On fibrations related to real spectra

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We consider real spectra, collections of $\mathbb{Z}/(2)$–spaces indexed over $\mathbb{Z} \oplus \mathbb{Z}\alpha$ with compatibility conditions. We produce fibrations connecting the homotopy fixed points and the spaces in these spectra. We also evaluate the map which is the analogue of the forgetful functor from complex to reals composed with complexification. Our first fibration is used to connect the real $2^{n+2}(2^n - 1)$–periodic Johnson–Wilson spectrum $ER(n)$ to the usual $2(2^n - 1)$–periodic Johnson–Wilson spectrum, $E(n)$. Our main result is the fibration $\Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n)$, where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.

1 Introduction

In 1968, Landweber [11] introduced the idea of a real complex cobordism by taking the homotopy fixed points of complex cobordism under complex conjugation. A few years later this theory was studied again by Araki and Murayama [1; 2; 3]. Recently there has been a flurry of activity around this theory by Hu and Kriz [4; 5; 6; 7; 8; 9].

In [6], Hu and Kriz produce real versions, $ER(n)$, of the Johnson–Wilson spectra $E(n)$ (see Johnson and Wilson [10]) and compute their homotopy. The homotopy of $E(n)$ is $\mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n^{\pm 1}]$. $E(n)$ is periodic of period $|v_n| = 2(2^n - 1)$, $ER(n)$ is periodic of period $\left|v_n^{2^{n+1}}\right| = 2^{n+2}(2^n - 1)$, and the construction gives maps of spectra: $ER(n) \to E(n)$.

In the case $n = 1$ this is just the map $KO_{(2)} \to KU_{(2)}$ and Wood identified the fibre as $\Sigma KO_{(2)}$.

The main purpose of this paper is to identify the fibre of $ER(n) \to E(n)$, producing the fibration:

$$(1) \qquad \Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n).$$

where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. These fibrations should make these theories much more accessible.

Let $\mathbb{E}$ be a real spectrum as defined in [6]. In particular, $\mathbb{E}$ is given by a collection of pointed $\mathbb{Z}/(2)$–spaces $\mathbb{E}_V$ indexed by the representation ring $RO(\mathbb{Z}/(2))$ of the group $\mathbb{Z}/(2)$. Recall that $RO(\mathbb{Z}/(2)) = \mathbb{Z} \oplus \mathbb{Z}\alpha$, where α is the sign representation. Moreover, we require that the spaces $\mathbb{E}_V$ be compatible in the following sense.

Given a representation U, and a pointed $\mathbb{Z}/(2)$–space X, let $\Omega^U X$ denote the space $\mathrm{Map}_*(S^U, X)$, where S^U is the one-point compactification of U. The space $\Omega^U X$ has an induced diagonal action of the group $\mathbb{Z}/(2)$. For the spectrum $\mathbb{E}$, we require the existence of a family of equivariant homeomorphisms $\alpha_{U,V}\colon \Omega^U \mathbb{E}_{U \oplus V} \to \mathbb{E}_V$, that satisfy obvious compatibility.

A multiplicative real spectrum $\mathbb{E}$ is one that admits a multiplication preserving the real structure (see [6]).

Example 1.1 The real complex bordism spectrum $\mathbb{MU}$ is defined as follows. Let $MU(n)$ denote the Thom space of the universal bundle over $BU(n)$. Complex conjugation induces an action of $\mathbb{Z}/(2)$ on $MU(n)$. Define $\mathbb{MU}_V$ as the space $\varinjlim_n \Omega^{n(1+\alpha)-V} MU(n)$ for $V \in RO(\mathbb{Z}/(2))$. Notice that $n(1 + \alpha) - V$ is a well defined representation of $\mathbb{Z}/(2)$ for sufficiently large values of n. It is left to the reader to verify that $\mathbb{MU}$ has the properties of a multiplicative real spectrum.

Example 1.2 The Brown–Peterson spectrum has a real analogue $\mathbb{BP}$. The real Johnson–Wilson spectra $\mathbb{E}(n)$ may also be defined along similar lines [6]. These spectra are in fact multiplicative real spectra. $\mathbb{E}(1)$ is 2–localized real K–theory of Atiyah [6].

