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Homotopy groups of homotopy fixed point spectra associated to E_n

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We compute the $\mathrm{mod}(p)$ homotopy groups of the continuous homotopy fixed point spectrum $E_2^{hH_2}$ for $p > 2$, where E_n is the Landweber exact spectrum whose coefficient ring is the ring of functions on the Lubin–Tate moduli space of lifts of the height n Honda formal group law over $\mathbb{F}_{p^n}$, and H_n is the subgroup $W\mathbb{F}_{p^n}^{\times} \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ of the extended Morava stabilizer group G_n. We examine some consequences of this related to Brown–Comenetz duality and to finiteness properties of homotopy groups of $K(n)_*$–local spectra. We also indicate a plan for computing $\pi_*(E_n^{hH_n} \wedge V(n-2))$, where $V(n-2)$ is an E_{n*}–local Toda complex.

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Introduction

Let E_n denote the Landweber exact spectrum with coefficient ring

$$E_{n*} = W\mathbb{F}_{p^n}[\![u_1, \dots, u_{n-1}]\!][u, u^{-1}],$$

where $W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$ of p^n elements, and whose BP_*–algebra structure map $r\colon BP_* \to E_{n*}$ is given by

$$r(v_i) = \begin{cases} u_i u^{1-p^i} & i < n \\ u^{1-p^n} & i = n \\ 0 & i > n \end{cases}$$

where $v_i \in BP_*$ is the i^{th} Hazewinkel generator. In particular, each u_i has degree 0 and u has degree -2. E_n is a commutative ring spectrum, and Morava theory tells us that the group of ring automorphisms of E_n is isomorphic to the profinite group $G_n = S_n \rtimes \mathrm{Gal}$, where S_n denotes the group of (not necessarily strict) isomorphisms of the height n Honda formal group law over $\mathbb{F}_{p^n}$, and Gal is the Galois group of $\mathbb{F}_{p^n}/\mathbb{F}_p$. A priori, G_n acts on E_n only in the stable category, but Hopkins and Miller (later improved by Goerss and Hopkins) proved that this can be made an honest action in an appropriate point set category of spectra (see Goerss and Hopkins [9] and Rezk [14]). "Continuous homotopy fixed point spectra" may also be constructed [7]: if G is

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a closed subgroup of G_n, the continuous homotopy G fixed point spectrum will be denoted by E_n^{hG}; if G is finite, this spectrum agrees with the ordinary homotopy fixed point spectrum. Moreover, $E_n^{hG_n} \simeq L_{K(n)}S^0$, the $K(n)_*$–localization of S^0, E_n^{hG} has the expected functorial properties, and there is a strongly convergent "continuous homotopy fixed point spectral sequence"

$$H_c^*(G, E_n^* X) \Rightarrow (E_n^{hG})^* X$$

for any spectrum X. ($H_c^*(G, E_n^* X)$ denotes the continuous cohomology of G with coefficients in the profinite G–module $E_n^* X$.)

The hope of this paper is to make some headway towards the computation of $\pi_* E_n^{hG}$, for G a closed subgroup of G_n. At first sight, this program seems impossible: the formulas for the action of (most elements of) G_n on E_{n*} are extremely complicated (see Devinatz and Hopkins [6]), making the computation of $H_c^*(G, E_{n*})$ apparently inaccessible. However,

$$H_c^*(G_n, N) = \mathrm{Ext}^*_{\mathrm{Map}_c(G, E_{n*})}(E_{n*}, N),$$

where $(E_{n*}, \mathrm{Map}_c(G, E_{n*}))$ is the complete Hopf algebroid defined using the action of G on E_{n*} (see for example Devinatz [5]). Since $\mathrm{Map}_c(G, E_{n*})$ is a quotient of

$$\mathrm{Map}_c(G_n, E_{n*}) = E_{n*} \mathbin{\widehat{\otimes}}_{BP_*} BP_*BP \mathbin{\widehat{\otimes}}_{BP_*} E_{n*} \equiv E^{\wedge}_{n*} E_n,$$

one may try to use the Hopf algebroid structure maps in BP_*BP together with several Bockstein spectral sequences to go from, for example, $H_c^*(G, E_{n*}/I_n)$ to $H_c^*(G, E_{n*})$. As usual, I_n is the maximal ideal $(p, u_1, \ldots, u_{n-1})$ in E_{n*}.

Let $H_n = W\mathbb{F}_{p^n}^\times \rtimes \mathrm{Gal} \subset G_n$, where $W\mathbb{F}_{p^n}^\times$ is the subgroup of S_n consisting of the diagonal matrices (see Section 1), and let $M(p)$ denote the mod(p) Moore spectrum. We compute $\pi_*\big(E_2^{hH_2} \wedge M(p)\big)$ for all primes $p > 2$ (Theorem 3.8). Of course, $\pi_* L_{K(2)} M(p)$ is known (Shimomura [15; 16]) for $p > 2$, so it is unclear if our computation yields any new homotopy information. Our computation is, however, much simpler and already indicates the necessity of "p–adic suspensions" in the Gross–Hopkins work on Brown–Comenetz duality (Remark 3.9). Moreover, we believe that computations such as $\pi_*\big(E_n^{hH_n} \wedge V(n-2)\big)$—recall that the Toda complex $V(n-2)$ exists E_{n*}–locally whenever p is sufficiently large compared to n—should be accessible to more skilled calculators.

