Classifying spaces of compact Lie groups that are p–compact for all prime numbers

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We consider a problem on the conditions of a compact Lie group G that the loop space of the p–completed classifying space be a p–compact group for a set of primes. In particular, we discuss the classifying spaces BG that are p–compact for all primes when the groups are certain subgroups of simple Lie groups. A survey of the p–compactness of BG for a single prime is included.

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A p–compact group (see Dwyer–Wilkerson [8]) is a loop space X such that X is $\mathbb{F}_p$–finite and that its classifying space BX is $\mathbb{F}_p$–complete (see Andersen–Grodal–Møller–Viruel [2] and Dwyer–Wilkerson [11]). We recall that the p–completion of a compact Lie group G is a p–compact group if $\pi_0(G)$ is a p–group. Next, if $C(\rho)$ denotes the centralizer of a group homomorphism ρ from a p–toral group to a compact Lie group, according to [8, Theorem 6.1], the loop space of the p–completion $\Omega(BC(\rho))^\wedge_p$ is a p–compact group.

In a previous article [19], the classifying space BG is said to be *p–compact* if $\Omega(BG)^\wedge_p$ is a p–compact group. There are some results for a special case. A survey is given in Section 1. It is well-known that, if Σ_3 denotes the symmetric group of order 6, then $B\Sigma_3$ is not 3–compact. In fact, for a finite group G, the classifying space BG is p–compact if and only if G is p–nilpotent. Moreover, we will see that BG is p–compact toral (see Ishiguro [20]) if and only if the compact Lie group G is p–nilpotent (see Henn [14]). For the general case, we have no group theoretical characterization, though a few necessary conditions are available. This problem is also discussed in the theory of p–local groups (see Broto, Levi and Oliver [6; 7]) from a different point of view.

We consider the p–compactness of BG for a set of primes. Let Π denote the set of all primes. For a non–empty subset $\mathbb{P}$ of Π, we say that BG is *$\mathbb{P}$–compact* if this space is p–compact for any $p \in \mathbb{P}$. If G is connected, then $\Omega(BG)^\wedge_p \simeq G^\wedge_p$ for any prime p, and hence BG is Π–compact. The connectivity condition, however, is not necessary. For instance, the classifying space of each orthogonal group $O(n)$ is also Π–compact. Since $\pi_0(O(n)) = \mathbb{Z}/2$ is a 2–group, $BO(n)$ is 2–compact, and for any

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odd prime p, the p–equivalences $BO(2m) \simeq_p BO(2m+1) \simeq_p BSO(2m+1)$ tell us that $BO(n)$ is Π–compact.

Next let $\mathbb{P}(BG)$ denote the set of primes p such that BG is p–compact. In [20] the author has determined $\mathbb{P}(BG)$ when G is the normalizer NT of a maximal torus T of a connected compact simple Lie group K with Weyl group $W(K)$. Namely

$$\mathbb{P}(BNT) = \begin{cases} \Pi & \text{if } W(K) \text{ is a 2–group,} \\ \{p \in \Pi \mid |W(K)| \not\equiv 0 \mod p\} & \text{otherwise.} \end{cases}$$

Other examples are given by a subgroup $H \cong SU(3) \rtimes \mathbb{Z}/2$ of the exceptional Lie group G_2 and its quotient group $\Gamma_2 = H/(\mathbb{Z}/3)$.

$$\begin{array}{ccccc} \mathbb{Z}/3 & = & \mathbb{Z}/3 & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ SU(3) & \longrightarrow & H & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow & & \| \\ PU(3) & \longrightarrow & \Gamma_2 & \longrightarrow & \mathbb{Z}/2 \end{array}$$

A result of [19] implies that $\mathbb{P}(BH) = \Pi$ and $\mathbb{P}(B\Gamma_2) = \Pi - \{3\}$.

In this paper we explore some necessary and sufficient conditions for a compact Lie group to be Π–compact. First we consider a special case. We say that BG is $\mathbb{P}$–*compact toral* if for each $p \in \mathbb{P}$ the loop space $\Omega(BG)^\wedge_p$ is expressed as an extension of a p–compact torus $T^\wedge_p$ by a finite p–group π so that that there is a fibration $(BT)^\wedge_p \longrightarrow (BG)^\wedge_p \longrightarrow B\pi$. Obviously, if BG is $\mathbb{P}$–compact toral, the space is $\mathbb{P}$–compact. A necessary and sufficient condition that BG be p–compact toral is given in [20]. As an application, we obtain the following:

Theorem 1 *Suppose G is a compact Lie group, and G_0 denotes its connected component with the identity. Then BG is Π–compact toral if and only if the following two conditions hold:*

(a) *G_0 is a torus T, and the group $G/G_0 = \pi_0 G$ is nilpotent.*

(b) *T is a central subgroup of G.*

For a torus T and a finite nilpotent group γ, the product group $G = T \times \gamma$ satisfies conditions (a) and (b). Thus BG is Π–compact toral. Proposition 2.2 will show, however, that a group G with BG being Π–compact toral need not be a product group.

Next we ask if BH is $\mathbb{P}$–compact when H is a subgroup of a simple Lie group G. For $\mathbb{P} = \Pi$, the following result determines certain types of (G, H_0) where H_0 is the connected component of the identity. We have seen the cases of $(G, H) = (G, NT)$ when $W(G) = NT/T$ is a 2–group, and of $(G, H) = (G_2, SU(3) \rtimes \mathbb{Z}/2)$ which is considered as a case with $(G, H_0) = (G_2, A_2)$. Recall that the Lie algebra of $SU(n+1)$ is simple of type A_n, and the Lie group $SU(3)$ is of A_2–type (see Bourbaki [4]).