We will use the notation $\mathbb{ER}_V$ to denote the homotopy fixed points of the $\mathbb{Z}/(2)$–action on $\mathbb{E}_V$. Notice that for a fixed $V \in RO(\mathbb{Z}/(2))$, the collection of spaces $\{\mathbb{ER}_{n+V}, n \in \mathbb{Z}\}$ form a spectrum in the usual sense. We shall abuse notation and refer to the spectra $\{\mathbb{ER}_{n+V}, n \in \mathbb{Z}\}$ and $\{\mathbb{E}_{n+V}, n \in \mathbb{Z}\}$ as the spectra $\mathbb{ER}_V$ and $\mathbb{E}_V$ respectively. The purpose of this paper is to relate $\mathbb{ER}_V$ to $\mathbb{E}_V$ via a fibration. Of particular interest to us will be the case when the spectrum $\mathbb{E}$ is $\mathbb{E}(n)$. We need the following result which we assume is well known to the experts:

Proposition 1.3 *There are fibrations of spectra:*

$$\mathbb{ER}_{V-\alpha} \xrightarrow{a} \mathbb{ER}_V \xrightarrow{\iota} \mathbb{E}_V, \qquad \mathbb{E}_V \xrightarrow{1+\sigma} \mathbb{ER}_V \xrightarrow{a} \mathbb{ER}_{V+\alpha},$$

where the map a is induced by the map $a\colon S^0 \longrightarrow S^\alpha$ given by the inclusion of the poles. The map ι is the standard inclusion, and the map $(1+\sigma)$ is a lift of the Norm map on $\mathbb{E}_V$. Moreover, if $\mathbb{E}$ is a multiplicative real spectrum, then $\mathbb{ER}_V$ is a $\mathbb{ER}_0$–module spectrum for all V, and the above fibrations are fibrations of $\mathbb{ER}_0$–module spectra.

Remark 1.4 On the level of individual spaces we have fibrations

$$\mathbb{ER}_{m+(n-1)\alpha} \to \mathbb{ER}_{m+n\alpha} \to \mathbb{E}_{m+n\alpha}.$$

This is a great help to computations as we hope to demonstrate in a future paper.

Observe that the spaces $\mathbb{E}_{V-1} = \Omega\mathbb{E}_V$ and $\mathbb{E}_{V-\alpha} = \Omega^\alpha\mathbb{E}_V$ are homeomorphic. (Actually, $\mathbb{E}_{m+n\alpha}$ and $\mathbb{E}_{m'+n'\alpha}$ are the same when $m+n = m'+n'$.) In the statement of the next theorem, we will use this homeomorphism to identify the two spaces. Note, however that the action of $\mathbb{Z}/(2)$ on the two spaces is different. If we let σ denote the action of the generator of $\mathbb{Z}/(2)$ on $\mathbb{E}_{V-1}$, and $\tilde{\sigma}$ the action on $\mathbb{E}_{V-\alpha}$, then the two actions are related via $\tilde{\sigma} = \sigma\alpha_* = -\sigma$.

Now consider the boundary map. This map ∂ is defined as the map $\mathbb{E}_{V-1} \to \mathbb{ER}_{V-\alpha}$ given by looping back the first fibration above composed with the map $\mathbb{ER}_{V-\alpha} \to \mathbb{E}_{V-\alpha}$ given by the inclusion of the fixed points. Therefore

$$\partial\colon \mathbb{E}_{V-1} \longrightarrow \mathbb{E}_{V-\alpha}.$$

We have the following proposition.

Proposition 1.5 *Let $\mathbb{E}_{V-1}$ be identified with the space $\mathbb{E}_{V-\alpha}$ as explained above. Then the map ∂ is given by $\partial = Id - \sigma = Id + \tilde{\sigma}$.*

The standard example of this result is the composition $KU \to KO \to KU$ and this is just a generalization of it. The boundary is the composition of two maps. The first can be thought of as forgetting the complex structure and looking only at the underlying real structure. The next map can be thought of as complexification. This boundary map comes in useful in calculations we hope will appear in a future paper.

Our primary interest is the case when $\mathbb{E} = \mathbb{E}(n)$. The following theorem uses the computation of the homotopy of $\mathbb{ER}(n)$ given in [6].

