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Milnor operations and the generalized Chern character

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We have shown that the n–th Morava K–theory $K^*(X)$ for a CW–spectrum X with action of Morava stabilizer group G_n can be recovered from the system of some height–$(n+1)$ cohomology groups $E^*(Z)$ with G_{n+1}–action indexed by finite subspectra Z. In this note we reformulate and extend the above result. We construct a symmetric monoidal functor $\mathcal{F}$ from the category of $E^\vee_*(E)$–precomodules to the category of $K_*(K)$–comodules. Then we show that $K^*(X)$ is naturally isomorphic to the inverse limit of $\mathcal{F}(E^*(Z))$ as a $K_*(K)$–comodule.

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Dedicated to Professor Nishida on the occasion of his 60th birthday

1 Introduction

From the chromatic point of view, the complex K–theory is a height–1 cohomology theory and the ordinary rational cohomology is a height–0 cohomology theory. As geometric aspects, the rational cohomology is defined by means of differential forms, and the K–theory is defined by means of vector bundles. The classical Chern character associates to a complex vector bundle the sum of exponentials of formal roots of the total Chern polynomial. It may be regarded as a multiplicative natural transformation from the K–theory to the rational cohomology, that is to say, from a height–1 cohomology to a height–0 cohomology. There is a height–2 cohomology theory, which is called the elliptic cohomology. Conjecturally, the elliptic cohomology may also have a geometric interpretation analogous to the rational cohomology and the K–theory. A generalization of Chern character to the elliptic cohomology has been considered by Miller [15]. The idea is that the formal group law on the moduli stack of elliptic curves is degenerate to the multiplicative formal group law when it is restricted around a cusp. Miller's elliptic character is a multiplicative natural transformation from the elliptic cohomology to the K–theory with coefficients in the formal Laurent series ring, hence from a height–2 theory to a height–1 theory.

The elliptic character may be regarded as the q–expansion map of modular forms parametrized by spaces. The q–expansion is the Fourier expansion of modular forms

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at a cusp, which associates a formal Laurent series with variable $q = \exp(2\pi\sqrt{-1}\tau)$. The q–expansion map has been extended at more general algebraic setting, and it has been shown that it has a very good property, which is called q–expansion principle (cf Deligne–Rapoport [3] and Katz [11]). In particular, the q–expansion map is injective, and hence the modular forms are controlled by their q–expansion. The analogous property on the elliptic character has been studied at odd primes by Laures [13]. At the prime 2, there is a more elaborate cohomology theory related to elliptic curves and modular forms. It is defined by the spectrum *tmf* of topological modular forms, which is introduced by Mike Hopkins. The q–expansion map (evaluation at the Tate curve) also induces a ring spectrum map from *tmf* to $K[\![q]\!]$. In [14] Laures studied the $K(1)$–local topological modular forms at the prime 2, and discussed the relationship between the q–expansion map, Witten genus and $MO\langle 8\rangle$–orientation of *tmf*.

A generalization of Chern character to higher chromatic level has been considered by Ando, Morava and Sadofsky [1] under geometric background. Their generalized Chern character is a multiplicative natural transformation from $(n+1)$–th Morava E–theory E_{n+1} to the n–th Morava E–theory with coefficients in some big Cohen ring. In [23] we studied the degeneration of formal group law, which is used to construct their Chern character. By using the results in [23] we refined their generalized Chern character in [22]. Then we were able to control it algebraically. In this note we reformulate and extend some results in [22].

Let $\mathcal{S}$ be the stable homotopy category of p–local spectra for some prime p. It is known that there is a filtration of full subcategories of $\mathcal{S}$, which corresponds to the height filtration of the moduli space of one-dimensional commutative formal group laws; see Devinatz, Hopkins and Smith [5], Hopkins and Smith [7], Morava [17] and Ravenel [19]. The layers of this filtration are equivalent to the $K(n)$–local categories, where $K(n)$ is the n–th Morava K–theory. Hence it is considered that the stable homotopy category $\mathcal{S}$ is built from $K(n)$–local categories. In fact, the chromatic convergence theorem (cf Ravenel [19]) says that for a p–local finite spectrum X, the natural tower $\cdots \to L_{n+1}X \to L_nX \to \cdots \to L_0X$ recovers X, that is, the homotopy inverse limit of the tower is homotopy equivalent to X. Furthermore, the chromatic splitting conjecture (cf Hovey [8]) implies that the p–completion of a finite spectrum X is a direct summand of the product $\prod_n L_{K(n)}X$. This means that it is not necessarily to reconstruct the tower but it is sufficient to know all $L_{K(n)}X$ to obtain some information about X. In [8] Hovey observed that the weak form of the chromatic splitting conjecture should imply many interesting results. The weak form means that the canonical map $L_n(S^0)^\wedge_p \to L_nL_{K(n+1)}S^0$ is a split monomorphism. In [16, Remark 3.1(i)] Minami indicated that the weak form implies that there is a natural map ρ_X for a finite spectrum X from the $K(n+1)$–localization $L_{K(n+1)}X$

to the $K(n)$–localization $L_{K(n)}X$ such that the following diagram commutes:

$$
\begin{tikzcd}
& X \arrow[dl, "\eta_{K(n+1)}"'] \arrow[dr, "\eta_{K(n)}"] & \\
L_{K(n+1)}X \arrow[rr, "\rho"] & & L_{K(n)}X
\end{tikzcd}
\tag{1}
$$

where $\eta_K(n)\colon X \to L_{K(n)}X$ and $\eta_K(n+1)\colon X \to L_{K(n+1)}X$ are the localization maps.

In [22] we considered the modulo I_n version of the algebraic analogue of the diagram (1). The Morava E–theory E_n defines a functor from the $K(n)$–local category to the category of twisted E_{n*}–G_n–modules, where G_n is the n–th extended Morava stabilizer group. The Adams–Novikov spectral sequence based on E_n–theory has its E_2–term $H_c^{**}(G_n; E_{n*}(X))$ and converges to $\pi_*(L_{K(n)}X)$ strongly if X is finite, where $H_c^{**}(G_n; -)$ is the continuous cohomology group of G_n. Hence the category of twisted E_{n*}–G_n–modules can be considered as an algebraic approximation of the $K(n)$–local category.

Let BP_* be the Brown–Peterson spectrum at an odd prime p and I_n the invariant prime ideal generated by $p, v_1, \dots, v_{n-1}$. There is a commutative ring spectrum $E = E_{n+1}/I_n$, which is a complex oriented cohomology theory with coefficient ring $E_* = E_{n+1,*}/I_n = \mathbf{F}[u_n][u^{\pm 1}]$. We denote by $K^*(-)$ a variant of n–th Morava K–theory with coefficient $\mathbf{F}[w^{\pm 1}]$. The modulo I_n–version of the algebraic analogue of the diagram (1) is

$$
\begin{tikzcd}
& \mathcal{S}^f \arrow[dl, "E"'] \arrow[dr, "K"] & \\
\mathcal{M}^f_{n+1} \arrow[rr, "\mu"] & & \mathcal{M}^f_n
\end{tikzcd}
\tag{2}
$$

where $\mathcal{S}^f$ is the stable homotopy category of finite spectra, $\mathcal{M}^f_{n+1}$ is the category of finitely generated twisted E_*–G_{n+1}–modules and $\mathcal{M}_n$ is the category of finitely generated twisted K_*–G_n–modules. So the question is: Does there exist a functor μ from $\mathcal{M}^f_{n+1}$ to $\mathcal{M}^f_n$ which makes the diagram (2) commute?

In [23] we constructed a Galois extension L of the quotient field of $\mathbf{F}[u_n]$, over which the formal group F_{n+1} associated with E is nicely isomorphic to the Honda group law H_n. By using this result, we constructed a natural transformation Θ from E–cohomology to K–cohomology with coefficients in L in [22]:

$$\Theta\colon E^*(X) \longrightarrow L^*(X).$$

This is regarded as a generalized Chern character since it is a multiplicative natural transformation from the height–$(n+1)$ cohomology E to the height–n cohomology K with coefficients in L. Then it is shown that Θ is equivariant with respect to the action of G_{n+1}, and Θ induces $L_* \otimes_{E_*} E^*(Z) \cong L_* \otimes_{K_*} K^*(Z)$, a natural isomorphism of $\mathcal{G}$–modules for any finite spectrum Z, where $\mathcal{G} = \Gamma \ltimes (S_n \times S_{n+1})$ and $L_* = L[u^{\pm 1}]$. By these results, we have shown that there is a natural isomorphism of twisted K_*–G_n–modules:

$$K^*(X) \cong \varprojlim_Z H^0(S_{n+1}; L_* \otimes_{E_*} E^*(Z))$$

for any CW-spectrum X where Z ranges over finite subspectra of X [22, Corollary 4.3]. Hence if we set the functor $\mu(-) = H^0(S_{n+1}; L_* \otimes_{E_*} -)$, it makes the diagram (2) commute.

Essentially, the twisted K_*–G_n–module structure gives $K^*(X)$ the stable cohomology operations except for Milnor operations. So in this note we would like to extend the above result in the form which includes the action of Milnor operations. Note that the twisted K_*–G_n–module structure on $K^*(X)$ with Milnor operations is equivalent to the $K_*(K)$–comodule structure. In this note we construct a symmetric monoidal functor $\mathcal{F}$ from the category of profinite $E^\vee_*(E)$–precomodules to the category of profinite $K_*(K)$–comodules. Roughly speaking, a profinite $E^\vee_*(E)$–precomodule is a filtered inverse limit of finitely generated E_*–module

$$M = \varprojlim M/M_\lambda$$

such that M^c has a complete $E^\vee_*(E)$–comodule structure, where

$$M^c = \varprojlim M/M_\lambda + \mathfrak{m}^i M.$$

For a profinite $E^\vee_*(E)$–precomodule M, there is a natural twisted E_*–G_{n+1}–module structure on M. Hence $M \widehat{\otimes}_{E_*} L_*$ is a twisted L_*–G_{n+1}–module. We set

$$\mathcal{F}(M) = H^0(S_{n+1}; M \widehat{\otimes}_{E_*} L_*).$$

Theorem A (Corollary 6.5) *The functor $\mathcal{F}$ extends to a symmetric monoidal functor from the category of profinite $E^\vee_*(E)$–precomodules to the category of profinite $K_*(K)$–comodules.*

For a spectrum X we denote by $\Lambda(X)$ the category whose objects are maps $Z \xrightarrow{u} X$ with Z finite. We associate to X a cofiltered system $\mathbb{E}^*(X) = \{E^*(Z)\}$ and $\mathbb{K}^*(X) = \{K^*(Z)\}$ indexed by $\Lambda(X)$. Then $E^*(X) = \varprojlim \mathbb{E}^*(X)$ and $K^*(X) = \varprojlim \mathbb{K}^*(X)$.

The following is the main theorem of this note.

Theorem B (Theorem 6.11) *For any spectrum X, the generalized Chern character Θ induces a natural isomorphism of cofiltered system of $K_*(K)$–comodules:*

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

If X is a space, then this is an isomorphism of systems of $K_*(K)$–comodule algebras.

