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Divisibility of characteristic numbers

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We use homotopy theory to define certain rational coefficients characteristic numbers with integral values, depending on a given prime number q and positive integer t. We prove the first nontrivial *degree formula* and use it to show that existence of morphisms between algebraic varieties for which these numbers are not divisible by q give information on the degree of such morphisms or on zero cycles of the target variety.

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1 Introduction

Given a complex vector bundle V on a smooth, compact complex manifold M of dimension m, we can associate a collection of integers $\{n_I\}_I$ which are bundle isomorphisms invariant. These numbers are obtained as follows: let $I = (i_1, i_2, \dots, i_r)$ and $f_I(c_j) = c_1^{i_1}(V)c_2^{i_2}(V)\dots c_r^{i_r}(V)$ be a monomial of degree m in the Chern classes of V (we set c_j to have degree $2j$). Then n_I is defined to be $\langle f_I(c_j), [M]_{\mathbf{H}_{\mathbb{Z}}}\rangle$, that is the Kronecker pairing of $f_I(c_j)$ with the integral coefficients fundamental homology class of M. We will deal with numbers associated with two special bundles: one is the tangent bundle T_M, and the other is the normal bundle ν_i to a closed embedding $i\colon M \hookrightarrow \mathbb{R}^N$ for N large enough. By the Thom–Pontryagin construction, if $[M]$ denotes the complex cobordism class of M, **MU** the complex cobordism spectrum and h is its Hurewicz homomorphism $\pi_*(\mathbf{MU}) \to H_*(\mathbf{MU}, \mathbb{Z})$, then the coefficients of the polynomial generators of $H_*(\mathbf{MU}, \mathbb{Z})$ in the expression of $h[M]$ are certain numbers $\langle s_{(\alpha_1,\alpha_2,\dots,\alpha_r)}(\nu_i), [M]_{\mathbf{H}_{\mathbb{Z}}}\rangle$ associated to ν_i. These numbers are sometimes referred to as *the characteristic numbers of* M. Notice that, by definition, $T_M + \nu_i = 0$ in $K_0(M)$, the Grothendieck group of vector bundles on M. Thus such numbers depend just on M (in fact on the complex cobordism class of M). Since the image of h is completely known (see Stong [9]), we get information on such numbers, for instance, if $s_{(\alpha_1,\alpha_2,\dots,\alpha_r)}(M)$ is the coefficient of $b_1^{\alpha_1}b_2^{\alpha_2}\dots b_r^{\alpha_r} \in H_*(\mathbf{MU}, \mathbb{Z})$ in $h[M]$, then we know that p always divides $s_{(0,0,\dots,1)}(M)$, where the 1 entry is in the (p^t-1)th place, for any prime number p and positive integer t. We will denote such an integer as $s_{p^t-1}(M)$. More in general, for each choice of polynomial generators of $\pi_*(\mathbf{MU})$,

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the coefficients of $[M]$ are integers and can be expressed in function of characteristic numbers of M itself, which can be written as a rational linear combination of Chern numbers (eg expression (3) for algebraic surfaces). This way one gets divisibility properties of Chern numbers, much like, by the Grothendieck–Riemann–Roch formula, we deduce divisibility properties of the Todd numbers. In a sense, such divisibility is maximal and the varieties which make it maximal enjoy surprising properties. In the case of $s_{p^t-1}(M)$, Voevodsky remarked that the algebraic varieties for which that number is *not* divisible by p^2 are very special. Given a smooth algebraic variety X over a field k we can talk about *characteristic numbers* by considering the polynomials in the Chern classes of the tangent bundle of X, $s_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(\nu_i) = s'_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(T_X)$. They are represented by zero dimensional cycles, that is integral linear combinations $z = \sum_u n_u z_u$, where $z_u\colon \operatorname{Spec} k_u \to X$ are (closed) points of X (that is, k_u are finite field extensions of k). An integer can be associated to z: the *degree* of z (see Definition 2.2), denoted with $\deg(z)$. The integer $\deg s'_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(T_X)$ plays the role of $s_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(M)$. This statement can be made more precise: if X admits a closed embedding $X \hookrightarrow \mathbb{P}^N$ (such varieties are said *projective*) and k admits an embedding in the complex numbers, then $\deg s'_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(T_X)$ equals $s_{(\alpha_1,\alpha_2,\ldots,\alpha_r)}(X(\mathbb{C}))$, where $X(\mathbb{C})$ is the topological space defined by the same equations as X except that we see them having coefficients in the complex numbers by means of one of the embeddings of k in $\mathbb{C}$. It can be proved that these integers do not depend on the field embedding. Under these assumption on the base field k, Voevodsky noted that, if p^2 does not divide $s_{p^t-1}(Y)$ and $f\colon Y \to X$ is any algebraic morphism to a smooth, projective variety X of nonzero dimension less or equal than Y, one of the following must hold: (i) f is surjective and $\dim(X) = \dim(Y) = p^t - 1$ or (ii) there exists a closed point $\operatorname{Spec} L \to X$ with p not dividing $[L : k]$. This result follows from a sharper statement, now known as *degree formula*: keeping the same assumptions, we have that $(1/p)s_{p^t-1}(Y) \equiv \deg(f)(1/p)s_{p^t-1}(X) \mod p$, or else X has a point not divisible by p. This manuscript is based on a talk I gave at the 2003 International Conference of Algebraic Topology held in Kinosaki, Japan in honor of Goro Nishida's 60th birthday. We will go through Voevodsky's original idea which gave rise to such formula by using homotopy theory to define characteristic numbers with rational coefficients and discuss Rost's use of this formula on quadrics proving results originally due to Hoffmann and Izhboldin (see Merkurjev [6]). Degree formulae involving each a different set of characteristic numbers, with various *obstruction ideals* associated to them, have also been derived by Rost (top Segre numbers) [6] and Merkurjev [7] over any field, by the author in [4] over perfect fields and by Levine and Morel [5] over fields of characteristic zero. All of them can be used to yield proofs of the results of Hoffmann and Izhboldin and as indicated by Rost.