Theorem 2 *Suppose a connected compact Lie group G is simple. Suppose also that H is a proper closed subgroup of G with $\mathrm{rank}(H_0) = \mathrm{rank}(G)$, and that the map $BH \longrightarrow BG$ induced by the inclusion is p–equivalent for some p. Then the following hold:*

(a) *If the space BH is Π–compact, (G, H_0) is one of the following types:*

$$(G, H_0) = \begin{cases} (G, T_G) & \text{for } G = A_1 \text{ or } B_2(= C_2) \\ (B_n, D_n) & \\ (C_2, A_1 \times A_1) & \\ (G_2, A_2) & \end{cases}$$

where T_G is the maximal torus of G.

(b) *For any odd prime p, all above types are realizable. Namely, there are G and H of types as above such that BH is Π–compact, together with the p–equivalent map $BH \longrightarrow BG$. When $p = 2$, any such pair (G, H) is not realizable.*

We make a remark about covering groups. Note that if $\alpha \longrightarrow \widetilde{G} \longrightarrow G$ is a finite covering, then α is a central subgroup of $\widetilde{G}$. For a central extension $\alpha \longrightarrow \widetilde{G} \longrightarrow G$ and a subgroup H of G, we consider the following commutative diagram:

$$\begin{array}{ccccc} \alpha & \longrightarrow & \widetilde{G} & \longrightarrow & G \\ \| & & \uparrow & & \uparrow \\ \alpha & \longrightarrow & \widetilde{H} & \longrightarrow & H \end{array}$$

Obviously the vertical map $H \longrightarrow G$ is the inclusion, and $\widetilde{H}$ is the induced subgroup of $\widetilde{G}$. We will show that the pair (G, H) satisfies the conditions of Theorem 2 if and only if its cover $(\widetilde{G}, \widetilde{H})$ satisfies those of Theorem 2. Examples of the type $(G, H_0) = (B_n, D_n)$, for instance, can be given by $(SO(2n+1), O(2n))$ and the double cover $(Spin(2n+1), Pin(2n))$.

For the case $(G, H_0) = (G_2, A_2)$, we have seen that H has a finite normal subgroup $\mathbb{Z}/3$, and that for its quotient group Γ_2 the classifying space $B\Gamma_2$ is p–compact if and only if $p \neq 3$. So $\mathbb{P}(B\Gamma_2) \neq \Pi$. The following result shows that this is the only case. Namely, if Γ is such a quotient group for $(G, H_0) \neq (G_2, A_2)$, then $\mathbb{P}(B\Gamma) = \Pi$.

Theorem 3 *Let (G, H) be a pair of compact Lie groups as in Theorem 2. For a finite normal subgroup ν of H, let Γ denote the quotient group H/ν. If $(G, H_0) \neq (G_2, A_2)$, then $B\Gamma$ is Π–compact.*

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1 A survey of the p–compactness of BG

We summarize work of earlier articles [19; 20] together with some basic results, in order to introduce the problem of p–compactness. For a compact Lie group G, the classifying space BG is p–compact if and only if $\Omega(BG)^\wedge_p$ is $\mathbb{F}_p$–finite. So it is a mod p finite H–space. The space $B\Sigma_3$ is not p–compact for $p = 3$. We notice that $\Omega(B\Sigma_3)^\wedge_3$ is not a mod 3 finite H–space, since the degree of the first non–zero homotopy group of $\Omega(B\Sigma_3)^\wedge_3$ is not odd. Actually there is a fibration $\Omega(B\Sigma_3)^\wedge_3 \longrightarrow (S^3)^\wedge_3 \longrightarrow (S^3)^\wedge_3$ (see Bousfield and Kan [5]).

First we consider whether BG is p–compact toral, as a special case. When G is finite, this is the same as asking if BG is p–compact. Note that, for a finite group π, the classifying space $B\pi$ is an Eilenberg–MacLane space $K(\pi, 1)$. Since $(BT)^\wedge_p$ is also Eilenberg–MacLane, for BG being p–compact toral, the n–th homotopy groups of $(BG)^\wedge_p$ are zero for $n \geq 3$. A converse to this fact is the following.

Theorem 1.1 [20, Theorem 1] *Suppose G is a compact Lie group, and X is a p–compact group. Then we have the following:*

(i) *If there is a positive integer k such that $\pi_n((BG)^\wedge_p) = 0$ for any $n \geq k$, then BG is p–compact toral.*

(ii) *If there is a positive integer k such that $\pi_n(BX) = 0$ for any $n \geq k$, then X is a p–compact toral group.*

This theorem is also a consequence of work of Grodal [12; 13]

A finite group γ is p–nilpotent if and only if γ is expressed as the semidirect product $\nu \rtimes \gamma_p$, where ν is the subgroup generated by all elements of order prime to p, and where γ_p is the p–Sylow subgroup. The group Σ_3 is p–nilpotent if and only if $p \neq 3$. Recall that a fibration of connected spaces $F \longrightarrow E \longrightarrow B$ is said to be preserved by the p–completion if $F^\wedge_p \longrightarrow E^\wedge_p \longrightarrow B^\wedge_p$ is again a fibration. When $\pi_0(G)$ is a p–group, a result of Bousfield and Kan [5] implies that the fibration $BG_0 \longrightarrow BG \longrightarrow B\pi_0 G$ is preserved by the p–completion, and BG is p–compact.

We have the following necessary and sufficient conditions that BG be p–compact toral.