The organization of this note is as follows. In Section 2.1 we summarize well-known results on generalized cohomology theories which are Landweber exact over $P(n)$. In Section 2.2 we review our main result in [23] on degeneration of formal group laws. In Section 2.3 we review on the construction of the generalized Chern character.

In Section 3.1 we study the category of complete Hausdorff filtered modules and the action of a profinite group on a complete module. In Section 3.2 we recall complete Hopf algebroids and their comodules.

In Section 4.1 we describe the structure of complete Hopf algebroid $C(G, R^c)$, where $C(G, R^c)$ is the ring of all continuous functions from a profinite group G to an even-periodic complete local ring R^c. In Section 4.2 we show the well-known fact that the category of complete $C(G, R^c)$–comodules is equivalent to the category of complete twisted R^c–G–modules. Usually, we use and study the category of complete twisted $\mathbf{F}_{p^n}$–G_n–modules. In Section 4.3 we show that there is no essential difference between the category of complete twisted $\mathbf{F}$–G_n–modules and the category of complete twisted $\mathbf{F}_{p^n}$–G_n–modules. In Section 4.4 we reformulate a result of [22]. We construct a symmetric monoidal functor $\mathcal{F}$ from the category of profinite $C(G_{n+1}, E^c_*)$–precomodules to the category of profinite $C(G_n, K_*)$–comodules, and show that there is a natural isomorphism between $\mathcal{F}(\mathbb{E}^*(X))$ and $\mathbb{K}^*(X)$ as systems of $C(G_n, K_*)$–comodules.

In Section 5.1 we define a complete co-operation ring $A^\vee_*(A)$ for $A = E_{n+k}/I_n$, and study a $A^\vee_*(A)$-(pre)comodule algebra structure on the A–cohomology of the projective space $\mathbf{C}P^\infty$ and the lens space $S^{2p^n-1}/(\mathbf{Z}/p)$. In Section 5.2 we study a twisted E_*–G_{n+1}–module structure on the exterior algebra Λ_{E_*} and show that $\mathcal{F}(\Lambda_{E_*}) \cong \Lambda_{K_*}$ as twisted K_*–G_n–modules. In Section 5.3 we define Milnor operations Q^A_i for a Λ_{A_*}–comodule M. In Section 5.4 we study $A^\vee_*(A)$–comodule structures in terms of C_{A_*}–comodule structures and Λ_{A_*}–comodule structures. We show that an $A^\vee_*(A)$–comodule structure is equivalent to a C_{A_*}–comodule structure and a Λ_{A_*}–comodule structure which satisfy some compatibility condition.

In Section 6.1 we extend the symmetric monoidal functor $\mathcal{F}$ from the category of profinite $E^\vee_*(E)$–precomodules to the category of profinite $K_*(K)$–comodules. In Section 6.2 we prove the main theorem.

In this note p shall be an odd prime except for Section 2.2, $\mathbf{F}$ a finite field containing $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n+1}}$, and Gal the Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. We think a group G acts on a ring R from the right and denote by r^g the right action of $g \in G$ on $r \in R$. For a power series $\alpha(X) = \sum \alpha_i X^i \in R[\![X]\!]$, we set $\alpha^g(X) = \sum \alpha_i^g X^i$ for $g \in G$. An R–module means a left R–module if nothing else is indicated.

2 The generalized Chern character

2.1 Landweber exact theories over $P(n)$

Definition 2.1 A graded commutative ring R_* is said to be even-periodic if R_* is concentrated in even degrees and R_2 contains a unit in R_*. A multiplicative generalized cohomology theory $h^*(-)$ is said to be even-periodic if the coefficient ring $h_* = h^*(\mathrm{pt})$ is even-periodic.

For a spectrum X, we denote by $\Lambda(X)$ the category whose objects are maps $Z \xrightarrow{u} X$ such that Z is finite, and whose morphisms are maps $Z \xrightarrow{v} Z'$ such that $u'v = u$. Then $\Lambda(X)$ is an essentially small filtered category.

Definition 2.2 Let $h^*(-)$ be a generalized cohomology theory. For a spectrum X, we define a filtration on $h^*(X)$ indexed by $\Lambda(X)$ as

$$F^Z h^*(X) = \mathrm{Ker}(h^*(X) \longrightarrow h^*(Z))$$

for $Z \in \Lambda(X)$. We call this filtration the profinite filtration and the resulting topology the profinite topology.

Remark 2.3 If $h^*(-)$ is even-periodic and the degree–0 coefficient ring h_0 is a complete Noetherian local ring, then $h^*(Z)$ is a finitely generated h_*–module for all $Z \in \Lambda(X)$, and the canonical homomorphism

$$h^*(X) \longrightarrow \varprojlim_{Z \in \Lambda(X)} h^*(Z)$$

is an isomorphism. This implies that $h^*(X)$ is complete Hausdorff with respect to the profinite topology.

Let BP be the Brown–Peterson spectrum at an odd prime p, whose coefficient ring is given by $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_i| = 2(p^i - 1)$. Let I_n be the invariant prime ideal generated by $p, v_1, \ldots, v_{n-1}$. There is a commutative BP–algebra spectrum $P(n)$, whose coefficient ring is $P(n)_* = BP_*/I_n$. In particular, $P(0) = BP$. Let

$\mathcal{X} = \mathbf{C}P^\infty$ the infinite dimensional complex projective space, and $\mathcal{Y} = S^{2p^n-1}/(\mathbf{Z}/p)$ the lens space of dimension $2p^n-1$, where $\mathbf{Z}/p$ is the cyclic group of order p acting on the unit sphere S^{2p^n-1} in $\mathbf{C}^{p^n}$ by standard way. These spaces are important test spaces to stable cohomology operations of complex oriented cohomology theories (cf [2, Section 14]). The $P(n)$–cohomology of $\mathcal{X}$ and $\mathcal{Y}$ are given as follows:

$$\begin{aligned} P(n)^*(\mathcal{X}) &= P(n)_*[\![x]\!], \\ P(n)^*(\mathcal{Y}) &= \Lambda(y) \otimes P(n)_*[x]/(x^{p^n}), \end{aligned}$$

where $x \in P(n)^2(\mathcal{X})$ is the orientation class and $y \in P(n)^1(\mathcal{Y})$.

Let F be a p–typical formal group law over a commutative ring R. By universality of the p–typical formal group law F_{BP} associated to BP, there is a unique ring homomorphism $f\colon BP_* \to R$ such that F is the base change of F_{BP} by f. If $f(v_i) = 0$ for $0 \le i < n$, then f induces a ring homomorphism $\overline{f}\colon P(n)_* \to R$. In this case we say that a p–typical formal group law F is of strict height at least n. Hence $P(n)_*$ is the universal ring of p–typical formal group law of strict height at least n. We say that a ring homomorphism $P(n)_* \to R$ is Landweber exact over $P(n)$, if the sequence $v_n, v_{n+1}, \dots$ is regular in R. In this case, the functor $R_* \otimes_{P(n)_*} P(n)_*(-)$ is a generalized homology theory by Landweber–Yagita exact functor theorem [12; 25], where R_* is the even-periodic commutative ring $R[u^{\pm 1}]$ with $|u| = -2$. Furthermore, if R is a complete Noetherian local ring, then

$$R^*(X) = \varprojlim_{\Lambda(X)} (R_* \otimes_{P(n)_*} P(n)^*(Z))$$

is a generalized cohomology theory.

Let $\mathbf{F}$ be a finite field which contains the finite fields $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n+1}}$. Let $E_n^*(-)$ be a variant of Morava E–theory whose coefficient ring is given by

$$E_{n*} = W(\mathbf{F})[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}],$$

where $W(\mathbf{F})$ is the ring of Witt vectors with coefficients in $\mathbf{F}$. The grading is given by $u_i = 0$ for $1 \le i < n$ and $|u| = -2$. Then the degree–0 formal group law F_n associated to E_n is a universal deformation of the Honda group law H_n of height n over $\mathbf{F}$. For $0 \le k \le n$, there is a commutative multiplicative cohomology theory $(E_n/I_k)^*(-)$ whose coefficient ring is just E_{n*}/I_k, where I_k is the invariant prime ideal $(p, u_1, \dots, u_{k-1})$.

We define even-periodic graded commutative rings E_* and K_* as follows:

$$\begin{aligned} E_* &= \mathbf{F}[\![u_n]\!][u^{\pm 1}], \\ K_* &= \mathbf{F}[w^{\pm 1}], \end{aligned}$$

where the gradings are given by $|u_n| = 0, |w| = |u| = -2$. The ring homomorphisms $P(n)_* \to E_*$ given by $v_n \mapsto u_n u^{-(p^n-1)}, v_{n+1} \mapsto u^{-(p^{n+1}-1)}, v_i \mapsto 0$ for $i > n+1$, and $P(n)_* \to K_*$ given by $v_n \mapsto w^{-(p^n-1)}, v_i \mapsto 0$ for $i > n$, make E_* and K_* Landweber exact $P(n)_*$–algebras, respectively. Hence

$$\begin{aligned} E^*(X) &= \varprojlim \, (E_* \otimes_{P(n)_*} P(n)^*(Z)), \\ K^*(X) &= \varprojlim \, (K_* \otimes_{P(n)_*} P(n)^*(Z)), \end{aligned}$$

define generalized cohomology theories. Note that there are no limit one problems since the degree–0 subrings are complete Noetherian local rings, respectively. The cohomology theory $K^*(-)$ is a variant of Morava K–theory and the associated degree–0 formal group law is the Honda group law H_n of height n over $\mathbf{F}$. Since the cohomology theory $E^*(-)$ is $(E_{n+1}/I_n)^*(-)$, the associated degree–0 formal group law is the base change of F_{n+1} to $\mathbf{F}[\![u_n]\!]$.

We set
$$\begin{aligned} x_E &= 1 \otimes x \in E^0(\mathcal{X}), \\ y_E &= 1 \otimes y \in E^1(\mathcal{Y}), \\ x_K &= 1 \otimes x \in K^0(\mathcal{X}), \\ y_K &= 1 \otimes y \in K^1(\mathcal{Y}). \end{aligned}$$

Then we have
$$\begin{aligned} E^*(\mathcal{X}) &\cong E_*[\![x_E]\!], \\ E^*(\mathcal{Y}) &\cong \Lambda(y_E) \otimes E_*[x_E]/(x_E^{p^n}), \\ K^*(\mathcal{X}) &\cong K_*[\![x_K]\!], \\ K^*(\mathcal{Y}) &\cong \Lambda(y_K) \otimes K_*[x_K]/(x_K^{p^n}). \end{aligned}$$

2.2 Degeneration of formal groups

In this subsection we review some results in [23]. In this subsection p is any prime number. Let $E_{n+1,0}$ be the degree–0 coefficient ring of the variant of Morava E–theory E_{n+1}:

$$E_{n+1,0} = W(\mathbf{F})[\![u_1, \ldots, u_n]\!].$$

The associated degree–0 formal group law F_{n+1} is a universal deformation of the Honda group laws H_{n+1} of height $n+1$ over $\mathbf{F}$. The extended Morava stabilizer group $G_{n+1} = \mathrm{Gal} \ltimes S_{n+1}$ is the automorphism group of F_{n+1} in some generalized sense (cf Strickland [21] and Torii [23]), where Gal is the Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ and S_{n+1} is the n–th Morava stabilizer group. Note that S_{n+1} is the automorphism group of H_{n+1} in the usual sense. The extended Morava stabilizer group G_{n+1} is a profinite group and acts on $E_{n+1,0}$ continuously, where the topology of $E_{n+1,0}$ is given by the adic-topology. Since the ideal $I_n = (p, u_1, \dots, u_{n-1})$ of $E_{n+1,0}$ is stable under the action, G_{n+1} also acts on the quotient ring $E_{n+1,0}/I_n = \mathbf{F}[\![u_n]\!]$ continuously.

