*Geometry & Topology Monographs 10 (2007) 319–332* 

# On the homotopy groups of $E(n)$–local spectra with unusual invariant ideals

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Let $E(n)$ and $T(m)$ for nonnegative integers $n$ and $m$ denote the Johnson–Wilson and the Ravenel spectra, respectively. Given a spectrum whose $E(n)_*$–homology is $E(n)_*(T(m))/(v_1,\dots,v_{n-1})$, then each homotopy group of it estimates the order of each homotopy group of $L_nT(m)$. We here study the $E(n)$–based Adams $E_2$–term of it and present that the determination of the $E_2$–term is unexpectedly complex for odd prime case. At the prime two, we determine the $E_\infty$–term for $\pi_*(L_2T(1)/(v_1))$, whose computation is easier than that of $\pi_*(L_2T(1))$ as we expect.

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## 1 Introduction

In [4], Ravenel has constructed the homotopy associative commutative ring spectrum $T(m)$ as a summand of $p$–component of the Thom spectrum associated with the map $\Omega SU(p^m)\to BU$. It is extensively used in [4, Section 7] to compute the homotopy groups of spheres in terms of "the method of infinite descent". The Adams–Novikov $E_2$–term converging to the stable homotopy groups $\pi_*(T(m))$ is described by use of the Hopf algebroid $(BP_*,\Gamma(m+1))$ (cf [4, Definition 7.1.1]). In particular, the 0–th line is

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*,BP_*)=\mathbb{Z}_{(p)}[v_1,\dots,v_m]\subset BP_*=\mathbb{Z}_{(p)}[v_1,\dots],$$

and the more the value of $m$, the more primitives we obtain. Since $v_k$ for $1\le k\le m$ is a permanent cycle of the spectral sequence, we obtain spectra $T(m)/(v_k)$ and $T(m)/(v_k,v_l)$ for $1\le k,l\le m$ (see Lemma 3.7.) Here $T(m)/J$ for an ideal $J$ of $BP_*$ denotes a spectrum such that $BP_*(T(m)/J)=BP_*/J$.

Let $BP\langle n\rangle$ denote the Johnson–Wilson ring spectrum with $BP\langle n\rangle_*=\mathbb{Z}_{(p)}[v_1,\dots,v_n]$ and put $E(n)=v_n^{-1}BP\langle n\rangle$ as usual. Then we have the $E(n)$–based Adams spectral sequence $E_r^{s,t}(X)\Rightarrow\pi_*(L_nX)$ for a spectrum $X$, whose $E_2$–term is $E_2^*(X)=\mathrm{Ext}^*_{E(n)_*(E(n))}(E(n)_*,E(n)_*(X))$. Here $L_n$ denotes the Bousfield localization functor with respect to $E(n)$. Note that $BP_*(T(m))=BP_*[t_1,\dots,t_m]\subset BP_*[t_1,\dots]=$

*Published: 18 April 2007* DOI: 10.2140/gtm.2007.10.319

$BP_*(BP)$. In order to study the $E_2$–term for a spectrum $X$ with $E(n)_*(X) = E(n)_*/J[t_1,\dots,t_m]$ for an ideal $J$ of $E(n)_*$, we introduce the generalized Johnson–Wilson spectrum $E_m(n) = v_n^{-1}BP\langle n+m\rangle$. Then

$$\Sigma(n,m) = E_m(n)_* \otimes_{BP_*} BP_*[t_{m+1}, t_{m+2}, \dots] \otimes_{BP_*} E_m(n)_*$$

is a Hopf algebroid over $E_m(n)_*$, and the $E(n)$–based Adams $E_2$–term $E_2^*(X)$ is isomorphic to $\mathrm{Ext}^*_{\Sigma(n,m+1)}(E_m(n)_*, E_m(n)_*/J)$, which we denote $\mathrm{Ext}^*(E_m(n)_*/J)$, by a similar change-of-rings theorem of Hovey and Sadofsky [1].

Consider $J_n$ be the sequence $v_1, v_2, \dots, v_{n-1}$. Then $T(m)/(J_n)$ exists if $n \le 2$ as commented above. Besides, if $L_nT(m)/J$ exists, then the $E(n)$–based Adams $E_2$–term for $\pi_*(L_nT(m)/J)$ is isomorphic to an Ext group $\mathrm{Ext}^*(E(n)_*/J)$. Consider the long exact sequence of Ext groups associated to the short exact sequence

$$0 \longrightarrow E_m(n)_*/(J_n) \longrightarrow p^{-1}E_m(n)_*/(J_n) \longrightarrow E_m(n)_*/(p^\infty, J_n) \longrightarrow 0.$$

Since $\mathrm{Ext}^*(p^{-1}E_m(n)_*/(J_n)) = \mathbb{Q}$, Corollary 4.5 implies our first theorem:

**Theorem 1.1** *The* Ext *group* $\mathrm{Ext}^0(E_m(n)_*/(J_n))$ *is isomorphic to* $\mathbb{Z}_{(p)}$*, and the group* $E_2^1(E_m(n)_*/(J_n))$ *is isomorphic to the direct sum of the cyclic module over the ring* $\mathbb{Z}_{(p)}[v_n^{\pm1}, v_{n+1}, \dots, v_m]$ *generated by*

$$\frac{v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}$$

*of order* $p^{1+\nu(e_k)}$ *with* $\nu(e_k) = \min\{\nu(e_1), \dots, \nu(e_n)\}$*, where the integer* $\nu(\ell)$ *denotes the maximal power of* $p$ *that divides* $\ell$*.*

