

Interactions of strings and equivariant homology theories

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We introduce the notion of the space of parallel strings with partially summable labels, which can be viewed as a geometrically constructed group completion of the space of particles with labels. We utilize this to construct a machinery which produces equivariant generalized homology theories from such simple and abundant data as partial monoids.

[55N20](#), [55N91](#); [55P47](#)

1 Introduction

In [6] we attached to any pair of a Euclidean space V and a partial abelian monoid M a space $C(V, M)$ whose points are pairs consisting of a finite subset c of V and a map $a: c \rightarrow M$, but (c, a) is identified with (c', a') if $c \subset c'$, $a'|_c = a$, and $a'(v) = 0$ for $v \notin c$. Any such pair (c, a) can be identified with the set consisting of “labeled particles” $(v, a(v))$, $v \in c$. Suppose V is an orthogonal G -module for some finite group G and M admits a G -action compatible with partial sum operations. Then $C(V, M)$ is a G -space with respect to the G -action

$$g(c, a) = (gc, gag^{-1}), \quad g \in G, (c, a) \in C(V, M).$$

Let $I(\mathbb{R})$ be the space of finite disjoint unions of bounded intervals in the real line. Then $I(\mathbb{R})$ is a partial abelian monoid with partial sum operation given by superimposition. Let us denote $I(V, M) = C(V, I(\mathbb{R}) \wedge M)$ for any partial abelian monoid with G -action M . Observe that under the correspondence

$$a: c \rightarrow I(\mathbb{R}) \quad \mapsto \quad \bigcup_{v \in c} \{v\} \times a(v) \subset V \times \mathbb{R}$$

any map from a finite subset of V to $I(\mathbb{R})$ can be identified with a finite disjoint union of bounded subsets of the form $\{v\} \times J \subset V \times \mathbb{R}$, where J is a bounded interval. We call such $\{v\} \times J$ a *string* over v . Thus $I(V, M)$ can be regarded as the space consisting of finite sets of pairwise disjoint labeled strings whose members over the same point in V has the same label in M .

The aim of this paper is to show that if V is sufficiently large then there is a G -equivariant group completion map $C(V, M) \rightarrow I(V, M)$ and also that the correspondence $X \mapsto \pi_n I(V, X \wedge M)$, $n \geq 0$, extends to an $RO(G)$ -graded generalized homology theory.

To state the precise results, let $\text{Top}(G)$ be the category of all pointed G -spaces and all pointed maps with G acting on maps by conjugation. In [5] we have shown that any G -equivariant continuous functor $T: \text{Top}(G) \rightarrow \text{Top}(G)$ such that $T(*) = *$ is associated with pairings $X \wedge TY \rightarrow T(X \wedge Y)$, $TX \wedge Y \rightarrow T(X \wedge Y)$ natural in both X and Y . Therefore, T preserves G -homotopies and there is a natural transformation $S^W \wedge T(X) \rightarrow T(S^W \wedge X)$ for any orthogonal G -module W , where S^W is the one point compactification of W .

Suppose V is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of G over the real number fields. Such a G -module V is called a G -universe. Now the main results can be stated as follows.

Theorem 1.1 *There is a diagram consisting of maps of Hopf G -spaces*

$$C(V, M) \xleftarrow{\lambda} I_+(V, M) \xrightarrow{\rho} I(V, M)$$

satisfying the conditions below.

- (1) λ is a G -homotopy equivalence.
- (2) ρ is an equivariant group completion, that is to say, it restricts to a group completion map $I_+(V, M)^H \rightarrow I(V, M)^H$ for every subgroup H of G .

Theorem 1.2 *The correspondence $X \rightarrow I(V, X \wedge M)$ is a G -equivariant continuous functor of $\text{Top}(G)$ into itself and we have the following:*

- (1) *For any orthogonal G -module W the natural map*

$$I(V, X \wedge M) \rightarrow \Omega^W I(V, \Sigma^W X \wedge M)$$

adjoint to $S^V \wedge I(V, X \wedge M) \rightarrow I(V, S^W \wedge X \wedge M)$ is a weak G -equivalence.

- (2) *There exists an $RO(G)$ -graded homology theory $h_\bullet^G(-)$ such that*

$$h_n^G(X) = \pi_n I(V, X \wedge M)^G$$

holds for any X and $n \geq 0$.

