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Determination of the multiplicative nilpotency of self-homotopy sets

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The semigroup of the homotopy classes of the self-homotopy maps of a finite complex which induce the trivial homomorphism on homotopy groups is nilpotent. We determine the nilpotency of these semigroups of compact Lie groups and finite Hopf spaces of rank 2. We also study the nilpotency of semigroups for Lie groups of higher rank. Especially, we give Lie groups with the nilpotency of the semigroups arbitrarily large.

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Introduction

Let $[X,Y]$ denote the based homotopy classes of maps from X to Y. When $X=Y$, the self-homotopy set $[X,X]$ is a monoid by the binary operation induced by composition of maps. In this paper we deal with a subset $\mathcal{Z}^n(X)$ which consists of elements inducing the trivial homomorphism on homotopy groups in dimensions $\le n$, where n is a natural number or ∞. $\mathcal{Z}^n(X)$ is a multiplicative subset of $[X,X]$, though it has no unit element and is merely a semigroup in general. With respect to this binary operation the (multiplicative) nilpotency is defined (see Section 1). For a finite complex X, it is known by Arkowitz–Maruyama–Stanley [1] that $\mathcal{Z}^n(X)$ is nilpotent for a sufficiently large integer n and the nilpotency is bounded above by other invariants such as the cone length or the killing length of X. On the other hand, lower bounds for the nilpotency of $\mathcal{Z}^n(X)$ or more desirably the precise value of it have not been studied except for a few cases, see [1]. Our purposes are to know the nilpotency for compact Lie groups or finite H–spaces of low rank and to obtain lower bounds of the nilpotency for more general cases. We will determine the nilpotency of $\mathcal{Z}^n(X)$ when X is a finite 1–connected H–space of rank 2. We will also determine the nilpotency of $\mathcal{Z}^n(X)$ in the rank 3 cases where $X=SU(4)$ or $Sp(3)$. Incidentally the nilpotencies are equal to 2 in these cases. However this is not the case for Lie groups of higher rank. Actually, we will give the lower bounds of the nilpotency for $SU(n)$ or $Sp(n)$ and show that the nilpotency could be arbitrarily large for these spaces.

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We briefly review the sections. In Section 1 we recall some basic definition and the work of Arkowitz–Maruyama–Stanley [1] and Maruyama [5]. In Section 2 we first compute the nilpotency of $\mathcal{Z}^*(X)$ when X is a 1–connected compact Lie group of rank 2, then apply the result to the case where X is a 1–connected finite H–space of rank 2. In Section 3 we find the nilpotency of $SU(4)$ and $Sp(3)$. To this end we use S Oka's work on the structures of self-homotopy sets of $SU(4)$ and $Sp(3)$. In Section 4 we derive a property of the nilpotency of the rationalization of an H–space which is the key to the proof of the theorem on the nilpotency of $SU(n)$ and $Sp(n)$ mentioned above.

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1 Preliminaries

In this section we fix our notation and recall some results in [1]. For spaces X and Y, let $\mathcal{Z}^n(X,Y)$ denote the subset of $[X,Y]$ consisting of all homotopy classes $\alpha \in [X,Y]$ such that $\alpha_* = 0$: $\pi_i(X) \to \pi_i(Y)$ for $i \leq n$. If $n = \infty$ we write $\mathcal{Z}^\infty(X,Y)$. We also write $\mathcal{Z}(X,Y)$ for $\mathcal{Z}^{\dim X}(X,Y)$ if $n = \dim X$. Finally we write $\mathcal{Z}^n(X)$ for $\mathcal{Z}^n(X,X)$ and $\mathcal{Z}^\infty(X)$ for $\mathcal{Z}^\infty(X,X)$.

$\mathcal{Z}^n(X)$ is a semigroup by the binary operation induced by composition of maps.

Definition 1.1 If there exists an integer $t \geq 1$ such that $a_1 \circ a_2 \circ \cdots \circ a_t = 0$ for all $a_1, a_2, \ldots, a_t \in \mathcal{Z}(X)$, then $\mathcal{Z}(X)$ is called nilpotent. The smallest such t is called the nilpotency of $\mathcal{Z}(X)$ and written $t(X)$. Similarly we define the nilpotency of $\mathcal{Z}^\infty(X)$ and denote it by $t_\infty(X)$.

Clearly $t_\infty(X) \leq t(X)$. In [1] it is shown that if X is a finite complex, then $t(X)$ and thus $t_\infty(X)$ are finite and the following inequalities allow us to know about the upper bounds for the nilpotency.

