Spaces of real polynomials with common roots

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Let $RX^l_{k,n}$ be the space consisting of all $(n+1)$--tuples $(p_0(z), \ldots, p_n(z))$ of monic polynomials over $\mathbb{R}$ of degree $k$ and such that there are at most $l$ roots common to all $p_i(z)$. In this paper, we prove a stable splitting of $RX^l_{k,n}$.

55P15; 55P35, 58D15

1 Introduction

Let $\text{Rat}_k(\mathbb{C} P^n)$ denote the space of based holomorphic maps of degree $k$ from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C} P^n$. The basepoint condition we assume is that $f(\infty) = [1, \ldots, 1]$. Such holomorphic maps are given by rational functions:

$$\text{Rat}_k(\mathbb{C} P^n) = \{(p_0(z), \ldots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \text{ of degree } k \text{ and such that there are no roots common to all } p_i(z)\}.$$

There is an inclusion $\text{Rat}_k(\mathbb{C} P^n) \hookrightarrow \Omega^2_k \mathbb{C} P^n \simeq \Omega^2 S^{2n+1}$. Segal [6] proved that the inclusion is a homotopy equivalence up to dimension $k(2n-1)$. Later, the stable homotopy type of $\text{Rat}_k(\mathbb{C} P^n)$ was described by Cohen et al [2; 3] as follows. Let $\Omega^2 S^{2n+1} \simeq \sqrt{\sum_{1 \leq q \leq k} D_q(S^{2n-1})}$ be Snaith’s stable splitting of $\Omega^2 S^{2n+1}$. Then

$$\text{(1-1) } \text{Rat}_k(\mathbb{C} P^n) \simeq \bigvee_{q=1}^k D_q(S^{2n-1}).$$

In Kamiyama [4], (1–1) was generalized as follows. We set

$$X^l_{k,n} = \{(p_0(z), \ldots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \text{ of degree } k \text{ and such that there are at most } l \text{ roots common to all } p_i(z)\}.$$

In particular, $X^0_{k,n} = \text{Rat}_k(\mathbb{C} P^n)$. Let

$$J^l(S^{2n}) \simeq S^{2n} \cup e^{4n} \cup e^{6n} \cup \ldots \cup e^{2ln} \subset \Omega S^{2n+1}$$

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.227
be the \(l\)th stage of the James filtration of \(\Omega S^{2n+1}\), and let \(W^l(S^{2n})\) be the homotopy theoretic fiber of the inclusion \(J^l(S^{2n}) \hookrightarrow \Omega S^{2n+1}\). We generalize Snaith’s stable splitting of \(\Omega^2 S^{2n+1}\) as follows:

\[
W^l(S^{2n}) \simeq \bigvee_{1 \leq q} D_q S^l(S^{2n}).
\]

Then we have a stable splitting

\[
X^l_{k,n} \simeq \bigvee_{q=1}^k D_q S^l(S^{2n}).
\]

The purpose of this paper is to study the real part \(RX^l_{k,n}\) of \(X^l_{k,n}\) and prove a stable splitting of this. More precisely, let \(RX^l_{k,n}\) be the subspace of \(X^l_{k,n}\) consisting of elements \((p_0(z), \ldots, p_n(z))\) such that each \(p_i(z)\) has real coefficients. Our main results will be stated in Section 2. Here we give a theorem which generalizes (1–1). Since the homotopy type of \(RX^0_{k,1}\) is known (see Example 2.1 (iii)), we assume \(n \geq 2\). In this case, there is an inclusion

\[RX^0_{k,n} \hookrightarrow \Omega S^n \times \Omega^2 S^{2n+1}.
\]

(See Lemma 3.1.)

**Theorem 1.1** For \(n \geq 2\), we define the weight of stable summands in \(\Omega S^n\) as usual, but those in \(\Omega^2 S^{2n+1}\) we define as being twice the usual one. Then \(RX^0_{k,n}\) is stably homotopy equivalent to the collection of stable summands in \(\Omega S^n \times \Omega^2 S^{2n+1}\) of weight \(\leq k\). Hence,

\[
RX^0_{k,n} \simeq \bigvee_{p+2q \leq k} \Sigma^{p(n-1)} D_q (S^{2n-1}) \vee \bigvee_{p=1}^k S^{p(n-1)}.
\]

This paper is organized as follows. In Section 2 we state the main results. We give a stable splitting of \(RX^l_{k,n}\) in Theorem A and Theorem B. In order to prove these theorems, we also consider a space \(Y^l_{k,n}\), which is an open set of \(RX^l_{k,n}\). We give a stable splitting of \(Y^l_{k,n}\) in Proposition C. In Section 3 we prove Proposition C. In Section 4 we prove Theorem A and Theorem B.