Even when a complete calculation of $\pi_* E_n^{hG}$ is unattainable, partial information can lead to interesting consequences. For example, it is a long-standing conjecture that $\pi_* L_{K(n)} S^0$ is a module of finite type over the p–adic integers $\mathbb{Z}_p$. (This conjecture is known to be true for $n = 1$, and, if $p \geq 3$, for $n = 2$ Shimomura and Wang

[17], Shimomura and Yabe [18]. The reader may also find Hovey and Strickland [12, Theorem 15.1] interesting, where it is shown that $\pi_* L_{K(2)} S^0$ is *not* of finite type—at least when $p \geq 5$—if the grading is taken to be over the Picard group of invertible spectra in the $K(2)_*$–local category.) By a thick subcategory argument—see Devinatz [3] for a discussion of this in the E_{n*}–local category—if $\pi_* L_{K(n)} X$ is of finite type for some X in the E_{n*}–local category, then $\pi_* L_{K(n)} Y$ is of finite type for any finite Y such that

$$\{m \leq n : K(m)_* Y \neq 0\} \subset \{m \leq n : K(m)_* X \neq 0\}.$$

This in turn only requires that we prove that $\pi_*(E_n^{hG} \wedge X)$ is of finite type for some closed subgroup G of G_n for which there exists a chain

$$G = K_0 \lhd K_1 \lhd \cdots \lhd K_t = G_n$$

of closed subgroups. Indeed, assume inductively that $\pi_*(E_n^{hK_i} \wedge X)$ is of finite type. Then, since K_{i+1}/K_i is a p–adic analytic profinite group (see Dixon, du Sautoy, Mann and Segal [8, Theorem 9.6]), we have that $H_c^*\big(K_{i+1}/K_i, \pi_*(E_n^{hK_i} \wedge Y)\big)$ is also of finite type. (This follows from the fact that any p–adic analytic profinite group is of type $p - FP_\infty$ in the language of Symonds and Weigel [20].) But, in an earlier paper [4], we constructed a strongly convergent spectral sequence

$$H_c^*\big(K_{i+1}/K_i, \pi_*\big(E_n^{hK_i} \wedge X\big)\big) \Rightarrow \pi_*(E_n^{hK_{i+1}} \wedge X)$$

and showed that its E_∞ term has a horizontal vanishing line. This implies that the group $\pi_*\big(E_n^{hK_{i+1}} \wedge X\big)$ is of finite type and hence, by induction, so is $\pi_* L_{K(n)} X = \pi_*\big(E_n^{hG_n} \wedge X\big)$.

These considerations are unfortunately not applicable to $G = H_n$, since the normalizer of H_n in G_n is H_n, and, moreover, the group $\pi_*\big(E_2^{hH_2} \wedge M(p)\big)$ is not even of finite type. Yet it is, in some sense, "almost" of finite type (see Section 4), although the significance of this property is not clear.

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1 $H_c^*(H_n, E_{n*}/I_n)$ and its Hopf algebroid description

Recall that the group S_n may be described in several ways. If Γ_n denotes the height n Honda formal group law over $\mathbb{F}_{p^n}$, then S_n consists of all formal power series of the form $\sum_{i\geq 0}^{\Gamma_n} b_i x^{p^i}$ with each $b_i \in \mathbb{F}_{p^n}$ and $b_i \neq 0$. The ring of endomorphisms of Γ_n may also be described as the ring obtained by adjoining an indeterminate S—which corresponds to the endomorphism $f(x) = x^p$—to $W\mathbb{F}_{p^n}$ along with the relations $S^n = p$ and $Sw = w^\sigma S$, where $\sigma\colon W\mathbb{F}_{p^n} \to W\mathbb{F}_{p^n}$ denotes the Frobenius automorphism. The automorphism $\sum_{i\geq 0}^{\Gamma_n} b_i x^{p^i}$ corresponds to the element $\sum_{i=0}^{n-1} a_i S^i$ with

$$a_i = \sum_{k\geq 0} e(b_{i+nk}) p^k,$$

where $e(b)$ is the multiplicative representative of b in $W\mathbb{F}_{p^n}$. The subgroup $W\mathbb{F}_{p^n}^\times$ of S_n is then the group of automorphisms with $a_i = 0$ for all $i > 0$. In terms of matrices, S_n is the subgroup of $GL_n(W\mathbb{F}_{p^n})$ consisting of matrices of the form

$$\begin{bmatrix} a_0 & pa_{n-1} & pa_{n-2} & \cdots & pa_1 \\ a_1^{\sigma^{-1}} & a_0^{\sigma^{-1}} & pa_{n-1}^{\sigma^{-1}} & \cdots & pa_2^{\sigma^{-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & pa_{n-1}^{\sigma^{-(n-2)}} \\ a_{n-1}^{\sigma^{-(n-1)}} & a_{n-2}^{\sigma^{-(n-1)}} & a_{n-3}^{\sigma^{-(n-1)}} & \cdots & a_0^{\sigma^{-(n-1)}} \end{bmatrix},$$

and $W\mathbb{F}_{p^n}^\times$ is the subgroup of diagonal matrices in S_n.