These theorems enable us to construct equivariant generalizations of several popular homology theories. For example, consider the simplest case $M = S^0$. Then $C(V, X)$ is the usual configuration space, and hence its group completion $I(V, X)$ is weakly G -equivalent to the equivariant infinite loop space $\Omega^V \Sigma^V X$ by [1, Theorem (1.18)]. Thus we obtain the G -equivariant stable homotopy theory in this case. On the other hand, if we take arbitrary positive integers as labels then we obtain an $RO(G)$ -graded homology theory extending the ordinary homology $\widetilde{H}_n(X/G, \mathbb{Z})$. (Compare Lewis, May and McClure [2].) K -theory type examples also occur from our method, which will be discussed in a future paper.

2 Partial abelian monoids with G -action

Definition 2.1 A pointed G -space M is called a partial abelian monoid with G -action, or G -partial monoid for short, if for every $n \geq 0$ there are G -invariant subsets M_n of M^n and G -maps

$$M_n \rightarrow M, \quad (a_1, \dots, a_n) \mapsto a_1 + \dots + a_n$$

satisfying the conditions below.

- (1) $M_0 \rightarrow M$ is the inclusion of the basepoint 0 of M .
- (2) $M_1 \rightarrow M$ is the identity of M .
- (3) Let J_1, \dots, J_r be disjoint subsets of $\{1, \dots, n\}$ such that $J_1 \cup \dots \cup J_r = \{1, \dots, n\}$, and let (a_1, \dots, a_n) be an element of M^n such that $(a_j)_{j \in J_k}$ belongs to M_{n_k} , where n_k is the cardinality of J_k . Then $(a_1, \dots, a_n) \in M_n$ if and only if $(\sum_{j \in J_1} a_j, \dots, \sum_{j \in J_r} a_j) \in M_r$, and we have

$$a_1 + \dots + a_n = \sum_{j \in J_1} a_j + \dots + \sum_{j \in J_r} a_j$$

if either side of the equation makes sense.

Among the examples we have the following:

- (1) Let M be a G -invariant subset of a topological abelian group on which G acts through group homomorphisms. Suppose M contains the unit 0. Then M is a G -partial monoid with respect to the subsets

$$M_n = \{(a_1, \dots, a_n) \in M^n \mid a_1 + \dots + a_n \in M\}.$$

More generally, any G -invariant subset of a G -partial monoid that contains 0 is again a G -partial monoid.

- (2) Any pointed G -space X is a G -partial monoid with respect to folding maps $X_n = X \vee \cdots \vee X \rightarrow X$. In fact, this is a special case of the previous example, as X is a G -invariant subset of the infinite symmetric product $\mathrm{SP}^\infty X$.
- (3) Let V be an infinite dimensional real inner product space on which G acts through linear isometries. Then the Grassmannian $\mathrm{Gr}(V)$ of finite-dimensional subspaces of V is a G -partial monoid with respect to the inner direct sum operation $\mathrm{Gr}(V)_n \rightarrow \mathrm{Gr}(V)$, where $\mathrm{Gr}(V)_n$ is defined to be the subset consisting of those (W_1, \dots, W_n) such that $W_i \perp W_j$ if $i \neq j$.

Definition 2.2 For given G -partial monoids M and N , their smash product $M \wedge N$ is a G -partial monoid such that $(M \wedge N)_n$ is the subset consisting of those n -tuples that can be summed up to an element of $M \wedge N$ by using the distributivity relations:

$$\begin{aligned} c_1 \wedge d + \cdots + c_k \wedge d &= (c_1 + \cdots + c_k) \wedge d, & (c_1, \dots, c_k) \in M_k \\ c \wedge d_1 + \cdots + c \wedge d_l &= c \wedge (d_1 + \cdots + d_l), & (d_1, \dots, d_l) \in N_l \end{aligned}$$

Example 2.3 If X is a pointed G -space and M is a G -partial monoid, then $X \wedge M$ is a G -partial monoid such that

$$(X \wedge M)_n = X \wedge M_n$$

holds for every $n \geq 0$.

For any orthogonal G -module V , the labeled configuration space $C(V, M)$ is a G -partial monoid with respect to the partial sum operations

$$C(V, M)_n \rightarrow C(V, M), \quad ((c_1, a_1), \dots, (c_n, a_n)) \mapsto (\bigcup c_i, \bigcup a_i).$$

