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Toward a fundamental groupoid for the stable homotopy category

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This very speculative sketch suggests that a theory of fundamental groupoids for tensor triangulated categories could be used to describe the ring of integers as the singular fiber in a family of ring-spectra parametrized by a structure space for the stable homotopy category, and that Bousfield localization might be part of a theory of 'nearby' cycles for stacks or orbifolds.

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Introduction

One of the motivations for this paper comes from John Rognes' Galois theory for structured ring spectra. His account [37] ends with some very interesting remarks about analogies between classical primes in algebraic number fields and the non-Euclidean primes of the stable homotopy category, and I try here to develop a language in which these analogies can be restated as the assertion that Waldhausen's *unfolding*

$$\mathrm{spec}\,\mathbb{Z} \to \mathrm{spec}\, S$$

of the integers in the category of brave new rings (or E_∞ ring-spectra, or commutative S–algebras) leads to the existence of commutative diagrams of the form

$$\begin{array}{ccc} \mathrm{spec}\, \mathbf{o}_{\overline{\mathbb{Q}}_p} & \longrightarrow & \mathrm{spec}\, L^{MU}_{K(n)} MU \\ \downarrow {\scriptstyle \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} & & \downarrow {\scriptstyle D^\times} \\ \mathrm{spec}\, \mathbb{Z}_p & \longrightarrow & \mathrm{spec}\, L_{K(n)} S \ . \end{array}$$

The vertical arrow on the left is the Galois cover defined by the ring of integers in an algebraic closure of the p–adic rationals, but to make sense of the right-hand side would require, among other things, a good theory of structure objects (analogous to the prime ideal spectra of commutative algebra) for some general class of tensor triangulated categories. In this direction I have mostly hopes and analogies, summarized in Section 3.

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The main result of the first section below, however, is that local classfield theory implies the existence of an interesting system of group homomorphisms

$$\rho\colon \mathrm{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to D^\times$$

which can plausibly be interpreted as the maps induced on the fundamental groups of these hypothetical structure objects. [The group on the left is Weil's technical variant of the Galois group; its topology (see Lichtenbaum [32]) is slightly subtler than the usual one.]

This example is local: it depends on a choice of the prime p. But Rognes [37, Section 12.2.1] has global results as well; in particular, he identifies the ringspectrum $S[BU]$ as the Hopf algebra of functions on an analog $\mathrm{Gal}_{\mathrm{hot}}$ of a Galois group for MU, regarded as an (inseparable) algebraic closure of S. This seems to be in striking agreement with work of Connes and Marcolli [10], Cartier [6], and Deligne and Goncharov [14] on a remarkable 'cosmic' generalization of Galois theory, involving a certain motivic Galois group $\mathrm{Gal}_{\mathrm{mot}}$: there is a very natural morphism

$$S[BU] \to H\mathbb{Z} \otimes \mathrm{QSymm}_*$$

of Hopf algebra objects, QSymm_* being a certain graded Hopf algebra of quasisymmetric functions (see Hazelwinkel [21]), defined by the inclusion of the symmetric in the quasisymmetric functions, and in Section 2 I suggest that a quotient of this map defines a representation

$$\mathrm{Gal}_{\mathrm{mot}} \to \mathrm{Gal}_{\mathrm{hot}}$$

(at least, over $\mathbb{Q}$) which conjecturally plays the role of the homomorphism induced on fundamental groups by a 'geometric realization' construction [35], sending the derived category of mixed Tate motives to some category of complex-oriented spectra.

The paper ends with a very impressionistic (fauvist?) discussion of a possible theory of fundamental groupoids for (sufficiently small) tensor triangulated categories, which might be flexible enough to encompass both these examples.

I am indebted to many mathematicians for conversations about this material over the years, most recently A Ne'eman, T Torii, and W Dwyer. It is a particular pleasure to dedicate this paper to Goro Nishida, in recognition of his broad vision of the importance of group actions, at many levels, in topology.

1 Some local Galois representations

This section summarizes some classical local number theory. The first two subsections define a system $\mathrm{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to D^\times$ of representations of certain Galois-like groups in the units of suitable p–adic division algebras; then 1.3 sketches a conjecture about the structure of the absolute inertia group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{nr}})$, which says, roughly, that these systems are nicely compatible.

1.0 A *local* field is a commutative field, with a nontrivial topology in which it is locally compact. The reals or complexes are examples, but I will be concerned here mostly with totally disconnected cases, in particular the fields of characteristic zero obtained as non-Archimedean completions of algebraic number fields. These are finite extensions L of $\mathbb{Q}_p$, for some Euclidean prime p; the topology defines a natural equivalence class of valuations, with the elements algebraic over $\mathbb{Z}_p$ as (local) valuation ring.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of an algebraic closure of the rationals acts on the set of prime ideals in the ring of algebraic integers in $\mathbb{Q}$, with orbits corresponding to the classical primes; the corresponding isotropy groups can be identified with the Galois groups $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the algebraic closures of the p–adic rationals. These isotropy groups preserve the valuation rings, and hence act on their residue fields, defining (split) exact sequences

$$1 \to \mathbf{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}} \to 0$$

of profinite topological groups. The cokernel is the closure of a dense subgroup $\mathbb{Z}$ generated by the Frobenius automorphism $\sigma\colon x \mapsto x^p$ of $\overline{\mathbb{F}}_p$, and Weil observed that in some contexts it is more natural to work with the pullback extension

$$1 \to \mathbf{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z} \to 0\ ,$$

the so-called Weil group, which is now only locally compact. [These groups are defined much more generally by Tate [41] but we won't need that here.]

1.1.1 The work of Lubin and Tate defines very interesting representations of these groups as automorphisms of certain one-dimensional formal groups. Honda's logarithm

$$\log_q(T) = \sum_{i \geq 0} p^{-i} T^{q^i}$$

(with $q = p^n$) defines a formal group law

$$F_q(X,Y) = \log_q^{-1}(\log_q(X) + \log_q(Y)) \in \mathbb{Z}_p[\![X,Y]\!]$$

(it's not obvious that its coefficients are integral!) whose endomorphism ring contains, besides the elements

$$T \mapsto [a](T) = \log_q^{-1}(a \log_q(T))$$

defined by multiplication in the formal group by a p–adic integer a, the endomorphism

$$T \mapsto [\omega](T) = \log_q^{-1}(\omega \log_q(T)) = \omega T$$

defined by multiplication by a $(q-1)$st root of unity ω. In fact the full ring of endomorphisms of $\mathbb{F}_q$ can be identified with the Witt ring $W(\mathbb{F}_q)$, which can also be described as the algebraic integers in the unramified extension field $\mathbb{Q}_q$ obtained from $\mathbb{Q}_p$ by adjoining ω.

