Geometry & Topology Monographs 10 (2007) 187–193 **187**

Hypersurface representation and the image of the double S^3–transfer

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We study the image of a transfer homomorphism in the stable homotopy groups of spheres. Actually, we show that an element of order 8 in the 18 dimensional stable stem is in the image of a double transfer homomorphism, which reproves a result due to P J Eccles that the element is represented by a framed hypersurface. Also, we determine the image of the transfer homomorphism in the 16 dimensional stable stem.

55R12; 55Q45

1 Introduction and result

Let $\nu^* \in \pi_{18}^s(S^0)$ be an element of order 8 in the stable homotopy groups of spheres. Throughout the paper, $\pi_i(X)$ (resp. $\pi_i^s(X)$) denotes the homotopy group (resp. the stable homotopy group) of a space X, and we use the notations of Toda [10] for elements of $\pi_*^s(S^0)$. Then, using a generators $\nu \in \pi_3^s(S^0) \cong \mathbb{Z}/24$ and $\sigma \in \pi_7^s(S^0) \cong \mathbb{Z}/240$, ν^* is represented by the Toda bracket $\langle \sigma, 2\sigma, \nu \rangle = -\langle \sigma, \nu, \sigma \rangle$ with no indeterminacy.

As is known, ν^* is not in the image of the homomorphism $J_0\colon \pi_{18}(SO) \to \pi_{18}^s(S^0)$ induced by the stable J–map $J\colon SO \to \Omega^\infty\Sigma^\infty S^0$. We shall show that ν^* is in the image of the homomorphism $J_1\colon \pi_{19}(\Sigma SO) \to \pi_{18}^s(S^0)$ induced by the adjoint map $\Sigma SO \to \Omega^\infty\Sigma^\infty S^1$ to J. Actually, we prove that ν^* is in a double S^3–transfer image which is a subgroup of $\mathrm{Im}(J_1)$.

Eccles [3] has made clear that $\mathrm{Im}(J_1)$ consists of elements represented by framed hypersurfaces. Such study has also done by Rees [9]. Our result gives the following.

Theorem (Eccles [3, page 168]) *ν^* is representable by a framed hypersurface of dimension 18.*

Here, a framed hypersurface of dimension 18 is a closed manifold of dimension 18 embedded in S^{19} and framed in S^N for sufficiently large N, and the theorem means

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.187

that ν^* is the framed cobordism class of such a framed hypersurface when we regard $\pi_{18}^s(S^0)$ as the framed cobordism group Ω_{18}^{fr}.

The Bott map $B\colon \Sigma^3 BSp \to SO$ is the adjoint map to the homotopy equivalence $BSp \times \mathbb{Z} \to \Omega^3 SO$, where $SO = \bigcup_n SO(n)$ is the rotation group and BSp is the classifying space of the symplectic group $Sp = \bigcup_n Sp(n)$. Let $\mathbb{H}P^\infty = \bigcup_n \mathbb{H}P^n$ be the infinite dimensional quaternionic projective space. Then, $\mathbb{H}P^\infty = BSp(1)$, and there is an inclusion map $i\colon \mathbb{H}P^\infty \to BSp$. We define a map t_H as the composition

$$t_H\colon \Sigma^3 \mathbb{H}P^\infty \xrightarrow{i} \Sigma^3 BSp \xrightarrow{B} SO \xrightarrow{J} \Omega^\infty \Sigma^\infty S^0.$$

We denote by $t_H^s\colon \mathbb{H}P^\infty \to S^{-3}$ a stable map adjoint to t_H. It is not certain whether t_H^s is stably homotopic to the stable map constructed by the Boardman's transfer construction [1] on the principal S^3–bundle over $\mathbb{H}P^\infty$. However, since t_H has the following property in common with the Boardman's S^3–transfer map, we call t_H and t_H^s the S^3–transfer maps.

