Geometry & Topology Monographs 10 (2007) 245–279

Goodwillie towers and chromatic homotopy: an overview

NICHOLAS J KUHN

This paper is based on talks I gave in Nagoya and Kinosaki in August of 2003. I survey, from my own perspective, Goodwillie's work on towers associated to continuous functors between topological model categories, and then include a discussion of applications to periodic homotopy as in my work and the work of Arone–Mahowald.

55P43, 55P47, 55N20; 18G55

1 Introduction

About two decades ago, Tom Goodwillie began formulating his calculus of homotopy functors as a way to organize and understand arguments being used by him and others in algebraic K–theory. Though it was clear early on that his general theory offered a new approach to the concerns of classical homotopy, and often shed light on older approaches, it is relatively recently that its promise has been begun to be realized. This has been helped by the recent publication of the last of Goodwillie's series [14; 15; 16], and by the support of many timely new results in homotopical algebra and localization theory allowing his ideas to be applied more widely.

At the Workshop in Algebraic Topology held in Nagoya in August 2002, I gave a series of three talks entitled 'Goodwillie towers: key features and examples', in which I reviewed the aspects of Goodwillie's work that I find most compelling for homotopy theory. A first goal of this paper is to offer a written account of my talks. As in my talks, I focus on towers associated to functors, ie the material of [16]. As 'added value' in this written version, I include some fairly extensive comments about the general model category requirements for running Goodwillie's arguments.

At the Conference on Algebraic Topology held in Kinosaki just previous to the workshop, I discussed a result of mine [29] that says that Goodwillie towers of functors of spectra split after periodic localization. This is one of a number of ways discovered so far in which Goodwillie calculus interacts beautifully with homotopy as organized by the chromatic point of view; another is the theorem of Greg Arone and Mark Mahowald [4]. A second goal is to survey these results as well, and point to directions for the future.

Published: 29 January 2007 DOI: 10.2140/gtm.2007.10.245

The paper is organized as follows.

In Section 2, I describe the major properties of Goodwillie towers associated to continuous functors from one topological model category to another. In Section 3, I discuss model category prerequisites. The basic facts about cubical diagrams and polynomial functors are reviewed in Section 4. The construction of the Goodwillie tower of a functor is given in Section 5, and I sketch the main ideas behind the proofs that towers have the properties described in Section 2.

In Section 6, I discuss some of my favorite examples: Arone's model for the tower of the functor sending a space X to $\Sigma^\infty \operatorname{Map}(K, Z)$ [2], the tower for the functor sending a spectrum X to $\Sigma^\infty \Omega^\infty X$, the tower of the identity functor on the category of commutative augmented S–algebras, and tower for the identity functor on the category of topological spaces as analyzed by Johnson, Arone, Mahowald and Dwyer [23; 4; 3]. Besides organizing these in a way that I hope readers will find helpful, I have also included some remarks that haven't appeared elsewhere, eg I note (in Example 6.3) that the bottom of the tower for $\Sigma^\infty \Omega^\infty X$ can be used to prove the Kahn–Priddy Theorem, 'up to one loop'.

The long Section 7 begins with a discussion of how Goodwillie towers interact with Bousfield localization. Included is a simple example (see Example 7.4) that shows that the composite of homogeneous functors between spectra need not again be homogeneous. In the remainder of the section, I survey three striking results in which the Goodwillie towers discussed in Section 6 interact with chromatic homotopy theory: my theorems on splitting localized towers [29] and calculating the Morava K–theories of infinite loopspaces [30], and Arone and Mahowald's work on calculating the unstable v_n–periodic homotopy groups of spheres [4]. All of these relate to telescopic functors Φ_n from spaces to spectra constructed a while ago by Pete Bousfield and I [7; 27; 9] using the Nilpotence and Periodicity Theorems of Devinatz, Hopkins and Smith [11; 19]. This suggests that Goodwillie calculus can be used to further explore these curious functors. Included in this section, as an application of my work in [30], is an outline of a new way to possibly find a counterexample to the Telescope Conjecture.

The Kinosaki conference was on the occasion of Professor Nishida's 60th birthday, and I wish to both offer him my hearty congratulations, and thank him for his kind interest in my research over the years. Many thanks also to Noriko Minami and the other conference organizers for their hospitality.

This research was partially supported by a grant from the National Science Foundation.

2 Properties of Goodwillie towers

The basic problem that Goodwillie calculus is designed to attack is as follows. One has a homotopy functor

$$F\colon \mathcal{C} \to \mathcal{D}$$

between two categories in which one can do homotopy. One wishes to understand the homotopy type of $F(X)$, perhaps for some particular $X \in \mathcal{C}$.

Goodwillie's key idea is to use the *functoriality* as X varies, to construct a canonical *polynomial* resolution of $F(X)$ as a functor of X.

The first thing to specify is what is meant by categories in which one can do homotopy theory. In Goodwillie's papers, these are $\mathcal{T}$, the category of pointed topological spaces, or $\mathcal{S}$, an associated category of spectra (eg the S–modules of Elmendorf–Kriz–Mandell–May [13]), or variants of these, eg $\mathcal{T}_Y$, the category of spaces over and under a fixed space Y. But the arguments and constructions of Goodwillie [16] are written in a such a manner that they apply to situations in which $\mathcal{C}$ and $\mathcal{D}$ are suitably nice based model categories: in Section 3, we will spell out precisely what we mean.

Among all functors $F\colon \mathcal{C} \to \mathcal{D}$, some will be d*–excisive* (or *polynomial of degree at most* d). This will be carefully explained in Section 4.2; we note that a 0–excisive functor is one that is homotopically constant, a functor is 1–excisive if it sends homotopy pushout squares to homotopy pullback squares, and a $(d-1)$–excisive functor is also d–excisive.

Goodwillie's first theorem says that any F admits a canonical polynomial resolution.

Theorem 2.1 [16, Theorem 1.8] *Given a homotopy functor $F\colon \mathcal{C} \to \mathcal{D}$ there exists a natural tower of fibrations under $F(X)$,*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_2F(X) \arrow[d, "p_2"] \\
 & P_1F(X) \arrow[d, "p_1"] \\
F(X) \arrow[uur, "e_2"] \arrow[ur, "e_1"] \arrow[r, "e_0"] & P_0F(X),
\end{tikzcd}$$

such that

(1) $P_d F$ is d–excisive, and

(2) $e_d\colon F \to P_d F$ is the universal weak natural transformation to a d–excisive functor.

Let us explain what we mean by property (2). By a weak natural transformation $f\colon F \to G$, we mean a pair of natural transformations $F \xrightarrow{g} H \xleftarrow{h} G$ such that $H(X) \xleftarrow{h} G(X)$ is a weak equivalence for all X. Note that a weak natural transformation induces a well defined natural transformation between functors taking values in the associated homotopy category. Property (2) means that, given any d–excisive functor G, and natural transformation $f\colon F \to G$, there exists a weak natural transformation $g\colon P_d F \to G$ such that, in the homotopy category of $\mathcal{D}$,

$$\begin{tikzcd} F(X) \arrow[r, "e_d(X)"] \arrow[d, "f(X)"'] & P_d F(X) \arrow[dl, "g(X)"] \\ G(X) & \end{tikzcd}$$

commutes for all $X \in \mathcal{C}$, and any two such g agree.

A very useful property of the P_d construction is the following Lemma.

Lemma 2.2 *Given natural transformations $F \to G \to H$, if*

$$F(X) \to G(X) \to H(X)$$

is a fiber sequence for all X, then so is

$$P_d F(X) \to P_d G(X) \to P_d H(X).$$

Let $D_d F\colon \mathcal{C} \to \mathcal{D}$ be defined by letting $D_d F(X)$ be the homotopy fiber of $P_d F(X) \to P_{d-1} F(X)$. The lemma and theorem formally imply that $D_d F$ is *homogeneous of degree d*: it is d–excisive, and $P_{d-1} D_d F(X) \simeq *$ for all X.

When $\mathcal{D}$ is $\mathcal{T}$, Goodwillie discovered a remarkable fact: these fibers are canonically infinite loopspaces. For a general $\mathcal{D}$, we let $\mathcal{S}(\mathcal{D})$ be the associated category of '$\mathcal{D}$–spectra' (see Section 3), and Goodwillie's second theorem then goes as follows.

Theorem 2.3 [16, Theorem 2.1] *Let $F\colon \mathcal{C} \to \mathcal{D}$ be homogeneous of degree d. Then there is a naturally defined homogeneous degree d functor $F^{st}\colon \mathcal{C} \to \mathcal{S}(\mathcal{D})$, such that, for all $X \in \mathcal{C}$, there is a weak equivalence*

$$F(X) \simeq \Omega^\infty(F^{st}(X)).$$

The category $\mathcal{S}(\mathcal{D})$ is an example of a *stable* model category. In a manner similar to results in the algebra literature, Goodwillie relates homogenous degree d functors landing in a stable model category to symmetric multilinear ones. A functor $L\colon \mathcal{C}^d \to \mathcal{D}$ is d–linear if it is homogeneous of degree 1 in each variable, and is symmetric if L is invariant under permutations of the coordinates of $\mathcal{C}^d$. Goodwillie's third theorem goes as follows.

Theorem 2.4 [16, Theorem 3.5] *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a homogeneous functor of degree d with $\mathcal{D}$ a stable model category. Then there is a naturally defined symmetric d–linear functor $LF\colon \mathcal{C}^d \to \mathcal{D}$, and a weak natural equivalence*

$$(LF(X,\dots,X))_{h\Sigma_d} \simeq F(X).$$

If $F\colon \mathcal{C} \to \mathcal{D}$ is a homotopy functor with $\mathcal{C}$ and $\mathcal{D}$ either $\mathcal{T}$ or $\mathcal{S}$, let $C_F(d) = L(D_d F)^{st}(S,\dots,S)$, a spectrum with Σ_d–action. (Here S denotes either the 0–sphere S^0 or its suspension spectrum.) Goodwillie refers to $C_F(d)$ as the d^{th} Taylor coefficient of F due the following corollary of the last theorem.

Corollary 2.5 *In this situation, there is a weak natural transformation*

$$(C_F(d) \wedge X^{\wedge d})_{h\Sigma_d} \to (D_d F)^{st}(X)$$

that is an equivalence if either X is a finite complex, or F commutes with directed homotopy colimits up to weak equivalence.

As will be illustrated in the examples, these equivariant spectra have often been identified.

The theorems above are the ones I wish to stress in these notes, but I should say a little about convergence. In [15], Goodwillie carefully proves a generalized Blakers–Massey theorem, and uses it to study questions that are equivalent to the convergence of these towers in the cases when $\mathcal{C}$ is $\mathcal{T}$ or $\mathcal{S}$. In particular, many functors can be shown to be 'analytic', and an analytic functor F admits a 'radius of convergence' $r(F)$ with the property that the tower for $F(X)$ converges strongly for all $r(F)$–connected objects X. The number $r(F)$ is often known, as will be illustrated in the examples. A nice result from [15] reads as follows.

