Hypersurface representation and the image of the double $S^3$–transfer

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We study the image of a transfer homomorphism in the stable homotopy groups of spheres. Actually, we show that an element of order 8 in the 18 dimensional stable stem is in the image of a double transfer homomorphism, which reproves a result due to P J Eccles that the element is represented by a framed hypersurface. Also, we determine the image of the transfer homomorphism in the 16 dimensional stable stem.

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1 Introduction and result

Let $v^* \in \pi_{18}^s(S^0)$ be an element of order 8 in the stable homotopy groups of spheres. Throughout the paper, $\pi_i(X)$ (resp. $\pi_i^s(X)$) denotes the homotopy group (resp. the stable homotopy group) of a space $X$, and we use the notations of Toda [10] for elements of $\pi_i^s(S^0)$. Then, using a generators $v \in \pi_3^s(S^0) \cong \mathbb{Z}/24$ and $\sigma \in \pi_7^s(S^0) \cong \mathbb{Z}/240$, $v^*$ is represented by the Toda bracket $\langle \sigma, 2\sigma, v \rangle = -(\sigma, v, \sigma)$ with no indeterminacy.

As is known, $v^*$ is not in the image of the homomorphism $J_0: \pi_{18}(SO) \to \pi_{18}^s(S^0)$ induced by the stable $J$–map $J: SO \to \Omega^\infty \Sigma^\infty S^0$. We shall show that $v^*$ is in the image of the homomorphism $J_1: \pi_{19}(\Sigma SO) \to \pi_{18}^s(S^0)$ induced by the adjoint map $\Sigma SO \to \Omega^\infty \Sigma^\infty S^1$ to $J$. Actually, we prove that $v^*$ is in a double $S^3$–transfer image which is a subgroup of $\text{Im}(J_1)$.

Eccles [3] has made clear that $\text{Im}(J_1)$ consists of elements represented by framed hypersurfaces. Such study has also done by Rees [9]. Our result gives the following.

**Theorem** (Eccles [3, page 168]) $v^*$ is representable by a framed hypersurface of dimension 18.

Here, a framed hypersurface of dimension 18 is a closed manifold of dimension 18 embedded in $S^{19}$ and framed in $S^N$ for sufficiently large $N$, and the theorem means...
that \(v^*\) is the framed cobordism class of such a framed hypersurface when we regard \(\pi^3_18(S^0)\) as the framed cobordism group \(\Omega^{fr}_{18}\).

The Bott map \(B: \Sigma^3 BS^p \to SO\) is the adjoint map to the homotopy equivalence \(BS^p \times \mathbb{Z} \to \Omega^3 SO\), where \(SO = \bigcup_n SO(n)\) is the rotation group and \(BS^p\) is the classifying space of the symplectic group \(Sp = \bigcup_n Sp(n)\). Let \(\mathbb{H} \mathbb{P}^\infty = \bigcup_n \mathbb{H} P^n\) be the infinite dimensional quaternionic projective space. Then, \(\mathbb{H} \mathbb{P}^\infty = BS^p(1)\), and there is an inclusion map \(i: \mathbb{H} \mathbb{P}^\infty \to BS^p\). We define a map \(t_H\) as the composition

\[
t_H: \Sigma^3 \mathbb{H} P^\infty \xrightarrow{i} \Sigma^3 BS^p \xrightarrow{B} SO \xrightarrow{J} \Omega^\infty \Sigma^\infty S^0.
\]

We denote by \(t_H^s: \mathbb{H} \mathbb{P}^\infty \to S^{-3}\) a stable map adjoint to \(t_H\). It is not certain whether \(t_H^s\) is stably homotopic to the stable map constructed by the Boardman’s transfer construction [1] on the principal \(S^3\)-bundle over \(\mathbb{H} \mathbb{P}^\infty\). However, since \(t_H\) has the following property in common with the Boardman’s \(S^3\)-transfer map, we call \(t_H\) and \(t_H^s\) the \(S^3\)-transfer maps.

**Lemma 1**  The restriction \(t_H^s: S^7 \to \Omega^\infty \Sigma^\infty S^0\) of \(t_H\) to the bottom sphere represents \(\sigma \in \pi^3_7(S^0)\) up to sign.

