Geometry & Topology Monographs 10 (2007) 167–186

Higher homotopy commutativity and cohomology of finite H–spaces

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We study connected mod p finite A_p–spaces admitting AC_n–space structures with $n < p$ for an odd prime p. Our result shows that if $n > (p-1)/2$, then the mod p Steenrod algebra acts on the mod p cohomology of such a space in a systematic way. Moreover, we consider A_p–spaces which are mod p homotopy equivalent to product spaces of odd dimensional spheres. Then we determine the largest integer n for which such a space admits an AC_n–space structure compatible with the A_p–space structure.

55P45, 57T25; 55P48, 55S05

1 Introduction

In this paper, we assume that p is a fixed odd prime and that all spaces are localized at p in the sense of Bousfield–Kan [2].

In the paper [10], we introduced the concept of AC_n–space which is an A_n–space whose multiplication satisfies the higher homotopy commutativity of the n-th order. Then we showed that a mod p finite AC_n–space with $n \geq p$ has the homotopy type of a torus. Here by being mod p finite, we mean that the mod p cohomology of the space is finite dimensional. To prove it, we first studied the action of the Steenrod operations on the mod p cohomology of such a space. Then we showed that the possible cohomology generators are concentrated in dimension 1.

In the above argument, the condition $n \geq p$ is essential. In fact, any odd dimensional sphere admits an AC_{p-1}–space structure by [10, Proposition 3.8]. This implies that for any given exterior algebra, we can construct a mod p finite AC_{p-1}–space such that the mod p cohomology of it is isomorphic to the algebra.

On the other hand, if the A_{p-1}–space structure of the AC_{p-1}–space is extendable to an A_p–space structure, then the situation is different. For example, it is known that an odd dimensional sphere with an A_p–space structure does not admit an AC_{p-1}–space

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.167

structure except for S^1. In fact, an odd dimensional sphere S^{2m-1} admits an A_p–space structure if and only if $m|(p-1)$, and then it admits an AC_n–space structure compatible with the A_p–space structure if and only if $nm \leq p$ by [6, Theorem 2.4]. In particular, if $p=3$, then mod 3 finite A_3–space with AC_2–space structure means mod 3 finite homotopy associative and homotopy commutative H–space. Then by Lin [21], such a space has the homotopy type of a product space of S^1s and $Sp(2)$s.

In this paper, we study mod p finite A_p–spaces with AC_n–space structures for $n < p$. First we consider the case of $n > (p-1)/2$. In this case, we show the following fact on the action of the Steenrod operations:

Theorem A *Let p be an odd prime. If X is a connected* mod *p finite A_p–space admitting an AC_n–space structure with $n > (p-1)/2$, then we have the following:*

(1) *If $a \geq 0$, $b > 0$ and $0 < c < p$, then*

$$QH^{2p^a(pb+c)-1}(X;\mathbb{Z}/p) = \mathfrak{P}^{p^a t} QH^{2p^a(p(b-t)+c+t)-1}(X;\mathbb{Z}/p)$$

for $1 \leq t \leq \min\{b, p-c\}$ and

$$\mathfrak{P}^{p^a t} QH^{2p^a(pb+c)-1}(X;\mathbb{Z}/p) = 0$$

for $c \leq t < p$.

(2) *If $a \geq 0$ and $0 < c < p$, then*

$$\mathfrak{P}^{p^a t}\colon QH^{2p^a c-1}(X;\mathbb{Z}/p) \longrightarrow QH^{2p^a(tp+c-t)-1}(X;\mathbb{Z}/p)$$

is an isomorphism for $1 \leq t < c$.

In the above theorem, the assumption $n > (p-1)/2$ is necessary. In fact, (2) is not satisfied for the Lie group S^3 although S^3 admits an $AC_{(p-1)/2}$–space structure for any odd prime p as is proved in [6, Theorem 2.4].

Theorem A (1) has been already proved for a special case or under additional hypotheses: for $p=3$ by Hemmi [7, Theorem 1.1] and for $p \geq 5$ by Lin [19, Theorem B] under the hypotheses that the space admits an AC_{p-1}–space structure and the mod p cohomology is A_p–primitively generated (see Hemmi [8] and Lin [19]).

In the above theorem, we assume that the prime p is odd. However, if we consider the case $p=2$, then the condition $p > n > (p-1)/2$ is equivalent to $n=1$, which means that the space is just an H–space. Thus Theorem A can be considered as the odd prime version of Thomas [26, Theorem 1.1] or Lin [18, Theorem 1]. (Note that in their theorems they assumed that the mod 2 cohomology of the space is primitively generated, while we do not need such an assumption.)

By using Theorem A, we show the following result:

Theorem B *Let p be an odd prime. If X is a connected* mod p *finite A_p–space admitting an AC_n–space structure with $n > (p-1)/2$ and the Steenrod operations $\mathfrak{P}^j$ act on $QH^*(X;\mathbb{Z}/p)$ trivially for $j \geq 1$, then X is* mod p *homotopy equivalent to a torus.*

Next we consider the case of $n \leq (p-1)/2$. This includes the case $n = 1$, which means that the space is just a mod p finite A_p–space. For the cohomology of mod p finite A_p–spaces, we can show similar facts to Theorem A. For example, the results by Thomas [26, Theorem 1.1] and Lin [18, Theorem 1] mentioned above is for $p = 2$, and for odd prime p, many results are known (cf. [1], [5], [20]).

However, for odd primes in particular, those results have some ambiguities. In fact, there are many A_p–spaces with AC_n–space structures for some $n \leq (p-1)/2$ such that the Steenrod operations act on the cohomology trivially. In the next theorem, we determine n for which a product space of odd dimensional spheres to be an A_p–space with an AC_n–space structure.

Theorem C *Let X be a connected A_p–space* mod p *homotopy equivalent to a product space of odd dimensional spheres $S^{2m_1-1} \times \cdots \times S^{2m_l-1}$ with $1 \leq m_1 \leq \cdots \leq m_l$, where p is an odd prime. Then X admits an AC_n–space structure if and only if $nm_l \leq p$.*

By the results of Clark–Ewing [4] and Kumpel [17], there are many spaces satisfying the assumption of Theorem C. Moreover, we note that the above result generalizes [6, Theorem 2.4].

This paper is organized as follows: In Section 2, we first recall the modified projective space $\mathfrak{M}(X)$ of a finite A_p–space constructed by Hemmi [8]. Based on the mod p cohomology of $\mathfrak{M}(X)$, we construct an algebra $A^*(X)$ over the mod p Steenrod algebra which is a truncated polynomial algebra at height $p+1$ (Theorem 2.1). Next we introduce the concept of $\mathfrak{D}_n$–algebra and show that if X is an A_p–space with an AC_n–space structure, then $A^*(X)$ is a $\mathfrak{D}_n$–algebra (Theorem 2.6). Finally we prove the theorems in Section 3 by studying the action of the Steenrod algebra on $\mathfrak{D}_n$–algebras algebraically (Proposition 3.1 and Proposition 3.2).

