4. There is an Adams–Novikov differential

$$d_3(\alpha_k) = \alpha_1 \cdot \alpha_{k-2}\alpha_1^2.$$

The $(-2k+2)$–cell attaches to the $-2k$–cell in P^{-2k+2}_{-2k} with attaching map α_1. We conclude that

$$\alpha_{k-2}\alpha_1^2 \in R^{[3]}_{BP}\big(2^k\big).$$

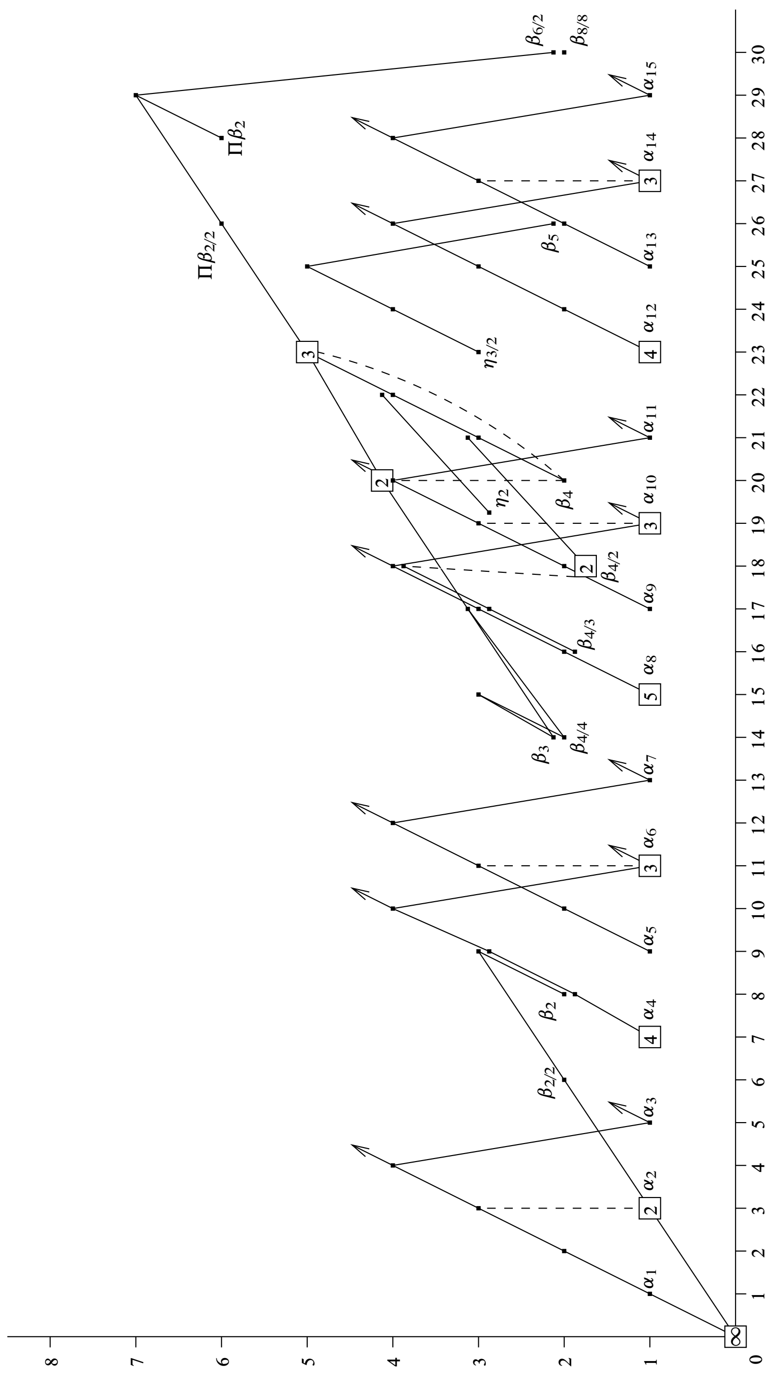


Figure 1: The Adams–Novikov spectral sequence at $p = 2$

The element $\alpha_{k-2}\alpha_1^2$ detects an element of the homotopy root invariant $R(2^k)$.

$k \equiv 4 \pmod 4$**:** The element α_k is a permanent cycle in the ANSS, and detects the element of order 2 in the image of J in the $(2k-1)$–stem. This is the root invariant $R\big(2^k\big)$.

9 The first two BP–filtered root invariants of $\alpha_{i/j}$

In this section we will compute $R^{[k]}_{BP}(\alpha_{i/j})$ for $k = 1, 2$ using the indeterminacy calculations of Section 7. We will then analyze the higher BP–filtered root invariants using the theorems of Section 6.

The following proposition gives the first few filtered root invariants of the $\alpha_{i/j}$. Since we actually use the homotopy root invariant to determine these filtered root invariants, this proposition gives no new information. However, we do see the indeterminacy group $A_{i/j}$ adding essential terms to the $BP \wedge \widetilde{BP}$–root invariant.

Proposition 9.1 *The low dimensional filtered root invariants of the elements $\alpha_{i/j}$ are given (up to a multiple in $\mathbb{Z}_{(2)}^\times$) by the following table.*

x	$R^{[1]}_{BP}(x)$	$R^{[2]}_{BP}(x)$
α_1	$\alpha_{2/2}$	$-$
$\alpha_{2/2}$	$\alpha_{4/4}$	$-$
α_2	$v_1\alpha_{4/4}$	$\alpha_{4/4}\alpha_1$

Proof Proposition 7.5, combined with Proposition 7.4, gives the following values for $R^{[1]}_{BP}$.

$$R^{[1]}_{BP}(\alpha_1) \doteq \widetilde{\beta}_1 + c_1 \cdot v_1\alpha_1$$
$$R^{[1]}_{BP}(\alpha_{2/2}) \doteq \widetilde{\beta}_{2/2} + c_2 \cdot x_7$$
$$R^{[1]}_{BP}(\alpha_2) \doteq \widetilde{\beta}_2 + c_3 \cdot v_1 x_7.$$