Proposition 2.6 [15, Proposition 5.1] *Let $F \to G$ be a natural transformation between analytic functors, and let r be the maximum of $r(F)$ and $r(G)$. If $F(X) \to G(X)$ is an equivalence for all X that are equivalent to high suspensions then it is an equivalence for all r–connected X.*

3 Model category prerequisites

References for model categories include Quillen's original 1967 lecture notes [41], Dwyer and Spalinski's 1995 survey article [12], and the more recent books by Hovey and Hirschhorn [20; 18].

3.1 Nice model categories

We will assume that $\mathcal{C}$ and $\mathcal{D}$ are either *simplicial* or *topological* based model categories. (See [18, Chapter 9] for a careful discussion of simplicial model categories and [13, Chapter VII] for topological model categories.) 'Based' means that the initial and final object are the same: we will denote this object by $*$.

As part of the structure of a based topological (or simplicial) model category $\mathcal{C}$, given $K \in \mathcal{T}$ and $X \in \mathcal{C}$, one has new objects in $\mathcal{C}$, $X \otimes K$ and $\mathrm{Map}(K, X)$ satisfying standard properties. This implies that $\mathcal{C}$ supports canonical homotopy limits and colimits: given a functor $\mathcal{X} : \mathcal{J} \to \mathcal{C}$ from a small category $\mathcal{J}$, $\mathrm{hocolim}_J \mathcal{X}$ and $\mathrm{holim}_J \mathcal{X}$ are defined as appropriate coends and ends:

$$\operatorname*{hocolim}_J \mathcal{X} = \mathcal{X} \otimes_{\mathcal{J}} E\mathcal{J}^{op} \quad \text{and} \quad \operatorname*{holim}_J \mathcal{X} = \mathrm{Map}^{\mathcal{J}}(E\mathcal{J}, \mathcal{X}),$$

where $E\mathcal{J}\colon \mathcal{J} \to \mathcal{T}$ is given by $E\mathcal{J}(j) = B(\mathcal{J} \downarrow j)_+$. (Thus these generalize the constructions $X \times_G EG$ and $\mathrm{Map}^G(EG, X)$ familiar from equivariant topology.) For details and basic properties, see [18, Chapter 18].

With such canonical homotopy limits and colimits, $\mathcal{C}$ will support a sensible theory of homotopy Cartesian and coCartesian cubes, as discussed in [15]: see Section 4.1 below. To know that certain explicit cubes in $\mathcal{C}$ are homotopy coCartesian, one also needs that $\mathcal{C}$ be left proper, and it seems prudent to require both $\mathcal{C}$ and $\mathcal{D}$ to be *proper*: the pushout of a weak equivalence by a cofibration is a weak equivalence, and dually for pullbacks.

$\mathcal{D}$ then needs a further axiom ensuring that the sequential homotopy colimit of homotopy Cartesian cubes is again homotopy Cartesian: assuming that $\mathcal{D}$ admits the (sequential) *small object argument* does the job: see Schwede [43, Section 1.3].

Examples 3.1 The following categories satisfy our hypotheses:

- $\mathcal{T}_Y$, the category of spaces over and under Y,
- $R-\mathcal{M}od$, the category of R–modules, where R is an E_∞ ring spectrum, a.k.a. commutative S–algebra [13],
- $R-\mathcal{A}lg$, the category of augmented commutative R–algebras,
- simplicial versions of all of these, eg spectra as in [10].

3.2 Spectra in model categories

Let $\mathcal{D}$ be a model category as above, and let ΣX denote $X \otimes S^1$, ie X tensored with the circle S^1. Trying to force the suspension $\Sigma\colon \mathcal{D} \to \mathcal{D}$ to be 'homotopy invertible' leads to a model category of spectra $\mathcal{S}(\mathcal{D})$ in the 'usual way': this has been studied carefully by Schwede [43] (following Bousfield–Friedlander [10]), Hovey [21], and Basterra–Mandell [5]. Roughly put, an object in $\mathcal{S}(\mathcal{D})$ will consist of a sequence of objects $X_0, X_1, X_2, \dots$ in $\mathcal{D}$, together with maps $\Sigma X_n \to X_{n+1}$. The point of this construction is that the model category structure $\mathcal{S}(\mathcal{D})$ has the additional property that it is *stable*: homotopy cofibration sequences in $\mathcal{S}(\mathcal{D})$ agree with the homotopy fibration sequences. The associated homotopy category will be triangulated.

As in the familiar case when $\mathcal{D} = \mathcal{T}$, there are adjoint functors

$$\Sigma^\infty\colon \mathcal{D} \to \mathcal{S}(\mathcal{D}) \text{ and } \Omega^\infty\colon \mathcal{S}(\mathcal{D}) \to \mathcal{D}.$$

If $\mathcal{D}$ is already stable these functors form a Quillen equivalence. For an arbitrary $\mathcal{D}$, this adjoint pair can take a surprising form, as the following example illustrates.

Example 3.2 In [5], the authors show that the category $\mathcal{S}(R\text{–}\mathcal{A}lg)$ is Quillen equivalent to $R - \mathcal{M}od$ so that $\Sigma^\infty\colon R\text{–}\mathcal{A}lg \to \mathcal{S}(R\text{–}\mathcal{A}lg)$ identifies with the Topological André–Quillen Homology functor[1] $TAQ\colon R\text{–}\mathcal{A}lg \to R\text{–}\mathcal{M}od$, and $\Omega^\infty\colon \mathcal{S}(R\text{–}\mathcal{A}lg) \to R\text{–}\mathcal{A}lg$ identifies with the functor sending an R–module M to the trivial augmented R–algebra $R \vee M$. (Partial results along these lines were also proved in [6; 43].)

3.3 Functors between model categories

Suppose $\mathcal{C}$ and $\mathcal{D}$ are nice topological model categories. There are couple of useful properties that a functor

$$F\colon \mathcal{C} \to \mathcal{D}$$

might have.

Firstly F will usually be *continuous*: for all X and Y in $\mathcal{C}$, the function

$$F\colon \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

should be continuous.

[1] To be precise, by $TAQ(B)$ we mean the Topological André–Quillen Homology of B with coefficients in the B–bimodule R.

If F is continuous, given $X \in \mathcal{C}$ and $K \in \mathcal{T}$, there is a natural assembly map

$$F(X) \otimes K \to F(X \otimes K) \tag{1}$$

defined by means of various adjunctions. The existence of these assembly maps implies that F will be a *homotopy* functor: a weak equivalence between fibrant cofibrant objects in $\mathcal{C}$ is carried by F to a weak equivalence in $\mathcal{D}$.

The second property that some functors F satisfy is that F commutes with filtered homotopy colimits, up to weak equivalence. A functor having this property has sometimes also been termed 'continuous', but Goodwillie [16] more cautiously uses the term *finitary* and so will we.

There are many interesting functors that are not finitary, as the next example shows.

Example 3.3 Let $L_E\colon \mathcal{S} \to \mathcal{S}$ be Bousfield localization of spectra with respect to a spectrum E. Then L_E is finitary exactly when the assembly map

$$L_E(S) \wedge X \to L_E(X)$$

is a weak equivalence for all spectra X. In other words, L_E is finitary exactly when it is smashing, a property that many interesting L_E's do not have.

Just to confuse the issue, we note that if L_E is regarded as taking values in the topological model category $L_E\mathcal{S}$, in which equivalences are E_*–isomorphisms and fibrant objects are E_*–local [13, Chapter VIII], then $L_E\colon \mathcal{S} \to L_E\mathcal{S}$ *is* finitary.

Finally, lets say a word about maps between functors. If $\mathcal{C}$ is not small, then it seems a bit daunting (set theoretically) to impose a model category structure on the class of functors $F\colon \mathcal{C} \to \mathcal{D}$. As an adequate fix for calculus purposes, we use the following terminology. Call a natural transformation $f\colon F \to G$ a weak equivalence, and write $F \underset{\sim}{\xrightarrow{f}} G$, if $f(X)\colon F(X) \to G(X)$ is a weak equivalence for all X in $\mathcal{C}$. By a weak natural transformation $f\colon F \to G$ we mean a pair of natural transformations of the form $F \underset{\sim}{\xleftarrow{g}} H \xrightarrow{h} G$ or $F \xrightarrow{h} H \underset{\sim}{\xleftarrow{g}} G$. We say that a diagram of weak natural transformations commutes if, after evaluation on any object X, the associated diagram commutes in the homotopy category of $\mathcal{D}$. Finally, we say that a diagram of functors $F \to G \to H$ a fiber sequence if $F(X) \to G(X) \to H(X)$ is a (homotopy) fiber sequence for all X.

4 Cubical diagrams and polynomial functors

4.1 Cubical diagrams

We review some of the theory of cubical diagrams; a reference is [15, Section 1].

Let S be a finite set. The power set of S, $\mathcal{P}(S) = \{T \subseteq S\}$, is a partially ordered set via inclusion, and is thus a small category. Let $\mathcal{P}_0(S) = \mathcal{P}(S) - \{\varnothing\}$ and let $\mathcal{P}_1(S) = \mathcal{P}(S) - \{S\}$.

Definitions 4.1 (a) A *d–cube* in $\mathcal{C}$ is a functor $\mathcal{X}\colon \mathcal{P}(S) \to \mathcal{C}$ with $|S| = d$.

(b) $\mathcal{X}$ is *Cartesian* if the natural map

$$\mathcal{X}(\varnothing) \to \operatorname*{holim}_{T \in \mathcal{P}_0(S)} \mathcal{X}(T)$$

is a weak equivalence.

(c) $\mathcal{X}$ is *coCartesian* if the natural map

$$\operatorname*{hocolim}_{T \in \mathcal{P}_1(S)} \mathcal{X}(T) \to \mathcal{X}(S)$$

is a weak equivalence.

(d) $\mathcal{X}$ is *strongly coCartesian* if $\mathcal{X}|_{\mathcal{P}(T)} : \mathcal{P}(T) \to \mathcal{C}$ is coCartesian for all $T \subseteq S$ with $|T| \geq 2$.

Often S will be the concrete set $\mathbf{d} = \{1, \ldots, d\}$.

Example 4.2 A 0–cube $\mathcal{X}(0)$ is Cartesian if and only if it is coCartesian if and only if $\mathcal{X}(0)$ is acyclic (ie weakly equivalent to the initial object $*$).

Example 4.3 A 1–cube $f\colon \mathcal{X}(0) \to \mathcal{X}(1)$ is Cartesian if and only if it is coCartesian if and only if f is an equivalence.

