Homotopy theoretical considerations of the Bauer–Furuta stable homotopy Seiberg–Witten invariants

MIKIO FURUTA
YUKIO KAMETANI
HIROFUMI MATSUE
NORIHIKO MINAMI

We show the "non-existence" results are essential for all the previous known applications of the Bauer–Furuta stable homotopy Seiberg–Witten invariants. As an example, we present a unified proof of the adjunction inequalities.

We also show that the nilpotency phenomenon explains why the Bauer–Furuta stable homotopy Seiberg–Witten invariants are not enough to prove 11/8–conjecture.

57R57, 55P91, 55P92; 55P42, 57M50

1 Background

Nowadays, there are many applications of the Bauer–Furuta stable homotopy Seiberg–Witten invariants (see the work of Bauer and Furuta [1; 2; 5]). Amongst of all, we single out the following:

(1) $10/8$–theorem (see Furuta [6]),

(2) Adjunction inequalities (see the work of Furuta, Kametani, Matsue and Minami [9; 8; 11]),

(3) Constructions of spin 4–manifolds without Einstein metric and computations of their Yamabe invariants (see the work of Ishida and LeBrun [12; 13; 15]).

However, there is some ramification amongst the proofs of these results. Actually, (1) is proven by reducing to "non-existence" results, whereas (2) and (3) are proven by reducing to "non-triviality" results. Furthermore, the techniques employed to show these "non-existence" and "non-triviality" results have been different so far, and experts have regarded them as rather independent results.

In this paper, we unify these two approaches and deliver the following message: To apply the Bauer–Furuta stable homotopy Seiberg–Witten invariants in the direction of

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.155

(1), (2), and (3), it suffices to prove the "non-existence" results (which is the standard way to attack $11/8$–conjecture ever since Furuta's celebrated paper [6]).

Now, there are a couple of advantages of our approach. First, we can generalize the known results for (2) and (3) to slightly wider classes of 4–manifolds. Second, in the course of our proof, we can conceptually recognize how the "nilpotency phenomenon" is responsible for the fact why we can *never* prove the $11/8$–conjecture affirmatively by the Bauer–Furuta stable homotopy Seiberg–Witten invariants.

The basic ingredient of the present paper was originally announced in the JAMI conference on Geometry and Physics at the Johns Hopkins University in March, 2002. The authors would like to thank JAMI for its hospitality. The fourth author was partially supported by Grant-in-Aid for Scientific Research No.13440020, Japan Society for the Promotion of Science.

2 Level and the $11/8$–conjecture

In this section we review the concept of level for free $\mathbb{Z}/2$–spaces, and recall the results of Stolz [19], Furuta [5], Furuta–Kametani [7] and others in this terminology.

Definition 2.1 (see Dai–Lam [3] and Dai–Lam–Peng [4]) For a free $\mathbb{Z}/2$–space X, define the level of X, which we denote by $\mathrm{level}(X)$, as follows:

$$\mathrm{level}(X) := \text{the smallest } n \text{ such that } [X, S^{n-1}]^{\mathbb{Z}/2} \neq \varnothing,$$

where S^{n-1} is endowed with the antipodal $Z/2$–action.

Example 2.2 (i) The classical Borsuk–Ulam theorem states $\mathrm{level}(S^m) = m+1$.

(ii) (see Stolz [19]) For $\mathbb{R}P^{2m-1} = S(\mathbb{C}^m)/\langle i^2\rangle$ with multiplication by i,

$$\mathrm{level}(\mathbb{R}P^{2m-1}) = \begin{cases} m+1 & \text{if } m \equiv 0, 2 \mod 8 \\ m+2 & \text{if } m \equiv 1,3,4,5,7 \mod 8 \\ m+3 & \text{if } m \equiv 6 \mod 8 \end{cases}$$

The concept of level comes into the picture of 4–manifolds because of the following consequence of the Bauer–Furuta stable homotopy Seiberg–Witten invariants for closed Spin 4–manifolds (see Furuta [6], Furuta–Kametani [7], and Section 6), via the "G–join theorem" (see Minami [17] and Schmidt [18]):

Theorem 2.3 *If there is a closed Spin 4–manifold with $k = -\frac{\operatorname{sign}(X)}{16}$ and $l = b_2^+(X)$, then*

$$l \geq \operatorname{level}(\mathbb{C}P^{2k-1}).$$

Here, via the unit sphere $S(\mathbb{H}^k)$ of the direct sum of the k–copies of the quaternions $\mathbb{H}$, $\mathbb{C}P^{2k-1} = S(\mathbb{H}^k)/\{\cos\theta + i\sin\theta \mid 0 \leq \theta \leq 2\pi\}$ is endowed with $\mathbb{Z}/2$–action induced by multiplication by $j \in \mathbb{H}$.

