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Nilpotency of the Bauer–Furuta stable homotopy Seiberg–Witten invariants

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We prove a nilpotency theorem for the Bauer–Furuta stable homotopy Seiberg–Witten invariants for smooth closed 4–manifolds with trivial first Betti number.

57R57, 55P91; 55P42, 57M50

Dedicated to Professor Goro Nishida for his 60th birthday

1 Introduction

As a byproduct of his celebrated work [5] on the Yukio Matsumoto $11/8$ birthday conjecture [8], the first author and S Bauer [2; 3] significantly refined the Seiberg–Witten invariants [11] for smooth closed Spin^c 4–manifolds. While the connected sum of two smooth connected Spin^c 4–manifolds with $b_2^+ \geq 2$ always yields trivial Seiberg–Witten invariants, these refined Seiberg–Witten invariants yielded non-trivial values for some connected sums.

In this paper, we investigate the behavior of these refined Seiberg–Witten invariants, which we call the Bauer–Furuta stable homotopy Seiberg–Witten invariants, with respect to the connected sum operations. For simplicity, we always assume $b_1 = 0$. Then our main theorem is the following:

Theorem 1.1 *Let X be a Spin^c 4–manifold with $b_2^+(X) \geq 1$ and $b_1(X) = 0$. Then, there is some large N such that, for any $n \geq N$, $\#^n X$, the n–fold connected sum of X with itself, has a trivial Bauer–Furuta stable homotopy Seiberg–Witten invariant for any Spin^c–structure c of $\#^n X$ and any orientation $o_{\#^n X}$ of $H^+(\#^n X)$.*

This is an easy consequence of the Devinatz–Hopkins–Smith nilpotency theorem [4; 7], which generalizes the earlier work of Nishida's nilpotency theorem [10]. Actually, the gluing theorem of S Bauer [1] translates the connected sum operation corresponding

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to the S^1–equivariant join operation, in which case the usual non-equivariant smash product may be used.

After recalling the basic notations and backgrounds in Section 2, we present the Bauer gluing theorem in Section 3 and the Devinatz–Hopkins–Smith nilpotency theorem in Section 4. Then, in Section 5, we prove the main theorem.

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The nilpotency is really the key philosophy of this paper, and we would like to dedicate this paper to Professor Goro Nishida.

2 Notation and background

We first set the following notations:

X	an oriented closed 4–manifold with $b_1(X) = 0$
c	a Spin^c–structure of X
$\mathrm{sign}(X)$	the signature of X
$H^+(X)$	the maximal positive definite subspace of $H^2(X,\mathbb{R})$
o_X	an orientation of $H^+(X)$
$b_2^+(X)$	the dimension of $H^+(X)$

We now put

$$m := \tfrac{1}{8}\big(c_1(L_c)^2 - \mathrm{sign}(X)\big) \qquad n := b_2^+(X)$$

and assume $m \geq 0$. Actually, we can always make $m \geq 0$ by changing the orientation of X, if necessary. Now the Bauer–Furuta stable homotopy Seiberg–Witten invariant $\widetilde{SW}(X, c, o_X)$ [3] is defined for the data (X, c, o_X) so that

(i) $SW(X, c, o_X) \in \{S(\mathbb{C}^m), S(\mathbb{R}^n)\}^{U(1)} :=$
$$\lim_{p,q\to\infty}[S(\mathbb{C}^{p+m} \oplus \mathbb{R}^q), S(\mathbb{C}^p \oplus \mathbb{R}^{q+n})]^{U(1)}$$

(ii) (via the G–Freudenthal suspension theorem by Hauschild [6], Namboodiri [9]) If $n = b_2^+(X) \geq 2$,

$$\begin{aligned}\{S(\mathbb{C}^m), S(\mathbb{R}^n)\}^{U(1)} &= \lim_{p,q\to\infty}[S(\mathbb{C}^{p+m} \oplus \mathbb{R}^q), S(\mathbb{C}^p \oplus \mathbb{R}^{q+n})]^{U(1)} \\ &\xleftarrow{\cong} \lim_{q\to\infty}[S(\mathbb{C}^m \oplus \mathbb{R}^q), S(\mathbb{R}^{q+n})]^{U(1)} \\ &= [\mathbb{CP}^{m-1} * S^{q-1}, S^{n-1} * S^{q-1}] = \pi_s^{n-1}(\mathbb{CP}^{m-1}_+),\end{aligned}$$

where $\dim \mathcal{M} = 2m - n - 1$.