We regard the formal group law F_{n+1} as being defined over $\mathbf{F}[\![u_n]\!]$ by the obvious base change. This situation is a kind of degeneration and a fundamental technique to study a degeneration is to investigate the monodromy representation. Let $M = \mathbf{F}((u_n))$ be the quotient field of $\mathbf{F}[\![u_n]\!]$ and M^{sep} its separable closure. Then the height of F_{n+1} on M is n. Hence the fibre of F_{n+1} over M^{sep} is isomorphic to H_n since the isomorphism classes of formal group laws over a separably closed field are classified by their height. The monodromy representation of F_{n+1} around the closed point gives the following homomorphism:

$$\mathrm{Gal}(M^{\mathrm{sep}}/M) = \pi_1(M) \longrightarrow \mathrm{Aut}(H_n) = S_n.$$

This homomorphism was studied by Gross in [6].

Let Φ be an isomorphism over M^{sep} between F_{n+1} and H_n:

$$\Phi(F_{n+1}(X,Y)) = H_n(\Phi(X), \Phi(Y)).$$

Let L be a separable algebraic extension of M obtained by adjoining all the coefficients of $\Phi(X)$. Then the above homomorphism $\mathrm{Gal}(M^{\mathrm{sep}}/M) \to S_n$ induces an isomorphism $\mathrm{Gal}(L/M) \xrightarrow{\cong} S_n$, and this extends to an isomorphism $\mathrm{Gal}(L/\mathbf{F}_p((u_n))) \cong G_n$. Let $\mathcal{G}$ be the semidirect product $\mathrm{Gal} \ltimes (S_n \times S_{n+1})$. Then $\mathcal{G}$ is a profinite group, and contains G_n and G_{n+1} as closed subgroups.

The following theorem is a main point of [23].

Theorem 2.4 [23, Section 2.4] *The profinite group $\mathcal{G}$ acts on the formal group law (F_{n+1}, L) in generalized sense. The action of the subgroup G_{n+1} is an extension of the action on $(F_{n+1}, \mathbf{F}[\![u_n]\!])$. The action of the subgroup G_n on (F_{n+1}, L) is the action of Galois group on L and the trivial action on F_{n+1}. Under the isomorphism Φ: $F_{n+1} \xrightarrow{\cong} H_n$, the induced action of $\mathcal{G}$ on (H_n, L) is encoded as the following two commutative diagrams. For $g \in G_{n+1}$, there is a commutative diagram*

(3)
$$\begin{array}{ccc} F_{n+1} & \xrightarrow{t_E(g)} & {F_{n+1}}^g \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi^g} \\ H_n & \xrightarrow{=} & H_n^g, \end{array}$$

where $t_E(g)(X)$ is an isomorphism from F_{n+1} to ${F_{n+1}}^g$ corresponding to g. For $h \in G_n$, there is a commutative diagram

$$
\begin{tikzcd}
F_{n+1} \arrow[r, "="] \arrow[d, "\Phi"'] & {F_{n+1}}^h \arrow[d, "\Phi^h"] \\
H_n \arrow[r, "t_K(h)"] & H_n^h,
\end{tikzcd}
\tag{4}
$$

where $t_K(h)(X)$ is the automorphism of H_n corresponding to h.

2.3 The generalized Chern character

In this subsection we review the generalized Chern character Θ constructed in [22].

The co-operation ring $P(n)_*(P(n))$ is isomorphic to

$$P(n)_*[t_1, t_2, \ldots] \otimes \Lambda(a_{(0)}, \ldots, a_{(n-1)})$$

as a left $P(n)_*$–algebra, where $|t_i| = 2(p^i - 1)$ and $|a_{(i)}| = 2p^i - 1$. In particular, $P(n)_*(P(n))$ is a free $P(n)_*$–module. Hence $(P(n)_*, P(n)_*(P(n)))$ is a Hopf algebroid over $\mathbf{F}_p$. By formalism of Boardman [2], there is a natural $P(n)_*(P(n))$–comodule structure on the completion $P(n)^*(X)^\wedge$ with respect to the profinite topology:

$$\rho\colon\ P(n)^*(X)^\wedge \longrightarrow P(n)_*(P(n))\,\widehat{\otimes}_{P(n)_*} P(n)^*(X)^\wedge.$$

The set of $\mathbf{F}_p$–algebra homomorphisms from $P(n)_*(P(n))/(a_{(0)}, \ldots, a_{(n-1)})$ to an even-periodic $\mathbf{F}_p$–algebra R_* is naturally identified with the set of triples (F, f, G), where F and G are p–typical formal group laws over R_0 with strict height at least n, and f is an isomorphism between them. Let L_* be an even-periodic E_*–algebra $L[u^{\pm 1}]$. By Theorem 2.4 and the above moduli interpretation of the ring $P(n)_*(P(n))/(a_{(0)}, \ldots, a_{(n-1)})$, there is a ring homomorphism $\theta\colon P(n)_*(P(n)) \to P(n)_*(P(n))/(a_{(0)}, \ldots, a_{(n-1)}) \to L_*$ such that the following diagram commutes:

$$
\begin{tikzcd}
P(n)_* \arrow[r, "H_n"] \arrow[d, "\eta_R"'] & K_* \arrow[d] \\
P(n)_*(P(n)) \arrow[r, "\theta"] & L_* \\
P(n)_* \arrow[r, "F_{n+1}"] \arrow[u, "\eta_L"] & E_*. \arrow[u]
\end{tikzcd}
$$

That is, θ corresponds to the triple (F_{n+1}, Φ, H_n) over L. For $Z \in \Lambda(X)$, by extending the natural ring homomorphism

$$\begin{aligned} P(n)^*(Z) &\xrightarrow{\rho} P(n)_*(P(n)) \otimes_{P(n)_*} P(n)^*(Z) \\ &\xrightarrow{\theta\otimes 1} L_* \otimes_{P(n)_*} P(n)^*(Z) \\ &\cong L^*(Z) \end{aligned}$$

to $E^*(Z) = E_* \otimes_{P(n)_*} P(n)^*(Z) \to L^*(Z)$, we obtain a multiplicative natural transformation

$$\Theta : E^*(X) \longrightarrow L^*(X), \tag{5}$$

which we call the generalized Chern character.

The following lemma is easily checked.

Lemma 2.5 $\Theta(x_E) = \Phi(x_K)$ *and* $\Theta(y_E) = 1 \otimes y_K$.

3 Complete Hopf algebroids

3.1 Complete modules

Let k be a commutative ring. We say that $(M, \{F^\lambda M\}_{\lambda\in\Lambda})$ is a filtered k–module if M is an k–module and $\{F^\lambda M\}_{\lambda\in\Lambda}$ is a family of k–submodules indexed by a (small) filtered category Λ. Then M can be given a linear topology. We denote by FMod_k the category of filtered k–modules and continuous homomorphisms. A filtered k–module $(M, \{F^\lambda M\}_{\lambda\in\Lambda})$ is said to be complete Hausdorff if the canonical homomorphism

$$M \to \varprojlim_{\Lambda} M/F^\lambda M$$

is an isomorphism. We denote by FMod_k^c the full subcategory of FMod_k whose objects are complete Hausdorff. We say that $(M, \{F^\lambda M\}_{\lambda\in\Lambda}) \in \mathrm{FMod}_k^c$ is a profinite k–module if $M/F^\lambda M$ is a finitely generated k–module for all $\lambda \in \Lambda$. We denote by ProFG_k the full subcategory of FMod_k^c whose objects are profinite.

Since FMod_k^c is a symmetric monoidal category with tensor product $\widehat{\otimes}_k$ and unit object k, we can define commutative monoid objects in FMod_k^c, that is, complete commutative k–algebras. We denote by FAlg_k^c the category of complete commutative k–algebras. For $R \in \mathrm{FAlg}_k^c$, we can define an R–module in FMod_k^c, and we denote by FMod_R^c the category of R–modules. For $R_1 \to R_2 \in \mathrm{FAlg}_k^c$, there is a base change functor

$$(-)\widehat{\otimes}_{R_1} R_2 \colon \mathrm{FMod}_{R_1}^c \longrightarrow \mathrm{FMod}_{R_2}^c.$$

If R is a complete Noetherian local k–algebra with maximal ideal $\mathfrak{m}$, then R with $\mathfrak{m}$–adic filtration can be regarded as an object in FAlg_k^c. We denote by R^c the k–module R with $\mathfrak{m}$–adic filtration, and simply by R the k–module R with trivial filtration $\{0\}$. Note that the base change $M \widehat{\otimes}_R R^c$ for $M \in \mathrm{FMod}_R^c$ is given by

$$\varprojlim M/(F^\lambda M + \mathfrak{m}^i M)$$

with inverse limit topology. Since M is isomorphic to the inverse limit of $M/\mathfrak{m}^i M$ for a finitely generated R–module M, we see that $M \widehat{\otimes}_R R^c \cong M$ as (abstract) R–modules for $M \in \mathrm{ProFG}_R$.

Example 3.1 Let $h^*(-)$ be a generalized cohomology theory and X a spectrum. We defined the profinite filtration on $h^*(X)$ in Remark 2.3. If $h^*(-)$ is even-periodic and the degree–0 coefficient ring h_0 is a complete Noetherian local ring, then $h^*(X)$ is a complete Hausdorff profinite h_*–module. Hence the cohomology theory $h^*(-)$ gives a functor from the stable homotopy category to ProFG_{h_*}.

Lemma 3.2 *Let $M \in \mathrm{FMod}_k^c$ and $\underline{M}$ the underlying k–module. If $M \in \mathrm{FMod}_R^c$, then $\underline{M}$ is an R–module in the usual sense.*

Proof The map $R \otimes M \to R \widehat{\otimes} M \to M$ gives an R–module structure on $\underline{M}$. □

Lemma 3.3 *If $M \in \mathrm{FMod}_R^c$, then for any open k–submodule M_λ there is an open R–submodule N such that $N \subset M_\lambda$.*

Proof The fact that $M \in \mathrm{FMod}_R^c$ implies that the map $R \widehat{\otimes} M \to M \to M/M_\lambda$ factors through $R \otimes M/M_\mu$ for some open k–submodule M_μ. Hence $R \cdot M_\mu \subset M_\lambda$. We take N as $R \cdot M_\mu$. Since $M_\mu \subset N$, N is an open R–submodule. This completes the proof. □

Corollary 3.4 *Let $M \in \mathrm{FMod}_R^c$. There is a fundamental (open) neighborhood system at 0 consisting of R–submodules.*

Corollary 3.5 *Let M and N be objects in FMod_R^c. Then*

$$M \underset{R}{\widehat{\otimes}} N \cong \varprojlim_{\lambda,\mu} (M/F^\lambda M) \otimes_R (N/F^\mu N),$$

where $\{F^\lambda M\}_\lambda$ and $\{F^\mu N\}_\mu$ are families of all open R–submodules of M and N, respectively.