1.1.2 By reducing the coefficients of F_q modulo p we obtain a formal group law $\bar{F}_q$ over $\mathbb{F}_p$, which admits

$$T \mapsto F(T) = T^p$$

as a further endomorphism. Honda's group law is a Lubin–Tate group (see Serre [38]) for the field $\mathbb{Q}_q$, and it can be shown that

$$[p](T) \equiv T^q \bmod p ;$$

in other words, $F^n = p$ in the ring of endomorphisms of $\bar{F}_q$. From this it is not hard to see that

$$\mathrm{End}_{\bar{\mathbb{F}}_p}(\bar{F}_q) = W(\mathbb{F}_q)\langle F\rangle/(F^n - p)$$

(the pointed brackets indicating a ring of noncommutative indeterminates, subject to the relation

$$a^\sigma F = Fa$$

for $a \in W(\mathbb{F}_q)$, with $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$ being the (other!) Frobenius endomorphism). This is the ring $\mathbf{o}_D$ of integers in the division algebra

$$D = \mathbb{Q}_q\langle F\rangle/(F^n - p)$$

with center $\mathbb{Q}_p$; its group $\mathbf{o}_D^\times$ of strict units is the full group of automorphisms of $\bar{F}_q$, and it is convenient to think of

$$1 \to \mathbf{o}_D^\times \to D^\times \to \mathbb{Z} \to 0$$

as a semidirect product, with quotient generated by F. Thus conjugation by F acts on $W(\mathbb{F}_q)^\times \subset \mathbf{o}_D^\times$ as $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$: the Galois action is encoded in the division algebra structure.

1.2.1 For each $n \geq 1$, a deep theorem of Weil and Shafarevich defines a homomorphism

$$ws_q\colon \mathrm{W}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p) \cong W(\mathbb{F}_q)^\times \ltimes \mathbb{Z} \to \mathbf{o}_D^\times \ltimes \mathbb{Z} = D^\times$$

of locally compact topological groups. It extends the local version of Artin's reciprocity law, which defines an isomorphism

$$L^\times \cong \mathrm{W}(L^{\mathrm{ab}}/L)$$

for any totally disconnected local field L (with L^{ab} its maximal abelian extension). When $L = \mathbb{Q}_q$ is unramified, the sequence

$$1 \to \mathrm{W}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_q) \to \mathrm{W}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p) \to \mathrm{W}(\mathbb{Q}_q/\mathbb{Q}_p) \to 0$$

is just the product

$$0 \to W(\mathbb{F}_q)^\times \times \mathbb{Z} \to W(\mathbb{F}_q)^\times \ltimes \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

of the elementary exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

with a copy of the units in $W(\mathbb{F}_q)$: in the semidirect product extension above, the generator $1 \in \mathbb{Z}$ acts on $W(\mathbb{F}_q)$ by σ, so its nth power acts trivially.

1.2.2 More generally, any Galois extension L of degree n over $\mathbb{Q}_p$ can be embedded in a division algebra of rank n with center $\mathbb{Q}_p$, as a maximal commutative subfield. When the division algebra D has invariant $1/n$ in the Brauer group $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{Q}_p$ (as it does in our case: this invariant equals the class, modulo $\mathbb{Z}$, of the p–order of an element (eg F) generating the maximal ideal of $\mathbf{o}_D$), the theorem [45, Appendix] of Weil and Shafarevich defines an isomorphism of the normalizer of $L^\times$ in $D^\times$ with the Weil group of L^{ab} over $\mathbb{Q}_p$.

[The maximal abelian extension L^{ab} is obtained by adjoining the p–torsion points of the Lubin–Tate group of L to the maximal unramified extension L^{nr}; the resulting field acquires an action of the group $\mathbf{o}_L^\times$ of units of L (by 'complex multiplication' on the Lubin–Tate group) together with an action of the automorphisms $\hat{\mathbb{Z}}$ of the algebraic closure of the residue field.

The point is that the Lubin–Tate group of L is *natural* in the etale topology: to be precise, any two Lubin–Tate groups for L become isomorphic over the completion $L^{\mathbf{nr}}$ of the maximal unramified extension $L^{\mathrm{nr}} = L \otimes_{W(k)} W(\bar{k})$ of L (see Serre [38, Section 3.7]). Since an automorphism of L over $\mathbb{Q}_p$ takes one Lubin–Tate group to another, the resulting group of automorphisms of 'the' Lubin–Tate group of L (as a completed Hopf algebra over $\mathbb{Z}_p$) is an extension of $\mathrm{Gal}(L/\mathbb{Q}_p)$ by $\mathbf{o}_L^\times \times \hat{\mathbb{Z}}$, that is, the profinite completion of $L^\times$ [34]. This extension is classified by an element of

$$H^2(\mathrm{Gal}(L/\mathbb{Q}_p), L^\times) \cong \mathbb{Z}/n\mathbb{Z}\ ,$$

which also classifies those algebras, simple with center $\mathbb{Q}_p$, which split after tensoring with L. The extension in question generates this group, by a fundamental result of local classfield theory; but the division algebra with invariant $1/n$ also generates this group, and the associated group extension is the normalizer of $L^\times$ in $D^\times$.]

1.2.3 The remarks in this subsection are a digression, but they will be useful in 3.3: Since the normalizer of $L^\times$ acts as generalized automorphisms of a Lubin–Tate group for L, then for every $g \in \mathrm{W}(L^{\mathrm{ab}}/\mathbb{Q}_p)$ there is a power series

$$[g](T) \in \mathbf{o}_{L^{\mathbf{nr}}}[\![T]\!]$$

satisfying $[g_0]([g_1](T)) = [g_0 g_1](T)$, compatible with the natural action of theWeil group $\mathrm{W}(L^{\mathrm{ab}}/\mathbb{Q}_p)$ on the ring of integers $\mathbf{o}_{L^{\mathbf{nr}}}$ in $L^{\mathbf{nr}}$. It follows that $[g](t) = \alpha(g)T + \cdots$ defines a *crossed* homomorphism

$$\alpha\colon \mathrm{W}(L^{\mathrm{ab}}/\mathbb{Q}_p) \to (L^{\mathbf{nr}})^\times ,$$

ie a map satisfying $\alpha(g_0 g_1) = \alpha(g_0) \cdot \alpha(g_1)^{g_0}$, the superscript denoting the action of the Weil group on $L^{\mathbf{nr}}$ through its quotient $\mathrm{W}(L^{\mathrm{nr}}/\mathbb{Q}_p)$. When $L = \mathbb{Q}_q$ this implies the existence of an extension of the identity homomorphism from $\mathbb{Q}_q^\times$ to itself, to a crossed homomorphism from $\mathbb{Q}_q^\times \ltimes \mathbb{Z}$ to $(L^{\mathbf{nr}})^\times$. A corollary is the existence of a representation $\mathbb{Q}_p^{\mathbf{nr}}(1)$ of $\mathrm{W}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p)$ on the completion $\mathbb{Q}_p^{\mathbf{nr}}$ extending the action of $W(\mathbb{F}_q)^\times$ by multiplication. [The completions in this construction are cumbersome, and might be unnecessary. Experts may know how to do without them, but I don't.]

1.3 The representations promised in the introduction are the compositions

$$\rho_q\colon \mathrm{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{W}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p) \to D^\times$$

with the second arrow coming from the Weil–Shafarevich theorem.

In the remainder of this section I will sketch a conjecture about the relations between these representations. The argument goes back to Serre's *Cohomologie Galoisienne*, and I believe that many people have thought along the lines below, but I don't know of any place in the literature where this is spelled out. It is based on a p–adic analog of a conjecture of Deligne, related to an older conjecture of Shafarevich (see Morava [33]; see also Furusho [15]).