Such a result is interesting because the celebrated Yukio Matsumoto November the 8th birthday conjecture [16] (commonly called "$11/8$–conjecture") predicts:

$$l \geq 3k$$

Thus, the determination of $\operatorname{level}(\mathbb{C}P^{2k-1})$ is clearly important. It is easy to see $\operatorname{level}(\mathbb{C}P^{2k-1}) \leq 3k$. On the other hand, since $\mathbb{R}P^{4k-1} = S(\mathbb{H}^k)/\{\pm 1\}$, there is an obvious $\mathbb{Z}/2$–map $\mathbb{R}P^{4k-1} \to \mathbb{C}P^{2k-1}$, and so

$$3k \geq \operatorname{level}(\mathbb{C}P^{2k-1}) \geq \operatorname{level}(\mathbb{R}P^{4k-1}) = \begin{cases} 2k+1 & \text{if } k \equiv 1 \mod 4 \\ 2k+2 & \text{if } k \equiv 2 \mod 4 \\ 2k+3 & \text{if } k \equiv 3 \mod 4 \\ 2k+1 & \text{if } k \equiv 4 \mod 4 \end{cases}$$

where we used Stolz' theorem.

Clearly, the interesting questions are: how large $\operatorname{level}(\mathbb{C}P^{2k-1})$ could be, and whether $\operatorname{level}(\mathbb{C}P^{2k-1})$ is ever shown to be $3k$.

However, John Jones realized $\operatorname{level}(\mathbb{C}P^{2k-1}) \neq 3k$ via explicit computations for some small k, and conjectured

$$\operatorname{level}(\mathbb{C}P^{2k-1}) = \begin{cases} 2k+2 & \text{if } k \equiv 1 \mod 4,\ k > 1 \\ 2k+2 & \text{if } k \equiv 2 \mod 4 \\ 2k+3 & \text{if } k \equiv 3 \mod 4 \\ 2k+4 & \text{if } k \equiv 4 \mod 4 \end{cases}$$

Now the current best result in this direction is the following:

Theorem 2.4 (Furuta–Kametani [7])

$$\operatorname{level}(\mathbb{C}P^{2k-1}) \geq \begin{cases} 2k+1 & \textit{if } k \equiv 1 \mod 4 \\ 2k+2 & \textit{if } k \equiv 2 \mod 4 \\ 2k+3 & \textit{if } k \equiv 3 \mod 4 \\ 2k+3 & \textit{if } k \equiv 4 \mod 4 \end{cases}$$

We note that the Bauer–Furuta stable homotopy Seiberg–Witten invariants are defined "stably," (see Section 6) and Furuta–Kametani [7] (also the original work of Furuta [6]) actually proved the "stable" version (which is actually equivalent to the above statement via the G–join theorem).

Whereas Stolz [19] and Furuta–Kametani [7] are "non-existence" results, other kinds of applications of the Bauer–Furuta stable homotopy Seiberg–Witten invariants (see Section 1 for more details) require "non-triviality" results.

To unify "non-existence" and "non-triviality" approaches, we generalize the concept of level in the next section.

3 Level, colevel, and their stable analogues

In this section, we present some very general definitions.

Definition 3.1 Fix a topological group G and a non-empty G–space A, and let X and Y be arbitrary G–spaces.