Theorem 6.4 implies that

(1) $$d_1\big(R^{[1]}_{BP}(\alpha_{i/j})\big) = 2 \cdot R^{[2]}_{BP}(\alpha_{i/j})$$

for the values of i and j we are considering. The known root invariants (see Mahowald–Ravenel [21]) of these elements are

$$R(\alpha_1) = R(\eta) \doteq \nu \qquad R(\alpha_{2/2}) = R(\nu) \doteq \sigma \qquad R(\alpha_2) = R(2\nu) \doteq \sigma\eta.$$

In the ANSS we have the following representatives of elements of the E_2–term.

$$\begin{aligned}\alpha_{2/2} &= \widetilde{\beta}_1 \\ \alpha_{4/4} &\equiv \widetilde{\beta}_{2/2} + x_7 \pmod 2\end{aligned}$$

We also have the following d_1–differentials in the ANSS.

$$\begin{aligned}d_1(x_7) \equiv d_1(\widetilde{\beta}_{2/2}) &\equiv 2\beta_{2/2} && \pmod 4 \\ d_1(\widetilde{\beta}_2) &\equiv 2\beta_2 && \pmod 4 \\ d_1(\widetilde{\beta}_2 + v_1 x_7) &\equiv 2\alpha_1\alpha_{4/4} && \pmod 4\end{aligned}$$

The only way these differentials can be compatible with Equation (1) and Theorem 6.1 is for the coefficients c_i to have the following values.

$$c_1 \equiv 0 \pmod 2, \qquad c_2 \equiv 1 \pmod 2, \qquad c_3 \equiv 1 \pmod 2. \qquad \square$$

The second filtered root invariants of the rest of the $\alpha_{i/j}$'s are given by the following proposition.

Proposition 9.2 *The filtered root invariant $R^{[2]}_{BP}(\alpha_{i/j})$ for $3i - j \ge 6$ contains (up to a multiple in $\mathbb{Z}^\times_{(2)}$) the element*

$$\beta_{i/j} + \begin{cases} c \cdot \alpha_1 \widetilde{\alpha}_{3i-j-1} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$

with $c \in \mathbb{Z}_{(2)}$. Here $\widetilde{\alpha}_k$ represents the ANSS element $\alpha_{k/l}$ with l maximal.

Proof Proposition 7.5, together with Proposition 7.4 gives

$$\widetilde{\beta}_{i/j} + c \cdot x \,\dot{\in}\, R^{[1]}_{BP}(\alpha_{i/j})$$

where c is an element of $\mathbb{Z}_{(2)}$ and $x \in BP_*\widetilde{BP}$ has the property that the Adams–Novikov differential $d_1(x)$ is given by

$$d_1(x) \doteq \begin{cases} 2\alpha_1\widetilde{\alpha}_{3i-j-1}, & \nu_2(i) \le 3i - j - 2,\ j \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We claim that the condition $\nu_2(i) \le 3i - j - 2$ is always satisfied for $3i - j \ge 6$ where i and j are such that $\alpha_{i/j}$ exists in the Adams–Novikov E_2–term. Indeed, for $\alpha_{i/j}$ to exist we must have $j \le \nu_2(i) + 2$, from which it follows that

$$3i - \nu_2(i) - 4 \le 3i - j - 2.$$

Therefore it suffices to show that $\nu_2(i) \leq 3i - \nu_2(i) - 4$, or equivalently $2\nu_2(i) \leq 3i - 4$. The latter is true for $i \geq 2$, and the condition $3i - j \geq 6$ in particular implies that $i \geq 2$. Thus the condition in the first case of Equation (2) may be simplified to simply read "j odd."

There are Adams–Novikov differentials

$$d_1(\widetilde{\beta}_{i/j}) \equiv 2\beta_{i/j} \pmod 4.$$

Theorem 6.4 applies to give

$$d_1\big(R^{[1]}_{BP}(\alpha_{i/j})\big) = 2 \cdot R^{[2]}_{BP}(\alpha_{i/j})$$

and the result follows. □

10 Higher BP and $H\mathbb{F}_2$–filtered root invariants of some v_1–periodic elements

We denote the Eilenberg–MacLane spectrum $H\mathbb{F}_2$ by H. We describe what happens first in the ASS, and then the ANSS. All of the algebraic root invariants used were taken from Bruner [5]. These algebraic root invariants coincide with the first non-trivial filtered root invariants by Theorem 6.6. The computations are summarized below. We will show in Section 11 that in each of these cases the top filtered root invariant successfully detects the homotopy root invariant through the use of Theorem 6.1.

$R(\overline{\alpha}_1)$ — ASS: The element $\overline{\alpha}_1 = \eta$ is detected by h_1. We have $h_2 \in R_{\mathrm{alg}}(h_1)$. The element h_2 detects ν.
ANSS: We have $\alpha_{2/2} \in R^{[1]}_{BP}(\overline{\alpha}_1)$. The element $\alpha_{2/2}$ also detects ν.

$R(\overline{\alpha}_1^2)$ — ASS: The algebraic root invariant is given by $h_2^2 \in R_{\mathrm{alg}}(h_1^2)$, and h_2^2 detects ν^2.

$R(\overline{\alpha}_1^3)$ — ASS: We have $h_2^3 \in R_{\mathrm{alg}}(h_1^3)$, which detects ν^3.

$R(\overline{\alpha}_{2/2})$ — ASS: The element $\overline{\alpha}_{2/2}$ is detected by h_2. The algebraic root invariant is given by $h_3 \in R_{\mathrm{alg}}(h_2)$. The element h_3 detects σ.
ANSS: The first filtered root invariant is given by $\alpha_{4/4} \in R^{[1]}_{BP}(\alpha_{2/2})$, and $\alpha_{4/4}$ detects σ.