This paper is dedicated to Professor Goro Nishida on his 60th birthday. The authors appreciate the referee for many useful comments.

2 Modified projective spaces

Stasheff [25] introduced the concept of A_n–space which is an H–space with multiplication satisfying higher homotopy associativity of the n-th order. Let X be a space and

$n \geq 2$. An A_n–form on X is a family of maps $\{M_i\colon K_i \times X^i \to X\}_{2\leq i\leq n}$ with the conditions of [25, I, Theorem 5], where $\{K_i\}_{i\geq 2}$ are polytopes called the associahedra. A space X having an A_n–form is called an A_n–space. From the definition, an A_2–space and an A_3–space are the same as an H–space and a homotopy associative H–space, respectively. Moreover, it is known that an A_∞–space has the homotopy type of a loop space.

Let X be an A_n–space. Then by Stasheff [25, I, Theorem 5], there is a family of spaces $\{P_i(X)\}_{1\leq i\leq n}$ called the projective spaces associated to the A_n–form on X. From the construction of $P_i(X)$, we have the inclusion $\iota_{i-1}\colon P_{i-1}(X) \to P_i(X)$ for $2 \leq i \leq n$ and the projection $\rho_i\colon P_i(X) \to P_i(X)/P_{i-1}(X) \simeq (\Sigma X)^{(i)}$ for $1 \leq i \leq n$, where $Z^{(i)}$ denotes the i-fold smash product of a space Z for $i \geq 1$.

For the rest of this section, we assume that X is a connected A_p–space whose mod p cohomology $H^*(X;\mathbb{Z}/p)$ is an exterior algebra

(2–1) $$H^*(X;\mathbb{Z}/p) \cong \Lambda(x_1,\dots,x_l) \qquad \text{with } \deg x_i = 2m_i - 1$$

for $1 \leq i \leq l$, where $1 \leq m_1 \leq \dots \leq m_l$.

Iwase [12] studied the mod p cohomology of the projective space $P_n(X)$ for $1 \leq n \leq p$. If $1 \leq n \leq p-1$, then there is an ideal $S_n \subset H^*(P_n(X);\mathbb{Z}/p)$ closed under the action of the mod p Steenrod algebra $\mathfrak{A}_p^*$ such that

(2–2) $$H^*(P_n(X);\mathbb{Z}/p) \cong T_n \oplus S_n \qquad \text{with } T_n = T^{[n+1]}[y_1,\dots,y_l],$$

where $T^{[n+1]}[y_1,\dots,y_l]$ denotes the truncated polynomial algebra at height $n+1$ generated by $y_i \in H^{2m_i}(P_n(X);\mathbb{Z}/p)$ with $\iota_1^* \dots \iota_{n-1}^*(y_i) = \sigma(x_i)$ for $1 \leq i \leq l$. He also proved a similar result for the mod p cohomology of $P_p(X)$ under an additional assumption that the generators $\{x_i\}_{1\leq i\leq l}$ are A_p–primitive (see Hemmi [8] and Iwase [12]).

Hemmi [8] modified the construction of the projective space $P_p(X)$ to get the algebra $T^{[p+1]}[y_1,\dots,y_l]$ also for $n = p$ without the assumption of the A_p–primitivity of the generators. Then he proved the following result:

Theorem 2.1 (Hemmi [8, Theorem 1.1]) *Let X be a simply connected A_p-space whose mod p cohomology $H^*(X;\mathbb{Z}/p)$ is an exterior algebra in* (2–1)*, where p is an odd prime. Then we have a space $\mathfrak{M}(X)$ and a map $\epsilon\colon \Sigma X \to \mathfrak{M}(X)$ with the following properties:*

(1) *There is a subalgebra $R^*(X)$ of $H^*(\mathfrak{M}(X);\mathbb{Z}/p)$ with*

$$R^*(X) \cong T^{[p+1]}[y_1,\dots,y_l] \oplus M,$$

where $y_i \in H^{2m_i}(\mathfrak{M}(X);\mathbb{Z}/p)$ are classes with $\epsilon^*(y_i) = \sigma(x_i)$ for $1 \le i \le l$ and $M \subset H^*(\mathfrak{M}(X);\mathbb{Z}/p)$ is an ideal with $\epsilon^*(M) = 0$ and $M \cdot H^*(\mathfrak{M}(X);\mathbb{Z}/p) = 0$.

(2) *$R^*(X)$ and M are closed under the action of $\mathfrak{A}_p^*$, and so*

$$A^*(X) = R^*(X)/M \cong T^{[p+1]}[y_1, \dots, y_l] \tag{2–3}$$

is an unstable $\mathfrak{A}_p^*$-algebra.

(3) *ϵ^* induces an $\mathfrak{A}_p^*$-module isomorphism:*

$$QA^*(X) \longrightarrow QH^{*-1}(X;\mathbb{Z}/p).$$

Next we recall the higher homotopy commutativity of H–spaces.

Kapranov [16] and Reiner–Ziegler [23] constructed special polytopes $\{\Gamma_n\}_{n\ge1}$ called the permuto–associahedra. Let $n \ge 1$. A partition of the sequence $\mathbf{n} = (1, \dots, n)$ of type $(t_1, \dots, t_m)$ is an ordered sequence $(\alpha_1, \dots, \alpha_m)$ consisting of disjoint subsequences α_i of $\mathbf{n}$ of length t_i with $\alpha_1 \cup \ldots \cup \alpha_m = \mathbf{n}$ (see Hemmi–Kawamoto [10] and Ziegler [28] for the full details of the partitions). By Ziegler [28, Definition 9.13, Example 9.14], the permuto–associahedron Γ_n is an $(n-1)$-dimensional polytope whose facets (codimension one faces) are represented by the partitions of $\mathbf{n}$ into at least two parts. Let $\Gamma(\alpha_1, \dots, \alpha_m)$ denote the facet of Γ_n corresponding to a partition $(\alpha_1, \dots, \alpha_m)$. Then the boundary of Γ_n is given by

$$\partial\Gamma_n = \bigcup_{(\alpha_1,\dots,\alpha_m)} \Gamma(\alpha_1, \dots, \alpha_m) \tag{2–4}$$

for all partitions $(\alpha_1, \dots, \alpha_m)$ of $\mathbf{n}$ with $m \ge 2$. If $(\alpha_1, \dots, \alpha_m)$ is of type $(t_1, \dots, t_m)$, then the facet $\Gamma(\alpha_1, \dots, \alpha_m)$ is homeomorphic to the product $K_m \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m}$ by the face operator $\epsilon^{(\alpha_1,\dots,\alpha_m)}\colon K_m \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m} \to \Gamma(\alpha_1, \dots, \alpha_m)$ with the relations of [10, Proposition 2.1]. Moreover, there is a family of degeneracy operators $\{\delta_j\colon \Gamma_i \to \Gamma_{i-1}\}_{1\le j\le i}$ with the conditions of [10, Proposition 2.3].