Now let S_n^0 be the p–Sylow subgroup of S_n consisting of strict automorphisms of Γ_n. There is a split extension

$$S_n^0 \to S_n \to \mathbb{F}_{p^n}^\times;$$

the map $S_n \to \mathbb{F}_{p^n}^\times$ is given by $\sum_{i=0}^{n-1} a_i S^i \mapsto \overline{a_0}$, and the splitting sends $a \in \mathbb{F}_{p^n}$ to $e(a) \in W\mathbb{F}_{p^n}^\times \subset S_n$. This map also gives us a splitting of the short exact sequence

$$0 \to W\mathbb{F}_{p^n}^0 \to W\mathbb{F}_{p^n}^\times \to \mathbb{F}_{p^n}^\times \to 0,$$

and hence an isomorphism $W\mathbb{F}_{p^n}^\times \to W\mathbb{F}_{p^n}^0 \times \mathbb{F}_{p^n}^\times$. Since the order of $\mathbb{F}_{p^n}^\times$ is prime to p, it follows that

$$H_c^*(W\mathbb{F}_{p^n}^\times, N) \xrightarrow{\approx} H_c^*(W\mathbb{F}_{p^n}^0, N)^{\mathbb{F}_{p^n}^\times}$$

whenever N is a discrete $\mathbb{Z}_p[\![W\mathbb{F}_{p^n}^\times]\!]$–module. Now suppose, in addition, that N is a $W\mathbb{F}_{p^n}$–module and H_n–module in such a way that the $W\mathbb{F}_{p^n}^\times$ action is $W\mathbb{F}_{p^n}$–linear and that $\sigma(cn) = c^\sigma \sigma(n)$ for all $c \in W\mathbb{F}_{p^n}$ and $n \in N$. It then follows from Devinatz [1, Lemma 5.4] that $H^i\big(\mathrm{Gal}, H_c^*(W\mathbb{F}_{p^n}^\times, N)\big) = 0$ for all $i > 0$, and hence

$$H_c^*(H_n, N) \xrightarrow{\approx} H_c^*(W\mathbb{F}_{p^n}^\times, N)^{\mathrm{Gal}}.$$

Now S_n acts on $E_{n*}/I_n = \mathbb{F}_{p^n}[u, u^{-1}]$ via $\mathbb{F}_{p^n}$–algebra homomorphisms, and the action on u is given by

$$\Big(\sum_{i=0}^{n-1} a_i S^i\Big)(u) = \overline{a_0} u, \tag{1–1}$$

where, once again, $\overline{a_0}$ is the $\mathrm{mod}(p)$ reduction of a_0. From this it follows that

$$H_c^*(W\mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}[u,u^{-1}]) = \mathbb{F}_{p^n}[v_n, v_n^{-1}] \otimes_{\mathbb{F}_{p^n}} H_c^*(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n}).$$

Moreover, since Gal acts trivially on v_n,

$$H_c^*(H_n, \mathbb{F}_{p^n}[u,u^{-1}]) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes H_c^*(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n})^{\mathrm{Gal}}.$$

It is also easy to compute $H_c^*(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n})^{\mathrm{Gal}}$. Let $g_i \in \mathrm{Hom}_c(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n})$ be defined by

$$g_i\Big(1 + \sum_{j\geq 1} e(c_j)p^j\Big) = c_1^{p^i} = c_1^{\sigma^i}, \tag{1–2}$$

$0 \leq i \leq n-1$. Since the Galois automorphisms $id, \sigma, \dots, \sigma^{n-1}$ are linearly independent over $\mathbb{F}_{p^n}$, so are the g_i's. Now, and for the rest of this section, assume that $p > 2$. Then

$$\mathbb{Z}_p^n \approx W\mathbb{F}_{p^n} \xrightarrow{\approx} W\mathbb{F}_{p^n}^0$$

via the map sending $x \in W\mathbb{F}_{p^n}$ to $\exp(px) = 1 + \sum_{j\geq 1} \frac{p^j x^j}{j!} \in W\mathbb{F}_{p^n}^0$, so that $H_c^*(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n})$ is the exterior algebra over $\mathbb{F}_{p^n}$ on n generators in $H_c^1(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n})$. This implies that these generators may be taken to be $g_0, g_1, \dots, g_{n-1}$. Each g_i is Galois invariant, so

$$H_c^*(H_n, \mathbb{F}_{p^n}[u,u^{-1}]) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes E(g_0, g_1, \dots, g_{n-1}). \tag{1–3}$$

Next consider the complete Hopf algebroid $\big(E_{n*}, \mathrm{Map}_c\big(W\mathbb{F}_{p^n}^0, E_{n*}\big)\big) \equiv (E_{n*}, \Sigma_n)$. We explicitly identify $\Sigma_n/I_n\Sigma_n$ as a quotient of $E_{n*}^\wedge E_n/I_n E_{n*}^\wedge E_n$ and give cobar representatives for

$$g_i \in H_c^{1,0}(W\mathbb{F}_{p^n}^0, \mathbb{F}_{p^n}[u,u^{-1}]) = \mathrm{Ext}^{1,0}_{\Sigma_n/I_n\Sigma_n}(\mathbb{F}_{p^n}[u,u^{-1}], \mathbb{F}_{p^n}[u,u^{-1}]).$$