Theorem 1.2 (Arkowitz, Maruyama and Stanley [1]) *If X is a 1–connected finite complex then,*

$$t_\infty(X) \leq t(X) \leq \mathrm{kl}_s(X) \leq \mathrm{cl}_s(X)$$

where $\mathrm{kl}_s(X)$ is the spherical killing length of X and $\mathrm{cl}_s(X)$ is the spherical cone length of X.

In this paper we deal with the spaces which have multiplications. When G is a group-like finite complex, $[G,G]$ is a nilpotent group (see Whitehead [12]) and $\mathcal{Z}^n(G)$ is a subgroup. There exists the following naturality property of localization which will be used in the proofs of the results in the later sections.

Proposition 1.3 (Maruyama [5]) *Let G be a group-like finite complex. Then with respect to the group structures induced from the multiplication of G, $\mathcal{Z}(G)_p \cong \mathcal{Z}(G_p)$ and $\mathcal{Z}^\infty(G)_p \cong \mathcal{Z}^\infty(G_p)$ for any prime p. Here X_p is the localization of X at p.*

2 The rank 2 case

In this section we consider simply connected compact Lie groups of rank 2 and related H–spaces.

Theorem 2.1 *Let G be a 1–connected compact Lie group of rank 2. Then*

$$t(G) = t_\infty(G) = 2.$$

Proof It is known that G is isomorphic to one of the Lie groups

$$S^3 \times S^3, SU(3), Sp(2), G_2.$$

We have the exact sequence

$$0 \to [S^3 \wedge S^3, S^3 \times S^3] \xrightarrow{q^*} [S^3 \times S^3, S^3 \times S^3] \to [S^3 \vee S^3, S^3 \times S^3] \to 0,$$

where $q\colon S^3 \times S^3 \to S^3 \wedge S^3$ is the projection map to the smash product. Since generally the projection $q\colon S^m \times S^n \to S^m \wedge S^n \simeq S^{m+n}$ belongs to $\mathcal{Z}^\infty(S^m \times S^n, S^{m+n})$ and by the above exact sequence we easily obtain

$$\mathcal{Z}^\infty(S^3 \times S^3) = \mathcal{Z}(S^3 \times S^3) = \operatorname{Im} q^* = \pi_6(S^3) \oplus \pi_6(S^3) = \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}.$$

Let $f_1, f_2 \in \mathcal{Z}^\infty(S^3 \times S^3)$ ($= \mathcal{Z}(S^3 \times S^3)$), then $f_1 \circ f_2 = 0$. We already know that $\mathcal{Z}^\infty(S^3 \times S^3)$ is not trivial. Thus $t_\infty(S^3 \times S^3) = t(S^3 \times S^3) = 2$. Now we turn to the other cases. Let G be one of our Lie groups other than $S^3 \times S^3$ and let

$$q_G^*\colon \pi_{\dim G}(G) \to \mathcal{Z}(G)$$

denote the induced map of $q_G\colon G \to S^{\dim G}$, the pinching map to the top cell. If $G = SU(3)$ or $Sp(2)$, then $\mathcal{Z}^\infty(G) = \mathcal{Z}(G) \neq 0$ by Maruyama [6] (isomorphic to $\mathbb{Z}_{12}, \mathbb{Z}_{120}$ respectively) and they coincide with $\operatorname{Im} q_G^*$. Let $f_1, f_2 \in \operatorname{Im} q_G^*$ and $f_1 = q_G^*(x_1)$, $f_2 = q_G^*(x_2)$, then $f_1 \circ f_2$ is trivial since in the composition

$$f_1 \circ f_2\colon G \xrightarrow{q_G} S^{\dim G} \xrightarrow{x_2} G \xrightarrow{q_G} S^{\dim G} \xrightarrow{x_1} G$$

x_2 is of finite order while $[S^{\dim G}, S^{\dim G}]$ is isomorphic to $\mathbb{Z}$. Therefore we obtain that $t(G) = t_\infty(G) = 2$ for $G = SU(3)$ and $Sp(2)$. Though it is shown that $\mathcal{Z}(G_2)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{21}$, $\mathcal{Z}^\infty(G_2)$ is not determined (for some partial results see [6]). However by [6] and Ōshima [10] $\mathcal{Z}(G_2)$ is generated by the elements of $\mathrm{Im}\, q_{G_2}^*$ and $[1, \alpha]$. Here $[1, \alpha]$ is the commutator of the identity map and some $\alpha \in [G_2, G_2]$ of infinite order. $[1, \alpha]$ is known to be of order 2. Thus $\mathcal{Z}^\infty(G_2)$ is not trivial as $[1, \alpha]$ is an element of $\mathcal{Z}^\infty(G_2)$. Let $f = q_{G_2}^*(x) \in \mathrm{Im}\, q_{G_2}^*(x \in \pi_{14}(G_2))$ and $g \in \mathcal{Z}(G_2)$, then $f \circ g = 0 = g \circ f$. For,

$$g \circ f \colon G_2 \xrightarrow{q_{G_2}} S^{14} \xrightarrow{x} G_2 \xrightarrow{g} G_2$$

is trivial since $g \in \mathcal{Z}(G_2)$.