Theorem 1.2 [20, Theorem 2] *Suppose G is a compact Lie group, and G_0 is the connected component with the identity. Then BG is p–compact toral if and only if the following conditions hold:*

(a) *G_0 is a torus T and $G/G_0 = \pi_0 G$ is p–nilpotent.*

(b) *The fibration $BT \longrightarrow BG \longrightarrow B\pi_0 G$ is preserved by the p–completion.*

Moreover, the p–completed fibration $(BT)^\wedge_p \longrightarrow (BG)^\wedge_p \longrightarrow (B\pi_0 G)^\wedge_p$ splits if and only if T is a central subgroup of G.

Next we consider the general case. What are the conditions that BG be p–compact? For example, for the normalizer NT of a maximal torus T of a connected compact Lie group K, it is well–known that $(BNT)^\wedge_p \simeq (BK)^\wedge_p$ if p does not divide the order of the Weyl group $W(K)$. This means that BNT is p–compact for such p. Using the following result, we can show the converse.

Proposition 1.3 [20, Proposition 3.1] *If BG is p–compact, then the following hold:*

(a) *$\pi_0 G$ is p–nilpotent.*

(b) *$\pi_1((BG)^\wedge_p)$ is isomorphic to a p–Sylow subgroup of $\pi_0 G$.*

The necessary condition of this proposition is not sufficient, even though the rational cohomology of $(BG)^\wedge_p$ is assumed to be expressed as a ring of invariants under the action of a group generated by pseudoreflections.

Theorem 1.4 [19, Theorem 1] *Let $G = \Gamma_2$, the quotient group of a subgroup $SU(3) \rtimes \mathbb{Z}/2$ of the exceptional Lie group G_2. For $p = 3$, the following hold:*

(1) *$\pi_0 G$ is p–nilpotent and $\pi_1((BG)^\wedge_p)$ is isomorphic to a p–Sylow subgroup of $\pi_0 G$.*

(2) *$(BG)^\wedge_p$ is rationally equivalent to $(BG_2)^\wedge_p$.*

(3) *BG is not p–compact.*

We discuss invariant rings and some properties of $B\Gamma_2$ and BG_2 at $p = 3$. Suppose G is a compact connected Lie group. The Weyl group $W(G)$ acts on its maximal torus T^n, and the integral representation $W(G) \longrightarrow GL(n, \mathbb{Z})$ is obtained (see Dwyer and Wilkerson [9; 10]). It is well–known that $K(BG) \cong K(BT^n)^{W(G)}$ and $H^*(BG; \mathbb{F}_p) \cong H^*(BT^n; \mathbb{F}_p)^{W(G)}$ for large p. Let $W(G)^*$ denote the dual representation of $W(G)$. Although the mod 3 reductions of the integral representations of $W(G_2)$ and $W(G_2)^*$ are not equivalent, there is $\psi \in GL(2, \mathbb{Z})$ such that $\psi W(G_2) \psi^{-1} = W(G_2)^*$ [19, Lemma 3]. Consequently, $K(BT^2; \mathbb{Z}^\wedge_3)^{W(G_2)} \cong K(BT^2; \mathbb{Z}^\wedge_3)^{W(G_2)^*}$. Since $K(B\Gamma_2; \mathbb{Z}^\wedge_3) \cong K(BT^2; \mathbb{Z}^\wedge_3)^{W(G_2)^*}$, we have the following result.

Theorem 1.5 [19, Theorem 3] *Let Γ_2 be the compact Lie group as in Theorem 1.4. Then the following hold:*

(1) *The* 3*–adic K-theory $K(B\Gamma_2;\mathbb{Z}_3^\wedge)$ is isomorphic to $K(BG_2;\mathbb{Z}_3^\wedge)$ as a λ–ring.*

(2) *Let Γ be a compact Lie group such that $\Gamma_0 = PU(3)$ and the order of $\pi_0(\Gamma)$ is not divisible by* 3*. Then any map from $(B\Gamma)_3^\wedge$ to $(BG_2)_3^\wedge$ is null homotopic. In particular $[(B\Gamma_2)_3^\wedge, (BG_2)_3^\wedge] = 0$.*

We recall that if a connected compact Lie group G is simple, the following results hold:

(1) For any prime p, the space $(BG)_p^\wedge$ has no nontrivial retracts (see Ishiguro [15]).

(2) Assume $|W(G)| \equiv 0 \bmod p$. If a self-map $(BG)_p^\wedge \longrightarrow (BG)^p$ is not null homotopic, it is a homotopy equivalence (see Møller [22]).

(3) Assume $|W(G)| \equiv 0 \bmod p$, and let K be a compact Lie group. If a map $f\colon (BG)_p^\wedge \longrightarrow (BK)_p^\wedge$ is trivial in mod p cohomology, then f is null homotopic (see Ishiguro [16]).

Replacing G by Γ_2 at $p = 3$, we will see that (3) still holds. On the other hand it is not known if (1) and (2) hold, though on the level of K-theory they do.

2 Π–compact toral groups

Recall that a finite group γ is *p–nilpotent* if and only if γ is expressed as the semidirect product $\nu \rtimes \gamma_p$, where the normal p–complement ν is the subgroup generated by all elements of order prime to p, and where γ_p is the p–Sylow subgroup. For such a group γ, we see $(B\gamma)_p^\wedge \simeq B\gamma_p$. For a finite group G, one can show that $\mathbb{P}(BG) = \{p \in \Pi \mid G \text{ is } p\text{–nilpotent}\}$. Consequently, if $G = \Sigma_n$, the symmetric group on n letters, then $\mathbb{P}(B\Sigma_2) = \Pi$, $\mathbb{P}(B\Sigma_3) = \Pi - \{3\}$, and $\mathbb{P}(B\Sigma_n) = \{p \in \Pi \mid p > n\}$ for $n \geq 4$.

