

Spaces of real polynomials with common roots

YASUHIKO KAMIYAMA

Let $RX_{k,n}^l$ be the space consisting of all $(n + 1)$ -tuples $(p_0(z), \dots, p_n(z))$ of monic polynomials over \mathbb{R} of degree k and such that there are at most l roots common to all $p_i(z)$. In this paper, we prove a stable splitting of $RX_{k,n}^l$.

55P15; 55P35, 58D15

1 Introduction

Let $\text{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree k from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^n$. The basepoint condition we assume is that $f(\infty) = [1, \dots, 1]$. Such holomorphic maps are given by rational functions:

$$\text{Rat}_k(\mathbb{C}P^n) = \{(p_0(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \\ \text{of degree } k \text{ and such that there are no roots common to all } p_i(z)\}.$$

There is an inclusion $\text{Rat}_k(\mathbb{C}P^n) \hookrightarrow \Omega_k^2 \mathbb{C}P^n \simeq \Omega^2 S^{2n+1}$. Segal [6] proved that the inclusion is a homotopy equivalence up to dimension $k(2n - 1)$. Later, the stable homotopy type of $\text{Rat}_k(\mathbb{C}P^n)$ was described by Cohen et al [2; 3] as follows. Let $\Omega^2 S^{2n+1} \simeq_s \bigvee_{1 \leq q} D_q(S^{2n-1})$ be Snaith's stable splitting of $\Omega^2 S^{2n+1}$. Then

$$(1-1) \quad \text{Rat}_k(\mathbb{C}P^n) \simeq_s \bigvee_{q=1}^k D_q(S^{2n-1}).$$

In Kamiyama [4], (1-1) was generalized as follows. We set

$$X_{k,n}^l = \{(p_0(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \\ \text{of degree } k \text{ and such that there are at most } l \text{ roots common to all } p_i(z)\}.$$

In particular, $X_{k,n}^0 = \text{Rat}_k(\mathbb{C}P^n)$. Let

$$J^l(S^{2n}) \simeq S^{2n} \cup e^{4n} \cup e^{6n} \cup \dots \cup e^{2ln} \subset \Omega S^{2n+1}$$

be the l th stage of the James filtration of ΩS^{2n+1} , and let $W^l(S^{2n})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n}) \hookrightarrow \Omega S^{2n+1}$. We generalize Snaith's stable splitting of $\Omega^2 S^{2n+1}$ as follows:

$$W^l(S^{2n}) \underset{s}{\simeq} \bigvee_{1 \leq q} D_q \xi^l(S^{2n}).$$

Then we have a stable splitting

$$X_{k,n}^l \underset{s}{\simeq} \bigvee_{q=1}^k D_q \xi^l(S^{2n}).$$

The purpose of this paper is to study the real part $RX_{k,n}^l$ of $X_{k,n}^l$ and prove a stable splitting of this. More precisely, let $RX_{k,n}^l$ be the subspace of $X_{k,n}^l$ consisting of elements $(p_0(z), \dots, p_n(z))$ such that each $p_i(z)$ has real coefficients. Our main results will be stated in Section 2. Here we give a theorem which generalizes (1–1). Since the homotopy type of $RX_{k,1}^0$ is known (see Example 2.1 (iii)), we assume $n \geq 2$. In this case, there is an inclusion

$$RX_{k,n}^0 \hookrightarrow \Omega S^n \times \Omega^2 S^{2n+1}.$$

(See Lemma 3.1.)

Theorem 1.1 *For $n \geq 2$, we define the weight of stable summands in ΩS^n as usual, but those in $\Omega^2 S^{2n+1}$ we define as being twice the usual one. Then $RX_{k,n}^0$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times \Omega^2 S^{2n+1}$ of weight $\leq k$. Hence,*

$$RX_{k,n}^0 \underset{s}{\simeq} \bigvee_{p+2q \leq k} \Sigma^{p(n-1)} D_q(S^{2n-1}) \vee \bigvee_{p=1}^k S^{p(n-1)}.$$

This paper is organized as follows. In Section 2 we state the main results. We give a stable splitting of $RX_{k,n}^l$ in Theorem A and Theorem B. In order to prove these theorems, we also consider a space $Y_{k,n}^l$, which is an open set of $RX_{k,n}^l$. We give a stable splitting of $Y_{k,n}^l$ in Proposition C. In Section 3 we prove Proposition C. In Section 4 we prove Theorem A and Theorem B.