(i) Denote the iterated join $*^k A$ inductively so that $*^0 A = A$, $*^k A = A * (*^{k-1} A)$. We also understand $(*^{-1} A) * Y = Y$. Then, set $\mathrm{link}_A(X, Y)$, *link* of X to Y with respect to A, by

$$\mathrm{link}_A(X,Y) := \begin{cases} \mathrm{Min}\{n \in \mathbb{Z}_{\geq 0} \mid [X, (*^{n-1} A) * Y]^G \neq \varnothing\} & \text{if } [X,Y]^G \neq \varnothing; \\ -\mathrm{Max}\{n \in \mathbb{N} \mid [(*^{n-1} A) * X, Y]^G \neq \varnothing\} & \text{if } [X,Y]^G = \varnothing, \end{cases}$$

where we set

$$\begin{cases} \mathrm{Min}\{n \in \mathbb{Z}_{\geq 0} \mid [X, (*^{n-1} A) * Y]^G \neq \varnothing\} & = +\infty \\ & \text{if } \{n \in \mathbb{Z}_{\geq 0} \mid [X, (*^{n-1} A) * Y]^G \neq \varnothing\} = \varnothing \\ -\mathrm{Max}\{n \in \mathbb{N} \mid [(*^{n-1} A) * X, Y]^G \neq \varnothing\} & = -\infty \\ & \text{if } [(*^{n-1} A) * X, Y]^G \neq \varnothing \text{ for any } n \in \mathbb{N}. \end{cases}$$

(ii) Furthermore, set $\mathrm{slink}_A(X, Y)$, *stable link* of X to Y with respect to A, by

$$\mathrm{slink}_A(X,Y) := \lim_{q \to \infty} \mathrm{link}\big(\big(*^{q-1} A\big) * X, \big(*^{q-1} A\big) * Y\big).$$

(iii) For extreme cases, set the *level*, *stable level*, *colevel*, and *stable colevel* with respect to A by:

$$\mathrm{level}_A(X) := \mathrm{link}_A(X, A) + 1 \qquad \mathrm{slevel}_A(X) := \mathrm{slink}_A(X, A) + 1$$
$$\mathrm{colevel}_A(Y) := -\mathrm{link}_A(A, Y) + 1 \qquad \mathrm{scolevel}_A(Y) := -\mathrm{slink}_A(A, Y) + 1$$

Remark 3.2 (i) Clearly, $\mathrm{slink}_A(X,Y) \le \mathrm{link}_A(X,Y)$. Consequently, $\mathrm{slevel}_A(X) \le \mathrm{level}_A(X)$ and $\mathrm{scolevel}_A(X) \ge \mathrm{colevel}_A(X)$.

(ii) When $G = \mathbb{Z}/2$ and $A = \mathbb{Z}/2$ with the free $\mathbb{Z}/2$–action, then $\mathrm{level}_{\mathbb{Z}/2}$ and $\mathrm{colevel}_{\mathbb{Z}/2}$ are respectively the classical level and the colevel in the sense of Dai and Lam [3] (and Section 2). This is because

$$*^{n-1}\mathbb{Z}/2 = *^{n-1}S(\mathbb{R}) = S(\mathbb{R}^n) = S^{n-1},$$

where $\mathbb{R}$ with the sign representation, S^{n-1} with the antipodal action.

(iii) Suppose X is a free $\mathbb{Z}/2$–space such that

$$\dim X \le 2 \cdot \mathrm{slevel}_{\mathbb{Z}/2}(X) - 1,$$

then $\mathrm{slevel}_{\mathbb{Z}/2}(X) = \mathrm{level}_{\mathbb{Z}/2}(X) = \mathrm{level}(X)$.

Actually, this is a direct consequence of the G–join theorem (see Schmidt [18] and Minami [17], and examples satisfying this condition include $S^m, \mathbb{R}P^{2m-1}, \mathbb{C}P^{2k-1}$ (cf Stolz [19] and Furuta [6]).

4 Level and "non-triviality"

We begin with the fundamental question which relates "non-triviality" problem to the concept of level.

Question 4.1 *When* $n := \mathrm{level}_{G/H}(X)$, *does the restriction* $[X, *^{n-1}G/H]^G \to [X, *^{n-1}G/H]^H$ *ever hit a constant map?*

Remark 4.2 (i) When $G = \mathbb{Z}/2$, $H = \{e\}$ and $n > \mathrm{level}_{G/H}(X)$, the non-empty image of the composite

$$[X, *^{n-2}G/H]^G \to [X, *^{n-1}G/H]^G \to [X, *^{n-1}G/H]^H$$

consists of the constant maps. In fact, this follows immediately from the triviality of the bottom arrow in the following commutative diagram:

$$\begin{array}{ccc}
[X, *^{n-2}G/H]^G & \longrightarrow & [X, *^{n-1}G/H]^G \\
\downarrow & & \downarrow \\
[X, *^{n-2}G/H]^H & \longrightarrow & [X, *^{n-1}G/H]^H \\
\| & & \| \\
[X, S^{n-2}] & \longrightarrow & [X, S^{n-1}]
\end{array}$$