2 Main results

We set
\[ Y_{k,n}^I = \{(p_0(z), \ldots, p_n(z)) \in RX_{k,n}^I : \text{there are no real roots common to all } p_i(z)\} \]

The spaces \( Y_{k,n}^I \) and \( RX_{k,n}^I \) are in the following relation:
\[
Y_{k,n}^I \supset Y_{k,n}^{k-1} \supset \cdots \supset Y_{k,n}^1 \supset \cdots \supset Y_{k,n}^0 = Y_{k,n}
\]
\[
RX_{k,n}^I \supset RX_{k,n}^{k-1} \supset \cdots \supset RX_{k,n}^1 \supset \cdots \supset RX_{k,n}^0 \supset RX_{k,n}^0
\]

where each subset is an open set. Moreover, \( Y_{k,n}^{2i+1} = Y_{k,n}^{2i} \). In fact, if \( \alpha \in H_+ \) (where \( H_+ \) is the open upper half-plane) is a root of a real polynomial, then so is \( \bar{\alpha} \in H_- \).

We have the following examples.

Example 2.1

(i) It is proved by Mostovoy [5] that \( Y_{k,1}^k \) consists of \( k + 1 \) contractible connected components.

(ii) The following result is proved by Vassiliev [7]. For \( n \geq 3 \), there is a homotopy equivalence \( Y_{k,n}^k \simeq J^k(S^{n-1}) \), where \( J^k(S^{n-1}) \) is as above the \( k \)th stage of the James filtration of \( \Omega S^n \). For \( n = 2 \), these spaces are stably homotopy equivalent.

(iii) It is proved by Segal [6] that
\[
RX_{k,1,0}^0 \simeq \bigsqcup_{q=0}^{k} \text{Rat}_{\min(q,k-q)}(\mathbb{C} P^1).
\]

(iv) \( RX_{k,n}^{k-1} \cong \mathbb{R}^k \times (\mathbb{R}^{kn})^* \) and \( RX_{k,n}^k \cong \mathbb{R}^{k(n+1)} \).

In fact, \( (p_0(z), \ldots, p_n(z)) \in RX_{k,n}^k \) is an element of \( RX_{k,n}^{k-1} \) if and only if \( p_i(z) \neq p_j(z) \) for some \( i, j \). Hence, the first homeomorphism holds.

Now we state our main results.

**Theorem A** For \( n \geq 1 \) and \( i \geq 0 \), there is a homotopy equivalence
\[
RX_{k,n}^{2i+1} \simeq X_{\left\lfloor \frac{k}{2} \right\rfloor, n}^{i},
\]
where \( \left\lfloor \frac{k}{2} \right\rfloor \) denotes as usual the largest integer \( \leq \frac{k}{2} \).
**Theorem B**  For \( n \geq 1 \) and \( i \geq 0 \), there is a stable homotopy equivalence

\[
RX_{k,n}^{2i} \simeq X_{k,n}^{\lfloor \frac{i}{2} \rfloor} \vee \sum_{p+2q \leq k-2i, 1 \leq p} \sum_{p}^{p(n-1)} D_q(S^{2n-1}) \vee \sum_{p=1}^{k-2i} S^{p(n-1)}.
\]

We study \( RX_{k,n}^l \) by induction with making \( l \) larger. Hence, the induction starts from \( RX_{k,n}^0 \). Recall that \( RX_{k,n}^0 = Y_{k,n}^0 \). We study \( Y_{k,n}^l \) by induction with making \( l \) smaller, where the initial condition is given in Example 2.1 (ii). In fact, we have the following proposition.