Under the cohomotopy Hurewicz map

$$\pi_s^{n-1}(\mathbb{CP}_+^{m-1}) \to H^{n-1}(\mathbb{CP}_+^{m-1}),$$

$SW(X, c, o_X)$ maps to the usual Seiberg–Witten invariant. Note that the difference of the top cell dimensions of $\mathbb{CP}_+^{m-1}$ and S^{n-1} is nothing but the dimension of the Seiberg–Witten moduli space $\mathcal{M}$:

$$\dim \mathcal{M} = 2(m-1) - (n-1) = 2m - n - 1.$$

3 Bauer's gluing theorem

For oriented closed 4–manifolds X_1, X_2, consider their connected sum, the oriented closed 4–manifold $X_1 \# X_2$. Then any Spin^c–structure c of $X_1 \# X_2$ and any orientation $o_{X_1 \# X_2}$ of $H^+(X_1 \# X_2)$, the maximal positive definite subspace of $H^2(X_1 \# X_2, \mathbb{R})$, are both joins of the corresponding quantities of X_1 and X_2. More explicitly, for $i = 1, 2$, there are Spin^c–structures c_i of X_i and orientations o_{X_i} of $H^+(X_i)$, the maximal positive definite subspace of $H^2(X_i, \mathbb{R})$, such that

$$c = c_1 \# c_2 \qquad o_{X_1 \# X_2} = o_{X_1} \# o_{X_2}$$

Thus, we may write

$$(X_1 \# X_2, c, o_{X_1 \# X_2}) = (X_1 \# X_2, c_1 \# c_2, o_{X_1} \# o_{X_2}) = (X_1, c_1, o_{X_1}) \# (X_2, c_2, o_{X_2}).$$

Now the Bauer gluing formula [1] describes the Bauer–Furuta stable homotopy Seiberg–Witten invariant of this connected sum:

Theorem 3.1 *The connected sum # corresponds to the join $*$:*

$$SW((X_1, c_1, o_{X_1}) \# (X_2, c_2, o_{X_2})) = SW(X_1, c_1, o_{X_1}) * SW(X_2, c_2, o_{X_2}),$$

where the join

$$*\colon \{S(\mathbb{C}^{2k_1}), S(\mathbb{R}^{l_1})\}^{U(1)} \times \{S(\mathbb{C}^{2k_1}), S(\mathbb{R}^{l_1})\}^{U(1)} \to \{S(\mathbb{C}^{2k_1+k_2}), S(\mathbb{R}^{l_1+l_2})\}^{U(1)}$$

is induced by the usual join operation $S(U) * S(V) = S(U \oplus V)$.

We note the dimension of the moduli space behaves as

$$\dim \mathcal{M}((X_1, c_1, o_{X_1}) \# (X_2, c_2, o_{X_2})) \\ = \dim \mathcal{M}(X_1, c_1, o_{X_1}) + \dim \mathcal{M}(X_2, c_2, o_{X_2}) + 1,$$

via the join operation. From this, we observe the usual Seiberg–Witten invariant of

$$(X_1, c_1, o_{X_1})\#(X_2, c_2, o_{X_2})$$

is 0, when $b^+(X_i) \geq 2$, $(i = 1, 2)$, However, as was first pointed out by Bauer [1], this is *not* the case for the Bauer–Furuta stable homotopy Seiberg–Witten invariant. Actually, most applications of the Bauer–Furuta stable homotopy Seiberg–Witten invariants reflect this phenomenon.

4 Nilpotency

We now change gears to hard-core homotopy theory. We recall the fundamental concept and the theorem of Hopkins–Smith [7].

Definition 4.1 [7, Definition 1.i] A map of spectra $f\colon F \to X$ is *smash nilpotent* if for $n \gg 0$ the induced map n–fold smash products $f^{(n)}\colon F^{(n)} \to X^{(n)}$ is null.

Theorem 4.2 [7, Theorem 3.iii] *A map $f\colon F \to X$ from a finite spectrum to a p–local spectrum is smash nilpotent if and only if $K(n)_* f = 0$ for all $0 \leq n \leq \infty$.*

This deep theorem generalizes their earlier work [4] with Devinatz, and specializes to Nishida's nilpotency on the stable homotopy groups of spheres [10] when F and X are both spheres of distinct dimensions.

5 Proof

Before the proof, we first consider some general properties of the Bauer–Furuta stable homotopy Seiberg–Witten invariant $SW(\#_{i=1}^n X_i, c, o_{\#_{i=1}^n X_i})$ of $\#_{i=1}^n X_i$.

First, as was discussed at the beginning of Section 3, we may always write

$$SW(\#_{i=1}^n X_i, c, o_{\#_{i=1}^n X_i}) = SW(\#_{i=1}^n (X_i, c_i, o_{X_i})) = *_{i=1}^n SW(X_i, c_i, o_{X_i})$$

for some choices of Spin^c–structure c_i and orientation o_{X_i} of $H^+(X_i)$, the maximal positive definite subspace of $H^2(X_i, \mathbb{R})$.

