Geometry & Topology Monographs 10 (2007) 117–129

The stable braid group and the determinant of the Burau representation

F R COHEN
J PAKIANATHAN

This article gives certain fibre bundles associated to the braid groups which are obtained from a translation as well as conjugation on the complex plane. The local coefficient systems on the level of homology for these bundles are given in terms of the determinant of the Burau representation.

De Concini, Procesi, and Salvetti [6] considered the cohomology of the nth braid group B_n with local coefficients obtained from the determinant of the Burau representation, $H^*(B_n; \mathbb{Q}[t^{\pm 1}])$. They show that these cohomology groups are given in terms of cyclotomic fields.

This article gives the homology of the stable braid group with local coefficients obtained from the determinant of the Burau representation. The main result is an isomorphism

$$H_*(B_\infty; \mathbb{F}[t^{\pm 1}]) \to H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{F})$$

for any field $\mathbb{F}$ where $\Omega^2 S^3\langle 3\rangle$ denotes the double loop space of the 3–connected cover of the 3–sphere. The methods are to translate the structure of $H_*(B_n; \mathbb{F}[t^{\pm 1}])$ to one concerning the structure of the homology of certain function spaces where the answer is computed.

20F14, 20F36, 20F40, 52C35, 55Q99; 12F99

1 Introduction

This article gives certain fibre bundles associated to the braid groups which are obtained from a natural action of translation, and conjugation on the complex plane. The local coefficient systems on the level of homology for these bundles are given in terms of the determinant of the Burau representation. Furthermore, the following are addressed:

(1) The total space of each of the fibre bundles is a $K(\pi, 1)$ with local coefficients in homology given by the determinant of the Burau representation. These are related to work of C De Concini, C Procesi, and M Salvetti [6] who analyze the cohomology of the braid groups with local coefficients given by the determinant

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.117

of the Burau representation. These bundles arise by pulling back a bundle given below which is obtained from an action of the integers on the complex plane $\mathbb{C}$ with the integral points $(n,0)$ deleted.

(2) The homology of the bundles obtained by the standard stablization of the braid groups is given by the homology for the 3–connected cover of the double loop space of the 3–sphere. Thus homology is considered here rather than cohomology as in the case of [6].

(3) The last section of this article is a short synopsis of a recurring relationship concerning the (co)homology of the braid groups with certain related local coefficient systems. Here the resulting groups are given by cyclotomic fields, as well as classical modular forms (see Furusawa–Tezuka–Yagita [8] and Cohen [5]). Several related problems are posed.

Let $\mathbb{F}$ denote a field. The ring of Laurent polynomials

$$\mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{F} = \mathbb{F}[t^{\pm 1}]$$

is given by the elements

$$\Sigma_{-L \leq i \leq L} a_i t^i = a_{-L}t^{-L} + a_{-L+1}t^{-L+1} + \cdots + a_0 + a_1 t^1 + \cdots + a_L t^L$$

for all $0 \leq L < \infty$.

Artin's presentation for nth braid group B_n (see Birman [1]) is given by generators σ_j with $1 \leq i \leq n-1$ with relations

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$, and
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Observe that the braid group acts on the complex numbers $\mathbb{C}$ by the formula

$$\sigma_i(z) = 1 + \chi(z)$$

where $\chi(z)$ denotes the complex conjugate of z. This action restricts to one on

(1) the space X which is the complement of the standard lattice $\mathbb{Z}+i\mathbb{Z}$, the Gaussian integers in $\mathbb{C}$, and

(2) the space Y which is the complement of the integral points $\{(\lambda,0) | \lambda \in \mathbb{Z}\}$ in $\mathbb{C}$.