2 Characteristic numbers and the spectrum Φ_1

Voevodsky defined a cohomology theory [12] on algebraic varieties $H^{*,*}(-,A)$ which is expected to play the role of singular cohomology in topology, as prescribed by the *Beilinson conjectures*. This cohomology theory is called *motivic cohomology* with coefficients in an abelian group A, and, when evaluated on an algebraic variety, it is equipped of a tautological structure of left module over the (bistable) motivic cohomology operations $\mathcal{A}_m^{**}$. In the case $A=\mathbb{Z}/p$ and perfect base field, Voevodsky proved that $\mathcal{A}_m^{**}$ contains the bigraded Hopf $\mathbb{Z}/p$ algebra $H^{*,*}(\operatorname{Spec} k,\mathbb{Z}/p)\otimes_{\mathbb{Z}/p}\mathcal{A}_{\mathrm{top}}^{*,*}:=\mathcal{A}^{**}$ where $\mathcal{A}_{\mathrm{top}}^{*,*}$ is the topological Steenrod algebra with an appropriate bigrading [11]. In particular, for a fixed prime p, we have the left multiplication

$$Q_t\cdot\colon H^{*,*}(-,\mathbb{Z}/p)\to H^{*+2p^t-1,*+p^t-1}(-,\mathbb{Z}/p)$$

by the Milnor operation $Q_t\in\mathcal{A}_{\mathrm{top}}^{*,*}$. In our approach we will use the triangulated category $\mathcal{SH}(k)$ that shares some good properties with the ordinary stable homotopy category. Adopting Voevodsky's definition of motivic cohomology, if k is a perfect field, $H^{*,*}(-,A)$ is a representable functor in $\mathcal{SH}(k)$. The representing object is denoted by $\mathbf{H}_A$ and is called the *motivic Eilenberg–MacLane spectrum with coefficients in A*. A fundamental fact of this cohomology theory is that, if k is a perfect field and X is a smooth scheme, $H^{2*,*}(X,A)$ is isomorphic to the Chow group $CH^*(X)\otimes_{\mathbb{Z}}A$. To any algebraic variety X we can associate a graded ring $CH^*(X)$. Each homogeneous component $CH^n(X)$ is defined as the quotient of the free group over the codimension n closed subvarieties in X modulo *rational equivalence*. The product of two classes Z_1 and Z_2 is represented by the closed subscheme obtained as the intersection of two representatives of Z_1 and Z_2 intersecting *properly*. If $\mathbb{P}V$ is the projectivization of a rank $n+1$ vector bundle $V\to X$, then

$$CH^*(\mathbb{P}V)\cong\frac{CH^*(X)[t]}{(p(t))} \tag{1}$$

as left $CH^*(X)$ algebra, where $p(t)$ is a monic degree n polynomial. If $p(t)=t^n+c_1t^{n-1}+\dots+c_{n-1}t+c_n$, we can let c_i to be the i th Chern class of V, which will be therefore represented by some integral coefficient linear combination of codimension i closed subvarieties of X.