$R(\overline{\alpha}_2)$ — ASS: The element $\overline{\alpha}_2$ is detected by h_0h_2, and $h_1h_3 \in R_{\mathrm{alg}}(h_0h_2)$ detects $\eta\sigma$.
ANSS: The second filtered root invariant is given by $\alpha_{4/4}\alpha_1 \in R^{[2]}_{BP}(\overline{\alpha}_2)$, which detects $\eta\sigma$.

$R(\overline{\alpha}_{4/4})$ ASS: The element $\overline{\alpha}_{4/4}$ is detected by h_3 in the ASS. The algebraic root invariant is given by $h_4 \in R_{\mathrm{alg}}(h_3)$, which coincides with the first filtered root invariant $R_H^{[1]}(h_3)$ by Theorem 6.6. The Hopf invariant 1 differential $d_2(h_4) = h_0 h_3^2$ allows one apply Theorem 6.4 and get $h_3^2 \in R_H^{[2]}(h_3)$. The element h_3^2 detects σ^2.

ANSS: We have the second filtered root invariant $\beta_{4/4} \in R_{BP}^{[2]}(\overline{\alpha}_{4/4})$, and $\beta_{4/4}$ detects σ^2.

$R(\overline{\alpha}_{4/3})$ ASS: The element $\overline{\alpha}_{4/3}$ is detected by $h_0 h_3$ in the ASS. The algebraic root invariant is $h_1 h_4 \in R_{\mathrm{alg}}(h_0 h_3)$, which detects η_4.

ANSS: The second filtered root invariant is $\beta_{4/3} + c\alpha_{8/5}\alpha_1 \in R_{BP}^{[2]}(\overline{\alpha}_{4/3})$, for $c \in \mathbb{Z}/2$. The element $\beta_{4/3}$ detects η_4. The value of c is irrelevant — the AHSS differentials of Mahowald [19] imply that if η_4 is in the root invariant $R(2\sigma)$ (which it is), then $\alpha_1\alpha_{8/5}$ is in the indeterminacy of this root invariant. In the AHSS for $\pi_*(P_{-10}^{\infty})$ the element $\overline{\alpha}_1\overline{\alpha}_{8/5}[-10]$ is the target of a differential supported by $\overline{\alpha}_5[-2]$.

$R(\overline{\alpha}_{4/2})$ ASS: The element $\overline{\alpha}_{4/2}$ is detected by $h_0^2 h_3$. The algebraic root invariant is $h_1^2 h_4 \in R_{\mathrm{alg}}(h_0^2 h_3)$, and this detects $\eta\eta_4$.

ANSS: The second filtered root invariant is given by $\beta_{4/2} \in R_{BP}^{[2]}(\overline{\alpha}_{4/2})$, where $c \in \mathbb{Z}/2$. The element $\beta_{4/2}$ is a permanent cycle but does not survive to the homotopy root invariant. We appeal to Theorem 6.5 to find a higher filtered root invariant. In P_{-13}^{-12} the -12–cell attaches to the -13–cell with degree 2 attaching map, and the -11–cell attaches to the -13–cell in P_{-13}^{-11} by η. There is a hidden extension in the ANSS given by $\overline{\alpha}_1 \cdot \overline{\alpha}_1\overline{\beta}_{4/3} = 2 \cdot \overline{\beta}_{4/2}$, which indicates that we have a higher filtered root invariant $\alpha_1\beta_{4/3} \in R_{BP}^{[3]}(\overline{\alpha}_{4/2})$. The element $\alpha_1\beta_{4/3}$ detects $\eta\eta_4$.

$R(\overline{\alpha}_4)$ ASS: The element $\overline{\alpha}_4$ is detected by $h_0^3 h_3$, with algebraic root invariant $h_1^3 h_4 \in R_{\mathrm{alg}}(h_0^3 h_3)$ which detects $\eta^2\eta_4$.

ANSS: The second filtered root invariant is given by $(\beta_4 + c\alpha_1\alpha_{10} \in R_{BP}^{[2]}(\overline{\alpha}_4)$. It turns out that $\alpha_1\alpha_{10}$ is in the indeterminacy of $R_{BP}^{[2]}(\overline{\alpha}_4)$, since in the AHSS for P_{-14}^{∞}, we have $d_2(\alpha_{10}[-12]) = \alpha_{10}\alpha_1[-14]$. Therefore we may as well set $c = 0$. The element β_4 corresponds to the element g in the ASS. The element β_4 is a permanent cycle, so we use Theorem 6.5 to look for a higher root invariant.

In P_{-15}^{-12} the -14–cell attaches to the -15–cell by the degree 2 map, and there is a Toda bracket $\langle P_{-15}^{-12}\rangle(-) \overset{\cap}{=} \langle 2, \alpha_1, -\rangle$.

There is a hidden extension in the ANSS given by $2\overline{\beta}_4 = (\overline{\beta}_{4/4}\overline{\beta}_{2/2})/2$, and in the ANSS there is a Toda bracket $(\beta_{4/4}\beta_{2/2})/2 \in \langle 2, \alpha_1, \beta_{4/3}\alpha_1^2\rangle$.

We conclude using Theorem 6.5 that we have the higher filtered root invariant $\beta_{4/3}\alpha_1^2 \in R_{BP}^{[4]}(\overline{\alpha}_4)$. The element $\beta_{4/3}\alpha_1^2$ detects $\eta^2\eta_4$.

$R(\overline{\alpha}_{4/4}\overline{\alpha}_1)$ ASS: The element $\overline{\alpha}_{4/4}\overline{\alpha}_1$ is detected by h_3h_1 with algebraic root invariant $h_4h_2 \in R_{\mathrm{alg}}(h_3h_1)$ which detects ν^*.