In [14], Henn provides a generalized definition of p–nilpotence for compact Lie groups. A compact Lie group G is *p–nilpotent* if and only if the connected component of the identity, G_0, is a torus; the finite group $\pi_0 G$ is p-nilpotent, and the cojugation action of the normal p–complement is trivial on T. We note that such a p–nilpotent group need not be semidirect product.

Let $\gamma = \pi_0 G$. Then, from the inclusion $\gamma_p \longrightarrow \gamma$, a subgroup G_p of G is obtained as follows:

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & \gamma \\ \| & & \uparrow & & \uparrow \\ T & \longrightarrow & G_p & \longrightarrow & \gamma_p \end{array}$$

A result of Henn [14] shows $(BG)^\wedge_p \simeq (BG_p)^\wedge_p$ if and only if the compact Lie group G is p–nilpotent.

Lemma 2.1 *A classifying space BG is p–compact toral if and only if the compact Lie group G is p–nilpotent.*

Proof If BG is p–compact toral, we see from [20, Theorem 2] that the fibration $BT \longrightarrow BG \longrightarrow B\pi_0 G$ is preserved by the p–completion. Let $\pi = \pi_0 G$. Then we obtain the following commutative diagram:

$$\begin{array}{ccccc} (BT)^\wedge_p & \longrightarrow & (BG)^\wedge_p & \longrightarrow & (B\pi)^\wedge_p \\ \| & & \uparrow & & \uparrow \\ (BT)^\wedge_p & \longrightarrow & (BG_p)^\wedge_p & \longrightarrow & (B\pi_p)^\wedge_p \end{array}$$

By [20, Theorem 2], the finite group π is p–nilpotent, so the map $(B\pi_p)^\wedge_p \longrightarrow (B\pi)^\wedge_p$ is homotopy equivalent. Thus $(BG)^\wedge_p \simeq (BG_p)^\wedge_p$,and hence the result of [14] implies that G is p–nilpotent. Conversely, if G is p–nilpotent, then the following commutative diagram

$$\begin{array}{ccccc} BT & \longrightarrow & BG & \longrightarrow & B\pi \\ \| & & \uparrow & & \uparrow \\ BT & \longrightarrow & BG_p & \longrightarrow & B\pi_p \end{array}$$

tells us that $BT \longrightarrow BG \longrightarrow B\pi$ is p–equivalent to the fibration

$$(BT)^\wedge_p \longrightarrow (BG_p)^\wedge_p \longrightarrow (B\pi_p)^\wedge_p.$$

From [20, Theorem 2], we see that BG is p–compact toral. □

Proof of Theorem 1 First suppose BG is Π–compact toral. Lemma 2.1 implies that G_0 is a torus T and $G/G_0 = \pi_0 G$ is p–nilpotent for any p. According to [20, Lemma 2.1], the group $\pi_0 G$ must be nilpotent. We notice that for each p the normal p–complement of $\pi_0 G$ acts trivially on T. Thus $\pi_0 G$ itself acts trivially on T, and T is a central subgroup of G. Conversely, assume that conditions (a) and (b) hold. According to [14, Proposition 1.3], we see that G is p–nilpotent for any p. Therefore BG is Π–compact toral. □

We will show that a group which satisfies conditions (a) and (b) of Theorem 1 need not be a product group. For instance, consider the quaternion group Q_8 in $SU(2)$. Recall that the group can be presented as $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1}\rangle$. Let $\rho\colon Q_8 \longrightarrow U(2)$ be a faithful representation given by the following:

$$\rho(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let S denote the center of the unitary group $U(2)$ and let G be the subgroup of $U(2)$ generated by $\rho(Q_8)$ and S. Then we obtain the group extension $S \longrightarrow G \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $S \cong S^1$, this group G satisfies conditions (a) and (b). On the other hand, we see that the non–abelian group G can not be a product group. This result can be generalized as follows:

Proposition 2.2 *Suppose $\rho\colon \pi \longrightarrow U(n)$ is a faithful irreducible representation for a non–abelian finite nilpotent group π. Let S be the center of the unitary group $U(n)$ and let G be the subgroup of $U(n)$ generated by $\rho(\pi)$ and S with group extension $S \longrightarrow G \longrightarrow \pi_0 G$. Then this extension does not split, and G satisfies conditions (a) and (b) of Theorem 1.*

Proof First we show that G satisfies conditions (a) and (b) of Theorem 1. Since π is nilpotent, so is the finite group $\pi_0 G \cong G/S$. Recall that the center of the unitary group $U(n)$ consists of scalar matrices, and is isomorphic to S^1. Thus we obtain the desired result.

Next we show that the group extension $S \longrightarrow G \longrightarrow \pi_0 G$ does not split. If this extension did split, then we would have $G \cong S \rtimes \pi_0 G$. Since the action of $\pi_0 G$ on the center S is trivial, it follows that G is isomorphic to the product group $S \times \pi_0 G$. Let $Z(\pi)$ denote the center of π. Since the representation $\rho\colon \pi \longrightarrow U(n)$ is irreducible and faithful, Schur's Lemma implies $S \cap \rho(\pi) = Z(\rho(\pi)) \cong Z(\pi)$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} S & \longrightarrow & G & \longrightarrow & \pi_0 G \\ \uparrow & & \uparrow & & \| \\ Z(\pi) & \longrightarrow & \pi & \xrightarrow{q} & \pi_0 G \end{array}$$

Regarding π as a subgroup of $G = S \times \pi_0 G$, an element $y \in \pi$ can be written as $y = (s, x)$ for $s \in S$ and $x \in \pi_0 G$. Notice that $\pi_0 G$ is nilpotent and this group has a non–trivial center, since π is non–abelian. The map $q\colon \pi \longrightarrow \pi_0 G$ is an epimorphism. Consequently we can find an element $y_0 = (s_0, x_0)$ where $s_0 \in S$ and x_0 is a non–identity element of $Z(\pi_0 G)$. This means that y_0 is contained in $Z(\pi)$, though $q(y_0)$ is a non–identity element. This contradiction completes the proof. □

3 Π–compact subgroups of simple Lie groups

We will need the following results to prove Theorem 2.