For the case where $n > m$, we have an example which has a similar result to the above theorem (cf Proposition 4.7):

**Proposition 1.2** *The* $E(2)$*–based Adams* $E_2$*–term* $E_2^0(T(1)/(v_1))$ *is isomorphic to* $\mathbb{Z}_{(p)}$ *and* $E_2^1(T(1)/(v_1))$ *is the direct sum of the cyclic module over* $\mathbb{Z}_{(p)}$ *generated by* $v_2^{sp^i}v_3^{tp^j}/p^{1+\min(i,j)}$ *of order* $p^{1+\min(i,j)}$*.*

In these cases, we did not determine $E_2^s$ for $s > 1$ since there is an obstruction, which comes from the generators known as $b_{i,j}$ (see (3–2)). This is what we did not expect. For $p = 2$, we have the relation $b_{i,j} = h_{i,j}^2$, which makes possible to compute for $s > 1$. Since the $E(2)$–based Adams differentials are read off from Mahowald and Shimomura [2], we obtain the $E_\infty$–term.

**Theorem 1.3** *Let $p = 2$. The $E(2)$–based Adams $E_\infty$–term for $\pi_*(L_2T(1)/(v_1))$ is isomorphic to $\mathbb{Z}_{(2)}$ if $s = 0$ and is isomorphic to the tensor product of $\Lambda(\rho_2)$ and the direct sum of*

(1) $\widetilde{v_2 A[h_{20}]}$, $v_3 B[h_{30}]/(h_{30}^3)$ *and* $v_3 B h_{30} h_{31}$ *whose elements are of order two,*

(2) $M^0$ *and* $M^1$.

*Here the modules are given in Section 5.*

In Section 2, we consider the Hopf algebroid $(E_m(n)_*, \Sigma(n, m+1))$ and show a variation of the change-of-rings theorem given in Hovey and Sadofsky [1]. In Section 3, we exhibit the formulas for the structure maps (the right unit $\eta_R$ and the diagonal maps $\Delta$). We then observe the existence of spectra of the form $T(m)/J$. Section 4 is devoted to prove Theorem 1.1 and Proposition 1.2. In Section 5, we determine the $E_\infty$–term for $\pi_*(L_2T(1)/(2^\infty, v_1))$. The homotopy groups $\pi_*(L_2T(1))$ is determined easily if $p$ is odd, and stays undetermined if $p = 2$. The result of this section is the first step to understand $\pi_*(L_2T(1))$ at the prime two.

**Acknowledgements** We wish to thank to the organizers of Nishida Conference held on August 2003 for making arrangement of the publication for the Proceedings. We are also grateful to Ippei Ichigi for reading the draft paper carefully and for pointing out some misprints.

## 2 A generalized Johnson–Wilson theory

Let $BP$ and $BP\langle n\rangle$ denote the Brown–Peterson and the Johnson–Wilson spectra characterized by $\pi_*(BP) = BP_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, \dots]$ and $\pi_*(BP\langle n\rangle) = BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP_*$ with $|v_n| = |t_n| = 2(p^n - 1)$. Then the $BP_*$–homology of $BP$ is $BP_*(BP) = BP_*[t_1, \dots, t_n, \dots]$, We put

$$E_m(n) = v_n^{-1} BP\langle n+m\rangle$$

for nonnegative integers $n$ and $m$. Then

$$E_m(n)_* = E(n)_*[v_{n+1}, \dots, v_{n+m}] \subset v_n^{-1} BP_*.$$

We notice that $E_0(n)$ is the localized Johnson–Wilson spectrum $E(n)$.

Let $\Gamma(m+1)$ (cf Ravenel [4, 7.1.1]) be the $BP_*(BP)$–comodule defined by

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots].$$

Then the pair $(BP_*, \Gamma(m+1))$ has the structure of the Hopf algebroid inherited from $(BP_*, BP_*(BP))$. Put

$$\Sigma_m(n,i) = E_m(n)_* \otimes_{BP_*} \Gamma(i) \otimes_{BP_*} E_m(n)_*.$$

In particular, we write

$$\Sigma(n,m+1) = \Sigma_m(n,m+1) = E_m(n)_* \otimes_{BP_*} \Gamma(m+1) \otimes_{BP_*} E_m(n)_*.$$

The pair $(E_m(n)_*, \Sigma_m(n,i))$ is a Hopf algebroid with the structure maps inherited from those of the Hopf algebroid $(BP_*, \Gamma(i))$ for all $i>0$. Consider the map between Hopf algebroids $(E_m(n)_*, \Sigma_m(n,1)) \longrightarrow (E_m(n)_*, \Sigma(n,m+1))$ induced from the projection from $BP_*(BP)$ to $\Gamma(m+1)$. The map is normal and that

$$E_m(n)_*(T(m)) = E_m(n)_* \square_{\Sigma(n,m+1)} \Sigma_m(n,1) \tag{2–1}$$

if $m>0$. Here, $T(m)$ denotes the Ravenel spectrum [4, 6.5.1], which is an associative commutative ring spectrum characterized by $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$. Since $\Sigma_m(n,1)$ is $E_m(n)_*(E_m(n))$, the change-of-rings theorem [4, A1.3.12] shows the following:

**Lemma 2.2** *There is an isomorphism*

$$\mathrm{Ext}_{E_m(n)_*(E_m(n))}(E_m(n)_*, E_m(n)_*(T(m))) = \mathrm{Ext}_{\Sigma(n,m+1)}(E_m(n)_*, E_m(n)_*).$$

**Remark 2.3** In general, equation (2–1) does not hold if we work on $E(n)_*E(n)$–comodules. For example, if we set $(n,i) = (2,3)$, then

$$\Sigma_0(2,3) = E(2)_*[t_3, t_4, \ldots]/(\eta_R(v_k) \colon k>2).$$

In the right hand side we have the relation $v_2 t_1^{p^2} \equiv v_2^p t_1 \mod (p)$ since $\eta_R(v_3) = 0$. On the other hand, we do not have any relation on $t_1$ in $E(2)_*T(2) = E(2)_*[t_1, t_2]$.

Since $E_m(n)_*$ is a free $E(n)_*$–module over the bases $v^E = v_{n+1}^{e_1} \cdots v_{n+m}^{e_m}$ for $E = (e_1, \ldots, e_m)$ with $e_k \geq 0$, there is a homotopy equivalence $E_m(n) = \bigvee_E \Sigma^{|E|} E(n)$. This shows that the $E(n)$–based and the $E_m(n)$–based Adams spectral sequences agrees from the $E_2$–term (cf Hovey and Sadofsky [1]).

## 3 Existence of some spectra

An ideal $I = (a_0, a_1, \ldots, a_{n-1})$ of $BP_*$ is called *invariant* if $\eta_R(a_i) \equiv a_i \mod (a_0, a_1, \ldots, a_{i-1})$ for each $0 \leq i < n$ as a $BP_*BP$–comodule. It is well known that if there is a spectrum $X$ such that $BP_*(X) = BP_*/I$, then $I$ is invariant. Consider now the Ravenel spectrum $T(m)$. Then the $E_2$–term of the Adams–Novikov spectral sequence for $\pi_*(W \wedge T(m))$ for a spectrum $W$ is isomorphic to an Ext group over the Hopf algebroid $(BP_*, \Gamma(m+1))$. We call an ideal $J = (w_0, w_1, \dots, w_{n-1})$ of $BP_*$ *unusual* if it is not invariant and $\eta_R(w_i) \equiv w_i \mod (w_0, w_1, \dots, w_{i-1})$ for each $0 \le i < n$ as a $\Gamma(m+1)$–comodule. In the same manner as above, if there is a spectrum $X$ such that $BP_*(X) = BP_*/J[t_1, \dots, t_m]$ for $m > 0$, then $J$ is invariant or unusual. In this section, we study the existence of a spectrum $X$ with $BP_*$–homology (resp. $E(n)_*$–homology)

$$BP_*(X) = BP_*/J[t_1, \dots, t_m] \quad (\text{resp. } E(n)_*(X) = E(n)_*/J[t_1, \dots, t_m])$$

for an unusual ideal $J$. We write $T(m)/J$ (resp. $L_n T(m)/J$) for such $X$.

The next lemma is verified by Hazewinkel's and Quillen's formulas (see Miller, Ravenel and Wilson [3, (1.1)–(1.3)]):

**Lemma 3.1** *Assume that $n \le m$. Let $J_n$ denote the ideal $(v_1, \dots, v_{n-1})$ of $BP_*$. Then the structure maps in $(BP_*, \Gamma(m+1))$ act as*

$$\begin{aligned}
\eta_R(v_k) &\equiv v_k && \text{for } n \le k \le m,\\
\eta_R(v_{m+k}) &\equiv v_{m+k} + p t_{m+k} && \text{for } 0 < k \le n,\\
\Delta(t_{m+k}) &\equiv t_{m+k} \otimes 1 + 1 \otimes t_{m+k} && \text{for } 0 \le k \le n,\\
\Delta(t_{m+n+1}) &\equiv t_{m+n+1} \otimes 1 + 1 \otimes t_{m+n+1} + v_n b_{m+1,n-1}
\end{aligned}$$

*mod $J_n$, where*

$$b_{i,j} = \left(t_i^{p^{j+1}} \otimes 1 + 1 \otimes t_i^{p^{j+1}} - (t_i \otimes 1 + 1 \otimes t_i)^{p^{j+1}}\right)/p. \tag{3–2}$$

By this lemma, we read off the behavior of the structure maps $\eta_R$ and $\Delta$ mod $J_n$ of the Hopf algebroid $(E_m(n)_*, \Sigma(n, m+1))$. For $n > m$, we only consider the case where $n = 2$ and $m = 1$.