Example 4.4 A 2–cube

$$\begin{tikzcd} \mathcal{X}(0) \arrow[r] \arrow[d] & \mathcal{X}(1) \arrow[d] \\ \mathcal{X}(2) \arrow[r] & \mathcal{X}(12) \end{tikzcd}$$

is Cartesian if it is a homotopy pullback square, and coCartesian if it is a homotopy pushout square.

Example 4.5 Strongly coCartesian d–cubes are equivalent to ones constructed as follows. Given a family of cofibrations $f(t)\colon X(0) \to X(t)$ for $1 \leq t \leq d$, let $\mathcal{X}\colon \mathbf{d} \to \mathcal{C}$ be defined by $\mathcal{X}(T) =$ the pushout of $\{f(t) \mid t \in T\}$. (Note that $\mathcal{X}(T)$ can be interpreted as the coproduct under $X(0)$ of $X(t), t \in T$.)

Critical to Goodwillie's constructions, is a special case of this last example.

Definition 4.6 If T is a finite set, and X is an object in $\mathcal{C}$, let $X * T$ be the homotopy cofiber of the folding map $\coprod_T X \to X$.

For $T \subseteq \mathbf{d}$, the assignment $T \mapsto X * T$ is easily seen to define a strongly coCartesian d–cube $\mathcal{X}$: if $X \to *$ factors as $X \xrightarrow{i} CX \xrightarrow{p} *$, with i a cofibration and p an acyclic fibration, then $\mathcal{X}$ agrees with the cube of the last example with $f(t) = i\colon X \to CX$ for all t.

In the special case when $\mathcal{C} = \mathcal{T}$, $X * T$ is the (reduced) join of X and T: the union of $|T|$ copies of the cone CX glued together along their common base X.

There is a very useful way to inductively identify Cartesian cubes. Note that the fibers of the vertical maps in a Cartesian 2–cube as in Example 4.4 form a Cartesian 1–cube as in Example 4.3. This generalizes to higher dimensional cubes as we now explain.

Regard $\mathbf{d}$ as the obvious subset of $\mathbf{d+1}$. Given an $(d+1)$–cube $\mathcal{X}\colon \mathcal{P}(\mathbf{d+1}) \to \mathcal{C}$, we define three associated d-cubes

$$\mathcal{X}_{\mathrm{top}}, \mathcal{X}_{\mathrm{bottom}}, \partial\mathcal{X}\colon \mathcal{P}(\mathbf{d}) \to \mathcal{C}$$

as follows. Let $\mathcal{X}_{\mathrm{top}}(T) = \mathcal{X}(T)$ and $\mathcal{X}_{\mathrm{bottom}}(T) = \mathcal{X}(T \cup \{n+1\})$. Then define $\partial\mathcal{X}(T)$ by taking homotopy fibers of the evident natural transformation between these:

$$\partial\mathcal{X}(T) = \mathrm{hofib}\{\mathcal{X}_{\mathrm{top}}(T) \to \mathcal{X}_{\mathrm{bottom}}(T)\}.$$

Lemma 4.7 *$\mathcal{X}$ is Cartesian if and only if $\partial\mathcal{X}$ is Cartesian.*

Lemma 4.8 *If $\mathcal{X}_{top}$ and $\mathcal{X}_{bottom}$ are Cartesian, so is $\mathcal{X}$.*

Remark 4.9 Dual lemmas hold for coCartesian cubes. One application of this is that if $\mathcal{C}$ is a *stable* model category, so that homotopy fibre sequences are the same as homotopy cofiber sequences, then $\mathcal{X}$ is Cartesian if and only if $\mathcal{X}$ is coCartesian.

4.2 Polynomial functors

Let $\mathcal{C}$ and $\mathcal{D}$ be topological or simplicial model categories as in Section 3.1.

Definition 4.10 $F\colon \mathcal{C} \to \mathcal{D}$ is called *d–excisive* or said to be *polynomial of degree at most d* if, whenever $\mathcal{X}$ is a strongly coCartesian $(d+1)$–cube in $\mathcal{C}$, $F(\mathcal{X})$ is a Cartesian cube in $\mathcal{D}$.

Example 4.11 F has degree 0 if and only if $F(X) \to F(*)$ is an equivalence for all $X \in \mathcal{C}$, ie F is homotopy constant.

Example 4.12 $F\colon \mathcal{C} \to \mathcal{D}$ is 1–excisive means that F takes pushout squares to pullback squares.

In the classical case when $\mathcal{C}$ and $\mathcal{D}$ are spaces or spectra, this implies that the functor sending X to $\pi_*(F(X))$ satisfies the Mayer–Vietoris property.

If F is also finitary, then Milnor's wedge axiom holds as well. Then there are spectra C_0 and C_1 such that $F(X) \simeq C_0 \vee (C_1 \wedge X)$ if $\mathcal{D} = \mathcal{S}$ and $F(X) \simeq \Omega^\infty(C_0 \vee (C_1 \wedge X))$ if $\mathcal{D} = \mathcal{T}$.

Remark 4.13 Without the finitary hypothesis, classifying 1–excisive functors seems very hard. Examples of 1–excisive functors of X from spectra to spectra include the localization functors $L_E X$ and functors of the form $\mathrm{Map}_{\mathcal{S}}(C, X)$ where $C \in \mathcal{S}$ is fixed.

The following proposition of Goodwillie constructs d–excisive functors out of d–variable 1–excisive functors.

Proposition 4.14 [15, Proposition 3.4] *If $L\colon \mathcal{C}^d \to \mathcal{D}$ is 1–excisive in each of the d–variables, then the functor sending X to $L(X, \dots, X)$ is d–excisive.*

Corollary 4.15 *In this situation, if L is symmetric, and $\mathcal{D}$ is a stable model category, then, given any subgroup G of the d^{th} symmetric group Σ_n, the functor sending X to $L(X, \dots, X)_{hG}$ is d–excisive.*

The various lemmas about identifying Cartesian cubes can be used to prove the next two useful lemmas.

Lemma 4.16 *If F is d–excisive, then F is c–excisive for all $c \geq d$.*

Lemma 4.17 *If $F \to G \to H$ is a fiber sequence of functors, and G and H are both d–excisive, then so is F.*

5 Construction of Goodwillie towers and the proof of the main properties

5.1 Construction of the tower and the proof of Theorem 2.1

If one is to construct a d–excisive functor $P_d F$, then $P_d F(\mathcal{X})$ needs to be Cartesian for for all strongly coCartesion $(d+1)$–cubes $\mathcal{X}$. The idea behind the construction of $P_d F$ is to force this condition to hold for certain strongly coCartesion $(d+1)$–cubes $\mathcal{X}$.

Fix an object $X \in \mathcal{C}$. As discussed above, for $T \subseteq \mathbf{d+1}$, the assignment $T \mapsto X * T$ defines a strongly coCartesian $(d+1)$–cube $\mathcal{X}$. For example, when $d+1 = 2$, one gets the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X. \end{tikzcd}$$

Definition 5.1 Let $T_d F \colon \mathcal{C} \to \mathcal{D}$ be defined by

$$T_d F(X) = \operatorname*{holim}_{T \in \mathbb{P}_0(\mathbf{d+1})} F(X * T).$$

Note that there is an evident natural transformation $t_d(F) \colon F \to T_d F$, and that this is an equivalence if F is d–excisive.

Definition 5.2 Let $P_d F \colon \mathcal{C} \to \mathcal{D}$ be defined by

$$P_d F(X) = \operatorname{hocolim}\Big\{ F(X) \xrightarrow{t_d(F)} T_d F(X) \xrightarrow{t_d(T_d F)} T_d T_d F(X) \to \cdots \Big\}.$$

Example 5.3 $T_1 F(X)$ is the homotopy pullback of

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X), \end{tikzcd}$$

Suppose that $F(*) \simeq *$. Then $F(CX) \simeq *$, so that $T_1 F(X)$ is equivalent to the homotopy pullback of

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & F(\Sigma X), \end{tikzcd}$$

which is $\Omega F(\Sigma X)$. It follows that there is a natural weak equivalence

$$P_1 F(X) \simeq \operatorname*{hocolim}_{n\to\infty} \Omega^n F(\Sigma^n X).$$

Example 5.4 Specializing the last example to the case when F is the identity functor Id: $\mathcal{D} \to \mathcal{D}$, we see that

$$P_1(\mathrm{Id})(X) \simeq \Omega^\infty \Sigma^\infty X.$$

If $\mathcal{D} = \mathcal{T}$, topological spaces, we see that $P_1(\mathrm{Id})(X) = QX$.

If $\mathcal{D} = R{-}\mathcal{A}lg$, we see that $P_1(\mathrm{Id})(B) \simeq R \vee TAQ(B)$ for an augmented commutative R–algebra B.

The proof of Theorem 2.1 amounts to checking that the P_d construction just defined has the two desired properties: $P_d F$ should always be d–excisive, and $F \to P_d F$ should be universal. Checking the first of these is by far the more subtle, and follows from the next lemma.

Lemma 5.5 (Goodwillie [16, Lemma 1.9]) *If F: $\mathcal{C} \to \mathcal{D}$ is a homotopy functor, and $\mathcal{X}$ is strongly coCartesian $(d+1)$–cube in $\mathcal{C}$, then there is a Cartesian $(d+1)$–cube $\mathcal{Y}$ in $\mathcal{D}$, such that $F(\mathcal{X}) \to T_d F(\mathcal{X})$ factors through $\mathcal{Y}$.*

The construction of $\mathcal{Y}$ is very devious. $\mathcal{Y}$ is (roughly) constructed to be the homotopy limit of $(d+1)$–cubes in $\mathcal{D}$ that are each seen to be Cartesian for the following reason: they are constructed by applying F to $(d+1)$–cubes in $\mathcal{C}$ formed by means of evident objectwise equivalences between two d–cubes.

In contrast, proving that $F \to P_d F$ is appropriately universal is much easier. Once one knows that $P_d F$ is d–excisive, universality amounts to checking the following two things:

(a) If F is d–excisive, then $e_d(F)$: $F \to P_d F$ is a weak equivalence, and

(b) $P_d(e_d(F)))$: $P_d F \to P_d P_d F$ is a weak equivalence.

These follow immediately from T_d–versions of these statements:

(a′) If F is d–excisive, then $t_d(F)$: $F \to T_d F$ is a weak equivalence, and

(b′) $P_d(t_d(F)))$: $P_d F \to P_d T_d F$ is a weak equivalence.

As was noted above, the first of these is clear. The second admits a fairly simple proof based on the commutativity of iterated homotopy inverse limits. Similar reasoning verifies the next lemma, which in turn implies Lemma 2.2, which said that P_d preserves fiber sequences.