Theorem 1.6 *There exist nontrivial elements $x(n) \in \pi_{\lambda(n)}(\mathbb{ER}(n)_0)$, where $\lambda(n)$ is the integer defined by $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$, such that one has a fibration of $\mathbb{ER}(n)_0$–module spectra:*

$$\Sigma^{\lambda(n)}\mathbb{ER}(n)_V \xrightarrow{x(n)} \mathbb{ER}(n)_V \xrightarrow{\iota} \mathbb{E}(n)_V.$$

Remark 1.7 An interesting special case of the above theorem is when $n = 1$, and $V = 0$. Note that $\mathbb{E}(1) = KU_{(2)}$, and hence $\mathbb{ER}(1) = KO_{(2)}$. Moreover, the element $x(1)$ is none other than η. Hence one reproduces a well-known result

$$\Sigma KO_{(2)} \xrightarrow{\eta} KO_{(2)} \longrightarrow KU_{(2)}.$$

More generally, fixing $V = 0$, we get the fibration (1).

Our dependence on the published work of Hu and Kriz is obvious. In addition, they recently informed us they can prove a generalization of our result. By working with the universal example, ie inverting v_n in MU, they can show that any theory with v_n inverted has the same fibration we have for $ER(n)$. The proof is the same. Note that this works for their version of real Morava K–theory in [6] making it a much more interesting theory but unfortunately still not a ring theory.

2   The fibrations

In this section we will show the existence of the two fibration given in the introduction.

Let S^α denote the one-point compactification of the one dimensional nontrivial representation of $\mathbb{Z}/(2)$. Notice that one has a $\mathbb{Z}/(2)$–equivariant cofibration:

$$(2) \qquad\qquad \mathbb{Z}/(2)^+ \longrightarrow S^0 \xrightarrow{\ a\ } S^\alpha$$

where the map $\mathbb{Z}/(2)^+ \to S^0$ is given by the pinch map. Let $\mathbb{E}$ be a real spectrum, and for the purposes of this section, let $\mathbb{E}_V$ denote the spectrum given by the collection of spaces $\{\mathbb{E}_{n+V}, n \in \mathbb{Z}\}$. Smashing the cofibration (2) yields a cofibration of equivariant spectra

$$(3) \qquad\qquad \mathbb{E}_V \wedge \mathbb{Z}/(2)^+ \longrightarrow \mathbb{E}_V \xrightarrow{\ a\ } \mathbb{E}_{V+\alpha}.$$

Notice that $\mathbb{Z}/(2)^+$ may be identified with $S^0 \vee S^0$, with the $\mathbb{Z}/(2)$ action given by the twist map. Under this identification, the pinch map $\mathbb{Z}/(2)^+ \to S^0$ corresponds to the fold map $S^0 \vee S^0 \to S^0$. Hence, in the category of spectra, $\mathbb{E}_V \wedge \mathbb{Z}/(2)^+$ may be identified with $\mathbb{E}_V \vee \mathbb{E}_V = \mathbb{E}_V \times \mathbb{E}_V$ with the $\mathbb{Z}/(2)$ action given by $\tilde{\sigma}(x, y) = (\sigma(y), \sigma(x))$, where σ denotes the generator of $\mathbb{Z}/(2)$. Furthermore, the pinch map $\mathbb{E}_V \wedge \mathbb{Z}/(2)^+ \to \mathbb{E}_V$ corresponds to the sum map $\mathbb{E}_V \times \mathbb{E}_V \xrightarrow{\ +\ } \mathbb{E}_V$.

Consider the twisted diagonal map

$$\Delta \colon \mathbb{E}_V \longrightarrow \mathbb{E}_V \times \mathbb{E}_V, \qquad \Delta(x) = (x, \sigma(x)).$$

Notice that $\tilde{\sigma}\Delta(x) = \Delta(x)$. From this it follows easily that Δ lifts to an equivalence $\mathbb{E}_V \to (\mathbb{E}_V \times \mathbb{E}_V)^{h\mathbb{Z}/(2)}$. Putting these results together, we get the following proofs.

