Homotopy groups of homotopy fixed point spectra associated to $E_n$

Ethan S Devinatz

We compute the mod($p$) homotopy groups of the continuous homotopy fixed point spectrum $E_n^{hH_n}$ for $p > 2$, where $E_n$ is the Landweber exact spectrum whose coefficient ring is the ring of functions on the Lubin–Tate moduli space of lifts of the height $n$ Honda formal group law over $\mathbb{F}_{p^n}$, and $H_n$ is the subgroup $W_{\mathbb{F}_{p^n}} \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ of the extended Morava stabilizer group $G_n$. We examine some consequences of this related to Brown–Comenetz duality and to finiteness properties of homotopy groups of $K(n)$–local spectra. We also indicate a plan for computing $\pi_* (E_n^{hH_n} \wedge V(n-2))$, where $V(n-2)$ is an $E_n$–local Toda complex.

55Q10, 55T25

Introduction

Let $E_n$ denote the Landweber exact spectrum with coefficient ring

$$E_n = W[\mathbb{F}_{p^n}[u_1, \ldots, u_{n-1}][u, u^{-1}]],$$

where $W[\mathbb{F}_{p^n}]$ denotes the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$ of $p^n$ elements, and whose $BP_*$–algebra structure map $r: BP_* \to E_n*$ is given by

$$r(v_i) = \begin{cases} u_i u^{1-p^i} & i < n \\ u^{1-p^n} & i = n \\ 0 & i > n \end{cases}$$

where $v_i \in BP_*$ is the $i^{th}$ Hazewinkel generator. In particular, each $u_i$ has degree 0 and $u$ has degree $-2$. $E_n$ is a commutative ring spectrum, and Morava theory tells us that the group of ring automorphisms of $E_n$ is isomorphic to the profinite group $G_n = S_n \rtimes \text{Gal}$, where $S_n$ denotes the group of (not necessarily strict) isomorphisms of the height $n$ Honda formal group law over $\mathbb{F}_{p^n}$, and Gal is the Galois group of $\mathbb{F}_{p^n}/\mathbb{F}_p$. A priori, $G_n$ acts on $E_n$ only in the stable category, but Hopkins and Miller (later improved by Goerss and Hopkins) proved that this can be made an honest action in an appropriate point set category of spectra (see Goerss and Hopkins [9] and Rezk [14]). “Continuous homotopy fixed point spectra” may also be constructed [7]: if $G$ is...
a closed subgroup of $G_n$, the continuous homotopy $G$ fixed point spectrum will be denoted by $E_n^{hG}$; if $G$ is finite, this spectrum agrees with the ordinary homotopy fixed point spectrum. Moreover, $E_n^{hG} \simeq L_{K(n)}S^0$, the $K(n)_*$–localization of $S^0$, $E_n^{hG}$ has the expected functorial properties, and there is a strongly convergent “continuous homotopy fixed point spectral sequence”

$$H^*_c(G, E^*_n X) \Rightarrow (E^{hG}_n)^* X$$

for any spectrum $X$. ($H^*_c(G, E^*_n X)$ denotes the continuous cohomology of $G$ with coefficients in the profinite $G$–module $E^*_n X$.)

The hope of this paper is to make some headway towards the computation of $\pi_* E_n^{hG}$, for $G$ a closed subgroup of $G_n$. At first sight, this program seems impossible: the formulas for the action of (most elements of) $G_n$ on $E^*_n$ are extremely complicated (see Devinatz and Hopkins [6]), making the computation of $H^*_c(G, E^*_n)$ apparently inaccessible. However,

$$H^*_c(G_n, N) = \text{Ext}^*_{Map_c(G, E^*_n)}(E^*_n, N),$$

where $(E^*_n, \text{Map}_c(G, E^*_n))$ is the complete Hopf algebroid defined using the action of $G$ on $E^*_n$ (see for example Devinatz [5]). Since Map$_c(G, E^*_n)$ is a quotient of $\text{Map}_c(G_n, E^*_n) = E^*_n \hat{\otimes} \text{BP}_* \text{BP}_* \text{BP} \hat{\otimes} \text{BP}_* E^*_n \equiv E^*_n \bar{E}_n,$

one may try to use the Hopf algebroid structure maps in $\text{BP}_* \text{BP}$ together with several Bockstein spectral sequences to go from, for example, $H^*_c(G, E^*_n/I^n)$ to $H^*_c(G, E^*_n)$. As usual, $I_n$ is the maximal ideal $(p, u_1, \ldots, u_{n-1})$ in $E^*_n$.