Lemma 1 *The restriction $t'_H\colon S^7 \to \Omega^\infty \Sigma^\infty S^0$ of t_H to the bottom sphere represents $\sigma \in \pi_7^s(S^0)$ up to sign.*

Let $\mu\colon \Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0 \to \Omega^\infty \Sigma^\infty S^0$ be the multiplication on the infinite loop space $\Omega^\infty \Sigma^\infty S^0$ given by composition, and $\mu'\colon \Sigma \Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0 \to \Omega^\infty \Sigma^\infty S^1$ the adjoint map to μ. Then, we set

$$t_H(2) = \mu' \circ \Sigma(t_H \wedge t_H)\colon \Sigma^7 \mathbb{H}P^\infty \wedge \mathbb{H}P^\infty \to \Omega^\infty \Sigma^\infty S^1,$$

and call $t_H(2)$ a double S^3–transfer map. Then, our main result may be stated as follows:

Theorem 2 *There exists an element $a \in \pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$ satisfying*

$$t_H(2)_*(\Sigma^7 a) = \nu^* \quad in \ \ \pi_{18}^s(S^0).$$

Applying a result due to Eccles and Walker [4], we have $\mathrm{Im}(t_H(2)_*) \subset \mathrm{Im}(J_1)$ (see Proposition 6 and Corollary7). Hence, it turns out that ν^* is represented by a framed hypersurface of dimension 18 as is stated in the first theorem.

In Carlisle et al [2], effective use of the S^1–transfer homomorphism $\tau\colon \pi_{i-1}^s(\mathbb{C}P^\infty) \to \pi_i^s(S^0)$ shows that certain elements are represented by framed hypersurfaces. In this respect, ν^* is not in the image of the double S^1–transfer homomorphism $\tau_2\colon \pi_{16}^s(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \to \pi_{18}^s(S^0)$. In fact, the image of τ_2 in $\pi_{18}^s(S^0)$ is equal to $\mathbb{Z}/4\{2\nu^*\}$ by Imaoka [5, Theorem 10].

By Toda [10], $\pi^s_{16}(S^0) = \mathbb{Z}/2\{\eta^*\} \oplus \mathbb{Z}/2\{\eta \circ \rho\}$. Here, $\eta \in \pi^s_1(S^0) \cong \mathbb{Z}/2$ is a generator, η^* is an element in the Toda bracket $\langle \sigma, 2\sigma, \eta \rangle$ and ρ is a generator of the image of $J_0\colon \pi_{15}(SO) \to \pi^s_{15}(S^0)$. Mahowald [6] has constructed an important family $\eta_i \in \pi^s_{2^i}(S^0)$ for $i \geq 3$, and $\eta^* \equiv \eta_4 \pmod{\eta \circ \rho}$.

By Lemma 1, we see the following:

Lemma 3 $\mathrm{Im}(t_H(2)_*) = 0$ *in* $\pi^s_{16}(S^0)$.

Mukai [8, Theorem 2] has shown that both ν^* and η^* are in the image of the S^1–transfer homomorphism $\tau\colon \pi^s_{i-1}(\mathbb{C}P^\infty) \to \pi^s_i(S^0)$. Morisugi [7, Corollary E] has shown that all Mahowald's elements are in the image of the S^3–transfer homomorphism $\pi^s_*(Q_\infty) \to \pi^s_*(S^0)$ given for the quaternionic quasi-projective space $Q_\infty = \bigcup_n Q_n$.

In contrast with these, we have the following:

Theorem 4 $\mathrm{Im}((t^s_H)_*\colon \pi^s_{13}(\mathbb{H}P^\infty) \to \pi^s_{16}(S^0)) = \mathbb{Z}/2\{\eta \circ \rho\}$, *and thus* $\eta^* \notin \mathrm{Im}(t^s_H)_*$.

We also remark the following noted by Eccles [3], where $J_2\colon \pi_{18}(\Sigma^2 SO) \to \pi^s_{16}(S^0)$ is the homomorphism induced by the map $\Sigma^2 SO \to \Omega^\infty \Sigma^\infty S^2$ adjoint to the stable J–map.

Proposition 5 $\eta^* \in \mathrm{Im}(J_2)$.

In Section 2 we prove Lemma 1 and Theorem 2, and in Section 3 we prove Lemma 3, Theorem 4 and Proposition 5.

2 Proof of Theorem 2

We first prove Lemma 1.