**Lemma 3.3** *The structure maps in $(BP_*, \Gamma(2))$ acts as*

$$\begin{aligned}
&\eta_R(v_i) \equiv v_i + p t_i \quad \text{for } i = 2 \text{ and } 3,\\
&\eta_R(v_4) \equiv v_4 + v_2 t_2^{p^2} + p t_4 + v_2 c_{21} - \eta_R(v_2)^{p^2} t_2,\\
&\eta_R(v_5) \equiv v_5 + v_3 t_2^{p^3} + v_2 t_3^{p^2} + p t_5 + v_2 c_{31} + v_3 c_{22} - \eta_R(v_3)^{p^2} t_2 - \eta_R(v_2)^{p^3} t_3,\\
&\quad \Delta(t_i) \equiv t_i \otimes 1 + 1 \otimes t_i \quad \text{for } i = 2 \text{ and } 3,\\
&\quad \Delta(t_4) \equiv t_4 \otimes 1 + 1 \otimes t_4 + t_2 \otimes t_2^{p^2} + v_2 b_{21},\\
&\quad \Delta(t_5) \equiv t_5 \otimes 1 + 1 \otimes t_5 + t_3 \otimes t_2^{p^3} + t_2 \otimes t_3^{p^2} + v_2 b_{31} + v_3 b_{22}
\end{aligned}$$

mod $(v_1)$, *where* $c_{i,j} = p^{-1}(v_i^{p^{j+1}} - \eta_R(v_i^{p^{j+1}}))$. *In particular,* $b_{i,j} \equiv t_i^{2^j} \otimes t_i^{2^j}$ mod $(2)$ *for* $p = 2$.

We consider the Adams–Novikov spectral sequence

$$E_2^{*,*}(X) = \mathrm{Ext}_{BP_*(BP)}^{*,*}(BP_*, BP_*(X)) \quad \Longrightarrow \quad \pi_*(X). \tag{3–4}$$

By the change-of-rings theorem [4, A1.3.12], we have an isomorphism

$$E_2^*(T(m)/I_n) = \mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n). \tag{3–5}$$

Hereafter we use the abbreviation:

$$\mathrm{Ext}_\Gamma(A, -) = \mathrm{Ext}_\Gamma(-) \quad \text{for a Hopf algebroid } (A, \Gamma).$$

Lemma 3.1 implies the following:

**Lemma 3.6** *For* $0 \le k \le m$,

$$v_{n+k} \in E_2^0(T(m)/I_n) = \mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n),$$

*where* $I_n = (p) + J_n$.

**Lemma 3.7** *Let* $M$ *be a* $T(m)$*–module spectrum. If* $\alpha$ *and* $\beta \in E_2(T(m))$ *are permanent cycles in the spectral sequence* (3–4)*, then there exist spectra of the form* $M/(\alpha^a)$ *and* $M/(\alpha^a, \beta^b)$ *for positive integers* $a$ *and* $b$*. In particular, we have* $T(m)/(v_k^a)$ *and* $T(m)/(v_i^a, v_j^b)$ *for* $i, j, k < m+2$.

**Proof** Since $M$ is a $T(m)$–module spectrum, the elements $\alpha$ and $\beta$ yield the self maps on $M$, which we also denote by $\alpha$ and $\beta$. Now $M/(\alpha^a)$ is a cofiber of the self map $\alpha^a$, and the $M/(\alpha^a, \beta^b)$ is obtained by use of Verdier's axiom on the equation $\alpha^a\beta^b = \beta^b\alpha^a$ in $[M, M]_*$.

Since the reduced comodule $\overline{\Gamma(m+1)}$ is $(2p^{m+1}-3)$–connected, we have the vanishing line $E_2^{s,t}(T(m)) = 0$ for $t < 2s(p^{m+1}-1)$ by (3–5). It follows that $v_k \in E_2^*(T(m))$ in Lemma 3.6 is permanent if $k < m+2$. □

The existence of a spectrum with $BP_*$–homology $BP_*/I_n$ is problematic and we still have little information for such a spectrum, which we usually call the $((n-1)$st) Smith–Toda spectrum and is denoted by $V(n-1)$ (eg Smith [6], Toda [7] and Ravenel [4]). For $n \le 3$, it is shown that $V(n)$ exists if and only if $p > 2n$. On the other hand, $L_nV(n-1)$ exists if $n^2+n < 2p$ [5]. The smash products $T(m)$ and these Smith–Toda spectra show the following:

**Proposition 3.8** *If* $p > 2n$, $T(m)/I_n$ *exists, and if* $n^2+n < 2p$, $L_nT(m)/I_n$ *exists.*

# 4 $\mathrm{Ext}^s_{\Sigma(n,m+1)}(E_m(n)_*/J_n)$ for small $s$

In this section, let $J_n$ denote the sequence $v_1, \dots, v_{n-1}$ of elements of $E_m(n)_*$. Applying Ext to the short exact sequence

$$0 \longrightarrow E_m(n)_*/(p, J_n) \xrightarrow{1/p} E_m(n)_*/(p^\infty, J_n) \xrightarrow{p} E_m(n)_*/(p^\infty, J_n) \longrightarrow 0,$$

we have the long exact sequence of Ext groups with connecting homomorphism

$$\delta\colon \mathrm{Ext}^r(E_m(n)_*/(p^\infty, J_n)) \longrightarrow \mathrm{Ext}^{r+1}(E_m(n)_*/(p, J_n)). \tag{4–1}$$

By [4, Theorem 6.5.6], we know the structure of $\mathrm{Ext}(E_m(n)_*/(p, J_n))$, which means that $\mathrm{Ext}(E_m(n)_*/(J_n))$ is a computable object.