Lemma 3.6 *Let $M \in \mathrm{FMod}^c_R$. Then M is an R^c–module compatible with given R–module structure if and only if for any open R–submodule N there is a nonnegative integer i such that $\mathfrak{m}^i M \subset N$.*

Proof If M is an R^c–module compatible with given R–module structure, then there is a continuous map $R^c \widehat{\otimes}_R M \longrightarrow M$, which makes M an R^c–module. Then the map $R^c \widehat{\otimes}_R M \to M \to M/N$ factors through $R/\mathfrak{m}^i \otimes_R M/N'$ for some i and some open R–submodule N'. This implies that $\mathfrak{m}^i M \subset \mathfrak{m}^i M + N' \subset N$.

If for any open R–submodule N there is i such that $\mathfrak{m}^i M \subset N$, then there are compatible maps $R^c \widehat{\otimes}_R M \to R/\mathfrak{m}^i \otimes_R M/N \to M/N$, which induce a continuous map $R^c \widehat{\otimes}_R M \to M$. This map defines an R^c–module structure on M compatible with given R–module structure. □

Lemma 3.7 *Let $M \in \mathrm{FMod}^c_k$. If a profinite group G acts on M continuously as k–module homomorphisms, then for any open submodule M_λ there is an open submodule N such that $G \cdot N \subset M_\lambda$.*

Proof For any $g \in G$, there are an open submodule N_g of M and an open neighborhood U_g of g such that $U_g \cdot N_g \subset M_\lambda$. Since G is compact, $G = U_{g_1} \cup \cdots \cup U_{g_n}$. Take an open submodule N such that $N \subset N_{g_1} \cap \cdots \cap N_{g_n}$. Then for any $g \in G$, $g \in U_{g_i}$ for some i, and for any $x \in N \subset N_{g_i}$, $g \cdot x \in U_{g_i} \cdot N_{g_i} \subset M_\lambda$. Hence we obtain that $G \cdot N \subset M_\lambda$. □

Corollary 3.8 *Let $M \in \mathrm{FMod}^c_k$. If a profinite group G acts on M continuously as k–modules homomorphisms, then for any open submodule M_λ, there is an open G–submodule N such that $N \subset M_\lambda$.*

Proof By Lemma 3.7, there is an open submodule N' such that $G \cdot N' \subset M_\lambda$. Let N be the submodule generated by $G \cdot N'$. Then N is a G–submodule and $N \subset M_\lambda$. Since $N' \subset N$, N is an open submodule. This completes the proof. □

Corollary 3.9 *Let $M \in \mathrm{FMod}^c_k$ and G a profinite group. Suppose that G acts on M continuously as k–module homomorphisms. There is a fundamental (open) neighborhood system at 0 consisting of G–submodules.*

Theorem 3.10 *Let G be a profinite group acting on R^c continuously as k–algebra homomorphisms, and M a complete twisted R^c–G–module. For any open R–submodule M_λ, there is an open R–G–submodule N such that $N \subset M_\lambda$.*

Proof By Corollary 3.8, there is an open k–G–submodule N' such that $N' \subset M_\lambda$. Let N be the R–submodule generated by N'. Then N is an open G–submodule such that $N \subset M_\lambda$. □

Corollary 3.11 *Let M be a complete twisted R^c–G–module. Then there is a fundamental (open) neighborhood system at 0 consisting of R–G–submodules.*

3.2 Complete Hopf algebroids and complete precomodules

Let A and Γ be objects in FAlg_k^c. We suppose that there are maps in FAlg_k^c:

$$\begin{array}{rcll} \eta_R\colon & A & \longrightarrow & \Gamma, \\ \eta_L\colon & A & \longrightarrow & \Gamma, \\ \chi\colon & \Gamma & \longrightarrow & \Gamma, \\ \varepsilon\colon & \Gamma & \longrightarrow & A. \end{array}$$

If the maps $(\eta_R, \eta_L, \chi, \varepsilon)$ satisfy the usual Hopf algebroid relations [18, Appendix 1], then we say that the pair (A, Γ) is a complete Hopf algebroid over k. A Hopf algebroid is a complete Hopf algebroid with discrete topology. Since $P(n)_*(P(n))$ is free over $P(n)_*$, $P(n)_*(P(n))$ is a Hopf algebroid, hence, a complete Hopf algebroid over $\mathbf{F}_p$.

Let $A \to B$ be a map in FAlg_k^c. We set

$$\Gamma_B := B \widehat{\otimes}_A \Gamma \widehat{\otimes}_A B.$$

Then (B, Γ_B) is a complete Hopf algebroid over k as usual.

Example 3.12 Let $h^*(-)$ be an even-periodic Landweber exact theory over $P(n)$ such that h_0 is a complete Noetherian local ring. Put

$$\Gamma(h) = h_*^c \widehat{\otimes}_{P(n)_*} P(n)_*(P(n)) \widehat{\otimes}_{P(n)_*} h_*^c.$$

Then $(h_*^c, \Gamma(h))$ is a complete Hopf algebroid over $\mathbf{F}_p$.

An object $M \in \mathrm{FMod}_A^c$ is said to be a complete Γ–comodule if there is a continuous map $\rho\colon M \to \Gamma \widehat{\otimes}_A M$ such that obvious co-associativity and co-unity diagrams commute.

Definition 3.13 Let R be a complete Noetherian local ring, and (R^c, Γ) a complete Hopf algebroid over k. An object $M \in \mathrm{FMod}_R^c$ is said to be a complete Γ–precomodule if there is a continuous map

$$\rho\colon M \longrightarrow \Gamma \widehat{\otimes}_R M$$

such that the following two conditions are satisfied:

(1) For any open R–submodule M_λ of M, there is an open R–submodule M_μ such that the map $M \xrightarrow{\rho} \Gamma\widehat{\otimes}_R M \to \Gamma\widehat{\otimes}_R M/M_\lambda$ factors through M/M_μ. When the above condition is satisfied, ρ induces a continuous map $\rho^c\colon M^c \to \Gamma\widehat{\otimes}_{R^c} M^c$.

(2) The continuous map ρ^c makes M^c a complete Γ–comodule.

Furthermore, if M is a complete Hausdorff commutative R–algebra and ρ^c is a map of complete R^c–algebras, then M is said to be a complete Γ–precomodule algebra.

For Z finite, the coaction map $P(n)^*(Z) \to P(n)_*(P(n)) \otimes_{P(n)_*} P(n)^*(Z)$ induces a natural continuous map

$$h^*(Z) \longrightarrow \Gamma(h)\widehat{\otimes}_{h_*} h^*(Z).$$

Proposition 3.14 Let $h^*(-)$ be an even-periodic Landweber exact theory over $P(n)$ such that h_0 is a complete Noetherian local ring. Then $h^*(Z)$ has a natural $\Gamma(h)$–precomodule structure for finite Z. Furthermore, if Z is a finite CW-complex, then $h^*(Z)$ is a $\Gamma(h)$–precomodule algebra.

Proof Since $h^*(Z)$ is discrete if Z is finite, the condition (1) is trivial. Actually, $h^*(Z)$ is a $h_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} h_*$–comodule. Hence $h^*(Z)^c = h_*^c \widehat{\otimes}_{h_*} h^*(Z)$ is $\Gamma(h)$–comodule. If Z is a finite CW-complex. it is easy to see that $h^*(Z)$ is a $\Gamma(h)$–precomodule algebra. □

4 Complete Hopf algebroid $C(G, R_*^c)$

4.1 The Hopf algebroid structure of $C(G, R_*^c)$

Let k be a commutative ring and R_* an even-periodic graded commutative k–algebra such that the degree–0 subring R_0 is a complete local ring with maximal ideal $\mathfrak{m}_0$. We denote by R_*^c a graded topological ring R_* with $\mathfrak{m}$–adic topology, where $\mathfrak{m} = \mathfrak{m}_0 R$. Let G be a profinite group, which continuously acts on R_*^c as k–algebra automorphisms from the right. Let $C = C(G, R_*^c)$ be the set of all continuous maps from G to R_*^c. Then C is an even-periodic commutative ring from the ring structure on R_*^c. It is known that the pair (R_*^c, C) is a graded complete Hopf algebroid over k. In this section we describe the structure of (R_*^c, C) (cf [9, Section 6.3]).

First, note that there is an isomorphism of commutative rings

$$C = C(G, R_*^c) \cong \varprojlim_i C(G, R_*/\mathfrak{m}^i),$$

where $C(G, R_*/\mathfrak{m}^i)$ is the ring of all continuous map from G to the discrete ring $R_*/\mathfrak{m}^i$. We give the inverse limit topology to C, where $C(G, R_*/\mathfrak{m}^i)$ is discrete. The projection $R^c_* \times G \to R^c_*$ gives a continuous ring homomorphism $\eta_R\colon R^c_* \to C$. By the ring homomorphism $k \to R^c_* \stackrel{\eta_R}{\to} C$, we regard C as a commutative k–algebra. The action $R^c_* \times G \to R^c_*$ gives a continuous ring homomorphism $\eta_R\colon R^c_* \to C$, which is a k–algebra homomorphism.

Let $C(G \times G, R^c_*)$ be the ring of all continuous maps from $G \times G$ to R^c_*. Then $C(G \times G, R^c_*)$ is a complete commutative k–algebra as in C.

Let G be a profinite group. We denote by $C(G, M)$ the set of all continuous maps from G to $M \in \mathrm{FMod}^c_k$. Then it can be given a k–module structure on $C(G, M)$ from the k–module structure on M. There is an isomorphism of k–modules

$$C(G, M) \cong \varprojlim_N \varinjlim_U F(G/U, M/N),$$

where $F(G/U, M/N)$ is the set of all maps from G/U to M/N, N ranges over all open submodules of M, and U ranges over all open normal subgroup of G. We regard $C(G, M)$ as an object in FMod^c_k by inverse limit topology.

Lemma 4.1 *For a profinite group G and $M \in \mathrm{FMod}^c_k$, there is a natural isomorphism in FMod^c_k:*

$$C(G, k)\widehat{\otimes}_k M \cong C(G, M).$$

Proof We have an isomorphism

$$C(G, k)\widehat{\otimes} M \cong \varprojlim \varinjlim F(G/U, k) \otimes M/N.$$

Since G/U is a finite set, $F(G/U, k) \otimes M/N \cong F(G/U, M/N)$. Hence we see that $C(G, k)\widehat{\otimes} M \cong \varprojlim\varinjlim F(G/U, M/N) \cong C(G, M)$. □

Let $m\colon C \times C \to C(G \times G, R^c_*)$ be a map given by $m(\alpha, \beta)(g_1, g_2) = \alpha(g_1)^{g_2}\beta(g_2)$ for $\alpha, \beta \in C, g_1, g_2 \in G$. The map m induces an isomorphism of complete commutative k–algebras:

$$C \mathop{\widehat{\otimes}}_{R^c_*} C \xrightarrow{\cong} C(G \times G, R^c_*).$$

We define a map ψ by

$$\psi\colon C \stackrel{\widetilde{\psi}}{\to} C(G \times G, R^c_*) \cong C \mathop{\widehat{\otimes}}_{R^c_*} C,$$

where $\widetilde{\psi}$ is the map induced by the multiplication $G \times G \to G$. Then we can check that ψ is a continuous k–algebra homomorphism.

