On the E^1–term of the gravity spectral sequence

DAI TAMAKI

The author constructed a spectral sequence strongly converging to $h_*(\Omega^n\Sigma^n X)$ for any homology theory in [13]. In this note, we prove that the E^1–term of the spectral sequence is isomorphic to the cobar construction, and hence the spectral sequence is isomorphic to the classical cobar-type Eilenberg–Moore spectral sequence based on the geometric cobar construction from the E^1–term. Similar arguments can be also applied to its variants constructed in [14].

55T20; 55P48

1 Introduction

In [13], the author introduced a filtration $\{F_{-s}\mathcal{C}_n(j)\}_{s\geq 0}$ on the space of little cubes $\mathcal{C}_n(j)$ with

$$\emptyset = F_{-j-1}\mathcal{C}_n(j) \subset \cdots F_{-s-1}\mathcal{C}_n(j) \subset F_{-s}\mathcal{C}_n(j) \subset \cdots \\ \subset F_{-1}\mathcal{C}_n(j) = F_0\mathcal{C}_n(j) = \mathcal{C}_n(j).$$

This is called the gravity filtration. By using the Snaith splitting,

$$\Omega^n\Sigma^n X \underset{S}{\simeq} \bigvee_{j=1}^{\infty} \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$

we obtain a stable filtration on $\Omega^n\Sigma^n X$ and hence a spectral sequence for computing $h_*(\Omega^n\Sigma^n X)$. This spectral sequence is called the gravity spectral sequence. The author proved in [13] the following:

Theorem 1.1 *For any homology theory $h_*(-)$, the gravity filtration induces a spectral sequence strongly converging to $h_*(\Omega^n\Sigma^n X)$.*

When h_* is multiplicative and $h_*(\Omega^{n-1}\Sigma^n X)$ is $h_*(*)$–flat, we have

$$E^2 \cong \mathrm{Cotor}^{h_*(\Omega^{n-1}\Sigma^n X)}(h_*(*), h_*(*)).$$

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Later in [14], the author constructed spectral sequences for fibrations related to iterated Freudenthal suspensions by adopting the gravity filtration.

The Eilenberg–Moore spectral sequence constructed by the (algebraic or geometric) cobar construction (see Eilenberg–Moore [4; 5], Rector [7] and Dwyer [2; 3]), the so-called cobar-type Eilenberg–Moore spectral sequence, also has its E^2–term isomorphic to Cotor. A natural question is if the gravity spectral sequence is isomorphic to the cobar-type Eilenberg–Moore spectral sequence.

In the case of the cobar-type Eilenberg–Moore spectral sequence for the path-loop fibration

$$\Omega X \longrightarrow PX \longrightarrow X,$$

the E^1–term is given by the (algebraic) cobar construction, ie

$$E^1_{-s,*} \cong \left(\Sigma^{-1}\widetilde{h}_*(X)\right)^{\otimes s},$$

if h_* is multiplicative and $h_*(X)$ is $h_*(*)$–flat.

The purpose of this note is to prove the following theorem.

Theorem 1.2 *The E^1–term of the gravity spectral sequence [13] is isomorphic to the cobar construction, ie*

$$E^1_{-s,*} \cong \left(\Sigma^{-1}\widetilde{h}_*(\Omega^{n-1}\Sigma^n X)\right)^{\otimes s}$$

as chain complexes, if h_* is multiplicative and $h_*(\Omega^{n-1}\Sigma^n X)$ is $h_*(*)$–flat. Hence the gravity spectral sequence is isomorphic to the classical cobar-type Eilenberg–Moore spectral sequence as spectral sequences.

This result is useful for practical applications of the gravity spectral sequence and simplifies the arguments in the computation performed in the last section of [14].

This result also gives us a "geometric model" for the cobar differential. Note that the spectral sequence splits into a direct sum of small spectral sequences each of which is induced from the filtration on $\mathcal{C}_n(j)$ for $j = 1, 2, \cdots$. By definition, the d^1 differential on $\mathcal{C}_n(j)$ part is given by the following composition

$$\begin{aligned} F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) &\simeq F_{-s}\mathcal{C}_n(j) \cup CF_{-s-1}\mathcal{C}_n(j) \\ &\longrightarrow \Sigma F_{-s-1}\mathcal{C}_n(j) \longrightarrow \Sigma F_{-s-1}\mathcal{C}_n(j)/F_{-s-2}\mathcal{C}_n(j) \end{aligned}$$

where the NDR representation of $(F_{-s}\mathcal{C}_n(j), F_{-s-1}\mathcal{C}_n(j))$, which has an explicit description in terms of the centers and radii of little cubes, gives the first homotopy equivalences.

In particular, as the referee pointed out, in the case of $n = 2$, the d^1 differential is given in terms of shuffles for smash products of ΣX. Note that the decomposition of the permutation representation

$$\mathbb{R}^j \cong \left\{(b_1, \cdots, b_j) \in \mathbb{R}^j \,\middle|\, \sum b_k = 0\right\} \oplus \{(t, \cdots, t) \mid t \in \mathbb{R}\}$$

allows us to associate a map

$$s\colon (\Sigma X)^{\wedge j} \longrightarrow (\Sigma X)^{\wedge j}$$

to an element s of the group ring $\mathbb{Z}[\Sigma_j]$ which induces the action of s in homology. Let $s_{i,j} \in \mathbb{Z}[\Sigma_{i+j}]$ be the sum of all (i, j)–shuffles.

With this notation, we see that, when $n = 2$, the d^1 differential

$$d^1_{s,*}\colon E^1_{-s,*} \cong (\Sigma^{-1} T(\widetilde{h}_*(\Sigma X)))^{\otimes s} \longrightarrow (\Sigma^{-1} T(\widetilde{h}_*(\Sigma X)))^{\otimes s+1} \cong E^1_{-s-1,*}$$

is induced by the following map:

$$\sum_{m=1}^{s} \sum_{\ell=1}^{j_m-1} 1 \wedge \cdots \wedge s_{\ell, j_m-\ell} \wedge \cdots \wedge 1\colon (\Sigma X)^{\wedge j_1} \wedge \cdots \wedge (\Sigma X)^{\wedge j_s} \longrightarrow \bigvee_{(k_1,\cdots,k_{s+1})} (\Sigma X)^{\wedge k_1} \wedge \cdots \wedge (\Sigma X)^{\wedge k_{s+1}}, \tag{1}$$

where the wedge sum in the range is taken over $(s + 1)$–tuples of positive integers $(k_1, \cdots, k_{s+1})$ with

$$k_1 + \cdots + k_{s+1} = j_1 + \cdots + j_s.$$

This paper is organized as follows: we recall the definition of the gravity filtration together with the basic properties of the little cubes operad in Section 2. The E^1–term of the gravity filtration is analyzed in the first half of Section 3 and then it is proved that the E^1–term and the first differential of the gravity spectral sequence coincides with those of the classical Eilenberg–Moore spectral sequence in the rest of Section 3. In Section 4, we recall Rector's and Larry Smith's constructions in order to prove that their spectral sequences are isomorphic to the gravity spectral sequence from the E^1–terms in Section 5.

Acknowledgements This paper should be considered as a complement to [13], which was a part of the author's PhD thesis supervised by Fred Cohen. The author is grateful to him for asking about the E^1–term of the gravity spectral sequence. The result in this paper would not have come out without his persuasion. The author would also like to thank the referee for pointing out that the d^1 differential for $\Omega^2\Sigma^2 X$ could be written in terms of shuffles on $(\Sigma X)^{\wedge j}$.

2 The gravity filtration

Let us first recall the construction of the gravity spectral sequence in [13].

Definition 2.1 A little n–cube is an embedding

$$c\colon [-1,1]^n \hookrightarrow [-1,1]^n$$

of the form
$$c = \ell_1 \times \cdots \times \ell_n$$

where each ℓ_i is an orientation preserving affine embedding

$$\ell_i\colon [-1,1] \longrightarrow [-1,1].$$

The space of little n–cubes (with compact-open topology) is denoted by $\mathcal{C}_n(1)$.

Note that a 1–cube c is determined by its center $C = c(0)$ and radius $R = c(1) - c(0)$.

Definition 2.2 The space of j little n–cubes $\mathcal{C}_n(j)$ is defined by

$$\mathcal{C}_n(j) = \left\{(c_1,\cdots,c_j) \in \mathcal{C}_n(1)^j \;\middle|\; \operatorname{Im} c_i \cap \operatorname{Im} c_k = \varnothing \ (i \neq k)\right\},$$

where $\operatorname{Im} c_i = c_i((-1,1)^n)$. The symmetric group of j letters Σ_j acts on $\mathcal{C}_n(j)$ by permuting cubes.

Note that, for any finite set S, the space of n–cubes indexed in S, $\mathcal{C}_n(S)$, is defined. Thus $\mathcal{C}_n$ can be regarded as a contravariant functor

$$\mathcal{C}_n\colon \textbf{Finite Sets} \longrightarrow \textbf{Spaces}$$

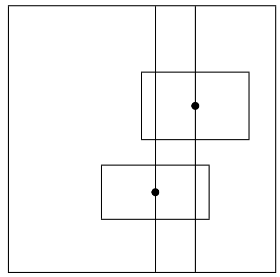
from the category of finite sets and injective maps to the category of topological spaces.

The gravity filtration on $\mathcal{C}_n(j)$ is defined as follows.

Definition 2.3 Let $F_0\mathcal{C}_n(j) = \mathcal{C}_n(j)$. For $s \geq 1$, define

$(c_1,\cdots,c_j) \in F_{-s}\mathcal{C}_n(j) \Longleftrightarrow$ We need to decompose the set $\{c_1,\cdots,c_j\}$ into at least s disjoint subsets in order to make each group "stable under gravity".

We say a collection of little n–cubes $\{c_{i_1},\cdots,c_{i_k}\}$ is stable under gravity if the center of the first coordinate of each cube is contained in the images of the first coordinate of other cubes. For example, the two 2-cubes in the picture below are stable under gravity.



In other words, the vertical hyperplane through the center of a cube must intersect with the interior of other cubes.

It is more useful to describe the above filtration in terms of functions which measure overlaps of the first coordinates of cubes.

Definition 2.4 Let b be a little 1–cube with center C and radius R. For $x \in (-1,1)$, define

$$d(x,b) = \frac{2R - ||C+R-x| - |C-R-x||}{2R}.$$

For $c_1, c_2 \in \mathcal{C}_n(1)$, define

$$\mathrm{dis}(c_1,c_2) = \min\{d(c_1'(0),c_2'), d(c_2'(0),c_1')\},$$

where c_i' is the first coordinate of c_i.