Lemma 5.6 *Given natural transformations $F \to G \to H$, if*

$$F(X) \to G(X) \to H(X)$$

is a fiber sequence for all X, then so is

$$T_d F(X) \to T_d G(X) \to T_d H(X).$$

5.2 Delooping homogeneous functors and Theorem 2.3

The most surprising property of Goodwillie towers is stated in Theorem 2.3. This says that, for $d > 0$, homogeneous d–excisive functors are infinitely deloopable. To show this, Goodwillie proves his beautiful key lemma, which says that $P_d F(X) \to P_{d-1} F(X)$ is always a principal fibration if F is *reduced*: $F(*) \simeq *$.

Lemma 5.7 *[16, Lemma 2.2] Let $d > 0$, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a reduced functor. There exists a homogeneous degree d functor $R_d F\colon \mathcal{C} \to \mathcal{D}$ fitting into a fiber sequence of functors*

$$P_d F \to P_{d-1} F \to R_d F.$$

Iteration of the R_d construction leads to Theorem 2.3: if F is homogeneous of degree d, then we can let $F^{st}(X)$ be the spectrum with r^{th} space $R_d^r F(X)$.

The proof of Lemma 5.7 is yet another clever manipulation of categories related to cubes. As an indication of how this might work, we sketch how one can construct a homotopy pullback square

$$\begin{tikzcd}
T_d F(X) \arrow[r] \arrow[d] & K_d F(X) \arrow[d] \\
T_{d-1} F(X) \arrow[r] & Q_d F(X)
\end{tikzcd}$$

with $K_d F(X) \simeq *$, in the case when $d = 2$.

One needs to look at how one passes from $\mathcal{P}_0(2)$ to $\mathcal{P}_0(3)$. In pictures, $\mathcal{P}_0(2)$ looks like

$$\begin{tikzcd}[row sep=small]
 & & 1 \\
2 \arrow[r, dash] & 12, \arrow[ur, dash] &
\end{tikzcd}$$

while $\mathcal{P}_0(3)$ looks like

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 1 \arrow[dd, dash] \\
2 \arrow[rr, dash] \arrow[dd, dash] & & 12 \arrow[ur, dash] \arrow[dd, dash] & \\
 & 3 \arrow[rr, dash] \arrow[dl, dash] & & 13 \arrow[dl, dash] \\
23 \arrow[rr, dash] & & 123. &
\end{tikzcd}$$

Now we decompose the poset $\mathcal{P}_0(3)$ as

(2)
$$\begin{tikzcd}
\mathcal{A}\cap\mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{A} \arrow[r] & \mathcal{A}\cup\mathcal{B} = \mathcal{P}_0(3),
\end{tikzcd}$$

where $\mathcal{A}$ is

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 3 \arrow[rr, dash] \arrow[dl, dash] & & 13 \arrow[dl, dash] \\
23 \arrow[rr, dash] & & 123, &
\end{tikzcd}$$

and $\mathcal{B}$ is

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 1 \arrow[dd, dash] \\
2 \arrow[rr, dash] \arrow[dd, dash] & & 12 \arrow[ur, dash] \arrow[dd, dash] & \\
 & & & 13 \arrow[dl, dash] \\
23 \arrow[rr, dash] & & 123, &
\end{tikzcd}$$

so that $\mathcal{A}\cap\mathcal{B}$ is

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 13 \arrow[dl, dash] \\
23 \arrow[rr, dash] & & 123. &
\end{tikzcd}$$

The decomposition of posets (2) induces a homotopy pullback diagram

$$\begin{tikzcd}
\operatorname{holim}_{T\in\mathcal{P}_0(3)} F(X*T) \arrow[r] \arrow[d] & \operatorname{holim}_{T\in\mathcal{A}} F(X*T) \arrow[d] \\
\operatorname{holim}_{T\in\mathcal{B}} F(X*T) \arrow[r] & \operatorname{holim}_{T\in\mathcal{A}\cap\mathcal{B}} F(X*T).
\end{tikzcd}$$

The top left corner is $T_2F(X)$, by definition. As $\mathcal{P}_0(2)$ is cofinal in $\mathcal{B}$, the bottom left corner is equivalent to $T_1F(X)$. Finally $\mathcal{A}$ has initial object $\{3\}$, so that the upper left corner is contractible:

$$\operatorname*{holim}_{T\in\mathcal{A}} F(X*T) \simeq F(X*\{3\}) = F(CX) \simeq *.$$

5.3 Cross effects and the proof of Theorem 2.4

Definition 5.8 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We define $cr_d F\colon \mathcal{C}^d \to \mathcal{D}$, the d^{th} *cross effect* of F, to be the the functor of d variables given by

$$(cr_d F)(X_1, \dots, X_d) = \operatorname{hofib}\Big\{F\Big(\bigvee_{i \in \mathbf{d}} X_i\Big) \to \operatorname*{holim}_{T \subset \mathcal{P}_0(d)} F\Big(\bigvee_{i \in \mathbf{d}-T} X_i\Big)\Big\}.$$

The d–cube sending T to $\bigvee_{i \in \mathbf{d}-T} X_i$ is easily seen to be strongly coCartesian; letting $d = 2$ for example, the square

$$\begin{tikzcd} X_1 \vee X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd}$$

is weakly equivalent to the evidently coCartesian square

$$\begin{tikzcd} X_1 \vee X_2 \arrow[r] \arrow[d] & CX_1 \vee X_2 \arrow[d] \\ X_1 \vee CX_2 \arrow[r] & CX_1 \vee CX_2. \end{tikzcd}$$

It follows that if F is $(d-1)$–excisive, then $cr_d F(X_1, \dots, X_d) \simeq *$ for all X_i. A similar argument [16, Lemma 3.3] shows that if F is d–excisive, then $cr_d F$ is 1–excisive in each of its variables.

More cleverly, Goodwillie establishes a converse of sorts when $\mathcal{D}$ is a stable model category [16, proof of Proposition 3.4]: if F is d–excisive, and $cr_d F(X_1, \dots, X_d) \simeq *$ for all X_i, then F is $(d-1)$–excisive. This has the following useful consequence.

Lemma 5.9 *If $\mathcal{D}$ is a stable model category, a natural map $\alpha : F \to G$ between homogeneous functors of degree d will be an equivalence if and only if $cr_d\alpha\colon cr_d F \to cr_d G$ is an equivalence.*

Another property of $cr_d F\colon \mathcal{C}^d \to \mathcal{D}$ that is easy to see is that it is *reduced*:

$$cr_d F(X_1, \dots, X_d) \simeq *$$

if $X_i \simeq *$ for some i.

A permutation of $\mathbf{d}$, $\sigma \in \Sigma_d$, induces an evident isomorphism

$$\sigma_*\colon cr_d F(X_1, \dots, X_d) \to cr_d F(X_{\sigma(1)}, \dots, X_{\sigma(d)}),$$

satisfying $(\sigma \circ \tau)_* = \sigma_* \circ \tau_*$: a functor of d variables with this structure is called *symmetric*.

Definition 5.10 Let $L_d F\colon \mathcal{C}^d \to \mathcal{D}$ be the functor obtained from $cr_d F$ by applying P_1 to each variable. Thus we have

$$L_d F(X_1, \dots, X_d) \simeq \operatorname*{hocolim}_{n_i \to \infty} \Omega^{n_1 + \cdots + n_d} cr_d F(\Sigma^{n_1} X_1, \dots, \Sigma^{n_d} X_d).$$

$L_d F$ will always be symmetric and d–linear, and if F is d–excisive, then the natural map $cr_d F \to L_d F$ is an equivalence.

If G is a finite group, let $G - \mathcal{D}$ denote the category of objects in $\mathcal{D}$ with a G–action. Given $Y \in G - \mathcal{D}$, we let $Y_{hG} = Y \otimes_G EG$ and $Y^{hG} = \mathrm{Map}^G(EG, Y)$ denote the associated homotopy quotient and fixed point objects in $\mathcal{D}$.

Definition 5.11 Let $\Delta_d F\colon \mathcal{C} \to \Sigma_d - \mathcal{D}$ be defined by

$$\Delta_d F(X) = L_d F(X, \dots, X).$$

A more precise version of Theorem 2.4 is the following.

Theorem 5.12 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a homotopy functor, with $\mathcal{D}$ a stable model category. Then there is a natural weak equivalence*

$$\Delta_d F(X)_{h\Sigma_d} \simeq D_d F(X).$$

If F is d–excisive then $\Delta_d F(X)$ can be identified with $(cr_d F)(X, \dots, X)$. In this case, one gets a natural transformation

$$\alpha_d(X)\colon (\Delta_d F)(X)_{h\Sigma_d} \to F(X)$$

defined to be the composite

$$(\Delta_d F)(X)_{h\Sigma_d} \to F\Big(\bigvee_{i=1}^{d} X\Big)_{h\Sigma_d} \to F(X).$$

Here the second map is induced by the fold map $\bigvee_{i=1}^{d} X \to X$. The theorem is then proved by verifying that $cr_d(\alpha_d)$ is an equivalence, and appealing to Lemma 5.9.

We indicate how Corollary 2.5 follows from Theorem 5.12. The assembly map for F induces an assembly map

$$(\Delta_d F(X) \otimes K^{\wedge d})_{h\Sigma_d} \to \Delta_d F(X \otimes K)_{h\Sigma_d},$$

for $X \in \mathcal{C}$ and $K \in \mathcal{T}$.

If $\mathcal{C}$ is $\mathcal{T}$, $\mathcal{D}$ is $\mathcal{S}$, and $X = S$, then this reads

$$(C_F(d) \wedge K^{\wedge d})_{h\Sigma_d} \to \Delta_d F(K)_{h\Sigma_d},$$

where $C_F(d) = \Delta_d F(S)$. By construction, this map is the identity if $K = S$, and it follows that it will be an equivalence for all finite K, or all K under the additional hypothesis that F is finitary. A similar argument holds if both $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{S}$: here the assembly map can be constructed for all $K \in \mathcal{S}$.

When the domain category $\mathcal{C}$ is also stable, and $\mathcal{D} = \mathcal{S}$, there is an elegant addendum to Theorem 5.12 essentially due to R McCarthy [37].

Given $Y \in G - \mathcal{S}$, there is a natural norm map $N(Y)\colon Y_{hG} \to Y^{hG}$ satisfying the property that $N(Y)$ is an equivalence if Y is a finite free G–CW spectrum. As in [29], we let the Tate spectrum of Y, $\mathcal{T}_G(Y)$, be the cofiber.

Proposition 5.13 [29] *Let $F\colon \mathcal{C} \to \mathcal{S}$ be any homotopy functor, with $\mathcal{C}$ stable. For all $d \geq 1$, there is a homotopy pullback diagram*

$$\begin{tikzcd} P_d F(X) \arrow[r] \arrow[d] & (\Delta_d F(X))^{h\Sigma_d} \arrow[d] \\ P_{d-1} F(X) \arrow[r] & \mathcal{T}_{\Sigma_d}(\Delta_d F(X)). \end{tikzcd}$$

6 Examples

6.1 Suspension spectra of mapping spaces

Fix a finite C.W. complex K. Let $\mathrm{Map}_{\mathcal{T}}(K, X)$ be the space of based continuous maps from K to a space X. Similarly, given a spectrum Y, let $\mathrm{Map}_{\mathcal{S}}(K, Y)$ be the evident function spectrum.