Let $\chi\colon C \to C$ be the map given by $\chi(\alpha)(g) = \alpha(g^{-1})^g$ for $\alpha \in C, g \in G$. Then it is easy to see that χ is a continuous k–algebra automorphism. Let $\varepsilon\colon C \to R_*^c$ be the map given by $\varepsilon(\alpha) = \alpha(e)$ for $\alpha \in C$, where e is the identity element of G. Then it is also easy to see that ε is a continuous k–algebra homomorphism.

Theorem 4.2 (cf Hovey [9, Section 6.3]) *The pair (R_*^c, C) with $(\eta_R, \eta_L, \psi, \chi, \varepsilon)$ is a graded complete Hopf algebroid over k.*

Remark 4.3 Let $C(G,k)$ be the ring of all continuous maps from G to k. There is an isomorphism $C \cong C(G,k)\widehat{\otimes}_k R_*^c$ of complete k–algebras by Lemma 4.1, and $C(G,k)$ is a Hopf algebra over k by Theorem 4.2. The right action of G on R_*^c gives R_*^c a graded (left) $C(G,k)$–comodule algebra structure. Let $\rho\colon R_*^c \to C(G,k)\widehat{\otimes}_k R_*^c$ be the comodule algebra structure map. In this situation we can construct a split Hopf algebroid $(R_*^c, C(G,k)\widehat{\otimes}_k R_*^c)$. In fact, $\rho = \eta_L$ under the above isomorphism, and the graded complete Hopf algebroid (R_*^c, C) is isomorphic to $(R_*^c, C(G,k)\widehat{\otimes}_k R_*^c)$.

4.2 Twisted modules

In this subsection we show that there is an equivalence of symmetric monoidal categories between the category of complete C–comodules and the category of complete twisted R_*^c–G–modules.

Definition 4.4 A complete Hausdorff filtered R_*^c–module M is said to be a complete twisted (right) R_*^c–G–module if G acts on M continuously (from the right) such that $(am)g = a^g \cdot (m)g$ for all $m \in M, a \in R_*, g \in G$.

Remark 4.5 *The category of complete twisted R_*^c–G–modules is a symmetric monoidal category under complete tensor product $\widehat{\otimes}_{R_*^c}$ and unit object R_*^c.*

Definition 4.6 A complete Hausdorff filtered R_*^c–module M is said to be a complete (left) C–comodule if there is a continuous left R_*^c–module homomorphism $\rho_M\colon M \to C \widehat{\otimes}_{R_*^c} M$, which makes co-associativity and co-unity diagrams commute.

Remark 4.7 *The category of complete C–comodules is a symmetric monoidal category under complete tensor product $\widehat{\otimes}_{R_*^c}$ and unit object R_*^c.*

Lemma 4.8 *For a complete (left) C–comodule M, there is a natural complete twisted (right) R^c_*–G–module structure on M.*

Proof We denote by $\mathrm{ev}(g)\colon C \to R^c_*$ the evaluation map at $g \in G$. Then the map

$$M \longrightarrow C \underset{R^c_*}{\widehat{\otimes}} M \xrightarrow{\mathrm{ev}(g)\otimes 1} R^c_* \underset{R^c_*}{\widehat{\otimes}} M \cong M$$

defines a twisted R_*–G–module structure on M. Hence it is sufficient to show that the action map $M \times G \to M$ is continuous.

Let N be an open R–submodule of M. Then $\mathfrak{m}^i M \subset N$ for some i, since M is an R^c_*–module. In this case, $C(G, R/\mathfrak{m}^i) \otimes_R M/N \cong C(G, M/N)$. Then there is an open R–submodule N', which makes the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & C\widehat{\otimes}M \arrow[d] \\ M/N' \arrow[r] & C(G, M/N). \end{tikzcd}$$

We note that for any element of M/N' the image under the bottom arrow factors through $F(G/U, M/N)$ for some open normal subgroup U of G. The above commutative diagram gives us the following commutative diagram:

$$\begin{tikzcd} M \times G \arrow[r] \arrow[d] & M \arrow[d] \\ M/N' \times G \arrow[r] & M/N. \end{tikzcd}$$

By the above remark, the bottom arrow is continuous. Hence top arrow is also continuous. This completes the proof. □

Lemma 4.9 *Let M be an R^c_*–module and G a profinite group. Then there is an isomorphism of left R^c_*–modules*

$$C \underset{R^c_*}{\widehat{\otimes}} M \cong C(G, M),$$

where the left R^c_*–module structure on $C\widehat{\otimes}M$ comes from η_L of C, and the left R^c_*–module structure on $C(G, M)$ is given by $(r \cdot f)(g) = r^g \cdot f(g)$ for $r \in R$, $f \in C(G, M)$, $g \in G$.

Proof There is an isomorphism $C \cong C(G,k)\widehat{\otimes}_k R^c_*$ by Lemma 4.1. Hence $C\widehat{\otimes}_{R^c_*} M$ is isomorphic to $C(G,k)\widehat{\otimes}_k M$. Then we have

$$\begin{aligned} C(G,k)\widehat{\otimes}_k M &\cong \varprojlim_N \varinjlim_U F(G/U,k)\otimes_k M/N \\ &\cong \varprojlim_N \varinjlim_U F(G/U, M/N) \\ &\cong \varprojlim_N C(G, M/N) \\ &\cong C(G,M). \end{aligned}$$

It is easy to check that this isomorphism respects the left R^c_*–module structures. □

Lemma 4.10 *For a complete twisted (right) R^c_*–G–module M, there is a natural complete (left) C–comodule structure on M.*

Proof The G–module structure map $M\times G\to M$ gives a map $M\to C(G,M)$. By Lemma 4.9, we obtain a map $M\to C\widehat{\otimes} M$. If this map is continuous, it is easy to check that it defines a complete C–comodule structure on M. Hence it is sufficient to show that $M\to C(G,M)$ is continuous. For any open submodule N of M and $g\in G$, there are open submodule N_g of M and an open neighborhood U_g of g such that $N_g\cdot U_g\subset N$. Since G is compact, $G=U_{g_1}\cup\cdots\cup U_{g_n}$. Let N' be an open submodule such that $N'\subset N_{g_1}\cap\cdots\cap N_{g_n}$. Then $N'\cdot G\subset N$. Hence the map $M\times G\to M\to M/N$ factors through $M/N'\times G$. This implies that the map $M\to C(G,M)\to C(G,M/N)$ factors through M/N'. Hence $M\to C(G,M)$ is continuous. This completes the proof. □

Theorem 4.11 *There is an equivalence of symmetric monoidal categories between the category of complete twisted (right) R^c_*–G–modules and the category of complete (left) C–comodules.*

Proof By Lemma 4.8 and Lemma 4.10, there is an equivalence of categories between the category of complete C–comodules and the category of complete twisted R^c_*–G–modules. It is easy to check that this equivalence respects the symmetric monoidal structures. □

4.3 Remark on twisted modules

In this subsection we let $G=\mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)\ltimes S_n$. Usually, G is called the n–th extended Morava stabilizer group, and it is important to study the category of complete twisted $\mathbf{F}_{p^n}$–G–modules. In this subsection we compare the category of complete twisted $\mathbf{F}_{p^n}$–G–modules and the category of complete twisted $\mathbf{F}$–G_n–modules.

There is an exact sequence:

$$1 \to \mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n}) \longrightarrow G_n \longrightarrow G \to 1.$$

Hence, for a twisted $\mathbf{F}$–G_n–module M, the submodule $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ invariant over $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})$ is a twisted $\mathbf{F}_{p^n}$–G–module. Conversely, for a twisted $\mathbf{F}_{p^n}$–G–module N, we can give $\mathbf{F}\otimes_{\mathbf{F}_{p^n}} N$ an obvious twisted $\mathbf{F}$–G_n–module structure.

Lemma 4.12 *For a finite dimensional twisted $\mathbf{F}$–G_n–module M, $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ is a finite dimensional twisted $\mathbf{F}_{p^n}$–G–module, and $\mathbf{F}\otimes_{\mathbf{F}_{p^n}} M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ is isomorphic to M as a twisted $\mathbf{F}$–G_n–module.*

Proof Let $m = \dim_{\mathbf{F}} M$. We obtain an isomorphism $M \cong \mathbf{F}\otimes_{\mathbf{F}_{p^n}} M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ as twisted $\mathbf{F}$–$\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})$–modules, since we have $H^1(\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n}); \mathrm{GL}_m(\mathbf{F})) = \{1\}$ (cf Serre [20, Proposition X.1.3]). By the above exact sequence, $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ is a twisted $\mathbf{F}_{p^n}$–G–module, and we see that this is an isomorphism of twisted $\mathbf{F}$–G_n–modules. □

Remark 4.13 *Since the cardinality of M is finite, the action of G_n on M and the action of G on $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ are continuous by Lemma 4.20.*

Let M be a profinite twisted $\mathbf{F}$–G_n–module. By Corollary 3.11, we can take a fundamental neighborhood system $\{F^\lambda M\}$ at 0 consisting of open $\mathbf{F}$–G_n–submodules. Then

$$M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})} \cong \varprojlim (M/F^\lambda M)^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})},$$

and $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ is a profinite twisted $\mathbf{F}_{p^n}$–G–module with filtration $F^\lambda(M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)})$, where $F^\lambda(M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})})$ is the kernel of the map $M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})} \to (M/F^\lambda M)^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$. Conversely, for a profinite twisted $\mathbf{F}_{p^n}$–G–module N, we can give $\mathbf{F}\widehat{\otimes}_{\mathbf{F}_{p^n}} N$ an obvious profinite twisted $\mathbf{F}$–G_n–module structure. By Lemma 4.12 and Remark 4.13, we obtain the following proposition.

Proposition 4.14 *The functor $M \mapsto M^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{p^n})}$ gives an equivalence of symmetric monoidal categories between the category of profinite twisted $\mathbf{F}$–G–modules and the category of profinite twisted $\mathbf{F}_{p^n}$–G–modules. The quasi-inverse of this functor is given by $N \mapsto \mathbf{F}\widehat{\otimes}_{\mathbf{F}_{p^n}} N$.*

By Proposition 4.14, there is no essential difference between profinite twisted $\mathbf{F}$–G_n–modules and profinite twisted $\mathbf{F}_{p^n}$–G–modules.

4.4 Reformulation

In this section we reformulate the results in [22]. Set

$$\begin{aligned} C_{E_*} &= C(G_{n+1}, E_*^c), \\ C_{K_*} &= C(G_n, K_*). \end{aligned}$$

Then (E_*^c, C_{E_*}) and (K_*, C_{K_*}) are graded complete Hopf algebroids over $\mathbf{F}_p$ by Theorem 4.2.