Definition 2.5 For $S \subset \{1,\cdots,j\}$ and $\mathbf{c} \in \mathcal{C}_n(j)$, define

$$\mathrm{OL}(\mathbf{c},S) = \min\{\mathrm{dis}(c_k,c_\ell) | k,\ell \in S\}.$$

For a partition $P : S_1 \amalg \cdots \amalg S_s = \{1,\cdots,j\}$, define

$$\mathrm{MOL}(\mathbf{c},P) = \min_k\{\mathrm{OL}(\mathbf{c},S_k)\}.$$

And for $\mathbf{c} \in F_{-s}\mathcal{C}_n(j)$, define

$$u_s(\mathbf{c}) = \max\{\mathrm{MOL}(\mathbf{c},P) | P : \text{ partition into } s \text{ subsets}\}.$$

Recall that the configuration space of j points in $\mathbb{R}^n$, $F(\mathbb{R}^n,j)$, is Σ_j–equivariantly homotopy equivalent to $\mathcal{C}_n(j)$. And it is possible and seems more natural to define a filtration on $F(\mathbb{R}^n,j)$ by the number of distinct first coordinates. In fact, this filtration on $F(\mathbb{R}^n,j)$ was the origin of the gravity filtration on $\mathcal{C}_n(j)$. The following fact is an essential difference between these two filtrations and is proved in [13].

Lemma 2.6 *The map*

$$u_s\colon F_{-s}\mathcal{C}_n(j) \longrightarrow [0,1]$$

is continuous and

$$u_s^{-1}(0) = F_{-s-1}\mathcal{C}_n(j).$$

Furthermore there exists a homotopy

$$h_s\colon F_{-s}\mathcal{C}_n(j) \times I \longrightarrow F_{-s}\mathcal{C}_n(j)$$

with which $(h_s, \widetilde{u}_s)$ *is an NDR representation for* $(F_{-s}\mathcal{C}_n(j), F_{-s-1}\mathcal{C}_n(j))$*, where* $\widetilde{u}_s = m \circ u_s$ *and the function*

$$m\colon [0,1] \longrightarrow [0,1]$$

is defined by

$$m(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1. \end{cases}$$

The following property of u_q is not used in [13], but turns out to be very useful for identifying the E^1–term.

Lemma 2.7 $u_s(\mathbf{c}) = 1$ *if and only if* $\mathbf{c}$ *consists of* s *piles of cubes each of which consists of cubes whose centers are lined up on a single vertical line (hyperplane).*

By using the base point relation, we can form a single space by gluing $\mathcal{C}_n(j) \times_{\Sigma_j} X^j$ together.

Definition 2.8 For a pointed space X, define

$$C_n(X) = \left(\coprod_{j=1}^{\infty} \mathcal{C}_n(j) \times_{\Sigma_j} X^j \right) \Big/ \sim$$

where the relation $\sim$ is given by

$$(c_1, \cdots, c_j; x_1, \cdots, x_j) \sim (c_1, \cdots, \widehat{c_i}, \cdots, c_j; x_1, \cdots, \widehat{x_i}, \cdots, x_j),$$

if $x_i = *$.

$C_n(X)$ is an approximation to $\Omega^n \Sigma^n X$ up to a weak equivalence.

Theorem 2.9 (Approximation Theorem [6]) *For a path-connected space* X *with nondegenerate base point, we have the following natural weak equivalence*

$$C_n(X) \underset{w}{\simeq} \Omega^n \Sigma^n X.$$

Unfortunately the gravity filtration is not compatible with the base point relation in the definition of $C_n(X)$. Fortunately, however, we can introduce a stable filtration on $C_n(X)$, thanks to the following famous theorem.

Theorem 2.10 (Snaith Splitting [10]) *For a path-connected space X with a nondegenerate base point, we have the following natural weak equivalence in the stable homotopy category,*

$$C_n(X) \underset{S}{\simeq} \bigvee_{j=1}^{\infty} \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}. \tag{2}$$

Definition 2.11 We define

$$F_{-s}C_n(X) = \bigvee_{j=1}^{\infty} F_{-s}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}.$$

This can be regarded as a filtration on $\Omega^n\Sigma^n X$ in the stable homotopy category.

We use the following notations:

$$D_j^{(n)}(X) = \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$
$$F_{-s}D_j^{(n)}(X) = F_{-s}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$

This stable filtration gives a spectral sequence strongly converging to $h_*(\Omega^n\Sigma^n X)$ for any homology theory $h_*(-)$, whose E^1–term is given by

$$\begin{aligned} E^1_{-s,t} &= h_{-s+t}(F_{-s}C_n(X), F_{-s-1}C_n(X)) \\ &\cong \bigoplus_{j\geq 1} h_{-s+t}\left(F_{-s}D_j^{(n)}(X), F_{-s-1}D_j^{(n)}(X)\right) \\ &\cong \bigoplus_{j\geq 1} \widetilde{h}_{-s+t}\left(F_{-s}D_j^{(n)}(X)/F_{-s-1}D_j^{(n)}(X)\right) \end{aligned}$$

since $\left(F_{-s}D_j^{(n)}(X), F_{-s-1}D_j^{(n)}(X)\right)$ is an NDR pair if the base point of X is nondegenerate, thanks to Lemma 2.6.

3 Decomposition of cubes

Let $h_*(-)$ be a homology theory satisfying the strong form of Künneth isomorphism for $\Omega^{n-1}\Sigma^n X$. Then the E^1–term of the classical Eilenberg–Moore spectral sequence for the path-loop fibration on $\Omega^{n-1}\Sigma^n X$ is isomorphic to

$$\begin{aligned} E^1_{-s,t} &\cong \widetilde{h}_t\left(\left(\Omega^{n-1}\Sigma^n X\right)^{\wedge s}\right) \\ &\cong \widetilde{h}_t\left(C_{n-1}(\Sigma X)^{\wedge s}\right). \end{aligned}$$

Under the Snaith splitting, we have

$$\widetilde{h}_t\left(C_{n-1}(\Sigma X)^{\wedge s}\right) \cong \bigoplus_{\sum i_k = j} \widetilde{h}_t\left(D^{(n-1)}_{i_1}(\Sigma X)\wedge\cdots\wedge D^{(n-1)}_{i_s}(\Sigma X)\right).$$

Thus, with the notations in Section 2, all we want to do is to find a natural homotopy equivalence

$$\Sigma^s F_{-s}D^{(n)}_j(X)/F_{-s-1}D^{(n)}_j(X) \simeq \bigvee_{\sum i_k = j} D^{(n-1)}_{i_1}(\Sigma X)\wedge\cdots\wedge D^{(n-1)}_{i_s}(\Sigma X)$$

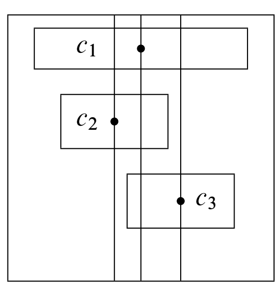
or a Σ_j–equivariant homotopy equivalence

$$\Sigma^s F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) \simeq \bigvee_{\substack{\amalg S_k = \\ \{1,\cdots,j\}}} \left(\mathcal{C}_{n-1}(S_1)_+\wedge S^{|S_1|}\right)\wedge\cdots\wedge\left(\mathcal{C}_{n-1}(S_s)_+\wedge S^{|S_s|}\right)$$

where $\mathcal{C}_n(S)$ is the space of little n–cubes indexed by the set S and the wedge sum on the right hand side runs over all partitions of $\{1,\cdots,j\}$ into nonempty k subsets.

However this is not easy. To understand the difficulty, let us try to define a map

$$\Sigma^2 F_{-2}\mathcal{C}_n(3)/F_{-3}\mathcal{C}_n(3) \longrightarrow \bigvee_{\substack{S_1\amalg S_2 = \\ \{1,2,3\}}} \left(\mathcal{C}_{n-1}(S_1)_+\wedge S^{|S_1|}\right)\wedge\left(\mathcal{C}_{n-1}(S_2)_+\wedge S^{|S_2|}\right).$$



Forget about the suspension coordinates. Consider the cubes in the above picture. This element belongs to $F_{-2}\mathcal{C}_n(3) - F_{-3}\mathcal{C}_n(3)$ and there are two ways to decompose it into two collections of cubes stable under gravity, ie

$$\{1,2,3\} = \{1,2\} \amalg \{3\}$$
$$\{1,2,3\} = \{1,3\} \amalg \{2\}.$$

Thus a canonical map we obtain is

$$\Sigma^2 F_{-2}\mathcal{C}_n(3)/F_{-3}\mathcal{C}_n(3) \longrightarrow \prod_{\substack{S_1 \amalg S_2 = \\ \{1,2,3\}}} \left(\mathcal{C}_{n-1}(S_1)_+ \wedge S^{|S_1|}\right) \wedge \left(\mathcal{C}_{n-1}(S_2)_+ \wedge S^{|S_2|}\right),$$

not into $\bigvee$.

More generally, taking all possible decompositions would give the following map

$$\Sigma^s F_{-s}\mathcal{C}_n(j) \Big/ F_{-s-1}\mathcal{C}_n(j) \longrightarrow \prod_{\substack{\amalg S_k = \\ \{1,\cdots,j\}}} \left(\mathcal{C}_{n-1}(S_1)_+ \wedge S^{|S_1|}\right) \wedge \cdots \wedge \left(\mathcal{C}_{n-1}(S_s)_+ \wedge S^{|S_s|}\right).$$

We need to compress the image of this map into

$$\bigvee_{\amalg S_k = \{1,\cdots,j\}} \left(\mathcal{C}_{n-1}(S_1)_+ \wedge S^{|S_1|}\right) \wedge \cdots \wedge \left(\mathcal{C}_{n-1}(S_s)_+ \wedge S^{|S_s|}\right).$$

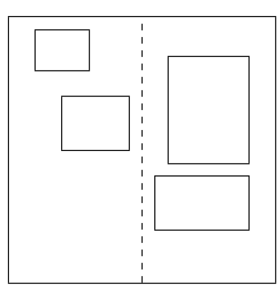
To this end, our idea is to deform $\Sigma^s F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j)$ into a smaller space of "decomposable cubes".

G Dunn introduced the notion of decomposable cubes in [1] and proved a decomposition of the little n–cubes operad

$$\mathcal{C}_n \simeq \underbrace{\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_1}_{n}.$$

In our case, we need the notion of horizontally decomposable cubes.

Definition 3.1 Let $\mathcal{D}^s_n(j)$ be the subset of $\mathcal{C}_n(j)$ consisting of cubes which are horizontally decomposable into s collections (as in the picture below).



More precisely, let
$$i_1\colon \mathcal{C}_1(j) \hookrightarrow \mathcal{C}_n(j)$$
be the inclusion of the first coordinate given by the multiplication of the identity $(n-1)$–cube.