By using the permuto–associahedra, Hemmi and Kawamoto [10] introduced the concept of AC_n–form on A_n–spaces.

Let X be an A_n–space whose A_n–form is given by $\{M_i\}_{2\le i\le n}$. An AC_n–form on X is a family of maps $\{Q_i\colon \Gamma_i \times X^i \to X\}_{1\le i\le n}$ with the following conditions:

$$Q_1(*, x) = x. \tag{2–5}$$

$$\begin{aligned} &Q_i(\epsilon^{(\alpha_1,\dots,\alpha_m)}(\sigma, \tau_1, \dots, \tau_m), x_1, \dots, x_i) \\ &\quad = M_m(\sigma, Q_{t_1}(\tau_1, x_{\alpha_1(1)}, \dots, x_{\alpha_1(t_1)}), \dots, Q_{t_m}(\tau_m, x_{\alpha_m(1)}, \dots, x_{\alpha_m(t_m)})) \end{aligned} \tag{2–6}$$

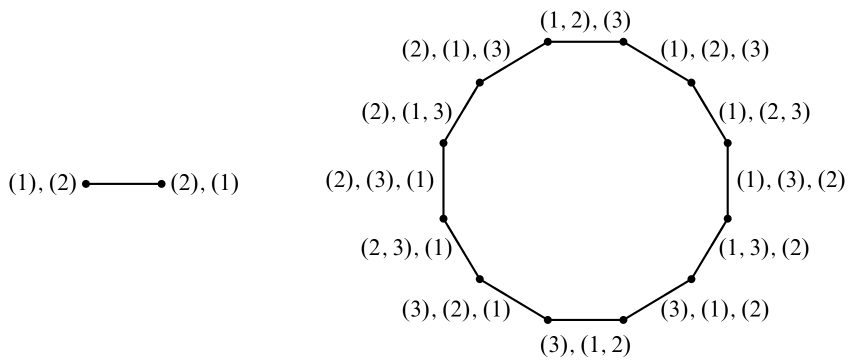

Figure 1: Permuto–associahedra Γ_2 and Γ_3

for a partition $(\alpha_1, \dots, \alpha_m)$ of $\mathbf{i}$ of type $(t_1, \dots, t_m)$.

(2–7) $Q_i(\tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = Q_{i-1}(\delta_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$

for $1 \le j \le i$.

By [10, Example 3.2 (1)], an AC_2–form on an A_2–space is the same as a homotopy commutative H–space structure since $Q_2\colon \Gamma_2 \times X^2 \to X$ gives a commuting homotopy between xy and yx for $x, y \in X$ (see Figure 2). Let us explain an AC_3–form on an A_3–space. Assume that X is an A_3–space admitting an AC_2–form. Then by using the associating homotopy $M_3\colon K_3 \times X^3 \to X$ and the commuting homotopy $Q_2\colon \Gamma_2 \times X^2 \to X$, we can define a map $\widetilde{Q}_3\colon \partial\Gamma_3 \times X^3 \to X$ which is illustrated by the right dodecagon in Figure 2. For example, the uppermost edge represents the commuting homotopy between xy and yx, and thus it is given by $Q_2(t, x, y)z$. On the other hand, the next right edge is the associating homotopy between $(xy)z$ and $x(yz)$ which is given by $M_3(t, x, y, z)$. Then X admits an AC_3–form if and only if $\widetilde{Q}_3$ is extended to a map $Q_3\colon \Gamma_3 \times X^3 \to X$. Moreover, if X is an H–space, then by [10, Example 3.2 (3)], the multiplication of the loop space ΩX on X admits an AC_∞–form.

Hemmi [6] considered another concept of higher homotopy commutativity of H–spaces. Let X be an A_n–space with the projective spaces $\{P_i(X)\}_{1 \le i \le n}$. Let $J_i(\Sigma X)$ be the i-th stage of the James reduced product space of ΣX and $\pi_i\colon J_i(\Sigma X) \to (\Sigma X)^{(i)}$ be the obvious projection for $1 \le i \le n$. A quasi C_n–form on X is a family of maps

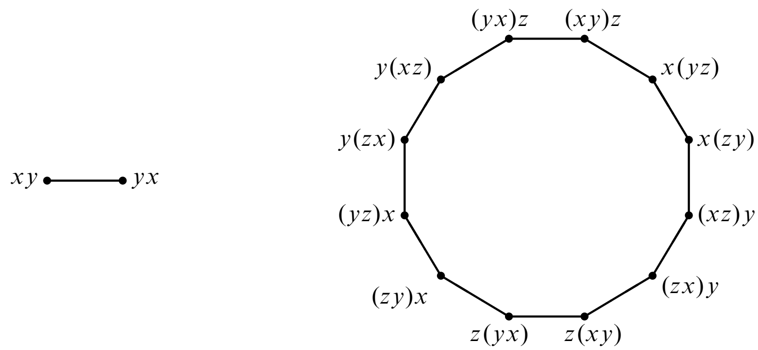

Figure 2: $Q_2(t, x, y)$ and $Q_3(\tau, x, y, z)$

$\{\psi_i\colon J_i(\Sigma X) \to P_i(X)\}_{1\le i\le n}$ with the following conditions:

(2–8) $\quad \psi_1 = 1_{\Sigma X}\colon \Sigma X \longrightarrow \Sigma X.$

(2–9) $\quad \psi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1}\psi_{i-1} \qquad$ for $2 \le i \le n$.

(2–10) $\quad \rho_i\psi_i \simeq \Big(\sum_{\sigma\in\Sigma_i} \sigma\Big)\pi_i \qquad$ for $1 \le i \le n$,

where the symmetric group Σ_i acts on $(\Sigma X)^{(i)}$ by the permutation of the coordinates and the summation on the right hand side is given by using the obvious co–H–structure on $(\Sigma X)^{(i)}$ for $1 \le i \le n$.

Hemmi and Kawamoto [10] proved the following result:

Theorem 2.2 (Hemmi–Kawamoto [10, Theorem A]) *Let X be an A_n–space for $n \ge 2$. Then we have the following:*

(1) *If X admits an AC_n–form, then X admits a quasi C_n–form.*

(2) *If X is an A_{n+1}–space admitting a quasi C_n–form, then X admits an AC_n–form.*

Remark 2.3 In the proof of Theorem 2.2 (2), we do not need the condition (2–10). In fact, the proof of Theorem 2.2 (2) shows that if X is an A_{n+1}-space and there is a family of maps $\{\psi_i\}_{1\le i\le n}$ with the conditions (2–8)–(2–9), then there is a family of maps $\{Q_i\}_{1\le i\le n}$ with the conditions (2–5)–(2–7).