$$f \circ g \colon G_2 \xrightarrow{g} G_2 \xrightarrow{q_{G_2}} S^{14} \xrightarrow{x} G_2$$

is also trivial as for the previous cases. Moreover $[1, \alpha] \circ h = [h, \alpha \circ h] = 0$ for any $h \in \mathcal{Z}(G_2)$, because $\alpha \circ h$ is an element of $\mathcal{Z}(G_2)$ and the group $\mathcal{Z}(G_2)$ is commutative. Let $f, f' \in \mathrm{Im}\, q_{G_2}^*$. By the above arguments

$$(f + [1, \alpha]) \circ (f' + [1, \alpha]) = f \circ (f' + [1, \alpha]) + [1, \alpha] \circ (f' + [1, \alpha]) = 0.$$

Thus we have shown that all the compositions of the elements of $\mathcal{Z}(G_2)$ are trivial. As was noted above $\mathcal{Z}^\infty(G_2)$ contains an nontrivial element $[1, \alpha]$, $t_\infty(G_2) > 1$ (actually $|\mathcal{Z}^\infty(G_2)|$ is greater than 42 see [6]). Therefore $t(G_2) = t_\infty(G_2) = 2$, and we obtain the result. □

The assertion of Theorem 2.1 also holds for finite H–spaces of rank 2.

Theorem 2.2 *Let X be a 1–connected finite H–space of rank 2. Then*

$$t(X) = t_\infty(X) = 2.$$

We will use the following lemma.

Lemma 2.3 *Let X be a finite nilpotent space. If $t(X_p) \le n$ for all prime numbers p, then $t(X) \le n$. The same is true for $t_\infty(X)$.*

Proof Let $a_1, \ldots, a_n$ be elements of $\mathcal{Z}(X)$. Then

$$(a_1 \circ \cdots \circ a_n)_p = 0$$

for any prime number p by the assumption. Thus $a_1 \circ \cdots \circ a_n$ is trivial by Hilton–Mislin–Roitberg [3, Corollary 5.12, Chapter II]. □

Proof of Theorem 2.2 By the classical result of Mimura, Nishida and Toda [7], a 1–connected finite H–space X of rank 2 is homotopy equivalent to one of $S^3 \times S^3$, $SU(3)$, E_k ($k = 0, 1, 3, 4, 5$), $S^7 \times S^7$ or $G_{2,b}$ $(-2 \leq b \leq 5)$. Here E_k is the principal S^3–bundle over S^7 with the characteristic class $k\omega \in \pi_7(BS^3)$, ω a generator, and $G_{2,b}$ is the principal S^3–bundle over the Stiefel manifold $V_{7,2}$ induced by a suitable map $f_b \colon V_{7,2} \to BS^3$. We note that $E_1 = Sp(2)$, $G_{2,0} = G_2$ and we have already shown the assertion for $SU(3)$, $Sp(2)$ and G_2 in Theorem 2.1. By definition we obtain

$$\begin{aligned} &(E_3)_p \simeq Sp(2)_p \text{ for } p \neq 3 && (E_3)_3 \simeq S^3_3 \times S^7_3 \\ &(E_4)_p \simeq Sp(2)_p \text{ for } p \neq 2 && (E_4)_2 \simeq S^3_2 \times S^7_2 \\ &(E_5)_p \simeq Sp(2)_p \text{ for all } p. && \end{aligned}$$

Let p be a prime number, then

$$(G_{2,b})_p \simeq (G_2)_p \quad \text{or} \quad (G_{2,b})_p \simeq S^3_p \times S^{11}_p$$

depending on p by [7]. Therefore if X is a 1–connected finite H–space of rank 2, then X_p is homotopy equivalent to G_p or $S^m_p \times S^n_p$ for each prime number p, where G is a 1–connected Lie group of rank 2 and $m, n \in \{3, 5, 7, 11\}$. It is easy to see that $t(S^m_p \times S^n_p) \leq 2$ for any prime p. On the other hand, $\mathcal{Z}(G_p) = \mathcal{Z}(G)_p$ by Proposition 1.3 and $\mathcal{Z}(G)_p \subset \mathcal{Z}(G)$ since $\mathcal{Z}(G)$ is a finite nilpotent group for a 1–connected compact Lie group G of rank 2. Thus $t(G_p) \leq 2$ for an arbitrary prime number p. Thus we obtain that if X is a 1–connected finite H–space of rank 2, then $t(X_p) \leq 2$ for all prime numbers p and hence $t(X) \leq 2$ by Lemma 2.3.