In addition, the spaces X, and Y are both $K(\pi,1)$'s for which π is a countably infinitely generated free group. Furthermore a choice of generators of $\pi_1(Y)$ is naturally indexed by powers of the indeterminate t in $\mathbb{Z}[t^{\pm 1}]$ with the abelianization of $\pi_1(Y)$ isomorphic to the underlying structure of $\mathbb{Z}[t^{\pm 1}]$ as an abelian group. One choice of generators of $\pi_1(X)$ is naturally indexed by the points in the standard integral lattice for the complex numbers.

The braid group acts on $\mathbb{Z}[t^{\pm 1}]$ where each σ_i acts via multiplication by $-t$. This action is obtained from the determinant of the classical Burau representation of σ_i,

$$b\colon B_n \to GL(n, \mathbb{Z}[t^{\pm 1}])$$

with $b(\sigma_i)$ given by the matrix

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ . & 0 & a_{i,i} & a_{i,i+1} & 0 & . \\ . & 0 & a_{i+1,i} & a_{i+1,i+1} & 0 & . \\ . & . & 0 & 0 & 1 & . \\ 0 & 0 & . & . & 0 & 1 \end{pmatrix}$$

where $a_{i,i} = 0$, $a_{i+1,i} = 1$, $a_{i,i+1} = t$, and $a_{i+1,i+1} = 1 - t$. Thus $\mathbb{Z}[t^{\pm 1}]$ is a B_n–module where σ_i acts by multiplication by the determinant d of $b(\sigma_i)$ with

$$d(b(\sigma_i)) = -t.$$

This module structure is precisely that given by De Concini, Procesi and Salvetti [6] after tensoring with the rational numbers.

The main result of [6] is a computation of the cohomology of the nth Artin braid group B_n with coefficients in $\mathbb{Q}[t^{\pm 1}]$ with the above action. As in [6], let $\mu_d(q)$ denote the cyclotomic polynomial in q having as roots the primitive dth roots of unity. Let R_q denote the B_{m+1}–module given by the action on the ring

$$R_q = \mathbb{Q}[q^{\pm 1}]$$

of Laurent polynomials over the rational numbers in the variable q which is defined by mapping each standard generator σ_i to the operator of multiplication by $-q$. Define

$$\{d\} = \mathbb{Q}[q^{\pm 1}]/\mu_d(q) = \mathbb{Q}[q]/\mu_d(q),$$

the cyclotomic field of d roots of unity.

Theorem 1.1 (De Concini–Procesi–Salvetti [6]) *If $hi = m + 1$ or $hi = m$ then*

$$H^{i(h-2)+1}(B_{m+1}, R_q) = \{h\}.$$

All other cohomology groups are trivial.

The main result of this article addresses the homology of the stable braid group B_∞ with coefficients in $\mathbb{Z}[t^{\pm 1}]$ rather than cohomology with analogous coefficients. The main point here is that there are natural geometric bundle interpretations of these results.

That is, consider the Borel construction or homotopy orbit space

$$EB_n \times_{B_n} Y,$$

the total space of a fibre bundle over $BB_n = K(B_n, 1)$ with fibre $Y = K(\pi_1(Y), 1)$. Consequently, the space $EB_n \times_{B_n} Y$ is a $K(\pi, 1)$, and is equipped with natural "stabilization" maps $EB_n \times_{B_n} Y \to EB_{n+1} \times_{B_{n+1}} Y$ with colimit denoted $EB_\infty \times_{B_\infty} Y$. The next proposition records the local coefficient system for the bundle projection $EB_n \times_{B_n} Y \to BB_n$.

Proposition 1.2 *Assume that $n \geq 2$. The action of B_n on $H_1(Y)$ specified by the local coefficient system in homology for the bundle projection $EB_n \times_{B_n} Y \to BB_n$ is given by multiplication by the determinant of the Burau representation: $\sigma_i(t^j) = -t^{j+1}$.*

The homology of the space $EB_\infty \times_{B_\infty} Y$ with coefficients in a field $\mathbb{F}$ is stated in the next theorem for which the space $\Gamma(S^1)$ denotes $S^1 \vee S^1 = K(F_2, 1)$ with F_2 a free group on two letters.