1.3.1 We need some basic facts:

(i) in order that a pro–p–group be free, it is necessary and sufficient that its cohomological dimension be ≤ 1 (see Serre [39, I.4.2, Corollary 2]), and

(ii) the maximal unramified extension of a local field with perfect residue field has cohomological dimension ≤ 1 [39, II.3.3, Ex c].

I will follow Serre's notation, which very similar to that used above. K is a field complete with respect to a discrete valuation, with residue field k, eg a finite extension of $\mathbb{Q}_p$. $K^{\rm nr}$ will denote its maximal unramified extension, K^s its (separable) algebraic closure, and $K^{\rm tr}$ will be the union of the tamely ramified Galois extensions (ie with Galois group of order prime to p) of K in K^s; thus

$$K^s \supset K^{\rm tr} \supset K^{\rm nr} \supset K .$$

Eventually K will be the quotient field $\mathbb{Q}_q$ of the ring $W(k)$ of Witt vectors for some $k = \mathbb{F}_q$ with $q = p^f$ elements, but for the moment we can be more general. We have

$$\mathrm{Gal}(K^{\rm tr}/K^{\rm nr}) \cong \varprojlim \mathbb{F}_{p^n}^\times \cong \prod_{l \neq p} \mathbb{Z}_l = \hat{\mathbb{Z}}(1)(\neg p)$$

[39, II.4, Ex 2a]; the term on the right denotes the component 'away from p' of the Tate representation of $\mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$, in which $1 \in \mathbb{Z}$ acts as multiplication by $q = \#(k)$ [39, II.5.6 Ex 1]. The kernel in the extension

$$1 \to \mathrm{Gal}(K^s/K^{\rm tr}) = P \to \mathrm{Gal}(K^s/K^{\rm nr}) \to \mathrm{Gal}(K^{\rm tr}/K^{\rm nr}) \to 1$$

is a pro–p–group [39, Ex 2b] closed in a group of cohomological dimension one (by assertion (ii)), hence itself of cohomological dimension one [39, I.3.3, Proposition 14], hence *free* by assertion (i).

The extension in question is the *inertia* group of K; it has a natural $\mathrm{Gal}(\bar{k}/k)$–action.

1.3.2 From now on I will assume that $K = \mathbb{Q}_q$ is unramified over $\mathbb{Q}_p$.

Conjecture I *The extension displayed immediately above splits:* $\mathrm{Gal}(K^s/K^{\rm nr}) \cong P \rtimes \hat{\mathbb{Z}}(1)(\neg p)$ *is a semidirect product.*

[Since the kernel and quotient have relatively prime order, this would be obvious if either were finite. This may be known to the experts.]

Let $W(k)_0^\times = (1 + pW(k))^\times$ be the group of those units of $W(k)$ congruent to 1 mod p: the logarithm defines an exact sequence

$$1 \to k^\times \to W(k)^\times \to W(k) \to k \to 0$$

taking $W(k)_0^\times$ isomorphically to $pW(k)$. Let

$$\mathbb{W}(\bar{k})_0^\times = \varprojlim \{W(k')_0^\times \mid k' \text{ finite} \subset \bar{k}\}$$

be the limit under norms; it is a compactification of $W(\bar{k})_0^\times$.

Conjecture II *The topological abelianization of P is naturally isomorphic to $\mathbb{W}(\bar{k})_0^\times$.*

Alternately: Lazard's group ring

$$\mathbb{Z}_p[\![P]\!] \cong \varprojlim \{\hat{T}(W(k')_0^\times) \mid k' \text{ finite } \subset \bar{k}\}$$

of P of is isomorphic to the limit (under norms) of the system of completed tensor algebras of $pW(k') \cong W(k')_0^\times$. It is thus (hypothetically) a kind of noncommutative Iwasawa algebra [39, I.1.5, Proposition 7, page 8].

If both these conjectures are true, then I can think of no natural way for $\hat{\mathbb{Z}}(1)(\neg p)$ to act on P, so I will go the rest of the way and conjecture as well that this action is *trivial*. I will also abbreviate the sum of these conjectures as the assertion that the absolute inertia group $\mathbf{I}$ of $\mathbb{Q}_p$ is the product of $\hat{\mathbb{Z}}(1)(\neg p)$ with the pro-free pro–p–group generated by $W(\bar{k})_0^\times \cong pW(\bar{k})$.

1.3.3 These conjectures seems to be compatible with other known facts of classfield theory. In particular, they would imply that

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_q) \cong \mathbf{I} \ltimes \hat{\mathbb{Z}}$$

with $1 \in \mathbb{Z}$ acting as $\sigma(x) = x^q$ on k. This abelianizes to $W(k)^\times \times \hat{\mathbb{Z}}$, agreeing with Artin's reciprocity law, and if $q_1 = q_0^m$ this would yield an exact sequence

$$1 \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{q_1}) \cong \mathbf{I} \ltimes \hat{\mathbb{Z}} \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{q_0}) \cong \mathbf{I} \ltimes \hat{\mathbb{Z}} \to \mathrm{Gal}(\mathbb{F}_{q_1}/\mathbb{F}_{q_0}) \cong \mathbb{Z}/m\mathbb{Z} \to 0\,.$$

Finally, the composition

$$\mathbf{I} \to \mathbf{I}_{\mathrm{ab}} \cong \mathbb{W}(\bar{k})_0^\times \times \hat{\mathbb{Z}}(1)(\neg p) \to W(\mathbb{F}_q)^\times$$

defines a compatible system of candidates for the quotient maps

$$\rho_q\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \cong \mathbf{I} \ltimes \hat{\mathbb{Z}} \to W(\mathbb{F}_q)^\times \ltimes \hat{\mathbb{Z}} \cong \mathrm{Gal}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p)\,.$$

2 Some more global representations

This section is concerned with two very global objects, each of which has some unfamiliar features. The first subsection is concerned with the suspension spectrum

$$S[BU] = \Sigma^\infty(BU_+)$$

of the classifying space for the infinite unitary group, and its interpretation (following Rognes [37]) as a Hopf algebra object in the category of spectra. The second subsection reviews some properties of the Hopf algebra of quasisymmetric functions, following Cartier [6] and Hoffman [23]. A remarkable number of the properties of the Hopf algebra of symmetric functions generalize to this context (see Hazelwinkel [22]), and I will try to keep this fact in focus. This Hopf algebra is conjectured to be closely related to a certain motivic group of interest in arithmetic geometry, and I have tried to say a little about that, in particular because it seems to overlap with recent work of Connes and Marcolli, discussed in 2.3.4.

2.1.1 Since BU is an infinite loopspace, $S[BU]$ becomes an E_∞ ringspectrum (or, in an alternate language, a commutative S–algebra); but BU is also a *space*, with a diagonal map, and this structure can be used to make this ringspectrum into a Hopf algebra in the category of S–modules.