ANSS: The root invariant of $\overline{\alpha}_{4/4}$ will turn out to be given by $\overline{\beta}_{4/4}$, so one might initially suspect that $\overline{\alpha}_{2/2}\overline{\beta}_{4/4}$ would detect the homotopy root invariant $R(\overline{\alpha}_{4/4}\overline{\alpha}_1)$. However, the -6–cell attaches to the -10–cell with attaching map ν. Thus in the AAHSS for P_{-10}, there is a differential $d_4(\beta_{4/4}[-6]) = \alpha_{2/2}\cdot\beta_{4/4}[-10]$. Looking at the attaching map structure of P_{-11}^{-6}, this differential actually pushes the root invariant to $\beta_{4/2,2} \in \langle\alpha_{2/2}, 2, \beta_{4/4}\rangle$, and this element detects ν^*.

$R(\overline{\alpha}_{4/4}\overline{\alpha}_1^2)$ ASS: The element $\overline{\alpha}_{4/4}\overline{\alpha}_1^2$ is detected by $h_3h_1^2$, with algebraic root invariant $h_4h_2^2 \in R_{\mathrm{alg}}(h_3h_1^2)$. which detects $\nu^*\nu$.

$R(\overline{\alpha}_5)$ ASS: The element $\overline{\alpha}_5$ is detected by Ph_1, with algebraic root invariant $h_2g \in R_{\mathrm{alg}}(Ph_1)$ which detects $\nu\overline{\kappa}$.

ANSS: The second filtered root invariant is given by $\beta_5 + c\alpha_{13}\alpha_1 \in R_{BP}^{[2]}(\overline{\alpha}_5)$, where $c \in \mathbb{Z}/2$. Theorem 6.4 applies to the differential $d_3(\beta_5 + c\alpha_{13}\alpha_1) = \alpha_1^2\eta_{3/2}$ to give the higher filtered root invariant $\alpha_1\eta_{3/2} \in R_{BP}^{[4]}(\overline{\alpha}_5)$. The element $\alpha_1\eta_{3/2}$ corresponds to the element $h_4c_0h_1$ in the ASS. Using the May spectral sequence, we see that there is a Massey product $h_2^2g \in \langle h_0, h_1, h_4c_0h_1\rangle$. The element h_2^2g is a permanent cycle which corresponds to the element $\Pi\beta_{2/2}$ in the ANSS. We conclude that in the ANSS there is a hidden Toda bracket $\overline{\alpha}_{2/2}\cdot\overline{\beta}_4\overline{\alpha}_1^3/8 = \overline{\Pi\beta_{2/2}} \in \langle 2, \overline{\alpha}_1, \overline{\alpha}_1\overline{\eta}_{3/2}\rangle$. In P_{-19}^{-15}, the -16–cell attaches to the -19–cell with the Toda bracket $\langle 2, \overline{\alpha}_1, -\rangle$ and the -15–cell attaches to the -19–cell with attaching map $\overline{\alpha}_{2/2}$. We conclude, using Theorem 6.5, that we have another higher filtered root invariant $\beta_4\alpha_1^3/8 \in R_{BP}^{[5]}(\overline{\alpha}_5)$. The element $\beta_4\alpha_1^3/8$ detects $\nu\overline{\kappa}$.

$R(\overline{\alpha}_5\overline{\alpha}_1)$ ASS: The element $\overline{\alpha}_5\overline{\alpha}_1$ is detected by Ph_1^2, with algebraic root invariant $r \in R_{\mathrm{alg}}(Ph_1^2)$. Theorem 6.6 implies that we have the filtered root invariant $r \in R_H^{[6]}(\overline{\alpha}_5\overline{\alpha}_1)$. The element r supports the Adams differential $d_3(r) = h_1d_0^2$. Thus, we may use Theorem 6.4 to deduce that there is a higher filtered root invariant $d_0^2 \in R_H^{[8]}(\overline{\alpha}_5\overline{\alpha}_1)$ which detects $\kappa^2 = \epsilon\overline{\kappa}$.

$R(\overline{\alpha}_5\overline{\alpha}_1^2)$ ASS: The element $\overline{\alpha}_5\overline{\alpha}_1^2$ is detected by Ph_1^3, with algebraic root invariant $h_1q \in R_{\mathrm{alg}}(Ph_1^3)$.

$R(\overline{\alpha}_{6/2})$ ASS: This element presents a very interesting story: it is an instance where the ASS seems to give us nothing yet the ANSS tells us the homotopy root invariant. The element $\overline{\alpha}_{6/2}$ corresponds to the element Ph_2 in the ASS, with algebraic root invariant $h_0^3 h_4^2 \in R_{\mathrm{alg}}(Ph_2)$. The element $h_0^3 h_4^2$ is killed by a differential in the ASS, and it is not clear what a candidate for the root invariant should be.
ANSS: The second filtered root invariant is given by $\beta_{6/2} \in R_{BP}^{[2]}(\overline{\alpha}_{6/2})$. There is an Adams–Novikov differential $d_5(\beta_{6/2}) = \alpha_{2/2} \cdot \Pi\beta_{2/2}$. Theorem 6.4 indicates that we have the higher filtered root invariant $\Pi\beta_{2/2} \in R_{BP}^{[6]}(\alpha_{6/2})$ which detects $\nu^2\overline{\kappa}$.

$R(\overline{\alpha}_6)$ ASS: The element $\overline{\alpha}_6$ is detected by Ph_2h_0, with algebraic root invariant $q \in R_{\mathrm{alg}}(Ph_2h_0)$.

11 Homotopy root invariants of some v_1–periodic elements

In this section we will use the filtered root invariant computations of Section 10 to compute some homotopy root invariants. These computations are summarized in the following theorem. The first few are well-known (see Mahowald–Ravenel [21]).