Let \(\mu: \Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0 \to \Omega^\infty \Sigma^\infty S^0\) be the multiplication on the infinite loop space \(\Omega^\infty \Sigma^\infty S^0\) given by composition, and \(\mu': \Sigma \Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0 \to \Omega^\infty \Sigma^\infty S^1\) the adjoint map to \(\mu\). Then, we set

\[
t_H(2) = \mu' \circ \Sigma(t_H \wedge t_H): \Sigma^7 \mathbb{H} P^\infty \wedge \mathbb{H} P^\infty \to \Omega^\infty \Sigma^\infty S^1,
\]

and call \(t_H(2)\) a double \(S^3\)-transfer map. Then, our main result may be stated as follows:

**Theorem 2**  There exists an element \(a \in \pi_{12}(\mathbb{H} P^\infty \wedge \mathbb{H} P^\infty)\) satisfying

\[
t_H(2)_*(\Sigma^7 a) = v^* \quad \text{in} \quad \pi^3_18(S^0).
\]

Applying a result due to Eccles and Walker [4], we have \(\text{Im}(t_H(2)_*) \subset \text{Im}(J_1)\) (see Proposition 6 and Corollary7). Hence, it turns out that \(v^*\) is represented by a framed hypersurface of dimension 18 as is stated in the first theorem.

In Carlisle et al [2], effective use of the \(S^1\)-transfer homomorphism \(\tau: \pi^3_{i-1}(\mathbb{C} P^\infty) \to \pi^3_i(S^0)\) shows that certain elements are represented by framed hypersurfaces. In this respect, \(v^*\) is not in the image of the double \(S^1\)-transfer homomorphism \(\tau_2: \pi^3_{16}(\mathbb{C} P^\infty \wedge \mathbb{C} P^\infty) \to \pi^3_{18}(S^0)\). In fact, the image of \(\tau_2\) in \(\pi^3_{18}(S^0)\) is equal to \(\mathbb{Z}/4\{2v^*\}\) by Imaoka [5, Theorem 10].
By Toda [10], \( \pi_3^6(S^0) = \mathbb{Z}/2\eta^* \oplus \mathbb{Z}/2(\eta \circ \rho) \). Here, \( \eta^* \) is an element in the Toda bracket \( \langle \sigma, 2\sigma, \eta \rangle \) and \( \rho \) is a generator of the image of \( J_0: \pi_{15}(SO) \to \pi_{15}^4(S^0) \). Mahowald [6] has constructed an important family \( \eta_i \in \pi_{2i}^4(S^0) \) for \( i \geq 3 \), and \( \eta^* \equiv \eta_4 \) (mod \( \eta \circ \rho \)).

By Lemma 1, we see the following:

**Lemma 3** \( \text{Im}(t_H^2) = 0 \) in \( \pi_1^6(S^0) \).

Mukai [8, Theorem 2] has shown that both \( v^* \) and \( \eta^* \) are in the image of the \( S^1 \)-transfer homomorphism \( \tau: \pi_{2i-1}^s(CP^\infty) \to \pi_i^s(S^0) \). Morisugi [7, Corollary E] has shown that all Mahowald’s elements are in the image of the \( S^3 \)-transfer homomorphism \( \pi_2^s(Q_\infty) \to \pi_3^s(S^0) \) given for the quaternionic quasi-projective space \( Q_\infty = \bigcup_n Q_n \).

In contrast with these, we have the following:

**Theorem 4** \( \text{Im}(t_H^2) = 0 \) in \( \pi_1^6(S^0) \).

We also remark the following noted by Eccles [3], where \( J_2: \pi_{18}(\Sigma^2 SO) \to \pi_4^6(S^0) \) is the homomorphism induced by the map \( \Sigma^2 SO \to \Omega^\infty \Sigma^\infty S^2 \) adjoint to the stable \( J \)-map.

**Proposition 5** \( \eta^* \in \text{Im}(J_2) \).

In Section 2 we prove Lemma 1 and Theorem 2, and in Section 3 we prove Lemma 3, Theorem 4 and Proposition 5.

### 2 Proof of Theorem 2

We first prove Lemma 1.