(ii) When $G=\mathbb{Z}/2,\ H=\{e\}$ and $X=S^{n-1}$ with the antipodal $\mathbb{Z}/2$–action, as was remarked in Example 2.2 (ii), the classical Borsuk–Ulam theorem states $\mathrm{level}_{G/H}(X)=n$. In this case, the other version of the classical Borsuk–Ulam theorem states that

$$\begin{array}{ccc} [X,*^{n-1}G/H]^G & \longrightarrow & [X,*^{n-1}G/H]^H \\ \| & & \| \\ [S^{n-1},S^{n-1}]^{\mathbb{Z}/2} & \longrightarrow & [S^{n-1},S^{n-1}] \end{array}$$

never hits a constant map.

(iii) Suppose the restriction

$$[X,*^{n-1}G/H]^G \to [X,*^{n-1}G/H]^H$$

never hit the constant map for $G=\mathbb{Z}/2,\ H=\{e\}$ and $X=\mathbb{C}P^{2k-1}$ with the $\mathbb{Z}/2$–action as in Theorem 2.3. Then the Bauer–Furuta stable homotopy Seiberg–Witten invariant (see Section 6), applied to closed Spin 4–manifolds M^4 with

$$b_1(M^4)=0, \qquad k=-\frac{\mathrm{sign}(M^4)}{16}, \qquad b_2^+(M^4)=\mathrm{level}(\mathbb{C}P^{2k-1}),$$

imply the following geometric consequences for M^4:

(1) (adjunction inequality, see the work of Furuta, Kametani, Matsue and Minami [14; 11; 9]) For any embedded oriented closed surface $\Sigma\subseteq M^4$,

$$|2g(\Sigma)-2|\geq[\Sigma]\cdot[\Sigma].$$

(2) (Ishida–LeBrun [15; 13; 12]) Non-existence of Einstein metrics and computations of the Yamabe invariants under some circumstances.

In this way, the concept of level, which arises naturally in "non-existence" problems, also show up "non-triviality" problems.

5 Main Theorem

We now state our main theorem, which partially answers Question 4.1.

Theorem 5.1 (i) *When* $n:=\mathrm{slink}_{G/H}(X,Y)$,

$$\lim_{q\to\infty}[X*\big(*^{q-1}\,G/H\big),Y*\big(*^{n+q-1}\,G/H\big)]^G \longrightarrow \lim_{q\to\infty}[X*\big(*^{q-1}\,G/H\big),Y*\big(*^{n+q-1}\,G/H\big)]^H$$

never hits a constant map.

(ii) *When* $n := \mathrm{link}_{G/H}(X, Y) \leq 0$,

$$[X * \left(*^{-n-1}\, G/H \right), Y]^G \to [X * \left(*^{-n-1}\, G/H \right), Y]^H$$

never hits a constant map.

Corollary 5.2 (i) *When* $l := \mathrm{slevel}_{G/H}(X)$,

$$\lim_{q\to\infty}[X * \left(*^{q-1}\, G/H \right), \left(*^{l+q-1}\, G/H \right)]^G \to \lim_{q\to\infty}[X * \left(*^{q-1}\, G/H \right), \left(*^{l+q-1}\, G/H \right)]^H$$

never hits a constant map.

(ii) *When* $c := \mathrm{colevel}_{G/H}(Y)$,

$$[\left(*^{c-1}\, G/H \right), Y]^G \to [\left(*^{c-1}\, G/H \right), Y]^H$$

never hits a constant map.

Proof of Theorem 5.1 For both (i) and (ii), it suffices to show $[X' * (G/H), Y']^G \neq \varnothing$ if the restriction map

$$[X', Y']^G \to [X', Y']^H \cong [X' \times G/H, Y']^G$$

hits a constant map. But, this follows easily from the following observations:

(1) The restriction $[X', Y']^G \to [X', Y']^H \cong [X' \times G/H, Y']^G$ is induced by the first projection $X' \times G/H \to X'$.