Let $H^*_n = W^\infty_{p^n} \rtimes \text{Gal} \subset G_n$, where $W^\infty_{p^n}$ is the subgroup of $S_n$ consisting of the diagonal matrices (see Section 1), and let $\hat{M}(p)$ denote the mod$(p)$ Moore spectrum. We compute $\pi_* \left(E^{hH_2}_2 \wedge M(p)\right)$ for all primes $p > 2$ (Theorem 3.8). Of course, $\pi_* L_{K(2)} \hat{M}(p)$ is known (Shimomura [15; 16]) for $p > 2$, so it is unclear if our computation yields any new homotopy information. Our computation is, however, much simpler and already indicates the necessity of “$p$–adic suspensions” in the Gross–Hopkins work on Brown–Comenetz duality (Remark 3.9). Moreover, we believe that computations such as $\pi_* \left(E^{hH_2}_n \wedge V(n-2)\right)$—recall that the Toda complex $V(n-2)$ exists $E^*_n$–locally whenever $p$ is sufficiently large compared to $n$—should be accessible to more skilled calculators.

Even when a complete calculation of $\pi_* E_n^{hG}$ is unattainable, partial information can lead to interesting consequences. For example, it is a long-standing conjecture that $\pi_* L_{K(n)} S^0$ is a module of finite type over the $p$–adic integers $\mathbb{Z}_p$. (This conjecture is known to be true for $n = 1$, and, if $p \geq 3$, for $n = 2$ Shimomura and Wang...
The author was partially supported by a grant from the NSF.

These considerations are unfortunately not applicable to \( G \) relations...—which corresponds to the endomorphism \( f(x) = x^p \) —to \( W F_{p^n} \) along with the relations \( S^n = p \) and \( Sw = w^a S \), where \( \sigma : W F_{p^n} \to W F_{p^n} \) denotes the Frobenius.
automorphism. The automorphism \( \sum_{i \geq 0} b_i x^{p^i} \) corresponds to the element \( \sum_{i=0}^{n-1} a_i S^i \) with

\[
a_i = \sum_{k \geq 0} e(b_{i+nk}) p^k,
\]

where \( e(b) \) is the multiplicative representative of \( b \) in \( W \mathbb{F}_{p^n} \). The subgroup \( W \mathbb{F}_{p^n}^\times \) of \( S_n \) is then the group of automorphisms with \( a_i = 0 \) for all \( i > 0 \). In terms of matrices, \( S_n \) is the subgroup of \( GL_n(W \mathbb{F}_{p^n}) \) consisting of matrices of the form

\[
\begin{pmatrix}
a_0 & pa_{n-1} & pa_{n-2} & \cdots & pa_1 \\
a_1 & a_0 & pa_{n-2} & \cdots & pa_2 \\
& & & & \\
& & & & \\
& & & & \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0
\end{pmatrix},
\]

and \( W \mathbb{F}_{p^n}^\times \) is the subgroup of diagonal matrices in \( S_n \).

Now let \( S_n^0 \) be the \( p \)-Sylow subgroup of \( S_n \) consisting of strict automorphisms of \( \Gamma_n \). There is a split extension

\[
S_n^0 \rightarrow S_n \rightarrow \mathbb{F}_{p^n};
\]

the map \( S_n \rightarrow \mathbb{F}_{p^n}^\times \) is given by \( \sum_{i=0}^{n-1} a_i S^i \mapsto \overline{a_0} \), and the splitting sends \( a \in \mathbb{F}_{p^n} \) to \( e(a) \in W \mathbb{F}_{p^n} \subset S_n \). This map also gives us a splitting of the short exact sequence

\[
0 \rightarrow W \mathbb{F}^0_{p^n} \rightarrow W \mathbb{F}_{p^n}^\times \rightarrow \mathbb{F}_{p^n}^\times \rightarrow 0,
\]

and hence an isomorphism \( W \mathbb{F}_{p^n}^\times \rightarrow W \mathbb{F}^0_{p^n} \times \mathbb{F}_{p^n}^\times \). Since the order of \( \mathbb{F}_{p^n}^\times \) is prime to \( p \), it follows that

\[
H^\bullet_c(W \mathbb{F}_{p^n}^\times, N) \xrightarrow{\approx} H^\bullet_c(W \mathbb{F}^0_{p^n}, N) \mathbb{F}_{p^n}^\times
\]

whenever \( N \) is a discrete \( \mathbb{Z}_p[W \mathbb{F}_{p^n}^\times] \)-module. Now suppose, in addition, that \( N \) is a \( W \mathbb{F}_{p^n}^\times \)-module and \( H_n \)-module in such a way that the \( W \mathbb{F}_{p^n}^\times \) action is \( W \mathbb{F}_{p^n}^\times \)-linear and that \( \sigma(cn) = e^\sigma \sigma(n) \) for all \( c \in W \mathbb{F}_{p^n}^\times \) and \( n \in N \). It then follows from Devinatz [1, Lemma 5.4] that \( H^i(\text{Gal}, H^\bullet_c(W \mathbb{F}_{p^n}^\times, N)) = 0 \) for all \( i > 0 \), and hence

\[
H^\bullet_c(H_n, N) \xrightarrow{\approx} H^\bullet_c(W \mathbb{F}_{p^n}^\times, N)^{\text{Gal}}.
\]