In [15], Goodwillie proved that the functor from spaces to spectra sending X to $\Sigma^\infty \mathrm{Map}_{\mathcal{T}}(K, X)$ is analytic with radius of convergence equal to the dimension of K.

In [2], Arone gave a very concrete model for the associated Goodwillie tower $\{P^K_*(X)\}$. The paper by Ahearn and Kuhn [1] includes further details about Arone's construction while building in extra structure.

Let $\mathcal{E}$ be the category with objects the finite sets $\mathbf{d}$, $d \geq 1$, and with morphisms the epic functions. $\mathcal{E}_d$ will denote the full subcategory with objects $\mathbf{c}$ with $c \leq d$.

Given a based space X, let $X^{\wedge}\colon \mathcal{E}^{op} \to \mathcal{T}$ be the functor sending $\mathbf{d}$ to $X^{\wedge d}$. Then Arone's model for $P_d^K\colon \mathcal{T} \to \mathcal{S}$ is given by

$$P_d^K(X) = \mathrm{Map}_{\mathcal{S}}{}^{\mathcal{E}_d}(K^{\wedge}, \Sigma^{\infty} X^{\wedge}),$$

the spectrum of natural transformations between the two contravariant functors of $\mathcal{E}_d$. The natural transformation

$$\Sigma^{\infty} \mathrm{Map}_{\mathcal{T}}(K, X) \to P_d^K(X)$$

is induced by sending $f\colon K \to X$ to $f^{\wedge}\colon K^{\wedge} \to X^{\wedge}$ and then stablizing.

A by product of this construction is that there is a homotopy pullback square of $\mathcal{S}$–modules which has some of the same flavor as Proposition 5.13:

$$\begin{tikzcd}
P_d^K(X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{S}}{}^{\Sigma_d}(K^{\wedge d}, \Sigma^{\infty} X^{\wedge d}) \arrow[d] \\
P_{d-1}^K(X) \arrow[r] & \mathrm{Map}_{\mathcal{S}}{}^{\Sigma_d}(\delta_d(K), \Sigma^{\infty} X^{\wedge d}),
\end{tikzcd}$$

where $\delta_d(K) \subset K^{\wedge d}$ denotes the fat diagonal.

Thus the dth fiber, $D_d^K(X)$, can be described as follows. Let $K^{(d)}$ denote $K^{\wedge d}/\delta_d(K)$. Then we have

$$\begin{aligned}
D_d^K(X) &= \mathrm{Map}_{\mathcal{S}}{}^{\Sigma_d}(K^{(d)}, \Sigma^{\infty} X^{\wedge d}) \\
&\simeq \mathrm{Map}_{\mathcal{S}}(K^{(d)}, \Sigma^{\infty} X^{\wedge d})_{h\Sigma_d} \\
&\simeq (\mathcal{D}(K^{(d)}) \wedge X^{\wedge d})_{h\Sigma_d}.
\end{aligned}$$

Here $\mathcal{D}(K^{(d)})$ denotes the equivariant S–dual of $K^{(d)}$, and the equivalences follow from the fact that $K^{(d)}$ is both finite and Σ_d–free away from the basepoint. It follows that the dth Taylor coefficient of the functor sending X to $\Sigma^{\infty} \mathrm{Map}_{\mathcal{T}}(K, X)$ is $\mathcal{D}(K^{(d)})$.

Remark 6.1 In [28], we observed that, when X is also a finite complex, the tower $P_*^K(X)$ also arises as by taking the S–dual of a natural filtration on the nonunital commutative S–algebra $\mathcal{D}(X) \otimes K$.

By Alexander duality, $\mathcal{D}(K^{(d)})$ can be identified an appropriate equivariant desuspension of the suspension spectrum of a configuration space. Specializing to the case when $K = S^n$, this takes the following concrete form. Let $\mathcal{C}(n, d)$ denote the space of d

distinct little n–cubes in a big n–cube (May [36]). Via a Thom–Pontryagin collapse, there is a very explicit duality map of Σ_d spaces [1]

$$\mathcal{C}(n,d)_+ \wedge S^{n(d)} \to S^{nd}.$$

One proof that Arone's model works when $K = S^n$ goes roughly as follows. Suppose $X = \Sigma^n Y$. One has the usual filtered configuration space model $C_n(Y)$ for $\Omega^n\Sigma^n Y$ [36]. Thus one has maps

$$\Sigma^\infty F_d C_n(Y) \to \Sigma^\infty \Omega^n \Sigma^n Y \to P_d^{S^n}(Y).$$

The nontriviality of the second map is proved by showing that the composite is an equivalence. By induction on d, it suffices to show that cr_d applied to this composite is an equivalence, and the verification of that leads back to the above explicit duality map.

A bonus corollary of this proof is that one also establishes a rather nice version of 'Snaith splitting': the tower strongly splits when $X = \Sigma^n Y$.

Example 6.2 One application of this comes from applying mod p cohomology to the tower. One obtains a spectral sequence of differential graded algebras $\{E_r^{s,t}(S^n, X)\}$ with

$$E_1^{d,*}(S^n, X) = H^*((\mathcal{C}(n,d)_+ \wedge (\Sigma^{-d}X)^{\wedge d})_{h\Sigma_d}; \mathbb{Z}/p))$$

and converging strongly to $H^*(\Omega^n X; \mathbb{Z}/p)$ if X is n–connected. This E_1 term is a known functor of $H^*(X;\mathbb{Z}/p)$. The differentials have not been fully explored, but seem to be partly determined by derived functors of destablization of unstable modules over the Steenrod algebra, as applied to the $\mathcal{A}$–module $\Sigma^{-n}H^*(X;\mathbb{Z}/p)$.

6.2 Suspension spectra of infinite loopspaces

The previous example can be used to determine the tower $\{P_d^{S^\infty}\}$ for the functor from spectra to spectra sending a spectrum X to $\Sigma^\infty\Omega^\infty X$.

Let X_n denote the nth space of the spectrum X. Then we have that $\Omega^n X_n \simeq \Omega^\infty X$ for all n, and the natural map

$$\operatorname*{hocolim}_{n\to\infty} \Sigma^{-n}\Sigma^\infty X_n \to X$$

is an equivalence. From this and the last example, one can deduce that the tower converges for 0–connected spectra X and that

$$\operatorname*{hocolim}_{n\to\infty} \Sigma^{-n} P_d^{S^n}(X_n) \simeq P_d^{S^\infty}(X).$$

As $\operatorname{hocolim}_{n\to\infty} \mathcal{C}(n,d)_+$ is a model for $E\Sigma_d+$, and this is weakly equivalent to S^0, it follows that the formula for the dth fiber is

$$D_d^{S^\infty}(X) \simeq X^{\wedge d}_{h\Sigma_d},$$

and thus the dth Taylor coefficient of the functor sending a spectrum X to $\Sigma^\infty\Omega^\infty X$ is the sphere spectrum S for all $d > 0$.

Finally, Proposition 5.13 specializes to say that for each $d > 0$ there is a pullback square

$$\begin{tikzcd} P_d^{S^\infty}(X) \arrow[r] \arrow[d] & (X^{\wedge d})^{h\Sigma_d} \arrow[d] \\ P_{d-1}^{S^\infty}(X) \arrow[r] & \mathcal{T}_{\Sigma_d}(X^{\wedge d}). \end{tikzcd}$$

Example 6.3 The tower begins

$$\begin{tikzcd} & P_2^{S^\infty}(X) \arrow[d, "p_1"] \\ \Sigma^\infty\Omega^\infty X \arrow[ur, "e_2"] \arrow[r, "e_1"] & X, \end{tikzcd}$$

where $e_1\colon \Sigma^\infty\Omega^\infty X \to X$ is adjoint to the identity on $\Omega^\infty X$. A formal consequence is that

$$\Omega^\infty p_1\colon \Omega^\infty P_2^{S^\infty}(X) \to \Omega^\infty X$$

admits a natural section.

The map p_1 fits into a natural cofibration sequence

$$P_2^{S^\infty}(X) \xrightarrow{p_1} X \to \Sigma(X\wedge X)_{h\Sigma_2}.$$

Specializing to the case when $X = S^{-1}$, this can be identified [29, Appendix] with the cofibration sequence

$$\Sigma^{-1}\mathbb{R}P_0^\infty \xrightarrow{t} S^{-1} \to \mathbb{R}P_{-1}^\infty,$$

where t is one desuspension of the Kahn–Priddy transfer, and $\mathbb{R}P_k^\infty$ denotes the Thom spectrum of k copies of the canonical line bundle over $\mathbb{R}P^\infty$.

Letting QZ denote $\Omega^\infty\Sigma^\infty Z$, we conclude that

$$\Omega^\infty tr\colon \Omega Q\mathbb{R}P_+^\infty \to \Omega QS^0$$

admits a section: a result 'one loop' away from the full strength of the Kahn–Priddy Theorem [25] at the prime 2. The odd prime version admits a similar proof using that, localized at a prime p, $P_d^{S^\infty}(S^{-1}) \simeq *$ for $1 < d < p$.

6.3 The identity functor for $\mathcal{A}lg$

Let $\mathcal{A}lg$ be the category of commutative augmented S–algebras. This is a model category in which weak equivalences and fibrations are determined by forgetting down to S–modules. More curious is that the coproduct of A and B is $A \wedge B$.

Let $\{P_d^{alg}\}$ denote the tower associated to the identity $\mathcal{I}\colon \mathcal{A}lg \to \mathcal{A}lg$. Given $B \in \mathcal{A}lg$, it is not too hard to deduce that the tower $\{P_d^{alg}(B)\}$ will strongly converge to B if $I(B)$ is 0–connected, where $I(B)$ denotes the 'augmentation ideal': the homotopy fiber of the augmentation $B \to S$.

Let $D_d^{alg}(B)$ be the fiber of $P_d^{alg}(B) \to P_{d-1}^{alg}(B)$. As already discussed in Example 5.4, $D_1^{alg}(B)$ can be identified with $TAQ(B)$, the Topological André–Quillen Homology of B with coefficients in the B–bimodule S.

The fact that coproducts in $\mathcal{A}lg$ correspond to smash products of S–modules leads to a simple calculation of the d^{th} cross effect of $\mathcal{I}$:

$$cr_d(\mathcal{I})(B_1, \dots, B_d) \simeq I(B_1) \wedge \cdots \wedge I(B_d).$$

From this, one gets a formula for $D_d^{alg}(B)$:

Theorem 6.4

$$D_d^{alg}(B) \simeq TAQ(B)^{\wedge d}_{h\Sigma_d}.$$

A proof of this in the spirit of this paper appears in [30]. See also Minasian [39].