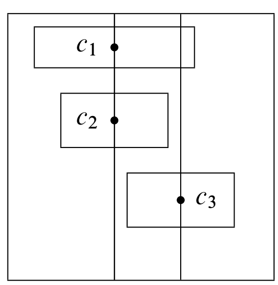
Namely $$i_1(c_1,\cdots,c_j) = (c_1 \times 1_{I^{n-1}},\cdots,c_j \times 1_{I^{n-1}}).$$

Then $\mathcal{D}_n^s(j)$ is the image of the following restriction of the operad structure map
$$\gamma\colon i_1(\mathcal{C}_1(s)) \times \left(\coprod_{\sum i_k = j} \mathcal{C}_n(i_1)\times\cdots\times\mathcal{C}_n(i_s)\right) \longrightarrow \mathcal{C}_n(j).$$

We have the following diagram:
$$\begin{tikzcd}
\mathcal{D}_n^s(j) \arrow[r, "\subset"] & F_{-s}\mathcal{C}_n(j) \\
\mathcal{D}_n^{s+1}(j) \arrow[u, "\cup"'] \arrow[r, "\subset"] & F_{-s-1}\mathcal{C}_n(j) \arrow[u, "\cup"']
\end{tikzcd}$$

We want to show that the inclusion gives a homotopy equivalence
$$\mathcal{D}_n^s(j)/\mathcal{D}_n^{s+1}(j)) \simeq F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j).$$

Note that if $\mathbf{c} = (c_1,\cdots,c_j) \in F_{-s}\mathcal{C}_n(j)$, then there are at least s cubes $c_{i_1},\cdots,c_{i_s}$ whose centers of the first coordinates are distinct.



Thus by shrinking the radii of the first coordinates of cubes, we can deform $F_{-s}\mathcal{C}_n(j)$ into $\mathcal{D}_n^s(j)$. The cubes in the above picture are in $F_{-2}\mathcal{C}_2(3)$ but not in $\mathcal{D}_2^2(3)$. A horizontal shrinking deforms the cubes into $\mathcal{D}_2^2(3)$.

Definition 3.2 Let
$$H\colon \mathcal{C}_n(j)\times[0,1) \longrightarrow \mathcal{C}_n(j)$$

be the homotopy which shrinks the radius of the first coordinate of each cube linearly without moving the center.

Define $$\sigma_s\colon F_{-s}\mathcal{C}_n(j) \longrightarrow [0,1)$$
by $$\sigma_s(\mathbf{c}) = \inf\{t \mid H(\mathbf{c},t) \in \mathcal{D}_n^s(j)\}.$$

Then obviously σ_s is continuous and gives the minimal amount of the radii of the first coordinates we need to shrink for those cubes in $F_{-s}\mathcal{C}_n(j)$ in order to compress them into $\mathcal{D}_n^s(j)$. For most cubes $\mathbf{c} \in F_{-s}\mathcal{C}_n(j)$, $H(\mathbf{c},\sigma_{q+1}(\mathbf{c}))$ is defined. However, if $\mathbf{c}$ consists of s piles of cubes each of which consists of cubes whose centers are lined up in a single vertical line (hyperplane), $H(\mathbf{c},\sigma_{q+1}(\mathbf{c}))$ squashes the cubes flat vertically. For those cubes we need to use $H(\mathbf{c},\sigma_s(\mathbf{c}))$. Namely, the amount of shrinking varies for different configurations of cubes.

Fortunately, we can distinguish those vertically aligned cubes by using the function

$$u_s\colon F_{-s}\mathcal{C}_n(j) \longrightarrow [0,1],$$

thanks to Lemma 2.7. Now the following gives us a homotopy we want

$$G(\mathbf{c},t) = H(\mathbf{c}, t(u_s(\mathbf{c})\sigma_s(\mathbf{c}) + (1-u_s(\mathbf{c}))\sigma_{s+1}(\mathbf{c}))).$$

Note that $$G(\mathbf{c},1) \in D_n^s(j)$$

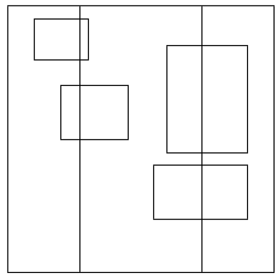
since $\sigma_s(\mathbf{c}) \le \sigma_{s+1}(\mathbf{c})$. Thus we have a Σ_j–equivariant homotopy equivalence

$$\mathcal{D}_n^q(j)/\mathcal{D}_n^{q+1}(j) \simeq F_{-q}\mathcal{C}_n(j)/F_{-q-1}\mathcal{C}_n(j).$$

Horizontally decomposable cubes decompose. Thus it is enough to prove the following homotopy equivalence:

$$\left(\mathcal{D}_n^1(j)/\mathcal{D}_n^2(j) \wedge_{\Sigma_j} X^{\wedge j}\right) \wedge S^1 \simeq D_j^{(n-1)}(\Sigma X).$$

Note that $\mathcal{D}_n^1(j)$ contains cubes that are not necessary for analyzing the filtration quotients. Namely, we don't need those cubes with $\operatorname{Im} c'_1 \cap \cdots \cap \operatorname{Im} c'_j = \varnothing$, where c'_i is the first coordinate of the cube c_i, as with the four cubes in the picture below.



In order to be more efficient, let us introduce yet another filtration.

Definition 3.3 Define $G_{-s}\mathcal{C}_n(j)$ to be the subset of $\mathcal{C}_n(j)$ consisting of cubes $(c_1, \cdots, c_j)$ which cannot be decomposed into $(s-1)$ collections of cubes each of which can be skewered by a vertical line (hyperplane) intersecting with each interior.

Then we have the homotopy equivalence

$$G_{-1}\mathcal{C}_n(j)/G_{-2}\mathcal{C}_n(j) \simeq \mathcal{D}_n^1(j)/\mathcal{D}_n^2(j),$$

and the scanning map

$$\mathrm{scan}_1\colon \left(G_{-1}\mathcal{C}_n(j)/G_{-2}\mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j}\right) \wedge \left(\Delta^1/\partial\Delta^1\right) \longrightarrow D_j^{(n-1)}(\Sigma X)$$

given by taking the intersection with the vertical hyperplane with the first coordinate $t \in \Delta^1$ is surjective. However, it is not easy to find a homotopy inverse to this map. We use the following auxiliary space instead.

Definition 3.4 Let $\mathcal{C}_n^\varepsilon(j)$ be the subset of $\mathcal{C}_n(j)$ consisting of cubes whose first coordinates have radius ε.

For those cubes in $G_{-1}\mathcal{C}_n(j) - G_{-2}\mathcal{C}_n(j)$, we can deform the radii in the horizontal direction freely and we have a homotopy equivalence

$$G_{-1}\mathcal{C}_n(j)/G_{-2}\mathcal{C}_n(j) \simeq G_{-1}\mathcal{C}_n^\varepsilon(j)/G_{-2}\mathcal{C}_n^\varepsilon(j)$$

for ε small enough.

Since the cubes in $\mathcal{C}_n^\varepsilon(j)$ are determined by their centers, we have the following homeomorphism

$$G_{-1}\mathcal{C}_n^\varepsilon(j)/G_{-2}\mathcal{C}_n^\varepsilon(j) \cong P_j^\varepsilon/dP_j^\varepsilon \wedge \mathcal{C}_{n-1}(j)_+,$$

where P_j^ε is the convex polytope in $[-1,1]^j$ given by

$$P_j^\varepsilon = \left\{(b_1, \cdots, b_j) \in [-1,1]^j \,\middle|\, |b_i - b_k| \le 2\varepsilon \text{ for any } i, k\right\}$$

and dP_j^ε is given by

$$dP_j^\varepsilon = \left\{(b_1, \cdots, b_j) \in [-1,1]^j \,\middle|\, |b_i - b_k| = 2\varepsilon \text{ for some } i,k \right\}.$$

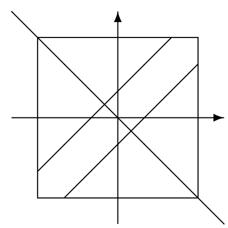
By projecting onto the hyperplane

$$V = \left\{(b_1, \cdots, b_j) \in [-1,1]^j \,\middle|\, \sum b_k = 0\right\}$$

we obtain a homotopy equivalence

$$P_j^\varepsilon / dP_j^\varepsilon \simeq P_j^\varepsilon \cap V / dP_j^\varepsilon \cap V.$$

The picture below illustrates the case $j = 2$.



$P_j^\varepsilon \cap V$ is a $(j-1)$–dimensional convex polytope (dual of permutohedron) and $dP_j^\varepsilon \cap V$ is its boundary. The decomposition of the permutation representation

$$\mathbb{R}^j \cong \left\{(b_1, \cdots, b_j) \in \mathbb{R}^j \,\middle|\, \sum b_k = 0\right\} \oplus \{(t, \cdots, t) \mid t \in \mathbb{R}\}$$

gives a Σ_j–equivariant homotopy equivalence

$$(P_j^\varepsilon \cap V / dP_j^\varepsilon \cap V) \wedge (\mathbb{R} \cup \{\infty\}) \simeq (S^1)^{\wedge j}$$

And we obtain a homotopy equivalence

$$\begin{aligned} \left(G_{-1}\mathcal{C}_n(j)/G_{-2}\mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j}\right) \wedge S^1 &\simeq (P_j^\varepsilon / dP_j^\varepsilon \wedge \mathcal{C}_{n-1}(j)_+ \wedge_{\Sigma_j} X^{\wedge j}) \wedge S^1 \\ &\simeq \mathcal{C}_{n-1}(j)_+ \wedge_{\Sigma_j} (S^1 \wedge X)^{\wedge j} \\ &= D_j^{(n-1)}(\Sigma X). \end{aligned}$$

This completes the proof of the identification of the E^1–term of the gravity filtration with the desired tensor algebra.

Let us consider d^1 next. What we have proved so far is the following fact.

$$\begin{aligned}
&\Sigma^s F_{-s}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}/F_{-s-1}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}\\
&= \Sigma^s F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j}\\
&\simeq \left(\bigvee_{S_1\amalg\cdots\amalg S_s=\{1,\cdots,j\}} \mathcal{C}_{n-1}(S_1)_+ \wedge S^{S_1} \wedge\cdots\wedge \mathcal{C}_{n-1}(S_s)_+ \wedge S^{S_s} \wedge X^{\wedge j}\right)\Big/\Sigma_j\\
&\simeq \left(\bigvee_{S_1\amalg\cdots\amalg S_s=\{1,\cdots,j\}} \mathcal{C}_{n-1}(S_1)_+ \wedge (\Sigma X)^{\wedge S_1} \wedge\cdots\wedge \mathcal{C}_{n-1}(S_s)_+ \wedge (\Sigma X)^{\wedge S_s}\right)\Big/\Sigma_j
\end{aligned}$$

where the wedge in the first homotopy equivalence runs over all decomposition of the set $\{1,\cdots,j\}$ into a disjoint of nonempty s subsets

$$S_1 \amalg\cdots\amalg S_s = \{1,\cdots,j\},$$

and for $S\subset\{1,\cdots,j\}$, we abuse the notation to denote $Y^{\wedge|S|}$ together with the action of the symmetric group by $Y^{\wedge S}$.