Theorem 1.3 *Assume that $n \geq 2$.*

(1) *There is an induced map*

$$\rho\colon EB_\infty \times_{B_\infty} Y \to \Gamma(S^1) \times \Omega^2 S^3\langle 3\rangle$$

which is a homology isomorphism for any field $\mathbb{F}$ (with trivial action) where $\Gamma(S^1)$ is a $K(F_2, 1)$. Thus the suspension of ρ, $\Sigma(\rho)$, is a homotopy equivalence.

(2) *There are isomorphisms*

$$H_*(B_\infty; \mathbb{F}) \oplus H_*(B_\infty; \mathbb{F}[t^{\pm 1}]) \to H_*(EB_\infty \times_{B_\infty} Y; \mathbb{F}),$$
$$H_*(B_\infty; \mathbb{F}) \oplus H_*(B_\infty; \mathbb{F}[t^{\pm 1}]) \to H_*(\Gamma(S^1); \mathbb{F}) \otimes H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{F}),$$
$$H_*(B_\infty; \mathbb{F}[t^{\pm 1}]) \to H_*(\Omega^2 S^3\langle 3\rangle; \mathbb{F}).$$

(3) *If $i \geq 1$, there are isomorphisms $H_{i+1}(EB_n \times_{B_n} Y; \mathbb{Q}) \to H_i(B_n; \mathbb{Q}[t^{\pm 1}])$ for $i > 0$.*

This theorem is proven by analyzing properties of fibre bundles described $EB_n \times_{B_n} Y$ in Section 2. In addition, the integer homology of $\Omega^2 S^3\langle 3\rangle$ is well-understood. For example, the reduced homology is entirely torsion for which the p–torsion is of order exactly p.

The mod–2 homology is a polynomial ring with generators x_i of degree i where i is either 2 or 2^{n-1} for all $n > 1$. The first Bockstein β satisfies $\beta(x_3) = x_2$ and $\beta(x_{2^n-1}) = x^2_{2^{n-1}-1}$ if $n > 2$. If p is an odd prime, the mod-p homology is a tensor product of (i) an exterior algebra with generators y_{2p^n-1} of degree $2p^n - 1$, $n \geq 1$ and (ii) a polynomial algebra with generators x_{2p^n-2} of degree $2p^n - 2$ for $n \geq 1$. The first Bockstein β satisfies $\beta(y_{2p^n-1}) = x_{2p^n-2}$. One reference is [3], but the structure as an algebra was first worked out by J C Moore.

Remark Filippo Callegaro has shown that the bundles $EB_n \times_{B_n} Y \to BB_n$ admit cross-sections. Thus if $i \geq 1$, there are isomorphisms

$$H_{i+1}(EB_n \times_{B_n} Y; \mathbb{Z}) \to H_{i+1}(B_n; \mathbb{Z}) \oplus H_i(B_n; \mathbb{Z}[t^{\pm 1}]).$$

In addition, let $\mathbb{Z}[t^{\pm 1}]^*$ denote the hom-dual of $\mathbb{Z}[t^{\pm 1}]$. The existence of a cross-section implies that there are isomorphisms

$$H^{i+1}(EB_n \times_{B_n} Y; \mathbb{Z}) \to H^{i+1}(B_n; \mathbb{Z}) \oplus H^i(B_n; \mathbb{Z}[t^{\pm 1}]^*).$$

Callegaro has also determined the precise homology of $EB_n \times_{B_n} Y$ in [2].

The authors would like to congratulate Goro Nishida on this happy occasion of his 60th birthday. His work has been very interesting as well as very stimulating for the authors, especially an early paper which greatly encouraged one of the authors. In addition, the authors would like to thank Filippo Callegaro as well as the referee for suggestions concerning this article. ,Cohen was partially supported by the National Science Foundation under Grant No. 9704410 and CNRS-NSF Grant No. 17149.