Lemma 3.1 *Let K be a compact Lie group, and let G be a connected compact Lie group. If $(BK)^\wedge_p \simeq (BG)^\wedge_p$ for some p, we have a group extension as follows:*

$$1 \longrightarrow W(K_0) \longrightarrow W(G) \longrightarrow \pi_0 K \longrightarrow 1$$

Proof It is well–known that $H^*((BG)^\wedge_p;\mathbb{Q}) = H^*((BT_G)^\wedge_p;\mathbb{Q})^{W(G)}$, and since $(BK)^\wedge_p \simeq (BG)^\wedge_p$, it follows that $H^*((BG)^\wedge_p;\mathbb{Q}) = H^*((BK)^\wedge_p;\mathbb{Q})$. Notice that $H^*((BK)^\wedge_p;\mathbb{Q}) = H^*((BK_0)^\wedge_p;\mathbb{Q})^{\pi_0 K} = (H^*((BT_{K_0})^\wedge_p;\mathbb{Q})^{W(K_0)})^{\pi_0 K}$. Galois theory for the invariant rings (see Smith [23]) tells us that $W(K_0)$ is a normal subgroup of $W(G)$ and that the quotient group $W(G)/W(K_0)$ is isomorphic to $\pi_0 K$. This completes the proof. □

Lemma 3.2 *For a compact Lie group K, suppose the loop space of the p–completion $\Omega(BK)^\wedge_p$ is a connected p–compact group. Then p doesn't divide the order of $\pi_0 K$.*

Proof Since BK is p–compact, $\pi_0 K$ is p–nilpotent. So, if π denotes a p–Sylow subgroup of $\pi_0 K$, then $(B\pi_0 K)^\wedge_p \simeq B\pi$. Notice that $(BK)^\wedge_p$ is 1–connected. Hence the map $(BK)^\wedge_p \longrightarrow (B\pi_0 K)^\wedge_p$ induced from the epimorphism $K \longrightarrow \pi_0 K$ is a null map. Consequently the p–Sylow subgroup π must be trivial. □

For $K = NT$, the normalizer of a maximal torus T of a connected compact simple Lie group, the converse of Lemma 3.2 is true, though it doesn't hold in general. Note that $\pi_0 \Gamma_2 = \mathbb{Z}/2$ and that $B\Gamma_2$ is not 3–compact [19].

Proof of Theorem 2 (1) Since $(BH)^\wedge_p \simeq (BG)^\wedge_p$ for some p, Lemma 3.1 says that the Weyl group $W(H_0)$ is a normal subgroup of $W(G)$. First we show that $W(H_0) \neq W(G)$. If $W(H_0) = W(G)$, the inclusion $H_0 \longrightarrow G$ induces the isomorphism $H^*(BH_0;\mathbb{Q}) \cong H^*(BG;\mathbb{Q})$, since $\mathrm{rank}(H_0) = \mathrm{rank}(G)$. Hence $BH_0 \simeq_0 BG$. Consequently if $\widetilde{H}_0$ and $\widetilde{G}$ denote the universal covering groups of H_0 and G respectively, then $\widetilde{H}_0 \cong \widetilde{G}$. The maps $B\widetilde{H}_0 \longrightarrow BH_0$ and $B\widetilde{G} \longrightarrow BG$ are rational equivalences. According to [18, Lemma 2.2], we would see that $H_0 = H = G$. Since H must be a proper subgroup of G, we obtain the desired result.

We now see that $W(H_0)$ is a proper normal subgroup of $W(G)$. If $W(H_0)$ is a nontrivial group, a result of Asano [3] implies that (G, H_0) is one of the following types:

$$(G, H_0) = \begin{cases} (B_n, D_n) \\ (C_n, A_1 \times \cdots \times A_1) \\ (G_2, A_2) \\ (F_4, D_4) \end{cases}$$

According to [20, Lemma 2.1 and Proposition 3.1], we notice $\pi_0 H = W(G)/W(H_0)$ is a nilpotent group since BH is Π–compact. Recall that $W(C_n)/W(A_1 \times \cdots \times A_1) \cong \Sigma_n$ and $W(F_4)/W(D_4) \cong \Sigma_3$. For $n \geq 3$, we notice that the symmetric group Σ_n is not nilpotent. Hence the group $W(G)/W(H_0)$ is nilpotent when $W(G)/W(H_0) \cong \mathbb{Z}/2$. Consequently, we see the following:

$$(G, H_0) = \begin{cases} (B_n, D_n) \\ (C_2, A_1 \times A_1) \\ (G_2, A_2) \end{cases}$$

It remains to consider the case that $W(H_0)$ is a trivial group. In this case $\pi_0 H = W(G)$, and $W(G)$ is a nilpotent group. From [20, Proposition 3.4], we see that $G = A_1$ or $B_2 (= C_2)$.