(2) Any constant map in $[X', Y']^H$ corresponds to a map factorizing the second projection $X' \times G/H \to G/H$ in $[X' \times G/H, Y']^G$.

(3) There is a homotopy push-out diagram

$$\begin{array}{ccc} X' \times G/H & \longrightarrow & G/H \\ \big\downarrow & & \big\downarrow \\ X' & \xrightarrow{i} & X' * (G/H). \end{array}$$

□

From Theorem 5.1 and the G–join theorem [17; 18] (see Remark 3.2 (ii)), we obtain the following important consequence.

Theorem 5.3 $G = \mathbb{Z}/2,\ H = \{e\},\ X$: *a free* $\mathbb{Z}/2$*–space such that*

$$\dim X \leq 2n - 1,$$

where $n = \mathrm{slevel}_{\mathbb{Z}/2}(X)$*. Then*

$$[X, S^{n-1}]^{\mathbb{Z}/2} \to [X, S^{n-1}]$$

never hit a constant map.

Theorem 5.3 has the following applications:

(1) It recovers the classical Borsuk–Ulam theorem (which was quoted as "the other version" in the sense of Remark 4.2 (ii)).

(2) It yields non-trivial family of elements of $\pi^*(\mathbb{R}P^{2m-1})$ together with Stolz' result.

(3) It offers some applications to 4–manifolds via the Bauer–Furuta Seiberg–Witten invariants, along the line of Remark 4.2 (iii).

To give some flavor, we single out a statement about adjunction inequalities, by applying Theorems 5.3 and 2.4:

Theorem 5.4 *Let* M^4 *be a closed Spin* 4*–manifolds* M^4 *with* $b_1(M^4) = 0$ *such that*

$$b_2^+(M^4) = \begin{cases} -\frac{\mathrm{sign}(M^4)}{8} + 1 & \text{if } -\frac{\mathrm{sign}(M^4)}{16} \equiv 1 \mod 4 \\ -\frac{\mathrm{sign}(M^4)}{8} + 2 & \text{if } -\frac{\mathrm{sign}(M^4)}{16} \equiv 2 \mod 4 \\ -\frac{\mathrm{sign}(M^4)}{8} + 3 & \text{if } -\frac{\mathrm{sign}(M^4)}{16} \equiv 3 \mod 4 \\ -\frac{\mathrm{sign}(M^4)}{8} + 3 & \text{if } -\frac{\mathrm{sign}(M^4)}{16} \equiv 4 \mod 4 \end{cases}$$

Then, for any embedded oriented closed surface $\Sigma \subseteq M^4$*,*

$$|2g(\Sigma) - 2| \geq [\Sigma] \cdot [\Sigma].$$

We remark the case $-\frac{\mathrm{sign}(M^4)}{16} = 1, 2, 3$ were already treated in more direct "non-triviality" approach (cf Kronheimer–Mrowka [14], and Furuta, Kametani, Matsue and Minami [11; 9]). On the other hand, we have shown it by reducing to a "non-existence" result: Theorem 2.4.

6 Nilpotency rules!

In this section, we explains the conceptual reason why we can never prove the $11/8$–conjecture affirmatively by use of the Bauer–Furuta stable homotopy Seiberg–Witten invariants.

For this purpose, we briefly recall the Furuta–Bauer stable homotopy Seiberg–Witten invariant [5; 2]. We begin with notations:

Let M^4 be an oriented closed 4–manifold with $b_1(M^4) = 0$, c a Spin^c–structure of M^4, and $[o_{M^4}]$ an orientation of $H^+(M^4)$, the maximal positive definite subspace of $H^2(M^4, \mathbb{R})$. Set

$$m := \frac{c_1(c)^2 - \mathrm{sign}(M^4)}{8}, \qquad n := b_2^+(M^4)$$

and *assume* $m \geq 0$, by changing the orientation of M^4 if necessary.