2 Bundles associated to the determinant of the Burau representation

Regard $\mathbb{Z}[t^{\pm 1}]$ as a B_n–module obtained from the determinant of the Burau representation as above. Let EG denote Milnor's contractible "universal space" on which the discrete group G acts freely and properly discontinuously. The purpose of the next lemma is to give elementary properties stated earlier.

Lemma 2.1 (1) *There is an action of B_n on the complex line $\mathbb{C}$ given by*

$$\sigma_j(z) = 1 + \chi(z).$$

(2) *This action of B_n on $\mathbb{C}$ restricts to an action on the spaces X and Y.*

(3) *The associated Borel constructions $EB_n \times_{B_n} W$, where W is either X, or Y are $K(\pi,1)$'s. In case $W = Y$, the group π is given by a group extension*

$$1 \to \pi_1(Y) \to \pi \to B_n \to 1$$

where $\pi_1(Y)$ is a free group with generators indexed by t^q for q in $\mathbb{Z}$ and where the action of B_n on $H_1(Y)$ is given by the action of the determinant of the Burau representation.

Proof An action of B_n on the complex numbers $\mathbb{C}$ is specified by the composite

$$B_n \xrightarrow{A} \mathbb{Z} \xrightarrow{\Theta} \mathrm{Top}(\mathbb{C})$$

where $\mathrm{Top}(\mathbb{C})$ is the homeomorphism group of $\mathbb{C}$,

$$\Theta(1)(z) = 1 + \chi(z)$$

and

$$A\colon B_n \to \mathbb{Z}$$

is given by abelianization with $A(\sigma_i) = 1$ for all i. Thus

$$\sigma_j(z) = 1 + \chi(z).$$

That this action restricts to an action on X, and Y follows at once as the standard integral lattice as well as the integral points $\{(n,0) | n \in \mathbb{Z}\}$ in $\mathbb{C}$ are preserved by this action. The second part follows.

Notice that there is a fibration

$$EB_n \times_{B_n} W \to BB_n$$

with fibre W. Since BB_n, X and Y are $K(\pi,1)'s$, so is the associated total space. Finally, the group extension arises by applying the fundamental group functor to the fibration $EB_n \times_{B_n} W \to BB_n$ where all spaces are $K(G,1)'s$. That the local coefficient system in homology for this fibration is given by the action of the determinant of the Burau representation is verified below in Lemma 2.2. The third part follows. □

Observe that the first homology group of X, $H_1(X)$, is a countably infinite direct sum of copies of the integers indexed by the points in the standard standard integral lattice in $\mathbb{C}$. Similarly, the first homology group of Y, $H_1(Y)$, is a countably infinite direct sum of copies of the integers indexed by the integers. Furthermore, a basis for $H_1(Y)$ is given by ι_n for n in $\mathbb{Z}$ for which ι_n is given by the image of a fixed fundamental cycle of a small circle embedded in Y with center $(n,0)$, and radius $1/2$.

The next lemma directly implies Proposition 1.2 concerning the local coefficient system in homology for the bundle projection $EB_n \times_{B_n} Y \to BB_n$.

Lemma 2.2 *There is an isomorphism of B_∞–modules*

$$\Theta\colon \mathbb{Z}[t^{\pm 1}] \to H_1(Y)$$

given by $\Theta(t^n) = \iota_n$ where ι_n is the homology class of a small circle in Y with center $(n,0)$, and radius $1/2$.