Let M be a profinite C_{E_*}–precomodule. Then M^c is a complete twisted E_*^c–G_{n+1}–module by Lemma 4.8. Note that $M^c = M$ as an abstract E_*–module. Since the G_{n+1} action on L is compatible with the G_{n+1}–action on M, G_{n+1} acts on $M \widehat{\otimes}_{E_*} L_*$, where $L_* = L[u^{\pm 1}]$ is regarded as a discrete module. We define $\mathcal{F}(M)$ to be the S_{n+1}–invariant submodule of $L_* \widehat{\otimes}_{E_*} M$:

$$\mathcal{F}(M) = H^0(S_{n+1}; L_* \underset{E_*}{\widehat{\otimes}} M).$$

We regard $K_* = \mathbf{F}[w^{\pm 1}]$ as a subring of $L_* = L[u^{\pm 1}]$ by $w = \Phi_0^{-1} u$. The following lemma was proved in [22, Lemma 4.2].

Lemma 4.15 $H^0(S_{n+1}; L_*) = K_*$.

Note that E_* with discrete topology is a C_{E_*}–precomodule.

Corollary 4.16 $\mathcal{F}(E_*) = K_*$.

Lemma 4.17 *Let M_L be a finite dimensional twisted L_*–G_{n+1}–module. Then the dimension of $H^0(S_{n+1}; M_L)$ over K_* is finite.*

Proof We prove the lemma by induction on the dimension of M_L. Suppose that $\dim M_L = 1$. If $H^0(S_{n+1}; M_L) = 0$, then it is okay. Suppose that $H^0(S_{n+1}; M_L) \neq 0$. Take a nonzero $a \in H^0(S_{n+1}; M_L)$. Then M_L is isomorphic to L_* as a twisted L_*–G_{n+1}–module. Hence this case follows from Lemma 4.15.

Suppose that $\dim M_L = n > 1$, and that the lemma is true for M'_L of dimension $< n$. If $H^0(S_{n+1}; M_L) = 0$, then it is okay. Suppose that $H^0(S_{n+1}; M_L) \neq 0$. Let $a \in H^0(S_{n+1}; M_L)$ be a nonzero element, and N_L the L_*–submodule generated by a. There is an exact sequence of K_*–modules:

$$0 \to H^0(S_{n+1}; N_L) \longrightarrow H^0(S_{n+1}; M_L) \longrightarrow H^0(S_{n+1}; M_L/N_L).$$

By hypothesis of the induction, the dimension of $H^0(S_{n+1}; M_L/N_L)$ is finite and $\dim H^0(S_{n+1}; N_L) = 1$. Hence we obtain that $\dim H^0(S_{n+1}; M_L)$ is finite. This completes the proof. □

Remark 4.18 More precisely, we see that $\dim_{K_*} H^0(S_{n+1}; N_L) \le \dim_{L_*} N_L$ by the proof of Lemma 4.17.

Corollary 4.19 *If M is a finitely generated discrete C_{E_*}–precomodule, then the dimension of $\mathcal{F}(M)$ over K_* is finite.*

The following lemma is fundamental on the topology of G_n.

Lemma 4.20 (cf Hovey [9, Theorem A.2]) *A subgroup of G_n is open if and only if its index in G_n is finite.*

Let $h \in G_n$ and σ the image of the projection $G_n \to \mathrm{Gal}$. For $g \in S_{n+1}$, $g\sigma = \sigma g^\sigma$ in G_{n+1}, and $gh = hg^\sigma$ in $\mathcal{G}$. Hence the following diagram commutes for all $g \in S_{n+1}$:

$$\begin{tikzcd} L_* \underset{E_*}{\widehat{\otimes}} M \arrow[r, "\sigma\otimes h"] \arrow[d, "g\otimes g"'] & L_* \underset{E_*}{\widehat{\otimes}} M \arrow[d, "g^\sigma\otimes g^\sigma"] \\ L_* \underset{E_*}{\widehat{\otimes}} M \arrow[r, "\sigma\otimes h"] & L_* \underset{E_*}{\widehat{\otimes}} M. \end{tikzcd}$$

This diagram induces an action of G_n on $\mathcal{F}(M)$, and it is easy to check that $\mathcal{F}(M)$ is a twisted K_*–G_n–module.

Lemma 4.21 *If M is a finitely generated discrete C_{E_*}–precomodule, then $\mathcal{F}(M)$ has a natural complete twisted K_*–G_n–module structure.*

Proof By Corollary 4.19, $\mathcal{F}(M)$ is a twisted K_*–G_n–module of finite dimension. Then the action of G_n is continuous by Lemma 4.20. □

If M is a complete C_{E_*}–precomodule, then there is a fundamental system $\{F^\lambda M\}$ of (open) neighborhoods at 0 consisting of E_*–G_{n+1}–submodules by Corollary 3.11. Hence there is an isomorphism

$$\mathcal{F}(M) \cong \varprojlim_{\lambda} \mathcal{F}(M/F^\lambda M).$$

We give $\mathcal{F}(M)$ the inverse limit topology. Note that this topology is independent of a choice of fundamental system of neighborhood at 0. Furthermore, if M is profinite, then $\mathcal{F}(M)$ is also profinite by Corollary 4.19, and complete twisted K_*–G_n–module by Lemma 4.21. Hence we obtain the following proposition.

Proposition 4.22 *$\mathcal{F}$ defines a symmetric monoidal functor from the category of profinite C_{E_*}–precomodules to the category of profinite C_{K_*}–comodules.*

Proof Since the construction of twisted K_*–G_n–module structure on $\mathcal{F}(M)$ is natural, we see that $\mathcal{F}$ defines a functor from the category of profinite C_{E_*}–precomodules to the category of profinite twisted K_*–G_n–modules, which is equivalent to the category of profinite C_{K_*}–comodules by Theorem 4.11. It is easy to check that the functor $\mathcal{F}$ respects the monoidal structures. □

Definition 4.23 Let $\mathcal{C}_{\mathbb{E}}^f$ (resp. $\mathcal{C}_{\mathbb{K}}^f$) be the category of finitely generated discrete C_{E_*}–precomodules (resp. C_{K_*}-(pre)comodules). Define $\mathcal{C}_{\mathbb{E}}$ (resp. $\mathcal{C}_{\mathbb{K}}$) to be the procategory of $\mathcal{C}_{\mathbb{E}}^f$ (resp. $\mathcal{C}_{\mathbb{K}}^f$), that is, the category of (small) cofiltered system of objects in $\mathcal{C}_{\mathbb{E}}^f$ (resp. $\mathcal{C}_{\mathbb{K}}^f$). These are symmetric monoidal categories.

For a finite Z, $E^*(Z)$ is a natural finitely generated discrete C_{E_*}–precomodule by Proposition 3.14. Furthermore, if Z is a finite CW-complex, then $E^*(Z)$ is a C_{E_*}–precomodule algebra.

Definition 4.24 We define $\mathbb{E}^*(X) \in \mathcal{C}_{\mathbb{E}}$ to be the system

$$\{E^*(Z)\}_{Z\in\Lambda(X)}$$

indexed by $\Lambda(X)$. We also define $\mathbb{K}^*(X) \in \mathcal{C}_{\mathbb{K}}$ by the same manner.

Then we have

$$\varprojlim \mathbb{E}^*(X) \cong E^*(X),$$

$$\varprojlim \mathbb{K}^*(X) \cong K^*(X),$$

as profinite $C_{\mathbb{E}_*}$–precomodules and profinite $C_{\mathbb{K}_*}$-(pre)comodules, respectively. By Corollary 4.19 and Lemma 4.21, we can extend the functor $\mathcal{F}$ from $\mathcal{C}_{\mathbb{E}}$ to $\mathcal{C}_{\mathbb{K}}$ by obvious way:

$$\mathcal{F}\colon \mathcal{C}_{\mathbb{E}} \longrightarrow \mathcal{C}_{\mathbb{K}}.$$

Note that $\mathcal{F}$ is a monoidal functor.

By [22, Theorem 4.1], the generalized Chern character (5)

$$\Theta\colon E^*(X) \longrightarrow L^*(X)$$

induces a natural isomorphism of twisted L_*–$\mathcal{G}$–modules:

$$L_* \otimes_{E_*} E^*(Z) \xrightarrow{\cong} L_* \otimes_{K_*} K^*(Z)$$

for finite Z. The following theorem is a reformulation of [22, Corollary 4.3].

Theorem 4.25 *For any spectrum X, the generalized Chern character Θ induces a natural isomorphism in $\mathcal{C}_{\mathbb{K}}$:*

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

If X is a space, then this is an isomorphism of cofiltered systems of finite C_{K_*}–comodule algebras.

5 Milnor operations

5.1 Complete co-operation ring

In this section we let $A = E_{n+k}/I_n$ for some $k \geq 0$. Hence $A = E$ if $k = 1$ and $A = K$ if $k = 0$. The coefficient ring $A_* = \mathbf{F}[\![u_n, \ldots, u_{n+k-1}]\!][u_A]$, $|u_A| = -2$ is a graded complete Noetherian local ring with maximal ideal $\mathfrak{m}_A = (u_n, \ldots, u_{n+k-1})$. Put $G_A = G_{n+k}$ and $C_{A_*} = C(G_A, A_*)$. We denote by $A \widehat{\wedge} A$ the $K(n+k)$–localization of $A \wedge A$. Since A is a commutative ring spectrum, So is $A \widehat{\wedge} A$. We define a graded commutative ring $A_*^\vee(A)$ to be $\pi_*(A \widehat{\wedge} A)$.

Since A is Landweber exact over $P(n)$, there is an isomorphism of commutative $\mathbf{F}_p$–algebras:

$$\pi_*(A \wedge A) \cong A_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} A_*. \tag{6}$$

Lemma 5.1 *There is an isomorphism of graded commutative $\mathbf{F}_p$–algebras*

$$A_*^\vee(A) \cong A_*^c \widehat{\otimes}_{P(n)_*} P(n)_*(P(n)) \widehat{\otimes}_{P(n)_*} A_*^c,$$

where A_*^c is a graded topological ring A_* with $\mathfrak{m}_A$–adic topology.

Proof By [10, Proposition 7.10(e)], we see that $A_*^\vee(A)$ is the I_{n+k}–adic completion of $\pi_*(A \wedge A)$. Hence the lemma follows from the isomorphism (6). □

By Lemma 5.1, we see that $A_*^\vee(A)$ has a graded complete Hopf algebroid structure induced from $P(n)_*(P(n))$. We say that $A_*^\vee(A)$ is the complete co-operation ring of A.

Let $\Lambda_{\mathbf{Z}}$ be the graded commutative algebra over $\mathbf{Z}$ generated by $a_{(i)}$ for $0 \leq i < n$, where the degree of $a_{(i)}$ is $2p^i - 1$. Hence $\Lambda_{\mathbf{Z}}$ is an exterior algebra. For an evenly graded commutative ring R_*, we set $\Lambda_{R_*} = R_* \otimes \Lambda_{\mathbf{Z}}$. There is an isomorphism of commutative $\mathbf{F}_p$–algebras

$$P(n)_*(P(n)) \cong P(n)_*[t_1, t_2, \ldots] \otimes \Lambda_{\mathbf{Z}},$$

where $|t_i| = 2(p^i - 1)$. Let $C_{P(n)_*}$ be the $P(n)_*$–subalgebra of $P(n)_*(P(n))$ generated by $t_1, t_2, \ldots$:

$$C_{P(n)_*} = P(n)_*[t_1, t_2, \ldots].$$

Then it is known that $C_{P(n)_*}$ is a sub-Hopf algebroid of $P(n)_*(P(n))$ (cf Wurgler [24]). Hence we can give $A^c_* \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A^c_*$ the induced graded complete Hopf algebroid structure.