Then the Furuta–Bauer stable homotopy Seiberg–Witten invariant $SW(M^4, c, o_{M^4})$ for the data (M^4, c, o_{M^4}) is defined so that

$$SW(M^4, c, o_{M^4}) \in \{S(\mathbb{C}^m), S(\mathbb{R}^n)\}^{U(1)} := \lim_{p,q\to\infty} [S(\mathbb{C}^{p+m} \oplus \mathbb{R}^q), S(\mathbb{C}^p \oplus \mathbb{R}^{q+n})]^{U(1)}$$

Suppose the Spin^c–structure c comes from a Spin–structure s, and set:

$$k := -\frac{\mathrm{sign}(M^4)}{16}, \qquad l := b^+,$$

where we *assume* $k \geq 0$, by changing the orientation of M^4, if necessary. We also prepare the following representation theoretic notations:

Pin_2	The closed subgroup of the quaternions $\mathbb{H}$, generated by j and $U(1) = \{\cos\theta + i \sin\theta \mid 0 \leq \theta < 2\pi\}$.
$\mathbb{H}$	The quaternions $\mathbb{H}$, regarded as a right Pin_2–module by the right $\mathrm{Pin}_2(\subset \mathbb{H})$ multiplication.
$\widetilde{\mathbb{R}}$	$\mathbb{R}$ regarded as a right Pin_2–module via the sign representation of $\{\pm 1\} \cong \mathrm{Pin}_2 / U(1)$.

Then the Furuta–Bauer stable homotopy Seiberg–Witten invariant $SW(M^4, s, o_{M^4})$ for the data (M^4, s, o_{M^4}) is defined so that

$$SW(M^4, s, o_{M^4}) \in \{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} := \lim_{p,q\to\infty} [S(\mathbb{H}^{p+k} \oplus \widetilde{\mathbb{R}}^q), S(\mathbb{H}^p \oplus \widetilde{\mathbb{R}}^{q+l})]^{\mathrm{Pin}_2}$$

Furthermore, if we regard a Spin–structure s as as Spin^c–structure c, then the forgetful map via $U(1) \subseteq \mathrm{Pin}_2$ induces the natural correspondence:

$$\begin{gathered} \{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} \to \{S(\mathbb{C}^{2k}), S(\mathbb{R}^l)\}^{U(1)} \\ SW(M^4, s, o_{M^4}) \mapsto SW(M^4, c, o_{M^4}) \end{gathered}$$

Now the key to relate the Bauer–Furuta stable homotopy Seiberg–Witten invariants to our discussions in the previous sections is the following observation, which follows from Furuta's original work [6] (which showed $l \geq 2k+1$) and the G–join theorem [17; 18]: If $\{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} \neq \varnothing$, then

$$\begin{array}{ccc} \{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} & \longrightarrow & \{S(\mathbb{C}^{2k}), S(\mathbb{R}^l)\}^{U(1)} \\ \| & & \| \\ [X, S^{l-1}]^{\mathbb{Z}/2} & \longrightarrow & [X, S^{l-1}], \end{array} \tag{6–1}$$

where $X = \mathbb{C}P^{2k-1}$ with the $\mathbb{Z}/2$–action as in Theorem 2.3.

Now we are ready to offer a conceptual explanation why we can never prove the $11/8$–conjecture by use of the Bauer–Furuta stable homotopy Seiberg–Witten invariants.

We first note we should show

$$l < 3k \implies \{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} = \varnothing. \tag{6–2}$$

to prove the $11/8$–conjecture via the Bauer–Furuta stable homotopy Seiberg–Witten invariants. Then consider the following key commutative diagram, whose horizontal arrows are induced by the $(N-1)$–fold iterated join map:

$$\begin{array}{ccc} \{S(\mathbb{H}^k), S(\widetilde{\mathbb{R}}^l)\}^{\mathrm{Pin}_2} & \xrightarrow{*^{N-1}} & \{S(\mathbb{H}^{Nk}), S(\widetilde{\mathbb{R}}^{Nl})\}^{\mathrm{Pin}_2} \\ \downarrow & & \downarrow \\ \{S(\mathbb{C}^{2k}), S(\mathbb{R}^l)\}^{U(1)} & \xrightarrow{*^{N-1}} & \{S(\mathbb{C}^{2Nk}), S(\mathbb{R}^{Nl})\}^{U(1)} \end{array}$$

Now, because of the Bauer–Furuta Seiberg–Witten invariants of a $K3$–surface with Spin–structure, we see

$$\{S(\mathbb{H}^1), S(\widetilde{\mathbb{R}}^3)\}^{\mathrm{Pin}_2} \neq \varnothing$$

Thus we apply the above key commutative diagram with $k = 1, l = 3, N \gg 0$. Then, because of the Nilpotency Theorem [10], the bottom horizontal arrow is the trivial map for sufficiently large N (actually, straight-forward computations show this is trivial for $N \geq 5$). Therefore, the right vertical map hits the constant map.