In algebraic geometry there is an analogue concept of what it is known as *characteristic numbers* in homotopy theory. To a n–tuple of nonnegative integers $\alpha=(\alpha_1,\dots,\alpha_n)$ with $0\le\alpha_1\le\alpha_2\le\dots\le\alpha_n$ we associate the symmetric polynomial $g_\alpha(\mathbf{t})$ in the variables $t_1,\dots,t_z$ for $z\ge\sum\alpha_i$. This is the unique symmetric polynomial having monomials with α_1 variables raised to the power 1, α_2 variables to the power 2 and so on. Let $f_\alpha(\sigma_1, \sigma_2, \ldots, \sigma_n)$ be the polynomial in the elementary symmetric functions $\sigma_i(\mathbf{t})$ such that $f_\alpha(\mathbf{t}) = g_\alpha(\mathbf{t})$.

Definition 2.1 Let M be an algebraic variety of pure dimension m and $V \to M$ be a vector bundle. For a n–tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\sum_{j=1}^n j\alpha_j = m$,

(1) set $s_\alpha(c_1, \ldots, c_n) = f_\alpha(\sigma_1, \ldots, \sigma_n)$ by formally replacing the variables σ_i with $c_i(V)$, the i th Chern class of V.

(2) The zero cycle $s_\alpha(V)$ is defined as $s_\alpha(c_1(V), c_2(V), \ldots c_n(V))$.

Definition 2.2 Given a zero dimensional cycle z in an algebraic variety X over a field k, that is, an integral coefficients linear combination

$$\sum_{i=1}^n \lambda_i \operatorname{Spec} L_i$$

of closed points $\operatorname{Spec} L_i \to X$, we define the *degree* of z to be the integer $\sum_{i=1}^n \lambda_i [L_i : k]$.

We recall that to each prime number q and positive integer t, there is a canonical motivic cohomological operation Q_t of degree $(2q^t-1, q^t-1)$. In the early version of Voevodsky's proof of the Milnor Conjecture [10], the operation Q_t appeared implicitly in an argument employing the homology theory $(\Phi_1)_{*,*}(-)$ represented by the object defined by the exact triangle

$$\Phi_1 \to \mathbf{H}_{\mathbb{Z}/q} \xrightarrow{Q_t} \Sigma^{2q^t-1, q^t-1} \mathbf{H}_{\mathbb{Z}/q}$$

The spectrum Φ_1 is related to the number $(1/q)s_{q^t-1}$. To show this, we are going to consider the spectrum $\mathbf{MGl} \in \mathcal{SH}(k)$, formally defined in the same way as in homotopy theory: $\mathbf{MGl}_n$ is the Thom space of the universal n–plane bundle over the infinite grassmanian Gr_n. The definition of the structure morphisms is slightly more subtle than in homotopy theory. The spectrum $\mathbf{MGl}$ shares the same nice ring object properties as $\mathbf{MU}$ does in classical homotopy theory. In particular, for any smooth, projective variety X of pure dimension n, there exists a canonical *fundamental homology class* $[X]_{\mathbf{MGl}} \in \mathbf{MGl}_{2n,n}(X)$ (see [1] or [4]). The graded group $H^{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$ has been entirely computed in [3] on characteristic zero fields and in [2], more generally, on perfect fields. Since $H^{i,j}(\mathbf{MGl}, \mathbb{Z}/q)$ is zero for $i > 2j$, for each r, there exists a map $e_1 \colon \mathbf{MGl} \to \Phi_1$, lifting the *Thom class*: $\mathbf{MGl} \to \mathbf{H}_{\mathbb{Z}/q}$.

Theorem 2.3 *Let $i\colon k \hookrightarrow \mathbb{C}$ be field and $p\colon X \to \operatorname{Spec} k$ be the structure morphism of a projective, smooth variety X of pure dimension. For each prime number q and positive integer t, let Φ_1 be the corresponding spectrum. Then there exists a choice of e_1 such that*

$$(e_1)_* p_*[X]_{\mathbf{MGl}} = \begin{cases} \deg(X) & \text{if } \dim(X) = 0 \\ -\frac{1}{q}\deg s_{q^t-1}(T_X) & \text{if } \dim(X) = q^t - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof Instead of considering X, we can argue on the algebraic variety $X^i_{\mathbb{C}}$ given by the limit of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{i^*} & \operatorname{Spec} k \end{array}$$