**Proof of Lemma 1** Let \( \xi \) be the canonical left \( \mathbb{H} \)–line bundle over \( \mathbb{H}P^\infty \), and put \( \tilde{\xi} = \xi - 1_{\mathbb{H}} \). Also, let \( \xi^* \) be the adjoint to the restriction of \( \tilde{\xi} \) to \( \mathbb{H}P^1 = S^4 \). Then, the restriction of the classifying map \( \varphi \) of \( \xi^* \otimes_{\mathbb{H}} \tilde{\xi} \) to the bottom sphere \( S^8 = S^4 \wedge S^4 \) represents a generator of \( \pi_8(BSO) \approx \mathbb{Z} \). As is known, the adjoint map \( \Sigma^4 \mathbb{H}P^\infty \to BSO \) to the composition \( B \circ i: \Sigma^3 \mathbb{H}P^\infty \to SO \) is homotopic to \( \varphi \). Since \( \sigma = J_0(\tau) \) for a generator \( \tau \in \pi_7(SO) \), \( t_H^2 \) represents \( \sigma \) up to sign.  

\[ \square \]
Let $H(m): \Sigma SO \wedge SO \to \Sigma SO$ be the map defined by the Hopf construction on the multiplication $m: SO \times SO \to SO$, and $J^*: \Sigma SO \to \Omega^\infty \Sigma^\infty S^1$ the adjoint map to the stable $J$–map $J: SO \to \Omega^\infty \Sigma^\infty S^0$. Then, Eccles and Walker have shown the following [4, Proposition 2.2], in which $\mu^i: \Sigma(\Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0) \to \Omega^\infty \Sigma^\infty S^1$ is the adjoint map to the multiplication $\mu: \Omega^\infty \Sigma^\infty S^0 \wedge \Omega^\infty \Sigma^\infty S^0 \to \Omega^\infty \Sigma^\infty S^0$.

**Proposition 6** The composition $J^* \circ H(m): \Sigma SO \wedge SO \to \Omega^\infty \Sigma^\infty S^1$ is homotopic to $\mu^i \circ J$. 

Recall that the homomorphism $J_1: \pi_i(\Sigma SO) \to \pi^s_{i-1}(S^0)$ is induced by $J^*$. Since $(J \wedge J) \circ (B \circ i \wedge B \circ i)$ is equal to $t_H \wedge t_H$, by composing it with the maps of Proposition 6, $J^* \circ H(m) \circ \Sigma(B \circ i \wedge B \circ i): \pi_1(\Sigma^3 \mathbb{H}P^\infty \wedge \Sigma^3 \mathbb{H}P^\infty) \to \Omega^\infty \Sigma^\infty S^1$ is homotopic to $t_H(2) = \mu^i \circ J(t_H \wedge t_H)$. Hence, we have the following.

**Corollary 7** $\text{Im}(t_H(2)_*) \subset \text{Im}(J_1)$ in $\pi^s_2(S^0)$.

Let $\beta_i \in H_4(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Then, we have $H_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty; \mathbb{Z}) = \mathbb{Z}\{\beta_1 \wedge \beta_2\} \oplus \mathbb{Z}\{\beta_2 \wedge \beta_1\}$.

**Lemma 8** There exists an element $a \in \pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$ satisfying $h(a) = \beta_1 \wedge \beta_2 - \beta_2 \wedge \beta_1$ for the Hurewicz homomorphism $h$. 

**Proof** Notice that $\pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$. Since $\mathbb{H}P^2 = S^4 \cup ve^8$, the 12–skeleton of $\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty$ has a cell structure $S^8 \cup ve^8$ ($e^{12} \cup e^{12}$). Thus, we have the following exact sequence:

$$\pi^s_{12}(S^8) = 0 \to \pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \xrightarrow{\psi} \pi^s_{12}(S^1 \wedge S^1) \xrightarrow{\psi} \pi^s_{14}(S^8).$$

Here, $\psi: \pi^s_{12}(S^1 \wedge S^1) = \mathbb{Z}\{t_1\} \oplus \mathbb{Z}\{t_2\} \to \pi^s_{14}(S^8) = \mathbb{Z}/24\{v\}$ satisfies $\psi(t_1) = \psi(t_2) = v$. Therefore, $\pi^s_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) = \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}$ for some elements $a$ and $b$ satisfying $\psi(a) = t_1 - t_2$ and $\psi(b) = 24t_2$. Then, $a$ is a required element up to sign, since we have $\{\psi_*(\beta_1 \wedge \beta_2), \psi_*(\beta_2 \wedge \beta_1)\} = \{h(t_1), h(t_2)\}$ up to sign. 