Proof of Lemma 1 Let ξ be the canonical left $\mathbb{H}$–line bundle over $\mathbb{H}P^\infty$, and put $\widetilde{\xi} = \xi - 1_{\mathbb{H}}$. Also, let $\widetilde{\xi_1}^*$ be the adjoint to the restriction of $\widetilde{\xi}$ to $\mathbb{H}P^1 = S^4$. Then, the restriction of the classifying map φ of $\widetilde{\xi}^*_1 \otimes_{\mathbb{H}} \widetilde{\xi}$ to the bottom sphere $S^8 = S^4 \wedge S^4$ represents a generator of $\pi_8(BSO) \cong \mathbb{Z}$. As is known, the adjoint map $\Sigma^4 \mathbb{H}P^\infty \to BSO$ to the composition $B \circ i\colon \Sigma^3 \mathbb{H}P^\infty \to SO$ is homotopic to φ. Since $\sigma = J_0(\iota_7)$ for a generator $\iota_7 \in \pi_7(SO)$, t'_H represents σ up to sign. □

Let $H(m)\colon \Sigma SO \wedge SO \to \Sigma SO$ be the map defined by the Hopf construction on the multiplication $m\colon SO \times SO \to SO$, and $J^*\colon \Sigma SO \to \Omega^\infty\Sigma^\infty S^1$ the adjoint map to the stable J–map $J\colon SO \to \Omega^\infty\Sigma^\infty S^0$. Then, Eccles and Walker have shown the following [4, Proposition 2.2], in which $\mu'\colon \Sigma(\Omega^\infty\Sigma^\infty S^0 \wedge \Omega^\infty\Sigma^\infty S^0) \to \Omega^\infty\Sigma^\infty S^1$ is the adjoint map to the multiplication $\mu\colon \Omega^\infty\Sigma^\infty S^0 \wedge \Omega^\infty\Sigma^\infty S^0 \to \Omega^\infty\Sigma^\infty S^0$.

Proposition 6 *The composition $J^* \circ H(m)\colon \Sigma SO \wedge SO \to \Omega^\infty\Sigma^\infty S^1$ is homotopic to $\mu' \circ \Sigma(J \wedge J)$.*

Recall that the homomorphism $J_1\colon \pi_i(\Sigma SO) \to \pi^s_{i-1}(S^0)$ is induced by J^*. Since $(J \wedge J) \circ (B \circ i \wedge B \circ i)$ is equal to $t_H \wedge t_H$, by composing it with the maps of Proposition 6, $J^* \circ H(m) \circ \Sigma(B \circ i \wedge B \circ i)\colon \Sigma(\Sigma^3\mathbb{H}P^\infty \wedge \Sigma^3\mathbb{H}P^\infty) \to \Omega^\infty\Sigma^\infty S^1$ is homotopic to $t_H(2) = \mu' \circ \Sigma(t_H \wedge t_H)$. Hence, we have the following.

Corollary 7 $\mathrm{Im}(t_H(2)_*) \subset \mathrm{Im}(J_1)$ *in* $\pi^s_i(S^0)$.

Let $\beta_i \in H_{4i}(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Then, we have $H_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty; \mathbb{Z}) = \mathbb{Z}\{\beta_1 \otimes \beta_2\} \oplus \mathbb{Z}\{\beta_2 \otimes \beta_1\}$.

Lemma 8 *There exists an element $a \in \pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$ satisfying $h(a) = \beta_1 \otimes \beta_2 - \beta_2 \otimes \beta_1$ for the Hurewicz homomorphism h.*

Proof Notice that $\pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$. Since $\mathbb{H}P^2 = S^4 \cup_\nu e^8$, the 12–skeleton of $\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty$ has a cell structure $S^8 \cup_{\nu\vee\nu} (e^{12} \cup e^{12})$. Thus, we have the following exact sequence:

$$\pi^s_{12}(S^8) = 0 \to \pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \xrightarrow{\psi} \pi^s_{12}(S^{12} \vee S^{12}) \xrightarrow{\varphi} \pi^s_{11}(S^8).$$

Here, $\varphi\colon \pi^s_{12}(S^{12} \vee S^{12}) = \mathbb{Z}\{\iota_1\} \oplus \mathbb{Z}\{\iota_2\} \to \pi^s_{11}(S^8) = \mathbb{Z}/24\{\nu\}$ satisfies $\varphi(\iota_1) = \varphi(\iota_2) = \nu$. Therefore, $\pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) = \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}$ for some elements a and b satisfying $\psi_*(a) = \iota_1 - \iota_2$ and $\psi_*(b) = 24\iota_2$. Then, a is a required element up to sign, since we have $\{\psi_*(\beta_1 \otimes \beta_2), \psi_*(\beta_2 \otimes \beta_1)\} = \{h(\iota_1), h(\iota_2)\}$ up to sign. □