In fact, for any weighted degree n homogeneous polynomial $a(x_u)$ (that is, we set the degree of x_u to be $2u$) and any field embedding $e\colon k \hookrightarrow L$, we have $\deg(a(c_u(T_X))) = \deg(a(c_u(T_{X_L})))$, and, in particular, this integer does not depend on the field embedding e. The advantage of considering $X^i_{\mathbb{C}}$ is that this algebraic variety has a canonical complex manifold associated: $X^{\text{top}} := X(\mathbb{C}, i)$, that has the points $\operatorname{Spec}\mathbb{C} \to X^i_{\mathbb{C}}$ as underlying set. We recall that under these assumptions there is a *topological realization functor* (see Morel and Voevodsky [8]) which induces morphisms $t_{\mathbb{C}}\colon \mathbf{MGl}_{2*,*}(X) \to \mathbf{MU}_{2*}(X^{\text{top}})$ and $t_{\mathbb{C}}\colon CH^*(X^i_{\mathbb{C}}) \otimes A = H^{2*,*}(X, A) \to H^{2*}(X^{\text{top}}, A)$. This morphisms are compatible with the Kronecker product in the sense that $\langle z, [X^i_{\mathbb{C}}]_{CH}\rangle = \langle t_{\mathbb{C}} z, [X^{\text{top}}]_H\rangle$ for $z \in CH_0(X^i_{\mathbb{C}})$. Since $\langle s_\alpha(T_{X^i_{\mathbb{C}}}), [X^i_{\mathbb{C}}]_{CH}\rangle = \deg(s_\alpha(T_{X^i_{\mathbb{C}}}))$, this shows that the usual topological characteristic numbers $\langle s_\alpha, [X^{\text{top}}]_H\rangle$ are independent on the field embedding $i\colon k \hookrightarrow \mathbb{C}$ and equal to the *algebraic* characteristic numbers in Definition 2.1. Let now fix a choice of $e_1\colon \mathbf{MGl} \to \Phi_1$ lifting the Thom class which induces $e_1^{\text{top}}\colon \mathbf{MU} \to \Phi_1^{\text{top}}$. The morphism $t_{\mathbb{C}}\colon (\Phi_1)_{2*,*}(\operatorname{Spec}\mathbb{C}) \to (\Phi_1^{\text{top}})_{2*}(\text{pt})$ is an isomorphism for any $*$, thus we will consider $p_*[X^{\text{top}}]_{\mathbf{MU}} \in \mathbf{MU}_{2*}(\text{pt})$. This ring is isomorphic to the Lazard ring $\mathbb{Z}[1, a_1, a_2, \dots]$ with generators that can be chosen to have the following properties: let h be the Hurewicz homomorphism, if $i \neq p^u - 1$ for any prime p and positive integer u, then $h(a_i) = b_i + \cdots$ and if $i = p^u - 1$, $h(a_i) = p(b_i + \cdots)$, where b_i are the indecomposable elements of $H_{2*}(\mathbf{MU}, \mathbb{Z})$. Using the classical Thom Pontryagin construction and the above discussion on characteristic numbers, we see that the coefficients of $b_\alpha = b_1^{\alpha_1} b_2^{\alpha_2} \dots b_m^{\alpha_m}$ in $h(p_*[X^{\text{top}}]_{\mathbf{MU}})$ are precisely $\deg s_\alpha(-T_{X_{\mathbb{C}}})$, where $-T_{X_{\mathbb{C}}}$ is the opposite virtual bundle of $T_{X_{\mathbb{C}}}$ in $K_0(X_{\mathbb{C}})$. Since $(\Phi_1^{\text{top}})_{2*}(\text{pt}) = 0$ unless $* = 0$