Proof That Θ is a homology isomorphism follows by inspection. Notice that σ_i sends t^n to $(-t)t^n = -t^{n+1}$ in $\mathbb{Z}[t^{\pm 1}]$. Furthermore, σ_i sends the small circle with center $(n,0)$, and radius $1/2$ homeomorphically to a small circle with center $(n+1,0)$, and radius $1/2$, but with the opposite orientation induced by complex conjugation. The lemma follows. □

3 Related bundles and their homological properties

If $n \geq 2$, the abelianization of B_n is isomorphic to $\mathbb{Z}$ with one choice of epimorphism given by $A\colon B_n \to \mathbb{Z}$ for $A(\sigma_i) = 1$, $i \geq 1$. The stablization maps $s\colon B_n \to B_{n+1}$ send each σ_i to σ_i for $1 \leq i \leq n$. These maps are compatible with abelianization via the following commutative diagram.

$$\begin{tikzcd}
B_n \arrow[r, "s"] \arrow[d, "A"] & B_{n+1} \arrow[r, "s"] \arrow[d, "A"] & B_\infty \arrow[d, "A"] \\
\mathbb{Z} \arrow[r, "\text{Identity}"] & \mathbb{Z} \arrow[r, "\text{Identity}"] & \mathbb{Z}
\end{tikzcd}$$

In addition, there is a map $\theta\colon K(B_\infty, 1) \to \Omega^2 S^3$ which induces a homology isomorphism with any simple field coefficients and with acyclic homotopy fibre denoted F (see Cohen [3; 4], May [9], Segal [10]). The map θ is defined via

(1) the action of the little 2–cubes [9] giving a map from the unordered configuration space $Conf(\mathbb{R}^2, n)/\Sigma_n \to \Omega^2_{(n)} S^2$ the path-component of $\Omega^2 S^2$ given by maps having degree n

(2) together with the homotopy equivalence

$$\Omega^2(\eta)\colon \Omega^2 S^3 \to \Omega^2_{(0)} S^2$$

where the map η denotes the Hopf map $\eta\colon S^3 \to S^2$, and

(3) passage to colimits.

Thus there is a homotopy commutative diagram where $f\colon \Omega^2 S^3 \to S^1$ induces an isomorphism on the level of fundamental groups.

$$\begin{array}{ccc}
K(B_\infty,1) & \xrightarrow{\theta} & \Omega^2 S^3 \\
\Big\downarrow{\scriptstyle BA} & & \Big\downarrow{\scriptstyle f} \\
K(\mathbb{Z},1) & \xrightarrow{\text{Identity}} & K(\mathbb{Z},1).
\end{array}$$

This information is recorded in the next lemma.

Lemma 3.1 (1) *There is a map $K(B_\infty,1) \to \Omega^2 S^3$ which induces a homology isomorphism, and with acyclic homotopy fibre F.*

(2) *The action of B_n on $\mathbb{Z}[t^{\pm 1}]$ factors through the abelianization map $A\colon B_n \to \mathbb{Z}$, and thus the local coefficient system in homology for the fibration*

$$EB_n \times_{B_n} Y \to BB_n$$

factors through the isomorphism $\pi_1 \Omega^2 S^3 \to \pi_1 S^1$.

Notice that there are several bundles described above over a base B where

(1) the base B runs over $BB_n = K(B_n,1)$, $B_\infty = K(B_\infty,1)$, $\Omega^2 S^3$, or S^1,

(2) these bundles all have fibre Y, and

(3) the action of the fundamental group of B on the homology of Y factors through the action of $\pi_1(S^1)$ on $H_*(Y)$.

In what follows below, the notation $\Gamma(B)$ is used generically to denote any of these bundles. Thus there is a bundle $\Gamma(S^1)$ over S^1 with fibre Y together with a commutative diagram of bundles obtained from pulling back the bundle over S^1.

$$\begin{array}{ccccccc}
Y & \xrightarrow{1} & Y & \xrightarrow{1} & Y & \xrightarrow{1} & Y \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} \\
\Gamma(BB_n) & \longrightarrow & \Gamma(BB_\infty) & \longrightarrow & \Gamma(\Omega^2 S^3) & \longrightarrow & \Gamma(S^1) \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\
BB_n & \longrightarrow & BB_\infty & \longrightarrow & \Omega^2 S^3 & \longrightarrow & S^1
\end{array}$$