Now (6–1) allows us to apply Theorem 5.3, which implies

$$\{S(\mathbb{H}^N), S(\widetilde{\mathbb{R}}^{3N-1})\}^{\mathrm{Pin}_2} \neq \varnothing.$$

Of course, in view of (6–2), this means a failure of proving the $11/8$–conjecture via the Bauer–Furuta stable homotopy Seiberg–Witten invariants.

References

[1] **S Bauer**, *A stable cohomotopy refinement of Seiberg–Witten invariants II* `arXiv:math.DG/0204267`

[2] **S Bauer**, **M Furuta**, *A stable cohomotopy refinement of Seiberg–Witten invariants I* `arXiv:math.DG/0204340`

[3] **Z D Dai**, **T Y Lam**, *Levels in algebra and topology*, Comment. Math. Helv. 59 (1984) 376–424 MR761805

[4] **Z D Dai**, **T Y Lam**, **C K Peng**, *Levels in algebra and topology*, Bull. Amer. Math. Soc. (N.S.) 3 (1980) 845–848 MR578376

[5] **M Furuta**, *Stable homotopy version of Seiberg–Witten invariant*, MPIM preprint MPIM1997-110 (1997) Available at `http://www.mpim-bonn.mpg.de/preprints/send?bid=658`

[6] **M Furuta**, *Monopole equation and the* $11/8$*–conjecture*, Math. Res. Lett. 8 (2001) 279–291 MR1839478

[7] **M Furuta**, **Y Kametani**, *Equivariant* e*–invariants and its applications*, preprint

[8] **M Furuta**, **Y Kametani**, **H Matsue**, *Spin 4-manifolds with signature* $=-32$, Math. Res. Lett. 8 (2001) 293–301 MR1839479

[9] **M Furuta**, **Y Kametani**, **H Matsue**, **N Minami**, *Stable-homotopy Seiberg–Witten invariants and Pin bordism*, , UTMS preprint 2000-28 (2000)

[10] **M Furuta**, **Y Kametani**, **N Minami**, *Nilpotency of the Bauer–Furuta stable homotopy Seiberg–Witten invariants*, preprint

[11] **M Furuta**, **Y Kametani**, **N Minami**, *Stable-homotopy Seiberg-Witten invariants for rational cohomology* $K3\#K3$*'s*, J. Math. Sci. Univ. Tokyo 8 (2001) 157–176 MR1818910

[12] **M Ishida**, **C LeBrun**, *Curvature, connected sums, and Seiberg–Witten theory* `arXiv:math.DG/0111228`

[13] **M Ishida**, **C LeBrun**, *Spin manifolds, Einstein metrics and differential topology* `arXiv:math.DG/0107111`

[14] **P B Kronheimer**, **T S Mrowka**, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994) 797–808 MR1306022

[15] **C LeBrun**, *Ricci curvature, minimal volumes, and Seiberg-Witten theory*, Invent. Math. 145 (2001) 279–316 MR1872548

[16] **Y Matsumoto**, *On the bounding genus of homology* 3*–spheres*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982) 287–318 MR672065

[17] **N Minami**, G*–join theorem – an unbased* G*–Freudenthal theorem, homotopy representations, and a Borsuk–Ulam theorem*, preprint

[18] **B Schmidt**, Diplomarbeit, University of Bielefeld (1997)

[19] **S Stolz**, *The level of real projective spaces*, Comment. Math. Helv. 64 (1989) 661–674 MR1023002

Department of Mathematical Sciences, University of Tokyo
Tokyo 153-8914, Japan

Department of Mathematics, Keio University
Yokohama 223-8522, Japan

Department of Economics, Seijo University
Tokyo 157-8511, Japan

Department of Mathematics, Nagoya Institute of Technology
Nagoya 466-8555, Japan

furuta@ms.u-tokyo.ac.jp, kametani@math.keio.ac.jp, matsue@seijo.co.jp, minami.norihiko@nitech.ac.jp

Received: 1 January 2005 Revised: 5 January 2006