and $* = q^t - 1$, we may assume $\dim(X) = n = q^t - 1$, because the statement of Theorem 2.3 in the case of zero dimensional X is essentially tautological. If $\dim(X) = q^t - 1$, then $p_*[X^{\text{top}}]_{\mathbf{MU}} = m a_{q^t-1} +$ decomposables, for an integer m. $h(p_*[X^{\text{top}}]_{\mathbf{MU}}) = mqb_{q^t-1} + \cdots$ and $(e_1^{\text{top}})_* a_{q^t-1} = 1 \in (\Phi_1^{\text{top}})_{2(q^t-1)}(\text{pt}) \cong \mathbb{Z}/q$ so $(e_1^{\text{top}})_* p_*[X^{\text{top}}]_{\mathbf{MU}} = (1/q)s_{q^t-1}(-T_X) + (e_1^{\text{top}})_*(\text{decomposables}) \mod q$. The decomposable elements of $\mathbf{MU}_{2(q^t-1)}(\text{pt})$ are divided in two classes: those whose Hurewicz image is not divisible by q and the others. If $f(a_i)$ is a monomial of the first kind, then $(e_1^{\text{top}})_* f(a_i)$ can be any value of $\mathbb{Z}/q$, if the lifting e_1 is chosen appropriately. In fact, by adding a suitable map $\mathbf{MU} \xrightarrow{c} \Sigma^{2(q^t-1)} \mathbf{H}^{\text{top}}_{\mathbb{Z}/q} \to \Phi_1^{\text{top}}$ to our choice of e_1^{top}, we can make $(e_1^{\text{top}})_* f(a_i)$ to be an arbitrary value. Just set $c\colon \mathbf{MU} \to \Sigma^{2(q^t-1)} \mathbf{H}^{\text{top}}_{\mathbb{Z}/q}$ to be a multiple of the dual of any b_α, whose coefficient in the expression of $h(f(a_i))$ is not divisible by q. In the case $f(a_i)$ is decomposable and $h(f(a_i))$ is divisible by q, then $(e_1)_* f(a_i) = 0$ for any choice of e_1. Thus, we have a choice of e_1^{top} with the property that $(e_1^{\text{top}})_* p_*[X^{\text{top}}]_{\mathbf{MU}} = (1/q)s_{q^t-1}(-T_X) \mod q$. Since $t_{\mathbb{C}}\colon (\Phi_1)_{2*,*}(\operatorname{Spec}\mathbb{C}) \to (\Phi_1^{\text{top}})_{2*}(\text{pt})$ is an isomorphism, to get the same result in the algebraic setting it suffices to consider the lifting $e_1 + c$. Finally, the equality $s_{q^t-1}(-T_X) = -s_{q^t-1}(T_X)$ follows from the fact that s_i are dual to indecomposable classes. This finishes the proof of the Theorem 2.3. □

The existence of a topological realization functor in the case of the base field k embedding in $\mathbb{C}$ enables us to use constructions we know to exist in topology, such as the Thom–Pontryagin. If we wish to prove Theorem 2.3 over a finite or, more in general, perfect field k, then the scheme of the proof is similar, but we cannot shift the argument to topology and we are forced to exclusively use algebraic homotopy categories. This involves the developement of such constructions of topological origin in this algebraic setting. The details of them are written in [4].

In general, there is no canonical choice of the polynomial generators a_i of $\mathbf{MU}_{2*}(\text{pt})$, but $(e_1)_* p_*[X]_{\mathbf{MGl}}$ does not depend on such choices, thus these numbers should be expressible only in function of e_1. This dependence appears just in presence of a_i with the property that $h(a_i)$ is not divisible by the prime number q we are considering. The classes a_i can be all chosen to be $p_*[M]_{\mathbf{MU}}$ for certain smooth projective complete intersections M. It follows that the coefficients of $h(a_i)$ are explicitely computable and so do the numbers $(e_1)_* p_*[X]_{\mathbf{MGl}}$ for each choice of e_1. For instance, working at the prime $q = 3$ and $t = 1$, we are looking at $\mathbf{MU}_4(\text{pt})$ which is generated as free abelian group by the classes a_2 and a_1^2 where a_2 can be canonically chosen as $p_*[M^2]_{\mathbf{MU}}$ for any smooth cubic surface M^2 in $\mathbb{P}^3$ and a_1 as $p_*[M^1]_{\mathbf{MU}}$ for any smooth quadric curve in $\mathbb{P}^2$. Carring out the computations we get, if X is a smooth, projective algebraic variety over a perfect field,

$$(2)\quad (e_1)_* p_*[X]_{\mathbf{MGl}} = \begin{cases} \deg(X) \mod 3 & \text{if } \dim(X) = 0 \\ -\left(\frac{1}{3} - \frac{\lambda}{2}\right) \deg s_{(0,1)}(T_X) & \\ \quad + \frac{\lambda}{4} \deg\left(s^2_{(1)}(T_X) - s_{(2)}(T_X)\right) \mod 3 & \text{if X is a surface} \\ 0 & \text{otherwise} \end{cases}$$

where λ can be any integer. This parameter reflects the dependence of these numbers by the choice of the lifting e_1. In particular, setting $\lambda = 0$, we recover the formula of Theorem 2.3. This unusual way of getting characteristic numbers of smooth and projective varieties, implicitly provides information on divisibility of characteristic numbers. In fact, any number obtained with this construction is necessarily a reduction modulo q of a linear combination with rational coefficients of characteristic numbers with integral values. In the above example, we conclude that

$$(3)\qquad -(1/3) + \lambda/2) \deg s_{(0,1)}(T_X) + (\lambda/4)(\deg(s^2_{(1)}(T_X) - s_{(2)}(T_X))$$

is always an integer for any integer λ. For $\lambda = 0$ we get that $\deg s_{(0,1)}(T_X)$ is always divisible by 3 for any smooth projective surface over a characteristic zero field.