First recall that the maps $\eta_L, \eta_R\colon E_{n*} \to \mathrm{Map}_c(G_n, E_{n*})$ are given by $\eta_R(x)(s) = x$, $\eta_L(x)(s) = s^{-1}x$. Since $W\mathbb{F}_{p^n}^0 \subset G_n$ acts trivially on $W\mathbb{F}_{p^n} \subset E_{n*}$, it follows that $\eta_R\big|_{W\mathbb{F}_{p^n}} = \eta_L\big|_{W\mathbb{F}_{p^n}}$ in Σ_n, so that Σ_n is a Hopf algebra over $W\mathbb{F}_{p^n}$ and is a quotient of

$$\begin{aligned} W\mathbb{F}_{p^n} \otimes_{\mathbb{Z}_p} \mathrm{Map}_c(S_n, E_{n*})^{\mathrm{Gal}} &= W\mathbb{F}_{p^n} \otimes_{\mathbb{Z}_p} (E_{n*}^{\mathrm{Gal}} \mathbin{\widehat{\otimes}}_{BP_*} BP_*BP \mathbin{\widehat{\otimes}}_{BP_*} E_{n*}^{\mathrm{Gal}}) \\ &= W\mathbb{F}_{p^n} \otimes_{\mathbb{Z}_p} (E_n^{\mathrm{Gal}})_*^\wedge E_n^{\mathrm{Gal}}, \end{aligned}$$

where E_n^{Gal} is the Landweber exact spectrum with coefficient ring

$$\mathbb{Z}_p[\![u_1, \dots, u_{n-1}]\!][u, u^{-1}].$$

Now let $u \equiv \eta_R(u)$ and $w \equiv \eta_L(u)$ in $(E_n^{\mathrm{Gal}})^{\wedge}_* E_n^{\mathrm{Gal}}$. By (1–1), we have that $u = w$ in $\Sigma_n/I_n\Sigma_n$. Moreover, the image of $t_j \in BP_*BP$ in $(E_n^{\mathrm{Gal}})^{\wedge}_* E_n^{\mathrm{Gal}}$—also denoted t_j—satisfies

$$t_j\Big(\sum_{i\geq 0}{}^{\Gamma_n} b_i x^{p^i}\Big) = u^{1-p^i} b_0^{-1} b_j \quad \mod I_n(E_n^{\mathrm{Gal}})^{\wedge}_* E_n^{\mathrm{Gal}}$$

(see Devinatz [1, Proposition 2.11]), and thus

$$\Sigma_n/I_n\Sigma_n = \mathbb{F}_{p^n}[u,u^{-1}][t_n, t_{2n}, \ldots]/J_n, \tag{1–4}$$

with

$$J_n = (t_n^{p^n} - v_n^{p^n-1} t_n, t_{2n}^{p^n} - v_n^{p^{2n}-1} t_{2n}, \ldots, t_{jn}^{p^n} - v_n^{p^{jn}-1} t_{jn}, \ldots).$$

Finally, let $g = v_n^{-1} t_n \in \Sigma_n$. These considerations imply that $g^{p^i} \in \Sigma_n/I_n\Sigma_n$ is a cobar representative for $g_i \in H^1(W\mathbb{F}^0_{p^n}, \mathbb{F}_{p^n})$.

2 The Bockstein spectral sequence

Fix a prime p and integer $n \geq 2$, and let N be a complete $\big(E_{n*}, \mathrm{Map}_c(H_n, E_{n*})\big)$–comodule. Write

$$H^*N \equiv H_c^*(H_n, N) = H_c^*(W\mathbb{F}^0_{p^n}, N)^{\mathbb{F}^\times_{p^n} \rtimes \mathrm{Gal}} = \mathrm{Ext}^*_{\Sigma_n}(E_{n*}, N)^{\mathbb{F}^\times_{p^n} \rtimes \mathrm{Gal}}.$$

The Bockstein spectral sequence we will use is defined by the exact couple

$$\begin{tikzcd}[column sep=small] H^*(E_{n*}/(p,u_1,\ldots,u_{n-2})) \arrow[dr] & & H^*(E_{n*}/(p,u_1,\ldots,u_{n-2})) \arrow[ll, "v_{n-1}"'] \\ & H^*(E_{n*}/I_n). \arrow[ur] & \end{tikzcd} \tag{2–1}$$

Truncated, this spectral sequence is isomorphic to the spectral sequence of the unrolled exact couple

$$\begin{tikzcd}[column sep=small] 0 \arrow[d] & H^*(E_{n*}/I_n) \arrow[l] \arrow[dr] & H^*(E_{n*}/(p,\ldots,u_{n-2},u_{n-1}^2)) \arrow[l] \arrow[dr] & \cdots \arrow[l] \\ H^*(E_{n*}/I_n) \arrow[ur] & H^*(E_{n*}/I_n) \arrow[ur, "v_{n-1}"'] & H^*(E_{n*}/I_n) \arrow[ur, "v_{n-1}^2"'] & \end{tikzcd}$$

Since $H^{**}(E_{n*}/(p,u_1,\ldots,u_{n-2},u_{n-1}^k))$ is finite in each bidegree, this spectral sequence converges strongly to

$$H^*(E_{n*}/(p,\ldots,u_{n-2})) = \lim_{\overleftarrow{j}} H^*(E_{n*}/(p,u_1,\ldots,u_{n-2},u_{n-1}^j)).$$

3 Computation of $\pi_*(E^{hH_2} \wedge M(p))$

In this section, we specialize the above spectral sequence to the case $n = 2,\ p > 2$. Write $\overline{\Sigma}_2 = \Sigma_2/p\Sigma_2$, and let $g = v_2^{-1}t_2 \in \overline{\Sigma}_2$ as in Section 1.