2 Main results

We set

$$Y_{k,n}^l = \{(p_0(z), \dots, p_n(z)) \in RX_{k,n}^l : \text{there are no real roots common to all } p_i(z)\}.$$

The spaces $Y_{k,n}^l$ and $RX_{k,n}^l$ are in the following relation:

$$\begin{array}{ccccccccccc} Y_{k,n}^k & \supset & Y_{k,n}^{k-1} & \supset & \dots & \supset & Y_{k,n}^l & \supset & \dots & \supset & Y_{k,n}^1 & = & Y_{k,n}^0 \\ \cap & & \cap & & & & \cap & & & & \cap & & \parallel \\ RX_{k,n}^k & \supset & RX_{k,n}^{k-1} & \supset & \dots & \supset & RX_{k,n}^l & \supset & \dots & \supset & RX_{k,n}^1 & \supset & RX_{k,n}^0 \end{array}$$

where each subset is an open set. Moreover, $Y_{k,n}^{2i+1} = Y_{k,n}^{2i}$. In fact, if $\alpha \in H_+$ (where H_+ is the open upper half-plane) is a root of a real polynomial, then so is $\bar{\alpha} \in H_-$.

We have the following examples.

Example 2.1

- (i) It is proved by Mostovoy [5] that $Y_{k,1}^k$ consists of $k + 1$ contractible connected components.
- (ii) The following result is proved by Vassiliev [7]. For $n \geq 3$, there is a homotopy equivalence $Y_{k,n}^k \simeq J^k(S^{n-1})$, where $J^k(S^{n-1})$ is as above the k th stage of the James filtration of ΩS^n . For $n = 2$, these spaces are stably homotopy equivalent.
- (iii) It is proved by Segal [6] that

$$RX_{k,1}^0 \simeq \coprod_{q=0}^k \text{Rat}_{\min(q,k-q)}(\mathbb{C}P^1).$$

- (iv) $RX_{k,n}^{k-1} \cong \mathbb{R}^k \times (\mathbb{R}^{kn})^*$ and $RX_{k,n}^k \cong \mathbb{R}^{k(n+1)}$.

In fact, $(p_0(z), \dots, p_n(z)) \in RX_{k,n}^k$ is an element of $RX_{k,n}^{k-1}$ if and only if $p_i(z) \neq p_j(z)$ for some i, j . Hence, the first homeomorphism holds.

Now we state our main results.

Theorem A For $n \geq 1$ and $i \geq 0$, there is a homotopy equivalence

$$RX_{k,n}^{2i+1} \simeq X_{[\frac{k}{2},n}^i,$$

where $[\frac{k}{2}]$ denotes as usual the largest integer $\leq \frac{k}{2}$.

Theorem B For $n \geq 1$ and $i \geq 0$, there is a stable homotopy equivalence

$$RX_{k,n}^{2i} \underset{s}{\simeq} X_{[\frac{k}{2},n]}^i \vee_{\Sigma^{2in}} \left(\bigvee_{\substack{p+2q \leq k-2i \\ 1 \leq p}} \Sigma^{p(n-1)} D_q(S^{2n-1}) \vee \bigvee_{p=1}^{k-2i} S^{p(n-1)} \right).$$

We study $RX_{k,n}^l$ by induction with making l larger. Hence, the induction starts from $RX_{k,n}^0$. Recall that $RX_{k,n}^0 = Y_{k,n}^0$. We study $Y_{k,n}^l$ by induction with making l smaller, where the initial condition is given in Example 2.1 (ii). In fact, we have the following proposition.

Proposition C

- (i) For $n \geq 2$, we define the weight of stable summands in ΩS^n as usual, but those in $W^i(S^{2n})$ we define as being twice the usual one. Then $Y_{k,n}^{2i}$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times W^i(S^{2n})$ of weight $\leq k$. Hence,

$$Y_{k,n}^{2i} \underset{s}{\simeq} \bigvee_{p+2q \leq k} \Sigma^{p(n-1)} D_q S^i(S^{2n}) \vee \bigvee_{p=1}^k S^{p(n-1)}.$$

- (ii) When $n = 1$, there is a homotopy equivalence

$$Y_{k,1}^{2i} \simeq \prod_{q=0}^k X_{\min(q,k-q),1}^i.$$

Note that Proposition C (ii) is a generalization of Example 2.1 (i) and (iii).

3 Proof of Proposition C

We study the space of continuous maps which contains $Y_{k,n}^k$ or $RX_{k,n}^0$. For simplicity, we assume $n \geq 2$. (The case for $n = 1$ can be obtained by slight modifications.) Each $f \in Y_{k,n}^k$ defines a map $f: S^1 \rightarrow \mathbb{R}P^n$, where $S^1 = \mathbb{R} \cup \infty$. Hence, there is a natural map

$$Y_{k,n}^k \rightarrow \Omega_{k \bmod 2} \mathbb{R}P^n \simeq \Omega S^n.$$

Example 2.1 (ii) implies that $Y_{k,n}^k$ is the $k(n-1)$ -skeleton of ΩS^n .