3 Divisibility of characteristic numbers and degree formulae

In a sense, the $\mathbb{Q}$ linear relations between such characteristic numbers are *maximal* among those with integral value: for instance, let fix a characteristic zero field k; if we consider the rational numbers $(1/qi) \deg s_{q^t-1}(T_X)$ for each integer $i \neq 1$ or -1 there exists a smooth, projective algebraic variety over k such that $(1/qi) \deg s_{q^t-1}(T_X)$ is a nonintegral rational number. The smooth, projective algebraic varieties Y, with the property that $(e_1)_* p_*[Y]_{\mathbf{MGl}}$ is not further divisible by the prime number q in question, are of very special kind. The existence of any (rational) morphism f from such Y to any smooth projective variety X gives information on algebraic cycles of X and the degree of f. Indeed one of the following two statements must hold: (i) the number $(e_1)_* p_*[X]_{\mathbf{MGl}}$ is nonzero and the degree of f is not divisible by q, or (ii) X has a closed point x with $[k(x) : k]$ not divisible by q. A *rational morphism* $Y \to X$ is a map $U \to X$ where $U \subset X$ is a dense open set. This is a consequence of the so called *degree formula*: let $t_n(W)$ denote $(e_1)_* p_*[W]_{\mathbf{MGl}}$ for a smooth, projective algebraic variety W of pure dimension n,

Theorem 3.1 *Let f be a rational morphism $Y \longrightarrow X$ between smooth, projective algebraic varieties of pure dimension n over a characteristic zero field k. Then the* *relation*

$$t_n(Y) = (\deg f)t_n(X) \tag{4}$$

holds in $(\mathbb{Z}/q)/I(X)$, *where* $I(X,q) = \langle [k(x):k] \mod q \backslash x :$ *closed point of* $X\rangle$ *if* $\dim(Y) = \dim(X) = q^t - 1$, *and* $I(X,q) = 0$ *otherwise.*

Proof We first assume that the morphism f is regular. Applying the homology long exact sequences originating from the exact triangle

$$\Sigma^{2(q^t-1),q^t-1}\mathbf{H}_{\mathbb{Z}/q} \longrightarrow \Phi_1 \longrightarrow \mathbf{H}_{\mathbb{Z}/q} \tag{5}$$

to the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow^{p_Y} & \downarrow p_X \\ & & \operatorname{Spec} k \end{array} \tag{6}$$

we obtain the commutative diagram of groups:

$$\begin{array}{ccccc} \cdots \to CH_{*-q^t+1}(Y)\otimes\mathbb{Z}/q & \longrightarrow & (\Phi_1)_{2*,*}(Y) & \longrightarrow & CH_*(Y)\otimes\mathbb{Z}/q \to \cdots \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \cdots \to CH_{*-q^t+1}(X)\otimes\mathbb{Z}/q & \longrightarrow & (\Phi_1)_{2*,*}(X) & \longrightarrow & CH_*(X)\otimes\mathbb{Z}/q \to \cdots \\ \downarrow p_{X*} & & \downarrow p_{X*} & & \downarrow p_{X*} \\ CH_{*-q^t+1}(\operatorname{Spec} k)\otimes\mathbb{Z}/q & \longrightarrow & (\Phi_1)_{2*,*}(\operatorname{Spec} k) & \longrightarrow & CH_*(\operatorname{Spec} k)\otimes\mathbb{Z}/q \end{array} \tag{7}$$