**Proposition C**

(i) For \( n \geq 2 \), we define the weight of stable summands in \( \Omega S^n \) as usual, but those in \( W_i S^{2n} \) we define as being twice the usual one. Then \( Y_{k,n}^{2i} \) is stably homotopy equivalent to the collection of stable summands in \( \Omega S^n \times W_i (S^{2n}) \) of weight \( \leq k \). Hence,

\[
Y_{k,n}^{2i} \simeq \bigvee_{p+2q \leq k} \sum_{p}^{p(n-1)} D_q \xi^i (S^{2n}) \vee \sum_{p=1}^{k} S^{p(n-1)}.
\]

(ii) When \( n = 1 \), there is a homotopy equivalence

\[
Y_{k,1}^{2i} \simeq \bigvee_{q=0}^{k} X_{\min(q,k-q)}^{i},
\]

Note that Proposition C (ii) is a generalization of Example 2.1 (i) and (iii).

### 3 Proof of Proposition C

We study the space of continuous maps which contains \( Y_{k,n}^k \) or \( RX_{k,n}^0 \). For simplicity, we assume \( n \geq 2 \). (The case for \( n = 1 \) can be obtained by slight modifications.) Each \( f \in Y_{k,n}^k \) defines a map \( f : S^1 \to \mathbb{R} P^n \), where \( S^1 = \mathbb{R} \cup \{\infty\} \). Hence, there is a natural map

\[
Y_{k,n}^k \to \Omega_{k \mod 2} \mathbb{R} P^n \simeq \Omega S^n.
\]

Example 2.1 (ii) implies that \( Y_{k,n}^k \) is the \( k(n-1) \)-skeleton of \( \Omega S^n \).
On the other hand, let $\text{Map}^T_k(CP^1, CP^n)$ be the space of continuous basepoint-preserving conjugation-equivariant maps of degree $k$ from $CP^1$ to $CP^n$. Then there is an inclusion

$$RX_{k,n}^0 \hookrightarrow \text{Map}^T_k(CP^1, CP^n).$$

**Lemma 3.1** For $n \geq 2$, $\text{Map}^T_k(CP^1, CP^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$.

**Proof** It is easy to see that $\text{Map}^T_k(CP^1, CP^n) \simeq \text{Map}_0^T(CP^1, CP^n)$.

Since $\text{Map}_0^T(CP^1, CP^n)$ can be thought as the space of maps

$$(D^2, S^1, *) \to (CP^n, \mathbb{R}P^n, *)$$

of degree 0, there is a fibration

$$\Omega^2 S^{2n+1} \to \text{Map}_0^T(CP^1, CP^n) \to \Omega S^n.$$ 

This is a pullback of the path fibration $\Omega^2 S^{2n+1} \to \Omega S^n \to \Omega S^{2n+1}$ by the map $\Omega f: \Omega S^n \to \Omega S^{2n+1}$, where $f: S^n \to S^{2n+1}$ is a lift of the inclusion $\mathbb{R}P^n \hookrightarrow CP^n$. Since $f$ is null homotopic, the fibration is trivial. This completes the proof of Lemma 3.1.

Hereafter, every homology is with $\mathbb{Z}/p$–coefficients, where $p$ is a prime. Recall that for $n \geq 2$, we have $H_*(\Omega S^n) \cong \mathbb{Z}/p[x_{n-1}]$. We define the weight of $x_{n-1}$ by $w(x_{n-1}) = 1$. On the other hand, we define the weight of an element of $H_*(X_{k,n}^i)$ as being twice the usual one. For example, let $y_2(l+1)n-1$ be the generator of $H_*(X_{k,n}^i)$ of least degree. The usual weight of $y_2(l+1)n-1$ is $l + 1$, but we reset $w(y_2(l+1)n-1) = 2(l + 1)$.

**Proposition 3.2** For $n \geq 2$, $H_*(Y_{k,n}^{2i})$ is isomorphic to the subspace of $H_*(\Omega S^n \times X_{k,n}^i)$ spanned by monomials of weight $\leq k$.

We prove the proposition from the following lemma.

**Lemma 3.3** We have the following long exact sequence:

$$\cdots \to H_*(Y_{k,n}^{2i-2}) \to H_*(Y_{k,n}^{2i}) \xrightarrow{\phi} H_{*-2i}R(X_{k-2i,n}^0) \to H_{*-1}(Y_{k,n}^{2i-2}) \to \cdots.$$
Proof In [4, Propositions 4.5 and 5.4], we constructed a similar long exact sequence from the fact that
\[ X^l_{k,n} - X^{l-1}_{k,n} = \mathbb{C}^l \times \text{Rat}_{k-1}(\mathbb{C}P^n), \]
where \( \mathbb{C}^l \times \text{Rat}_{k-1}(\mathbb{C}P^n) \) corresponds to the subspace of \( X^l_{k,n} \) consisting of elements \((p_0(z), \ldots, p_n(z))\) such that there are exactly \( l \) roots common to all \( p_i(z) \). The proposition is proved similarly using the fact that
\[ Y^{2i}_{k,n} - Y^{2i-2}_{k,n} \cong \text{SP}^i(H_+ \times R X^0_{k-2i,n}), \]
where \( \text{SP}^i(H_+) \) denotes the \( i \)th symmetric product of \( H_+ \).