On the other hand, let $\text{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n)$ be the space of continuous basepoint-preserving conjugation-equivariant maps of degree k from $\mathbb{C}P^1$ to $\mathbb{C}P^n$. Then there is an inclusion

$$RX_{k,n}^0 \hookrightarrow \text{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n).$$

Lemma 3.1 For $n \geq 2$, $\text{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$.

Proof It is easy to see that

$$\text{Map}_k^T(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \text{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n).$$

Since $\text{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n)$ can be thought as the space of maps

$$(D^2, S^1, *) \rightarrow (\mathbb{C}P^n, \mathbb{R}P^n, *)$$

of degree 0, there is a fibration

$$\Omega^2 S^{2n+1} \rightarrow \text{Map}_0^T(\mathbb{C}P^1, \mathbb{C}P^n) \rightarrow \Omega S^n.$$

This is a pullback of the path fibration $\Omega^2 S^{2n+1} \rightarrow P\Omega S^{2n+1} \rightarrow \Omega S^{2n+1}$ by the map $\Omega f: \Omega S^n \rightarrow \Omega S^{2n+1}$, where $f: S^n \rightarrow S^{2n+1}$ is a lift of the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. Since f is null homotopic, the fibration is trivial. This completes the proof of Lemma 3.1. \square

Hereafter, every homology is with \mathbb{Z}/p -coefficients, where p is a prime. Recall that for $n \geq 2$, we have $H_*(\Omega S^n) \cong \mathbb{Z}/p[x_{n-1}]$. We define the weight of x_{n-1} by $w(x_{n-1}) = 1$. On the other hand, we define the weight of an element of $H_*(\widetilde{X}_{k,n}^i)$ as being twice the usual one. For example, let $y_{2(l+1)n-1}$ be the generator of $\widetilde{H}_*(\widetilde{X}_{k,n}^l)$ of least degree. The usual weight of $y_{2(l+1)n-1}$ is $l+1$, but we reset $w(y_{2(l+1)n-1}) = 2(l+1)$.

Proposition 3.2 For $n \geq 2$, $H_*(Y_{k,n}^{2i})$ is isomorphic to the subspace of $H_*(\Omega S^n \times X_{k,n}^i)$ spanned by monomials of weight $\leq k$.

We prove the proposition from the following lemma.

Lemma 3.3 We have the following long exact sequence:

$$\dots \rightarrow H_*(Y_{k,n}^{2i-2}) \rightarrow H_*(Y_{k,n}^{2i}) \xrightarrow{\phi} H_{*-2i n}(RX_{k-2i,n}^0) \rightarrow H_{*-1}(Y_{k,n}^{2i-2}) \rightarrow \dots$$

Proof In [4, Propositions 4.5 and 5.4], we constructed a similar long exact sequence from the fact that

$$X_{k,n}^l - X_{k,n}^{l-1} = \mathbb{C}^l \times \text{Rat}_{k-l}(\mathbb{C}P^n),$$

where $\mathbb{C}^l \times \text{Rat}_{k-l}(\mathbb{C}P^n)$ corresponds to the subspace of $X_{k,n}^l$ consisting of elements $(p_0(z), \dots, p_n(z))$ such that there are exactly l roots common to all $p_i(z)$. The proposition is proved similarly using the fact that

$$Y_{k,n}^{2i} - Y_{k,n}^{2i-2} \cong \text{SP}^i(H_+) \times RX_{k-2i,n}^0,$$

where $\text{SP}^i(H_+)$ denotes the i th symmetric product of H_+ . □

Proof of Proposition 3.2 In order to prove Proposition 3.2 by induction, we introduce the following total order \leq to $Y_{k,n}^{2i}$ for $k \geq 1$ and $i \geq 0$: $Y_{k,n}^{2i} < Y_{k',n}^{2i'}$ if and only if

- (i) $k < k'$, or
- (ii) $k = k'$ and $i > i'$.