We will need the following congruences for our calculation of the differentials in the Bockstein spectral sequence.

Lemma 3.1 $v_2^{1-p^2}t_2^{p^2} = t_2 \mod v_1^{p+1}\overline{\Sigma}_2$.

Proof Begin with the formula (see Ravenel [13, Theorem A2.2.5])

$$\sum_{i,j\ge 0}{}^F t_i\eta_R(v_j)^{p^i} = \sum_{i,j\ge 0}{}^F v_i t_j^{p^i}$$

in BP_*BP/pBP_*BP, where F is the universal p–typical formal group law on BP_*. Up through power series degree p^4 we then have

$$\text{(3–2)}\quad v_1 +_F v_1^p t_1 +_F v_1^{p^2} t_2 +_F v_1^{p^3} t_3 +_F \eta_R(v_2) +_F t_1\eta_R(v_2)^p +_F t_2\eta_R(v_2)^{p^2}$$
$$= v_1 +_F v_1 t_1^p +_F v_1 t_2^p +_F v_1 t_3^p +_F v_2 +_F v_2 t_1^{p^2} +_F v_2 t_2^{p^2}$$

in $E_{2*}^\wedge E_2/pE_{2*}^\wedge E_2$. But $\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1$ in BP_*BP/pBP_*BP, and therefore, since $t_1 \in v_1\overline{\Sigma}_2$, $\eta_R(v_2) = v_2 \mod v_1^{p+1}\overline{\Sigma}_2$. t_3 is also in $v_1\overline{\Sigma}_2$; hence, $\mod v_1^{p+1}\overline{\Sigma}_2$, (3–2) reduces to

$$v_2^p t_1 +_F v_2^{p^2} t_2 = v_1 t_2^p +_F v_2 t_2^{p^2}.$$

The desired result follows immediately from this equation. □

Lemma 3.3 $t_1 = v_1 g^p - v_1^{p+2}v_2^{-1}g + v_1^{p+2}v_2^{-1}g^p \mod v_1^{2p+3}\overline{\Sigma}_2$.

Proof From Ravenel [13, Corollary 4.3.21],

$$\eta_R(v_3) = v_3 + v_2 t_1^{p^2} + v_1 t_2^p - v_2^p t_1 - v_1^{p^2} t_2 - v_1^p t_1^{1+p^2} + v_1^{p^2} t_1^{1+p} + v_1 w_1\big(v_2, v_1 t_1^p, -v_1^p t_1\big)$$

in BP_*BP/pBP_*BP, where $w_1(x,y,z) \equiv \frac{1}{p}[x^p + y^p + z^p - (x+y+z)^p]$. Hence

$$\text{(3–4)}\qquad 0 = v_1 t_2^p - v_2^p t_1 - v_1^2 v_2^{p-1} t_1^p + v_1^{p+1} v_2^{p-1} t_1 \mod v_1^{2p+3}\overline{\Sigma}_2,$$

and thus

$$t_1 = v_2^{-p} v_1 t_2^p \mod v_1^{p+2}\overline{\Sigma}_2.$$

Plug this relation for t_1 back into the last two terms of (3–4) to get

$$v_2^p t_1 = v_1 t_2^p - v_1^{p+2} v_2^{-p^2+p-1} t_2^{p^2} + v_1^{p+2} v_2^{-1} t_2^p \mod v_1^{2p+3}\overline{\Sigma}_2.$$

By the previous lemma,

$$v_1^{p+2} v_2^{-p^2+p-1} t_2^{p^2} = v_1^{p+2} v_2^{p-2} t_2 \mod v_1^{2p+3}\overline{\Sigma}_2.$$

We then get the desired result. □

The next propositions will allow us to compute the Bockstein differentials on $v_2^k \in H^0(\mathbb{F}_{p^2}[u, u^{-1}])$.

Proposition 3.5 *In* $\overline{\Sigma}_2 / v_1^{3p+3}\overline{\Sigma}_2$,

$$\eta_R(v_2^s) - v_2^s = \\ s v_2^s \big[v_2^{-1} v_1^{1+p} (g^{p^2} - g^p) + v_2^{-2} v_1^{2(1+p)} (g - g^p) + \tfrac{s-1}{2} v_2^{-2} v_1^{2(1+p)} (g^{p^2} - g^p)^2 \big].$$