Theorem 11.1 *We have the following table of homotopy root invariants (up to some multiple in $\mathbb{Z}_{(2)}^{\times}$).*

x	$R(x)$
$\overline{\alpha}_1 = \eta$	ν
$\overline{\alpha}_1^2 = \eta^2$	ν^2
$\overline{\alpha}_1^3 = \eta^3$	ν^3
$\overline{\alpha}_{2/2} = \nu$	σ
$\overline{\alpha}_2 = 2\nu$	$\sigma\eta$
$\overline{\alpha}_{4/4} = \sigma$	σ^2
$\overline{\alpha}_{4/3} = 2\sigma$	η_4
$\overline{\alpha}_{4/2} = 4\sigma$	$\eta\eta_4$

x	$R(x)$
$\overline{\alpha}_4 = 8\sigma$	$\eta^2\eta_4$
$\overline{\alpha}_{4/4}\overline{\alpha}_1 = \eta\sigma$	ν^*
$\overline{\alpha}_{4/4}\overline{\alpha}_1^2 = \sigma\eta^2$	$\nu\nu^*$
$\overline{\alpha}_5$	$\nu\overline{\kappa}$
$\overline{\alpha}_5\overline{\alpha}_1$	$\kappa^2 = \epsilon\overline{\kappa}$
$\overline{\alpha}_5\overline{\alpha}_1^2$	$\eta\overline{q}$
$\overline{\alpha}_{6/2}$	$\nu^2\overline{\kappa}$
$\overline{\alpha}_6$	$\overline{q}$

Some of these root invariants may have indeterminacy.

We pause to remark on the elements that show up as root invariants in this table. With the exception of ν, σ, $\sigma\eta$, and $\nu^3 = \sigma\eta^2$, all of these elements are v_2–periodic. This was shown by Mahowald in [18], and Hopkins and Mahowald in [11]. We remind the reader that due to an error in Davis–Mahowald [7], one must replace $8k$ with $32k$ in [18]. Recently, Hopkins and Mahowald have produced some v_2^{32}–self maps [10]. In particular, in [19, Problem 4] (see also [7]), a list of v_2–periodic elements in $\pi_*(S)$ are given which are the first few homotopy Greek letter elements β_i^h:

$$\nu, \nu^2, \nu^3, \eta^2\eta_4, \nu\overline{\kappa}, \epsilon\overline{\kappa}, \eta\overline{q}, \ldots$$

Our computations show that these elements appear as the iterated root invariants $R(R(2^i))$ for $i \leq 7$.

The rest of this section is devoted to proving Theorem 11.1. We use Corollary 6.2 to deduce the homotopy root invariants from our filtered root invariants. In our range, the first part of Corollary 6.2 is easier to check using the ANSS rather than the ASS, since there are fewer elements to check. However, we have only determined the BP–filtered root invariants for the elements in Adams–Novikov filtration 1. For the v_1–periodic elements in higher Adams–Novikov filtration, we must use our $H\mathbb{F}_2$–filtered root invariants and the ASS. Since the author's knowledge of the 2–primary ANSS does not include β_6, we also use the ASS to compute $R(\alpha_6)$. The second part of Corollary 6.2 is verified afterwards.

Our computations for the first part of Corollary 6.2 using BP–filtered root invariants and the ANSS are summarized in the following table.

x	$R_{BP}^{[k]}(x)$	$-N$	$\{\gamma_i[n_i]\}$	diff
$\overline{\alpha}_1$	$\alpha_{2/2}$	-3		
$\overline{\alpha}_{2/2}$	$\alpha_{4/4}$	-5	$\beta_{2/2}[-4]$	$\leftarrow \alpha_1[2]$
$\overline{\alpha}_2$	$\alpha_{4/4}\alpha_1$	-6		
$\overline{\alpha}_{4/4}$	$\beta_{4/4}$	-8		
$\overline{\alpha}_{4/3}$	$\beta_{4/3}$	-10	$\beta_{4/4}\alpha_1[-9]$	$\leftarrow \beta_{4/4}[-7]$
$\overline{\alpha}_{4/2}$	$\beta_{4/3}\alpha_1$	-11		
$\overline{\alpha}_4$	$\beta_{4/3}\alpha_1^2$	-12		
$\overline{\alpha}_5$	$\alpha_1^3\beta_4/8$	-15		
$\overline{\alpha}_{6/2}$	$\Pi\beta_{2/2}$	-16		

We explain the columns of the table:

x	The element we wish to compute the root invariant of.
$R_{BP}^{[k]}(x)$	The top BP–filtered root invariant of x, which we want to show detects $R(x)$ using Corollary 6.2.
$-N$	The cell of $P_{-\infty}^{\infty}$ which carries the filtered root invariant.

$\{\gamma_i[n_i]\}$ The collection of elements in the E_1–term of the AAHSS converging to $\mathrm{Ext}(BP_* P_{-N})$ which could detect the difference between the top filtered root invariant and the homotopy root invariant. We exclude elements of infinite α_1–towers, since these are always the source or target of a d_2–differential in the AAHSS.

diff Each element $\gamma_i[n_i]$ turns out to be ineligible to detect the difference, since it is the target the indicated AAHSS differential.

We now deal with the leftover elements using the $H\mathbb{F}_2$–filtered root invariants and the ASS. We have the following $H\mathbb{F}_2$–filtered root invariants which are permanent cycles in the ASS.

x	$R_H^{[k]}(x)$	$-N$
$\overline{\alpha}_{4/4}\overline{\alpha}_1$	$h_4 h_2$	-11
$\overline{\alpha}_{4/4}\overline{\alpha}_1^2$	$h_4 h_2^2$	-13
$\overline{\alpha}_5\overline{\alpha}_1$	d_0^2	-19
$\overline{\alpha}_5\overline{\alpha}_1^2$	$h_1 q$	-23
$\overline{\alpha}_6$	q	-22

Using Corollary 6.2, we see that to verify that these filtered root invariants detect homotopy root invariants, we must first check that there are no elements of $\pi_{t-1}(P_{-N})$ of Adams filtration greater than k which can detect the root invariant on a higher cell. We handle this on a case-by-case basis, with the aid of the computations of Mahowald in [16], and the computer Ext computations of Bruner [4]. In the following analysis, we omit the elements detected in the AHSS by v_1–periodic elements. These elements cannot be root invariants in the stems we are considering by the following lemma.