Lemma 5.2 *There is an isomorphism of graded complete Hopf algebroids over* $\mathbf{F}_p$*:*

$$(A^c_*, C_{A_*}) \cong (A^c_*, A^c_* \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A^c_*).$$

Proof We let $D_* = \pi_*(E_{n+k} \widehat{\wedge} E_{n+k})$, where $E_{n+k} \widehat{\wedge} E_{n+k}$ is the $K(n+k)$–localization of $E_{n+k} \wedge E_{n+k}$. Then $D_* \cong E^c_{n+k,*} \widehat{\otimes}_{BP_*} BP_*(BP) \widehat{\otimes}_{BP_*} E^c_{n+k,*}$. Hence $(E^c_{n+k,*}, D_*)$ is a graded complete Hopf algebroid over $\mathbf{Z}_{(p)}$. Furthermore, $(E^c_{n+k,*}, D_*) \cong (E^c_{n+k,*}, C(G_{n+k}, E^c_{n+k,*}))$ [4; 9]. Hence

$$D_*/I_n \cong A^c_* \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A^c_*$$

by Lemma 5.1. By Lemma 4.1, $C(G_{n+k}, E_{n+k,*}) \cong C(G_{n+k}, \mathbf{Z}) \widehat{\otimes} E_{n+k,*}$. This implies that $C(G_{n+k}, E_{n+k,*})/I_n \cong C(G_{n+k}, A_*)$. Hence we have the isomorphism $C_{A_*} \cong A^c_* \widehat{\otimes}_{P(n)_*} C_{P(n)_*} \widehat{\otimes}_{P(n)_*} A^c_*$. We can check that this isomorphism induces the desired isomorphism of Hopf algebroids. □

Corollary 5.3 *There is an isomorphism of graded complete commutative* $\mathbf{F}_p$*–algebras:*

$$A^\vee_*(A) \cong C_{A_*} \widehat{\otimes} \Lambda_{\mathbf{Z}}.$$

Recall that $A^*(X)^c = A^c_* \widehat{\otimes}_{A_*} A^*(X)$. The natural $P(n)_*(P(n))$–comodule structure on $P(n)^*(Z)$ gives a natural $A^\vee_*(A)$–comodule structure on $A^*(Z)^c$, for any finite spectrum Z. By Lemma 4.1 and Corollary 5.3, this induces an $A^\vee_*(A)$–comodule structure on $A^*(X)^c$:

$$\rho\colon A^*(X)^c \longrightarrow A^\vee_*(A) \widehat{\otimes}_{A^c_*} A^*(X)^c.$$

If X is a space, then ρ defines an $A^\vee_*(A)$–comodule algebra structure on $A^*(X)^c$.

In the following of this subsection we describe the comultiplication ψ on $a^A_{(i)}$. For $0 \le i < n$, we set

$$b^A_{(i)} = u_A^{p^i} \otimes a_{(i)}.$$

Then $|b^A_{(i)}| = -1$ for all i. In particular, $u_A = u$ if $A = E$ and $u_A = w$ if $A = K$. We put

$$t_A(X) = \sum_{i\geq 0}{}^{\eta_{L*}F_A}\; t_i^A X^{p^i} \in C_{A_*}[X]$$

and

$$b_A(X) = \sum_{i=0}^{n-1} b^A_{(i)} X^{p^i} \qquad \in A_*^\vee(A)[X],$$

where F_A is the base change of the universal deformation F_{n+k} on $E_{n+k,0}$ to $E_{n+k,0}/I_n$.

The comodule algebra structure map $\rho\colon A^*(\mathcal{X})^c \to A_*^\vee(A)\widehat{\otimes}_{A_*^c} A^*(\mathcal{X})^c$ on x_A is given by the following lemma (cf [2, Section 14]).

Lemma 5.4 $\psi_A(x_A) = t_A(x_A)$.

The comodule algebra structure map $\rho\colon A^*(\mathcal{Y})^c \to A_*^\vee(A)\widehat{\otimes}_{A_*^c} A^*(\mathcal{Y})^c$ on y_A is given by the following lemma (cf [2, Section 14]).

Lemma 5.5 $\rho(y_A) = 1\otimes y_A + b_A(x_A)$.

Let i_l and i_r be the left and right inclusion of $A_*^\vee(A)$ into $A_*^\vee(A)\widehat{\otimes}_{A_*^c} A_*^\vee(A)$. The comultiplication map ψ on $b^A_{(i)}$ is encoded in the following lemma.

Lemma 5.6 $\psi(b_A(X)) \equiv i_r(b_A)(X) + i_l(b_A)(i_r(t_A)(X)) \mod (X^{p^n})$.

Proof This follows from the fact that $(\psi\otimes 1)\rho(y_A) = (1\otimes\rho)\rho(y_A)$ and Lemma 5.4 and Lemma 5.5. □

Lemma 5.7 *Let F be a p–typical formal group law of strict height at least n over an $\mathbf{F}_p$–algebra R. Then for $a_i \in R$ $(0 \leq i < n)$,*

$$\sum_{i=0}^{n-1}{}^F\, a_i X^{p^i} \equiv a_0 X + a_1 X^p + \cdots + a_{n-1} X^{p^{n-1}} \mod (X^{p^n}).$$

Proof This follows from the fact that $F(X,Y) \equiv X + Y \mod (X,Y)^{p^n}$. □

The following theorem describes the structure of graded complete Hopf algebroid $(A_*^c, A_*^\vee(A))$.

Theorem 5.8 *The pair $(A^c_*, A^\vee_*(A))$ is a graded complete Hopf algebroid over $\mathbf{F}_p$. There is an extension of graded complete Hopf algebroids*

$$C_{A_*} \longrightarrow A^\vee_*(A) \longrightarrow \Lambda_{A^c_*},$$

where the algebra $\Lambda_{A^c_*} = A^c_* \otimes \Lambda(b^A_{(0)}, \ldots, b^A_{(n-1)})$ is an exterior Hopf algebra over A^c_* generated by primitive elements $b^A_{(i)}$ for $0 \le i < n$. The comultiplication ψ and the counit ε on $b^A_{(i)}$ for $0 \le i < n$ are given as follows:

$$\begin{aligned} \psi(b^A_{(i)}) &= 1 \otimes b^A_{(i)} + \sum_{j=0}^{i} b^A_{(j)} \otimes (t^A_{i-j})^{p^j}, \\ \varepsilon(b^A_{(i)}) &= 0. \end{aligned}$$

Proof The comultiplication ψ on $b^A_{(i)}$ is obtained by Lemma 5.5 and Lemma 5.7. □

5.2 Exterior algebras Λ_{E_*} and Λ_{K_*}

We can give $\Lambda_{A^c_*}$ a structure of right C_{A_*}–comodule algebra by

$$\rho^{\mathrm{op}}_{C,\Lambda}\colon \Lambda_{A^c_*} \xrightarrow{i_\Lambda} A^\vee_*(A) \xrightarrow{\psi} A^\vee_*(A) \widehat{\otimes}_{A^c_*} A^\vee_*(A) \xrightarrow{\pi_\Lambda \otimes \pi_C} \Lambda_{A^c_*} \widehat{\otimes}_{A^c_*} C_{A_*},$$

where i_Λ is the canonical inclusion, $\pi_C = 1_C \otimes \varepsilon_\Lambda$ and $\pi_\Lambda = \varepsilon_C \otimes 1_\Lambda$. Hence Λ_{A_*} is a profinite right C_{A_*}–precomodule algebra.

Lemma 5.9 $\rho^{\mathrm{op}}_{\Lambda,C}(b_A(X)) \equiv b_A(t_A(X)) \mod (X^{p^n})$.

Proof This follows from Lemma 5.6. □

The left A^c_*–module homomorphism $\mathrm{ev}(g) \circ \chi\colon C_{A_*} \to C_{A_*} \to A^c_*$ defines a right action of G_A on $\Lambda_{A^c_*}$ by

$$\Lambda_{A^c_*} \xrightarrow{\rho^{\mathrm{op}}_{\Lambda,C}} \Lambda_{A^c_*} \widehat{\otimes}_{A^c_*} C_{A_*} \xrightarrow{1 \otimes (\mathrm{ev}(g) \circ \chi)} \Lambda_{A^c_*}.$$

Then $\Lambda_{A^c_*}$ is a twisted A^c_*–G_A–module.

Corollary 5.10 *For $g \in G_A$, $b_A{}^g(X) \equiv b_A(t_A(g)^{-1}(X))$ mod (X^{p^n}).*

Proof This follows from Lemma 5.9 and $t_A(g^{-1})^g(X) = t_A(g)^{-1}(X)$. □

Since Λ_{E_*} is a twisted E_*–G_{n+1}–module, $\Lambda_{L_*} = L_* \otimes_{E_*} \Lambda_{E_*}$ is a twisted L_*–$\mathcal{G}$–module. We define $\widehat{b}(X) = \sum_{i=0}^{n-1} \widehat{b}_{(i)} X^{p^i} \in \Lambda_{L_*}[X]$ by

$$\widehat{b}(X) \equiv b_E(\Phi^{-1}(X)) \mod (X^{p^n}).$$

Lemma 5.11 *For any $g \in G_{n+1}$, $\widehat{b}^{\,g}(X) = \widehat{b}(X)$.*

Proof By definition and Corollary 5.10, $\widehat{b}^{\,g}(X) \equiv b_E \circ t(g)^{-1} \circ (\Phi^{-1})^g(X)$ mod (X^{p^n}). By the diagram (3) in Theorem 2.4, $\Phi^g \circ t(g)(X) = \Phi(X)$. This implies that $t(g)^{-1} \circ (\Phi^g)^{-1}(X) = \Phi^{-1}(X)$. Hence $\widehat{b}^{\,g}(X) \equiv b \circ \Phi^{-1}(X)$ mod (X^{p^n}). □

By Lemma 5.11, we see that the coefficients of $\widehat{b}(X)$ are invariant under the action of G_{n+1}.

Lemma 5.12 $\mathcal{F}(\Lambda_{E_*}) = K_* \otimes \Lambda(\widehat{b}_{(0)}, \dots, \widehat{b}_{(n-1)})$ *as a graded commutative ring.*

Proof We have $\widehat{b}_{(0)}, \dots, \widehat{b}_{n-1} \in \mathcal{F}(\Lambda_{E_*})$. Since $\widehat{b}_{(i)}$ is a linear combination of $b^E_{(i)}, \dots, b^E_{(n-1)}$, we see that $K_* \otimes \Lambda(\widehat{b}_{(0)}, \dots, \widehat{b}_{(n-1)}) \subset \mathcal{F}(\Lambda_{E_*})$. Then the lemma follows from the fact that $\dim_{K_*} \mathcal{F}(\Lambda_{E_*}) \le 2^n$ by Remark 4.18. □

Recall that $t_K(h)(X)$ is the automorphism $t_K(h)\colon H_n \longrightarrow H_n^h = H_n$ corresponding to $h \in G_n$.