Now \( S_n \) acts on \( E_n/I_n = \mathbb{F}_{p^n}[u, u^{-1}] \) via \( \mathbb{F}_{p^n}^\times \)-algebra homomorphisms, and the action on \( u \) is given by

\[
(1-1) \quad \left( \sum_{i=0}^{n-1} a_i S^i \right)(u) = \overline{a_0} u.
\]
where, once again, \( \overline{a_0} \) is the mod(\( p \)) reduction of \( a_0 \). From this it follows that
\[
H_c^*(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p[u, u^{-1}]) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes_{\mathbb{F}_p} H_c^*(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p).
\]
Moreover, since Gal acts trivially on \( v_n \),
\[
H_c^*(H_n, \mathbb{F}_p[u, u^{-1}]) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes H_c^*(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p)^{\text{Gal}}.
\]
It is also easy to compute \( H_c^*(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p)^{\text{Gal}} \). Let \( g_i \in \text{Hom}_c(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p) \) be defined by
\[
g_i \left( 1 + \sum_{j \geq 1} e(c_j) p^j \right) = c_1^{p^i} = c_1^{\sigma^i},
\]
where \( 0 \leq i \leq n-1 \). Since the Galois automorphisms \( \text{id}, \sigma, \ldots, \sigma^{n-1} \) are linearly independent over \( \mathbb{F}_p \), so are the \( g_i \)'s. Now, and for the rest of this section, assume that \( p > 2 \). Then
\[
\mathbb{F}_p^n \approx W \mathbb{F}_p^{0, \infty} \longrightarrow W \mathbb{F}_p^0
\]
via the map sending \( x \in W \mathbb{F}_p^0 \) to \( \exp(p x) = 1 + \sum_{j \geq 1} \frac{p^j x^j}{j!} \in W \mathbb{F}_p^0 \), so that \( H_c^*(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p) \) is the exterior algebra over \( \mathbb{F}_p \) on \( n \) generators in \( H_c^1(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p) \). This implies that these generators may be taken to be \( g_0, g_1, \ldots, g_{n-1} \). Each \( g_i \) is Galois invariant, so
\[
H_c^*(H_n, \mathbb{F}_p[u, u^{-1}]) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes E(g_0, g_1, \ldots, g_{n-1}).
\]
Next consider the complete Hopf algebroid \( (E_{n*}, \text{Map}_c(W \mathbb{F}_p^{0, \infty}, E_{n*})) \equiv (E_{n*}, \Sigma_n) \). We explicitly identify \( \Sigma_n/I_n \Sigma_n \) as a quotient of \( E_{n*}^\wedge E_n/I_n E_{n*}^\wedge E_n \) and give cobar representatives for
\[
g_i \in H_c^{1,0}(W \mathbb{F}_p^{0, \infty}, \mathbb{F}_p[u, u^{-1}]) = \text{Ex}^{1,0}_{\Sigma_n/I_n \Sigma_n}(\mathbb{F}_p[u, u^{-1}], \mathbb{F}_p[u, u^{-1}]).
\]
First recall that the maps \( \eta_L, \eta_R \colon E_{n*} \to \text{Map}_c(G_n, E_{n*}) \) are given by \( \eta_R(x)(s) = x, \eta_L(x)(s) = s^{-1} x \). Since \( W \mathbb{F}_p^{0, \infty} \subset G_n \) acts trivially on \( W \mathbb{F}_p^{0, \infty} \subset E_{n*} \), it follows that \( \eta_R|_{W \mathbb{F}_p^{0, \infty}} = \eta_L|_{W \mathbb{F}_p^{0, \infty}} \) in \( \Sigma_n \), so that \( \Sigma_n \) is a Hopf algebra over \( W \mathbb{F}_p^{0, \infty} \) and is a quotient of
\[
W \mathbb{F}_p^{0, \infty} \otimes_{\mathbb{F}_p} \text{Map}_c(S_n, E_{n*})^{\text{Gal}} = W \mathbb{F}_p^{0, \infty} \otimes_{\mathbb{F}_p} (E_{n*}^{\text{Gal}} \otimes_{BP_*} BP_* BP \otimes_{BP_*} E_{n*}^{\text{Gal}})
\]
\[
= W \mathbb{F}_p^{0, \infty} \otimes_{\mathbb{F}_p} (E_{n*}^{\text{Gal}} \otimes_{E_n^{\text{Gal}}} E_n^{\text{Gal}}),
\]
where \( E_n^{\text{Gal}} \) is the Landweber exact spectrum with coefficient ring
\[
\mathbb{Z}_p[u_1, \ldots, u_{n-1}][u, u^{-1}].
\]
Now let \( u = \eta_R(u) \) and \( w = \eta_L(u) \) in \((E_n^{\text{Gal}})^* E_n^{\text{Gal}}\). By (1–1), we have that \( u = w \) in \( \Sigma_n/I_n \Sigma_n \). Moreover, the image of \( t_j \in BP_* BP \) in \((E_n^{\text{Gal}})^* E_n^{\text{Gal}}\) also denoted \( t_j \) satisfies