Under the identification by the action of Σ_j, we obtain the wedge over all $j_1+\cdots+j_s = j$ and

$$\begin{aligned}
&\Sigma^s F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j}\\
&\qquad\simeq \bigvee_{j_1+\cdots+j_s=j} \mathcal{C}_{n-1}(j_1)_+ \wedge_{\Sigma_{j_1}} (\Sigma X)^{\wedge j_1} \wedge\cdots\wedge \mathcal{C}_{n-1}(j_s)_+ \wedge_{\Sigma_{j_s}} (\Sigma X)^{\wedge j_s}.
\end{aligned}$$

However in order to compute d^1, we should compute the map

$$\Sigma^s F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) \wedge X^{\wedge j} \longrightarrow \Sigma^{s+1} F_{-s-1}\mathcal{C}_n(j)/F_{-s-2}\mathcal{C}_n(j) \wedge X^{\wedge j}$$

before we take the quotient by the action of Σ_j. This "connecting homomorphism" is given by the composition

$$\begin{aligned}
(3)\quad F_{-s}\mathcal{C}_n(j)/F_{-s-1}\mathcal{C}_n(j) &\simeq F_{-s}\mathcal{C}_n(j) \cup CF_{-s-1}\mathcal{C}_n(j)\\
&\longrightarrow \Sigma F_{-s-1}\mathcal{C}_n(j) \longrightarrow \Sigma F_{-s-1}\mathcal{C}_n(j)/F_{-s-2}\mathcal{C}_n(j).
\end{aligned}$$

The first homotopy equivalence in the above sequence of maps is obtained from an NDR representation for the pair $(F_{-s}\mathcal{C}_n(j), F_{-s-1}\mathcal{C}_n(j))$. More precisely, for an NDR pair (X,A) with NDR representation (h,u) satisfying

$$h(x,s)\in A \quad \text{for } u(x)<s,$$

the map
$$\widetilde{h}\colon X/A \xrightarrow{\simeq} X\cup CA$$

defined by
$$\widetilde{h}([x]) = \begin{cases} h(x,1) & u(x)=1 \\ (h(x,1),u(x)) & u(x)<1 \end{cases}$$

is a homotopy equivalence (see Strøm [11; 12]). It is straightforward to check that the NDR representation $(h_s, \widetilde{u}_s)$ for $(F_{-s}\mathcal{C}_n(j), F_{-s-1}\mathcal{C}_n(j))$ satisfies the above Strøm condition.

Recall that we have proved that $F_{-s}\mathcal{C}_n(j)$ can be replaced with the space of little cubes whose first coordinates have a fixed small radius ε, $G_{-s}\mathcal{C}_n^{\varepsilon}(j)$. In the case of $s=1$, the map

$$G_{-1}\mathcal{C}_n^{\varepsilon}(j)/G_{-2}\mathcal{C}_n^{\varepsilon}(j) \longrightarrow \Sigma G_{-2}\mathcal{C}_n^{\varepsilon}(j)/G_{-3}\mathcal{C}_n^{\varepsilon}(j) \tag{4}$$

is given by shrinking the radii of the first coordinate. And the identification

$$\Sigma^2 G_{-2}\mathcal{C}_n^{\varepsilon}(j)/G_{-3}\mathcal{C}_n^{\varepsilon}(j) \simeq \bigvee_{S_1 \amalg S_2 = \{1,\cdots,j\}} \mathcal{C}_{n-1}(S_1)_+ \wedge S^{S^1} \wedge \mathcal{C}_{n-1}(S_2)_+ \wedge S^{S_2}$$

is given by measuring the distance of the centers of the first coordinates. Namely the component to which the image of an element $(c_1,\cdots,c_j) \in G_{-1}\mathcal{C}_n^{\varepsilon}(j)$ belongs under the above identification is determined by measuring the difference of the centers of the first coordinates. Under the identification

$$\Sigma G_{-1}\mathcal{C}_n^{\varepsilon}(j)/G_{-2}\mathcal{C}_n^{\varepsilon}(j) \simeq \mathcal{C}_{n-1}(j)_+ \wedge S^j$$

the map (4) can be identified with

$$\begin{aligned} \mathcal{C}_{n-1}(j)_+ \wedge S^j &\longrightarrow \mathcal{C}_{n-1}(j)_+ \wedge (S^1 \vee S^1)^{\wedge j} \\ &= \bigvee_{S_1 \amalg S_2 = \{1,\cdots,j\}} \mathcal{C}_{n-1}(S_1)_+ \wedge S^{S^1} \wedge \mathcal{C}_{n-1}(S_2)_+ \wedge S^{S_2}. \end{aligned}$$

The suspension coordinate in $\Sigma G_{-1}\mathcal{C}_n^{\varepsilon}(j)/G_{-2}\mathcal{C}_n^{\varepsilon}(j)$ determines the position in the first coordinate with which $(c_1,\cdots,c_j)$ is cut into two collections. Thus d^1 is given by taking all possible decompositions $\{1,\cdots,j\} = S_1 \amalg S_2$ and summing it up, before we divide by the action of Σ_j.

Therefore we see that (3) is given by taking all possible decomposition of indexing sets

$$\{1,\cdots,j\} = S_1 \amalg \cdots \amalg S_s$$

under the horizontal decomposition above.

On the other hand, it is well-known that, under the Snaith splitting, the coproduct on $C_{n-1}(\Sigma X)$ is given by

$$\begin{aligned}D_j^{(n-1)}(\Sigma X) &\longrightarrow D_j^{(n-1)}(\Sigma X \vee \Sigma X)\\ &= \bigvee_{j_1+j_2=j} \mathcal{C}_{n-1}(j)_+ \wedge_{\Sigma_{j_1}\times\Sigma_{j_2}} (\Sigma X)^{\wedge j_1} \wedge (\Sigma X)^{\wedge j_2}\\ &\longrightarrow \bigvee_{j_1+j_2=j} D_{j_1}^{(n-1)}(\Sigma X) \wedge D_{j_2}^{(n-1)}(\Sigma X).\end{aligned}$$

Thus the map induced by the composition (3) in homology coincides with the cobar differential.

Consider the case $n=2$. Recall that $\mathcal{C}_1(j) \simeq \Sigma_j$, Σ_j–equivariantly, and we have

$$D_j^{(1)}(\Sigma X) = \mathcal{C}_1(j)_+ \wedge_{\Sigma_j} (\Sigma X)^{\wedge j} \simeq (\Sigma X)^{\wedge j}.$$

Thus the d^1 differential in the case of $\Omega^2\Sigma^2 X$ is given by the map described in (1).

4 Constructions by Rector and Smith

In the previous section, we have seen that the E^1–term of the gravity spectral sequence is isomorphic to the E^1–term of the classical cobar-type Eilenberg–Moore spectral sequence. In order to finish the proof of Theorem 1.2, we need to compare the E^r–terms for $r \geq 2$.

A couple of ways are known to construct a spectral sequence for a diagram

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

with E^2–term $\qquad E^2 \cong \mathrm{Cotor}^{h_*(B)}(h_*(X), h_*(Y))$

for reasonably good homology theory $h_*(-)$. Our gravity spectral sequence is one of them. Rector's construction [7] is a generalization of the classical Eilenberg–Moore spectral sequence [4; 5]. A construction due to Larry Smith [8; 9] gives us a general framework for this kind of construction. We prove the remaining part of Theorem 1.2 by comparing the gravity spectral sequence with Rector's construction of the Eilenberg–Moore spectral sequence with an aid of Larry Smith's construction.

Let us briefly recall Rector's construction in [7]. Given a diagram

$$Y \xrightarrow{f} B \xleftarrow{p} X$$

we can form a cosimplicial space

$$\Omega^0(Y,B,X) \underset{\xleftarrow{\sigma^1}}{\overset{\xrightarrow{\delta^0}}{\xrightarrow{\delta^1}}} \Omega^1(Y,B,X) \underset{\substack{\xleftarrow{\sigma^1} \\ \xleftarrow{\sigma^2}}}{\overset{\substack{\xrightarrow{\delta^0} \\ \xrightarrow{\delta^1}}}{\xrightarrow{\delta^2}}} \Omega^2(Y,B,X)\cdots$$

where $$\Omega^n(Y,B,X) = Y \times B^n \times X$$

and the maps are defined by

$$\delta^i(y,b_1,\cdots,b_n,x) = \begin{cases} (y,f(y),b_1,\cdots,b_n,x) & \text{if } i=0 \\ (y,b_1,\cdots,b_i,b_i,\cdots,b_n,x) & \text{if } 1 \le i \le n \\ (y,b_1,\cdots,b_n,p(x),x) & \text{if } i=n \end{cases}$$

$$\sigma^i(y,b_1,\cdots,b_n,x) = (y,b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,x).$$

Rector defined a sequence of pointed cofibrations

$$\begin{aligned} \Omega_0 &\xrightarrow{\varphi_{-1}} \overline{\Omega}_{-1} \longrightarrow \Omega_{-1} \\ \Omega_{-1} &\xrightarrow{\varphi_{-2}} \overline{\Omega}_{-2} \longrightarrow \Omega_{-2} \\ &\vdots \\ \Omega_{-n+1} &\xrightarrow{\varphi_{-n}} \overline{\Omega}_{-n} \longrightarrow \Omega_{-n} \\ &\vdots \end{aligned} \tag{5}$$

directly from the cosimplicial cobar construction $\Omega^*(Y,B,X)$ as follows.

First define $$\overline{\Omega}_{-n} = \Omega^n(Y,B,X)/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^n.$$

The map δ^0 induces a well-defined map

$$\psi_{-n}\colon \overline{\Omega}_{-n+1} \longrightarrow \overline{\Omega}_{-n}.$$

It is easy to check that $\psi_{-n-1}\psi_{-n} = *$. Ω_{-n}'s are defined inductively on n. Define

$$\Omega_0 = \Omega^0(Y,B,X) = Y \times X.$$

Note that $\overline{\Omega}_0 = \Omega_0/\varnothing$. Let φ_{-1} be the composition

$$\Omega_0 \longrightarrow \Omega_0/\varnothing = \overline{\Omega}_0 \xrightarrow{\psi_{-1}} \overline{\Omega}_{-1}.$$

Let Ω_{-1} be the (reduced) mapping cone of φ_{-1}. Since $\psi_{-2}\psi_{-1} = *$, ψ_{-2} induces a well-defined map

$$\varphi_{-2}\colon \Omega_{-1} \longrightarrow \overline{\Omega}_{-2}.$$

More explicitly φ_{-2} is given by

$$\varphi_{-2}(x) = \begin{cases} \psi_{-2}(x) & \text{if } x \in \overline{\Omega}_{-1} \\ * & \text{otherwise.} \end{cases}$$

From this description, it is easy to show $\psi_{-3}\varphi_{-2} = *$.

Inductively we obtain a map

$$\varphi_{-n}\colon \Omega_{-n+1} \longrightarrow \overline{\Omega}_{-n}$$

with $\psi_{-n-1}\varphi_{-n} = *$. Define Ω_{-n} to be the mapping cone of φ_{-n}.

For any homology theory $h_*(-)$, the sequence of cofibrations (5) induces an exact couple

$$\begin{aligned} {}^R D^1_{-p,q} &= \widetilde{h}_q(\Omega_{-p}) \\ {}^R E^1_{-p,q} &= \widetilde{h}_q(\overline{\Omega}_{-p}) \end{aligned}$$

Rector proved the following theorem in [7].