Here $C(V, M)_n$ consists of those n -tuples $((c_i, a_i)) \in C(V, M)^n$ such that for every $x \in \bigcup c_i$ the sum $\sum_{i \in \Lambda(x)} a_i(x)$ exists, where $\Lambda(x) = \{i \mid x \in c_i\}$, and $\bigcup a_i$ denotes the map $x \mapsto \sum_{i \in \Lambda(x)} a_i(x)$. Moreover, if V is a G -universe then $C(V, M)$ is a homotopy associative and homotopy commutative Hopf G -space. To see this, let us consider the functor

$$P \mapsto A(P) = C(V, P \wedge M)$$

from finite pointed sets to pointed G -spaces. For each $p \in P$, let δ_p be the pointed map $P \rightarrow \mathbf{1} = \{0, 1\}$ such that $\delta_p^{-1}(1) = \{p\}$ if p is not the basepoint of P , and let δ_p be the constant map if p is the basepoint. Then the G -map

$$\delta: A(P) \rightarrow \mathrm{Map}_0(P, A(\mathbf{1})), \quad a \mapsto (p \mapsto A(\delta_p)(a))$$

has a G -homotopy inverse $\psi: \mathrm{Map}_0(P, A(\mathbf{1})) \rightarrow A(P)$ defined as follows.

Since V is a G -universe, there exist an embedding of $P - \{0\}$ into V^G and a G -linear isometry $V \times V \rightarrow V$. Hence we can construct a G -equivariant embedding of $(P - \{0\}) \times V$ into V . For any $f \in \text{Map}_0(P, A(\mathbf{1}))$ let us write $f(p) = (c(p), a(p))$ and put $\psi(f) = (\hat{c}, \hat{a}) \in A(P)$, where \hat{c} is the image of

$$\bigcup_{p \in P - \{0\}} \{p\} \times c(p)$$

under the G -equivariant embedding $(P - \{0\}) \times V \rightarrow V$ and $\hat{a}: \hat{c} \rightarrow P \wedge M$ is induced by the composite maps

$$c(p) \xrightarrow{a(p)} M = \mathbf{1} \wedge M \xrightarrow{\iota_p \wedge 1} P \wedge M$$

where ι_p is a pointed map $\mathbf{1} \rightarrow P$ such that $\iota_p(1) = p$.

Therefore, A is a G -equivariant Γ -space in the sense of Segal. Hence the following proposition holds.

Proposition 2.4 $C(V, M)$ is a homotopy associative and homotopy commutative Hopf G -space with unit $\emptyset \in C(V, M)^G$.

Note that Hopf G -space multiplication μ of $C(V, M)$ is given by the composite

$$C(V, M)^2 \xrightarrow[\simeq]{\psi} C(V, M \vee M) \xrightarrow{\nabla_*} C(V, M)$$

where ∇_* is induced by the folding map $M \vee M \rightarrow M$.

Definition 2.5 A G -partial monoid M is *homotopically invertible* if there exist a map of G -partial monoids $\tau: M \rightarrow M$, called a *homotopy inversion*, and a G -homotopy $h_t: M \rightarrow M^2$ ($0 \leq t \leq 1$) satisfying the conditions below.

- (1) For every $t \in [0, 1]$, h_t is a map of G -partial monoid.
- (2) $h_0 = (1, \tau)$, ie we have $h_0(a) = (a, \tau(a))$ for any $a \in M$.
- (3) h_1 factors through a map $h'_1: M \rightarrow M_2$ and the composite $M \xrightarrow{h'_1} M_2 \xrightarrow{\Sigma} M$ is G -homotopic through maps of G -partial monoids to the constant map.

Proposition 2.6 If V is a G -universe and if M is homotopically invertible then $C(V, M)$ is a grouplike Hopf G -space.

Proof Let $\tau_*: C(V, M) \rightarrow C(V, M)$ be the map induced by the homotopy inversion of M . To see that $C(V, M)$ is grouplike, it suffices to show that the composite

$$C(V, M) \xrightarrow{(1, \tau_*)} C(V, M)^2 \xrightarrow{\mu} C(V, M)$$

is G -homotopic to the constant map with value \emptyset . Let us regard $M \times M$ as a G -partial monoid such that $(M \times M)_n = M_n \times M_n$ for $n \geq 0$. Then we have a diagram of pointed G -spaces

$$\begin{array}{ccccc}
 C(V, M) & \xrightarrow{(1, \tau)_*} & C(V, M^2) & \xrightarrow{(p_{1*}, p_{2*})} & C(V, M)^2 \\
 h'_{1*} \downarrow & & \uparrow & & \simeq \uparrow \delta \\
 C(V, M_2) & \xlongequal{\quad} & C(V, M_2) & \xleftarrow{\quad} & C(V, M \vee M) \\
 & & \Sigma_* \downarrow & & \downarrow \nabla_* \\
 & & C(V, M) & \xlongequal{\quad} & C(V, M)
 \end{array}$$