Proof of Proposition 3.2 In order to prove Proposition 3.2 by induction, we introduce the following total order to \( Y^{2i}_{k,n} \) for \( k \geq 1 \) and \( i \geq 0 \):
\[
\begin{align*}
(1) & \quad k < k', \\
(2) & \quad k = k' \text{ and } i > i'.
\end{align*}
\]
By Example 2.1 (ii), Proposition 3.2 holds for \( Y^k_{k,n} \). Assuming that Proposition 3.2 holds for \( Y^{2i}_{k,n} \) and \( R X^0_{k-2i,n} \), we prove for \( Y^{2i-2}_{k,n} \). We have the following long exact sequence:
\[
\cdots \to H_*\left( X^{i-1}_{[\frac{1}{2}],n} \right) \to H_*\left( X^i_{[\frac{1}{2}],n} \right) \\

\xrightarrow{\psi} H_{*-2in}\left( \text{Rat}_{[\frac{1}{2}],-i}(\mathbb{C}P^n) \right) \to H_{*-1}\left( X^{i-1}_{[\frac{1}{2}],n} \right) \to \cdots.
\]
For \( n \geq 2 \), we consider the homomorphism
\[
1 \otimes \psi: H_*(\Omega S^n) \otimes H_*\left( X^i_{[\frac{1}{2}],n} \right) \to H_*(\Omega S^n) \otimes H_{*-2in}\left( \text{Rat}_{[\frac{1}{2}],-i}(\mathbb{C}P^n) \right).
\]
Restricting the domain to \( H_*(Y^{2i}_{k,n}) \), we obtain the homomorphism \( \phi \) in Lemma 3.3. Now it is easy to prove Proposition 3.2.

Proof of Proposition C (i) We construct a stable map from the right-hand side of Proposition C (i) to \( Y^{2i}_{k,n} \). Since our constructions are similar, we construct a stable map \( g_{p,q,i,n}: \Sigma^{p(n-1)} D q^i z (S^{2n}) \to Y^{2i}_{k,n} \). First, using the fact that \( R X^0_{1,n} \cong S^{n-1} \) (see Example 2.1 (iv)), there is a stable map \( f_{p,n}: S^{p(n-1)} \to R X^0_{p,n} \). Second, there is a stable section \( e_{q,i,n}: D q^i z (S^{2n}) \to X^i_{q,n} \). Third, there is an inclusion
\[
\eta_{q,i,n}: X^i_{q,n} \hookrightarrow Y^{2i}_{2q,n}.
\]
To construct this, we fix a homeomorphism $h : \mathbb{C} \to H_+$. For $(p_0(z), \ldots, p_n(z)) \in X^l_{q,n}$, we write $p_j(z) = \prod_{s=1}^{q} (z - \alpha_{s,j})$. Then we set

$$\eta_{q,i,n}(p_0(z), \ldots, p_n(z)) = \left( \prod_{s=1}^{q} (z - h(\alpha_{s,0}))(z - \overline{h(\alpha_{s,0}))}) \right) \cdots \left( \prod_{s=1}^{q} (z - h(\alpha_{s,n}))(z - \overline{h(\alpha_{s,n}))}) \right).$$

Now consider the following composite of maps

$$(3\rightarrow 3) \quad S^{p(n-1)} \times D^4_{q,i} (S^{2n}) \overset{f_{p,n} \times (\eta_{q,i,n} \circ \eta_{q,i,n})}{\longrightarrow} RX^0_{p,n} \times Y_{2q,n}^{2i} \overset{\mu}{\longrightarrow} Y_{p+2q,n}^{2i} \hookrightarrow Y_{k,n}^{2i}.$$ 

where $\mu$ is a loop sum which is constructed in the same way as in the loop sum $\text{Rat}_k(\mathbb{C}P^n) \times \text{Rat}_l(\mathbb{C}P^n) \to \text{Rat}_{k+l}(\mathbb{C}P^n)$ in Boyer–Mann [1]. We can construct $g_{p,q,i,n}$ from (3–3).