Theorem 4.1 *When $h_*(-)$ is the singular homology theory, the spectral sequence associated with the above exact couple is naturally isomorphic to the original Eilenberg–Moore spectral sequence constructed in [4; 5].*

On the other hand, in [8; 9], Larry Smith introduced another construction. The first step in Smith's approach is to consider a fibration $X \xrightarrow{p} B$ as an object in the category of spaces over B.

Definition 4.2 Let B be an arbitrary space. A space over B is a continuous map $f\colon X \longrightarrow B$. A pointed space over B is a pair of maps

$$\begin{aligned} f\colon X &\longrightarrow B \\ s\colon B &\longrightarrow X \end{aligned}$$

with $f \circ s = \mathrm{id}_B$. For simplicity, we denote this object by (f, s). Morphisms between these objects are obviously defined.

If $f\colon X \longrightarrow B$ is a space over B, we sometimes denote the "total space" X by $T(f)$. Similarly, for (f,s) a pointed space over B, we use the notation $T(f) = X$ if $f\colon X \longrightarrow B$.

The category of spaces over B is denoted by **Spaces**$/B$. The category of pointed spaces over B is denoted by $(\textbf{Spaces}/B)_*$.

When $B = *$, we simply denote **Spaces** and **Spaces**$_*$ for **Spaces**$/*$ and $(\textbf{Spaces}/*)_*$, respectively. They are the usual categories of spaces and pointed spaces, respectively.

Almost all important constructions and notions in **Spaces** or in **Spaces**$_*$ have analogies in **Spaces**$/B$ or in $(\textbf{Spaces}/B)_*$. To be self-contained, we record some of them used in the following section.

Definition 4.3 For morphisms $\varphi_0, \varphi_1\colon (f,s) \longrightarrow (g,t)$, a homotopy from φ_0 to φ_1 is a map

$$\varphi\colon T(f) \times I \longrightarrow T(g)$$

which fits into the following commutative diagrams, where $r \in [0,1]$.

$$\begin{tikzcd}
T(f)\times I \arrow[r, "\varphi"] \arrow[d] & T(g) \arrow[dd, "g"] \\
T(f) \arrow[d, "f"] & \\
B \arrow[r, "="] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
T(f)\times I \arrow[r, "\varphi"] & T(g) \\
T(f)\times\{r\} \arrow[u] & \\
B \arrow[u, "s"] \arrow[r, "="] & B \arrow[uu, "t"']
\end{tikzcd}$$

$$\begin{tikzcd}
T(f)\times\{0\} \arrow[r] \arrow[d, "\varphi_0"] & T(f)\times I \arrow[d, "\varphi"] & T(f)\times\{1\} \arrow[l] \arrow[d, "\varphi_1"] \\
T(g) \arrow[r, "="] & T(g) & T(g) \arrow[l, "="']
\end{tikzcd}$$

Definition 4.4 A morphism $\varphi\colon (f,s) \longrightarrow (g,t)$ in $(\textbf{Spaces}/B)_*$ is called a cofibration in $(\textbf{Spaces}/B)_*$ if it has the homotopy extension property with respect to the homotopy defined above.

Definition 4.5 For a morphism $\varphi\colon (f,s) \longrightarrow (g,t)$ in $(\mathbf{Spaces}/B)_*$, the (reduced) mapping cone of φ, denoted by $(C(\varphi), S_C(\varphi))$, is defined as follows.

$$T(C(\varphi)) = T(f)\times I \amalg T(g) \Bigg/ \begin{array}{ll} (x,0)\sim\varphi(x) & \text{for } x\in T(f) \\ (s(b),r)\sim t(b) & \text{for } b\in B, r\in I \\ (x,1)\sim(x',1) & \text{if } f(x)=f(x') \text{ for } x,x'\in T(f) \end{array}$$

The projection $C(\varphi)\colon T(C(\varphi)) \longrightarrow B$ is given by f on $T(f)\times I$ and by g on $T(g)$. The section $S_C(\varphi)\colon B \longrightarrow T(C(\varphi))$ is defined either by s or t which agree in $T(C(\varphi))$.

Definition 4.6 For an object $(f,s)\in(\mathbf{Spaces}/B)_*$ the (reduced) suspension of $(f,s)\in(\mathbf{Spaces}/B)_*$, denoted by $S(f,s)=(S(f),S(s))$, is defined by

$$T(S(f)) = T(f)\times I \Bigg/ \begin{array}{ll} (x,0)\sim(x',0) & \text{if } f(x)=f(x') \text{ for } x,x'\in T(f) \\ (x,1)\sim(x',1) & \text{if } f(x)=f(x') \text{ for } x,x'\in T(f) \\ (s(b),r)\sim(s(b),r') & \text{for } b\in B \text{ and } r,r'\in I \end{array}$$

and
$$S(f)(x,t) = f(x)$$
$$S(s)(b) = (s(b),0).$$

Definition 4.7 For objects (f,s) and (g,t) in $(\mathbf{Spaces}/B)_*$, the smash product of (f,s) and (g,t), denoted by $(f,s)\wedge(g,t) = (f\wedge_B g, s\wedge_B t)$, is defined by

$$T(f\wedge_B g) = T(f)\times_B T(g)/(x,t(b))\sim(s(b),y)$$
$$\text{if } f(x)=g(y)=b \text{ for } x\in T(f), y\in T(g), b\in B$$

and
$$(f\wedge_B g)(x,y) = f(x) = g(y)$$
$$(s\wedge_B t)(b) = (s(b),t(b))$$

Lemma 4.8 *If $(f,s)\xrightarrow{\varphi}(g,t)$ is a cofibration in $(\mathbf{Spaces}/B)_*$, then so is*

$$(h,u)\wedge(f,s)\xrightarrow{\mathrm{id}\wedge\varphi}(h,u)\wedge(g,t)$$

with cofiber $(h,u)\wedge(C(\varphi),S_C(\varphi))$.

We fix notations for forgetful functors and their adjoints among the above categories.

Definition 4.9 Consider the functor given by forgetting sections

$$F\colon (\mathbf{Spaces}/B)_* \longrightarrow \mathbf{Spaces}/B.$$

Its adjoint is denoted by

$$G\colon \textbf{Spaces}\,/B \longrightarrow (\textbf{Spaces}\,/B)_*$$

which is given, on total spaces, by

$$T(G(f)) = T(f) \amalg B.$$

The section is defined to be the identity map into the second component B.

Let
$$\Gamma\colon \textbf{Spaces}_* \longrightarrow (\textbf{Spaces}\,/B)_*$$

be the functor defined on the total spaces, by

$$T(\Gamma(X)) = X \times B.$$

The section is defined by the composition

$$B = \{*\} \times B \hookrightarrow X \times B$$

where $*$ is the base point of X. Its adjoint is the functor

$$\Phi\colon (\textbf{Spaces}\,/B)_* \longrightarrow \textbf{Spaces}_*$$

defined, for an object (f,s), by

$$\Phi(f,s) = T(f)/s(B).$$

Definition 4.10 For any nonnegative integer n, we denote $(S^n_B, s^n_B) = \Gamma(S^n)$.

With these functors, we can describe the definition of homotopy in $(\textbf{Spaces}\,/B)_*$ more compactly.

Lemma 4.11 *For morphisms $\varphi_0, \varphi_1\colon (f,s) \longrightarrow (g,t)$, a homotopy from φ_0 to φ_1 is a morphism*

$$\varphi\colon (f,s) \wedge_B \Gamma(I_+) \longrightarrow (g,t)$$

with the following commutative diagram

$$\begin{tikzcd}
(f,s) \wedge_B \Gamma(\{0\}_+) \arrow[r] \arrow[d, "\|"] & (f,s) \wedge_B \Gamma(I_+) \arrow[d, "\varphi"] & (f,s) \wedge_B \Gamma(\{1\}_+) \arrow[l] \arrow[d, "\|"] \\
(f,s) \arrow[r, "\varphi_0"] & (g,t) & (f,s) \arrow[l, "\varphi_1"']
\end{tikzcd}$$

The functor Φ has a very good property.

Lemma 4.12 *For any cofibration*

$$(f_0, s_0) \longrightarrow (f_1, s_1) \longrightarrow (f_2, s_2)$$

in $(\mathbf{Spaces}\,/B)_*$,

$$\Phi(f_0, s_0) \longrightarrow \Phi(f_1, s_1) \longrightarrow \Phi(f_2, s_2)$$

is a cofibration in $\mathbf{Spaces}_*$.

Thanks to this lemma, any (reduced) homology theory on $\mathbf{Spaces}_*$ naturally extends to $(\mathbf{Spaces}\,/B)_*$.

Definition 4.13 Let $\widetilde{h}_*(-)$ be any homology theory on $\mathbf{Spaces}_*$ and $(f,s)\colon X \longrightarrow B$ any object in $(\mathbf{Spaces}\,/B)_*$. Define

$$h_*^B(f,s) = \widetilde{h}_* \circ \Phi(f,s) = \widetilde{h}_*(X/s(B)).$$

Thus we have a covariant functor

$$h_*^B(-)\colon (\mathbf{Spaces}\,/B)_* \longrightarrow \mathbf{Graded\ Abelian\ Groups},$$

where **Graded Abelian Groups** denotes the category of graded Abelian groups. The functor $h_*^B(-)$ is referred to as the homology theory on $(\mathbf{Spaces}\,/B)_*$ associated with $\widetilde{h}_*(-)$.

Corollary 4.14 *For any cofibration in* $(\mathbf{Spaces}\,/B)_*$

$$(f_0, s_0) \longrightarrow (f_1, s_1) \longrightarrow (f_2, s_2)$$

and a homology theory $\widetilde{h}_*(-)$ *on* $\mathbf{Spaces}_*$, *we have a long exact sequence:*

$$\cdots \longrightarrow h_*^B(f_0, s_0) \longrightarrow h_*^B(f_1, s_1) \longrightarrow h_*^B(f_2, s_2) \xrightarrow{\partial} h_{*-1}^B(f_0, s_0) \longrightarrow \cdots$$

Now we are ready to recall the construction of a cobar-type Eilenberg–Moore spectral sequence by Larry Smith. His idea is to construct a spectral sequence out of a "filtration" in the category $(\mathbf{Spaces}\,/B)_*$, ie display. Smith made a lot of assumptions on (co)homology theory and the space B in his paper [9]. Most of his assumptions are for the existence of a display and the convergence of the spectral sequence. Since our purpose is to show Rector's construction and the gravity filtration on $\Omega^n\Sigma^n X$ give rise to displays, we do not need these assumptions. What we really need is the following.

Assumption 4.15 Throughout the rest of this section, $\widetilde{h}_*(-)$ denotes a (reduced) multiplicative homology theory. We also assume that the external product

$$\widetilde{h}_*(B_+) \otimes_{h_*} \widetilde{h}_*(X) \longrightarrow \widetilde{h}_*(B_+ \wedge X)$$

is an isomorphism for any pointed space X. This condition is satisfied, for example, if $\widetilde{h}_*(B_+)$ is h_*–flat.

This condition is necessary for the following definition.