In complex cobordism, the complete Chern class (see Adams [1])

$$c_t = \sum c_I \otimes t^I = \phi^{-1} s_t \phi(1) \in MU^*(BU_+) \otimes S_*$$

(where $S_* = \mathbb{Z}[t_i \mid i \geq 1]$ is the Landweber–Novikov algebra, with coaction s_t) represents a morphism $S[BU] \to MU \wedge MU$ of ringspectra. Indeed, algebra morphisms from $S[BU]$ to the Thom spectrum MU correspond to maps

$$\mathbb{C}P^\infty_+ = MU(1) \to MU$$

of spectra, and hence to elements of $MU^*(\mathbb{C}P^\infty_+)$. The coaction

$$MU \cong S \wedge MU \to MU \wedge MU$$

is also ring map, so the resulting composition

$$S[BU] \to MU = S \wedge MU \to MU \wedge MU$$

is a morphism of ring spectra. On the other hand the Thom isomorphism

$$\phi\colon MU^*(BU_+) \cong MU^*(MU)$$

satisfies

$$\phi^{-1} s_t \phi(1) = c_t = \prod_i \mathbf{e}_i^{-1} t(\mathbf{e}_i)$$

(where $t(\mathbf{e}) = \sum t_k \mathbf{e}^{k+1} \in MU^*(\mathbb{C}P^\infty_+) \otimes S_*$, with $\mathbf{e}$ being the Euler, or first Chern, class).

In fact this map is also a morphism of Hopf algebra objects:

$$s_{t'}(\Delta c_t) = c_{t' \circ t} \otimes c_{t' \circ t} \in MU^*(BU_+ \wedge BU_+) \otimes (S_* \otimes S_*) ,$$

so the diagram

$$\begin{array}{ccccc} S[BU] & \longrightarrow & S[BU]\wedge S[BU] & \longrightarrow & (MU\wedge MU)\wedge(MU\wedge MU) \\ \downarrow & & & & \downarrow \\ MU\wedge MU & & \longrightarrow & & (MU\wedge MU)\wedge_{MU}(MU\wedge MU) \end{array}$$

commutes.

We can thus think of $S[BU]$ as a kind of Galois group for the category of MU–algebras over S, or alternately for the category of complex-oriented multiplicative cohomology theories. In particular, if E is an MU–algebra, algebra maps from $S[BU]$ to E define elements of $E^0(\mathbb{C}P^\infty_+)$; thus the group $\mathrm{Aut}(E)$ of multiplicative automorphisms of E maps to algebra homomorphisms from $S[BU]$ to E. The adjoint construction thus sends $S[BU]$ to the spectrum of maps from $\mathrm{Aut}(E)$ (regarded naively, as a set) to E; but these maps can be regarded as E_*–valued functions on $\mathrm{Aut}(E)$, and hence as elements of the coalgebra E_*E.

Note that $S[BU]$ is large, so the existence of an honest dual object in the category of spectra may be problematic. The remark above implies that this perhaps nonexistent group object admits the etale groupschemes $\mathbf{o}_D^\times$ of Section 1 as subgroups.

2.1.2 The Landweber–Novikov algebra represents the group of invertible power series under composition, and it may be useful below to know that (since all one-dimensional formal groups over the rationals are equivalent) $(H_*(MU,\mathbb{Q}), H_*(MU\wedge MU,\mathbb{Q}))$ represents the transformation groupoid defined by this group acting on itself by translation. On the other hand $H_*(BU,\mathbb{Q})$ represents the group of formal power series with leading coefficient 1, under multiplication, and the induced morphism

$$\mathrm{spec}(H_*(MU,\mathbb{Q}), H_*(MU\wedge MU,\mathbb{Q})) \to \mathrm{spec}(H_*(S,\mathbb{Q}), H_*(BU,\mathbb{Q}))$$

of groupoidschemes sends the pair (g,h) of invertible series, viewed as a morphism from h to $g\circ h$, to the translated derivative $g'(h(t))$. This is essentially the construction which assigns to a formal group, its canonical invariant differential. The chain rule ensures that this is a homomorphism: we have $(g,h)\circ(k,g\circ h)=(k\circ g,h)$, while

$$g'(h(t))\cdot k'((g\circ h)(t)) = (k\circ g)'(h(t))\ .$$

I'm indebted to Neil Strickland for pointing out the advantage of working in this context with MUP, that is, the spectrum MU made periodic: its homotopy groups represent the functor which classifies formal group laws together with a coordinate, while $MUP\wedge MUP$ represents formal group laws with a *pair* of coordinates. The unit classified by $S[BU]$ is then just the ratio of these coordinates.

2.2.1 The $\mathbb{Z}$–algebra of symmetric functions manifests itself in topology as the integral homology of BU; it is a commutative and cocommutative Hopf algebra (with a canonical nondegenerate inner product [22] which, from the topological point of view, looks quite mysterious). Over $\mathbb{Q}$ it is the universal enveloping algebra of an abelian graded Lie algebra with one generator in each even degree.

The graded ring QSymm_* of *quasi* symmetric functions is the commutative Hopf algebra dual to the free associative algebra on (noncommutative!) generators Z_k of degree $2k$, with coproduct

$$\Delta Z_k = \sum_{i+j=k} Z_i \otimes Z_k \ ;$$

it is thus the universal enveloping algebra for a free graded Lie algebra $\mathfrak{f}_{\mathbb{Z}}$, with one generator in each (even) degree (see Hazelwinkel [21]). There is a natural monomorphism embedding the symmetric functions in the quasisymmetric functions (see Cartier [6, Section 2.4]), dual to the map on enveloping algebras defined by the homomorphism from the free Lie algebra to its abelianization. This then defines a morphism

$$S[BU] \to H\mathbb{Z} \otimes \mathrm{QSymm}_*$$

of (Hopf) ringspectra.

[In [3], which appeared after this paper was written, Baker and Richter show that the ringspectrum $S[\Omega\Sigma\mathbb{C}P^{\infty}_{+}]$ has many properties one might expect of a dual (but see the cautionary remarks in the paragraphs above) to the algebra of functions on $\mathrm{Gal}_{\mathrm{mot}}$.]

The group-valued functors represented by such Hopf algebras are very interesting, and have a large literature. In what follows I will simplify by tensoring everything with $\mathbb{Q}$, which will be general enough for anything I have to say. The formal (Magnus) completion of the $\mathbb{Q}$–algebra of noncommuting power series is dual to the algebra of functions on a free prounipotent (see Deligne [12, Section 9]) groupscheme $\mathfrak{F}$. In this context, a grading on a Lie algebra can be reinterpreted as an action of the multiplicative groupscheme $\mathbb{G}_m$, which sends an element x of degree d to $\lambda^d x$, where λ is a unit in whatever ring we're working with, so we can describe the map above as defining a morphism

$$\mathfrak{F} \ltimes \mathbb{G}_m \to \mathrm{Gal}_{\mathrm{hot}}$$

of group objects of some sort, over $\mathbb{Q}$.