Next we will show that $t_\infty(X) > 1$ for our spaces. Namely, $\mathcal{Z}^\infty(X) \neq 0$. First we consider the case where $X = S^7 \times S^7$. As in the proof of Theorem 2.1 $\mathcal{Z}^\infty(S^7 \times S^7) = \pi_{14}(S^7) \oplus \pi_{14}(S^7)$ which is isomorphic to $\mathbb{Z}_{120} \oplus \mathbb{Z}_{120}$ by Toda [11]. We should note that $[S^7 \times S^7, S^7 \times S^7]$ is a group despite that S^7 is not homotopy associative (see Mimura–Ōshima [8]) though we do not need the group structure for our purpose. Let X be an H–space. If $n \geq \dim X$, by a result of James [4] there exists a bijection

$$T \colon \mathcal{Z}^n(X) \to \mathcal{E}^n_\#(X) \tag{2–1}$$

defined by $f \to 1 + f$. Here $\mathcal{E}^n_\#(X)$ is the group of homotopy classes of self-homotopy equivalences which induce the identity map on $\pi_i(X)$ for $i \leq n$.

$$\mathcal{E}^\infty_\#(E_k)_5 \cong \mathcal{E}^\infty_\#((E_k)_5),$$

by [5]. Note that $(E_k)_5$ is homotopy equivalent to $S^3_5 \times S^7_5$. Namely E_k is 5-regular. The group $\mathcal{E}^\infty_\#(S^3_5 \times S^7_5)$ is easily shown to be isomorphic to $\mathbb{Z}_5$ (see [6] or [8]). Hence

$$\mathcal{E}^\infty_\#(E_k)_5 \cong \mathbb{Z}_5.$$

Thus by the bijection T, $\mathcal{Z}^\infty(E_k)$ is not trivial for $k = 0, 1, 3, 4, 5$. Similarly it is known that the spaces $G_{2,b}$ $(-2 \le b \le 5)$ are 7–regular, that is

$$(G_{2,b})_7 \simeq S^3_7 \times S^{11}_7.$$

Therefore we obtain

$$\mathcal{E}^\infty_\#(G_{2,b})_7 \cong \mathcal{E}^\infty_\#(S^3 \times S^{11})_7.$$

The group $\mathcal{E}^\infty_\#(S^3 \times S^{11})_7$ is not trivial ($\cong \mathbb{Z}_7$ see [6]). Thus $\mathcal{Z}^\infty(G_{2,b})$ is non-trivial by the same reason for E_k.

Consequently, $t_\infty(X) > 1$ for all the 1–connected finite H–spaces of rank 2. We complete the proof. □

Remark In the above proof we cannot use Proposition 1.3 directly to show that $t_\infty(X) > 1$ because the spaces are not necessarily homotopy associative.

3 $SU(4)$ and $Sp(3)$

In this section we consider rank 3 Lie groups $SU(4)$ and $Sp(3)$. The statement of our theorem is completely the same as that of Theorem 2.1, but its proof is more complicated.

Theorem 3.1 *$t(G) = t_\infty(G) = 2$ for $G = SU(4)$ and $Sp(3)$.*

Our arguments in this section depend heavily on Oka's results in [9].

Let C_φ be the mapping cone of $\varphi\colon X \to Y$, $q\colon C_\varphi \to \Sigma X$ the projection map. Recall that there exists an action of $[\Sigma X, C_\varphi]$ on $[C_\varphi, C_\varphi]$ induced by the coaction map $\ell\colon C_\varphi \to \Sigma X \vee C_\varphi$. Namely, for $\alpha \in [\Sigma X, C_\varphi]$ and $g \in [C_\varphi, C_\varphi]$, $\alpha \cdot g$ is the following composition:

$$C_\varphi \xrightarrow{\ell} \Sigma X \vee C_\varphi \xrightarrow{\alpha \vee g} C_\varphi \vee C_\varphi \xrightarrow{\nabla} C_\varphi$$

where ∇ is the folding map. The following lemma is well known.