By Example 2.1 (ii), Proposition 3.2 holds for $Y_{k,n}^k$. Assuming that Proposition 3.2 holds for $Y_{k,n}^{2i}$ and $RX_{k-2i,n}^0$, we prove for $Y_{k,n}^{2i-2}$. We have the following long exact sequence:

$$(3-1) \quad \dots \longrightarrow H_*\left(X_{\lfloor \frac{k}{2} \rfloor, n}^{i-1}\right) \longrightarrow H_*\left(X_{\lfloor \frac{k}{2} \rfloor, n}^i\right) \\ \xrightarrow{\psi} H_{*-2in}\left(\text{Rat}_{\lfloor \frac{k}{2} \rfloor - i}(\mathbb{C}P^n)\right) \xrightarrow{\theta} H_{*-1}\left(X_{\lfloor \frac{k}{2} \rfloor, n}^{i-1}\right) \longrightarrow \dots$$

For $n \geq 2$, we consider the homomorphism

$$1 \otimes \psi: H_*(\Omega S^n) \otimes H_*\left(X_{\lfloor \frac{k}{2} \rfloor, n}^i\right) \rightarrow H_*(\Omega S^n) \otimes H_{*-2in}\left(\text{Rat}_{\lfloor \frac{k}{2} \rfloor - i}(\mathbb{C}P^n)\right).$$

Restricting the domain to $H_*(Y_{k,n}^{2i})$, we obtain the homomorphism ϕ in Lemma 3.3. Now it is easy to prove Proposition 3.2. □

Proof of Proposition C (i) We construct a stable map from the right-hand side of Proposition C (i) to $Y_{k,n}^{2i}$. Since our constructions are similar, we construct a stable map $g_{p,q,i,n}: \Sigma^{p(n-1)} D_q \xi^i(S^{2n}) \rightarrow Y_{k,n}^{2i}$. First, using the fact that $RX_{1,n}^0 \simeq S^{n-1}$ (see Example 2.1 (iv)), there is a stable map $f_{p,n}: S^{p(n-1)} \rightarrow RX_{p,n}^0$. Second, there is a stable section $e_{q,i,n}: D_q \xi^i(S^{2n}) \rightarrow X_{q,n}^i$. Third, there is an inclusion

$$(3-2) \quad \eta_{q,i,n}: X_{q,n}^i \hookrightarrow Y_{2q,n}^{2i}$$

To construct this, we fix a homeomorphism $h : \mathbb{C} \xrightarrow{\cong} H_+$. For $(p_0(z), \dots, p_n(z)) \in X_{q,n}^i$, we write $p_j(z) = \prod_{s=1}^q (z - \alpha_{s,j})$. Then we set

$$\begin{aligned} \eta_{q,i,n}(p_0(z), \dots, p_n(z)) &= \left(\prod_{s=1}^q (z - h(\alpha_{s,0}))(z - \overline{h(\alpha_{s,0})}), \dots, \prod_{s=1}^q (z - h(\alpha_{s,n}))(z - \overline{h(\alpha_{s,n})}) \right). \end{aligned}$$

Now consider the following composite of maps

$$(3-3) \quad S^{p(n-1)} \times D_q \xi^i(S^{2n}) \xrightarrow{f_{p,n} \times (\eta_{q,i,n} \circ e_{q,i,n})} RX_{p,n}^0 \times Y_{2q,n}^{2i} \xrightarrow{\mu} Y_{p+2q,n}^{2i} \hookrightarrow Y_{k,n}^{2i},$$

where μ is a loop sum which is constructed in the same way as in the loop sum $\text{Rat}_k(\mathbb{C}P^n) \times \text{Rat}_l(\mathbb{C}P^n) \rightarrow \text{Rat}_{k+l}(\mathbb{C}P^n)$ in Boyer–Mann [1]. We can construct $g_{p,q,i,n}$ from (3-3).

Note that the stable map for Proposition C (i) is compatible with the homology splitting by weights. Using Proposition 3.2, it is easy to show that this map induces an isomorphism in homology, hence is a stable homotopy equivalence. This completes the proof of Proposition C (i). \square

Proof of Proposition C (ii) By a similar argument to the proof of Proposition 3.2, we can calculate $H_*(Y_{k,1}^{2i})$. Then we can construct an unstable map from the right-hand side of Proposition C (ii) to $Y_{k,1}^{2i}$ in the same way as in Proposition C (i). \square

4 Proof of Theorem A and Theorem B

Proposition 4.1 *The homologies of the both sides of Theorem A or Theorem B are isomorphic.*

Proof We prove the proposition about $RX_{k,n}^l$ by induction with making l larger. As in Lemma 3.3, there is a long exact sequence

$$\begin{aligned} \dots \longrightarrow H_*(RX_{k,n}^l) &\longrightarrow H_*(RX_{k,n}^{l+1}) \\ &\longrightarrow H_{*-(l+1)n}(RX_{k-(l+1),n}^0) \xrightarrow{\Theta} H_{*-1}(RX_{k,n}^l) \longrightarrow \dots \end{aligned}$$

This sequence is constructed from the following decomposition as sets

$$RX_{k,n}^{l+1} - RX_{k,n}^l = \coprod_{a+2b=l+1} \text{SP}^a(\mathbb{R}) \times \text{SP}^b(H_+) \times RX_{k-(l+1),n}^0$$

and the fact that $H_c^*(\mathbb{S}P^a(\mathbb{R})) = 0$ for $a \geq 2$, where H_c^* is the cohomology with compact supports.