To compute $\mathrm{Ext}(E_m(n)_*/(p^\infty, J_n))$, we redefine the class $h_{m+k,0}$ $(0 < k \le n)$ by

$$h_{i,0} = \left[\frac{\log(1 + pv_i^{-1}t_i)}{p}\right] = \left[\sum_{n>0}(-1)^{n-1}\frac{(pv_i^{-1}t_i)^n}{pn}\right]. \tag{4–2}$$

**Lemma 4.3** *For $0 < k \le n$, the connecting homomorphism $\delta$ in (4–1) acts for all $\ell$ as $\delta(h_{m+k,0}/p^\ell) = 0$.*

**Proof** It suffices to show that $ph_{m+k,0} = d(\log(v_{m+k}))$. By Lemma 3.1, we have $\eta_R(v_{m+k}) = v_{m+k} + pt_{m+k}$ for $0 < k \le n$, so the equation

$$\log(1 + pv_{m+k}^{-1}t_{m+k}) = \log(\eta_R(v_{m+k})) - \log(v_{m+k}) = d(\log(v_{m+k}))$$

holds. □

The element $v_{m+k}^{k+1}x$ is well-defined in $\Sigma(n, m+1)/(p^k)$, although the representative $x = \log(1 + pv_{m+k}^{-1}t_{m+k})/p$ of $h_{m+k,0}$ has negative exponents of $v_{m+k}$ in the coefficient.

An easy computation with Lemma 3.1 shows the following:

**Lemma 4.4** *Put $\nu(e_k) = \min\{\nu(e_1), \dots, \nu(e_n)\}$. Then we have*

$$\delta\left(\frac{v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}\right) = v_{m+1}^{e_1} \cdots v_{m+n}^{e_n} h_{m+k,0} + \cdots$$

*in $\mathrm{Ext}^1(E_m(n)_*/(p, J_n))$ up to unit. For $\nu$, see Theorem 1.1.*

**Corollary 4.5** $\mathrm{Ext}^0(E_m(n)_*/(p^\infty, J_n))$ *is the direct sum of*

(1) *the cyclic* $\mathbb{Z}_{(p)}[v_n^{\pm 1}, v_{n+1}, \dots, v_m]$*–module generated by*

$$\frac{v_{m+1}^{e_1} \cdots v_{m+n}^{e_n}}{p^{1+\nu(e_k)}}$$

*of order* $p^{1+\nu(e_k)}$ *with* $\nu(e_k) = \min\{\nu(e_1), \dots, \nu(e_n)\}$ *and*

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}[v_n^{\pm 1}, v_{n+1}, \dots, v_m]$.

**Example 4.6** For $m = n = 2$, we have

$$\delta\left(\frac{v_3^{sp^i} v_4^{tp^j}}{p^{1+\min(i,j)}}\right) = \begin{cases} v_3^{sp^i} v_4^{tp^j} h_{40} & \text{for } i > j \\ v_3^{sp^i} v_4^{tp^j} h_{30} & \text{for } i < j \\ v_3^{sp^i} v_4^{tp^j} (h_{30} + a h_{40}) & \text{for } i = j \end{cases}$$

in $\mathrm{Ext}^1(E_2(2)_*/(p, v_1))$ up to unit (where $a \in (\mathbb{Z}/(p))^\times$), and $\mathrm{Ext}^0(E_2(2)_*/(p^\infty, v_1))$ is the direct sum of

(1) the cyclic module over $\mathbb{Z}_{(p)}[v_2^{\pm}]$ generated by $v_3^{sp^i} v_4^{tp^j}/p^{1+\min(i,j)}$ of order $p^{1+\min(i,j)}$ and

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}[v_2^{\pm 1}]$.

In the computations for $\delta(h_{31})$ and $\delta(h_{41})$, the elements $b_{i,j}$ (cf Lemma 3.1) occur, which are hard to express in terms of generators appearing in [4, Theorem 6.5.6]. We observe that the specific property $b_{i,j} = h_{i,j}^2$ at $p = 2$ makes the computations easy.

We consider the spectrum $L_n T(m)/(J_n)$ for $(n, m) = (2, 1)$, which is the simplest case satisfying $n > m$, and compute $\mathrm{Ext}^s_{\Sigma(2,2)}(E_1(2)_*/(v_1))$ for $s < 2$ for an odd prime. We consider the case for $p = 2$ in the next Section 5. Since $p$ is odd, the condition of [4, Theorem 6.5.6] is always satisfied and $\mathrm{Ext}_{\Sigma(2,2)}(E_1(2)_*/(p, v_1))$ is obtained as

$$K(2)_*[v_3] \otimes \Lambda(h_{i,j} : 2 \le i \le 3,\ j \in \mathbb{Z}/2).$$

Starting from this, $\mathrm{Ext}^0_{\Sigma(2,2)}(E_1(2)_*/(p^\infty, v_1))$ is determined by computing the connecting homomorphism (4–1) for $(m, n) = (1, 2)$ as follows Corollary 4.5:

**Proposition 4.7** *For* $sp^i \in \mathbb{Z}$ *and* $tp^j \ge 0$*, we have*

$$\delta(v_2^{sp^i} v_3^{tp^j}/p^{1+\min(i,j)}) = \begin{cases} s v_2^{sp^i-1} v_3^{tp^j} h_{20} & \text{if } i < j, \\ t v_2^{sp^i} v_3^{tp^j-1} h_{30} & \text{if } i > j, \\ v_2^{sp^i-1} v_3^{tp^j-1}(s v_3 h_{20} + t v_2 h_{30}) & \text{if } i = j, \end{cases}$$

*and* $\mathrm{Ext}^0_{\Sigma(2,2)}(E_1(2)_*/(p^\infty, v_1))$ *is the direct sum of*

(1) *the cyclic* $\mathbb{Z}_{(p)}$*–module generated by* $v_2^{sp^i}v_3^{tp^j}/p^{1+\min(i,j)}$ *of order* $p^{1+\min(i,j)}$ *and*

(2) $\mathbb{Q}/\mathbb{Z}_{(p)}$.

## 5 The homotopy groups $\pi_*(L_2T(1)/(v_1))$ at the prime two

We begin with recalling the result of Mahowald and Shimomura [2]:

(5–1) $$\operatorname{Ext}(E_1(2)_*/(2,v_1)) = K(2)_*[v_3,h_{20}]\otimes\Lambda(h_{21},h_{30},h_{31},\rho_2)$$

where $\rho_2$ is the generator of degree $0$ represented by the cocycle $v_2^{-5}t_4+v_2^{-10}t_4^2$. We see that (4–2) for $p=2$ is also a cocycle with leading term $v_i^{-2}t_i^2$, and replace the representative cocycles by

$$h_{i,0}=[t_i] \quad \text{and} \quad h_{i,1}=\Big[\sum_{n>0}(-1)^{n-1}\frac{(2v_i^{-1}t_i)^n}{2n}\Big].$$

Setting $B=\mathbb{Z}/2[v_2^{\pm 2},v_3^2]$, we rewrite the right hand side of (5–1) as

$$B\otimes\Lambda(v_3)\otimes\Lambda(h_{21},h_{30},h_{31})\otimes\Lambda(v_2)\otimes\mathbb{Z}/2[h_{20}]\otimes\Lambda(\rho_2).$$

Since $$h_{21}h_{31}=v_2^{-1}v_3^2h_{20}h_{21}+v_2^2h_{30}^2+v_2h_{20}h_{31}$$

by [2, p 243 (1)], we replace $h_{21}h_{31}$ (resp. $h_{21}h_{30}h_{31}$) with $h_{30}^2$ (resp. $h_{30}^3$).

**Lemma 5.2** *As the* $\mathbb{Z}/2$*–module,* $\operatorname{Ext}(E_1(2)_*/(2,v_1))$ *is isomorphic to*

$$\begin{aligned} & A\otimes\Lambda(v_2)\otimes\mathbb{Z}/2[h_{20}]\otimes\Lambda(\rho_2)\\ \textit{where}\quad & A=B\otimes\Lambda(v_3)\otimes\big(\mathbb{Z}/2[h_{30}]/(h_{30}^4)\oplus\mathbb{Z}/2\{h_{21},h_{31}\}\otimes\Lambda(h_{30})\big)\\ \textit{and}\quad & B=\mathbb{Z}/2[a_2^{\pm1},a_3] \qquad \textit{with } a_i=v_i^2. \end{aligned}$$

**Lemma 5.3** *The connecting homomorphism* (4–1) *for* $(m,n)=(1,2)$ *acts as*

$$\delta(v_i^s/2)=v_i^{s-1}h_{i,0} \quad \text{and} \quad \delta(a_i^{2^ns}/2^{n+2})=a_i^{2^ns}h_{i,1} \qquad (i=2,3)$$

*for odd* $s$ *and* $n\ge 0$.

**Proof** It follows from

$$\begin{aligned} (v_i^s) &\equiv 2v_i^{s-1}t_i && \bmod (4),\\ d(a_i^{2^ns}) &\equiv 2^{n+2}v_i^{2^{n+1}s}(v_i^{-1}t_i+v_i^{-2}t_i^2) && \bmod (2^{n+3}). \end{aligned}$$ □

$\mathrm{Ext}(E_1(2)_*/(2,v_1))$ is decomposed into the following four summands tensoring with $\Lambda(\rho_2)$:

$$\begin{aligned}
&v_2 A \oplus \Lambda(v_2) \otimes A \otimes \mathbb{Z}/p(h_{20})h_{20}\\
&v_3 B \oplus \Lambda(v_3) \otimes B\{h_{30}, h_{30}^2, h_{30}^3\} \oplus v_3 h_{30} h_{31} B\\
&B \oplus B\,\{h_{21}, h_{31}\} \oplus v_3 h_{21} h_{30} B\\
&B h_{30}\,\{h_{21}, h_{31}\} \oplus v_3 B\,\{h_{21}, h_{31}\}
\end{aligned}$$