Proof Compute in $\overline{\Sigma}_2 / v_1^{3p+3}\overline{\Sigma}_2$:

$$\begin{aligned}
\eta_R(v_2^s) - v_2^s &= v_2^s [(v_2^{-1}\eta_R(v_2))^s - 1] \\
&= v_2^s [(1 + v_2^{-1} v_1 t_1^p - v_2^{-1} v_1^p t_1)^s - 1] \\
&= s v_2^s \big[v_2^{-1}(v_1 t_1^p - v_1^p t_1) + \tfrac{s-1}{2} v_2^{-2}(v_1 t_1^p - v_1^p t_1)^2 \big] \\
&= s v_2^s \big[v_2^{-1} v_1^{p+1} g^{p^2} - v_2^{-1} v_1^{p+1} g^p + v_2^{-2} v_1^{2(p+1)} (g - g^p) \\
&\qquad + \tfrac{s-1}{2} v_2^{-2} v_1^{2(p+1)} (g^{p^2} - g^p)^2 \big]
\end{aligned}$$

by the previous lemma. □

Proposition 3.6 *Suppose that* $p \not| s$, *that* $k \geq 0$, *and let*

$$b(sp^k) = sp^k - p^k - p^{k-1} - \cdots - 1.$$

Then there exists $z_{sp^k} \in E_{2*}$ *such that* $z_{sp^k} = v_2^{sp^k} \bmod I_2$ *and such that*

$$dz_{sp^k} \equiv \eta_R(z_{sp^k}) - z_{sp^k} = c v_2^{b(sp^k)} v_1^{(p^k + p^{k-1} + \cdots + 1)(p+1)} (g^p - g)$$

in $\overline{\Sigma}_2 / v_1^{(p^k + p^{k-1} + \cdots + p + 2)(p+1)} \overline{\Sigma}_2$, *for some* $c \in \mathbb{F}_p^\times$.

Proof Proceed by induction on k. If $k=0$, let $z_s = v_2^s$. Then by the preceding proposition,

$$\eta_R(z_s) - z_s = s v_2^{s-1} v_1^{p+1}(g^{p^2} - g^p) \bmod v_1^{2(p+1)}\overline{\Sigma}_2.$$

But $g^{p^2} = g \bmod v_1^{p+1}\overline{\Sigma}_2$, so we get the desired result.

Suppose now that $z_{sp^{k-1}}$ has been chosen. Then

$$d(z_{sp^{k-1}})^p = c v_2^{pb(sp^{k-1})} v_1^{(p^k+p^{k-1}+\cdots+p)(p+1)}(g^{p^2} - g^p) \qquad \big(\bmod v_1^{(p^k+p^{k-1}+\cdots+p^2+2p)(p+1)}\overline{\Sigma}_2\big).$$

Next consider $dv_2^{(s-1)p^k - p^{k-1} - \cdots - p+1}$. Since the exponent of v_2 is equal to $1 \bmod (p)$, we have

$$d(v_2^{b(sp^k)+2}) = v_2^{b(sp^k)+1} v_1^{1+p}(g^{p^2} - g^p) + v_2^{b(sp^k)} v_1^{2(1+p)}(g - g^p) \quad \bmod v_1^{3p+3}\overline{\Sigma}_2.$$

Then take

$$z_{sp^k} = (z_{sp^{k-1}})^p - c v_1^{(p^k+p^{k-1}+\cdots+p-1)(p+1)} v_2^{(s-1)p^k - p^{k-1} - \cdots - p+1}. \qquad \square$$

Corollary 3.7 *Suppose that $p \nmid s$ and $k \geq 0$. Up to multiplication by a unit in $\mathbb{F}_p$,*

$$d_{(p^k+p^{k-1}+\cdots+1)(p+1)} v_2^{sp^k} = v_2^{(s-1)p^k - p^{k-1} - \cdots - p-1}(g^p - g)$$

in the Bockstein spectral sequence (2–1). In particular, $v_2^t(g^p - g)$ is a boundary for all $t \in \mathbb{Z}$.

Now, since g^{p^i} may be regarded as an element of $H_c^1(H_n, \mathbb{F}_{p^n})$, and the inclusion

$$\mathbb{F}_{p^n} \longrightarrow (E_n)_0/(p, u_1, \ldots, u_{n-2})$$

induces a map

$$H_c^*(H_n, \mathbb{F}_{p^n}) \longrightarrow H^*(E_{n*}/(p_1, u_1, \ldots, u_{n-2})),$$

it follows that g^{p^i} is a permanent cycle in the Bockstein spectral sequence. Moreover,

$$H^*(\mathbb{F}_{p^2}[u, u^{-1}]) = \mathbb{F}_p[v_2, v_2^{-1}] \otimes E(g + g^p, g - g^p).$$

Using the preceding corollary, it's easy to read off the remaining differentials, yielding our main result.

Theorem 3.8 *If* $p \geq 3$

$$H^i(\mathbb{F}_{p^2}[\![u_1]\!][u,u^{-1}]) = \begin{cases} \mathbb{F}_p[v_1]\{1\} & i=0 \\ \mathbb{F}_p[v_1]\{\zeta\} \times \prod_{t\in\mathbb{Z}} \frac{\mathbb{F}_p[v_1]}{(v_1^{n_t})}\{c_t\} & i=1 \\ \prod_{t\in\mathbb{Z}} \frac{\mathbb{F}_p[v_1]}{(v_1^{n_t})}\{c_t\zeta\} & i=2 \end{cases}$$

as $\mathbb{F}_p[v_1]$*–modules, where* $\zeta = g+g^p$, c_t *reduces to* $v_2^t(g-g^p) \in H^1(\mathbb{F}_{p^2}[u,u^{-1}])$, *and*

$$n_t = \begin{cases} p+1 & t \not\equiv -1 \mod(p) \\ (p^i+p^{i-1}+\cdots+1)(p+1) & t=(s-1)p^i - p^{i-1} - \cdots - p - 1, \\ & s \not\equiv 0 \mod(p) \end{cases}$$