Now we give the definition of $\mathfrak{D}_n$–algebra:

Definition 2.4 Assume that A^* is an unstable $\mathfrak{A}_p^*$–algebra for a prime p. Let $n \geq 1$. Then A^* is called a $\mathfrak{D}_n$–algebra if for any $\alpha_i \in A^*$ and $\theta_i \in \mathfrak{A}_p^*$ for $1 \leq i \leq q$ with

$$\sum_{i=1}^{q} \theta_i(\alpha_i) \in DA^*, \tag{2–11}$$

there are decomposable classes $\nu_i \in DA^*$ for $1 \leq i \leq q$ with

$$\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in D^{n+1}A^*, \tag{2–12}$$

where DA^* and D^tA^* denote the decomposable module and the t-fold decomposable module of A^* for $t > 1$, respectively.

Remark 2.5 It is clear from Definition 2.4 that any unstable $\mathfrak{A}_p^*$–algebra is a $\mathfrak{D}_1$–algebra. On the other hand, for an A_p–space X which satisfies the assumption of Theorem 2.1 with $l \geq 1$, the unstable $\mathfrak{A}_p^*$–algebra $A^*(X)$ given in (2–3) cannot be a $\mathfrak{D}_p$–algebra since $\mathfrak{P}^m(\alpha) = \alpha^p \neq 0$ for $\alpha \in QA^{2m}(X)$ from the unstable condition of $\mathfrak{A}_p^*$ and $D^{p+1}A^*(X) = 0$ in (2–12).

To prove Theorem A and Theorem B, we need the following theorem:

Theorem 2.6 *Let p be an odd prime and $1 \leq n \leq p-1$. Assume that X is a simply connected A_p-space whose* mod p *cohomology $H^*(X;\mathbb{Z}/p)$ is an exterior algebra in* (2–1)*. If the multiplication of X admits a quasi C_n-form, then $A^*(X)$ is a $\mathfrak{D}_n$–algebra.*

We need the following result which is a generalization of Hemmi [6]:

Lemma 2.7 *Assume that X satisfies the same assumptions as Theorem 2.6. If $\alpha_i \in H^*(P_n(X);\mathbb{Z}/p)$ and $\theta_i \in \mathfrak{A}_p^*$ for $1 \leq i \leq q$ satisfy*

$$\sum_{i=1}^{q} \theta_i(\alpha_i) = a + b \qquad \text{with } a \in DH^*(P_n(X);\mathbb{Z}/p) \text{ and } b \in S_n,$$

then there are decomposable classes $\nu_i \in DH^*(P_n(X);\mathbb{Z}/p)$ for $1 \leq i \leq q$ with

$$\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) = b.$$

Proof We give an outline of the proof since the argument is similar to Hemmi [6, Lemma 4.8]. It is clear for $n=1$. If the result is proved for $n-1$, then by the same reason as [6, Lemma 4.8], we can assume $a \in D^n H^*(P_n(X);\mathbb{Z}/p)$.

Put $\mathfrak{U}_n = \widetilde{H}^*((\Sigma X)^{[n]};\mathbb{Z}/p)$, $\mathfrak{V}_n = QH^*(X;\mathbb{Z}/p)^{\otimes n}$ and

$$\mathfrak{W}_n = \bigoplus_{i=1}^{n} \widetilde{H}^*(X;\mathbb{Z}/p)^{\otimes i-1} \otimes DH^*(X;\mathbb{Z}/p) \otimes \widetilde{H}^*(X;\mathbb{Z}/p)^{\otimes n-i},$$

where $Z^{[n]}$ denotes the n-fold fat wedge of a space Z given by

$$Z^{[n]} = \{(z_1,\dots,z_n) \in Z^n \mid z_j = * \text{ for some } 1 \le j \le n\}.$$

Then we have a splitting as an $\mathfrak{A}_p^*$–module

$$\widetilde{H}^*((\Sigma X)^n;\mathbb{Z}/p) \cong \mathfrak{U}_n \oplus \mathfrak{V}_n \oplus \mathfrak{W}_n. \tag{2–13}$$

Let $\mathfrak{R}_n\colon \widetilde{H}^*(X;\mathbb{Z}/p)^{\otimes n} \to H^*(P_n(X);\mathbb{Z}/p)$ denote the following composite:

$$\widetilde{H}^*(X;\mathbb{Z}/p)^{\otimes n} \xrightarrow{\sigma^{\otimes n}} H^*(\Sigma X;\mathbb{Z}/p)^{\otimes n} \cong H^*((\Sigma X)^{(n)};\mathbb{Z}/p) \xrightarrow{\rho_n^*} H^*(P_n(X);\mathbb{Z}/p).$$

Then by [6, Theorem 3.5], there are $\widetilde{a} \in \mathfrak{V}_n$ and $\widetilde{b} \in \mathfrak{W}_n$ with $a = \mathfrak{R}_n(\widetilde{a})$ and $b = \mathfrak{R}_n(\widetilde{b})$.

Now we set $\lambda_n^*(\alpha_i) = c_i + d_i + e_i$ with respect to the splitting (2–13) for $1 \le i \le q$, where $\lambda_n\colon (\Sigma X)^n \to P_n(X)$ denotes the composite of ψ_n with the obvious projection $\omega_n\colon (\Sigma X)^n \to J_n(\Sigma X)$. From the same reason as Hemmi [6, Lemma 4.8], we have

$$\sum_{i=1}^{q} \theta_i(d_i) = \sum_{\tau\in\Sigma_n} \tau(\widetilde{a}) = \lambda_n^*(a),$$

and so

$$\lambda_n^*\biggl(\sum_{i=1}^{q} \theta_i(\mathfrak{R}_n(d_i))\biggr) = \sum_{\tau\in\Sigma_n} \tau\biggl(\sum_{i=1}^{q} \theta_i(d_i)\biggr) = n! \sum_{\tau\in\Sigma_n} \tau(\widetilde{a}) = n!(\lambda_n^*(a)),$$

which implies

$$a = \frac{1}{n!}\sum_{i=1}^{q} \theta_i(\mathfrak{R}_n(d_i)) \tag{2–14}$$

by [6, Lemma 4.7]. If we put

$$\nu_i = \frac{1}{n!}\mathfrak{R}_n(d_i) \in D^n H^*(P_n(X);\mathbb{Z}/p)$$

for $1 \le i \le q$, then by (2–14),

$$\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) = b,$$

which completes the proof. □

Proof of Theorem 2.6 From the construction of the space $\mathfrak{M}(X)$ in Hemmi [8, Section 2], we have a space $\mathfrak{N}(X)$ and the following homotopy commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & P_{p-2}(X) & \xrightarrow{\xi} & \mathfrak{N}(X) & \xrightarrow{\eta} & \mathfrak{M}(X) \\
 & & & & \Vert & & \zeta\big\downarrow & & \big\downarrow\kappa \\
P_n(X) & \xrightarrow[\iota_n]{} & \cdots & \xrightarrow[\iota_{p-3}]{} & P_{p-2}(X) & \xrightarrow[\iota_{p-2}]{} & P_{p-1}(X) & \xrightarrow[\iota_{p-1}]{} & P_p(X).
\end{array} \tag{2–15}$$