(2) We first show that, for any odd prime p, all types of the pairs are realized for some G and H. To begin with, we consider the case (G, T_G) for $G = A_1$. Take $(G, H) = (SO(3), O(2))$. Since $\pi_0(O(2)) = \mathbb{Z}/2$ and $BO(2) \simeq_p BSO(3)$ for odd prime p, the space $BO(2)$ is Π–compact. In the case $G = B_2$, take $(G, H) = (G, NT_G)$ for $G = Spin(5)$. Then $\pi_0 H$ is a 2–group and $BNT_G \simeq_p BG$ for odd prime p, and hence BNT_G is Π–compact.

In the case of (B_n, D_n), take $(G, H) = (SO(2n+1), O(2n))$. Since $\pi_0(O(2n)) = \mathbb{Z}/2$ and $BO(2n) \simeq_p BSO(2n+1)$ for odd prime p, the space $BO(2n)$ is Π–compact. For $(C_2, A_1 \times A_1)$, take $G = Sp(2)$ and $H = (Sp(1) \times Sp(1)) \rtimes \mathbb{Z}/2\langle a \rangle$ where $a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in Sp(2)$. For complex numbers z and w, we see that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}.$$

Thus the action of $\mathbb{Z}/2\langle a \rangle$ is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We note that

$$W(Sp(2)) = D_8 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Consequently $\pi_0 H$ is a 2–group and $BH \simeq_p BG$ for odd prime p, and hence BH is Π–compact. Finally, for (G_2, A_2), as mentioned in the introduction, take $G = G_2$ and $H = SU(3) \rtimes \mathbb{Z}/2$. Then BH is Π–compact.

It remains to consider the case $p = 2$. Note that $|W(G)/W(H_0)|$ for each of such (G, H_0)'s is a power of 2. Lemma 3.1 implies that the finite group $\pi_0 H$ must be a 2–group. Lemma 3.2 says that $|\pi_0 H|$ is not divisible by 2, since $(BH)^\wedge_2 \simeq (BG)^\wedge_2$. Thus H is connected, and hence $H = G$. This completes the proof. □

Any proper closed subgroup of G which includes the normalizer NT satisfies the assumption of Theorem 2. So, this theorem shows, once again, that almost all BNT are not Π–compact [20]. Furthermore, for any connected compact Lie group G, it is well-known that $(BNT)^\wedge_p \simeq (BG)^\wedge_p$ if p does not divide the order of the Weyl group $W(G)$, hence BNT is p–compact for such p. The converse is shown in [20].

Lemma 3.3 *Let $\alpha \longrightarrow \widetilde{G} \longrightarrow G$ be a central extension of compact Lie groups. Then BG is p–compact if and only if $B\widetilde{G}$ is p–compact.*

Proof First assume that BG is p–compact. Since $\alpha \longrightarrow \widetilde{G} \longrightarrow G$ is a central extension, the fibration $B\alpha \longrightarrow B\widetilde{G} \longrightarrow BG$ is principal. Thus we obtain a fibration $B\widetilde{G} \longrightarrow BG \longrightarrow K(\alpha, 2)$. The base space is 1–connected, so the fibration is preserved by the p–completion, and hence we obtain the fibration

$$(B\alpha)^\wedge_p \longrightarrow (B\widetilde{G})^\wedge_p \longrightarrow (BG)^\wedge_p.$$

Since the loop spaces $\Omega(B\alpha)^\wedge_p$ and $\Omega(BG)^\wedge_p$ are $\mathbb{F}_p$–finite, so is $\Omega(B\widetilde{G})^\wedge_p$. Thus $B\widetilde{G}$ is p–compact.

Conversely we assume that $B\widetilde{G}$ is p–compact. Consider the fibration

$$\Omega(BG)^\wedge_p \longrightarrow (B\alpha)^\wedge_p \longrightarrow (B\widetilde{G})^\wedge_p.$$

Since the map $(B\alpha)^\wedge_p \longrightarrow (B\widetilde{G})^\wedge_p$ is induced from the inclusion $\alpha \hookrightarrow \widetilde{G}$, it is a monomorphism of p–compact groups. Hence its homotopy fiber $\Omega(BG)^\wedge_p$ is $\mathbb{F}_p$–finite, and therefore BG is p–compact. □

Corollary 3.4 *Let $\alpha \longrightarrow \widetilde{G} \longrightarrow G$ be a central extension of compact Lie groups, and let H be a subgroup of G so that there is the commutative diagram:*

$$\begin{array}{ccccc} \alpha & \longrightarrow & \widetilde{G} & \longrightarrow & G \\ \| & & \uparrow & & \uparrow \\ \alpha & \longrightarrow & \widetilde{H} & \longrightarrow & H \end{array}$$

Then the pair (G, H) satisfies the conditions of Theorem 2 if and only if so does the pair $(\widetilde{G}, \widetilde{H})$.

Proof Lemma 3.3 implies that BH is Π–compact if and only if $B\widetilde{H}$ is Π–compact. It is clear that $\mathrm{rank}(H_0)=\mathrm{rank}(G)$ if and only if $\mathrm{rank}(\widetilde{H}_0)=\mathrm{rank}(\widetilde{G})$. Finally we see $(BH)^\wedge_p \simeq (BG)^\wedge_p$ if and only if $(B\widetilde{H})^\wedge_p \simeq (B\widetilde{G})^\wedge_p$ from the following commutative diagram of fibrations:

$$\begin{array}{ccccc}
(B\alpha)^\wedge_p & \longrightarrow & (B\widetilde{G})^\wedge_p & \longrightarrow & (BG)^\wedge_p \\
\| & & \uparrow & & \uparrow \\
(B\alpha)^\wedge_p & \longrightarrow & (B\widetilde{H})^\wedge_p & \longrightarrow & (BH)^\wedge_p
\end{array}$$