With respect to each summand, we construct a long exact sequence in Lemma 5.4, Lemma 5.5 and Lemma 5.6. We often use the replacement

$$h_{31} = [v_3^{-1} t_3 + v_3^{-2} t_3^2] = v_3^{-1} h_{30} + \cdots .$$

If we define $P_i$ $(i \geq 0)$ and $Q_j$ $(j > 0)$ by

$$\begin{aligned}
P_i &= \mathbb{Z}_{(2)}\big\{a_2^{2^i s} a_3^{2^j t} : 0 \leq j \leq i, 0 \neq s \in \mathbb{Z}, t \geq 0\big\},\\
Q_j &= \mathbb{Z}_{(2)}\big\{a_2^{2^i s} a_3^{2^j t} : 0 \leq i < j, s \in \mathbb{Z}, t > 0\big\},
\end{aligned}$$

then we decompose $B$ into

$$B = \Big(\bigoplus_{i\geq 0} P_i\Big) \oplus \Big(\bigoplus_{j>0} Q_j\Big).$$

Define $M^0$ and $M^1$ by

$$\begin{aligned}
M^0 &= \left(\bigoplus_{i\geq 0} P_i\left\{\frac{1}{2^{i+2}}\right\}\right) \oplus \left(\bigoplus_{j>0} Q_j\left\{\frac{1}{2^{j+2}}\right\}\right) \oplus \mathbb{Q}/\mathbb{Z}_{(2)},\\
M^1 &= \left(\bigoplus_{i\geq 0} P_i\left\{\frac{h_{21}}{2^{i+2}}\right\}\right) \oplus \left(\bigoplus_{j>0} Q_j\left\{\frac{h_{31}}{2^{j+2}}\right\}\right).
\end{aligned}$$

Then we have the following results:

**Lemma 5.4** *We have two long exact sequences*

$$\begin{tikzcd}
B \arrow[r, tail] & M^0 \arrow[r, "2"] & M^0 \arrow[dll, two heads] \\
B\{h_{21}, h_{31}\} \arrow[r] & M^1 \arrow[r, "2"] & M^1 \arrow[dll, two heads, "\delta"] \\
v_3 h_{21} h_{30} B & &
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
v_3B\{h_{21},h_{31}\} \arrow[r, tail] & (v_3/2)B\{h_{21},h_{31}\} \arrow[r, "2"] & (v_3/2)B\{h_{21},h_{31}\} \arrow[dll, two heads, "\delta"] \\
Bh_{30}\{h_{21},h_{31}\}. & &
\end{tikzcd}$$

**Proof** In the first sequence the connecting homomorphism acts as:

$$\delta(a_2^{2^is}a_3^{2^jt}/2^{2+\min(i,j)}) = \begin{cases} a_2^{2^is}a_3^{2^jt}h_{21} & (i<j) \\ a_2^{2^is}a_3^{2^jt}h_{31} & (i>j) \\ a_2^{2^is}a_3^{2^jt}(h_{21}+h_{31}) & (i=j) \end{cases}$$

We also see that $\delta(a_2^{2^is}a_3^{2^jt}h_{31}/2^{i+2})$ for $i<j$, $\delta(a_2^{2^is}a_3^{2^jt}h_{21}/2^{j+2})$ for $i>j$, and $\delta(a_2^{2^is}a_3^{2^it}h_{21}/2^{i+2})$ are equal to $a_2^{2^is}a_3^{2^jt}h_{21}h_{31}$. Replacing $h_{31}$ with $v_3^{-1}h_{30}+\cdots$, we have the first sequence. The second sequence is obvious. □

**Lemma 5.5** *We have a long exact sequence*

$$\begin{tikzcd}
v_3B \arrow[r, tail] & (v_3/2)B \arrow[r, "2"] & (v_3/2)B \arrow[dll, "\delta"] \\
h_{30}B\otimes\Lambda(v_3) \arrow[r] & (v_3h_{30}/2)B \arrow[r, "2"] & (v_3h_{30}/2)B \arrow[dll, "\delta"] \\
\begin{matrix} h_{30}^2B\otimes\Lambda(v_3) \\ \oplus(v_3h_{30}h_{31})B \end{matrix} \arrow[r] & \begin{matrix} (v_3h_{30}^2/2)B \\ \oplus(v_3h_{30}h_{31}/2)B \end{matrix} \arrow[r, "2"] & \begin{matrix} (v_3h_{30}^2/2)B \\ \oplus(v_3h_{30}h_{31}/2)B \end{matrix} \arrow[dll, two heads, "\delta"] \\
h_{30}^3B\otimes\Lambda(v_3). & &
\end{tikzcd}$$

**Proof** It follows from

$$\delta(a_2^{2^is}a_3^{2^jt}v_3h_{30}^k/2) = a_2^{2^is}a_3^{2^jt}h_{30}^{k+1} \qquad \text{for } 0\le k\le 2,$$
$$\delta(a_2^{2^is}a_3^{2^jt}v_3h_{30}h_{31}/2) = a_2^{2^is}a_3^{2^jt}h_{30}^2h_{31} = a_2^{2^is}a_3^{2^jt-1}v_3h_{30}^3+\cdots$$ □

**Lemma 5.6** *We have a long exact sequence*

$$\begin{tikzcd}
v_2A \arrow[r, tail] & (v_2/2)A \arrow[r, "2"] & (v_2/2)A \arrow[dll, "\delta"] \\
h_{20}A\otimes\Lambda(v_2) \arrow[r] & (v_2h_{20}/2)A \arrow[r, "2"] & (v_2h_{20}/2)A \arrow[dll, "\delta"] \\
h_{20}^2A\otimes\Lambda(v_2) \arrow[r] & (v_2h_{20}^2/2)A \arrow[r, "2"] & \cdots .
\end{tikzcd}$$