in which p_{1*} and p_{2*} are induced by the projections $M^2 \rightarrow M$ onto the first and the second factors, respectively, and unnamed arrows are induced by the inclusions of G -partial monoids. Clearly, the right hand side squares are commutative, and the upper left square commutes up to G -homotopy. Since δ has a G -homotopy inverse ψ and since ψ restricts to a G -homotopy inverse to the map $C(V, M \vee M) \rightarrow C(V, M^2)$ induced by the inclusion $M \vee M \rightarrow M^2$, all the arrows constituting the upper right square are G -homotopy equivalences. Thus we have

$$\mu(1, \tau_*) = \nabla_* \psi(p_{1*}, p_{2*})(1, \tau)_* \simeq \Sigma_* h'_{1*} \simeq \emptyset. \quad \square$$

3 The space of strings with labels

As usual, the symbols $[a, b]$, $[a, b)$, $(a, b]$, (a, b) represent bounded intervals in the real line, and $b - a$ is called the length of the interval. The space of intervals $I(\mathbb{R})$ consists of those unions $P = J_1 \cup \dots \cup J_r$ of finite number of pairwise disjoint bounded intervals. It is topologized in such a way that such operations as isotopy moves, concatenation of two disjoint intervals that have a connected union (eg $[a, c] \cup [c, b] = [a, b]$), and deletion of a half-open interval when its length tends to 0 are all continuous. Let $I(\mathbb{R})_n$ be the subset of $I(\mathbb{R})^n$ consisting of those n -tuples (P_1, \dots, P_n) that are pairwise disjoint. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to these $I(\mathbb{R})_n$ and maps

$$I(\mathbb{R})_n \rightarrow I(\mathbb{R}), \quad (P_1, \dots, P_n) \mapsto P_1 \cup \dots \cup P_n.$$

Details are given in Okuyama [3], where $I(\mathbb{R})$ is denoted by $I_1(S^0)$.

Lemma 3.1 $I(\mathbb{R})$ is a homotopically invertible partial abelian monoid.

Proof Given a bounded interval J , let τJ denote the complement of the boundary of $-J$ in its closure. To be more explicit, we put

$$\tau[a, b] = (-b, -a), \quad \tau(a, b) = [-b, -a], \quad \tau[a, b) = [-b, -a), \quad \tau(a, b] = (-b, -a].$$

Then the correspondence $J \mapsto \tau J$ extends to an involution τ of $I(\mathbb{R})$

$$J_1 \cup \dots \cup J_r \mapsto \tau J_r \cup \dots \cup \tau J_1.$$

Let $\alpha: \mathbb{R} \rightarrow (0, 1)$ be an order preserving homeomorphism and let

$$\alpha_t(s) = (1 - t)s + t\alpha(s)$$

for $t \in [0, 1]$ and $s \in \mathbb{R}$. Since $\alpha_t: \mathbb{R} \rightarrow \mathbb{R}$ is an embedding, it induces a map of partial monoids $I(\alpha_t): I(\mathbb{R}) \rightarrow I(\mathbb{R})$ for every t , and hence we can define a homotopy $h_t: I(\mathbb{R}) \rightarrow I(\mathbb{R})^2$ by

$$h_t(P) = (I(\alpha_t)(P), \tau I(\alpha_t)(P)).$$

Clearly, h_t is a map of partial monoids and we have $h_0 = (1, \tau)$ because $I(\alpha_0)$ is the identity. On the other hand, h_1 maps $I(\mathbb{R})$ into $I(\mathbb{R})_2$ because $I(\alpha)(P)$ is contained in $(0, 1)$ and hence is disjoint from $\tau I(\alpha)(P) \subset (-1, 0)$. Finally, we can define a homotopy $\Sigma h_1 \simeq \emptyset$ by moving $I(\alpha)(P)$ to negative direction and $\tau I(\alpha)(P)$ to positive direction, simultaneously, so that the strings J in $I(\alpha)(P)$ meet with τJ at the origin and the resulting half-open intervals eventually vanish. \square

Let $I(\mathbb{R})_+$ be the subset of $I(\mathbb{R})$ consisting of those $J_1 \cup \dots \cup J_r$ such that every J_i is a closed interval. Clearly, $I(\mathbb{R})_+$ is a partial submonoid of $I(\mathbb{R})$.

Definition 3.2 Given an orthogonal G -module V and a G -partial monoid M , let

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M), \quad I_+(V, M) = C(V, I(\mathbb{R})_+ \wedge M).$$

For any G -partial monoid M , $I(\mathbb{R}) \wedge M$ is a homotopically invertible G -partial monoid with homotopy inversion $\tau \wedge 1$. Thus we have the following proposition.