2.2.2 In arithmetic geometry there is currently great interest in a groupscheme

$$\mathrm{Gal}_{\mathrm{mot}} = \mathfrak{F}_{\mathrm{odd}} \ltimes \mathbb{G}_m$$

which is conjectured to be (isomorphic to) the motivic Galois group of a certain Tannakian category, that of mixed Tate motives over $\mathbb{Z}$ (see Deligne and Goncharov [12; 14]). Usually $\mathfrak{F}_{\mathrm{odd}}$ is taken to be the prounipotent group defined by the free graded $\mathbb{Q}$–Lie algebra $\mathfrak{f}_{\mathrm{odd}}$ on generators of degree $4k+2$ (hence 'odd' according to the algebraists' conventions), with $k \geq 1$; but there are reasons to allow $k = 0$ as well. We can regard $\mathfrak{F}_{\mathrm{odd}}$ as a subgroupscheme of $\mathfrak{F}$, by regarding $\mathfrak{f}_{\mathrm{odd}}$ as a subalgebra of $\mathfrak{f}$.

A Tannakian category is, roughly, a suitably small k–linear abelian category with tensor product and duality – such as the category of finite-dimensional linear representations of a proalgebraic group over a field k. Indeed, the main theorem of the subject (see Deligne [13]) asserts that (when k has characteristic zero) any Tannakian category is of this form; then the relevant group is called the motivic group of the category. Present technology extracts mixed Tate motives from a certain triangulated category of (pieces of) algebraic varieties, constructed as a subcategory of the tensor triangulated category of more general motives (see Deligne and Goncharov [14] and Voevodsky [44]).

There are more details in Section 3 below, but one of the points of this paper is that the language of such tensor categories can be quite useful in more general circumstances. For example, the category of complex-oriented multiplicative cohomology theories behaves very much like (a derived category of) representations, with $S[BU]$ as its motivic group. Away from the prime two, the fibration

$$SO/SU_+ \to BSU_+ \to BSO_+$$

(defined by the forgetful map from $\mathbb{C}$ to $\mathbb{R}$) splits (even as maps of infinite loopspaces). Over the rationals, this is almost trivial: it corresponds to the splitting of the graded abelian Lie algebra with one generator in each even degree, into a sum of two such Lie algebras, with generators concentrated in degrees congruent to 0 and 2 mod 4 respectively. The composition (the first arrow is the projection of the H–space splitting, and the second corresponds to the abelianization of a graded free Lie algebra).

$$S[BU] \to S[SO/SU] \wedge S[\mathbb{C}P^\infty] \to H\mathbb{Q} \otimes \mathrm{QSymm}_{\mathrm{odd}}$$

is the candidate, promised in the introduction, for a natural representation

$$\mathrm{Gal}_{\mathrm{mot}} \to \mathrm{Gal}_{\mathrm{hot}}$$

(over $\mathbb{Q}$, of course!).

2.2.3 The degree two $(= 2(2 \cdot 0 + 1))$ generator in $\mathfrak{F}_{\mathrm{odd}}$ is closely related to the $S[\mathbb{C}P^\infty]$ factor in the ring decomposition above; both correspond to exceptional cases that deserve some explanation.

The symmetric *functions* are defined formally as an inverse limit of rings of symmetric *polynomials* in finitely many variables $x_n,\ n \geq 1$. There are thus interesting maps from the symmetric functions to other rings, defined by assigning interesting values to the x_n; but because infinitely many variables are involved, issues of convergence can arise. For example: if we send x_n to $1/n$, the nth power sum p_n maps to $\zeta(n) \in \mathbb{R} \ldots$ as long as $n > 1$, for $s = 1$ is a pole of $\zeta(s)$.

This is a delicate matter (see Cartier [6, Section 2.7] and Hoffman [23]), and it turns out to be very natural to send p_1 to Euler's constant γ. In fact this homomorphism extends, to define a homomorphism from QSymm to $\mathbb{R}$, whose image is sometimes called the ring MZN_* of 'multizeta numbers'; it has a natural grading. It is classical that for any positive integer n,

$$\zeta(2n) = -\tfrac{1}{2} B_{2n}\, \gamma_{2n}(2\pi i) \in \mathbb{Q}(\pi)\ ,$$

(where γ_k denotes the kth divided power) and in some contexts it is natural to work with the even-odd graded subring of $\mathbb{C}$ obtained by adjoining an invertible element $(2\pi i)^{\pm}$ to MZN_*. It is known (see Hain [20, Section 4]) that the multizeta numbers are periods of algebraic integrals, and that the ring generated by all such periods is a Hopf algebra, closely related to the algebra of functions on the (strictly speaking, still hypothetical) motivic group of all motives over $\mathbb{Q}$ (see Kontsevich [30]); but it is thought that Euler's constant is probably not a period. Nevertheless, from the homotopy-theoretic point of view presented here, it appears quite naturally.

2.3.1 These multizeta numbers may play some universal role in the general theory of asymptotic expansions; in any case, they appear systematically in renormalization theory. Connes and Marcolli, building on earlier work of Connes and Kreimer [8; 9], Broadhurst, and others, have put this in a Galois-theoretic framework. This subsection summarizes some of their work.

Classical techniques [of Bogoliubov, Parasiuk, Hepp, and Zimmerman] in physics have achieved an impressive level of internal consistency, but to mathematicians they lack conceptual coherence. Starting with a suitable Lagrangian density, Connes and Kreimer define a graded Lie algebra $\mathfrak{g}_*$ generated by a class of Feynman graphs naturally associated to the interactions encoded by the Lagrangian. They interpret the BPHZ dimensional regularization procedure as a Birkhoff decomposition for loops in the associated prounipotent Lie group $\mathcal{G}$, and construct a universal representation of this group in the formal automorphisms of the line at the origin, yielding a formula for a reparametrized coupling constant which eliminates the divergences in the (perturbative) theory defined by the original Lagrangian.

Dimensional regularization involves regarding the number d of space-time dimensions as a special value of a complex parameter; divergences are interpreted as poles at its physically significant value. Renormalization is thus expressed in terms much like the extraction of a residue, involving a simple closed curve encircling the relevant value of d (the source of the loop in $\mathcal{G}$ mentioned above). Connes and Marcolli [10] reformulate such data in geometric terms, involving flat connections on a $\mathbb{G}_m$–equivariant $\mathcal{G}$–bundle over a certain 'metaphysical' (not their terminology!) base space B. From this they define a Tannakian category (of flat, equisingular vector bundles with connection over B), and identify its motivic group as $\mathfrak{F} \ltimes \mathbb{G}_m$. Their constructions define a representation of the motivic group in $\mathcal{G}$, and hence in the group of reparametrizations of the coupling constant.

2.3.2 Besides the complex deformation of the space-time dimension, the base space B encodes information about the mass scale. It is a (trivial, but not naturally trivialized, see the end of [11, Section 2.13]) principal bundle

$$\mathbb{G}_m \to B \to \Delta$$

over a complex disk Δ centered around the physical dimension $d \in \mathbb{C}$. [This disk is treated as infinitesimal but there may be some use in thinking of it as the complement of infinity in $\mathbb{C}P_1$.] The renormalization group equations [10, Section 2.9] are reformulated in terms of the (mass-rescaling) $\mathbb{G}_m$–action on the principal bundle $\mathcal{G} \times B$ (the grading on $\mathfrak{g}_*$ endows $\mathcal{G}$ with a natural $\mathbb{G}_m$–action), leading to the existence of a unique gauge-equivalence class of flat $\mathbb{G}_m$–equivariant connection forms $\lambda \in \Omega^1(B, \mathfrak{g})$, which are equisingular in the sense that their restrictions to sections of B (regarded as a principal bundle over Δ) which agree at $0 \in \Delta$ are mutually (gauge) equivalent.