**Proof of Theorem 2** Let $a \in \pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty)$ be the element in Lemma 8. Since $\pi_{12}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi^s_{12}(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$ and $t_H(2)_* = (t_H^t \wedge t_H^t)_*$ for the suspension isomorphism $\Sigma^\infty: \pi_1(\Sigma \mathbb{H}P^\infty \wedge \Sigma \mathbb{H}P^\infty) \to \pi_1(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$, we identify $a$ with $\Sigma^\infty a \in \pi^s_{12}(\mathbb{H}P^2 \wedge \mathbb{H}P^2)$. Then, it is sufficient to show $(t_H^t \wedge t_H^t)_*(a) = \nu^*$ up to sign.
The following diagram is stably homotopy commutative up to sign:

\[
\begin{array}{cccccc}
S^{12} & \overset{a}{\longrightarrow} & \mathbb{H}P^2 \wedge \mathbb{H}P^2 & \overset{t_H^d \wedge 1}{\longrightarrow} & S^{-3} \wedge \mathbb{H}P^2 & \overset{1 \wedge t_H^d}{\longrightarrow} & S^{-6} \\
\downarrow \simeq & & \downarrow 1 \wedge q & & \downarrow q \\
S^4 \wedge S^8 & \overset{i \wedge 1}{\longrightarrow} & \mathbb{H}P^2 \wedge S^8 & \overset{t_H^d \wedge 1}{\longrightarrow} & S^{-3} \wedge S^8,
\end{array}
\]

where \( q \) is the collapsing map to the top cell. The left hand square is stably homotopy commutative because \((1 \wedge q)_*(a)\) is a generator of \( \pi_4^s(\mathbb{H}P^2) \) by Lemma 8. Hence, \( q \circ (t_H^d \wedge 1) \circ a \simeq (t_H^d \circ i) \wedge 1 \simeq \sigma \) by Lemma 1, and thus \((t_H^d \wedge 1) \circ a\) is a coextension of \( \sigma \). On the other hand, since \( t_H^d : \mathbb{H}P^2 \to S^{-3} \) is an extension of \( \sigma : S^4 \to S^{-3} \) by Lemma 1,

\[
(t_H^d \wedge t_H^d)_*(a) = (1 \wedge t_H^d)_*(i \wedge 1)_*(a) \in \langle \sigma, v, \sigma \rangle = \{-v^*\}
\]

by definition of the Toda bracket. Thus, we have completed the proof.

3 Transfer image in \( \pi_{16}^s(S^0) \)

We remark that

\[
\pi_{17}(\Sigma^7 \mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi_{10}(\mathbb{H}P^\infty \wedge \mathbb{H}P^\infty) \cong \pi_{10}^s(S^8) = \mathbb{Z}/2\{\eta^2\}.
\]

Then, using Lemma 1,

\[
t_H(2)_*(\eta^2) = ((t_H^d)_*(\eta))^2 = (\sigma \circ \eta)^2 = 0,
\]

and thus we have Lemma 3.

Lastly, we prove Theorem 4 and Proposition 5.

**Proof of Theorem 4** As mentioned in Section 1, \( \pi_{16}^s(S^0) = \mathbb{Z}/2\{\eta^*\} \oplus \mathbb{Z}/2\{\eta \circ \rho\} \) by Toda [10], and we shall show

\[
\text{Im}(t_H^d)_* : \pi_{13}^s(\mathbb{H}P^\infty) \to \pi_{16}^s(S^0)) = \mathbb{Z}/2\{\eta \circ \rho\}.
\]

Notice that \( \pi_{13}^s(\mathbb{H}P^\infty) \cong \pi_{13}^s(\mathbb{H}P^3) \) and the attaching map \( \varphi : S^{11} \to \mathbb{H}P^2 \) of the top cell in \( \mathbb{H}P^3 \) is stably a coextension of \( 2v \in \pi_3^s(S^0) \). We consider the following exact sequence:

\[
\pi_{14}^s(S^{12}) \overset{\varphi_\ast}{\longrightarrow} \pi_{13}^s(\mathbb{H}P^2) \overset{i_\ast}{\longrightarrow} \pi_{13}^s(\mathbb{H}P^3) \overset{q_\ast}{\longrightarrow} \pi_{13}^s(S^{12}) \overset{\varphi^\ast}{\longrightarrow} \pi_{12}^s(\mathbb{H}P^2),
\]