By Theorem 2.1 (2) and [8, page 593], we have that $M \subset R^*(X)$ is closed under the action of $\mathfrak{A}_p^*$ with $\eta^*(M) = 0$, which implies that $\eta^*|_{R^*(X)}\colon R^*(X) \to H^*(\mathfrak{N}(X); \mathbb{Z}/p)$ induces an $\mathfrak{A}_p^*$–homomorphism $\mathfrak{F}\colon A^*(X) = R^*(X)/M \to H^*(\mathfrak{N}(X); \mathbb{Z}/p)$. Then by applying the mod p cohomology to the diagram (2–15), we have the following commutative diagram of unstable $\mathfrak{A}_p^*$–algebras and $\mathfrak{A}_p^*$–homomorphisms:

$$\begin{array}{ccccc}
A^*(X) & \xrightarrow{\mathfrak{F}} & H^*(\mathfrak{N}(X); \mathbb{Z}/p) & \xleftarrow{\zeta^*} & H^*(P_{p-1}(X); \mathbb{Z}/p) \\
 & & \xi^*\big\downarrow & & \big\downarrow\iota_{p-2}^* \\
 & & H^*(P_{p-2}(X); \mathbb{Z}/p) & = & H^*(P_{p-2}(X); \mathbb{Z}/p) \\
 & & & & \big\downarrow\iota_{p-3}^* \\
 & & & & \vdots \\
 & & & & \big\downarrow\iota_n^* \\
 & & & & H^*(P_n(X); \mathbb{Z}/p).
\end{array}$$

First we assume $1 \le n \le p-2$. Put $\mathfrak{G}_n(\alpha_i) = \beta_i$ for $1 \le i \le q$, where $\mathfrak{G}_n\colon A^*(X) \to H^*(P_n(X); \mathbb{Z}/p)$ is the composite given by $\mathfrak{G}_n = \iota_n^* \dots \iota_{p-3}^* \xi^* \mathfrak{F}$. Then by applying $\mathfrak{G}_n$ to (2–11), we have

$$\sum_{i=1}^{q} \theta_i(\beta_i) \in DH^*(P_n(X); \mathbb{Z}/p),$$

and so by Lemma 2.7, there are decomposable classes $\widetilde{\nu}_i \in DH^*(P_n(X);\mathbb{Z}/p)$ for $1 \le i \le q$ with

$$\sum_{i=1}^{q} \theta_i(\widetilde{\alpha}_i - \widetilde{\nu}_i) = 0. \tag{2–16}$$

If we choose decomposable classes $\nu_i \in DA^*(X)$ to satisfy $\mathfrak{G}_n(\nu_i) = \widetilde{\nu}_i$ for $1 \le i \le q$, then by (2–16),

$$\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in D^{n+1}A^*(X),$$

which completes the proof in the case of $1 \le n \le p-2$.

Next let us consider the case of $n = p-1$. Put $\mathfrak{F}(\alpha_i) = \widetilde{\alpha}_i \in H^*(\mathfrak{R}(X);\mathbb{Z}/p)$ for $1 \le i \le q$. Then we have

$$\sum_{i=1}^{q} \theta_i(\widetilde{\alpha}_i) \in DH^*(\mathfrak{R}(X);\mathbb{Z}/p).$$

By [8, Proposition 5.2], we see that $\mathfrak{F}(A^*(X))$ is contained in $\zeta^*(H^*(P_{p-1}(X);\mathbb{Z}/p))$, and so we can choose $\beta_i \in H^*(P_{p-1}(X);\mathbb{Z}/p)$ and $a \in DH^*(P_{p-1}(X);\mathbb{Z}/p)$ with $\zeta^*(\beta_i) = \widetilde{\alpha}_i$ and

$$\zeta^*(a) = \sum_{i=1}^{q} \theta_i(\widetilde{\alpha}_i)$$

for $1 \le i \le q$. Then we can set

$$\sum_{i=1}^{q} \theta_i(\beta_i) = a + b$$

with $\zeta^*(b) = 0$, and by [8, Lemma 5.1], we have $b \in S_{p-1}$. By Lemma 2.7, there are decomposable classes $\mu_i \in DH^*(P_{p-1}(X);\mathbb{Z}/p)$ for $1 \le i \le q$ with

$$\sum_{i=1}^{q} \theta_i(\beta_i - \mu_i) = b.$$

Let $\nu_i \in DA^*(X)$ with $\mathfrak{F}(\nu_i) = \zeta^*(\mu_i)$ for $1 \le i \le q$. Then we have

$$\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in D^p A^*(X),$$

which implies the required conclusion. This completes the proof of Theorem 2.6. □

3 Proofs of Theorem A and Theorem B

In this section, we assume that A^* is an unstable $\mathfrak{A}_p^*$–algebra which is the truncated polynomial algebra at height $p+1$ given by

(3–1) $$A^* = T^{[p+1]}[y_1, \dots, y_l] \qquad \text{with } \deg y_i = 2m_i$$

for $1 \le i \le l$, where $1 \le m_1 \le \cdots \le m_l$. Moreover, we choose the generators $\{y_i\}$ to satisfy

(3–2) $$\mathfrak{P}^1(y_i) \in DA^* \text{ or } \mathfrak{P}^1(y_i) = y_j \qquad \text{for some } 1 \le j \le l.$$

The above is possible by the same argument as Hemmi [5, Section 4].

First we prove the following result:

Proposition 3.1 *Suppose that A^* is a $\mathfrak{D}_n$–algebra and $1 \le i \le l$. If $\mathfrak{P}^1(y_i)$ contains the term y_j^t for some $1 \le j \le l$ and $1 \le t \le n$, then $y_j = \mathfrak{P}^1(y_k)$ for some $1 \le k \le l$.*

Proof If $t = 1$, then by (3–2), the result is clear. Let t be the smallest integer with $1 < t \le n$ such that the term y_j^t is contained in $\mathfrak{P}^1(y_{i'})$ for some $1 \le i' \le l$. Then by (3–2), we have $\mathfrak{P}^1(y_{i'}) \in DA^*$. Since A^* is a $\mathfrak{D}_n$–algebra, there is a decomposable class $\nu \in DA^*$ with $\mathfrak{P}^1(y_{i'} - \nu) \in D^{n+1}A^*$. This implies that $\mathfrak{P}^1(\nu)$ contains the term y_j^t, and so there is one of the generators $y_{i''}$ of (3–1) for $1 \le i'' \le l$ such that $\mathfrak{P}^1(y_{i''})$ contains the term y_j^s for some $1 \le s < t$. Then we have a contradiction, and so $t = 1$. This completes the proof. □

In the proof of Theorem A, we need the following result:

Proposition 3.2 *Let p be an odd prime. If A^* is a $\mathfrak{D}_n$–algebra with $n > (p-1)/2$, then the indecomposable module QA^* of A^* satisfies the following:*

(1) *If $a \ge 0$, $b > 0$ and $0 < c < p$, then*

(3–3) $$QA^{2p^a(pb+c)} = \mathfrak{P}^{p^a t} QA^{2p^a(p(b-t)+c+t)}$$

for $1 \le t \le \min\{b, p-c\}$ and

(3–4) $$\mathfrak{P}^{p^a t} QA^{2p^a(pb+c)} = 0 \qquad \text{in } QA^{2p^a(p(b+t)+c-t)}$$

for $c \le t < p$.