A key result [10, Theorem 2.25, Section 2.13] characterizes such forms λ in terms of a graded element $\beta_* \in \mathfrak{g}_*$ corresponding to the beta-function of renormalization group theory. In local coordinates ($z \in \Delta$ near the basepoint, $u \in \mathbb{G}_m$) we can write

$$\lambda(z,u) = \lambda_0(z,u) \cdot dz + \lambda_1(z,u) \cdot u^{-1} du$$

with coefficients $\lambda_i \in \mathbb{C}\{u, z\}[z^{-1}]$ allowed singularities at $z = 0$; but flatness and equivariance imply that these coefficients determine each other. The former condition [10, Equation 2.166] can be stated as

$$\partial_z \lambda_1 = H\lambda_0 - [\lambda_0, \lambda_1]$$

where the grading operator

$$H = (u\partial_u)|_{u=1}$$

is the infinitesimal generator of the $\mathbb{G}_m$–action; it sends u^k to ku^k. Regularity of λ_0 at $u = 0$ implies that

$$\lambda_0 = H^{-1}[\partial_z \lambda_1 + [H^{-1}\partial_z \lambda_1, \lambda_1] + \cdots]$$

is determined, at least formally, by knowledge of λ_1 in the fiber direction. Connes and Marcolli's solution [10, Theorems 2.15 and 2.18, Equation 2.173] of the renormalization group equations imply that

$$\lambda_1(u, z) = -z^{-1} u^*(\beta)$$

for some unique $\beta \in \mathfrak{g}$. This characterizes a universal (formal) flat equisingular connection $\lambda(\beta)$ on B, related to the universal singular frame of [10, Section 2.14].

2.3.3 The Lie algebra $\mathbb{V}$ of the group of formal diffeomorphisms of the line at the origin has canonical generators $v_k = u^{k+1}\partial_u$ satisfying $[v_k, v_l] = (l-k)v_{k+1}$, $k, l \geq 1$, so a graded module with an action of such operators defines a flat equisingular vector bundle over B with $\beta_* = v_*$. A commutative ringspectrum E with $S[BU]$–action defines a multiplicative complex-oriented cohomology theory, and in particular possesses an MU–module structure. On rationalized homotopy groups the morphism

$$E \to E \wedge S \to E \wedge MU$$

defines an S_*–comodule structure map

$$E^*_{\mathbb{Q}} \to E^*_{\mathbb{Q}} \otimes_{MU^*_{\mathbb{Q}}} MU^*MU_{\mathbb{Q}} = E^*_{\mathbb{Q}} \otimes S_*$$

and thus an action of the group of formal diffeomorphisms; differentiating this action assigns $E^*_{\mathbb{Q}}$ an action of $\mathbb{V}$. The smash product of two complex-oriented spectra is another such thing, and the functor from $S[BU]$–representations to $\mathbb{V}$ representations takes this product to the usual tensor product of Lie algebra representations. This, together with the universal connection constructed above, defines a monoidal functor from $S[BU]$–representations to the Tannakian category of flat equisingular vector bundles over B of [10, Section 2.16], and thus a (rational) representation of its motivic group in $S[BU]$.

2.3.4 Connes and Marcolli note that their motivic group is isomorphic to the motivic group for mixed Tate motives over the Gaussian integers (which has generators in all degrees, unlike that for mixed Tate motives over the rationals), but they do not try to make this isomorphism canonical. It is striking to me that their formulas [10, Equation 2.137] actually take values in the group of *odd* formal diffeomorphisms. Presumably $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ acts naturally on the motivic group of mixed Tate motives over the Gaussian integers, and it seems conceivable that these constructions might actually yield a representation of $\mathfrak{F}_{\rm odd} \ltimes \mathbb{G}_m$ in the group of odd formal diffeomorphisms [33]; but I have no hard evidence for this.

Note that the function field $\mathbb{Q}(\Gamma(z), z)$ admits the endomorphism

$$\tau(z) = z + 1$$

making it into a difference field, with a flow defined by

$$\exp(t\tau)\ \Gamma(z) = (1-t)^{-z}\Gamma(s)$$

satisfying

$$\exp(t\tau)\ z = (1-t)^{-z}\ ;$$

as well – but beware, this group does not act multiplicatively! Taylor expansion at the origin defines a morphism

$$\Gamma(z)^{-1} \mapsto z\ \exp\Bigl(-\sum_{k\geq 1}\frac{\zeta(k)}{k}z^k\Bigr) \colon \mathbb{Q}(\Gamma(z), z) \to \mathbb{R}((z))$$

of fields, compatible with this flow; this can be interpreted as a Tannakian fiber functor, from some category of vector bundles over the sphere endowed with a shift operator. Its motivic group is closely related to the automorphisms of the transcendental extension of $\mathbb{Q}$ generated by the odd zeta-values.

Dimensional regularization replaces certain divergent Feynman integrals with expressions involving Gamma-functions. This 'asymptotic point' sends these function-field expressions to formal power series with coefficients in the field generated over the rationals by zeta-values.

3 Towards fundamental groupoids of tensor triangulated categories

3.0 A Tannakian category $\mathcal{A}$ is a k–linear abelian category, where k is a field, possessing a coherently associative and commutative tensor product $\otimes$ (see Deligne [13, Section 2.5]); moreover, $\mathcal{A}$ should be small enough: its objects should be of finite length, its Hom-objects should be finite-dimensional over k, the endomorphism ring of the identity object for the tensor product should be k, and it should admit a good internal duality [14, Section 2.12]. The specifically Tannakian data, however, consists of a nontrivial exact k–linear functor

$$\omega\colon \mathcal{A} \to (k - \text{Vect})$$

which is monoidal in the sense that

$$\omega(X \otimes Y) \cong \omega(X) \otimes_k \omega(Y) .$$

Let $\mathrm{Aut}_\omega(k)$ be the group of multiplicative automorphisms of ω, ie of natural transformations of k–module-valued functors from ω to itself, which commute with these multiplicativity isomorphisms. More generally, the multiplicative automorphisms

$$A \mapsto \mathrm{Aut}_\omega(A) = \mathrm{Aut}(\omega \otimes_k A)$$

of $\omega \otimes_k A$ define a group-valued functor on the category of commutative k–algebras.

It sometimes happens (eg when k is of characteristic zero) that this functor is *representable* by a suitable Hopf algebra $\mathcal{H}$, ie

$$\mathrm{Aut}(\omega \otimes_k A) \cong \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{H}, A) ;$$

then ω lifts to a functor from $\mathcal{A}$ to the category of finite-dimensional representations of the affine (pro)algebraic group represented by $\mathcal{H}$, and it may be that we can use this lift to identify $\mathcal{A}$ with such a category of representations [13, Section 7]. In any case, when the k–groupscheme

$$\mathrm{spec}\,\mathcal{H} = \pi_1(\mathrm{spec}\,\mathcal{A}, \omega)$$

exists, it is natural to think of it as a (‘motivic’) kind of fundamental group for the Tannakian category, with the ‘fiber functor’ ω playing the role of basepoint.