Proof of Theorem 2 Let $a \in \pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$ be the element in Lemma 8. Since $\pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi^s_{12}(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$ and $t_H(2)_* = (t^s_H \wedge t^s_H)_*\Sigma^\infty$ for the suspension isomorphism $\Sigma^\infty\colon \pi_{19}(\Sigma^7\mathbb{H}P^2 \wedge \mathbb{H}P^2) \to \pi^s_{12}(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$, we identify a with $\Sigma^\infty a \in \pi^s_{12}(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$. Then, it is sufficient to show $(t^s_H \wedge t^s_H)_*(a) = \nu^*$ up to sign.

The following diagram is stably homotopy commutative up to sign:

$$\begin{tikzcd}
S^{12} \arrow[r, "a"] \arrow[d, "\simeq"] & \mathbb{H}P^2 \wedge \mathbb{H}P^2 \arrow[r, "t_H^s \wedge 1"] \arrow[d, "1\wedge q"] & S^{-3} \wedge \mathbb{H}P^2 \arrow[r, "1\wedge t_H^s"] \arrow[d, "q"] & S^{-6} \\
S^4 \wedge S^8 \arrow[r, "i\wedge 1"] & \mathbb{H}P^2 \wedge S^8 \arrow[r, "t_H^s \wedge 1"] & S^{-3} \wedge S^8,
\end{tikzcd}$$

where q is the collapsing map to the top cell. The left hand square is stably homotopy commutative because $(1\wedge q)_*(a)$ is a generator of $\pi_4^s(\mathbb{H}P^2)$ by Lemma 8. Hence, $q \circ (t_H^s \wedge 1)\circ a \simeq (t_H^s \circ i)\wedge 1 \simeq \sigma$ by Lemma 1, and thus $(t_H^s \wedge 1)\circ a$ is a coextension of σ. On the other hand, since $t_H^s\colon \mathbb{H}P^2 \to S^{-3}$ is an extension of $\sigma\colon S^4 \to S^{-3}$ by Lemma 1,

$$(t_H^s \wedge t_H^s)_*(a) = (1\wedge t_H^s)_*(t_H^s \wedge 1)_*(a) \in \langle \sigma, \nu, \sigma\rangle = \{-\nu^*\}$$

by definition of the Toda bracket. Thus, we have completed the proof. □

3 Transfer image in $\pi_{16}^s(S^0)$

We remark that

$$\pi_{17}(\Sigma^7\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi_{10}^s(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi_{10}^s(S^8) = \mathbb{Z}/2\{\eta^2\}.$$

Then, using Lemma 1,

$$t_H(2)_*(\eta^2) = ((t_H^s)_*(\eta))^2 = (\sigma\circ\eta)^2 = 0,$$

and thus we have Lemma 3.

Lastly, we prove Theorem 4 and Proposition 5.

Proof of Theorem 4 As mentioned in Section 1, $\pi_{16}^s(S^0) = \mathbb{Z}/2\{\eta^*\} \oplus \mathbb{Z}/2\{\eta\circ\rho\}$ by Toda [10], and we shall show

$$\mathrm{Im}((t_H^s)_*\colon \pi_{13}^s(\mathbb{H}P^\infty) \to \pi_{16}^s(S^0)) = \mathbb{Z}/2\{\eta\circ\rho\}.$$

Notice that $\pi_{13}^s(\mathbb{H}P^\infty) \cong \pi_{13}^s(\mathbb{H}P^3)$ and the attaching map $\varphi\colon S^{11} \to \mathbb{H}P^2$ of the top cell in $\mathbb{H}P^3$ is stably a coextension of $2\nu \in \pi_3^s(S^0)$. We consider the following exact sequence:

$$\pi_{14}^s(S^{12}) \xrightarrow{\varphi_*} \pi_{13}^s(\mathbb{H}P^2) \xrightarrow{i_*} \pi_{13}^s(\mathbb{H}P^3) \xrightarrow{q_*} \pi_{13}^s(S^{12}) \xrightarrow{\varphi_*} \pi_{12}^s(\mathbb{H}P^2),$$

where i is the inclusion $\mathbb{H}P^2 \subset \mathbb{H}P^3$ and q is the collapsimg map $\mathbb{H}P^3 \to S^{12}$ to the top cell.