(2) *If $a \ge 0$ and $0 < c < p$, then*

(3–5) $$\mathfrak{P}^{p^a t}\colon QA^{2p^a c} \longrightarrow QA^{2p^a(tp+c-t)}$$

is an isomorphism for $1 \le t < c$.

Proof First we consider the case of $a=0$. Let us prove (1) by downward induction on b. If b is large enough, then the result is clear since $QA^{2(pb+c)}=0$. Assume that y_j is one of the generators of (3–1) for $1\le j\le l$ and $\deg y_j=2(pb+c)$ with $b>0$ and $0<c<p$. By inductive hypothesis, we can assume that if $f>b$ and $0<g<p$, then

$$QA^{2(pf+g)}=\mathfrak{P}^t QA^{2(p(f-1)+g+1)} \tag{3–6}$$

for $1\le t\le\min\{f,p-g\}$. If we put

$$\beta=\frac{1}{pb+c}\mathfrak{P}^{pb+c-1}(y_j)\in A^{2(p(pb+c-1)+1)},$$

then by (3–6), we have

$$\beta-\mathfrak{P}^{p-1}(\gamma)\in DA^*$$

for some $\gamma\in QA^{2p(p(b-1)+1)}$. Since A^* is a $\mathfrak{D}_n$–algebra,

$$\beta-\frac{1}{pb+c}\mathfrak{P}^{pb+c-1}(\mu)-\mathfrak{P}^{p-1}(\gamma-\nu)\in D^{n+1}A^* \tag{3–7}$$

for some decomposable classes $\mu\in DA^{2(pb+c)}$ and $\nu\in DA^{2p(p(b-1)+1)}$. If we apply $\mathfrak{P}^1$ to (3–7), then $y_j^p=\mathfrak{P}^1(\xi)$ for some $\xi\in D^{n+1}A^*$ since $\mathfrak{P}^{pb+c}(\mu)=\mu^p=0$ in A^* and $\mathfrak{P}^1\mathfrak{P}^{p-1}=p\mathfrak{P}^p=0$. Then for some generator y_i, $\mathfrak{P}^1(y_i)$ must contain some y_j^t with $1\le t\le p$ and $t+n=p$. By the assumption of $n>(p-1)/2$, we have $1\le t\le n$, which implies that $y_j=\mathfrak{P}^1(y_k)$ for some $1\le k\le l$ by Proposition 3.1. By iterating this argument, we have (3–3).

Now (3–4) follows from (3–3). In fact, if y_j is a generator in (3–1) with $\deg y_j=2(pb+c)$ for some $b>0$ and $0<c<p$, then we show that $\mathfrak{P}^c(y_j)=0$. If $b+c<p$, then by (3–3), we have $y_j=\mathfrak{P}^b(\kappa)$ for $\kappa\in QA^{2(b+c)}$, which implies that

$$\mathfrak{P}^c(y_j)=\mathfrak{P}^c\mathfrak{P}^b(\kappa)=\binom{b+c}{b}\kappa^p=0 \tag{3–8}$$

in $QA^{2p(b+c)}$. On the other hand, if $p\le b+c$, then by (3–3), we have $y_j=\mathfrak{P}^{p-c}(\zeta)$ for $\zeta\in QA^{2p(b+c-p+1)}$, and so

$$\mathfrak{P}^c(y_j)=\mathfrak{P}^c\mathfrak{P}^{p-c}(\zeta)=\binom{p}{c}\mathfrak{P}^p(\zeta)=0. \tag{3–9}$$

Next we show (2) with $a=0$. We only have to show that $\mathfrak{P}^{c-1}$ is a monomorphism on QA^{2c}. Let y_j be a generator in (3–1) such that $\deg y_j=2c$ with $0<c<p$. Suppose

contrarily that $\mathfrak{P}^{c-1}(y_j) = 0$ in $QA^{2(c-1)p+1}$. Since A^* is a $\mathfrak{D}_n$–algebra, we have that

$$\mathfrak{P}^{c-1}(y_j - \mu) \in D^{n+1}A^{2(c-1)p+1}$$

for some decomposable class $\mu \in DA^{2c}$. Then by a similar argument to the proof of (1), we have that $y_j = \mathfrak{P}^1(y_k)$ for some $1 \leq k \leq l$ with $\deg y_k = 2(c-p+1)$, which is impossible for dimensional reasons. This completes the proof of Proposition 3.2 in the case of $a = 0$.

Let I denote the ideal of A^* generated by y_i with $m_i \not\equiv 0 \bmod p$. Then for dimensional reasons and by (3–8) and (3–9), we see that I is closed under the action of $\mathfrak{A}_p^*$, which implies that A^*/I is an unstable $\mathfrak{A}_p^*$–algebra given by

$$A^*/I = T^{[p+1]}[y_{i_1}, \ldots, y_{i_q}] \qquad \text{with } m_{i_d} \equiv 0 \bmod p$$

for $1 \leq d \leq q$. Set $m_{i_d} = ph_d$ with $h_d \geq 1$ for $1 \leq d \leq q$. Let B^* denote the truncated polynomial algebra at height $p+1$ given by

$$B^* = T^{[p+1]}[z_1, \ldots, z_q] \qquad \text{with } \deg z_d = h_d$$

for $1 \leq d \leq q$. If we define a map $\widetilde{\mathfrak{L}}\colon \{y_{i_1}, \ldots, y_{i_q}\} \to B^*$ by $\widetilde{\mathfrak{L}}(y_{i_d}) = z_d$ for $1 \leq d \leq q$, then $\widetilde{\mathfrak{L}}$ is extended to an isomorphism $\mathfrak{L}\colon A^*/I \to B^*$. Moreover, B^* admits an unstable $\mathfrak{A}_p^*$–algebra structure by the action $\mathfrak{P}^r(z_d) = \mathfrak{L}(\mathfrak{P}^{pr}(y_{i_d}))$ for $r \geq 1$. Then we can show that B^* is a $\mathfrak{D}_n$–algebra concerning this structure since so is A^*. From the above arguments, we have the required results for B^* in the case of $a = 0$, which implies that A^* satisfies the required results for $a = 1$. By repeating these arguments, we can show that A^* satisfies the desired conclusions of Proposition 3.2 for any $a \geq 0$. This completes the proof. □