\[
 t_j \left( \sum_{i \geq 0} \Gamma_i b_i x^p^i \right) = u^{1-p^i} b_0^{-1} b_j \mod I_n(E_n^{\text{Gal}})^* E_n^{\text{Gal}}
\]

(see Devinatz [1, Proposition 2.11]), and thus

\[
(1–4) \quad \Sigma_n/I_n \Sigma_n = [F_{p^n}[u, u^{-1}][t_n, t_2n, \ldots]]/J_n,
\]

with

\[
J_n = (t_n^{p^n} - v_n^{-p^n-1} t_n, t_2n^{p^n} - v_n^{p^n-1} t_2n, \ldots, t_j^{p^n} - v_n^{p^n-j} t_j, \ldots).
\]

Finally, let \( g = v_n^{-1} t_n \in \Sigma_n \). These considerations imply that \( g x^p^j \in \Sigma_n/I_n \Sigma_n \) is a cobar representative for \( g \in H^1(W^0[p^n, F]) \).

2 The Bockstein spectral sequence

Fix a prime \( p \) and integer \( n \geq 2 \), and let \( N \) be a complete \((E_n^*, \text{Map}_c(H_n, E_n^*))\)–comodule. Write

\[
 H^* N \equiv H^*_c(H_n, N) = H^*_c(W^0[p^n, N])^{F_{p^n} \times \text{Gal}} = \text{Ext}_{\Sigma_n}^*(E_n^*, N)^{F_{p^n} \times \text{Gal}}.
\]

The Bockstein spectral sequence we will use is defined by the exact couple

\[
(2–1) \quad H^*(E_n^*/(p, u_1, \ldots, u_{n-2})) \xrightarrow{v_{n-1}} H^*(E_n^*/(p, u_1, \ldots, u_{n-2})) \xrightarrow{H^*(E_n^*/I_n)}
\]

Truncated, this spectral sequence is isomorphic to the spectral sequence of the unrolled exact couple

\[
\begin{array}{ccc}
0 & \xleftarrow{H^*(E_n^*/I_n)} & H^*(E_n^*/(p, u_1, \ldots, u_{n-2}, u_{n-1}^2)) \\
H^*(E_n^*/I_n) & \xleftarrow{v_{n-1}} & H^*(E_n^*/(p, \ldots, u_{n-2}, u_{n-1}^2)) \\
& \xleftarrow{v_{n-1}^2} & \cdots
\end{array}
\]

Since \( H^*(E_n^*/(p, u_1, \ldots, u_{n-2}, u_{n-1}^2)) \) is finite in each bidegree, this spectral sequence converges strongly to

\[
H^*(E_n^*/(p, u_1, \ldots, u_{n-2})) = \lim_{i} H^*(E_n^*/(p, u_1, \ldots, u_{n-2}, u_{n-1}^j)).
\]
3 Computation of $\pi_*(E^{hH^2 \wedge M}(p))$

In this section, we specialize the above spectral sequence to the case $n = 2$, $p > 2$. Write $\overline{\Sigma} = \Sigma_2/p\Sigma_2$, and let $g = v_2^{-1}t_2 \in \overline{\Sigma}$ as in Section 1.

We will need the following congruences for our calculation of the differentials in the Bockstein spectral sequence.

**Lemma 3.1** $v_2^{1-p^2}t_2^{p^2} = t_2 \mod v_1^{p+1}\overline{\Sigma}$.

**Proof** Begin with the formula (see Ravenel [13, Theorem A2.2.5])

$$\sum_{i,j \geq 0} F_t i_j \eta_R(v_j)^{p^i} = \sum_{i,j \geq 0} F_t i_j$$

in $BP_*BP/pBP_*BP$, where $F$ is the universal $p$–typical formal group law on $BP_*$. Up through power series degree $p^4$ we then have

$$(3-2) \quad v_1 + F v_1^p t_1 + F v_1^{p^2} t_2 + F v_1^{p^3} t_3 + F \eta_R(v_2) + F t_1 \eta_R(v_2) + F t_2 \eta_R(v_2)^p + F t_2 \eta_R(v_2)^p^2$$