For each $i = 1, \ldots, n$, let f_i be a representative of

$$SW(X_i, c_i, o_{X_i}) \in \{S(\mathbb{C}^{a_i}), S(\mathbb{R}^{b_i})\}^{U(1)}$$

where

$$a_i = \tfrac{1}{8}\big(c_1(L_{c_i})^2 - \mathrm{sign}(X_i)\big), \qquad b_i = b_2^+(X_i).$$

To simplify our notations, we express every map "unstably" in this section. For instance, we write

$$f_i\colon S(\mathbb{C}^{a_i}) \to S(\mathbb{R}^{b_i}),$$

though the general precise expression is of the form $f_i\colon S(\mathbb{C}^{a_i} \oplus \mathbb{R}^q) \to S(\mathbb{R}^{b_i+q})$ (cf Section 2(i)). However, such simplified notations shall not cause any problem here.

When $b_2^+(X_i) \geq 2$ for each $i = 1, \ldots, n$, f_i admits a non-equivariant counterpart as in Section 2(ii). We denote it also "unstably" as

$$\widetilde{f_i}\colon \mathbb{CP}^{a_i-1} \to S^{b_i-1},$$

though the general precise expression is of the form

$$\widetilde{f_i}\colon \mathbb{CP}^{a_i-1} * S^{q-1} \to S^{b_i-1} * S^{q-1} \qquad \text{(cf Section 2(ii))}$$

Then a non-equivariant counterpart $\widetilde{f}$ of a representative $\bigwedge_{i=1}^n \widetilde{f_i}$ of

$$SW(\#_{i=1}^n X_i, c, o_{\#_{i=1}^n X_i}) = SW(\#_{i=1}^n (X_i, c_i, o_{X_i})) = *_{i=1}^n SW(X_i, c_i, o_{X_i})$$

shows up at the top row of the following commutative diagram (which is written "unstably" to simplify the notations):

$$\begin{tikzcd}
\mathbb{CP}^{\sum a_i - 1} \arrow[rr, "\widetilde{f}"] \arrow[dd, "/\left((S^1)^n/\triangle S^1\right)"'] & & S^{\sum b_i - 1} \arrow[dd, equal] \arrow[dr, equal] & \\
& S\left(\oplus_{i=1}^n \mathbb{C}^{a_i}\right) \arrow[ul, "/\triangle S^1"'] \arrow[rr, "S(\oplus_{i=1}^n f_i)"] \arrow[dd, equal] & & S\left(\oplus_{i=1}^n \mathbb{R}^{b_i}\right) \arrow[dd, equal] \\
*_{i=1}^n \mathbb{CP}^{a_i-1} \arrow[rr, "*_{i=1}^n \widetilde{f_i}"] \arrow[dd, "\cong"'] & & *_{i=1}^n S^{b_i-1} \arrow[dd, "\cong"] \arrow[dr, equal] & \\
& *_{i=1}^n S\left(\mathbb{C}^{a_i}\right) \arrow[ul, "/(S^1)^n"'] \arrow[rr, "*_{i=1}^n S(f_i)"] & & *_{i=1}^n S\left(\mathbb{R}^{b_i}\right) \\
\Sigma^{n-1}\left(\bigwedge_{i=1}^n \mathbb{CP}^{a_i-1}\right) \arrow[rr, "\Sigma^{n-1}\left(\bigwedge_{i=1}^n \widetilde{f_i}\right)"'] & & \Sigma^{n-1}\left(\bigwedge_{i=1}^n S^{b_i-1}\right) &
\end{tikzcd}$$

From this commutative diagram, we see immediately that, to show the Bauer–Furuta stable homotopy Seiberg–Witten invariant

$$SW(\#_{i=1}^{N} X_i, c, o_{\#_{i=1}^{N} X_i}) = SW(\#_{i=1}^{N}(X_i, c_i, o_{X_i})) = *_{i=1}^{N} SW(X_i, c_i, o_{X_i})$$

is trivial, it is enough to show that the bottom row

$$\Sigma^{n-1}\big(\textstyle\bigwedge_{i=1}^{n} \widetilde{f_i}\big)\colon \Sigma^{n-1}\big(\textstyle\bigwedge_{i=1}^{n} \mathbb{CP}^{a_i-1}\big) \to \Sigma^{n-1}\big(\textstyle\bigwedge_{i=1}^{n} S^{b_i-1}\big)$$

is non-equivariantly (and, of course, stably) trivial.

Now we are ready to prove the theorem.