Assuming that the proposition holds for $l \leq 2i + 1$, we determine $H_*(RX_{k,n}^{2i+2})$. The homomorphism Θ is given as follows. Note that Theorem B is equivalent to

$$(4-1) \quad RX_{k,n}^{2i} \simeq_s X_{[\frac{k}{2}],n}^i \vee \Sigma^{(2i+1)n-1} (RX_{k-2i-1,n}^0 \vee S^0).$$

From inductive hypothesis, we have

$$(4-2) \quad H_{*-(2i+2)n}(RX_{k-2i-2,n}^0) \cong H_{*-(2i+2)n}(\text{Rat}_{[\frac{k}{2}]-(i+1)}(\mathbb{C}P^n)) \oplus \widetilde{H}_{*-(2i+2)n}(\Sigma^{n-1} RX_{k-2i-3,n}^0 \vee S^{n-1})$$

and

$$(4-3) \quad H_{*-1}(RX_{k,n}^{2i+1}) \cong H_{*-1}(X_{[\frac{k}{2}],n}^i).$$

Recall the homomorphism θ in (3-1) with i replaced by $i + 1$. Then $\Theta: (4-2) \rightarrow (4-3)$ is given by mapping the first summand by θ and the second summand by 0. Hence, $H_*(RX_{k,n}^{2i+2})$ is isomorphic to the homology of the right-hand side of (4-1) with i replaced by $i + 1$.

By a similar argument, we can determine $H_*(RX_{k,n}^{2i+1})$ inductively by assuming the truth of the proposition for $l \leq 2i$. This completes the proof of Proposition 4.1. \square

Finally, we construct an unstable map (resp. a stable map) from the right-hand side of Theorem A (resp. (4-1)) to $RX_{k,n}^{2i+1}$ (resp. $RX_{k,n}^{2i}$). First, the unstable map from the right-hand side of Theorem A or the first stable summand in (4-1) is essentially the inclusion

$$X_{q,n}^i \xrightarrow{\eta_{q,i,n}} Y_{2q,n}^{2i} \subset RX_{2q,n}^{2i},$$

where $\eta_{q,i,n}$ is defined in (3-2). Next, the stable map from the second stable summand in (4-1) is constructed in the same way as in $g_{p,q,i,n}$ (see (3-3)) using the fact that $RX_{2i+1,n}^{2i} \simeq S^{(2i+1)n-1}$ (see Example 2.1 (iv)). This completes the proofs of Theorem A and Theorem B. \square

References

- [1] **C P Boyer, B M Mann**, *Monopoles, nonlinear σ models, and two-fold loop spaces*, Comm. Math. Phys. 115 (1988) 571–594 MR933456
- [2] **F R Cohen, R L Cohen, B M Mann, R J Milgram**, *The topology of rational functions and divisors of surfaces*, Acta Math. 166 (1991) 163–221 MR1097023

- [3] **F R Cohen, R L Cohen, B M Mann, R J Milgram**, *The homotopy type of rational functions*, Math. Z. 213 (1993) 37–47 MR1217669
- [4] **Y Kamiyama**, *Polynomial model for homotopy fibers associated with the James construction*, Math. Z. 237 (2001) 149–180 MR1836776
- [5] **J Mostovoy**, *Spaces of rational loops on a real projective space*, Trans. Amer. Math. Soc. 353 (2001) 1959–1970 MR1813601
- [6] **G Segal**, *The topology of spaces of rational functions*, Acta Math. 143 (1979) 39–72 MR533892
- [7] **V A Vassiliev**, *Complements of discriminants of smooth maps: topology and applications*, Translations of Mathematical Monographs 98, American Mathematical Society, Providence, RI (1992) MR1168473 Translated from the Russian by B Goldfarb

*Department of Mathematics, University of the Ryukyus
Nishihara-Cho, Okinawa 903-0213, Japan*

kamiyama@sci.u-ryukyu.ac.jp

Received: 13 August 2004 Revised: 9 May 2005