Proof of the second fibration in Proposition 1.3.   Taking homotopy fixed points of (3) yields another fibration. If we identify $(\mathbb{E}_V \wedge \mathbb{Z}/(2)^+)^{h\mathbb{Z}/(2)}$ with $\mathbb{E}_V$, then the map $(\mathbb{E}_V \wedge \mathbb{Z}/(2)^+)^{h\mathbb{Z}/(2)} \to (\mathbb{E}_V)^{h\mathbb{Z}/(2)}$ is a lift of $(1 + \sigma)$. $\qquad\square$

Proof of the first fibration in Proposition 1.3. For the second fibration, one considers the Spanier–Whitehead dual of (2):

$$S^{-\alpha} \xrightarrow{a} S^0 \longrightarrow \mathbb{Z}/(2)^+$$

where the map $S^0 \to \mathbb{Z}/(2)^+ = S^0 \vee S^0$ corresponds to the diagonal. Smashing with $\mathbb{E}_V$ yields an equivariant fibration

$$\mathbb{E}_{V-\alpha} \xrightarrow{a} \mathbb{E}_V \longrightarrow \mathbb{E}_V \times \mathbb{E}_V.$$

Taking homotopy fixed points of this fibration and making the identifications described earlier, we get the remaining fibration:

$$\mathbb{E}\mathbb{R}_{V-\alpha} \xrightarrow{a} \mathbb{E}\mathbb{R}_V \xrightarrow{\iota} \mathbb{E}_V.$$

To complete the proof one simply observes that all the above constructions respect the $\mathbb{E}\mathbb{R}_0$–module structure if $\mathbb{E}$ is a multiplicative real spectrum. $\qquad\square$

3 The boundary map

In this section, we analyse the boundary map for the above fibrations. This map ∂ is defined as the composite of the map $\mathbb{E}_{V-1} \to \mathbb{E}\mathbb{R}_{V-\alpha}$ given by looping back the fibration constructed in the previous section, and the map $\mathbb{E}\mathbb{R}_{V-\alpha} \to \mathbb{E}_{V-\alpha}$ given by the inclusion of the fixed points. Therefore

$$\partial\colon \mathbb{E}_{V-1} \longrightarrow \mathbb{E}_{V-\alpha}.$$

The map ∂ may be explicitly constructed as follows. Consider the composite equivariant map given by the fold map followed by the pinch map:

$$(4) \qquad\qquad \mathbb{Z}/(2)^+ \wedge S^\alpha \xrightarrow{f} S^\alpha \xrightarrow{p} \mathbb{Z}/(2)^+ \wedge S^1.$$

Notice that the Spanier-Whitehead dual of the pinch map $p\colon S^\alpha \to \mathbb{Z}/(2)^+ \wedge S^1$ is the difference map $(-)\colon Z/(2)^+ \wedge S^{-1} \to S^{-\alpha}$. Taking the Spanier-Whitehead dual of the composite (4) yields

$$\mathbb{Z}/(2)^+ \wedge S^{-1} \xrightarrow{(-)} S^{-\alpha} \xrightarrow{\Delta} \mathbb{Z}/(2)^+ \wedge S^{-\alpha}.$$

On smashing the above with $\mathbb{E}_V$, we obtain the composite map

$$\Delta(-)\colon \mathbb{E}_{V-1} \times \mathbb{E}_{V-1} \xrightarrow{(-)} \mathbb{E}_{V-\alpha} \xrightarrow{\Delta} \mathbb{E}_{V-\alpha} \times \mathbb{E}_{V-\alpha}.$$

From the previous section, we can see that there is a commutative diagram

$$
\begin{array}{ccc}
\mathbb{E}_{V-1} & \xrightarrow{\ \partial\ } & \mathbb{E}_{V-\alpha} \\
\downarrow{\scriptstyle (1,\sigma)} & & \downarrow{\scriptstyle (1,\tilde{\sigma})} \\
\mathbb{E}_{V-1} \times \mathbb{E}_{V-1} & \xrightarrow{\ \Delta(-)\ } & \mathbb{E}_{V-\alpha} \times \mathbb{E}_{V-\alpha}
\end{array}
$$

where σ denotes the $\mathbb{Z}/(2)$–action on $\mathbb{E}_{V-1}$, and $\tilde{\sigma}$ denotes the $\mathbb{Z}/(2)$–action on $\mathbb{E}_{V-\alpha}$. Recall that the spaces $\mathbb{E}_{V-1} = \Omega\mathbb{E}_V$ and $\mathbb{E}_{V-\alpha} = \Omega^\alpha\mathbb{E}_V$ are homeomorphic and the above two actions are related via $\tilde{\sigma} = \sigma\alpha_*$. Since α_* is homotopic to the inversion, we have $\tilde{\sigma} = -\sigma$. From a diagram chase we get $\partial(x) = x - \sigma(x) = x + \tilde{\sigma}(x)$.