Recall that there is a homotopy equivalence

$$\Omega^2 S^3 \to \Omega^2 S^3\langle 3\rangle \times S^1$$

(where $S\langle 3\rangle$ denotes the 3–connected cover of the 3–sphere, and where this decomposition is not multiplicative). Since $\Omega^2 S^3\langle 3\rangle$ is simply-connected, the projection $p\colon \Omega^2 S^3\langle 3\rangle \times S^1 \to S^1$ induces an isomorphism on the level of fundamental groups.

Theorem 3.2 *There are homotopy equivalences*

(1) $\Omega^2 S^3\langle 3\rangle \times \Gamma(S^1) \to \Gamma(\Omega^2 S^3\langle 3\rangle \times S^1)$,

(2) $\Gamma(\Omega^2 S^3) \to \Gamma(\Omega^2 S^3\langle 3\rangle \times S^1)$, *and*

(3) $\Gamma(\Omega^2 S^3) \to \Omega^2 S^3\langle 3\rangle \times \Gamma(S^1)$.

Furthermore, the following properties are satisfied.

(4) *The map of bundles*

$$\Gamma(BB_\infty) \to \Gamma(\Omega^2 S^3)$$

induces a homology isomorphism with any simple coefficients.

(5) *With simple field coefficients $\mathbb{F}$, there are isomorphisms*

$$H_*(\Gamma(S^1);\mathbb{F}) \otimes_{\mathbb{F}} H_*(\Omega^2 S^3\langle 3\rangle;\mathbb{F}) \to H_*(\Gamma(\Omega^2 S^3);\mathbb{F}).$$

(6) *The space $\Gamma(S^1)$ is a $K(\pi,1)$ where π is a free group on 2 generators.*

(7) *If $i \geq 1$, there are isomorphisms $H_{i+1}(EB_n \times_{B_n} Y;\mathbb{Q}) \to H_i(B_n;\mathbb{Q}[t^{\pm 1}])$ for $i > 0$.*

Observe that Theorem 1.3 follow at once from Theorem 3.2 which is proven next.

Proof Notice that if $\pi\colon A \times B$ denotes the natural projection map with $F \to E \to B$ a fibre bundle, then the pullback to $A \times B$ is homeomorphic to a product $A \times E$. Thus statements (1)–(3) follow.

Next, observe that there is a map of fibrations

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[d] & Y \arrow[r, "1"] \arrow[d, "i"] & Y \arrow[d, "i"] \\
F \arrow[r] \arrow[d, "1"] & \Gamma(BB_\infty) \arrow[r] \arrow[d, "p"] & \Gamma(\Omega^2 S^3) \arrow[d, "p"] \\
F \arrow[r] & BB_\infty \arrow[r] & \Omega^2 S^3
\end{tikzcd}$$

where F, an acyclic space, is the homotopy theoretic fibre of the homology isomorphism $BB_\infty \to \Omega^2 S^3$. Since the homology of F is trivial, the Serre spectral sequence for the fibration $\Gamma(BB_\infty) \to \Gamma(\Omega^2 S^3)$ collapses, and the induced map is an isomorphism in homology with any simple coefficients. Part (4) follows.

To work out the homology of $\Gamma(\Omega^2 S^3)$, consider the following morphism of fibrations

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "i"] & Y \arrow[d, "i"] \\
\Omega^2 S^3\langle 3\rangle \arrow[r] \arrow[d, "1"] & \Gamma(\Omega^2 S^3) \arrow[r] \arrow[d, "p"] & \Gamma(S^1) \arrow[d, "p"] \\
\Omega^2 S^3\langle 3\rangle \arrow[r] & \Omega^2 S^3 \arrow[r] & S^1.
\end{tikzcd}$$