$$= v_1 + F v_1 t_1^p + F v_1 t_2^p + F v_1 t_3^p + F v_2 + F v_2 t_1^p + F v_2 t_2^p$$

in $E_2^*E_2/pE_2^*E_2$. But $\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1$ in $BP_*BP/pBP_*BP$, and therefore, since $t_1 \in v_1 \overline{\Sigma}$, $\eta_R(v_2) = v_2 \mod v_1^{p+1}\overline{\Sigma}$. $t_3$ is also in $v_1 \overline{\Sigma}$; hence, mod $v_1^{p+1}\overline{\Sigma}$, (3–2) reduces to

$$v_2^p t_1 + F v_2^p t_2 = v_1 t_2^p + F v_2 t_2^p.$$

The desired result follows immediately from this equation. \hfill \square

**Lemma 3.3** $t_1 = v_1 g^p - v_1^{p+2} v_2^{-1} g + v_1^{p+2} v_2^{-1} g^p \mod v_1^{2p+3}\overline{\Sigma}$.

**Proof** From Ravenel [13, Corollary 4.3.21],

$$\eta_R(v_3) = v_3 + v_2 t_1^{p^2} + v_1 t_2^{p^2} - v_2 t_1 - v_1 t_2 - v_1^{p^2} t_1^{1+p^2} + v_1^{p^2} t_1^{1+p} + v_1 w_1(v_2, v_1 t_1^p, -v_1^p t_1)$$

in $BP_*BP/pBP_*BP$, where $w_1(x, y, z) \equiv \frac{1}{p}[x^p + y^p + z^p - (x + y + z)^p]$. Hence

$$(3-4) \quad 0 = v_1 t_2^p - v_2 t_1 - v_1^2 v_2 t_1^{p-1} + v_1^{p+1} v_2^{p-1} t_1 \mod v_1^{2p+3}\overline{\Sigma},$$

and thus

$$t_1 = v_2^p v_1 t_2^p \mod v_1^{p+2}\overline{\Sigma}.$$
Plug this relation for $t_1$ back into the last two terms of (3–4) to get

$$v_2^p t_1 = v_1 t_2^p - v_1^{p+2} v_2^{-p^2+p-1} t_2^p + v_1^{p+2} v_2^{-1} t_2^p \mod v_1^{2p+3} \Sigma_2.$$ 

By the previous lemma,

$$v_1^{p+2} v_2^{-p^2+p-1} t_2^p = v_1^{p+2} v_2^{-2} t_2 \mod v_1^{2p+3} \Sigma_2.$$ 

We then get the desired result. □

The next propositions will allow us to compute the Bockstein differentials on $v_2^k \in H^0(\mathbb{F}_p[u, u^{-1}]).$

**Proposition 3.5** In $\Sigma_2/v_1^{3p+3} \Sigma_2$,

$$\eta_R(v_2^s) - v_2^s = sv_2^s \left[ v_2^{-1} v_1^{1+p} (g^{p^2} - g^p) + v_2^{-2} v_1^{2(1+p)} (g - g^p) + \frac{s-1}{2} v_2^{-2} v_1^{2(1+p)} (g^{p^2} - g^p)^2 \right].$$

**Proof** Compute in $\Sigma_2/v_1^{3p+3} \Sigma_2$:

$$\eta_R(v_2^s) - v_2^s = v_2^s [(v_2^{-1} \eta_R(v_2))^s - 1] = v_2^s [ (1 + v_2^{-1} v_1 t_1^p - v_2^{-1} v_1 t_1)^s - 1] = sv_2^s [ v_2^{-1} (v_1 t_1^p - v_1 t_1) + \frac{s-1}{2} v_2^{-2} (v_1 t_1^p - v_1 t_1)^2 ] = sv_2^s [ v_2^{-1} v_1^{p+1} g^{p^2} - v_2^{-1} v_1^{p+1} g^p + v_2^{-2} v_1^{2(p+1)} (g - g^p) + \frac{s-1}{2} v_2^{-2} v_1^{2(p+1)} (g^{p^2} - g^p)^2 ]$$

by the previous lemma. □

**Proposition 3.6** Suppose that $p \nmid s$, that $k \geq 0$, and let

$$b(sp^k) = sp^k - p^k - p^{k-1} - \ldots - 1.$$ 

Then there exists $z_{sp^k} \in E_2$ such that $z_{sp^k} = v_2^{sp^k} \mod I_2$ and such that

$$dz_{sp^k} \equiv \eta_R(z_{sp^k}) - z_{sp^k} = cv_2^{b(sp^k)} v_1^{(p^k+p^{k-1}+\ldots+1)(p+1)} (g^p - g)$$

in $\Sigma_2/v_1^{(p^k+p^{k-1}+\ldots+1+2)(p+1)} \Sigma_2$, for some $c \in \mathbb{F}_p$. 