Proposition 3.3 *If V is a G -universe then $I(V, M)$ is grouplike for any M .*

4 Proof of Theorem 1.1

To establish a relation between $I(V, M)$ and $C(V, M)$, let us choose a linear embedding $e: \mathbb{R} \rightarrow V^G$ and a G -linear isometry $l: V \times V \rightarrow V$. Then we can define

$$\lambda: I_+(V, M) \rightarrow C(V, M)$$

to be the map which sends a finite set of labeled strings $\{(\{v_i\} \times J_i, a_i)\}$ to the set of labeled particles $\{(l(v_i, e(\hat{J}_i)), a_i)\}$, where \hat{J}_i is the middle point of the closed interval J_i . Note that $(v_i, e(\hat{J}_i))$ are pairwise distinct, hence so are $l(v_i, e(\hat{J}_i))$.

Proposition 4.1 $\lambda: I_+(V, M) \rightarrow C(V, M)$ is a G -homotopy equivalence of Hopf G -spaces.

Proof Since λ is natural with respect to M , it extends to a map of G -equivariant Γ -spaces. This, of course, implies that λ is a map of Hopf G -spaces.

To see that λ is a G -homotopy equivalence, let $\gamma: C(V, M) \rightarrow I_+(V, M)$ be a pointed G -map which sends a finite set of labeled particles $\{(v_i, a_i)\}$ to the set of labeled strings $\{(\{v_i\} \times [-1, 1], a_i)\}$. Then we have

$$\gamma\lambda(\{(\{v_i\} \times J_i, a_i)\}) = \{(\{l(v_i, e(\hat{J}_i))\} \times [-1, 1], a_i)\}$$

and we can define a G -homotopy $\gamma\lambda \simeq 1$ by

$$(\gamma\lambda)_t(\{(\{v_i\} \times J_i, a_i)\}) = \begin{cases} \{(\{l(v_i, e_{2t}(\hat{J}_i))\} \times \mathcal{I}_{2t}(J_i), a_i)\}, & 0 \leq t \leq 1/2 \\ \{(\{l_{2t-1}(v_i)\} \times J_i, a_i)\}, & 1/2 \leq t \leq 1 \end{cases}$$

where

- (1) $e_t: \mathbb{R} \rightarrow V^G$ is a linear map $s \mapsto (1-t)e(s)$.
- (2) If $J = [a, b]$ then $\mathcal{I}_t(J) = [ta - (1-t), tb + (1-t)]$. Thus $\{\mathcal{I}_t(J)\}$ is a continuous family of closed intervals such that $\mathcal{I}_0(J) = [-1, 1]$ and $\mathcal{I}_1(J) = J$.
- (3) $\{l_t\}$ is a continuous family of G -linear isometries $V \rightarrow V$ such that $l_0(v) = l(v, 0)$ and l_1 is the identity of V . (Such a family certainly exists because the space of G -linear isometries $V \rightarrow V$ is contractible if V is a G -universe.)

On the other hand, we can define a G -homotopy $\lambda\gamma \simeq 1$ by

$$(\lambda\gamma)_t(\{(v_i, a_i)\}) = \{(l_t(v_i), a_i)\}. \quad \square$$

Now let $\rho: I_+(V, M) \rightarrow I(V, M)$ be the map induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$. To complete the proof of [Theorem 1.1](#), we need to show that

$$(4-1) \quad \rho^H: I_+(V, M)^H \rightarrow I(V, M)^H$$

is a group completion for every subgroup H of G . Since V is an H -universe for any subgroup H of G , we need only consider the case $H = G$. But then we have:

Lemma 4.2 $I_+(V, M)^G \rightarrow I(V, M)^G$ is a group completion for a G -partial monoid M if so is $I_+(\mathbb{R}^\infty, M) \rightarrow I(\mathbb{R}^\infty, M)$ for all partial abelian monoids M .