Now we prove Theorem A as follows:

Proof of Theorem A By Browder [3, Theorem 8.6], $H^*(X;\mathbb{Z}/p)$, the mod p cohomology, is an exterior algebra in (2–1). Let $\widetilde{X}$ be the universal cover of X. From the proof of [10, Lemma 3.9], we have that $\widetilde{X}$ is a simply connected A_p–space admitting an AC_n–form. It is enough to prove Theorem A for $\widetilde{X}$ since $X \simeq \widetilde{X} \times T$ for a torus T by Kane [14, page 24]. By Theorem 2.2 and Theorem 2.6, we have that $A^*(\widetilde{X})$ is a $\mathfrak{D}_n$–algebra. Then by Theorem 2.1 (3) and Proposition 3.2, we have the required conclusion. This completes the proof of Theorem A. □

By using Theorem A, we prove Theorem B as follows:

Proof of Theorem B We proceed by using a similar way to the proof of [6, Theorem 1.1]. Since $X \simeq \widetilde{X} \times T$ for a torus T as in the proof of Theorem A, the Steenrod operations $\mathfrak{P}^j$ act trivially on $QH^*(\widetilde{X};\mathbb{Z}/p)$ for $j \geq 1$. By Theorem A, if $QH^{2m-1}(\widetilde{X};\mathbb{Z}/p) \neq 0$, then $m = p^a$ for some $a \geq 1$, and so the mod p cohomology of $\widetilde{X}$ is an exterior algebra in (2–1), where $m_i = p^{a_i}$ with $a_i \geq 1$ for $1 \leq i \leq l$.

Let $P_{p-1}(\widetilde{X})$ be the $(p-1)$-th projective space of $\widetilde{X}$. Then by (2–2), there is an ideal $S_{p-1} \subset H^*(P_{p-1}(\widetilde{X});\mathbb{Z}/p)$ closed under the action of $\mathfrak{A}_p^*$ with

$$H^*(P_{p-1}(\widetilde{X});\mathbb{Z}/p)/S_{p-1} \cong T^{[p]}[y_1,\dots,y_l],$$

where $T^{[p]}[y_1,\dots,y_l]$ is the truncated polynomial algebra at height p generated by $y_i \in H^{2p^{a_i}}(P_{p-1}(\widetilde{X});\mathbb{Z}/p)$ with $\iota_1^* \dots \iota_{p-2}^*(y_i) = \sigma(x_i) \in H^{2p^{a_i}}(\Sigma\widetilde{X};\mathbb{Z}/p)$. Moreover, we have that the composite

$$H^t(P_p(\widetilde{X});\mathbb{Z}/p) \xrightarrow{\iota_{p-1}^*} H^t(P_{p-1}(\widetilde{X});\mathbb{Z}/p) \longrightarrow T^{[p]}[y_1,\dots,y_l] \tag{3–10}$$

is an isomorphism for $t < 2p^{a_1+1}$ and an epimorphism for $t < 2(p^{a_1+1}+p^{a_1}-1)$ by [6, page 106, (4.10)]. As in [6, page 106, (4.11)], we can show

$$\operatorname{Im}\mathfrak{P}^{p^{a_1}} \cap H^t(P_p(\widetilde{X});\mathbb{Z}/p) = 0 \tag{3–11}$$

for $t \leq 2p^{a_1+1}$. In fact, by (3–10) and for dimensional reasons, we have

$$\operatorname{Im}\beta \cap H^t(P_p(\widetilde{X});\mathbb{Z}/p) = 0$$
$$\operatorname{Im}\mathfrak{P}^1 \cap H^t(P_p(\widetilde{X});\mathbb{Z}/p) = 0$$

for $t \leq 2p^{a_1+1}$, which implies (3–11) by Liulevicius [22] or Shimada–Yamanoshita [24]. Now we can choose $w_1 \in H^{2p^{a_1}}(P_p(\widetilde{X});\mathbb{Z}/p)$ with $\iota_{p-1}^*(w_1) = y_1$ by (3–10), and we have $w_1^p = \mathfrak{P}^{p^{a_1}}(w_1) = 0$ by (3–11). Then we have a contradiction by using the same argument as the proof of [6, Theorem 1.1], and so $\widetilde{X}$ is contractible, which implies that X is a torus. This completes the proof of Theorem B. □

To show Theorem C, we need the following definition:

Definition 3.3 Assume that X is an A_n–space and Y is a space.

(1) An AC_n–form on a map $\phi\colon Y \to X$ is a family of maps $\{R_i\colon \Gamma_i \times Y^i \to X\}_{1\leq i\leq n}$ with the conditions $R_1(*,y) = \phi(y)$ for $y \in Y$ and (2–6)–(2–7).

(2) A quasi C_n–form on a map $\kappa\colon \Sigma Y \to \Sigma X$ is a family of maps $\{\zeta_i\colon J_i(\Sigma Y) \to P_i(X)\}_{1\leq i\leq n}$ with the conditions $\zeta_1 = \kappa$ and (2–9).

By using the same argument as the proof of Theorem 2.2 (1), we can prove the following result:

Theorem 3.4 *Assume that X is an A_n–space, Y is a space and $\phi\colon Y \to X$ is a map. Then any AC_n–form on ϕ induces a quasi C_n–form on $\Sigma\phi$.*

Now we prove Theorem C as follows:

Proof of Theorem C First we show that if X admits an AC_n–form, then $nm_l \le p$.

We prove by induction on n. If $n=1$, then the result is proved by Hubbuck–Mimura [11] and Iwase [13, Proposition 0.7]. Assume that the result is true for $n-1$. Then by inductive hypothesis, we have $(n-1)m_l \le p$. Now we assume that X admits an AC_n–form with

$$(n-1)m_l \le p < nm_l. \tag{3–12}$$

Then we show a contradiction.

Let $\widetilde{X}$ be the universal covering space of X. Then $\widetilde{X}$ is a simply connected A_p–space mod p homotopy equivalent to

$$S^{2m_1-1} \times \cdots \times S^{2m_l-1} \qquad \text{with } 1 < m_1 \le \cdots \le m_l \tag{3–13}$$

and the multiplication of $\widetilde{X}$ admits an AC_n–form by [10, Lemma 3.9]. Now we can set that

$$A^*(\widetilde{X}) = T^{[p+1]}[y_1, \dots, y_l] \qquad \text{with } \deg y_i = 2m_i$$

for $1 \le i \le l$, where $1 < m_1 \le \cdots \le m_l \le p$. By Theorem 2.2 and Theorem 2.6, $A^*(\widetilde{X})$ is a $\mathfrak{D}_n$–algebra.