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Proof  Proceed by induction on \( k \). If \( k = 0 \), let \( z_s = v_2^s \). Then by the preceding proposition,
\[ \eta R (z_s) - z_s = s v_2^{s-1} v_1^{p+1} (g p^2 - g p) \mod v_1^{2(p+1) \Sigma_2} \]
But \( g p^2 = g \mod v_1^{p+1} \Sigma_2 \), so we get the desired result.

Suppose now that \( z_{sp^{k-1}} \) has been chosen. Then
\[
d(z_{sp^{k-1}}) = c v_2^{b(s p^{k-1})} v_1^{p^k + p^{k-1} + \ldots + p} (p+1) (g p^2 - g p) \mod v_1^{(p^k + p^{k-1} + \ldots + p^2 + 2p)(p+1) \Sigma_2}.
\]

Next consider \( d(v_2^{(s-1)p^k - p^{k-1} - \ldots - p+1}) \). Since the exponent of \( v_2 \) is equal to \( 1 \mod p \),
we have
\[
d(v_2^{b(s p^k) + 2}) = v_2^{b(s p^k) + 1} v_1^{p^k + p^{k-1} + \ldots + p} (p+1) (g p^2 - g p) + v_2^{b(s p^k)} v_1^{2(1+p)} (g - g p) \mod v_1^{p + 3p^3 \Sigma_2}.
\]
Then take
\[
z_{sp^k} = (z_{sp^{k-1}}) - c v_1^{p^k + p^{k-1} + \ldots + p-1} v_2^{(s-1)p^k - p^{k-1} - \ldots - p+1}.
\]

Corollary 3.7  Suppose that \( p \nmid s \) and \( k \geq 0 \). Up to multiplication by a unit in \( \mathbb{F}_p \),
\[
d(p^k + p^{k-1} + \ldots + p) v_2^{s p^k} = v_2^{(s-1)p^k - p^{k-1} - \ldots - p-1} (g p - g)
\]
in the Bockstein spectral sequence \( (2-1) \). In particular, \( v_2^t (g p - g) \) is a boundary for all \( t \in \mathbb{Z} \).

Now, since \( g p^t \) may be regarded as an element of \( H^1_c (H_n, \mathbb{F}_p^n) \), and the inclusion
\[
\mathbb{F}_p^n \longrightarrow (E_n)_0 / (p, u_1, \ldots, u_{n-2})
\]
induces a map
\[
H^*_c (H_n, \mathbb{F}_p^n) \longrightarrow H^*_c (E_n, (p, u_1, \ldots, u_{n-2})),
\]
it follows that \( g p^t \) is a permanent cycle in the Bockstein spectral sequence. Moreover,
\[
H^*_c (\mathbb{F}_p^n[u, u^{-1}]) = \mathbb{F}_p[v_2, v_2^{-1}] \otimes E(g + g p^t, g - g p^t).
\]
Using the preceding corollary, it’s easy to read off the remaining differentials, yielding our main result.
Theorem 3.8 If $p \geq 3$

$$H^i(\mathbb{F}_p[u_1][u,u^{-1}]) = \begin{cases} \mathbb{F}_p[u_1][1] & i = 0 \\ \mathbb{F}_p[u_1][\xi] \times \prod_{t \in \mathbb{Z}} \mathbb{F}_p[u_1]\{c_t\} & i = 1 \\ \prod_{t \in \mathbb{Z}} \mathbb{F}_p[u_1]\{c_t\} & i = 2 \end{cases}$$

as $\mathbb{F}_p[u_1]$–modules, where $\xi = g + g^p$, $c_t$ reduces to $v_2^t(g - g^p) \in H^1(\mathbb{F}_p[u,u^{-1}])$, and

$$n_t = \begin{cases} p + 1 & t \neq -1 \mod(p) \\ (p^t + p^{t-1} + \cdots + 1)(p+1) & t = (s-1)p^{t-1} - \cdots - p - 1, \\ s \neq 0 \mod(p) \end{cases}$$

By sparseness,

$$\pi_{t-s}(E^h_2 \wedge M(p)) \approx H^{s,t}(\mathbb{F}_p[u_1][u,u^{-1}]).$$

Remark 3.9 Let $I_n$ denote the Brown–Comenetz dual of $L_n S^0$, the $E_n^*$–localization of $S^0$. $I_n$ is characterized by

$$\pi_0 F(X, I_n) = [X, I_n]_0 = \text{Hom}(\pi_0 L_n X, \mathbb{Q}/\mathbb{Z}(p))$$

for any spectrum $X$. In [10] (see also Strickland [19]), Gross and Hopkins establish a remarkable relationship between Brown–Comenetz and Spanier–Whitehead duality: they prove that if $p$ is sufficiently large compared to $n \geq 2$, and if $X$ is a $K(n-1)_*$–acyclic finite complex with $pE_n X = 0$ and with $v_n$ self-map $\Sigma^2 p^N(p^n-1)X \rightarrow X$,