Definition 4.16 Given any pointed space (f,s) over B, define a structure of left $h_*^B(S_B^0, s_B^0)$–comodule on $h_*^B(f,s)$ by the composition

$$\begin{aligned} h_*^B(f,s) \cong \widetilde{h}_*(\Phi(f,s)) \xrightarrow{\Delta_*} \widetilde{h}_*(\Phi(f,s)\wedge\Phi(f,s)) \xrightarrow{\Phi(*)\wedge \mathrm{id}} & \widetilde{h}_*(\Phi(S_B^0,s_B^0)\wedge\Phi(f,s)) \\ = & \widetilde{h}_*(B_+\wedge\Phi(f,s)) \\ \cong & \widetilde{h}_*(B_+)\otimes_{h_*}\widetilde{h}_*(\Phi(f,s)) \\ = & h_*^B(S_B^0,s_B^0)\otimes_{h_*} h_*^B(f,s). \end{aligned}$$

Similarly $h_*^B(f,s)$ also has a structure of right $h_*^B(S_B^0,s_B^0)$–comodule.

Definition 4.17 Let (f,s) be an object in $(\mathbf{Spaces}/B)_*$. An h_*^B–display of (f,s) is a sequence of cofibrations

$$\begin{array}{ccccc} (f,s) & \xrightarrow{\alpha_0} & (h_0,u_0) & \xrightarrow{\beta_{-1}} & (f_{-1},s_{-1}) \\ (f_{-1},s_{-1}) & \xrightarrow{\alpha_{-1}} & (h_{-1},u_{-1}) & \xrightarrow{\beta_{-2}} & (f_{-2},s_{-2}) \\ & & \vdots & & \\ (f_{-i},s_{-i}) & \xrightarrow{\alpha_{-i}} & (h_{-i},u_{-i}) & \xrightarrow{\beta_{-i-1}} & (f_{-i-1},s_{-i-1}) \\ & & \vdots & & \end{array}$$

satisfying the following two conditions.

(1) $h_*^B(h_{-i},u_{-i})$ is a flat h_*–module and an injective $h_*^B(S_B^0,s_B^0)\cong h_*(B_+)$–comodule for each i.

(2) $\alpha_{-i\,*}\colon h_*^B(f_{-i},s_{-i}) \longrightarrow h_*^B(h_{-i},u_{-i})$ is a monomorphism.

Suppose $\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}$ is an h_*^B–display of (f,s). Let (g,t) be another pointed space over B. By Lemma 4.8, smashing with (g,t) preserves cofibrations and we have cofibrations:

$$\begin{array}{ccccc} (g,t)\wedge(f,s) & \xrightarrow{\mathrm{id}\wedge\alpha_0} & (g,t)\wedge(h_0,u_0) & \xrightarrow{\mathrm{id}\wedge\beta_{-1}} & (g,t)\wedge(f_{-1},s_{-1}) \\ (g,t)\wedge(f_{-1},s_{-1}) & \xrightarrow{\mathrm{id}\wedge\alpha_{-1}} & (g,t)\wedge(h_{-1},u_{-1}) & \xrightarrow{\mathrm{id}\wedge\beta_{-2}} & (g,t)\wedge(f_{-2},s_{-2}) \\ & & \vdots & & \\ (g,t)\wedge(f_{-i},s_{-i}) & \xrightarrow{\mathrm{id}\wedge\alpha_{-i}} & (g,t)\wedge(h_{-i},u_{-i}) & \xrightarrow{\mathrm{id}\wedge\beta_{-i-1}} & (g,t)\wedge(f_{-i-1},s_{-i-1}) \\ & & \vdots & & \end{array}$$

Thus we obtain an exact couple by applying $h^B_*(-)$.

Definition 4.18 Define

$$D^1_{-p,q}((f,s),(g,t)) = h^B_{-p}((f_{-p},s_{-p}) \wedge (g,t))$$
$$E^1_{-p,q}((f,s),(g,t)) = h^B_{-p}((h_{-p},u_{-p}) \wedge (g,t)).$$

The cobar spectral sequence or the Künneth spectral sequence defined by an h^B_*–display

$$\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}$$

is the spectral sequence denoted $\{E^r_{*,*}((f,s),(g,t))\}$ associated with this exact couple.

In order to identify the E^2–term, it is important to use a special kind of display. To see this, let $(f,s),(g,t) \in (\mathbf{Spaces}/B)_*$ and

$$\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}$$

be an h^B_*–display. Since $h^B_*(h_{-i},u_{-i})$ is flat over h_*,

$$\begin{aligned}\widetilde{h}_*(\Phi(h_{-i},u_{-i}) \wedge \Phi(g,t)) &\cong \widetilde{h}_*(\Phi(h_{-i},u_{-i})) \otimes_{h_*} \widetilde{h}_*(\Phi(g,t)) \\ &= h^B_*(h_{-i},u_{-i}) \otimes_{h_*} h^B_*(g,t).\end{aligned}$$

Note that the following composition is trivial:

$$\begin{aligned} h^B_*((h_{-i},u_{-i}) \wedge (g,t)) \quad &\longrightarrow \quad \widetilde{h}_*(\Phi(h_{-i},u_{-i}) \wedge \Phi(g,t)) \\ &\cong \quad h^B_*(h_{-i},u_{-i}) \otimes_{h_*} h^B_*(g,t) \\ &\xrightarrow{\psi\otimes\mathrm{id}-\mathrm{id}\otimes\varphi} h^B_*(h_{-i},u_{-i}) \otimes_{h_*} h^B_*(S^0_B,s^0_B) \otimes_{h_*} h^B_*(g,t).\end{aligned}$$

where
$$\begin{aligned}\Delta_*\colon h^B_*(S^0_B,s^0_B) &\longrightarrow h^B_*(S^0_B,s^0_B) \otimes_{h_*} h^B_*(S^0_B,s^0_B) \\ \psi\colon h^B_*(h_{-i},u_{-i}) &\longrightarrow h^B_*(h_{-i},u_{-i}) \otimes_{h_*} h^B_*(S^0_B,s^0_B) \\ \varphi\colon h^B_*(g,t) &\longrightarrow h^B_*(S^0_B,s^0_B) \otimes_{h_*} h^B_*(g,t)\end{aligned}$$

are coalgebra and comodule structure maps, respectively. Thus we obtain a map

$$h^B_*((h_{-i},u_{-i}) \wedge (g,t)) \longrightarrow \mathrm{Ker}(\psi \otimes \mathrm{id} - \mathrm{id} \otimes \varphi).$$

By the definition of cotensor product

$$\mathrm{Ker}(\psi \otimes \mathrm{id} - \mathrm{id} \otimes \varphi) = h^B_*(h_{-i},u_{-i}) \Box_{h^B_*(S^0_B,s^0_B)} h^B_*(g,t).$$

It is convenient to assume the resulting map

$$\Psi\colon h^B_*((h_{-i},u_{-i}) \wedge (g,t)) \longrightarrow h^B_*(h_{-i},u_{-i}) \Box_{h^B_*(S^0_B,s^0_B)} h^B_*(g,t)$$

is an isomorphism.

Definition 4.19 Let (f,s) and (g,t) be pointed spaces over B. An h_*^B–display of (f,s)

$$\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}$$

is said to be injective with respect to (g,t) if

$$\Psi\colon h_*^B((h_{-i},u_{-i})\wedge(g,t)) \longrightarrow h_*^B(h_{-i},u_{-i})\Box_{h_*^B(S_B^0,s_B^0)}h_*^B(g,t)$$

is an isomorphism.

The following lemma is immediate from the definition.

Lemma 4.20 *If $\{E^r_{*,*}((f,s)\wedge(g,t)),d^r\}$ is a cobar spectral sequence defined by an injective h_*^B–display of (f,s) with respect to (g,t), then we have*

$$\begin{aligned}E^2_{*,*}((f,s)\wedge(g,t)) &\cong \mathrm{Cotor}^{h_*^B(S_B^0,s_B^0)}(h_*^B(f,s),h_*^B(g,t))\\ &= \mathrm{Cotor}^{\widetilde{h}_*(B+)}(\widetilde{h}_*(\Phi(f,s)),\widetilde{h}_*(\Phi(g,t))).\end{aligned}$$

This lemma suggests that cobar spectral sequences defined by using injective displays are isomorphic to each other from the E^2–term on. In fact, this is the case.

Theorem 4.21 *Let (f,s) and (g,t) be pointed spaces over B. Let*

$$\begin{gathered}\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}\\ \{(f'_{-i},s'_{-i}),(h'_{-i},u'_{-i})\}\end{gathered}$$

be injective h_*^B–displays for (f,s) with respect to (g,t). Let $\{E^r\}$ and $\{'E^r\}$ be the cobar spectral sequences defined by the first and the second display, respectively. Then we have an isomorphism of spectral sequences for $r\geq 2$,

$$E^r \cong {}'E^r.$$

Proof See pp 119–120 of [9]. Smith proved this fact by finding an intermediate display

$$\{(\overline{f}_{-i},\overline{s}_{-i}),(\overline{h}_{-i},\overline{u}_{-i})\}$$

and maps

$$\{(f_{-i},s_{-i}),(h_{-i},u_{-i})\}\longrightarrow\{(\overline{f}_{-i},\overline{s}_{-i}),(\overline{h}_{-i},\overline{u}_{-i})\}\longleftarrow\{f'_{-i},s'_{-i}),(h'_{-i},u'_{-i})\}.$$

The existence of such a display in our case is essentially proved in pp 112–113 of the same paper. □

5 Comparing spectral sequences

Let us compare Rector's construction for

$$* \longrightarrow \Omega^{n-1}\Sigma^n X \longleftarrow P\Omega^{n-1}\Sigma^n X$$

with the construction by the gravity filtration. We proved that the E^1–terms of spectral sequences are isomorphic as chain complexes in Section 3. In order to show that these spectral sequences are isomorphic from the E^2–term, it is enough to show that they both give rise to injective displays.

We first prove the following general fact.

Theorem 5.1 *Consider the pullback diagram:*

$$\begin{tikzcd} Y \times_B X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "f"] & B \end{tikzcd}$$

If $\widetilde{h}_*(B_+)$ is h_*–flat and p is a fibration, the spectral sequence induced from Rector's geometric cobar construction for this pullback diagram is isomorphic to Smith's spectral sequence from the E^2–term on.

Let $p\colon X \longrightarrow B$ be a fibration. Using the functor

$$G\colon \mathbf{Spaces}/B \longrightarrow (\mathbf{Spaces}/B)_*$$

we obtain an object $G(p)$ in $(\mathbf{Spaces}/B)_*$. Let $f\colon Y \longrightarrow B$ be a continuous map. In the following we construct a display for $G(p)$

$$\begin{array}{ccccc} G(p) & \longrightarrow & (\omega_{-1}, u_{-1}) & \longrightarrow & (p_{-1}, s_{-1}) \\ (p_{-1}, s_{-1}) & \longrightarrow & (\omega_{-2}, u_{-2}) & \longrightarrow & (p_{-2}, s_{-2}) \\ & & \vdots & & \end{array}$$

so that the filtration on $Y \times_B X$ induced by this display is the same as Rector's cosimplicial construction.