A nice corollary of this formula says the following.

Corollary 6.5 *If A and B in $\mathcal{A}lg$ have 0–connected augmentation ideals, then an algebra map $f\colon A \to B$ is an equivalence if $TAQ(f)$ is.*

The converse of this corollary — that $TAQ(f)$ is an equivalence if f is — is true even without connectivity hypotheses: see, eg [28]. Without any hypotheses implying convergence, one has that if $TAQ(f)$ is an equivalence, so is $\widehat{f}\colon \widehat{A} \to \widehat{B}$, where $\widehat{A}$ denotes the homotopy inverse limit of the tower for A [30].

Example 6.6 This tower overlaps in an interesting way with the one for $\Sigma^\infty \Omega^\infty X$ discussed above, and the corollary leads to a simple proof of a highly structured version of the classical stable splitting [24] of QZ, for a connected space Z.

It is well known that $\Sigma^\infty(\Omega^\infty X)_+$ is an E_∞ ring spectrum. Otherwise put, we can regard $\Sigma^\infty(\Omega^\infty X)_+$ as an object in $\mathcal{A}lg$. It is not hard to see that $TAQ(\Sigma^\infty(\Omega^\infty X)_+)$ is equivalent to the connective cover of X, and there is an equivalence

$$S \vee P_d^{S^\infty}(X) \simeq P_d^{alg}(\Sigma^\infty(\Omega^\infty X)_+)$$

for connective spectra X.

Another object in $\mathcal{A}lg$ is $\mathbb{P}(X)$, the free commutative S–algebra generated by X. As an S–module,

$$\mathbb{P}(X) \simeq \bigvee_{d=0}^{\infty} X^{\wedge d}_{h\Sigma_d},$$

and it is not hard to compute that $TAQ(\mathbb{P}(X)) \simeq X$.

The stable splitting of QZ gets proved as follows. The inclusion

$$\eta(Z)\colon Z \to QZ$$

induces a natural map in $\mathcal{A}lg$

$$s(Z)\colon \mathbb{P}(\Sigma^\infty Z) \to \Sigma^\infty(QZ)_+.$$

The construction of s makes it quite easy to verify that $TAQ(s(Z))\colon \Sigma^\infty Z \to \Sigma^\infty Z$ is the identity. The above corollary then implies that $s(Z)$ is an equivalence in $\mathcal{A}lg$, and thus in $\mathcal{S}$, for connected spaces Z.

A more detailed discussion of this appears in [30].

6.4 The identity functor for $\mathcal{T}$

Let $\{P_d\}$ denote the tower of the identity functor on $\mathcal{T}$. Given a space Z, let $D_d(Z)$ denote the fiber of $P_d(Z) \to P_{d-1}(Z)$, and then let $D_d^{st}(Z) \in \mathcal{S}$ be the infinite delooping provided by Theorem 2.3.

Goodwillie's estimates [15] show that the tower $\{P_d(Z)\}$ will strongly converge to Z when Z is simply connected. With a little more care, one can show that this is still true if Z is just nilpotent.[2] Thus, for nilpotent Z, one gets a strongly convergent 2nd quadrant spectral sequence converging to $\pi_*(Z)$, with $E^1_{-d,*} = \pi_*(D_d^{st}(Z))$.

[2] Thanks to Greg Arone for this observation.

Johnson [23] and Arone with collaborators Mahowald and Dwyer [4; 3] have identified the spectra $D_d^{st}(Z)$ thusly:

$$D_d^{st}(Z) \simeq (\mathcal{D}(\Sigma K_d) \wedge Z^{\wedge d})_{h\Sigma_d},$$

where K_d is the unreduced suspension of the classifying space of the poset of nontrivial partitions of $\mathbf{d}$, and $\mathcal{D}$ denotes the equivariant S–dual, as before.

Using this model, the discovery of Arone and Mahowald [4] is that when Z is an odd dimensional sphere, these spectra are very special spectra that were known previously. To state the theorem, we need some notation.

Let p be a prime. Let $m\rho_k$ denote the direct sum of m copies of the reduced real regular representation of $V_k = (\mathbb{Z}/p)^k$. Then $GL_k(\mathbb{Z}/p)$ acts on the Thom space $(BV_k)^{m\rho_k}$. Let $e_k \in \mathbb{Z}_{(p)}[GL_k(\mathbb{Z}/p)]$ be any idempotent in the group ring representing the Steinberg module, and then let $L(k,m)$ be the associated stable summand of $(BV_k)^{m\rho_k}$:

$$L(k,m) = e_k(BV_k)^{m\rho_k}.$$

The spectra $L(k,0)$ and $L(k,1)$ agree with spectra called $M(k)$ and $L(k)$ in the literature: see eg [40; 26; 32].

Collecting results from [4] and [3, Theorem 1.9, Corollary 9.6], one has the following theorem.

Theorem 6.7 *Let m be an odd natural number.*

(1) $D_d^{st}(S^m) \simeq *$ *if d is not a power of a prime.*

(2) *Let p be a prime. $D_{p^k}^{st}(S^m) \simeq \Sigma^{m-k} L(k,m)$, and thus has (only) p–torsion homotopy if $k > 0$.*

(3) $H^*(L(k,m);\mathbb{Z}/p)$ *is free over the subalgebra $\mathcal{A}(k-1)$ of the Steenrod algebra. As a function of k, the connectivity of $L(k,m)$ has a growth rate like p^k.*

Thus the associated spectral sequences for computing the unstable homotopy groups of odd spheres coverges exponentially quickly, and begins from stable information about spectra of roughly the same complexity as the suspension spectra of classifying spaces of elementary abelian p–groups.

Remark 6.8 When $m = 1$, one gets a spectral sequence converging to the known graded group $\pi_*(S^1)$, with $E^1_{-k,*} = \pi_*(L(k))$. Comparison with my work on the Whitehead Conjecture [26; 32] suggests that $E^2 = E^\infty$. Greg Arone and I certainly believe this, but a rigorous proof has yet to be nailed down.

As discussed near the end of the next section, the properties listed in the theorem have particularly beautiful consequences for computing the periodic unstable homotopy groups of odd dimensional spheres.

7 Interactions with periodic homotopy

For topologists who study classical unstable and stable homotopy theory, a major development of the past two decades has been the organization of these subjects via the chromatic filtration associated to the Morava K–theories.

One of the most unexpected aspects of Goodwillie towers is that they interact with the chromatic aspects of homotopy in striking ways. In this section, I survey, in inverse order of when they were proved, three different theorems of this sort.

7.1 Goodwillie towers and homology isomorphisms

There are a couple of useful general facts about how Bousfield localization relates to Goodwillie towers.

Let E_* be a generalized homology theory. A map $f\colon X \to Y$ of spaces or spectra is called an E_**–isomorphism* if $E_*(f)$ is an isomorphism. A natural transformation $f\colon F \to G$ between functors $F, G\colon \mathcal{C} \to \mathcal{S}$ is an E_**–isomorphism* if $f(X)$ is for all $X \in \mathcal{C}$. Then we have

Proposition 7.1 [31, Corollary 2.4] *If $F : \mathcal{C} \to \mathcal{S}$ is finitary and $f\colon X \to Y$ is an E_*–isomorphism then so are $D_d F(f)\colon D_d F(X) \to D_d F(Y)$ and $P_d F(f)\colon P_d F(X) \to P_d F(Y)$ for all d.*

Proposition 7.2 [29, Lemma 6.1] *If a natural transformation $f\colon F \to G$ between functors $F, G\colon \mathcal{C} \to \mathcal{S}$ is an E_*–isomorphism then so are $D_d f\colon D_d F \to D_d G$ and $P_d f\colon P_d F \to P_d G$ for all d.*

Both of these follow by observing that the various constructions defining P_d and D_d preserve E_*–isomorphisms.

The next example illustrates that the finitary hypothesis in Proposition 7.1 is needed.

Example 7.3 Consider $L_{H\mathbb{Z}/p}\colon \mathcal{S} \to \mathcal{S}$, which is a homogeneous functor of degree 1. Then $H\mathbb{Z}/p^\infty$ is $H\mathbb{Q}_*$–acyclic (ie $H\mathbb{Z}/p^\infty \to *$ is an $H\mathbb{Q}_*$–equivalence), but $L_{H\mathbb{Z}/p}(H\mathbb{Z}/p^\infty) = \Sigma H\mathbb{Z}_p$ is not.

For an application of Proposition 7.1 to the homology of mapping spaces, see [31]. Proposition 7.2 is crucially used in proofs of the two theorems discussed in the next two subsections.

We end this subsection with some observations related to the phenomenon illustrated in the last example.

If $F: \mathcal{C} \to \mathcal{S}$ is homogeneous of degree d, the functor $L_E F: \mathcal{C} \to \mathcal{S}$ will always again be d–excisive, but need no longer be homogeneous.

Example 7.4 Let $F: \mathcal{S} \to \mathcal{S}$ be defined by $F(X) = (X \wedge X)_{h\Sigma_2}$. The composite functor $L_E F$ will be 2–excisive, but need no be longer homogenous, even when restricted to finite spectra. Indeed, a simple calculation shows that

$$P_1(L_E F)(S) = \text{hocofib}\{L_E S \wedge \mathbb{R}P^\infty \to L_E \mathbb{R}P^\infty\}.$$

This can easily be nonzero. For example, when E is mod 2 K–theory, one has that $L_E \mathbb{R}P^\infty = L_E S$, as the transfer $\mathbb{R}P^\infty \to S$ is an $K\mathbb{Z}/2_*$–isomorphism. It follows that $P_1(L_E F)(S)$ has nonzero rational homology.

As a fix for this problem, we have the next lemma, which follows from Proposition 7.2.

Lemma 7.5 *If $F: \mathcal{C} \to \mathcal{S}$ is homogeneous of degree d, then $P_{d-1}(L_E F)$ is E_*–acyclic. Otherwise said, $D_d(L_E F) \to L_E F$ is an E_*–isomorphism.*

7.2 Goodwillie towers and periodic localization

We will consider two families of periodic homology theories.

Fixing a prime p, $K(n)_*$ is the nth Morava K–theory.

To define the second family, recall that a p–local finite complex M is of *type n* if M is $K(m)_*$–acyclic for $m < n$, but is not $K(n)_*$–acyclic. If M is of type n, then M admits a v_n*–self map*, a $K(n)_*$–isomorphism

$$v: \Sigma^d M \to M.$$

We let $T(n)$ denote the telescope of v. A consequence of the Nilpotence and Periodicity Theorems of Devanitz, Hopkins, and Smith [11; 19; 42] is that the associated Bousfield localization functor $L_{T(n)}: \mathcal{S} \to \mathcal{S}$ is independent of the choice of both the complex and self map. Also, we recall that $T(n)_*$–acyclics are $K(n)_*$–acyclic; thus the associated localization functors are related by $L_{K(n)} \simeq L_{K(n)} L_{T(n)}$.