Lemma 3.2 (Hilton [2, Theorem 15.7]) *Let $\alpha \in [\Sigma X, C_\varphi]$ and $g, h \in [C_\varphi, C_\varphi]$ be arbitrary elements. If C_φ is an H–space, then*

$$\alpha \cdot g = q^*(\alpha) + g \text{ and } h \circ (q^*(\alpha) + g) = h \circ q^*(\alpha) + h \circ g,$$

where $+$ denotes the addition induced by the H–structure of C_φ.

Now we consider the case where $G = SU(4)$. As noted by Oka [9, (2.2)], there exists a homotopy equivalence as follows.

$$SU(4)/SU(4)^7 \to \Sigma K \vee S^{15},$$

where $SU(4)^7$ is the 7–skeleton of $SU(4)$ and $K = (S^7 \vee S^9) \cup e^{11}$. We denote by

$$\pi_1\colon SU(4) \to \Sigma K$$

the projection map. We have the following maps (homomorphisms):

$$q^*_{SU(4)}\colon \pi_{15}(SU(4)) \to [SU(4), SU(4)]$$
$$\pi_1^*\colon [\Sigma K, SU(4)] \to [SU(4), SU(4)]$$

Lemma 3.3 *$\mathcal{Z}(SU(4))$ and $\mathcal{Z}^\infty(SU(4))$ are generated by elements of* $\mathrm{Im}\, q^*_{SU(4)} \cup \mathrm{Im}\pi_1^*$. *In particular they are abelian groups.*

Proof We show our claim is true for $\mathcal{Z}(SU(4))$ since $\mathcal{Z}^\infty(SU(4))$ is a subgroup of $\mathcal{Z}(SU(4))$. Let $\mathcal{E}_*(X)$ denote the the group of homotopy classes of self-homotopy equivalences which induce the identity map on the integral homology groups of X. Since $H^*(SU(4))$ is isomorphic to the exterior algebra $\Lambda_{\mathbb{Z}}(x_3, x_5, x_7)$, we have

$$\mathcal{E}^n_\#(SU(4)) \subset \mathcal{E}_*(SU(4)).$$

Here $n \geq 7$. By [9, Theorem 2.4, Thorem 8.3], $\mathcal{E}_*(SU(4))$ is generated by elements

$$q^*_{SU(4)}(x) + 1_{SU(4)} \quad \text{and} \quad \pi_1^*(y) + 1_{SU(4)},$$

where $x \in \pi_{15}(SU(4))$ and $y \in [\Sigma K, SU(4)]$. We easily see that

$$(q^*_{SU(4)}(x) + 1_{SU(4)})^n = q^*_{SU(4)}(nx) + 1_{SU(4)},$$
$$(\pi_1^*(y) + 1_{SU(4)})^n = \pi_1^*(ny) + 1_{SU(4)}$$

for $n \in \mathbb{Z}$, (cf [9]). Moreover,

$$(\pi_1^*(y) + 1_{SU(4)}) \circ (q^*_{SU(4)}(x) + 1_{SU(4)}) = q^*_{SU(4)}(y \circ \pi_1 \circ x + x) + \pi_1^*(y) + 1_{SU(4)},$$

by Lemma 3.2, and

$$(q^*_{SU(4)}(x) + 1_{SU(4)}) \circ (\pi_1^*(y) + 1_{SU(4)}) = q^*_{SU(4)}(x) + \pi_1^*(y) + 1_{SU(4)}.$$

The second equality follows from

$$q_{SU(4)} \circ (\pi_1^*(y) + 1_{SU(4)}) = q_{SU(4)}.$$

Therefore by the bijection T given in (2–1) $\mathcal{Z}(SU(4))$ is generated by some elements of $\mathrm{Im}\, q^*_{SU(4)}$ and $\mathrm{Im}\, \pi_1^*$.

It is known that $\mathrm{Im}\, q^*_{SU(4)}$ is in the center of $[SU(4), SU(4)]$ by [12]. Therefore $\mathcal{Z}(SU(4))$ is an abelian group since $\mathrm{Im}\, \pi_1^*$ is abelian. □

Now we prove the main theorem in this section.