$$F(X, I_n) \simeq \Sigma^\alpha L_n DX,$$

where $\alpha$ is any integer with

$$\alpha = 2p^N(p^n - 1/p - 1) + n^2 - n \mod(2p^N(p^n - 1)).$$

(As usual, $DX$ denotes the Spanier–Whitehead dual of $X$.) There is, however, no integer $\alpha$ for which (3–10) is satisfied for all $X$. This contrasts with the situation when $n = 1$: here we have $I_1 \simeq \Sigma^2 L_1(S^0_p)$ (if $p > 2$), where $S^0_p$ denotes $S^0$ completed at $p$, and thus $F(X, I_1) \simeq \Sigma^2 L_1 DX$ whenever $X$ is a rationally acyclic finite spectrum.

Historically, it was Shimomura’s calculation [15] of $\pi_*(L_2 M)$ which shattered the hope that $I_2$ might also be an integral suspension of $L_2(S^0_p)$. Our calculation of $\pi_*(E^h_2 \wedge M(p))$ yields this result as well; a sketch of the proof follows.
Suppose there existed an integer $c$ with
\[(3–11) \quad F(M(p, v_1^k), I_2) \simeq \Sigma^c L_2 DM(p, v_1^k)\]
for a cofinal set of $k$, where $M(p, v_1^k)$ denotes a finite spectrum with
\[BP_*M(p, v_1^k) = BP_*/(p, v_1^k).\]
In addition, we may assume that
\[DM(p, v_1^k) \simeq \Sigma^{-2k(p-1)-2} M(p, v_1^k).\]
Let
\[E_2* M(p, v_1^k) \simeq \text{Hom}(E_2* M(p, v_1^k), \mathbb{Q}/\mathbb{Z}(p)),\]
and recall that
\[\Sigma^4 (E_2* M(p, v_1^k)) \simeq E_2* F(M(p, v_1^k), I_2)\]
as modules over $E_2*$ and $G_2$. (See Strickland [19, Proposition 17] or Devinatz [2]
for $p \geq 5$; note, however, that we are using Strickland’s equivalent definition of
$E_2* M(p, v_1^k)$.) Then (3–11) implies that
\[E_2* M(p, v_1^k) \simeq \Sigma^{-2k(p-1)-6} E_2* M(p, v_1^k),\]
and, by the theory of Poincaré pro–$p$ groups (cf. Devinatz and Hopkins [6, Sections 5,
6]), there is a map
\[H^{2, 6+2k(p-1)-c}(E_2* M(p, v_1^k)) \to \mathbb{Q}/\mathbb{Z}(p)\]
such that
\[H^i(E_2* M(p, v_1^k)) \otimes H^{2-i}(E_2* M(p, v_1^k)) \to H^2(E_2* M(p, v_1^k)) \to \mathbb{Q}/\mathbb{Z}(p)\]
is a perfect pairing. Hence there must exist a $d_k \in H^{2, 6+2(p-1)-c}(E_2* M(p, v_1^k))$, for each $k$, such that $v_1^{k-1} d_k \neq 0$. But the computation of $H^2(\mathbb{F}_p[u][u, u^{-1}])$ together
with the exact sequence
\[H^2(\mathbb{F}_p[u][u, u^{-1}]) \xrightarrow{v_1^k} H^2(\mathbb{F}_p[u][u, u^{-1}]) \to H^2(E_2* M(p, v_1^k)) \to 0\]
shows that this is impossible.

4 Some remarks on finiteness

In this section, we work in the $E_n*$–local stable category, so that by a finite spectrum,
we mean an object of the thick subcategory generated by $L_n S^0$.

Let $G$ be a closed subgroup of $G_n$, $n \geq 1$. 
Proposition 4.1  Let $Y$ be a $K(n-1)_*-$acyclic finite spectrum. Then $\pi_*(E_n^{hG} \wedge Y)$ is of finite type (as a graded abelian group).

Proof  The proof is just as we argued in the Introduction: use the strongly convergent spectral sequence

$$H_c^{**}(G, E_{n*}Y) \Rightarrow \pi_*(E_n^{hG} \wedge Y)$$

whose $E_\infty$ term has a horizontal vanishing line. Since $E_{n*}Y$ is of finite type, so is $H_c^{**}(G, E_{n*}Y)$. The horizontal vanishing line then implies that $\pi_*(E_n^{hG} \wedge Y)$ is also of finite type.