Recall that Rector's spectral sequence is induced from the cofibration sequences

$$\begin{array}{ccccc} & & \vdots & & \\ \Omega_{-\ell+1} & \xrightarrow{\varphi_{-\ell}} & \overline{\Omega}_{-\ell} & \longrightarrow & \Omega_{-\ell} \\ & & \vdots & & \end{array}$$

while Smith's spectral sequence is induced from cofibration sequences:

$$\begin{array}{rcl}
\Phi(G(f)\wedge_B G(p)) & \longrightarrow \Phi(G(f)\wedge_B(\omega_{-1},u_{-1})) \longrightarrow & \Phi(G(f)\wedge_B(p_{-1},s_{-1})) \\
\Phi(G(f)\wedge_B(p_{-1},s_{-1})) & \longrightarrow \Phi(G(f)\wedge_B(\omega_{-2},u_{-2})) \longrightarrow & \Phi(G(f)\wedge_B(p_{-2},s_{-2})) \\
& \vdots &
\end{array}$$

But it is not difficult to find $(p_{-\ell},s_{-\ell})$ and $(\omega_{-\ell},u_{-\ell})$ with

$$\begin{aligned}
\Omega_{-\ell} &\simeq \Phi(G(f)\wedge_B(p_{-\ell},s_{-\ell})) \\
\overline{\Omega}_{-\ell} &\simeq \Phi(G(f)\wedge_B(\omega_{-\ell},u_{-\ell})).
\end{aligned}$$

It is natural to expect that the process of constructing such a display is very similar to that of the cofibration sequences of Rector's. We first define $(\omega_{-\ell},u_{-\ell})$ and then, inductively, $(p_{-\ell},s_{-\ell})$.

Definition 5.2 For $\ell\geq 0$, define

$$T(\omega_{-\ell}) = B\times(B^{\ell}\times X/\operatorname{Im}\delta^1\cup\cdots\cup\operatorname{Im}\delta^{\ell})$$

where the maps δ^i for $i=1,\cdots,\ell$ are the maps in the geometric cobar construction with $Y=*$. The map

$$\omega_{-\ell}\colon T(\omega_{-\ell})\longrightarrow B$$

is just the projection onto the first factor. The section

$$u_{-\ell}\colon B\longrightarrow T(\omega_{-\ell})$$

is defined by $u_{-\ell}(b)=(b,*)$.

As is the case of the geometric cobar construction, "δ^0" induces a map

$$\psi_{-\ell}\colon(\omega_{-\ell},u_{-\ell})\longrightarrow(\omega_{-\ell-1},u_{-\ell-1})$$

defined by $$\psi_{-\ell}(b;b_1,\cdots,b_\ell,x)=(b;b,b_1,\cdots,b_\ell,x).$$

We also have $\psi_{-\ell-1}\psi_{-\ell}=*$.

We need to check the following lemma.

Lemma 5.3 *Suppose $f\colon Y\longrightarrow B$ is surjective. Then for $\ell\geq 0$, we have a homeomorphism*

$$\overline{\Omega}_{-\ell}\cong\Phi(G(f)\wedge(\omega_{-\ell},u_{-\ell}))$$

making the following diagram commutative.

$$
\begin{tikzcd}
\overline{\Omega}_{-\ell} \arrow[r, "\cong"] \arrow[d, "\psi_{-\ell}"] & \Phi(G(f) \wedge (\omega_{-\ell}, u_{-\ell})) \arrow[d, "\Phi(1\wedge\psi_{-\ell})"] \\
\overline{\Omega}_{-\ell-1} \arrow[r, "\cong"] & \Phi(G(f) \wedge (\omega_{-\ell-1}, u_{-\ell-1}))
\end{tikzcd} \tag{6}
$$

Proof By the definition of the smash product in $\mathbf{Spaces}_* / B$.

$$
\begin{aligned}
T(G(f) \wedge (\omega_{-\ell}, u_{-\ell})) &= T(G(f)) \times_B T(\omega_{-\ell})/(y, u_{-\ell}(b)) \sim (b, x) \ \text{ if } f(y) = b) \\
&= (Y \amalg B) \times_B B \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell})/ \\
&\qquad\qquad\qquad\qquad (y, f(y), *) \sim (f(y), x) \\
&= Y \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell}) \\
&\quad \amalg B \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell})/(y, *) \sim (f(y), x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Phi(T(G(f) \wedge (\omega_{-\ell}, u_{-\ell}))) \\
&= \left. \frac{Y \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell}) \amalg B \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell})}{(y, *) \sim (f(y), x)} \right/ B \times * \\
&= Y \times (B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell})/Y \times * \\
&= Y \times B^{\ell} \times X/\operatorname{Im}\delta^1 \cup \cdots \cup \operatorname{Im}\delta^{\ell} \\
&= \overline{\Omega}_{-\ell}.
\end{aligned}
$$

With this description of the homeomorphism, it is easy to check the diagram (6) commutes. □

We can deform any continuous map $f\colon Y \longrightarrow B$ into a surjective map up homotopy. This does not change the homotopy type of the pullback by f since p is a fibration.

In the following, we always assume that $f\colon Y \longrightarrow B$ is surjective.

Corollary 5.4 *For each ℓ, if $\widetilde{h}_*(B_+)$ is h_*–flat, we have an isomorphism*

$$h_*^B(G(f) \wedge (\omega_{-\ell}, u_{-\ell})) \cong h_*^B(G(f)) \square_{h_*^B(S_B^0, s_B^0)} h_*^B(\omega_{-\ell}, u_{-\ell}).$$

Thus the sequence of cofibrations

$$\{(p_{-\ell+1}, s_{-\ell+1}) \longrightarrow (\omega_{-\ell}, u_{-\ell}) \longrightarrow (p_{-\ell}, s_{-\ell})\}_{\ell=1,2,\cdots}$$

just constructed is an injective h_*^B–display of $G(p)$ with respect to $G(f)$.

Proof Rector proved in [7] that

$$\begin{aligned}\widetilde{h}_*(\overline{\Omega}_{-\ell}) &\cong \widetilde{h}_*(Y_+)\Box_{\widetilde{h}_*(B_+)}\widetilde{h}_*(\Phi(\omega_{-\ell},u_{-\ell}))\\ &\cong h^B_*(G(f))\Box_{h^B_*(S^0_B,s^0_B)}h^B_*(\omega_{-\ell},u_{-\ell}).\end{aligned}$$

Therefore the display $\{(p_{-\ell},s_{-\ell}),(\omega_{-\ell},u_{-\ell})\}$ is an injective display. $\square$

Let us prove that the spectral sequence induced by the above display is identical to Rector's spectral sequence.

Let $\alpha_{-1}\colon G(p)\longrightarrow(\omega_{-1},u_{-1})$ be the map in $(\mathbf{Spaces}\,/B)_*$ defined on the total space by the following composition.

$$\begin{aligned}T(G(p)) = X \amalg B &\xrightarrow{p\times\mathrm{id}_X\amalg\mathrm{id}_B} B\times X\amalg B = B\times X\amalg B\times\{*\}\\ &= B\times X/\phi = T(\omega_0)\xrightarrow{\psi_{-1}} T(\omega_{-1}).\end{aligned}$$

Let (p_{-1},s_{-1}) be the cofiber of α_{-1}. Since $\psi_{-1}\psi_{-2}=*$, ψ_{-2} induces a map

$$\alpha_{-2}\colon(p_{-1},s_{-1})\longrightarrow(\omega_{-2},u_{-2})$$

with $\psi_{-3}\alpha_{-2}=*$. Inductively on ℓ, we can define $(p_{-\ell+1},s_{-\ell+1})$ and a map

$$\alpha_{-\ell}\colon(p_{-\ell+1},s_{-\ell+1})\longrightarrow(\omega_{-\ell},u_{-\ell})$$

with $$\psi_{-\ell-1}\alpha_{-\ell}=*.$$

By the commutativity of the diagram (6) and Lemma 4.12, we have an equivalence of cofibrations

$$\begin{tikzcd}\Omega_{-\ell+1} \arrow[r] \arrow[d] & \Phi(G(f)\wedge(p_{-\ell+1},s_{-\ell+1})) \arrow[d]\\ \overline{\Omega}_{-\ell} \arrow[r] \arrow[d] & \Phi(G(f)\wedge(\omega_{-\ell},u_{-\ell})) \arrow[d]\\ \Omega_{-\ell} \arrow[r] & \Phi(G(f)\wedge(p_{-\ell},s_{-\ell}))\end{tikzcd}$$

This completes the proof of Theorem 5.1.

It remains to show that the gravity filtration gives rise to an injective display. Before we investigate the gravity spectral sequence, we consider a more general situation. Let

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{*\} \arrow[r] & B. \end{tikzcd}$$

be a pullback diagram of pointed spaces. The construction of the Eilenberg–Moore spectral sequence in Section 4 is based on the notion of display which is a filtration on $G(p)$ in the category of pointed spaces over B. As we have seen in Section 4, a display induces a stable decreasing filtration on F, and hence a spectral sequence abutting to the homology of F. In some cases, however, the total space E itself has a stable decreasing filtration which induces a stable decreasing filtration on F. In fact, under some conditions, a decreasing filtration on E defines a display for $G(p)$ which induces the same stable filtration on F as the one induced by the filtration on E. To be more precise, consider the following data and conditions.

(1) A pullback diagram of pointed spaces

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{*\} \arrow[r, "\iota_0"] & B \end{tikzcd}$$

(2) A decreasing filtration on E

$$\cdots \subset F_{-q-1}E \subset F_{-q}E \subset \cdots \subset F_{-1}E \subset F_0E = E$$

in which each inclusion

$$F_{-q-1}E \xrightarrow{i_{-q-1}} F_{-q}E$$

is a pointed cofibration

(3) In the induced decreasing filtration on F

$$\cdots \subset F_{-q-1}F \subset F_{-q}F \subset \cdots \subset F_{-1}F \subset F_0F = F,$$

where $F_{-q}F = F_{-q}E \cap F$, each inclusion is also a pointed cofibration

Proposition 5.5 *Given the above data, let $p_{-q}\colon F_{-q}E \longrightarrow B$ be the restriction of p on $F_{-q}E$. If*

$$\widetilde{h}_*(F_{-q}E/F_{-q-1}E) \cong \widetilde{h}_*(F_{-q}F/F_{-q-1}F) \otimes_{h_*} \widetilde{h}_*(B_+)$$

as $\widetilde{h}_*(B_+)$–comodules, then

$$\begin{array}{ccccc}
G(p) & \longrightarrow & C(G(i_{-1})) & \longrightarrow & \Sigma G(p_{-1}) \\
\Sigma G(p_{-1}) & \longrightarrow & \Sigma C(G(i_{-2})) & \longrightarrow & \Sigma^2 G(p_{-2}) \\
 & & \vdots & & \\
\Sigma^q G(p_{-q}) & \longrightarrow & \Sigma^q C(G(i_{-q-1})) & \longrightarrow & \Sigma^{q+1} G(p_{-q-1}) \\
 & & \vdots & &
\end{array}$$

is an injective h_*^B–display of $G(p)$ with respect to $G(\iota_0)$. The stable filtration on F induced by this display coincides with the one induced directly from the filtration on E.