**Proof** Notice that each exponent of $v_2$ in $(v_2h_{20}^k/2)A$ is odd. Since we have $d(x)=0$ for $x \in A$ in the cobar complex, we have

$$d(v_2^{2s+1}v_3^t x) = d(v_2^{2s+1}v_3^t) \otimes x.$$

We see that

$$d(v_2^{2s+1}v_3^t) = \begin{cases} 2v_2^{2s}v_3^{2n}t_2 + \cdots & \text{for } t = 2n, \\ d(v_2^{2s+1}v_3^t) = 2v_2^{2s}v_3^{2n}(v_3t_2 + v_2t_3) + \cdots & \text{for } t = 2n+1. \end{cases}$$

In both cases we obtain

$$\delta\left(\frac{v_2^{2s+1}v_3^t x}{2}\right) = v_2^{2s}v_3^t h_{20} x$$

replacing $v_3h_{20}$ by $v_3h_{20} = [v_3t_2 + v_2t_3]$ only for the case $t = 2n+1$. □

By the above three lemmas, we obtain the chart of differentials

$$\begin{tikzcd}
\boxed{v_3B} \arrow[r] & h_{30}B & & & \\
& \boxed{v_3h_{30}B} \arrow[r] & h_{30}^2B & & \\
& & \boxed{v_3h_{30}^2B} \arrow[r] & h_{30}^3B & \\
& & \boxed{v_3h_{30}h_{31}B} \arrow[r] & v_3h_{30}^3B &
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{B} \arrow[r] \arrow[dr] & h_{21}B \arrow[dr] & \\
& \boxed{h_{31}B} \arrow[r] & v_3h_{21}h_{30}B
\end{tikzcd}$$

$$\boxed{v_3B\{h_{21}, h_{31}\}} \longrightarrow h_{30}B\{h_{21}, h_{31}\}$$

$$\begin{tikzcd}
\boxed{v_2A} \arrow[r] & h_{20}A & & \\
& \boxed{v_2h_{20}A} \arrow[r] & h_{20}^2A & \\
& & \boxed{v_2h_{20}^2A} \arrow[r] & h_{20}^3A \\
& & \vdots &
\end{tikzcd}$$

Thus we conclude the following:

**Lemma 5.7** $\mathrm{Ext}_{\Sigma(2,2)}(E_1(2)_*, E_1(2)_*/(2^\infty, v_1))$ *is the tensor product of* $\Lambda(\rho_2)$ *and the direct sum of*

(1) $v_2A[h_{20}]$, $v_3B[h_{30}]/(h_{30}^3)$ *and* $v_3Bh_{30}h_{31}$ *whose elements are of order two,*

(2) $M^0$ *and* $M^1$*.*

Let $E_\infty^*(X)$ for a spectrum $X$ denote the $E_\infty$–term of the $E(2)$–based Adams spectral sequence converging to the homotopy groups $\pi_*(L_2X)$.

**Theorem 5.8** *The* $E_\infty$*–term* $E_\infty^*(L_2T(1)/(2^\infty, v_1))$ *is the tensor product of* $\Lambda(\rho_2)$ *and the direct sum of*

(1) $\widetilde{v_2A[h_{20}]}$, $v_3B[h_{30}]/(h_{30}^3)$ *and* $v_3Bh_{30}h_{31}$ *whose elements are of order two,*

(2) $M^0$ *and* $M^1$*,*

*where* $\widetilde{v_2A[h_{20}]}$ *denotes the module*

$$\big(\mathbb{Z}/2[v_2^{\pm 2}, v_3^4]\otimes\Lambda(v_3)\otimes\big(\mathbb{Z}/2[h_{30}]/(h_{30}^4)\oplus\mathbb{Z}/2\{h_{21}, h_{31}\}\otimes\Lambda(h_{30})\big)\big)[h_{20}]/(h_{20}^3).$$

**Proof** In [2], the differentials of $E(2)$–based Adams spectral sequence for $L_2T(1)/I_2$ (written as $D$ in [2]) are determined as

$$d_3(v_3)=0 \quad\text{and}\quad d_3(v_3^k)=v_2^2v_3^{k-2}h_{20}^3 \quad \text{for } 2\le k\le 3,$$

and $d_3(v_3^kx)=d_3(v_3^k)x$ for $x=h_{20}$, $h_{21}$, $h_{30}$ and $h_{31}$. Note that for each element $wa_3^{2t+1}\in v_2A[h_{20}]$, we see that

$$d_3(wa_3^{2t+1}/2)=wa_3^{2t}h_{20}^3/2\in v_2A[h_{20}].$$

This shows the structure of $\pi_*(L_2T(1)/(2^\infty, v_1))$, since it has a horizontal vanishing line. □

**Proof of Theorem 1.3** Consider the cofiber sequence

$$T(1)/(v_1) \longrightarrow T(1)/(v_1)\wedge S\mathbb{Q} \longrightarrow T(1)/(2^\infty, v_1).$$

Then the homotopy groups of $T(m)/(v_1)\wedge S\mathbb{Q}$ and $T(1)/(2^\infty, v_1)$ are determined in [4, Corollary 6.5.6] and Theorem 5.8, respectively. □

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Received: 31 August 2004 Revised: 16 September 2005