Since $\mathbb{H}P^2 \approx S^4 \cup_\nu e^8$, we have

$$\pi_{12}^s(\mathbb{H}P^2) = \mathbb{Z}/2\{(i_0)_*(\overline{\nu})\} \oplus \mathbb{Z}/2\{(i_0)_*(\epsilon)\}$$
$$\text{and} \quad \pi_{13}^s(\mathbb{H}P^2) = \mathbb{Z}/2\{(i_0)_*(\mu)\} \oplus \mathbb{Z}/2\{(i_0)_*(\eta\circ\epsilon)\}$$

for the bottom inclusion $i_0\colon S^4 \to \mathbb{H}P^2$, where

$$\pi_8^s(S^0) = \mathbb{Z}/2\{\overline{\nu}\} \oplus \mathbb{Z}/2\{\epsilon\} \quad \text{and} \quad \pi_9^s(S^0) = \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta\circ\epsilon\} \oplus \mathbb{Z}/2\{\nu^3\}.$$

Since φ is a coextension of 2ν, we have $\varphi_*(\eta) = \varphi\circ\eta \in (i_0)_*(\langle \nu, 2\nu, \eta\rangle) \equiv (i_0)_*(\epsilon)$ (mod $i_*(\overline{\nu})$). Thus, $\varphi_*\colon \pi_{13}^s(S^{12}) \to \pi_{12}^s(\mathbb{H}P^2)$ is a monomorphism. Similarly, $\varphi_*(\eta^2) = \varphi\circ\eta^2$ is contained in $(i_0)_*(\langle \nu, 2\nu, \eta^2\rangle) \equiv (i_0)_*(\eta\circ\epsilon)$ (mod $(i_0)_*(\nu^3)$). Hence, we have $\pi_{13}^s(\mathbb{H}P^\infty) \cong \pi_{13}^s(\mathbb{H}P^3) = \mathbb{Z}/2\{(i_0)_*(\mu)\}$.

Since $t_H^s\colon \Sigma^3\mathbb{H}P^\infty \to S^0$ is an extension of $\sigma\colon S^7 \to S^0$ by Lemma 1, we conclude that $\mathrm{Im}((t_H^s)_*\colon \pi_{13}^s(\mathbb{H}P^\infty) \to \pi_{16}^s(S^0)) = \mathbb{Z}/2\{\sigma\circ\mu\} = \mathbb{Z}/2\{\eta\circ\rho\}$. □

Proof of Proposition 5 As mentioned in Section 1,

$$\eta^* \in \langle \sigma, 2\sigma, \eta\rangle \subset \pi_{16}^s(S^0),$$

and the indeterminacy of $\langle \sigma, 2\sigma, \eta\rangle$ is $\mathbb{Z}/2\{\eta\circ\rho\}$.

Recall that $\sigma = J_0(\iota_7) \in \pi_7^s(S^0)$ up to sign for a generator $\iota_7 \in \pi_7(SO) \cong \mathbb{Z}$. By Toda [10, Proposition 5.15, (5.16), Proposition 5.1], there exist elements $\sigma' \in \pi_{14}(S^7)$ and $\eta_{16} \in \pi_{17}(S^{16})$ which suspend to 2σ and η, respectively. Then, $\iota_7\circ\sigma' = 0$ since $\pi_{14}(SO) = 0$. Also, by [10, Theorem 7.1], $(\Sigma^2\sigma')\circ\eta_{16} = 0$. Hence, an unstable Toda bracket $\{\Sigma^2\iota_7, \Sigma^2\sigma', \eta_{16}\}$ is defined in $\pi_{18}(\Sigma^2 SO)$, and it satisfies

$$J_2(\{\Sigma^2\iota_7, \Sigma^2\sigma', \eta_{16}\}) \cap \langle \sigma, 2\sigma, \eta\rangle \neq \phi.$$

Hence, η^* or $\eta^* + \eta\circ\rho$ is in $\mathrm{Im}(J_2)$. But, since $\eta\circ\rho \in \mathrm{Im}(J_0)$ and $\mathrm{Im}(J_0) \subset \mathrm{Im}(J_2)$, we have the required result. □

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Received: 24 August 2004 Revised: 5 July 2005