4 The case of $\mathbb{E}(n)$

We recall the computation (via the Borel spectral sequence) of the homotopy of $\mathbb{BPR}$ given in [6], and described in the form we need in [4]. We will reproduce the Borel spectral sequence with $\mathbb{BP}$ replaced by $\mathbb{E}(n)$. The E_2–term of the Borel spectral sequence for $\mathbb{E}(n)$ is given by

$$
E_2 = \mathbb{Z}_{(2)}[v_k, v_n^{\pm 1}, a, \sigma^{\pm 2}]/(2a), \quad n > k \geq 0, \quad v_0 = 2.
$$

The bidegrees of the generators are given by

$$
|a| = -\alpha, \quad |v_k| = (2^k - 1)(1 + \alpha), \quad |\sigma^2| = 2(\alpha - 1).
$$

The differentials are given by comparing with the Borel spectral sequence converging to the homotopy of $\mathbb{BPR}$. In particular, the elements v_k and a are permanent cycles, and the nontrivial differentials are

$$
d_{2^{k+1}-1}(\sigma^{-2^k}) = v_k a^{2^{k+1}-1}, \quad 0 < k \leq n.
$$

Using the methods of [6], [4], we notice that the E_∞–term for the homotopy of $\mathbb{ER}(n)$ is given by the following ring:

$$
\mathbb{Z}_{(2)}[v_k\sigma^{l2^{k+1}}, a, v_n^{\pm 1}, \sigma^{\pm 2^{n+1}}]/I, \quad n > k \geq 0, \quad l \in \mathbb{Z}
$$

where I is the ideal generated by the relations:

$$
v_0 = 2,
$$
$$
a^{2^{k+1}-1}v_k\sigma^{l2^{k+1}} = 0,
$$
$$
v_m\sigma^{l2^{m+1}}.v_k\sigma^{s2^{m-k}2^{k+1}} = v_k.v_m\sigma^{(l+s)2^{m+1}} \quad m \geq k.
$$

The bidegrees of the generators are given by

$$|a| = -\alpha, \quad \left|v_k \sigma^{l2^{k+1}}\right| = (2^k - 1)(1 + \alpha) + l2^{k+1}(\alpha - 1).$$

Comparing with the homotopy of $\mathbb{BPR}$, we notice that there are no extension problems, and so the above is in fact isomorphic to the homotopy of $\mathbb{ER}(n)$.

Now consider the element

$$y(n) = v_n^{2^n-1}\sigma^{-2^{n+1}(2^{n-1}-1)}, \quad y(n) \in \pi_{\lambda(n)}(\mathbb{ER}(n)_{-\alpha})$$

where $\lambda(n)$ is the integer $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. The element $y(n)$ is clearly invertible in the above ring. Hence we get the following claim.

Claim 4.1 *Multiplication by the element $y(n)$ yields an equivalence of $\mathbb{ER}_0$–module spectra:*

$$\Sigma^{\lambda(n)}\mathbb{ER}(n)_V \xrightarrow{y(n)} \mathbb{ER}(n)_{V-\alpha}.$$

We define the element $x(n)$ to be the element

$$x(n) = a.y(n), \quad x(n) \in \pi_{\lambda(n)}(\mathbb{ER}(n)_0).$$

This claim, along with the first fibration given in Proposition 1.3 yields the proof of Theorem 1.6.

Remark 4.2 The spectrum $\mathbb{ER}(n)_0$ is periodic with period $2^{n+2}(2^n - 1)$ generated by the homotopy element $v_n^{2^{n+1}}\sigma^{-2^{n+1}(2^n-1)}$.

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