3.1 Here are some illustrative examples, and variations on this theme:

3.1.1 The category of local systems (ie of finite-dimensional flat k–vector spaces) over a connected, suitably locally connected topological space X is Tannakian: a basepoint $x \in X$ defines an exact functor, which sends the system to its fiber over x, and we recover the fundamental group $\pi_1(X, x)$ (or, more precisely, its ‘envelope’ or best approximation by a proalgebraic group over k) as its automorphism object (see Toen [42]).

3.1.2 The functor which assigns to a finite-dimensional $\mathbb{Q}$–vector space V, the p–adic vector space $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, defines a Tannakian structure on $\mathbb{Q}-\mathrm{Vect}$, with $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, regarded as a profinite groupscheme over $\mathbb{Q}_p$, as its motivic fundamental group.

3.1.3 One possible variation involves fiber functors taking values in categories more general than vector spaces over a field. The monoidal category of even-odd graded (or ‘super’) vector spaces is an important example. When this works, one gets a Hopf algebra object (corresponding to a ‘super’ groupscheme) in the enriched category;

the motivic group in this case is the multiplicative group μ_2 of square roots of unity (suitably interpreted [13, Section 8.19]).

3.1.4 The automorphisms of cohomology with $\mathbb{F}_p$ coefficients, which is defined not on an abelian category but on a tensor triangulated one, eg finite spectra, is a more exotic and perhaps more compelling example: the even-degree cohomology is a representation of the groupscheme defined by the functor

$$A \mapsto \{\sum_{k\geq 0} a_k T^{p^k} = a(T) \in A[\![T]\!] \mid a_0 \in A^\times\} ;$$

its Hopf algebra is dual to the algebra defined by Steenrod's reduced pth powers.

The group operation here is composition of power series. The Hopf algebra dual to the full Steenrod algebra can be recovered by working with μ_2–graded (super) vector spaces. Note that the groupscheme defined above contains the 'torus'

$$A \mapsto A^\times = \mathrm{Hom}_{k-\mathrm{alg}}(k[t_0^{\pm 1}], A)$$

as the subgroup of series of the form $a_0 T$, $a_0 \in A^\times$; this action by the multiplicative groupscheme $\mathbb{G}_m$ allows us to recover the (even part of) the grading on the cohomology in 'intrinsic' terms. However, we're not as lucky here as in the preceding case: the stable homotopy category (at p) is not the same as the category of modules over the Steenrod algebra. The Adams spectral sequence tells us that we have to take higher extensions into consideration.

3.1.5 These motivic fundamental groups are in a natural sense functorial [13, Section 8.15]: Given a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_0 & \xrightarrow{\eta} & \mathcal{A}_1 \\ \downarrow{\scriptstyle \omega_0} & & \downarrow{\scriptstyle \omega_1} \\ (k_0 - \mathrm{Vect}) & \xrightarrow{\tilde{\eta}} & (k_1 - \mathrm{Vect}) \end{array}$$

of Tannakian categories $\mathcal{A}_i$, fiber functors ω_i, and fields $k_0 \to k_1$, with η exact k_0–linear multiplicative and $\tilde{\eta} = - \otimes_{k_0} k_1$, there is a natural homomorphism

$$\eta^*\colon \pi_1(\mathrm{spec}\, \mathcal{A}_1, \omega_1) \to \pi_1(\mathrm{spec}\, \mathcal{A}_0, \omega_0) \times_{k_0} k_1$$

of groupschemes over k_1 constructed as follows: if $\alpha\colon \omega_1 \cong \omega_1$ is a multiplicative automorphism of ω_1 (perhaps after some base extension which I won't record), then

$$\alpha \circ \eta\colon \tilde{\eta} \circ \omega_0 = \omega_1 \circ \eta \cong \omega_1 \circ \eta = \tilde{\eta} \circ \omega_0$$

is an element of $\pi_1(\mathrm{spec}\,\mathcal{A}_0, \omega_0)$, pulled back by $\tilde{\eta}$ to become a groupscheme over k_1.

As 3.1.4 suggests, it is tempting to push these constructions in various directions; in particular, if $\mathcal{T}$ is a suitably small tensor triangulated category (ie a triangulated category with tensor product satisfying reasonable axioms (for example that tensoring with a suitable object Σ represents suspension) [27]), and ω is a multiplicative homological functor (taking distinguished triangles to long exact sequences) with values in k–vector spaces, we can consider its multiplicative automorphisms, as in the Tannakian case; and we might hope that if $\mathcal{T}$ is some kind of derived category of $\mathcal{A}$, then we could try to reconstruct $\mathcal{T}$ in terms of a derived category of representations of the automorphism group of the homological fiber functor ω (see Franke [15], Neeman [39]). (If we ask that the automorphisms behave reasonably under suspension, then the automorphism group will contain a torus encoding gradings, as in 3.1.4.)

3.2 What seems to be missing from this picture is a compatible understanding of π_0. If $\mathfrak{p}$ is a prime ideal in a commutative noetherian ring A then $A/\mathfrak{p}$ is a domain with quotient field $Q(A/\mathfrak{p})$, and the composite

$$\omega_{\mathfrak{p}}\colon A \to A/\mathfrak{p} \to Q(A/\mathfrak{p}) = k(\mathfrak{p})$$

determines $\mathfrak{p}$, so the prime ideal spectrum $\mathrm{spec}\,A$ is a set of equivalence classes of homomorphisms from A to (varying) fields k. Such a homomorphism lifts to define a triangulated functor $\otimes^L_A k(\mathfrak{p})$ from the derived category of A–modules to the derived category of vector spaces over $k(\mathfrak{p})$, taking (derived) tensor products to (derived) tensor products, and (as in the case of ordinary cohomology above) we can consider the functor of automorphisms associated to such a generalized point. G–equivariant stable homotopy theory is another example of a category with a plentiful supply of natural 'points', corresponding to conjugacy classes of closed subgoups of G.

It would be very useful if we could construct, for suitable $\mathcal{T}$, a fundamental groupoid $\boldsymbol{\pi}(\mathrm{spec}\,\mathcal{T})$ with objects corresponding to equivalence classes of multiplicative homological fiber functors ω, and morphisms coming from $\pi_1(\mathrm{spec}\,\mathcal{T}, \omega)$. The lattice of Bousfield localizations (see Hovey and Palmieri [26]) of a triangulated category is in some ways analogous to the Boolean algebra of subsets of the spectrum of an abelian category (see Balmer [4], Hopkins [24], Krause [31] and Neeman [36]) but we do not seem to understand, in any generality, how to identify in it a sublattice corresponding to the open sets of a reasonable topology. Moreover, there seem to be subtle finiteness issues surrounding this question: constructing a good π_1 at ω may require identifying a suitable subcategory of ω–finite or coherent objects (see Christensen–Hovey [7] and Hovey–Strickland [28, Section 8.6]), and we might hope that a reasonable topology on spec $\mathcal{T}$ would suggest a natural theory of adeles (see Kapranov [29]) associated to chains of specializations.