Proof Let \mathcal{F} be a family of orbit types and let $C(V, M)_{\mathcal{F}}$ be the subspace of $C(V, M)$ consisting of those $(c, v) \in C(V, M)$ such that $c \in V_{\mathcal{F}}$, where $V_{\mathcal{F}} = \{v \in V \mid G_v \in \mathcal{F}\}$. If \mathcal{F}_1 and \mathcal{F}_2 are families such that $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2 - \mathcal{F}_1$ consists of only one conjugacy class (H) then we have a fibration sequence

$$C(V, M)_{\mathcal{F}_1}^G \rightarrow C(V, M)_{\mathcal{F}_2}^G \rightarrow C(V, M)_{(H)}^G.$$

Therefore, we see that $I_+(V, M)^G \rightarrow I(V, M)^G$ is a group completion if and only if so are $I_+(V, M)_{(H)}^G \rightarrow I(V, M)_{(H)}^G$, by arguing as in Section 6 of Caruso and Waner [1]. But we have

$$C(V, M)_{(H)}^G \cong C(V^H, M^H)_{(H)}^{NH} \simeq C(\mathbb{R}^\infty, EW(H) \wedge_{W(H)} M^H).$$

It follows that $I_+(V, M)^G \rightarrow I(V, M)^G$ is a group completion if so are

$$I_+(\mathbb{R}^\infty, EW(H) \wedge_{W(H)} M^H) \rightarrow I(\mathbb{R}^\infty, EW(H) \wedge_{W(H)} M^H). \quad \square$$

In order to prove the lemma in the non equivariant case we need a CW-monoid replacement for $I(\mathbb{R}^\infty, M)$. For any M let $|S_\bullet M|$ be the realization of the total singular complex of M regarded as a partial abelian monoid such that

$$|S_\bullet M|_n = |S_\bullet M_n| \subset |S_\bullet M|^n \quad (n \geq 0).$$

Let $D(M)$ be the classifying space of the permutative category $\mathcal{Q}(|S_\bullet M|)$ whose space of objects is $\coprod_{p \geq 0} |S_\bullet M|^p$ and whose morphisms from $(a_i) \in |S_\bullet M|^p$ to $(b_j) \in |S_\bullet M|^q$ are given by a map of finite sets $\theta: \{1, \dots, p\} \rightarrow \{1, \dots, q\}$ such that $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ hold. Then $D(M)$ is a CW-monoid since it is homeomorphic to the realization of the diagonal simplicial set $[n] \mapsto N_n \mathcal{Q}(S_n M)$. Moreover, there is a natural weak equivalence of Hopf spaces $\Phi: D(M) \rightarrow C(\mathbb{R}^\infty, M)$. (For details, see Shimakawa [7, Section 2.4].) Thus to prove Lemma 4.2 we need only show the following

Proposition 4.3 *The natural map $D(I(\mathbb{R})_+ \wedge M) \rightarrow D(I(\mathbb{R}) \wedge M)$ induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$ is a group completion.*

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids $f: D \rightarrow D'$ let $B(D, D')$ denote the realization of the category $\mathcal{B}(D, D')$ whose space of objects is D' and whose space of morphisms is the product $D \times D'$, where $(d, d') \in D \times D'$ is regarded as a morphism from d' to $f(d) \cdot d'$. Then there is a sequence of maps

$$D' = B(0, D') \rightarrow B(D, D') \rightarrow B(D, 0) = BD$$

induced by the maps $0 \rightarrow D$ and $D' \rightarrow 0$ respectively. Observe that BD is the standard classifying space of the monoid D and $B(D, D)$ is contractible when f is the identity.

In particular, let us take $D = D(I(\mathbb{R})_+ \wedge M)$ and $D' = D(I(\mathbb{R}) \wedge M)$, and let $i: D \rightarrow D'$ be the monoid map induced by the inclusion $I(\mathbb{R})_+ \rightarrow I(\mathbb{R})$. Then there is a commutative diagram

$$(4-2) \quad \begin{array}{ccccc} D & \longrightarrow & B(D, D) & \longrightarrow & BD \\ i \downarrow & & \downarrow B(1,i) & & \parallel \\ D' & \longrightarrow & B(D, D') & \longrightarrow & BD \end{array}$$

in which the upper and the lower sequences are associated with the identity and the inclusion $i: D \rightarrow D'$, respectively.

Lemma 4.4 *The natural map $D \rightarrow \Omega BD$ is a group completion.*

This follows from the fact that D is a homotopy commutative monoid.

Lemma 4.5 *The lower sequence in the diagram (4-2) is a homotopy fibration sequence with contractible total space.*

Proposition 4.3 is deduced from this, because $D \rightarrow D'$ is equivalent to the group completion map $D \rightarrow \Omega BD$ under the equivalence $D' \simeq \Omega BD$.

Proof of Lemma 4.5 By Proposition 3.3, $D' = D(I(\mathbb{R}) \wedge M)$ is grouplike with homotopy inverse induced by the homotopy inversion $\tau \wedge 1$. Hence D acts on D' through homotopy equivalences, and the diagram

$$\begin{array}{ccc} D' & \longrightarrow & B(D, D') \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B(D, 0) \end{array}$$

is homotopy cartesian by Proposition 1.6 of Segal [4]. This implies that the lower sequence in the diagram (4-2) is a homotopy fibration sequence.