where \( i \) is the inclusion \( \mathbb{H}P^2 \subset \mathbb{H}P^3 \) and \( q \) is the collapsing map \( \mathbb{H}P^3 \to S^{12} \) to the top cell.
Since $\mathbb{H}P^2 \approx S^4 \cup_{\psi} e^8$, we have
\[
\pi^S_{12}(\mathbb{H}P^2) = \mathbb{Z}/2\langle (i_0)_*(\overline{v}) \rangle \oplus \mathbb{Z}/2\langle (i_0)_*(\epsilon) \rangle
\]
and
\[
\pi^S_{13}(\mathbb{H}P^2) = \mathbb{Z}/2\langle (i_0)_*(\mu) \rangle \oplus \mathbb{Z}/2\langle (i_0)_*(\eta \circ \epsilon) \rangle
\]
for the bottom inclusion $i_0: S^4 \to \mathbb{H}P^2$, where
\[
\pi^S_8(S^0) = \mathbb{Z}/2\{\overline{v}\} \oplus \mathbb{Z}/2\{\epsilon\} \quad \text{and} \quad \pi^S_8(S^0) = \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{v^3\}.
\]
Since $\psi$ is a coextension of $2\nu$, we have $\varphi_*(\eta) = \psi \circ \eta \in (i_0)_*(\langle v, 2v, \eta \rangle) \equiv (i_0)_*(\epsilon) \mod i_0(\overline{v})$. Thus, $\varphi_*: \pi^S_{13}(S^{12}) \to \pi^S_{12}(\mathbb{H}P^2)$ is a monomorphism. Similarly, $\varphi_*(\eta^2) = \psi \circ \eta^2$ is contained in $(i_0)_*(\langle v, 2v, \eta^2 \rangle) \equiv (i_0)_*(\eta \circ \epsilon) \mod (i_0)_*(v^3))$. Hence, we have $\pi^S_{13}(\mathbb{H}P^\infty) \cong \pi^S_{13}(\mathbb{H}P^3) = \mathbb{Z}/2\langle (i_0)_*(\mu) \rangle$.

Since $\iota^*_{H^*}: \Sigma^3 \mathbb{H}P^\infty \to S^0$ is an extension of $\sigma: S^7 \to S^0$ by Lemma 1, we conclude that $\text{Im}(\iota^*_{H^*}) = \pi^S_{13}(\mathbb{H}P^\infty) \to \pi^S_{16}(S^0)) = \mathbb{Z}/2\{\sigma \circ \mu\} = \mathbb{Z}/2\{\eta \circ \rho\}$. □

Proof of Proposition 5 As mentioned in Section 1,
\[\eta^* \in \langle \sigma, 2\sigma, \eta \rangle \subset \pi^S_{16}(S^0),\]
and the indeterminacy of $\langle \sigma, 2\sigma, \eta \rangle$ is $\mathbb{Z}/2\{\eta \circ \rho\}$.

Recall that $\sigma = J_0(\iota_7) \in \pi^S_2(S^0)$ up to sign for a generator $\iota_7 \in \pi_7(SO) \cong \mathbb{Z}$. By Toda [10, Proposition 5.15, (5.16), Proposition 5.1], there exist elements $\alpha', \beta' \in \pi_{14}(S^3)$ and $\eta_16 \in \pi_{17}(S^{16})$ which suspend to $2\sigma$ and $\eta$, respectively. Then, $\iota_7 \circ \sigma' = 0$ since $\pi_{14}(SO) = 0$. Also, by [10, Theorem 7.1], $(\Sigma^2 \alpha') \circ \eta_16 = 0$. Hence, an unstable Toda bracket $\{\Sigma^2 \iota_7, \Sigma^2 \alpha', \eta_16\}$ is defined in $\pi_{18}(\Sigma^2 SO)$, and it satisfies
\[J_{2}(\{\Sigma^2 \iota_7, \Sigma^2 \alpha', \eta_16\}) \cap \langle \sigma, 2\sigma, \eta \rangle \neq \phi.\]
Hence, $\eta^* + \eta \circ \rho$ is in $\text{Im}(J_2)$. But, since $\eta \circ \rho \in \text{Im}(J_0)$ and $\text{Im}(J_0) \subset \text{Im}(J_2)$, we have the required result. □

References


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