3.3 This is all quite vague; perhaps some examples will be helpful.

The structure map $S \to H\mathbb{Z}$ defines a monoidal pullback functor

$$- \wedge H\mathbb{Z}\colon (\text{Spectra}) \to D(\mathbb{Z} - \text{Mod}) \,;$$

simply regarding a $\mathbb{Z}$–algebra as an S–algebra is not monoidal: $H(A \otimes B)$ is not the same as $HA \wedge HB$.

3.3.1 Let $\hat{K}^*_p(-) = K^*(-) \otimes \mathbb{Z}_p$ denote classical complex K–theory, regarded as a functor on finite spectra and p–adically completed. The Adams operations, suitably normalized, define an action of $\mathbb{Z}_p^\times$ by stable multiplicative automorphisms (see Sullivan [40]), and the Chern character isomorphism

$$\hat{K}^*_p(-) \otimes \mathbb{Q} \to H^*(-, \mathbb{Q}_p)$$

identifies its eigenspaces with the graded components defined by ordinary cohomology.

This can be expressed in the language developed above, as follows: the fiber functor $\omega_0 = \hat{K}_p \otimes \overline{\mathbb{Q}}_p$ on finite spectra is ordinary cohomology in disguise, and so has automorphism group $\mathbb{G}_m \ltimes \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, as does

$$\omega_1\colon M \mapsto H^+(M \otimes \overline{\mathbb{Q}}_p)\colon D(\mathbb{Z} - \text{Mod}) \to (\overline{\mathbb{Q}}_p - \text{Vect}) \,.$$

The induced morphism

$$\mathbb{G}_m \ltimes \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{G}_m \ltimes \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

of motivic fundamental groups sends $(u, g) \in \mathbb{G}_m \ltimes \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $(g_{\text{ab}}u, g)$, where

$$g \mapsto g_{\text{ab}}\colon \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_{\text{ab}} \to \mathbb{Z}_p^\times$$

is the p–adic cyclotomic character; in Hopf algebra terms, this corresponds to the homomorphism

$$u \mapsto 1 - t\colon \mathbb{Q}_p[u^{\pm 1}] \to \mathbb{Z}_p[\![t]\!][\mu_{p-1}] \otimes \mathbb{Q} \cong \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \otimes \mathbb{Q}$$

of Iwasawa theory.

3.3.2 This has a conjectural generalization to the spectra obtained by specializing E_n, with

$$E_n^*(S) = \mathbb{Z}_p[\![v_1, \ldots, v_{n-1}]\!][u^{\pm 1}]$$

at $v_i \mapsto 0,\ 1 \le i \le n-1$. [These spectra correspond to the Honda formal group of 1.1.1; they have some kind (see Goerss and Hopkins [16]) of multiplicative structure. It is convenient to call the associated cohomology functors $K(n)^*(-;\mathbb{Z}_p)$.]

It is natural [35] to expect that the group of multiplicative automorphisms of this functor maps, under reduction modulo p, to the normalizer of the maximal torus $\mathbb{Q}_q^\times$ in $D^\times$, as in 1.2.1; in other words, precisely those automorphisms of $K(n)^*(-;\mathbb{F}_p)$ which lift to automorphisms of the formal group law of $K(n)^*(-;\mathbb{Z}_p)$ lift to multiplicative automorphisms of the whole functor. It would follow from this, that

$$K(n)^*(-;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^{\mathbf{nr}} \cong \oplus_{k \in \mathbb{Z}} H^*(-;\mathbb{Q}_p^{\mathbf{nr}}(1)^{\otimes k}) ,$$

where $\mathbb{Q}_p^{\mathbf{nr}}(1)$ is the representation of $\mathrm{Gal}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p)$ (and thus of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$) constructed in 1.2.3. The results of the preceding section would then generalize, with the role of the cyclotomic character now being played by the crossed homomorphism α.

3.4.1 This suggests that the remarks above about π_0 might be slightly naive: π_0 might make better sense as a small diagram category than as a topological space. Equivariant stable homotopy theory provides further evidence for this, as do derived categories of representations of quivers in groups.

The conjecture above asserts that the functor

$$A \mapsto \mathrm{Aut}(K(n)^*(-;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A)\colon\ (\textit{flat } A\text{–Mod}) \to (\mathrm{Groups})$$

is (i) representable (ii) by $\mathrm{Gal}(\mathbb{Q}_q^{\mathrm{ab}}/\mathbb{Q}_p)$. Enlarging the useable class of fiber functors in this way, allowing values in (say) modules over discrete valuation rings, would lead to the existence of specialization homomorphisms such as

$$\pi_1(\operatorname{spec} \mathbb{Z}_p) \to \pi_1(\operatorname{spec} S, K(n)^*(-;\mathbb{Z}_p)) \to \pi_1(\operatorname{spec} S, K(n)^*(-;\mathbb{F}_p)) .$$

3.4.2 This paper is so much a wish list, that while we're at it we might as well ask for a way to associate automorphism groups to suitable open subsets or subdiagrams of our hypothetical π_0, together with homomorphisms between them defined by inclusions. One of the beauties of Grothendieck's original account of the fundamental group is his theory of specialization [19, Section X.2.3] which, as he notes, has no immediate analog in pure algebraic topology. There is evidence (see Ando, Morava and Sadofsky [2] and Torii [43, Section 4]) that some generalization of his theory to a derived category context could accomodate a very general form of the theory of vanishing cycles:

Classically $i\colon Y \to X$ is the inclusion of a closed subscheme, and $j\colon X - Y \to X$ is the inclusion of its complement; for example X might be an object over the spectrum of a discrete valuation ring, with Y the fiber above the closed point, and $X - Y$ the fiber above the generic point. In terms of derived categories and the various functors relating them, Bousfield localization along Y is the functor

$$L_Y = j_* j^* \colon D(X) \to D(X)\,,$$

see Bökstedt–Neeman [5, Section 6] and Hopkins–Gross [25]; it is related to the Grothendieck local cohomology functor $i_* i^!$ by an exact triangle

$$\cdots \to i_* j^! \to \mathrm{Id} \to j_* j^* \to \ldots$$

[since we're working in derived categories, I won't bother to indicate that all the functors have been suitably (left or right) derived].

The vanishing cycle functor (see Grothendieck [17; 18, Section 2.2] and Consani [11]) is the related composition

$$i^* j_* \colon D(X - Y) \to D(Y)\,;$$

although for our purposes it might be better to think of this as the *nearby* cycles functor. The two constructions are related by the commutative diagram

$$\begin{array}{ccc} D(X) & \xrightarrow{\;j_* j^*\;} & D(X) \\ \big\downarrow{\scriptstyle j^*} & & \big\downarrow{\scriptstyle i^*} \\ D(X-Y) & \xrightarrow{\;i^* j_*\;} & D(Y)\,. \end{array}$$

in other words the nearby cycles functor is roughly Bousfield localization restricted to the open stratum. It would be nice to have an equivariant version of this, or (in another language) a form adapted to stacks [36].

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