Lemma 11.2 *Suppose that $\gamma[n] \in \pi_j(S^n)$ is an element of the E_1–term of the AHSS for $\pi_j(P_{-\infty}^{\infty})$ where γ is a v_1–periodic element. Then $\gamma[n]$ is either the source or target of a non-trivial AHSS differential unless we are in one of the following cases (in which case we do not know whether $\gamma[n]$ is the source or target of a differential).*

- $\gamma = \nu$ *and* $j \equiv 0 \pmod 4$
- $\gamma = B_i$ *with* $j \equiv 6 \pmod 8$ *and* $i \leq \nu_2(j+2) - 2$
- $\gamma = \sigma$, $\sigma\eta$, *or* $\sigma\eta^2$ *with* $j \equiv 6 \pmod 8$ *(The behavior of these elements is slightly anomalous, due to the presence of* ν^2, ϵ, *and* $\eta\epsilon$*.)*

Here B_i is i^{th} generator of the image of the J–homomorphism. Therefore, if $i = 4a+b$ with $0 \le b \le 3$, we have

$$B_i = \begin{cases} \widetilde{\alpha}_{4a}\overline{\alpha}_1^2, & b=0 \\ \widetilde{\alpha}_{4a+2}, & b=1 \\ \widetilde{\alpha}_{4(a+1)}, & b=2 \\ \widetilde{\alpha}_{4(a+1)}\overline{\alpha}_1, & b=3 \end{cases}$$

where $\widetilde{\alpha}_k$ is the element $\overline{\alpha}_{k/l}$ with l maximal.

Proof Mahowald [17, Theorem 4.6] states that you can lift the differentials from the J–homology modified AHSS to the (double suspension) EHP spectral sequence. The proofs of the announcements in [17] are the subject of [19]. Since the EHP spectral sequence maps to the AHSS for $\pi_*(P^\infty)$, the differentials in the AHSS follow. We then get the result for the AHSS for $P_{-\infty}^{\infty}$ by transporting our differentials with James periodicity. □

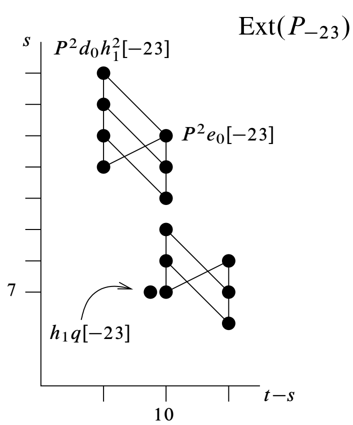
The first part of Corollary 6.2 on the remaining elements is verified as follows.

$\overline{\alpha}_{4/4}\overline{\alpha}_1$: According to the tables of Mahowald [16], the only elements of $\pi_7(P_{-11})$ of Adams filtration greater than 2 are v_1–periodic.

$\overline{\alpha}_{4/4}\overline{\alpha}_1^2$: According to the tables of [16], there are no elements of $\pi_8(P_{-13})$ of Adams filtration greater than 3.

$\overline{\alpha}_5\overline{\alpha}_1$: Examining the tables of [16], the only elements of $\pi_9(P_{-19})$ of Adams filtration greater than 8 have trivial image in $\pi_9(P_{-18})$. Therefore, none of them can detect a root invariant carried by a cell above the -19 cell.

$\overline{\alpha}_5\overline{\alpha}_1^2$: Examining the tables of Bruner [4], we find the following pattern of generators in $\mathrm{Ext}(H_*P_{-23})$.

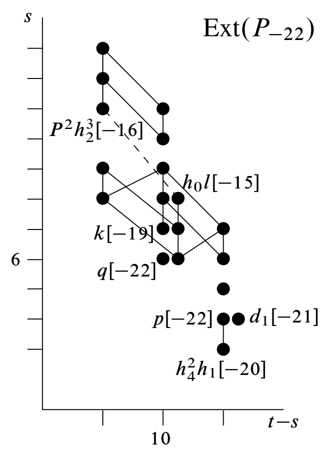


Some of the elements are labeled with their AAHSS names. These names were deduced from the AAHSS differentials computed in [16]. The Adams differentials originating from the elements in Adams filtration 6 and 7 may be deduced by extrapolating differentials computed in [16] using h_0, h_1, and h_2 multiplication. The inclusion of the bottom cell

$$S^{-23} \to P_{-23}$$

induces the differential $d_2(P^2e_0[-23]) = P^2d_0h_1^2[-23]$. The rest of the differentials are then forced by h_0 multiplication. We deduce that the only elements of $\pi_{10}(P_{-23})$ of Adams filtration greater than 7 are the v_1–periodic elements.

$\overline{\alpha}_6$: From the tables of Bruner [4], we find the following portion of $\mathrm{Ext}(H_*P_{-22})$.



All of the d_2 differentials shown are extrapolated from differentials in the charts of Mahowald [16] using h_0, h_1, and h_2–multiplication. The remaining two elements that could detect elements of higher Adams filtration, $h_0l[-15]$ and $k[-19]$, must be handled with care. We make the following claims, which combine to show there are no classes in $\pi_{10}(P_{-22})$ of Adams filtration greater than 6 which could detect the root invariant of $\overline{\alpha}_6$.

(1) There is an Adams differential $d_3(h_0l[-22]) = P^2h_2^3[-16]$ (as indicated by a dashed line in the chart).