By sparseness,

$$\pi_{t-s}(E_2^{hH_2} \wedge M(p)) \approx H^{s,t}(\mathbb{F}_{p^2}[\![u_1]\!][u,u^{-1}]).$$

Remark 3.9 Let I_n denote the Brown–Comenetz dual of L_nS^0, the E_{n*}–localization of S^0. I_n is characterized by

$$\pi_0F(X,I_n) = [X,I_n]_0 = \mathrm{Hom}(\pi_0L_nX, \mathbb{Q}/\mathbb{Z}_{(p)})$$

for any spectrum X. In [10] (see also Strickland [19]), Gross and Hopkins establish a remarkable relationship between Brown–Comenetz and Spanier–Whitehead duality: they prove that if p is sufficiently large compared to $n \geq 2$, and if X is a $K(n-1)_*$–acyclic finite complex with $pE_{n*}X = 0$ and with v_n self-map $\Sigma^{2p^N(p^n-1)}X \to X$, then

$$F(X,I_n) \simeq \Sigma^\alpha L_nDX, \tag{3–10}$$

where α is any integer with

$$\alpha = 2p^{nN}(p^n-1/p-1) + n^2 - n \mod(2p^N(p^n-1)).$$

(As usual, DX denotes the Spanier–Whitehead dual of X.) There is, however, no integer α for which (3–10) is satisfied for *all* X. This contrasts with the situation when $n=1$: here we have $I_1 \simeq \Sigma^2L_1(S_p^0)$ (if $p>2$), where S_p^0 denotes S^0 completed at p, and thus $F(X,I_1) \simeq \Sigma^2L_1DX$ whenever X is a rationally acyclic finite spectrum.

Historically, it was Shimomura's calculation [15] of π_*L_2M which shattered the hope that I_2 might also be an integral suspension of $L_2(S_p^0)$. Our calculation of $\pi_*(E_2^{hH_2} \wedge M(p))$ yields this result as well; a sketch of the proof follows.

Suppose there existed an integer c with

$$F(M(p,v_1^k),I_2)\simeq \Sigma^c L_2 DM(p,v_1^k) \tag{3–11}$$

for a cofinal set of k, where $M(p,v_1^k)$ denotes a finite spectrum with

$$BP_*M(p,v_1^k)=BP_*/(p,v_1^k).$$

In addition, we may assume that

$$DM(p,v_1^k)\simeq \Sigma^{-2k(p-1)-2}M(p,v_1^k).$$

Let

$$E_{2*}M(p,v_1^k)^{\sim}=\mathrm{Hom}(E_{2*}M(p,v_1^k),\mathbb{Q}/\mathbb{Z}_{(p)}),$$

and recall that

$$\Sigma^4(E_{2*}M(p,v_1^k)^{\sim})\approx E_{2*}F(M(p,v_1^k),I_2)$$

as modules over E_{2*} and G_2. (See Strickland [19, Proposition 17] or Devinatz [2] for $p\geq 5$; note, however, that we are using Strickland's equivalent definition of $E_{2*}M(p,v_1^k)^{\sim}$.) Then (3–11) implies that

$$E_{2*}M(p,v_1^k)^{\sim}\approx \Sigma^{c-2k(p-1)-6}E_{2*}M(p,v_1^k),$$

and, by the theory of Poincaré pro–p groups (cf. Devinatz and Hopkins [6, Sections 5, 6]), there is a map

$$H^{2,6+2k(p-1)-c}(E_{2*}M(p,v_1^k))\to \mathbb{Q}/\mathbb{Z}_{(p)}$$

such that

$$H^i(E_{2*}M(p,v_1^k))\otimes H^{2-i}(E_{2*}M(p,v_1^k))\to H^2(E_{2*}M(p,v_1^k))\to \mathbb{Q}/\mathbb{Z}_{(p)}$$

is a perfect pairing. Hence there must exist a $d_k\in H^{2,6+2(p-1)-c}(E_{2*}M(p,v_1^k))$, for each k, such that $v_1^{k-1}d_k\neq 0$. But the computation of $H^2(\mathbb{F}_{p^2}[\![u_1]\!][u,u^{-1}])$ together with the exact sequence

$$H^2(\mathbb{F}_{p^2}[\![u_1]\!][u,u^{-1}])\xrightarrow{v_1^k} H^2(\mathbb{F}_{p^2}[\![u_1]\!][u,u^{-1}])\to H^2(E_{2*}M(p,v_1^k))\to 0$$

shows that this is impossible.

4 Some remarks on finiteness

In this section, we work in the E_{n*}–local stable category, so that by a finite spectrum, we mean an object of the thick subcategory generated by L_nS^0.

Let G be a closed subgroup of G_n, $n\geq 1$.