The class $f_*(e_1)_*[Y]_{\mathbf{MGl}} - (\deg f)(e_1)_*[X]_{\mathbf{MGl}} \in (\Phi_1)_{2n,n}(X)$ lies in the image of some class in $CH_{n-q^t+1}(X)\otimes\mathbb{Z}/q$, because $f_*[Y]_{CH} = (\deg f)[X]_{CH}$ and $(e_1)_*[W]_{\mathbf{MGl}} = [W]_{CH}$ for any choice of e_1. On the other hand,

$$p_{X*}(f_*(e_1)_*[Y]_{\mathbf{MGl}} - (\deg f)(e_1)_*[X]_{\mathbf{MGl}}) = t_n(Y) - (\deg f)t_n(X)$$

and $p_{X*}\colon CH_0(X)\otimes\mathbb{Z}/q \to CH_0(\operatorname{Spec} k)\otimes\mathbb{Z}/q$ has $I(X,q)$ as image. Since $CH_{*-q^t+1}(\operatorname{Spec} k)\otimes\mathbb{Z}/q \to (\Phi_1)_{2*,*}(\operatorname{Spec} k)$ is always injective, the statement of Theorem 3.1 for regular morphisms follows. The case of rational morphism is treated by decomposing the rational morphism f as the diagram

$$\begin{array}{ccc} Y' & & \\ \downarrow g & \searrow^{f'} & \\ Y & \overset{f}{\dashrightarrow} & X \end{array}$$

where both g and f' are regular morphisms and g is a birational equivalence. Then we apply the regular version of the theorem to g and f' and use that $I(X)$ is birational invariant for smooth projective varieties. □

Theorem 2.3 along with the description of the numbers given in Theorem 3.1, yields a formula refined enough to prove some results about quadrics derived by Rost (see Merkurjev [6]).

Theorem 3.2 *Let Q be a smooth projective quadric of dimension $d \geq 2^m - 1$ and let X be a smooth variety over k such that $I(X,2) = 0$ and admitting a rational map $f\colon Q \dashrightarrow X$. Then,*

(1) $\dim(X) \geq 2^m - 1$*;*

(2) *if* $\dim(X) = 2^m - 1$*, there is a rational map* $X \dashrightarrow Q$ *.*

Proof Choose a subquadric $Q' \subset Q$ of dimension $2^m - 1$ so that the restriction of f to Q' is a rational morphism. We will start with proving the theorem by assuming that $t_{2^m-1}(Q') \neq 0 \mod 2$ and then we will show this assertion. For the first part we can use Rost's proof, which we rewrite here. If $\dim(X) < 2^m - 1$, then we consider the composition f'

$$Q' \overset{f}{\dashrightarrow} X \longrightarrow X \times \mathbb{P}_k^{2^m-1-\dim(X)} \tag{8}$$

to which we can apply the degree formula of Theorem 3.1. Notice that $I(X) = I(X \times \mathbb{P}^i)$ for any i. Since $\deg(f') = 0$, to show the first statement, it suffices to prove that $t_{2^m-1}(Q') \neq 0$. The same formula implies that $\deg(f)$ is odd. In the case the dimension of X equals $2^m - 1$, this can be rephrased by saying that the quadric $Q'' = Q' \times_k \operatorname{Spec} k(X)$ has a point over $k(X)$ of odd degree (namely its generic point as variety over $k(X)$). In turn, this is equivalent for Q'' to become isotropic over an odd degree field extension of $k(X)$. By Springer's Theorem then Q'' is isotropic over $k(X)$, that is, there exists a rational morphism $g\colon X \dashrightarrow Q''$. The rational morphism of the second statement of the theorem is the composition

$$X \overset{g}{\dashrightarrow} Q'' \xrightarrow{\text{can}} Q' \hookrightarrow Q$$

To prove that $t_{2^m-1}(Q') \neq 0$ for a smooth, projective quadric Q', we use Theorem 2.3 and the properties of the classes s_i: (a) $s_i(A+B) = s_i(A) + s_i(B)$ for two virtual bundles A and B and (b) $s_i(L) = c_1(L)^i$ for a line bundle L. By definition, the quadric Q' is determined by a degree two homogeneous equation in 2^m variables, thus its tangent bundle fits in the short exact sequence

$$0 \to T_{Q'} \to j^* T_{\mathbb{P}^{2^m}} \to j^* \mathcal{O}(2) \to 0 \tag{9}$$

where $j\colon Q' \hookrightarrow \mathbb{P}^{2^m}$ is the closed embedding. Property (a) implies that

$$s_i(T_{Q'}) = s_i(j^* T_{\mathbb{P}^{2^m}}) - s_i(j^* \mathcal{O}(2)).$$

Property (b) yields

$$s_i(j^* \mathcal{O}(2)) = j^* s_i(\mathcal{O}(2)) = j^* c_1^i(\mathcal{O}(2)) = j^* 2^i c_1^i \mathcal{O}(1) = 2^i Q' \cap c_1^i \mathcal{O}(1).$$