(2) The element $k[-19]$ is a non-trivial permanent cycle which detects an element $\gamma \in \pi_{10}(P_{-22})$.

(3) The image of γ in $\pi_{10}(P_{-21})$ cannot agree with the image of $\overline{\alpha}_6$ under the composite $S^{-1} \to P_{-\infty}^{\infty} \to P_{-21}$.

Proof of (1) In the ASS for $\pi_*(S^0)$, there is a differential

$$d_2(h_0 l) = Pe_0 h_2^2.$$

In the AAHSS for $\mathrm{Ext}(H_* P_{-22})$, there is a differential

$$d_6(Pe_0h_1[-16]) = \langle Pe_0h_1, h_2, h_1\rangle[-22] = Pe_0h_2^2[-22].$$

However, in the ASS for $\pi_*(S^0)$, there is a differential

$$d_2(Pe_0h_1) = P^2h_2^3.$$

We conclude that in the E_3–term of the ASS for $\pi_*(P_{-22})$, the elements $Pe_0h_2^2[-22]$ and $P^2h_2^3[-16]$ have been equated. Thus, the element $h_0l[-22]$ must kill the element $P^2h_2^3[-16]$.

Proof of (2) The generator of $\mathrm{Ext}(H_*P_{-22})$ in $(t-s,s) = (11,5)$ cannot support a d_2 killing $k[-19]$ because it does not support non-trivial h_0 multiplication. The elements p, d_1, and $h_4^2h_1$ of $\mathrm{Ext}(\mathbb{F}_2)$ detect homotopy elements $\overline{p}$, $\overline{d}_1$, and $\eta\theta_4$ in $\pi_*(S)$. These elements are easily seen to extend to elements $\overline{p}[-22]$, $\overline{d}_1[-21]$, and $\eta\theta_4[-20]$ of $\pi_{11}(P_{-22}^{-20})$. The elements $p[-22]$, $d_1[-21]$, and $h_4^2h_1[-20]$ detect the images of $\overline{p}[-22]$, $\overline{d}_1[-21]$, and $\eta\theta_4[-20]$ under the inclusion

$$P_{-22}^{-20} \to P_{-22}.$$

Therefore, the elements $p[-22]$, $d_1[-21]$ and $h_4^2h_1[-20]$ must be permanent cycles. There are no other elements of $\mathrm{Ext}(H_*P_{-22})$ which can kill $k[-19]$.

Proof of (3) Let $\gamma \in \pi_{10}(P_{-22})$ be detected by the permanent cycle $k[-19]$. Let ν_N denote the composite

$$\pi_*(S^{-1}) \to \pi_*(P_{-\infty}) \to \pi_*(P_N).$$

Let γ' be the image of γ in $\pi_{10}(P_{-21})$. Since the element $k[-19]$ is non-trivial in $\mathrm{Ext}(H_*P_{-21})$, the element γ' has Adams filtration 7. We must show that γ' cannot equal $\nu_{-21}(\overline{\alpha}_6)$. Let $\overline{q} \in \pi_{32}(S^0)$ be the element detected by q. Examining the ASS for S^0 (see Mahowald–Tangora [24], and Ravenel [27]), we see that the element $\overline{q}$ extends to an element $\delta = \overline{q}[-22]$ in $\pi_{10}(P_{-25})$. Since the algebraic root invariant of Ph_2h_0 is q, the element $\nu_{-25}(\overline{\alpha}_6)$ is detected in $\mathrm{Ext}(H_*P_{-22})$ by $q[-22]$. Since $q[-22]$ also detects δ, the sum $\zeta = \nu_{-25}(\overline{\alpha}_6) + \delta$ has Adams filtration greater than 6. Consulting the charts in [4], we find that there is one generator in $\mathrm{Ext}^{s,t}(H_*P_{-25})$ for $(t-s,s) = (10,7)$, which is detected in the AAHSS by the element $h_1q[-23]$. The image of ζ in $\pi_{10}(P_{-21})$ is $\nu_{-21}(\overline{\alpha}_6)$. Since the image of $h_1q[-23]$ in $\mathrm{Ext}(H_*P_{-21})$

is zero, we deduce that $\nu_{-21}(\overline{\alpha}_6)$ is of Adams filtration greater than 7. Thus $\nu_{-21}(\overline{\alpha}_6)$ cannot equal γ', since γ' has Adams filtration 6.

We have verified the first part of Corollary 6.2. We now must fulfill the second part of Corollary 6.2. Suppose that we are given an element $x \in \pi_*(S)$ and we want to see that y is an element of $R(x)$ for some $y \in \pi_*(S)$, where the root invariant is carried by the $-N$–cell. Suppose that y was detected by a filtered root invariant and that we have already verified the first part of Corollary 6.2. Then the root invariant of x is y if we can show that the image of the element y under the inclusion of the bottom cell

$$\pi_*(S^{-N}) \to \pi_*(P_{-N})$$

is nontrivial.

Lemma 11.3 *Let E be either $H\mathbb{F}_2$ or BP and suppose that $\widetilde{y} \in \mathrm{Ext}(E_*)$ detects y in the E–ASS. It suffices to show the following two things:*

(1) *The element $\widetilde{y}[-N]$ is not the target of a differential in the AAHSS.*

(2) *This element of $\mathrm{Ext}(E_* P_{-N})$ which $\widetilde{y}[-N]$ detects is not the target of an E–ASS differential.*

We have the following convenient proposition.