Proof (of Theorem 3.1) By Lemma 3.3 an element of $\mathcal{Z}(SU(4))$ is of the form $z = f + g$ with $f \in \mathrm{Im}\, q^*_{SU(4)}$ and $g \in \mathrm{Im}\, \pi_1^*$. Since f induces the trivial map on $\pi_i(SU(4))$, $i \leq 15$, g is an element of $\mathcal{Z}(SU(4))$. Therefore $g \circ f = 0$. We also have $f \circ g = 0$ easily. Let f_1, f_2 be elements of $\mathrm{Im}\, q^*_{SU(4)}$, then $f_1 \circ f_2 = 0$. Now $[\Sigma K, SU(4)]$ is a finite group and $\pi_{1*}\colon [\Sigma K, SU(4)] \to [\Sigma K, \Sigma K]$ is a homomorphism. We have the following isomorphism by [9, Lemma 3.3]:

$$[\Sigma K, \Sigma K] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore $g_1 \circ g_2 = 0$ for $g_1, g_2 \in \mathrm{Im}\, \pi_1^*$. Let $h\colon SU(4) \to SU(4)$ be any map, then we have $(f+g) \circ h = f \circ h + g \circ h$, since the addition is defined by the group structure of $SU(4)$. Moreover by Lemma 3.2 $h \circ (f+g) = h \circ f + h \circ g$. Here $f \in \mathrm{Im}\, q^*_{SU(4)}$ and $g \in \mathrm{Im}\, \pi_1^*$ as above. Consequently, the composition of any two elements $z = f_1 + g_1$, $z' = f_2 + g_2$ of $\mathcal{Z}(SU(4))$ is trivial, where $f_1, f_2 \in \mathrm{Im}\, q^*_{SU(4)}$ and $g_1, g_2 \in \mathrm{Im}\, \pi_1^*$. Therefore we have obtained that $t(SU(4)) \leq 2$ and thus $t_\infty(SU(4)) \leq 2$ (recall that $t_\infty(SU(4)) \leq t(SU(4))$).

Next we will show that $1 < t_\infty(SU(4))$, that is, $\mathcal{Z}^\infty(SU(4))$ is not trivial. It is known that $SU(4)_3$ is homotopy equivalent to $Sp(2)_3 \times S^5_3$. Therefore $\mathcal{Z}^\infty(Sp(2)_3) \subset \mathcal{Z}^\infty(SU(4)_3)$. As was mentioned in the proof of Theorem 2.1, $\mathcal{Z}^\infty(Sp(2))$ is isomorphic to $\mathbb{Z}_{120}$, and hence $\mathcal{Z}^\infty(SU(4)_3)$ is nontrivial. Therefore by Proposition 1.3, $\mathcal{Z}^\infty(SU(4))$ is also nontrivial.

The proof for $Sp(3)$ is parallel to that of $SU(4)$ by using [9, Theorem 2.5, Theorem 4.3]. We can show that $t(Sp(3)) \leq 2$ as in the $SU(4)$ case. To show that $t(Sp(3)) = 2$, we use the equivalence $Sp(3)_7 \simeq (S^3 \times S^7 \times S^{11})_7$ and the nontriviality of $\mathcal{Z}^\infty((S^3 \times S^7 \times S^{11})_7)$. This nontriviality is obtained by the existence of an essential map:

$$S^3 \times S^7 \times S^{11} \to S^3 \times S^{11} \to S^{14} \xrightarrow{\alpha_1} S^3 \to S^3 \times S^7 \times S^{11}$$

where α_1 is a generator of $\pi_{14}(S^3)_7 \cong \mathbb{Z}_7$ and other maps are the canonical projections and the inclusions. □

4 The lower bounds for classical groups

In this section we give a lower bound for $t_*(G)$ when G is $SU(n)$ or $Sp(n)$. We should admit that it is a crude one, but it gives us the theorem which states that $t_\infty(G)$ could be arbitrarily large for classical groups.

Proposition 4.1 *Let X be a homotopy associative finite H–space, then $t(X) \geq t(X_0)$ and $t_\infty(X) \geq t_\infty(X_0)$.*

Proof The rational cohomology ring $H^*(X;\mathbb{Q})$ is isomorphic to the exterior algebra $\Lambda_{\mathbb{Q}}(x_1, \cdots x_r)$ on primitive elements x_i with $\dim x_i = n_i$ odd. Let n denote $t(X)$ and $a_1, \ldots, a_n$ be elements of $\mathcal{Z}(X_0)$. We will show that $a_1 \circ \cdots \circ a_n = 0$. As X_0 is homotopy equivalent to the product of the Eilenberg-MacLane spaces, the elements of $[X_0, X_0]$ are determined by their induced maps on cohomology groups. Actually $\mathcal{Z}(X_0)$ is isomorphic to the module generated by the decomposable elements of degree $\{\dim x_i\}$. We define a basis for $\mathcal{Z}(X_0)$ as follows. Let $\{x_{i_1} x_{i_2} \cdots x_{i_j}\}$ with $i_1 < i_2 < \cdots < i_j$ be the basis for the module of decomposable elements of $H^*(X;\mathbb{Q})$. Let $\{i_1 < i_2 < \cdots < i_j\}$ be a set of subsets of $\{1, \ldots, r\}$ such that