By statement (4), there is a homotopy equivalence $\Gamma(\Omega^2 S^3) \to \Omega^2 S^3\langle 3\rangle \times \Gamma(S^1)$. Since the natural map $\Gamma(BB_\infty) \to \Gamma(\Omega^2 S^3)$ induces a homology isomorphism with any simple field coefficients $\mathbb{F}$, there is an induced map

$$\Gamma(BB_\infty) \to \Omega^2 S^3\langle 3\rangle \times \Gamma(S^1)$$

which induces a homology isomorphism with any simple field coefficients $\mathbb{F}$. Part (5) follows.

To finish part (6), the space $\Gamma(S^1)$ will be shown to be homotopy equivalent to a bouquet of two circles. Notice that $\Gamma(S^1)$ is given by

$$\mathbb{R} \times_{\mathbb{Z}} Y.$$

Let V denote the subspace of $\mathbb{C}$ given by the union of circles with centers at $(n,0)$ for integers n, and with radii all equal to $1/2$. Notice that V is a subspace of Y, and that the natural inclusion $V \to Y$ is both a homotopy equivalence, and invariant with respect to the action of $\mathbb{Z}$ for which a generator sends z to $1+\chi(z)$. In addition, this action of $\mathbb{Z}$ on V is free, and properly discontinuous. Thus $E(S^1)$ is homotopy equivalent to the orbit space

$$V/\mathbb{Z}.$$

Observe that the orbit space $V/\mathbb{Z}$ is the identification space of a circle with one pair of antipodal points identified. Thus $\Gamma(S^1)$ is $K(F_2,1)$ where F_2 is a free group with two generators.

To check statement 7, notice that if $i \geq 1$, there are isomorphisms $H_{i+1}(EB_n \times_{B_n} Y;\mathbb{Q}) \to H_i(B_n;\mathbb{Q}[t^{\pm 1}])$ for $i > 0$ by inspection of the Leray spectral sequence for the natural projection

$$EB_n \times_{B_n} Y \to BB_n,$$

together with the fact that $H_i(B_n;\mathbb{Q}) = \{0\}$ for $i > 1$.

The theorem follows. □

Proposition 3.3 *There are isomorphisms*

$$H_*(\Gamma(BB_\infty);\mathbb{F}) \to H_*(B_\infty;\mathbb{F}) \oplus H_{*-1}(B_\infty;\mathbb{F}[t^{\pm 1}]).$$

Proof Recall the previous commutative diagram

$$\begin{tikzcd} \Gamma(BB_\infty) \arrow[r] \arrow[d, "p"] & \Gamma(\Omega^2 S^3) \arrow[d, "p"] \\ BB_\infty \arrow[r] & \Omega^2 S^3, \end{tikzcd}$$

together with the homotopy equivalence $S^1 \times \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3$. The proposition follows from the collapse of the Serre spectral sequence for the fibrations p. □

Corollary 3.4 *There are isomorphisms*

$$H_*(B_\infty;\mathbb{F}[t^{\pm 1}]) \to H_*(\Omega^2 S^3\langle 3\rangle;\mathbb{F}).$$

4 Remarks

This section lists answers for the cohomology of the braid groups with various natural choices of local coefficient systems. The common theme here is that the answers are given by classical number theoretic constructions.

(1) The work of De Concini, Procesi, and Salvetti [6] gives that cyclotomic fields arise as the cohomology of the braid groups with coefficients in the determinant of the Burau representation.