To compute $s_i(j^* T_{\mathbb{P}^{2^m}})$ we use the short exact sequence of vector bundles over $\mathbb{P}^{2^m}$

$$0 \to T_{\mathbb{P}^{2^m}} \to \oplus_{2^m+1} \mathcal{O}(1) \to \mathcal{O} \to 0$$

We get

$$\begin{aligned} s_i(j^* T_{\mathbb{P}^{2^m}}) &= s_i(j^* \oplus_{2^m+1} \mathcal{O}(1)) - s_i(j^* \mathcal{O}) \\ &= (2^m+1) j^* c_1^i \mathcal{O}(1) = (2^m+1) Q' \cap c_1^i \mathcal{O}(1). \end{aligned}$$

Let now $i = 2^m - 1$ and compute the degree of the zero cycle that we get

$$\deg(s_{2^m-1}(T_{Q'})) = (2^m+1) \deg(Q' \cap c_1^i \mathcal{O}(1)) - 2^{2^m-1} \deg(Q' \cap c_1^i \mathcal{O}(1))$$

Since $c_1 \mathcal{O}(1)$ is the class of an hyperplane in $\mathbb{P}^{2^m}$, we have that $\deg(Q' \cap c_1^i \mathcal{O}(1)) = 2$. Hence we conclude that

$$\begin{aligned} t_{2^m-1}(Q') &= -(1/2) \deg(s_{2^m-1}(T_{Q'})) \mod 2 \\ &= -(2^m + 1 - 2^{2^m-1}) \mod 2 \\ &= 1 \mod 2 \end{aligned}$$

which completes the proof. □

Corollary 3.3 (Hoffmann) *Let Q_1, Q_2 be two anisotropic quadrics. If $\dim(Q_1) \geq 2^m - 1$ and Q_2 is isotropic over $k(Q_2)$, then $\dim(Q_2) \geq 2^m - 1$.*

Corollary 3.4 (Izhboldin) *Let Q_1, Q_2 be two anisotropic quadrics. If $\dim(Q_1) = \dim(Q_2) = 2^m - 1$ and Q_2 is isotropic over $k(Q_1)$, then Q_1 is isotropic over $k(Q_2)$.*

We may wonder now, if this process of obtaining degree formulae may be generalized to other spectra. The answer to this question is positive: the spectra Φ_r for $r > 1$ constructed on characteristic zero fields [3] and on perfect fields [2], can be used to define new rational characteristic numbers for each r and choice of lifting $e_r\colon \mathbf{MGl} \to \Phi_r$, each of them appearing in certain degree formulae [4].

Summing up, we first used homotopy theory to define certain numbers $(e_1)_* p_*[X]_{\mathbf{MGl}}$, expressed them as rational characteristic numbers with integral values (Theorem 2.3) and showed that they satisfy a degree formula (Theorem 3.1). Lastly, we computed them for smooth quadrics and used this degree formula to prove a result of Rost, at least under the assumption on the base field k of being of characteristic zero.

References

[1] **S Borghesi**, *Algebraic K–theories and the higher degree formula*, PhD thesis, Northwestern University (2000) Available at http://www.math.uiuc.edu/K-theory/0412/

[2] **S Borghesi**, *Algebraic Morava K–theory spectra over perfect fields*, preprint (2002) Available at http://geometria.sns.it/

[3] **S Borghesi**, *Algebraic Morava K–theories*, Invent. Math. 151 (2003) 381–413 MR1953263

[4] **S Borghesi**, *The degree formulae*, preprint (2004) Available at http://geometria.sns.it/

[5] **M Levine**, **F Morel**, *Algebraic cobordism I*, preprint (2002) Available at http://www.math.uiuc.edu/K-theory/0547/

[6] **A Merkurjev**, *Degree formula*, preprint (2000) Available at http://www.mathematik.uni-bielefeld.de/~rost/chain-lemma.html

[7] **A Merkurjev**, *Steenrod operations and degree formulas*, J. Reine Angew. Math. 565 (2003) 13–26 MR2024643

[8] **F Morel**, **V Voevodsky**, *$\mathbb{A}^1$–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999) 45–143 (2001) MR1813224

[9] **R E Stong**, *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J. (1968) MR0248858

[10] **V Voevodsky**, *The Milnor conjecture*, preprint (1996) Available at http://www.math.uiuc.edu/K-theory/0170/

[11] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. (2003) 1–57 MR2031198

[12] **V Voevodsky**, **A Suslin**, **E M Friedlander**, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies 143, Princeton University Press, Princeton, NJ (2000) MR1764197

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