First we consider the case of $m_l < p$. Let J be the ideal of $A^*(\widetilde{X})$ generated by y_i for $1 \le i \le l-1$. Then we see that

$$\mathfrak{P}^1(y_i) \notin J \qquad \text{for some } 1 \le i \le l. \tag{3–14}$$

In fact, if we assume that $\mathfrak{P}^1(y_i) \in J$ for any $1 \le i \le l$, then $\mathfrak{P}^1(y_l) \in J$ and $\mathfrak{P}^1(J) \subset J$. This implies that

$$y_l^p = \mathfrak{P}^{m_l}(y_l) = \frac{1}{m_l!}(\mathfrak{P}^1)^{m_l}(y_l) \in J,$$

which is a contradiction, and so we have (3–14). Then for dimensional reasons and by (3–12),

$$2(n-1)m_l < \deg \mathfrak{P}^1(y_i) < 2(n+1)m_l,$$

which implies that $\mathfrak{P}^1(y_i)$ contains the term ay_l^n with $a \neq 0$ in $\mathbb{Z}/p$ by (3–14). By Proposition 3.1, we have $y_l \in \mathfrak{P}^1 QA^{2(m_l-p+1)}(\widetilde{X})$, which causes a contradiction since $m_l < p$.

Next let us consider the case of $m_l = p$. In this case, (3–12) is equivalent to $n = 2$, and so $\widetilde{X}$ is assumed to have an AC_2–form. Then from the same arguments as above, we have that $A^*(\widetilde{X})$ is a $\mathfrak{D}_2$–algebra. By Kanemoto [15, Lemma 3], there is a generator $y_k \in QA^{2(p-1)}(\widetilde{X})$ for some $1 \leq k < l$. Let K be the ideal of $A^*(\widetilde{X})$ generated by y_i with $i \neq k$. From the same reason as (3–14), we see that $\mathfrak{P}^1(y_i) \notin K$ for some $1 \leq i \leq l$. Then for dimensional reasons, we see that $\mathfrak{P}^1(y_i)$ contains the term by_k^2 with $b \neq 0$ in $\mathbb{Z}/p$. By Proposition 3.1, we have a contradiction, and so $\widetilde{X}$ does not admit an AC_2–form.

Next we show that if $nm_l \leq p$, then X admits an AC_n–form. Since it is clear for $n = 1$ or $m_l = 1$, we can assume that $nm_l < p$. Let Y denote the wedge sum of spheres given by

$$Y = (S^{2m_1-1} \vee \ldots \vee S^{2m_l-1})_{(p)}$$

with the inclusion $\phi\colon Y \to X$. First we construct an AC_n–form $\{R_i\colon \Gamma_i \times Y^i \to X\}_{1\leq i\leq n}$ on $\phi\colon Y \to X$.

Suppose inductively that $\{R_i\}_{1\leq i<t}$ are constructed for some $t \leq n$. Then the obstructions for the existence of R_t belong to the following cohomology groups for $j \geq 1$:

$$H^{j+1}(\Gamma_t \times Y^t, \partial\Gamma_t \times Y^t \cup \Gamma_t \times Y^{[t]}; \pi_j(X)) \cong \widetilde{H}^{j+2}((\Sigma Y)^{(t)}; \pi_j(X)) \tag{3–15}$$

since $\Gamma_t \times Y^t/(\partial\Gamma_t \times Y^t \cup \Gamma_t \times Y^{[t]}) \simeq \Sigma^{t-1}Y^{(t)}$. This implies that (3–15) is non-trivial only if j is an even integer with $j < 2p-2$ since

$$\Sigma Y \simeq (S^{2m_1} \vee \ldots \vee S^{2m_l})_{(p)}$$

and $tm_l \leq nm_l < p$. On the other hand, according to Toda [27, Theorem 13.4], $\pi_j(X) = 0$ for any even integer j with $j < 2p-2$ since X is given by (3–13). Thus (3–15) is trivial for all j, and we have a map R_t. This completes the induction, and we have an AC_n–form $\{R_i\}_{1\leq i\leq n}$ on $\phi\colon Y \to X$.

Since X is an H–space, there is a map $\beta\colon \Omega\Sigma X \to X$ with $\beta\alpha \simeq 1_X$, where $\alpha\colon X \to \Omega\Sigma X$ denotes the adjoint of $1_{\Sigma X}\colon \Sigma X \to \Sigma X$. Moreover, ΣY is a retract of ΣX, and so we have a map $\nu\colon \Sigma X \to \Sigma Y$ with $\nu(\Sigma\phi) \simeq 1_{\Sigma Y}$. Put $\lambda = \beta\theta\colon X \to X$, where $\theta\colon X \to \Omega\Sigma X$ denotes the adjoint of $(\Sigma\phi)\nu$. Then we see that λ induces an isomorphism on the mod p cohomology, and so λ is a mod p homotopy equivalence.

By Theorem 3.4, there is a quasi C_n–form $\{\zeta_i\colon J_i(\Sigma Y) \to P_i(X)\}_{1\leq i\leq n}$ on $\Sigma\phi\colon \Sigma Y \to \Sigma X$. Let $\xi_i\colon J_i(\Sigma X) \to P_i(X)$ be the map defined by $\xi_i = \zeta_i J_i(\nu(\Sigma\lambda^{-1}))$ for

$1 \le i \le n$, where $\lambda^{-1}\colon X \to X$ denotes the homotopy inverse of λ. Then the family $\{\xi_i\}_{1\le i\le n}$ satisfies that $\xi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1}\xi_{i-1}$ for $2 \le i \le n$ and $\xi_1 = (\Sigma\phi)\nu(\Sigma\lambda^{-1}) = \chi(\Sigma\theta)(\Sigma\lambda^{-1})$, where $\chi\colon \Sigma\Omega\Sigma X \to \Sigma X$ is the evaluation map. Since $\iota_1(\Sigma\beta) \simeq \iota_1\chi\colon \Sigma\Omega\Sigma X \to P_2(X)$ by Hemmi [9, Lemma 2.1], we have $\xi_2|_{\Sigma X} = \iota_1\xi_1 \simeq \iota_1$. Let $\psi_i\colon J_i(\Sigma X) \to P_i(X)$ be the map defined by $\psi_1 = 1_{\Sigma X}$ and $\psi_i = \xi_i$ for $2 \le i \le n$. Then the family $\{\psi_i\}_{1\le i\le n}$ satisfies (2–8)–(2–9). By Theorem 2.2 (2) and Remark 2.3, we have an AC_n–form $\{Q_i\colon \Gamma_i \times X^i \to X\}_{1\le i\le n}$ on X with (2–5)–(2–7). This completes the proof of Theorem C. □

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Received: 14 August 2004 Revised: 19 September 2005