Proof By the construction, $\{\Sigma^q G(p_{-q}), \Sigma^q C(G(i_{-q-1}))\}$ is a display. Consider the map of cofibrations:

$$\begin{tikzcd}
\Phi(G(p_{-q-1})) \arrow[r] \arrow[d, "\parallel"'] & \Phi(G(p_{-q})) \arrow[r] \arrow[d, "\parallel"'] & \Phi(C(G(i_{-q-1}))) \arrow[d, dotted] \\
F_{-q-1}E_+ \arrow[r] & F_{-q}E_+ \arrow[r] & F_{-q}E/F_{-q-1}E
\end{tikzcd}$$

Since the left and the middle vertical map are homotopy equivalences, the induced map in the right is also a homotopy equivalence. Thus

$$\Phi(C(G(i_{-q-1}))) \simeq F_{-q}E/F_{-q-1}E.$$

On the other hand, since

$$F_{-q}E \times_B \{*\} = F_{-q}F,$$

we have
$$G(p_{-q}) \wedge G(\iota_0) = G(p_{-q} \circ j_{-q}),$$

where $j_{-q}\colon F_{-q}F \hookrightarrow F_{-q}E$ is the inclusion. Therefore

$$\begin{aligned}
C(G(i_{-q-1})) \wedge G(\iota_0) &\simeq C\left(G(p_{-q-1}) \wedge G(\iota_0) \xrightarrow{G(i_{-q-1})} G(p_{-q-1}) \wedge G(\iota_0)\right) \\
&\simeq C\left(G(p_{-q-1} \circ j_{-q}) \longrightarrow G(p_{-q} \circ j_{-q})\right).
\end{aligned}$$

Thus the following chain of equalities holds:

$$\begin{aligned} h_*^B(\Sigma^q C(G(i_{-q-1})) \wedge G(\iota_0)) &= h_*^B(\Sigma^q C(G(p_{-q-1} \circ j_{-q}) \longrightarrow G(p_{-q} \circ j_{-q}))) \\ &= \widetilde{h}_*(\Sigma^q(F_{-q}F/F_{-q-1}F)) \\ &\cong \widetilde{h}_*(\Sigma^q(F_{-q}E/F_{-q-1}E))\Box_{\widetilde{h}_*(B_+)} h_* \\ &= h_*^B(\Sigma^q G(p_{-q}))\Box_{h_*^B(S_B^0, s_B^0)} h_*^B(G(*)) \end{aligned}$$

This proves that $\{\Sigma^q G(p_{-q}), \Sigma^q C(G(i_{-q-1}))\}$ is an injective h_*^B–display of $G(p)$ with respect to $G(*)$. □

Corollary 5.6 *Under the assumption in the above proposition, the spectral sequence defined by the filtration on* F

$$E^1_{-q,*} = \widetilde{h}_*(F_{-q}F/F_{-q-1}F) \Longrightarrow \widetilde{h}_*(F)$$

is isomorphic to the classical Eilenberg–Moore spectral sequence for the pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B \end{tikzcd}$$

from the E^2*–term on.*

Let us apply this fact to the path-loop fibration,

$$\Omega^n\Sigma^n X \longrightarrow P\Omega^{n-1}\Sigma^n X \longrightarrow \Omega^{n-1}\Sigma^n X,$$

namely the pullback diagram

$$\begin{tikzcd} \Omega^n\Sigma^n X \arrow[r] \arrow[d] & P\Omega^{n-1}\Sigma^n X \arrow[d] \\ \{*\} \arrow[r] & \Omega^{n-1}\Sigma^n X \end{tikzcd} \tag{7}$$

or its little cube model

$$\begin{tikzcd} C_n(X) \arrow[r] \arrow[d] & E_n(CX, X) \arrow[d] \\ * \arrow[r] & C_{n-1}(\Sigma X) \end{tikzcd}$$

due to May [6]. Let us recall the definition of $E_n(CX, X)$.

Definition 5.7 Let (Y, B) be a pointed pair. We define a subspace $\mathcal{E}_n(j; Y, B)$ of $\mathcal{C}_n(j) \times Y^j$ as follows.

$$(c_1, \cdots, c_j; y_1, \cdots, y_j) \in \mathcal{E}_n(j; Y, B) \iff \text{if } y_k \notin B \text{ then } c_k \text{ can be extended to the right,}$$

where by "extend to the right", we mean the following: for a cube

$$c = (f_1, \cdots, f_n)\colon (-1, 1)^n \longrightarrow (-1, 1)^n,$$

let $f_1'(t)$ be the interval with $f_1'(-1) = f_1(-1)$ and $f_1'(1) = 1$. Let $\widetilde{c} = (f_1', f_2, \cdots, f_n)$. For $\mathbf{c} = (c_1, \cdots, c_j) \in \mathcal{C}_n(j)$, we say c_k, for $1 \le k \le j$, can be extended to the right if the image of $\widetilde{c}_k$ does not intersect with the images of other cubes, $\mathrm{Im}\,\widetilde{c}_k \cap \mathrm{Im}\, c_l = \phi$ for $l \neq k$.

By restricting the defining relation of $C_n(Y)$ to $\amalg \mathcal{C}_n(j) \times_{\Sigma_j} Y^j$, we define

$$E_n(Y, B) = (\coprod_j \mathcal{E}_n(j; Y, B)/\Sigma_j)/\sim .$$

For $k > 0$, we define

$$\mathcal{F}_k E_n(Y, B) = \mathrm{Im}(\coprod_{j=1}^{k} \mathcal{E}_n(j; Y, B)/\Sigma_j \longrightarrow E_n(Y, B))$$

and

$$\widetilde{\mathcal{E}}_n(j; Y, B) = \mathcal{F}_j E_n(Y, B)/\mathcal{F}_{j-1} E_n(Y, B).$$

These constructions have the following properties.

Theorem 5.8 *Under the same condition as above, the sequence*

$$C_n(X) \longrightarrow E_n(CX, X) \longrightarrow C_{n-1}(\Sigma X)$$

is a quasifibration which is weakly homotopy equivalent to the path-loop fibration

$$\Omega^n \Sigma^n X \longrightarrow P\Omega^{n-1}\Sigma^n X \longrightarrow \Omega^{n-1}\Sigma^n X.$$

The stable splitting theorem (Theorem 2.10) generalizes.

Theorem 5.9 *Under the same condition as above, we have the following natural stable homotopy equivalences*

$$E_n(CX, X) \underset{S}{\simeq} \bigvee_{j=1}^{\infty} \widetilde{\mathcal{E}}_n(j; CX, X)/\Sigma_j. \tag{8}$$

With these theorems, we can replace the pullback diagram (7) by the diagram

$$\begin{tikzcd} \bigvee_{j=1}^{\infty} \mathcal{C}_n(j) \wedge_{\Sigma_j} X^{\wedge j} \arrow[r] \arrow[d] & \bigvee_{j=1}^{\infty} \widetilde{\mathcal{E}}_n(j;CX,X)/\Sigma_j \arrow[d] \\ \{*\} \arrow[r] & \bigvee_{j=1}^{\infty} \mathcal{C}_{n-1}(j) \wedge_{\Sigma_j} (\Sigma X)^{\wedge j} \end{tikzcd}$$

which is stably homotopy equivalent to (7).

In [13], the author defined a Σ_j–equivariant decreasing filtrations on $\widetilde{\mathcal{E}}_n(j;CX,X)$ which is compatible with the gravity filtration on $\mathcal{C}_n(j)$. Thus we have a filtration on

$$\bigvee_{j=1}^{\infty} \widetilde{\mathcal{E}}(j;CX,X)/\Sigma_j$$

which induces the gravity filtration on

$$\bigvee_{j=1}^{\infty} \mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}.$$

It is essentially proved in [13] that these filtrations satisfy the condition in Proposition 5.5, hence, by Corollary 5.6, the gravity spectral sequence in [13] is isomorphic to the classical Eilenberg–Moore spectral sequence from the E^2–term on.

In order to see this is the case, we record the basic properties of the filtrations proved in [13].

Proposition 5.10 *The above filtrations on $\mathcal{C}_n(j)$ and $\widetilde{\mathcal{E}}_n(j;CX,X)$*

$$\phi = F_{-j-1}\mathcal{C}_n(j) \subset F_{-j}\mathcal{C}_n(j) \subset \cdots \subset F_{-1}\mathcal{C}_n(j) = F_0\mathcal{C}_n(j) = \mathcal{C}_n(j)$$

$$\begin{aligned} \phi = F_{-j-1}\widetilde{\mathcal{E}}_n(j;CX,X) \subset F_{-j}\widetilde{\mathcal{E}}_n(j;CX,X) \subset \cdots \\ \subset F_{-1}\widetilde{\mathcal{E}}_n(j;CX,X) \subset F_0\widetilde{\mathcal{E}}_n(j;CX,X) \end{aligned}$$

satisfy the following properties.

(1) *The inclusions*

$$F_{-q-1}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j} \subset F_{-q}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$
$$F_{-q-1}\widetilde{\mathcal{E}}_n(j;CX,X) \subset F_{-q}\widetilde{\mathcal{E}}_n(j;CX,X)$$

are pointed cofibrations for each q.

(2) *The inclusion*
$$\mathcal{C}_n(j)_+ \wedge X^{\wedge j} \subset \widetilde{\mathcal{E}}_n(j; CX, X)$$
is filtration preserving.

(3) *Define*
$$F_{-q}C_n(X) = \bigvee_{j=1}^{\infty} F_{-q}\mathcal{C}_n(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$
$$F_{-q}E_n(CX, X) = \bigvee_{j=1}^{\infty} F_{-q}\widetilde{\mathcal{E}}_n(j; CX, X)/\Sigma_j.$$

Then for each q we have a stable homotopy equivalence:
$$F_{-q}E_n(CX, X)/F_{-q-1}E_n(CX, X) \underset{S}{\simeq} F_{-q}C_n(X)/F_{-q-1}C_n(X) \wedge \Omega^{n-1}\Sigma^n X$$

From these facts it is clear that the gravity filtration on
$$\bigvee_{j=1}^{\infty} \widetilde{\mathcal{E}}_n(j; CX, X)/\Sigma_j$$
satisfies the condition in Proposition 5.5. Thus we obtain the following remaining part of Theorem 1.2.

Corollary 5.11 *If $h_*(\Omega^{n-1}\Sigma^n X)$ is h_*–flat, the gravity spectral sequence in [13] is isomorphic to the classical Eilenberg–Moore spectral sequence from the E^2–term on.*

Remark 5.12 The same argument works for spectral sequences constructed in [14].

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Department of Mathematical Sciences, Shinshu University
Matsumoto 390-8621, Japan

rivulus@math.shinshu-u.ac.jp

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