Proposition 4.1 *Let Y be a $K(n-1)_*$–acyclic finite spectrum. Then $\pi_*(E_n^{hG} \wedge Y)$ is of finite type (as a graded abelian group).*

Proof The proof is just as we argued in the Introduction: use the strongly convergent spectral sequence

$$H_c^{**}(G, E_{n*}Y) \Rightarrow \pi_*(E_n^{hG} \wedge Y)$$

whose E_∞ term has a horizontal vanishing line. Since $E_{n*}Y$ is of finite type, so is $H_c^{**}(G, E_{n*}Y)$. The horizontal vanishing line then implies that $\pi_*(E_n^{hG} \wedge Y)$ is also of finite type. □

Now suppose $n \geq 2$ and X is a $K(n-2)_*$–acyclic finite spectrum with v_{n-1} self-map ν. Let $X(\nu^k)$ denote the cofiber of $\nu^k \colon \Sigma^{k|\nu|} X \to X$, and let $X(\nu^\infty)$ denote the cofiber of $X \to \nu^{-1}X$, so that

$$X(\nu^\infty) = \operatorname*{holim}_{\to k} \Sigma^{-k|\nu|} X(\nu^k).$$

There are also canonical maps $X(\nu^k) \to X(\nu^{k-1})$ and $X \to \operatorname{holim}_{\leftarrow k} X(\nu^k)$. We will need the following well-known result (cf. Hovey [11, Section 2]).

Lemma 4.2 *If Z is any (E_{n*}–local) spectrum, the map*

$$Z \wedge X \to \operatorname*{holim}_{\leftarrow k} Z \wedge X(\nu^k)$$

is the $K(n)_*$–localization of $Z \wedge X$.

Proposition 4.3 *$\nu^{-1}\pi_*(E_n^{hG} \wedge X)$ is countable if and only if $\pi_*(E_n^{hG} \wedge X)$ is of finite type.*

Proof Proposition 4.1 implies that $\pi_*(E_n^{hG} \wedge X(\nu^\infty))$ is countable, and therefore $\nu^{-1}\pi_*(E_n^{hG} \wedge X)$ is countable if and only if $\pi_*(E_n^{hG} \wedge X)$ is countable. But

$$E_n^{hG} \wedge X \simeq \operatorname*{holim}_{\leftarrow k} E_n^{hG} \wedge X(\nu^k);$$

it therefore again follows from Proposition 4.1 that $\pi_i(E_n^{hG} \wedge X)$ is profinite and is thus countable if and only if it's finite. □

Remark 4.4 The chromatic splitting conjecture (see Hovey [11]) actually identifies $\nu^{-1}(E_n^{hG_n} \wedge X) = \nu^{-1}L_{K(n)}X$ as $L_{n-1}X \vee \Sigma^{-1}L_{n-1}X$.

Although $\pi_*(E_2^{hH_2} \wedge M(p))$ is not of finite type, this proposition suggests to us the sense in which it is "almost" of finite type. The details follow.

We will consider graded modules over the graded ring $\mathbb{F}_p[v]$, where v has positive even (unless $p=2$) degree, satisfying the following two conditions:

(i) M is complete in the sense that $M = \lim_{\leftarrow i} M/v^i M$.

(ii) M/vM is an $\mathbb{F}_p$ vector space of finite type.

Proposition 4.5 *Let X and v be as above and suppose that $p\colon X \to X$ is trivial. Then $\pi_*(E_n^{hG} \wedge X)$ is an $\mathbb{F}_p[v]$–module satisfying conditions (i) and (ii).*

Proof Since

$$\frac{\pi_*(E_n^{hG} \wedge X)}{v^k \pi_*(E_n^{hG} \wedge X)} \hookrightarrow \pi_*(E_n^{hG} \wedge X(v^k)),$$

we have the requisite finiteness. Moreover, it follows from the commutative diagram

$$\begin{tikzcd}
\pi_*(E_n^{hG} \wedge X) \arrow[r, "\approx"] \arrow[d] & \lim_{\leftarrow k} \pi_*(E_n^{hG} \wedge X(v^k)) \\
\lim_{\leftarrow k} \frac{\pi_*(E_n^{hG} \wedge X)}{v^k \pi_*(E_n^{hG} \wedge X)} \arrow[ur, hook] &
\end{tikzcd}$$

that $\pi_*(E_n^{hG} \wedge X)$ is complete. □

Torii [21, Proposition 4.10] shows that such a module M may be written as

$$M \approx \prod_\alpha \Sigma^{n_\alpha} \mathbb{F}_p[v] \times \prod_\beta \Sigma^{m_\beta} \mathbb{F}_p[v]/(v^{i_\beta}). \tag{4–6}$$

If M is of finite type, then the n_α's are bounded below and

$$v^{-1} M \approx v^{-1} \prod_\alpha \Sigma^{n_\alpha} \mathbb{F}_p[v] = \bigoplus_\alpha \Sigma^{n_\alpha} \mathbb{F}_p[v, v^{-1}].$$

In general, the torsion submodule T of M is a submodule of $\prod_\beta \Sigma^{m_\beta} \mathbb{F}_p[v]/(v^{i_\beta})$; its closure $\overline{T}$ is equal to $\prod_\beta \Sigma^{m_\beta} \mathbb{F}_p[v]/(v^{i_\beta})$. Let us say that M is *essentially of finite rank* if there are only a finite number of α in the decomposition (4–6); that is, if and only if $M/\overline{T}$ is a finitely generated $\mathbb{F}_p[v]$–module.

Our main theorem shows that $\pi_*(E_2^{hH_2} \wedge M(p))$ is essentially of finite rank for $p > 2$. We do not know, however, whether this property is generic; that is, whether, given G, $\pi_*(E_n^{hG} \wedge X)$ is essentially of finite rank for all X satisfying the hypotheses of Proposition 4.5 if it is for one such X with $K(n-1)_* X \neq 0$.

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