The main theorem of [29] says that Goodwillie towers of functors from spectra to spectra always split after applying $L_{T(n)}$.

Theorem 7.6 [29, Theorem 1.1] *Let $F\colon \mathcal{S} \to \mathcal{S}$ be any homotopy functor. For all primes p, $n \geq 1$, and $d \geq 1$, the natural map*

$$p_d(X)\colon P_d F(X) \to P_{d-1} F(X)$$

admits a natural homotopy section after applying $L_{T(n)}$.

Corollary 7.7 *Let $F\colon \mathcal{S} \to \mathcal{S}$ be polynomial of degree less than d and $G\colon \mathcal{S} \to \mathcal{S}$ homogeneous of degree d. Then any natural transformation $f\colon F \to L_{T(n)}G$ will be null.*

The corollary follows formally from the theorem using Lemma 7.5: we leave verifying this as an exercise for the reader. The theorem is proved by combining Proposition 5.13 and Proposition 7.2 with the following vanishing theorem about Tate homology.

Theorem 7.8 [29, Theorem 1.5] *For all finite groups G, primes p, and $n \geq 1$,*

$$L_{T(n)} \mathcal{T}_G(L_{T(n)} S) \simeq *.$$

In [29], I manage to first reduce the proof of the theorem to the case when $G = \Sigma_p$. There are familiar 'inverse limits of Thom spectra' models for $L_{T(n)}\mathcal{T}_{\Sigma_p}(L_{T(n)}S)$. Using these, the equivalence $L_{T(n)}\mathcal{T}_{\Sigma_p}(L_{T(n)}S) \simeq *$ can be shown to be equivalent to the case when $X = S$ of the following statement about the Goodwillie tower of $\Sigma^\infty \Omega^\infty X$.

Theorem 7.9 [29, Theorem 3.7]

$$\operatorname*{holim}_k \Sigma^k L_{T(n)} P_p^{S^\infty}(\Sigma^{-k} X) \to L_{T(n)} X$$

admits a homotopy section.

This theorem follows immediately from the existence of the natural section $\eta_n(X)$ of $L_{T(n)} e_1(X)\colon L_{T(n)} \Sigma^\infty \Omega^\infty X \to L_{T(n)} X$ to be discussed in the next subsection.

Remark 7.10 A weaker version of Theorem 7.8 with $K(n)$ replacing $T(n)$ appears in work by Greenlees, Hovey, and Sadofsky [17; 22], and certainly inspired my thinking, if not my proof. Theorem 7.8 when $G = \mathbb{Z}/2$ is equivalent to the main theorem of Mahowald–Shick [35].

7.3 The periodic homology of infinite loopspaces

Using the full strength of the Periodicity Theorem, Bousfield and I have constructed 'telescopic functors' as in the next theorem.

Theorem 7.11 *[7; 27; 9] For all p and $n > 0$, there exists a functor $\Phi_n\colon \mathcal{T} \to \mathcal{S}$ such that $\Phi_n(\Omega^\infty X) \simeq L_{T(n)}X$. Furthermore $\Phi_n(Z)$ is always $T(n)$–local.*

Some further nice properties of Φ_n will be discussed in the next subsection: see Proposition 7.16. Here we note the following corollary.

Corollary 7.12 *After applying $L_{T(n)}$, the natural transformation*

$$e_1(X)\colon \Sigma^\infty\Omega^\infty X \to X$$

admits a section

$$\eta_n(X)\colon L_{T(n)}X \to L_{T(n)}\Sigma^\infty\Omega^\infty X.$$

The section is defined by applying Φ_n to the natural map

$$\eta(\Omega^\infty X)\colon \Omega^\infty X \to Q\Omega^\infty X.$$

Remark 7.13 η_n is unique up to 'tower phantom' behavior in the following sense: for all d, the composite

$$L_{T(n)}X \xrightarrow{\eta_n(X)} L_{T(n)}\Sigma^\infty\Omega^\infty X \xrightarrow{L_{T(n)}e_d(X)} L_{T(n)}P_d^{S^\infty}(X)$$

is the unique natural section of $L_{T(n)}P_d^{S^\infty}(X) \xrightarrow{L_{T(n)}p_d(X)} L_{T(n)}X$. Here uniqueness is a consequence of Corollary 7.7.

In [30], I use η_n to prove a splitting result in a manner similar to Example 6.6. The natural transformation

$$\eta_n(X)\colon X \to L_{T(n)}\Sigma^\infty\Omega^\infty X$$

induces a map of commutative augmented $L_{T(n)}S$–algebras

$$s_n(X)\colon L_{T(n)}\mathbb{P}(X) \to L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+.$$

As in Example 6.6, $s_n(X)$ has been constructed so that it is easy to see that

$$TAQ(s_n(X))\colon L_{T(n)}X \to L_{T(n)}X$$

is homotopic to the identity, and one learns that $s_n(X)$ induces an equivalence of localized Goodwillie towers. Because the towers have been localized with respect to a nonconnected homology theory, the convergence of localized towers is problematic. However, one can easily deduce the first statement of the next theorem, and starting from this, I was able to establish the rest.

Theorem 7.14 [30] *For all $X \in \mathcal{S}$,*

$$s_n(X)_*\colon K(n)_*(\mathbb{P}(X)) \to K(n)_*(\Omega^\infty X)$$

is monic, and fits into a chain complex of commutative $K(n)_*$–Hopf algebras

$$K(n)_*(\mathbb{P}(X)) \xrightarrow{s_{n*}} K(n)_*(\Omega^\infty X) \to \bigotimes_{j=0}^{n+1} K(n)_*(K(\pi_j(X), j)).$$

This sequence of Hopf algebras is exact if X is $T(m)_*$–acyclic for all $0 < m < n$, and only if X is $K(m)_*$–acyclic for all $0 < m < n$.

Note that the two acyclicity conditions on X are empty if $n = 1$. They agree if $n = 2$, by the truth of the Telescope Conjecture when $n = 1$.

Recall that the Telescope Conjecture asserts that a $K(n)_*$–acyclic spectrum will always be $T(n)_*$–acyclic, and is believed to be not true for $n \geq 2$. Peter May has remarked that maybe the theorem could be used to disprove it. I end this subsection by describing one way this might go.

If Z is a connected space, let $j_n(Z)\colon L_{T(n)}\mathbb{P}(Z) \to L_{T(n)}\mathbb{P}(Z)$ be the composite

$$L_{T(n)}\mathbb{P}(Z) \xrightarrow{s_n(\Sigma^\infty Z)} L_{T(n)}\Sigma^\infty QZ \xrightarrow{L_{T(n)}s(Z)^{-1}} L_{T(n)}\mathbb{P}(Z).$$

Here we have written $\mathbb{P}(Z)$ for $\mathbb{P}(\Sigma^\infty Z)$.

The theorem says that if Z is $T(m)_*$–acyclic for all $0 < m < n$, and only if Z is $K(m)_*$–acyclic for all $0 < m < n$, there is a short exact sequence of $K(n)_*$–Hopf algebras

$$K(n)_*(\mathbb{P}(Z)) \xrightarrow{j_{n*}} K(n)_*(\mathbb{P}(Z)) \to \bigotimes_{j=0}^{n+1} K(n)_*(K(\pi_j^S(Z), j)).$$

It appears that for some Z, a calculation of both $K(n)_*(\mathbb{P}(Z))$ and j_{n*} may be accessible. If one could find a $K(n-1)_*$–acyclic space[3] Z, and explicit calculation showed that the above sequence is *not* exact, it would follow that Z would not be $T(m)_*$–acyclic for some $0 < m < n$. The first example to check is when $p = 2$, $n = 3$, and $Z = K(\mathbb{Z}/2, 3)$: it is known that this space is $K(2)_*$–acyclic, but it is unknown whether or not it is $T(2)_*$–acyclic.

[3] If a space Z is $K(n-1)_*$–acyclic, then it is $K(m)_*$–acyclic for all $m < n$ by Bousfield [8].

7.4 The periodic homotopy groups of odd dimensional spheres

Let $v\colon \Sigma^d M \to M$ be a $K(n)_*$–isomorphism of a space M whose suspension spectrum is a finite compex of type n. If Z is a space, one can use v to define periodic homotopy groups by letting

$$v^{-1}\pi_*(Z;M) = \operatorname*{colim}_r [\Sigma^{rd}M, Z]_*.$$

It is clear that these behave well with respect to fibration sequences in the Z variable. These can be similarly defined for spectra, and it is evident that there is an isomorphism $v^{-1}\pi_*(\Omega^\infty X;M) = v^{-1}\pi_*(X;M)$.

The direct limit appearing in the definition suggests that these functors of spaces do not necessarily commute with holimits of towers. However Arone and Mahowald note that the properties listed in Theorem 6.7 imply that the tower for an odd dimensional sphere leads to a convergent spectral sequence with only a finite number of infinite loop fibers for computing periodic homotopy. More precisely, they show prove the following.

Theorem 7.15 *Let m be odd. With (M,v) as above, the natural map*

$$v^{-1}\pi_*(S^m;M) \to v^{-1}\pi_*(P_{p^n}S^m;M)$$

is an isomorphism.

Bousfield notes that periodic homotopy can be computed using the telescopic functor Φ_n.

Proposition 7.16 (Bousfield [9, Theorem 5.3(ii) and Corollary 5.10(ii)]) *There are natural isomorphisms*

$$v^{-1}\pi_*(Z;M) \simeq [M, \Phi_n(Z)]_*.$$

Furthermore, given $f\colon Y \to Z$, $v^{-1}\pi_*(f;M)$ is an isomorphism if and only if $\Phi_n(f)\colon \Phi_n(Y) \to \Phi_n(Z)$ is a weak equivalence.

One can now deduce the following theorem.

Theorem 7.17 *When m is odd, there is a spectral sequence which converges to $v^{-1}\pi_*(S^m;M)$ with*

$$E^1_{-k,*} = \begin{cases} [M, L_{T(n)}L(k,m)]_{*+k-m} & \text{for } 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

This follows by assembling various results from above. By Theorem 7.15 and Theorem 6.7(1), applying $v^{-1}\pi_*(\ \ ;M)$ to the tower $P_d(S^m)$, the tower of the identity evaluated at S^m, yields a spectral sequence with

$$E^1_{-k,*} = v^{-1}\pi_*(D_{p^k}(S^m);M)$$

for $0 \le k \le n$ and 0 otherwise. But this rewrites:

$$\begin{aligned} v^{-1}\pi_*(D_{p^k}(S^m);M) &= [M, \Phi_n(\Omega^\infty D^{st}_{p^k}(S^m)]_* && \text{(by Proposition 7.16)} \\ &= [M, L_{T(n)} D^{st}_{p^k}(S^m)]_* && \text{(by Theorem 7.11)} \\ &= [M, L_{T(n)} L(k,m)]_{*+k-m} && \text{(by Theorem 6.7(2))}. \end{aligned}$$

Similarly, applying Φ_n to the tower $P_d(S^m)$, yields the next theorem.