$$\dim x_{i_1} + \dim x_{i_2} + \cdots + \dim x_{i_j} = \dim x_i .$$

Let $\wedge\colon \prod_{k=1}^{j} K(\mathbb{Q}, n_{i_k}) \to \wedge_{k=1}^{j} K(\mathbb{Q}, n_{i_k})$ be the projection to the smash product. Then the map $f_{i_1 i_2 \cdots i_j}$ is defined by the composition

$$X_0 \to \prod_{k=1}^{j} K(\mathbb{Q}, n_{i_k}) \xrightarrow{\wedge} \wedge_{k=1}^{j} K(\mathbb{Q}, n_{i_k}) \to K(\mathbb{Q}, n_i) \to X_0,$$

where the first and the last maps in the composition are the projection and inclusion maps, the third map is the map corresponding to the cohomology element $x_{i_1} x_{i_2} \cdots x_{i_j}$. Now we assume that $a_1 \circ \cdots \circ a_n$ is not trivial. Hence $(a_1 \circ \cdots \circ a_n)^*(x_k) \neq 0$ for some x_k. We have

$$(a_1 \circ \cdots \circ a_{i-1})^*(x_k) = \Sigma t_j x_{j_1} x_{j_2} \cdots x_{j_\ell},$$

for $i \leq n$, where t_j are nonzero rational numbers. Thus

$$a_i^*(x_{j_1} x_{j_2} \cdots x_{j_\ell})$$

is nontrivial for some $x_{j_1} x_{j_2} \cdots x_{j_\ell}$. It follows that for each x_{j_t} there exist decomposable elements $x_{s_1} x_{s_2} \cdots x_{s_k}$ such that $\dim x_{j_t} = \dim x_{s_1} x_{s_2} \cdots x_{s_k}$. Therefore the maps $f_{s_1 s_2 \cdots s_k}$ are defined for $\{x_{s_1} x_{s_2} \cdots x_{s_k}\}$, and $a_i^*(x_{j_1} x_{j_2} \cdots x_{j_\ell})$ is $(\Sigma r_{s_1 s_2 \cdots s_k} f_{s_1 s_2 \cdots s_k})^*(x_{j_1} x_{j_2} \cdots x_{j_\ell})$, where $r_{s_1 s_2 \cdots s_k}$ are rational numbers. From the nontriviality of $(a_1 \circ \cdots \circ a_n)^*(x_k)$, we obtain the nontrivial iterated composition :

$$g_1 \circ \cdots \circ g_n \in \mathcal{Z}(X_0)$$

such that $g_i = \Sigma f_{i_1 i_2 \cdots i_j}$ for each g_i. Let $m_1, \dots, m_n$ be nonzero integers. Since we see that

$$(m_1 g_1 \circ \cdots \circ m_n g_n)^*(x_k) = d x_{i_1} x_{i_2} \cdots x_{i_j}$$

for some $x_{i_1} x_{i_2} \cdots x_{i_j}$ with $i_1 < i_2 < \cdots < i_j$ and nontrivial integer d, thus the composition $m_1 g_1 \circ \cdots \circ m_n g_n$ is essential. Here we should note that $(m_1 g_1 \circ \cdots \circ m_n g_n)^*(x_k)$ is not equal to $m_1 m_2 \cdots m_n (g_1 \circ \cdots \circ g_n)^*(x_k)$ in general. As the homomorphism $\mathcal{Z}(X) \to \mathcal{Z}(X_0)$ is the localization by Proposition 1.3, $m_i g_i$ is an element of $\mathcal{Z}(X)$ for some nonzero integer m_i. So, we can find a nontrivial composition $m_1 g_1 \circ \cdots \circ m_n g_n$ with $m_i g_i \in \mathcal{Z}(X)$, this is a contradiction, hence we obtain that $t(X) \geq t(X_0)$.

Since $\mathcal{Z}(X_0) = \mathcal{Z}^\infty(X_0)$ in our case, we can show that $t_\infty(X) \geq t_\infty(X_0)$ similarly. □

Now we apply Proposition 4.1 to the classical groups $SU(n)$ and $Sp(n)$.