Proposition 11.4 *If z is a v_1–torsion element of $\mathrm{Ext}^{2,j}(BP_*)$ with $j \equiv 0 \pmod 2$, then in the AAHSS for $\mathrm{Ext}(BP_* P_{2m})$ the element $z[2m]$ cannot be the target of a differential.*

Proof The only elements which can support AAHSS differentials that hit $z[2m]$ are those of the form

$$1[2k-1] \qquad \text{or} \qquad \alpha_{i/j}[2k].$$

We only need to consider elements in $t - s \equiv 1 \pmod 2$, since $z[2m]$ is in $t - 2 \equiv 0 \pmod 2$. The differentials given in Propositions 7.2 and 7.3 tell us that these elements either kill or are killed by other v_1–periodic elements. □

Corollary 11.5 *The second filtered root invariant $\beta_{i/j} \in R^{[2]}_{BP}(\alpha_{i/j})$ always satisfies the second part of Corollary 6.2.*

We now finish the proof of Theorem 11.1 by verifying the second part of Corollary 6.2 each of our elements.

For the root invariants of the elements

$$\overline{\alpha}_{4/4}, \quad \overline{\alpha}_{4/3}, \quad \overline{\alpha}_6$$

we simply invoke Corollary 11.5. We mention that $\overline{q}$ is detected in the ANSS by the element β_6.

For the root invariants of the elements

$$\overline{\alpha}_1, \quad \overline{\alpha}_1^2, \quad \overline{\alpha}_1^3, \quad \overline{\alpha}_{2/2}, \quad \overline{\alpha}_2, \quad \overline{\alpha}_{4/4}\overline{\alpha}_1, \quad \overline{\alpha}_{4/4}\overline{\alpha}_1^2, \quad \overline{\alpha}_{4/2}, \quad \overline{\alpha}_4, \quad \overline{\alpha}_5, \quad \overline{\alpha}_{6/2}$$

we look at the tables in [16] to see that the required elements are non-zero in the homotopy of the stunted projective spaces.

The root invariant of $\overline{\alpha}_5\overline{\alpha}_1$ requires special treatment. We recall that we have the algebraic root invariant $d_0^2 \in R_{\mathrm{alg}}(\overline{\alpha}_5\overline{\alpha}_1)$ carried by the -19–cell. We must verify that the image of $\epsilon\overline{\kappa} = \kappa^2$ under the map

$$\pi_*(S^{-19}) \to \pi_*(P_{-19})$$

is non-zero. The problem is that when we look at Mahowald's computations [16] we see that $d_0^2[-19]$ is killed in the AAHSS. Indeed, we have the algebraic Atiyah–Hirzebruch differential

$$d_6^{\mathrm{AAHSS}}(i[-13]) = d_0^2[-19].$$

However, we have the Adams differential

$$d_2^{\mathrm{ASS}}(i) = Pd_0h_0.$$

Therefore, in the E_3–term of the ASS for P_{-19}, the elements $Pd_0h_0[-13]$ and $Pd_0h_1[-14]$ have been equated, so we may conclude that the combination of the AAHSS and ASS differentials implies that the image of $\epsilon\overline{\kappa}$ under the inclusion of the bottom cell is detected by $Pd_0h_1[-14]$. Mahowald's tables [16] indicate that this element is non-zero in $\pi_*(P_{-19})$.

For the purposes of determining the root invariant of

$$\overline{\alpha}_5\overline{\alpha}_1^2$$

we must determine whether the image of $\eta\overline{q}$ under the map

$$\pi_*(S^{-23}) \to \pi_*(P_{-23})$$

is non-zero. Examining the tables of Bruner [4], we see that the element $h_1q[-23]$ is non-trivial in $\mathrm{Ext}(P_{-23})$. We therefore just need to check that it cannot be the target of a differential. There are three possible sources of a differential that would kill $h_1q[-23]$. These are represented by the elements

$$n[-20], \quad d_1[-21], \quad \text{and} \quad h_5h_2[-23].$$

But these elements are permanent cycles, as argued in the following lemmas.

Lemma 11.6 *The element $n[-20] \in \mathrm{Ext}^{5,16}(H_* P_{-23})$ is a permanent cycle.*

Proof We just need to show that the element $\overline{n} \in \pi_{11}\big(S^{-20}\big)$ extends over P_{-23}^{-20}. Since $2\overline{n} = 0$ and $\eta\overline{n} = 0$, it suffices to show that the Toda bracket

$$\langle 2, \eta, \overline{n} \rangle$$

contains 0. The element $\overline{n}$ is given by the Toda bracket

$$\overline{n} \in \langle \nu, \sigma, \overline{\kappa} \rangle$$

(see Mahowald–Tangora [24]). We have

$$\langle 2, \eta, \overline{n} \rangle = \langle 2, \eta, \langle \nu, \sigma, \overline{\kappa} \rangle \rangle \supseteq \langle 2, \eta, \nu, \sigma \rangle \overline{\kappa}.$$

However, the Toda bracket $\langle 2, \eta, \nu, \sigma \rangle$ lies in $\pi_{13}(S^0)$, hence it must be zero. □

Lemma 11.7 *The element $d_1[-21] \in \mathrm{Ext}^{4,15}(H_* P_{-23})$ is a permanent cycle.*

Proof The element $\overline{d}_1 \in \pi_{32}(S^0)$ extends over P_{-23}^{-21} to give an element which is detected by $d_1[-21]$, so we may conclude that $d_1[-21]$ is a permanent cycle. □

The author thanks W H Lin for supplying the proof of the following lemma.

Lemma 11.8 *The element $h_5 h_2[-23] \in \mathrm{Ext}^{2,13}(H_* P_{-23})$ is a permanent cycle.*

Proof The element $h_5 h_2$ supports a differential of the form $d_3(h_5 h_2) = h_0 p$ in the ASS for S^0. However, in the AAHSS for $\mathrm{Ext}\big(H_* P_{-23}^{-22}\big)$, the element $h_0 p[-23]$ is killed by $p[-22]$. The element $\overline{p} \in \pi_{33}(S^0)$ therefore extends to an element $\overline{p}[-22] \in \pi_{11}(P_{-23})$ which is detected by $h_5 h_2[-23]$ in the ASS. We conclude that $h_5 h_2[-23]$ is also a permanent cycle. □

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