Proof of Theorem 1.1 Since $b_2^+(X) \geq 1$, we see that $b_2^+(X \# X) \geq 2$. Thus we may assume $b_2^+(X) \geq 2$ from the beginning. Then we may apply the preceding consideration with $X_i = X$ for $i = 1, \ldots, n$.

Now the crucial, but very elementary, observation is that there are just finitely many choices of a Spin^c–structure c of X and an orientation o_X of $H^+(X)$. So, list all such structures as (X, c_j, o_{X_j}) $(j = 1, \ldots, s)$. Now, for any m, choose n so that $n \geq (m-1)s + 1$. Then, up to permutations of factors, the Bauer–Furuta stable homotopy Seiberg–Witten invariant

$$SW(\#_{i=1}^{n} X_i, c, o_{\#_{i=1}^{n} X_i})$$

of an arbitrary Spin^c–structure c of $\#_{i=1}^{n} X_i$ and arbitrary orientation $o_{\#_{i=1}^{n} X_i}$ of $H^+(o_{\#_{i=1}^{n} X_i})$ may be expressed in the form

$$(*^m f_j) * g$$

for some equivariant map g and a representative f_j of $SW(X, c_j, o_{X_j})$ for some $j \in \{1, \ldots, s\}$.

Thus, it remains to show any S^1–equivariant map

$$f_0\colon S(\mathbb{C}^a) \to S(\mathbb{R}^b)$$

with $a \geq 1, b \geq 2$ is S^1–equivariantly join nilpotent (join stably). Actually, if this is shown, there are some n_j so that $*^{n_j} f_j$ is S^1–equivariantly (join stably) nilpotent, and so we may simply take

$$N = (\max\{n_j, j = 1, \ldots, s\} - 1)s + 1$$

as N in the statement of Theorem 1.1.

To show $f_0\colon S(\mathbb{C}^a) \to S(\mathbb{R}^b)$ with $a \geq 1, b \geq 2$ is S^1–equivariantly join nilpotent (join stably), we may assume its non-equivariant counter part $\widetilde{f_0}\colon \mathbb{CP}^{a-1} \to S^{b-1}$

induces the trivial maps of the Morava K–theories $K(n)_*$ for all $n \geq 0$. This is because joining once will reduce to the case b even.

Thus, $\widetilde{f}_0 \colon \mathbb{CP}^{a-1} \to S^{b-1}$ is (non-equivariantly) smash-nilpotent (stably) thanks to the Hopkins–Smith Theorem 4.2.

Now by the commutative diagram at the beginning of this section with $a_i = a,\ b_i = b$ for any i, the claim follows. □

Note added in proof To show the smash-nilpotency of f_0, we can also use more familiar MU_*–theory, rather than the less familiar Morava K–theories $K(n)_*$ ($n \geq 0$). Actually, for $h_* = MU_*$ or $K(n)_*$, applying the Atiyah–Hirzebruch spectral sequence, we see the groups $h_*(X)$ are concentrated in even dimensions for X a complex projective space and concentrated in odd dimensions for an odd dimensional sphere. We then use the Hopkins–Smith Theorem 4.2 in the MU_* version.

References

[1] **S Bauer**, *On connected sums of four-dimensional manifolds*, SFB343 preprint Available at `http://www.mathematik.uni-bielefeld.de/sfb343/preprints/pr00001.pdf.gz`

[2] **S Bauer**, *A stable cohomotopy refinement of Seiberg–Witten invariants II*, Invent. Math. 155 (2004) 21–40 MR2025299

[3] **S Bauer**, **M Furuta**, *A stable cohomotopy refinement of Seiberg–Witten invariants I*, Invent. Math. 155 (2004) 1–19 MR2025298

[4] **E S Devinatz**, **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory I*, Ann. of Math. (2) 128 (1988) 207–241 MR960945

[5] **M Furuta**, *Monopole equation and the $\frac{11}{8}$–conjecture*, Math. Res. Lett. 8 (2001) 279–291 MR1839478

[6] **H Hauschild**, *Äquivariante Homotopie I*, Arch. Math. (Basel) 29 (1977) 158–165 MR0467774

[7] **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory II*, Ann. of Math. (2) 148 (1998) 1–49 MR1652975

[8] **Y Matsumoto**, *On the bounding genus of homology 3–spheres*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982) 287–318 MR672065

[9] **U Namboodiri**, *Equivariant vector fields on spheres*, Trans. Amer. Math. Soc. 278 (1983) 431–460 MR701504

[10] **G Nishida**, *The nilpotency of elements of the stable homotopy groups of spheres*, J. Math. Soc. Japan 25 (1973) 707–732 MR0341485

[11] **E Witten**, *Monopoles and four-manifolds*, Math. Res. Lett. 1 (1994) 769–796 MR1306021

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