Note that the stable map for Proposition C (i) is compatible with the homology splitting by weights. Using Proposition 3.2, it is easy to show that this map induces an isomorphism in homology, hence is a stable homotopy equivalence. This completes the proof of Proposition C (i).

**Proof of Proposition C (ii)** By a similar argument to the proof of Proposition 3.2, we can calculate $H_*(Y_{k,1}^{2i})$. Then we can construct an unstable map from the right-hand side of Proposition C (ii) to $Y_{k,1}^{2i}$ in the same way as in Proposition C (i).

4 Proof of Theorem A and Theorem B

**Proposition 4.1** The homologies of the both sides of Theorem A or Theorem B are isomorphic.

**Proof** We prove the proposition about $RX^l_{k,n}$ by induction with making $l$ larger. As in Lemma 3.3, there is a long exact sequence

$$\cdots \to H_*(RX^l_{k,n}) \to H_*(RX^{l+1}_{k,n}) \to H_{*-l+1}(RX^0_{k-(l+1),n}) \to \cdots$$

This sequence is constructed from the following decomposition as sets

$$RX^{l+1}_{k,n} \setminus RX^l_{k,n} = \bigsqcup_{a+2b = l+1} \text{SP}^a(\mathbb{R}) \times \text{SP}^b(H_+) \times RX^0_{k-(l+1),n}$$
and the fact that $H^a_c(S^a(\mathbb{R})) = 0$ for $a \geq 2$, where $H^a_c$ is the cohomology with compact supports.

Assuming that the proposition holds for $l \leq 2i + 1$, we determine $H^*_{s}(R^{2i+2}X_{k,n})$. The homomorphism $\Theta$ is given as follows. Note that Theorem B is equivalent to

\[(4-1) \quad R^{2i}X_{k,n} \simeq X^i_{[\frac{1}{2}],n} \vee \Sigma^{(2i+1)n-1} (R^{0}X^0_{k-2i-1,n} \vee S^0).\]

From inductive hypothesis, we have

\[(4-2) \quad H^{-(2i+2)n}(R^{0}X^0_{k-2i-2,n}) \cong H^{-(2i+2)n}(\text{Rat}_{[\frac{1}{2}]}(\Sigma^{n-1} R^{0}X^0_{k-2i-3,n} \vee S^{n-1}))\]

and

\[(4-3) \quad H^{-1}(R^{2i+1}X_{k,n}) \cong H^{-1}(X^i_{[\frac{1}{2}],n}).\]

Recall the homomorphism $\theta$ in (3-1) with $i$ replaced by $i + 1$. Then $\Theta: (4-2) \to (4-3)$ is given by mapping the first summand by $\theta$ and the second summand by $0$. Hence, $H^*_s(R^{2i+2}X_{k,n})$ is isomorphic to the homology of the right-hand side of (4-1) with $i$ replaced by $i + 1$.

By a similar argument, we can determine $H^*_s(R^{2i+1}X_{k,n})$ inductively by assuming the truth of the proposition for $l \leq 2i$. This completes the proof of Proposition 4.1.

Finally, we construct an unstable map (resp. a stable map) from the right-hand side of Theorem A (resp. (4-1)) to $R^{2i+1}X_{k,n}$ (resp. $R^{2i}X_{k,n}$). First, the unstable map from the right-hand side of Theorem A or the first stable summand in (4-1) is essentially the inclusion

\[X^i_{q,n} \xrightarrow{\eta_{q,i,n}} Y^{2i}_{2q,n} \subset R^{2i}X_{2q,n},\]

where $\eta_{q,i,n}$ is defined in (3-2). Next, the stable map from the second stable summand in (4-1) is constructed in the same way as in $g_{p,q,i,n}$ (see (3-3)) using the fact that $R^{2i+1}_{2i+1,n} \simeq S^{(2i+1)n-1}$ (see Example 2.1 (iv)). This completes the proofs of Theorem A and Theorem B.

References


Geometry & Topology Monographs, Volume 10 (2007)
Spaces of real polynomials with common roots


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Received: 13 August 2004 Revised: 9 May 2005