Now suppose $n \geq 2$ and $X$ is a $K(n-2)_*-$acyclic finite spectrum with $v_{n-1}$ self-map $v$. Let $X(v^k)$ denote the cofiber of $v^k: \Sigma^k[v]X \to X$, and let $X(v^\infty)$ denote the cofiber of $X \to v^{-1}X$, so that

$$X(v^\infty) = \lim_{\to k} \Sigma^{-k[v]}X(v^k).$$

There are also canonical maps $X(v^k) \to X(v^{k-1})$ and $X \to \lim_{\to k} X(v^k)$. We will need the following well-known result (cf. Hovey [11, Section 2]).

Lemma 4.2  If $Z$ is any $(E_{n*}-local)$ spectrum, the map

$$Z \wedge X \to \lim_{\to k} Z \wedge X(v^k)$$

is the $K(n)_*$-localization of $Z \wedge X$.

Proposition 4.3  $v^{-1}\pi_*(E_n^{hG} \wedge X)$ is countable if and only if $\pi_*(E_n^{hG} \wedge X)$ is of finite type.

Proof  Proposition 4.1 implies that $\pi_*(E_n^{hG} \wedge X(v^\infty))$ is countable, and therefore $v^{-1}\pi_*(E_n^{hG} \wedge X) = \lim_{\to k} E_n^{hG} \wedge X(v^k)$; it therefore again follows from Proposition 4.1 that $\pi_1(E_n^{hG} \wedge X)$ is profinite and is thus countable if and only if it’s finite.

Remark 4.4  The chromatic splitting conjecture (see Hovey [11]) actually identifies $v^{-1}(E_n^{hG} \wedge X) = v^{-1}L_{K(n)}X$ as $L_{n-1}X \vee \Sigma^{-1}L_{n-1}X$.
Although $\pi_*(E^h_{2} \land M(p))$ is not of finite type, this proposition suggests to us the sense in which it is “almost” of finite type. The details follow.

We will consider graded modules over the graded ring $\mathbb{F}_p[v]$, where $v$ has positive even (unless $p = 2$) degree, satisfying the following two conditions:

(i) $M$ is complete in the sense that $M = \lim_{\leftarrow i} M/v^i M$.

(ii) $M/vM$ is an $\mathbb{F}_p$ vector space of finite type.

**Proposition 4.5** Let $X$ and $\nu$ be as above and suppose that $p: X \to X$ is trivial. Then $\pi_*(E^h_{n} \land X)$ is an $\mathbb{F}_p[v]$–module satisfying conditions (i) and (ii).

**Proof** Since

$$
\pi_*(E^h_{n} \land X) \to v^k \pi_*(E^h_{n} \land X) \to \pi_*(E^h_{n} \land X(v^k))
$$

we have the requisite finiteness. Moreover, it follows from the commutative diagram

$$
\pi_*(E^h_{n} \land X) \approx \lim_{\leftarrow k} \pi_*(E^h_{n} \land X(v^k))
$$

that $\pi_*(E^h_{n} \land X)$ is complete. 

Torii [21, Proposition 4.10] shows that such a module $M$ may be written as

$$(4–6) \quad M \approx \prod_{\alpha} \mathbb{F}_p[v]^{n_{\alpha}} \times \prod_{\beta} \mathbb{F}_p[v]/(v^{i_{\beta}}).$$

If $M$ is of finite type, then the $n_{\alpha}$’s are bounded below and

$$v^{-1} M \approx v^{-1} \prod_{\alpha} \mathbb{F}_p[v]^{n_{\alpha}} = \bigoplus_{\alpha} \mathbb{F}_p[v, v^{-1}].$$

In general, the torsion submodule $T$ of $M$ is a submodule of $\prod_{\beta} \mathbb{F}_p[v]/(v^{i_{\beta}})$; its closure $\overline{T}$ is equal to $\prod_{\beta} \mathbb{F}_p[v]/(v^{i_{\beta}})$. Let us say that $M$ is *essentially of finite rank* if there are only a finite number of $\alpha$ in the decomposition $(4–6)$; that is, if and only if $M/\overline{T}$ is a finitely generated $\mathbb{F}_p[v]$–module.

*Geometry & Topology Monographs, Volume 10 (2007)*
Our main theorem shows that $\pi_*(E_{2}^{hH_2} \wedge M(p))$ is essentially of finite rank for $p > 2$. We do not know, however, whether this property is generic; that is, whether, given $G$, $\pi_*(E_{hG}^{h} \wedge X)$ is essentially of finite rank for all $X$ satisfying the hypotheses of Proposition 4.5 if it is for one such $X$ with $K(n-1)_*X \neq 0$.

References


Homotopy groups of homotopy fixed point spectra associated to $E_n$


Department of Mathematics, University of Washington
Seattle, Washington, USA

devinatz@math.washington.edu

http://www.math.washington.edu/~devinatz/

Received: 13 September 2004  Revised: 5 July 2005