This completes the proof. □

Lemma 3.5 *Let $M\longrightarrow K\longrightarrow L$ be a short exact sequence of groups. If ν is a normal subgroup of K, the kernel ν' of the composition $\nu\longrightarrow K\longrightarrow L$ is a normal subgroup of M.*

Proof We consider the following commutative diagram:

$$\begin{array}{ccc}
\nu' & \longrightarrow & M \\
\downarrow & & \downarrow \\
\nu & \longrightarrow & K \\
\downarrow & & \downarrow q \\
q(\nu) & \longrightarrow & L
\end{array}$$

For $x\in\nu'$ and $m\in M$, it follows that

$$\begin{aligned} q(mxm^{-1}) &= q(m)q(x)q(m^{-1}) \\ &= q(m)q(m)^{-1} = e \end{aligned}$$

Thus $mxm^{-1}\in\ker q$. Since $\nu'\subset\nu$, $M\subset K$, and $\nu\lhd K$, we see that $mxm^{-1}\in\nu$. So $mxm^{-1}\in\ker q\cap\nu=\nu'$, and therefore $\nu'\lhd M$. □

Proof of Theorem 3 First suppose $(G,H_0)=(B_n,D_n)$ or $(C_2,A_1\times A_1)$. Let ν' be the kernel of the composition $\nu\longrightarrow H\longrightarrow\pi_0 H$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\nu' & \longrightarrow & H_0 \\
\downarrow & & \downarrow \\
\nu & \longrightarrow & H \\
\downarrow & & \downarrow q \\
q(\nu) & \longrightarrow & \pi_0 H
\end{array}$$

Lemma 3.5 says that $\nu' \lhd H_0$. Since ν' is a finite normal subgroup of H_0, it is a finite 2–group. As we have seen in the proof of Theorem 2, $\pi_0 H = W(G)/W(H_0)$ is a 2–group, and hence so is $q(\nu)$. Consequently ν is a 2–group.

Now consider the following commutative diagram:

$$\begin{array}{ccccc} \nu' & \longrightarrow & \nu & \longrightarrow & q(\nu) \\ \downarrow & & \downarrow & & \downarrow \\ H_0 & \longrightarrow & H & \longrightarrow & \pi_0 H \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_0 & \longrightarrow & \Gamma & \longrightarrow & \pi_0 \Gamma \end{array}$$

Since $\pi_0\Gamma$ is a 2–group, the fibration $B\Gamma_0 \longrightarrow B\Gamma \longrightarrow B\pi_0\Gamma$ is preserved by the 2–completion (see Bousfield and Kan [5]). Hence $B\Gamma$ is 2–compact. Next, for odd prime p, we see that $(B\Gamma)^\wedge_p \simeq (BH)^\wedge_p$, since ν is a 2–group. We see also that G has no odd torsion and $H^*(BH;\mathbb{F}_p) = H^*(BH_0;\mathbb{F}_p)^{\pi_0 H} \cong H^*(BG;\mathbb{F}_p)$. Consequently the space $(B\Gamma)^\wedge_p$ is homotopy equivalent to $(BG)^\wedge_p$. Therefore $B\Gamma$ is Π–compact.

It remains to consider the case $(G, H_0) = (G, T_G)$ for $G = A_1$ or $G = B_2(= C_2)$. Since $H_0 = T_G$ and $H_0 \lhd H$, we see that H is a subgroup of the normalizer NT_G. Consider the following commutative diagram:

$$\begin{array}{ccccc} T_G & \longrightarrow & NT_G & \longrightarrow & W(G) \\ \| & & \uparrow & & \uparrow \\ T_G & \longrightarrow & H & \longrightarrow & \pi_0 H \end{array}$$

Since the map $BH \longrightarrow BG$ is p–equivalent for some p, it follows that $\pi_0 H = W(G)$. Consequently $H = NT_G$.

If ν is a finite normal subgroup of NT_G, then $B\nu$ is contained in the kernel of the map $(BG)^\wedge_p \simeq (BNT_G)^\wedge_p \longrightarrow (B\Gamma)^\wedge_p$. Since G is simple and $G \neq G_2$, according to [16; 17], the group ν is included in the center of G. Thus ν is a 2–group. Therefore $(B\Gamma)^\wedge_p \simeq (BNT_G)^\wedge_p \simeq (BG)^\wedge_p$ for odd prime p, and hence $B\Gamma$ is p–compact for such p. Finally we note that $W(G)$ is a 2–group, and hence $B\Gamma$ is 2–compact. □

We will discuss a few more results. Basically we have been looking at three Lie groups $H_0 \subset H \subset G$. The following shows a property of the (non–connected) middle group H.

Proposition 3.6 *Suppose G is a connected compact Lie group, and H is a proper closed subgroup of G with $\mathrm{rank}(H_0) = \mathrm{rank}(G)$. If the order of $\pi_0 H$ is divisible by a prime p, so is the order of $W(G)/W(H_0)$.*

Proof Assuming $|W(G)/W(H_0)| \not\equiv 0 \bmod p$, we will show $\pi_0 H \not\equiv 0 \bmod p$. Notice that we have the following commutative diagram

$$\begin{array}{ccccc} T & \longrightarrow & N_G T & \longrightarrow & W(G) \\ \| & & \uparrow & & \uparrow \\ T & \longrightarrow & N_{H_0} T & \longrightarrow & W(H_0), \end{array}$$

where the vertical maps are injective, since $\mathrm{rank}(H_0) = \mathrm{rank}(G)$. We recall, from Jackowski, McClure and Oliver [21], that the Sylow theorem for compact Lie groups G holds. Namely G contains maximal p–toral subgroups, and all of which are conjugate to $N_p T$, where $N_p(T)/T$ is a p–Sylow subgroup of $N(T)/T = W(G)$.