Theorem 7.18 *Let m be odd. The spectrum $\Phi_n(S^m)$ admits a finite decreasing filtration with fibers $L_{T(n)}\Sigma^{m-k}L(k,m)$ for $k = 0,\dots,n$.*

For a space Z, the v_n–periodic homotopy groups with integral coefficients are defined as follows.

Let L^f_{n-1} denote localization with respect to $T(0)\vee\cdots\vee T(n-1)$. Then $L^f_{n-1}X \simeq *$ if and only if X is the filtered homotopy colimit of finite spectra of type at least n, and it follows quite formally from this that L^f_{n-1} is smashing. (See eg Bousfield [9] and Miller [38].)

Let $C^f_{n-1}X$ denote the fiber of $X \to L^f_{n-1}X$. Again C^f_{n-1} is smashing. $C^f_{n-1}S$ can be realized as the homotopy colimit of a sequence of finite spectra of type n,

$$C_1 \to C_2 \to \cdots .$$

Fairly explicit ways of doing this are described in Mahowald–Sadofsky [34] and [27].

One then defines $v_n^{-1}\pi_*(Z)$ by letting

$$v_n^{-1}\pi_*(Z) = \operatorname*{colim}_r v^{-1}\pi_*(Z;\mathcal{D}(C_r)).$$

The results above show that alternatively this can be computed as

$$v_n^{-1}\pi_*(Z) = \pi_*(C^f_{n-1}\Phi_n(Z)).$$

A little diagram chasing shows that there is an equivalence

$$C^f_{n-1}L_{T(n)}X \simeq M^f_n X,$$

where $M^f_n X$ is the fiber of $L^f_n X \to L^f_{n-1}X$. ($M^f_n X$ is the n^{th} monocular part of X is the terminology of Bousfield [7].)

Analogous to Theorems 7.17 and 7.18 is the following.

Theorem 7.19 *When m is odd, there is a spectral sequence for computing $v_n^{-1}\pi_*(S^m)$ with*
$$E^1_{-k,*} = \begin{cases} \pi_{*+k-m}(M_n^f L(k,m)) & \text{for } 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.17, Theorem 7.18, and Theorem 7.19 can be improved. We describe how this goes for Theorem 7.18. The inclusion
$$S^1 \to \Omega^{m-1} S^m$$
induces a map of towers
$$P_d(S^1) \to \Omega^{m-1} P_d(S^m).$$
Taking homotopy fibers and localizing at p, one gets a tower converging to $\Omega^m_0 S^m$, with fibers $\Omega^\infty L(k)^{m-1}_1$, where $L(k)^{m-1}_1$ is the fiber of the natural map of spectra $L(k,1) \to L(k,m)$. Applying Φ_n to this tower, one deduces

Theorem 7.20 *Let m be odd. The spectrum $\Phi_n(S^m)$ admits a finite decreasing filtration with fibers $L_{T(n)}\Sigma^{m+1-k}L(k)^{m-1}_1$ for $k = 1, \dots, n$.*

Example 7.21 When $p = 2$, $L(1)^m_1 = \mathbb{R}P^m$. Specializing to $n = 1$, we learn that there is a weak equivalence
$$\Phi_1(S^{2k+1}) \simeq \Sigma^{2k+1} L_{K(1)}\mathbb{R}P^{2k}.$$
Specializing to $n = 2$, we learn that there is a fibration sequence of spectra
$$\Phi_2(S^{2k+1}) \to \Sigma^{2k+1} L_{T(2)}\mathbb{R}P^{2k} \to \Sigma^{2k} L_{T(2)} L(2)^{2k}_1.$$
The first of these is equivalent to Mahowald's theorem [33] that the James–Hopf map $\Omega^{2k} S^{2k+1} \to \Omega^\infty\Sigma^\infty\Sigma\mathbb{R}P^{2k}$ induces an isomorphism on v_1–periodic homotopy groups. Thus Mahowald's older work can be given a conceptual proof and put into a general context.

References

[1] **S T Ahearn**, **N J Kuhn**, *Product and other fine structure in polynomial resolutions of mapping spaces*, Algebr. Geom. Topol. 2 (2002) 591–647 MR1917068

[2] **G Arone**, *A generalization of Snaith-type filtration*, Trans. Amer. Math. Soc. 351 (1999) 1123–1150 MR1638238

[3] **G Z Arone**, **W G Dwyer**, *Partition complexes, Tits buildings and symmetric products*, Proc. London Math. Soc. (3) 82 (2001) 229–256 MR1794263

[4] **G Arone**, **M Mahowald**, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. 135 (1999) 743–788 MR1669268

[5] **M Basterra**, **M A Mandell**, *Homology and cohomology of E_∞ ring spectra*, Math. Z. 249 (2005) 903–944 MR2126222

[6] **M Basterra**, **R McCarthy**, *Γ–homology, topological André–Quillen homology and stabilization*, Topology Appl. 121 (2002) 551–566 MR1909009

[7] **A K Bousfield**, *Uniqueness of infinite deloopings for K–theoretic spaces*, Pacific J. Math. 129 (1987) 1–31 MR901254

[8] **A K Bousfield**, *On $K(n)$–equivalences of spaces*, from: "Homotopy invariant algebraic structures (Baltimore, MD, 1998)", Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 85–89 MR1718077

[9] **A K Bousfield**, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. 353 (2001) 2391–2426 MR1814075

[10] **A K Bousfield**, **E M Friedlander**, *Homotopy theory of Γ–spaces, spectra, and bisimplicial sets*, from: "Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II", Lecture Notes in Math. 658, Springer, Berlin (1978) 80–130 MR513569

[11] **E S Devinatz**, **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) 128 (1988) 207–241 MR960945

[12] **W G Dwyer**, **J Spaliński**, *Homotopy theories and model categories*, from: "Handbook of algebraic topology", North–Holland, Amsterdam (1995) 73–126 MR1361887

[13] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, American Mathematical Society, Providence, RI (1997) MR1417719

[14] **T G Goodwillie**, *Calculus. I. The first derivative of pseudoisotopy theory*, K–Theory 4 (1990) 1–27 MR1076523

[15] **T G Goodwillie**, *Calculus. II. Analytic functors*, K–Theory 5 (1991/92) 295–332 MR1162445

[16] **T G Goodwillie**, *Calculus. III. Taylor series*, Geom. Topol. 7 (2003) 645–711 MR2026544

[17] **J P C Greenlees**, **H Sadofsky**, *The Tate spectrum of v_n–periodic complex oriented theories*, Math. Z. 222 (1996) 391–405 MR1400199

[18] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, American Mathematical Society, Providence, RI (2003) MR1944041

[19] **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) 148 (1998) 1–49 MR1652975

[20] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI (1999) MR1650134

[21] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127 MR1860878

[22] **M Hovey**, **H Sadofsky**, *Tate cohomology lowers chromatic Bousfield classes*, Proc. Amer. Math. Soc. 124 (1996) 3579–3585 MR1343699

[23] **B Johnson**, *The derivatives of homotopy theory*, Trans. Amer. Math. Soc. 347 (1995) 1295–1321 MR1297532

[24] **D S Kahn**, *On the stable decomposition of* $\Omega^\infty S^\infty A$, from: "Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II", Lecture Notes in Math. 658, Springer, Berlin (1978) 206–214 MR513576

[25] **D S Kahn**, **S B Priddy**, *The transfer and stable homotopy theory*, Math. Proc. Cambridge Philos. Soc. 83 (1978) 103–111 MR0464230

[26] **N J Kuhn**, *A Kahn–Priddy sequence and a conjecture of G W Whitehead*, Math. Proc. Cambridge Philos. Soc. 92 (1982) 467–483 MR677471

[27] **N J Kuhn**, *Morava* K*–theories and infinite loop spaces*, from: "Algebraic topology (Arcata, CA, 1986)", Lecture Notes in Math. 1370, Springer, Berlin (1989) 243–257 MR1000381

[28] **N J Kuhn**, *The McCord model for the tensor product of a space and a commutative ring spectrum*, from: "Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)", Progr. Math. 215, Birkhäuser, Basel (2004) 213–236 MR2039768

[29] **N J Kuhn**, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. 157 (2004) 345–370 MR2076926

[30] **N J Kuhn**, *Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. 201 (2006) 318–378 MR2211532

[31] **N J Kuhn**, **G Arone**, *Mapping spaces and homology isomorphisms*, Proc. Amer. Math. Soc. 134 (2006) 1237–1248 MR2196061

[32] **N J Kuhn**, **S B Priddy**, *The transfer and Whitehead's conjecture*, Math. Proc. Cambridge Philos. Soc. 98 (1985) 459–480 MR803606

[33] **M Mahowald**, *The image of* J *in the* EHP *sequence*, Ann. of Math. (2) 116 (1982) 65–112 MR662118

[34] **M Mahowald**, **H Sadofsky**, v_n *telescopes and the Adams spectral sequence*, Duke Math. J. 78 (1995) 101–129 MR1328754

[35] **M Mahowald**, **P Shick**, *Root invariants and periodicity in stable homotopy theory*, Bull. London Math. Soc. 20 (1988) 262–266 MR931189

[36] **J P May**, *The geometry of iterated loop spaces*, Springer, Berlin (1972) MR0420610

[37] **R McCarthy**, *Dual calculus for functors to spectra*, from: "Homotopy methods in algebraic topology (Boulder, CO, 1999)", Contemp. Math. 271, Amer. Math. Soc., Providence, RI (2001) 183–215 MR1831354

[38] **H Miller**, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) 37 (1992) 383–389 MR1317588

[39] **V Minasian**, *André–Quillen spectral sequence for THH*, Topology Appl. 129 (2003) 273–280 MR1962984

[40] **S A Mitchell**, **S B Priddy**, *Stable splittings derived from the Steinberg module*, Topology 22 (1983) 285–298 MR710102

[41] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer, Berlin (1967) MR0223432

[42] **D C Ravenel**, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies 128, Princeton University Press, Princeton, NJ (1992) MR1192553

[43] **S Schwede**, *Spectra in model categories and applications to the algebraic cotangent complex*, J. Pure Appl. Algebra 120 (1997) 77–104 MR1466099

Department of Mathematics, University of Virginia
Charlottesville VA 22903, USA

njk4x@virginia.edu

Received: 13 October 2004 Revised: 12 July 2005