Theorem 4.2 *Let ℓ be a natural number. Then*

$$t_\infty(SU(n)) > \ell \text{ for } n \geq (3^\ell + 1)^2/2$$
$$t_\infty(Sp(n)) > \ell \text{ for } n \geq (2 \cdot 5^{2\ell} + 5^\ell + 1)/4.$$

Proof Recall that $SU(n)$ is rationally equivalent to

$$S_0^3 \times \cdots \times S_0^{2n-1}.$$

By Proposition 4.1 it suffices to construct a desired nontrivial composition in $\mathcal{Z}^\infty(S_0^3 \times \cdots \times S_0^{2n-1})$. To this end, we take the smash product for each successive 3^k spheres in the product space. We already have dealt with such a map in Proposition 4.1, that is f_i. However here we need a more careful consideration about dimensions. Now we assume that n is sufficiently large. We let

$$\wedge_{2i-1,2i+1,2i+3} \colon S_0^{2i-1} \times S_0^{2i+1} \times S_0^{2i+3} \to S_0^{6i+3}.$$

denote the projection map to the smash product. Then we take the product of these maps

$$\prod_{i=1}^{3^{\ell-1}} \wedge_{6i-3,6i-1,6i+1} \colon \prod_{i=1}^{3^\ell} S_0^{2i+1} \to \prod_{i=1}^{3^{\ell-1}} S_0^{18i-3}.$$

We define a map $a_1 \colon S_0^3 \times \cdots \times S_0^{2n-1} \to S_0^3 \times \cdots \times S_0^{2n-1}$ to be the following composition.

$$\prod_{i=1}^{n-1} S_0^{2i+1} \to \prod_{i=1}^{3^\ell} S_0^{2i+1} \xrightarrow{\Pi_{i=1}^{3^{\ell-1}} \wedge_{6i-3,6i-1,6i+1}} \prod_{i=1}^{3^{\ell-1}} S_0^{18i-3} \to \prod_{i=1}^{n-1} S_0^{2i+1}$$

where the first map is the projection and the third map is the inclusion.

Similarly we construct a_2 as follows:

$$\prod_{i=1}^{n-1} S_0^{2i+1} \to \prod_{i=1}^{3^{\ell-1}} S_0^{18i-3} \xrightarrow{\Pi_{i=1}^{3^{\ell-2}} \wedge_{54i-39,54i-21,54i-3}} \prod_{i=1}^{3^{\ell-2}} S_0^{162i-63} \to \prod_{i=1}^{n-1} S_0^{2i+1}.$$

We continue this process and finally we obtain a_ℓ:

$$\prod_{i=1}^{n-1} S_0^{2i+1} \to \prod_{i=1}^{3} S_0^{3^{\ell-1}((2i-1)3^{\ell-1}+2)} \to S_0^{3^{2\ell}+2\cdot 3^\ell} \to \prod_{i=1}^{n-1} S_0^{2i+1}.$$

Clearly a map $a_\ell \circ a_{\ell-1} \cdots \circ a_2 \circ a_1$ induces a nontrivial map on cohomology and moreover induces the trivial map on the homotopy groups. The construction is possible if $2n-1 \geq 3^{2\ell} + 2\cdot 3^\ell$. Namely, if $n \geq (3^\ell+1)^2/2$ then

$$t_\infty(SU(n)) > \ell.$$

This completes the claim for $SU(n)$. To prove the $Sp(n)$ case we can use the same methods for the $SU(n)$ case. For $Sp(n)$ this time we consider the projection maps to the smash products from successive 5^k spheres instead of 3^k spheres which were used in the proof of $SU(n)$. Then we obtain that if $n \geq (2\cdot 5^{2\ell} + 5^\ell + 1)/4$,

$$t_\infty(Sp(n)) > \ell. \qquad \square$$

Remark We can apply the similar arguments in the proof of the above theorem to other classical groups. We obtain

$$t_\infty(U(n)) > \ell \ \text{ for } \ n \geq (3^\ell+1)^2/2,$$

and

$$t_\infty(G(m)) > \ell \text{ for } \begin{cases} m \geq (2\cdot 5^{2\ell} + 5^\ell + 3)/2 & \text{if } m \text{ is odd} \\ m \geq (2\cdot 5^{2\ell} + 5^\ell + 5)/2 & \text{if } m \text{ is even,} \end{cases}$$

where ℓ is an integer, and $G(m) = SO(m), \mathrm{Spin}(m)$ or $O(m)$.

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