Suppose K is a p–toral subgroup of H. Since $|W(G)/W(H_0)| \not\equiv 0 \bmod p$, we see that K is a subgroup of H_0 up to conjugate. Consequently, the composite map $K \hookrightarrow H \longrightarrow \pi_0 H$ must be homotopy equivalent to a null map. Since $H \longrightarrow \pi_0 H$ is surjective, the p–part of $\pi_0 H$ is trivial. □

For each pair mentioned in the part (a) of Theorem 2, we note that $|W(G)/W(H_0)|$ is a power of 2. Proposition 3.6 says, for instance, that $\pi_0 H$ is a 2–group for any (G, H) such that $|W(G)/W(H_0)|$ is a power of 2. As an application, one can show that if H is a non–connected proper closed subgroup of $SO(3)$ with $H_0 = SO(2)$, then H is isomorphic to $O(2)$. A proof may use the fact that H is 2–toral, and that a maximal 2–toral subgroup in H is 2–stubborn [21]. A 2–compact version of this result also holds. Suppose X is a 2–compact group such that there are two monomorphisms of 2–compact groups $BSO(2)^\wedge_2 \longrightarrow BX$ and $BX \longrightarrow BSO(3)^\wedge_2$. Then, along the line of a similar argument, one can also show that BX is homotopy equivalent to $BO(2)^\wedge_2$ if X is not connected. In the case of X being connected, the classifying space BX is either $BSO(2)^\wedge_2$ or $BSO(3)^\wedge_2$.

In Theorem 2, Lie groups of type $(C_2, A_1 \times A_1)$ has been discussed. An example is given by $Sp(1) \times Sp(1) \subset (Sp(1) \times Sp(1)) \rtimes \mathbb{Z}/2 \subset Sp(2)$. The middle group can be regarded as the wreath product $Sp(1) \wr \Sigma_n$ for $n = 2$. We ask for what n and p its classifying space is p–compact. Note that $Sp(1) \wr \Sigma_n$ is a proper closed subgroup of $Sp(n)$.

Proposition 3.7 *Let $\Gamma(n)$ denote the wreath product $Sp(1) \wr \Sigma_n$. Then*

$$\mathbb{P}(B\Gamma(n)) = \begin{cases} \Pi & \text{if } n = 2 \\ \{p \in \Pi \mid p > n\} & \text{if } n \geq 3 \end{cases}$$

Proof When $n = 2$, the desired result has been shown in our proof of the part (b) of Theorem 2. Recall from [20] that if $B\Gamma(n)$ is p–compact, then $\pi_0 B\Gamma(n) = \Sigma_n$ must be p–nilpotent. For $n \geq 4$, it follows that Σ_n is p–nilpotent if and only if $p > n$. Since the group $\Gamma(n)$ includes the normalizer of a maximal torus of $Sp(n)$, we see $B\Gamma(n) \simeq_p BSp(n)$ if $p > n$. Thus $\mathbb{P}(B\Gamma(n)) = \{p \in \Pi \mid p > n\}$ for $n \geq 4$.

For $n = 3$, note that Σ_3 is p–nilpotent if and only if $p \neq 3$. So it remains to prove that $B\Gamma(3)$ is not 2–compact. We consider a subgroup H of $\Gamma(3)$ which makes the following diagram commutative:

$$\begin{array}{ccccc}
\prod^3 Sp(1) & = & \prod^3 Sp(1) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
H & \longrightarrow & \Gamma(3) & \longrightarrow & \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \| \\
\mathbb{Z}/3 & \longrightarrow & \Sigma_3 & \longrightarrow & \mathbb{Z}/2
\end{array}$$

The fibration $BH \longrightarrow B\Gamma(3) \longrightarrow B\mathbb{Z}/2$ is preserved by the completion at $p = 2$. Hence, if $B\Gamma(3)$ were 2–compact, the space $\Omega(BH)^\wedge_2$ would be a connected 2–compact group so that the cohomology $H^*(BH; \mathbb{Q}^\wedge_2)$ should be a polynomial ring, (see Dwyer and Wilkerson [8, Theorem 9.7]). Though $H^*\big(B\prod^3 Sp(1); \mathbb{Q}^\wedge_2\big)$ is a polynomial ring, its invariant ring $H^*(BH; \mathbb{Q}^\wedge_2) = H^*\big(B\prod^3 Sp(1); \mathbb{Q}^\wedge_2\big)^{\mathbb{Z}/3}$ is not a polynomial ring, since the group $\mathbb{Z}/3$ is not generated by reflections. This contradiction completes the proof. □

For $(G, H) = (Sp(n), Sp(1) \wr \Sigma_n)$, we note that (G, H_0) is a type of $(C_n, A_1 \times \cdots \times A_1)$. This is one of the cases that the Weyl group $W(H_0)$ is a normal subgroup of $W(G)$ (see Asano [3]) discussed in our proof of the part (a) of Theorem 2. Finally we talk about the only remaining case $(G, H_0) = (F_4, D_4)$. An example is given by $Spin(8) \subset Spin(8) \rtimes \Sigma_3 \subset F_4$. Let Γ denote the middle group $Spin(8) \rtimes \Sigma_3$. Then we can show that $\mathbb{P}(B\Gamma) = \{p \in \Pi \mid p > 3\}$. To show that $B\Gamma$ is not 2–compact, one might use the fact, (see Adams [1, Theorem 14.2]), that $W(F_4) = W(Spin(8)) \rtimes \Sigma_3$, and that its subgroup $W(Spin(8)) \rtimes \mathbb{Z}/3$ is not a reflection group.

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