It remains to prove that $B(D, D')$ is contractible. In [7], we proved this in the case where the partial monoid $X \wedge \pm M$ is strictly invertible and is generated by the elements of $X \wedge M$ and their inverses. But the argument given there still applies to the current case, once we make the following change in the notation.

Replace $X \wedge M$ and $X \wedge \pm M$ by $I_+(\mathbb{R}) \wedge M$ and $I(\mathbb{R}) \wedge M$, respectively, and for any $S = (P_j \wedge a_j) \in S_0(I(\mathbb{R}) \wedge M)^P$ put

$$S_+ = (P_j^+ \wedge a_j), \quad S_- = (P_j^- \wedge a_j), \quad \bar{S} = (\tau P_j \wedge a_j),$$

where P_j^+ and P_j^- are the unions of closed intervals and of open or half-open intervals contained in P_j , respectively. Note that we have $P_j = P_j^+ \cup P_j^-$ and $P_j^+ \in I_+(\mathbb{R})$. Also, for any S such that $S = S_-$ the path $[S] \rightarrow [\mathbf{0}^P]$ in $B(D, D')$ should be defined to be the composite

$$[S] \rightarrow [\bar{S}_+ \cdot S] \rightarrow [\overline{I(\alpha)_*(S)}_+ \cdot I(\alpha)_*(S)] \xrightarrow{\nabla} [\mathbf{0}^P]$$

where α is a homeomorphism $\mathbb{R} \cong (0, 1)$, $I(\alpha)_*(P_j \wedge a_j) = (I(\alpha)(P_j) \wedge a_j)$, and ∇ is induced by the homotopy

$$\tau I(\alpha)(P_j)^+ \wedge a_j + I(\alpha)(P_j) \wedge a_j = (\tau I(\alpha)(P_j)^+ \cup I(\alpha)(P_j)) \wedge a_j \simeq \emptyset \wedge a_j = 0.$$

(Compare the proof of [Lemma 3.1.](#)) □

5 Proof of [Theorem 1.2](#)

By a simplicial pointed G -space we shall mean a simplicial object in the category of pointed G -spaces and basepoint preserving G -maps. If X_\bullet is a simplicial pointed G -space then the basepoints of X_n form the simplicial set $*$. Let

$$\|X_\bullet\|' = \|X_\bullet\|/\|*\|.$$

Then the natural map $\|X_\bullet\| \rightarrow \|X_\bullet\|'$ is a G -homotopy equivalence, and the maps $\Delta^n \times X_n \rightarrow \|X_\bullet\|$ induce $\Delta_+^n \wedge X_n \rightarrow \|X_\bullet\|'$.

Let T be a G -equivariant continuous functor $\text{Top}(G) \rightarrow \text{Top}(G)$. Then any simplicial pointed G -space X_\bullet is associated with a G -map $\|T(X_\bullet)\|' \rightarrow T(\|X_\bullet\|')$ induced by the maps

$$\Delta^n \times T(X_n) \rightarrow \Delta_+^n \wedge T(X_n) \rightarrow T(\Delta_+^n \wedge X_n) \rightarrow T(\|X_\bullet\|').$$

The next proposition plays a key role in the proof of [Theorem 1.2](#).

Proposition 5.1 *Let $T: \text{Top}(G) \rightarrow \text{Top}(G)$ be a G -equivariant continuous functor. Suppose T satisfies the conditions below.*

(C1) $T(*) = *$.

(C2) *For any simplicial pointed G -space X_\bullet the natural map $\|T(X_\bullet)\|' \rightarrow T(\|X_\bullet\|')$ is a G -homotopy equivalence.*

- (C3) For any X and Y the map $T(X \vee Y) \rightarrow T(X) \times T(Y)$ induced by the projections $X \vee Y \rightarrow X$ and $X \vee Y \rightarrow Y$ is a G -homotopy equivalence.
- (C4) For any subgroup H the natural map $T(G/H_+ \wedge X) \rightarrow \text{Map}_0(G/H_+, T(X))$, whose adjoint $G/H_+ \wedge T(G/H_+ \wedge X) \rightarrow T(X)$ is induced by the pairing $G/H_+ \wedge G/H_+ \wedge X \rightarrow X$ which sends (s, s, x) to x and (s, t, x) ($s \neq t$) to the basepoint of X , is a G -homotopy equivalence.