(2) Classical work of Eichler [7], and Shimura [11] gives that

$$H^1(SL(2,\mathbb{Z});\mathrm{Sym}^k(V_2)\otimes\mathbb{R})$$

is the vector space of classical modular forms of weight $k+2$ where V_2 is the fundamental representation of $SL(2,\mathbb{Z})$, and $\mathrm{Sym}^k(V_2)$ denotes the k–fold symmetric power of V_2. The third braid group B_3 is the universal central extension of $SL(2,\mathbb{Z})$. An inspection of the natural spectral sequence gives similar results for B_n, $n=3,4$. For example $H^*(B_3;\mathrm{Sym}^k(V_2)\otimes\mathbb{R})$ is given by

$$H^*(SL(2,\mathbb{Z});\mathrm{Sym}^k(V_2)\otimes\mathbb{R})\otimes E[v]$$

where $E[v]$ denotes an exterior algebra with a single generator in degree 1. Natural extensions apply to braid groups with more strands.

(3) Work of Furusawa, Tezuka, and Yagita [8] gives a computation of the real cohomology of the classifying space of the diffeomorphism group of the torus. The answer is given in terms of classical modular forms.
Related work by Cohen [5] gives that the real cohomology for the n–stranded braid group of a k–punctured torus, $Br_n(S^1 \times S^1 - \{x_1, x_2, \cdots, x_k\})$ is given in terms of classical modular forms, while a similar result is obtained after tensoring with the natural sign representation. Fix a dimension d, and stabilize $H^d(Br_n(S^1 \times S^1 - \{x_1, x_2, \cdots, x_k\}; \mathbb{R})$ by letting k go to ∞. A direct check of the the dimension of the resulting cohomology groups is that of certain spaces of Jacobi forms as calculated by Eichler, and Zagier.

Two problems arise naturally in this context.

(1) What is the homology of $EB_n \times_{B_n} Y$?

(2) Consider a symplectic representation of $B_n \to Sp(2g, \mathbb{Z})$. What are the cohomology groups $H^*(B_n; \mathrm{Sym}^k(V_{2g}) \otimes \mathbb{R})$ for which the local coefficient system is given by symmetric powers of the symplectic representation ? It is natural to ask whether these admit interpretations analogous to those given by the Eichler-Shimura isomorphism.

References

[1] **J S Birman**, *Braids, links, and mapping class groups*, Ann. of Math. Studies 82, Princeton University Press, Princeton, N.J. (1974) MR0375281

[2] **F Callegaro**, *The homology of the Milnor fibre for classical braid groups*, Algebr. Geom. Topol. 6 (2006) 1903–1923

[3] **F Cohen**, *Homology of* $\Omega^{n+1}\Sigma^{n+1}X$, Bull. Amer. Math. Soc. 79 (1973) 1236–1241 (1974) MR0339176

[4] **F Cohen**, *The homology of* $\mathcal{C}_{n+1}$*–spaces,* $n > 0$, from: "The homology of iterated loop spaces", Lecture Notes in Math. 533, Springer (1976) 207–352

[5] **F R Cohen**, *On genus one mapping class groups, function spaces, and modular forms*, from: "Topology, geometry, and algebra: interactions and new directions (Stanford, CA, 1999)", Contemp. Math. 279, Amer. Math. Soc., Providence, RI (2001) 103–128 MR1850743

[6] **C De Concini**, **C Procesi**, **M Salvetti**, *Arithmetic properties of the cohomology of braid groups*, Topology 40 (2001) 739–751 MR1851561

[7] **M Eichler**, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. 67 (1957) 267–298 MR0089928

[8] **M Furusawa**, **M Tezuka**, **N Yagita**, *On the cohomology of classifying spaces of torus bundles and automorphic forms*, J. London Math. Soc. (2) 37 (1988) 520–534 MR939127

[9] **J P May**, *The geometry of iterated loop spaces*, Springer, Berlin (1972) MR0420610

[10] **G Segal**, *Configuration-spaces and iterated loop-spaces*, Invent. Math. 21 (1973) 213–221 MR0331377

[11] **G Shimura**, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo (1971) MR0314766

Department of Mathematics, University of Rochester
Rochester NY 14627, USA

cohf@math.rochester.edu, jonpak@math.rochester.edu

Received: 15 January 2004 Revised: 5 June 2005