Suppose further that $T(X)^H$ is grouplike for any X and any subgroup H of G . Then the following hold.

- (1) For any orthogonal G -module W the natural map $T(X) \rightarrow \Omega^W T(\Sigma^W X)$ adjoint to $S^W \wedge T(X) \rightarrow T(S^W \wedge X)$ is a weak G -homotopy equivalence.
- (2) The correspondence $X \mapsto \{\pi_n T(X)^G\}$ is extendible to an $RO(G)$ -graded equivariant homology theory defined on the category of pointed G -spaces.

Proof For any pointed G -space X let $E(X) = \Omega T(\Sigma X)$. If T satisfies (C1), (C2) and (C3) then by the equivariant version of [6, Theorem 2.12] the natural map $T(X) \rightarrow E(X)$ is a G -equivariant group completion and the sequence

$$E(A) \rightarrow E(X) \rightarrow E(X \cup CA)$$

associated with a pair of pointed G -spaces (X, A) is a G -fibration sequence up to weak G -equivalence. But $T(X) \rightarrow E(X) = \Omega T(\Sigma X)$ is a weak G -equivalence because $T(X)^H$ is grouplike for any subgroup H . Hence

$$T(A) \rightarrow T(X) \rightarrow T(X \cup CA)$$

is a G -fibration sequence up to weak G -equivalence. Moreover, T preserves G -homotopies because it is a G -equivariant continuous functor. Therefore, the correspondence $X \mapsto \{\pi_n T(X)^G\}$ determines a \mathbb{Z} -graded equivariant homology theory.

Let Γ_G be the full subcategory of $\text{Top}(G)$ consisting of finite pointed G -sets. To prove the assertions we need only show that the correspondence $S \mapsto T(S \wedge X)$ from Γ_G to $\text{Top}(G)$ is a special Γ_G -space in the sense of [5]. But this follows from the conditions (C3) and (C4). \square

Now let $T(X) = I(V, X \wedge M)$. We shall show that T satisfies the conditions (C1), (C2), (C3) and (C4). This of course proves [Theorem 1.2](#).

It is obvious that (C1) holds. (C2) is proved by the argument similar to the one used in the proof of [6, Theorem 3.2]. To prove (C3) let us define

$$T(X) \times T(Y) \rightarrow T(X \vee Y)$$

to be the composite

$$I(V, X \wedge M) \times I(V, Y \wedge M) \xrightarrow{(i_*, j_*)} I(V, (X \vee Y) \wedge M)^2 \xrightarrow{\mu} I(V, (X \vee Y) \wedge M)$$

where i_* and j_* are induced by the inclusions of X and Y into $X \vee Y$, respectively, and μ is the multiplication of the Hopf G -space $I(V, (X \vee Y) \wedge M)$. By using the fact that the space of G -linear isometries of V is contractible one can show that the map above gives a G -homotopy inverse to $T(X \vee Y) \rightarrow T(X) \times T(Y)$. Finally, to prove (C4) let us choose a G -embedding $G/H \rightarrow V$ and a G -linear isometry $l: V \times V \rightarrow V$. Then we can construct a G -homotopy inverse to the natural map $T(G/H_+ \wedge X) \rightarrow \text{Map}_0(G/H_+, T(X))$ by the following procedure:

- (1) For given $f: G/H_+ \rightarrow T(X)$ let us write

$$f(gH) = (c(gH), P(gH) \wedge a(gH))$$

where $c(gH) \subset V$, $P(gH): c(gH) \rightarrow I(\mathbb{R})$ and $a(gH): c(gH) \rightarrow X \wedge M$.

- (2) Let \tilde{c} be the image of the union $\bigcup \{gH\} \times c(gH)$ under the embedding

$$\iota: G/H \times V \subset V \times V \xrightarrow{l} V.$$

- (3) Define $\tilde{a}: \tilde{c} \rightarrow I(\mathbb{R}) \wedge G/H_+ \wedge X \wedge M$ by

$$\tilde{a}(\iota(gH, \xi)) = P(gH) \wedge gH \wedge a(gH)(\xi), \quad \xi \in c(gH).$$

- (4) Define $\rho: \text{Map}(G/H, T(X)) \rightarrow T(G/H_+ \wedge X)$ by $\rho(f) = (\tilde{c}, \tilde{a})$.

That ρ gives a G -homotopy inverse to $T(G/H_+ \wedge X) \rightarrow \text{Map}(G/H, T(X))$ follows, again, from the contractibility of the space of G -linear isometries of V .

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