Lemma 5.13 *For any $h \in G_n$, we have $\widehat{b}^{\,h}(X) = \widehat{b} \circ t_K(h)^{-1}(X)$.*

Proof By definition and the fact that G_n acts on L as Galois group, we have $\widehat{b}^{\,h}(X) \equiv b \circ (\Phi^{-1})^h(X)$ mod (X^{p^n}). By the diagram (4) in Theorem 2.4, $t_K(h) \circ \Phi(X) = \Phi^h(X)$. This implies that $(\Phi^h)^{-1}(X) = \Phi^{-1} \circ t_K(h)^{-1}(X)$. Hence the congruence $\widehat{b}^{\,h}(X) \equiv b \circ \Phi^{-1} \circ t_K(h)^{-1}(X)$ mod (X^{p^n}) holds. □

Theorem 5.14 *As a C_{K_*}–comodule, $\mathcal{F}(\Lambda_{E_*})$ is isomorphic to Λ_{K_*}.*

Proof The map $\widehat{b}_{(i)} \mapsto b^K_{(i)}$ gives an isomorphism of twisted K_*–G_n–modules by Corollary 5.10, Lemma 5.12 and Lemma 5.13. □

5.3 Milnor operations

Let $A = E_{n+k}/I_n$ for some $k \ge 0$. In this section we study Milnor operations in A. We abbreviate C_{A_*} to C and $\Lambda_{A^c_*}$ to Λ. In this section we discuss in the category of complete Hausdorff filtered A^c_*–modules. We recall that Λ is a Hopf algebra such that $b^A_{(i)}$ is primitive for all i. We take monomials of $b^A_{(i)}$ as a basis of Λ, and denote the dual of $b^A_{(i)}$ by Q^A_i in the dual basis. Then the monomials of Q^A_i form the dual basis. We call Q^A_i the Milnor operations.

Let M be a left Λ–comodule with comodule structure map ρ. Then the Milnor operation Q_i^A defines a A_*^c–module homomorphism as follows:

$$M \xrightarrow{\rho} \Lambda \widehat{\otimes} M \xrightarrow{Q_i^A \otimes 1_M} M.$$

We abbreviate this homomorphism also to Q_i^A. Note that we write the action of Q_i^A from the right: if $\rho(x) = 1 \otimes x + \sum_i a_{(i)}^A \otimes x_i + \cdots$, then $(x)Q_i^A = (-1)^{|x|+1} x_i$. There is a relation in the endomorphism ring of M for any i and j:

$$Q_i^A Q_j^A + Q_j^A Q_i^A = 0. \tag{7}$$

In particular, $Q_i^A Q_i^A = 0$. Conversely, if there are A_*^c–module homomorphisms Q_i^A for $0 \le i < n$ such that (7) holds, then we can construct a Λ–comodule structure on M, and this construction gives an equivalence of categories.

The category of complete Λ–comodules is symmetric monoidal under complete tensor product $\widehat{\otimes}_{A_*^c}$ and unit object A_*^c.

Lemma 5.15 *Let M and N be complete Λ–comodules. For any $x \in M$ and $y \in N$, $(x \otimes y)Q_i^A = x \otimes (y)Q_i^A + (-1)^{|y|}(x)Q_i^A \otimes y$ in $M \widehat{\otimes} N$.*

Proof Let $\rho_M(x) = 1 \otimes x + \sum_i a_{(i)}^A \otimes x_i + \cdots$ with $x_i = (-1)^{|x|+1}(x)Q_i^A$, and let $\rho_N(y) = 1 \otimes y + \sum_i a_{(i)}^A \otimes y_i + \cdots$ with $y_i = (-1)^{|y|+1}(y)Q_i^A$. Then

$$\rho_{M \widehat{\otimes} N}(x \otimes y) = 1 \otimes x \otimes y + (-1)^{|x|} \sum_i a_{(i)}^A \otimes x \otimes y_i + \sum_i a_{(i)}^A \otimes x_i \otimes y + \cdots .$$

Hence we have $(x \otimes y)Q_i^A = (-1)^{|x|+|y|+1}((-1)^{|x|} x \otimes y_i + x_i \otimes y)$, which equals $x \otimes (y)Q_i^A + (-1)^{|y|}(x)Q_i^A \otimes y$. □

We say that a natural endomorphism Q of complete A_*^c–modules is a derivation of odd degree with respect to exterior products if $(x \otimes y)Q = x \otimes (y)Q + (-1)^{|y|}(x)Q \otimes y$ for any $x \in M$ and $y \in N$. Hence the Milnor operations Q_i^A is a derivation of odd degree with respect to exterior products.

Let Λ^* be the dual module of Λ: $\Lambda^* = \mathrm{Hom}_{A_*^c}(\Lambda, A_*^c)$. Then Λ^* is also a Hopf algebra over A_*^c, and $\Lambda^* \cong A_*^c \otimes \Lambda(Q_0^A, \ldots, Q_{n-1}^A)$ such that Q_i^A are primitive for all i. Recall that Λ is a twisted A_*^c–G_A–module. We can also define a twisted A_*^c–G_A–module structure on Λ^* by

$$(\lambda)(\theta^g) = ((\lambda \cdot g^{-1})\theta)g,$$

for $\theta \in \Lambda^*, g \in G_A, \lambda \in \Lambda$.

Lemma 5.16 *For $g \in G_A$,*

$$(Q_i^A)^g = \sum_{j=i}^{n-1} t_{j-i}^A(g)^{p^i} Q_j^A.$$

Proof This follows from Corollary 5.10. □

Let M be a profinite $A_*^\vee(A)$–comodule. Then M is a twisted A_*^c–G_A–module and Λ–comodule. The following proposition gives us an interaction of the actions of G_A and Q_i^A on M.

Lemma 5.17 *Let M be a profinite $A_*^\vee(A)$–comodule. For $x \in M$ and $g \in G_A$,*

$$((x)Q_i^A)g = ((x)g)(Q_i^A)^g.$$

Proof By Lemma 5.16, we see that the map

$$\theta_1\colon A_*^\vee(A) \xrightarrow{\psi} A_*^\vee(A) \widehat{\otimes} A_*^\vee(A) \xrightarrow{(Q_i^A \circ \pi_\Lambda)\otimes 1} A_*^\vee(A) \xrightarrow{\mathrm{ev}(g)\circ \pi_C} A_*^c$$

is equal to the map

$$\theta_2\colon A_*^\vee(A) \xrightarrow{\psi} A_*^\vee(A) \widehat{\otimes} A_*^\vee(A) \xrightarrow{(\mathrm{ev}(g)\circ \pi_C)\otimes 1} A_*^\vee(A) \xrightarrow{(Q_i^A)^g \circ \pi_\Lambda} A_*^c.$$

Hence $((x)Q_i^A)g = (\theta_1 \circ \rho)(x) = (\theta_2 \circ \rho)(x) = ((x)g)(Q_i^A)^g$. □

These are all relations on the $A_*^\vee(A)$–comodule M between the G_A–action and the Λ^*–action. We give interpretation of these relations in terms of comodule structures in Section 5.4.

In the following lemma we show that a derivation of odd degree with respect to exterior products in the category of stable cohomology operations of $K^*(-)$ is characterized by the action on $y_K \in K^*(\mathcal{Y})$.

Lemma 5.18 *Let Q be an odd degree stable cohomology operation of $K^*(-)$. Suppose that Q is a derivation with respect to exterior product. Then Q is characterized by the action on $y_K \in K^1(\mathcal{Y})$.*

Proof A stable cohomology operation $Q \in K^*(K)$ is a derivation if and only if Q is primitive in $K^*(K)$. Since $K_*(K)$ is free over K_*, the primitive submodule $P(K^*(K))$ is the dual of the indecomposable quotient $Q(K_*(K))$ of the cooperation ring $K_*(K)$. Recall the isomorphism $K_*(K) \cong C_{K_*} \otimes_{K_*} \Lambda_{K_*}$. Then we have $Q(K_*(K)) \cong Q(C_{K_*}) \oplus Q(\Lambda_{K_*}) \cong Q(\Lambda_{K_*})$, and $Q(\Lambda_{K_*})$ is isomorphic to $K_*\{a^K_{(0)}, \dots, a^K_{(n-1)}\}$. Hence Q is a linear combination $\sum_{i=0}^{n-1} q_i Q_i^K$ with $q_i \in K_*$. Since we know that $Q_i^K(y_K) = {x_K}^{p^i}$, we have $Q(y_K) = \sum_{i=0}^{n-1} q_i {x_K}^{p^i}$ in $K^*(\mathcal{Y})$. Since ${x_K}^{p^i}$ for $0 \le i < n-1$ are linearly independent, this uniquely determines q_i. Hence Q is characterized by the action of y_K. $\square$

5.4 Complete $A_*^\vee(A)$–comodules

Let $A = E_{n+k}/I_n$ for some $k \ge 0$. In this section we give a description of complete $A_*^\vee(A)$–comodules in terms of C_{A_*}–comodule structure and $\Lambda_{A_*^c}$–comodule structure. In this section we discuss in the category of complete Hausdorff filtered A_*^c–modules, and abbreviate C_{A_*} to C and $\Lambda_{A_*^c}$ to Λ.

Let M be a complete $A_*^\vee(A)$–comodule with $\rho_M\colon M \to A_*^\vee(A)\widehat{\otimes} M$. By Theorem 5.8, $A_*^\vee(A) \cong C \widehat{\otimes} \Lambda$ as an $\mathbf{F}_p$–algebra, and there is an extension of complete Hopf algebroids:

$$C \longrightarrow A_*^\vee(A) \xrightarrow{\pi_\Lambda} \Lambda. \tag{8}$$

Hence M is a Λ–comodule by

$$\rho_{\Lambda,M} \xrightarrow{\rho_M} A_*^\vee(A)\widehat{\otimes} M \xrightarrow{\pi_\Lambda} \Lambda \widehat{\otimes} M.$$

The counit of Λ induces a morphism of Hopf algebroid $\pi_C\colon A_*^\vee(A) \to C$, which is a splitting of the above extension (8). Then M is also a C–comodule by

$$\rho_{C,M}\colon M \xrightarrow{\rho_M} A_*^\vee(A)\widehat{\otimes} M \xrightarrow{\pi_C} C \widehat{\otimes} M.$$

We recall that Λ is a (left) C–comodule algebra by the structure map

$$\rho_{C,\Lambda}\colon \Lambda \xrightarrow{i_\Lambda} C\widehat{\otimes}\Lambda \xrightarrow{\psi} (C\widehat{\otimes}\Lambda)\widehat{\otimes}(C\widehat{\otimes}\Lambda) \xrightarrow{\pi_\Lambda\otimes\pi_C} \Lambda\widehat{\otimes} C \xrightarrow{\tau} C\widehat{\otimes}\Lambda,$$

where i_Λ is the canonical inclusion and τ is given by $\lambda\otimes c \mapsto \chi(c)\otimes\lambda$. For a complete C–comodule M, we denote by $\rho_{C,\Lambda\widehat{\otimes} M}$ the C–comodule structure map of the tensor product of Λ and M.

Lemma 5.19 *Let M be a complete $A_*^\vee(A)$–comodule. Then $\rho_{\Lambda,M}$ is a morphism of C–comodules. In other words, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\rho_{\Lambda,M}"] \arrow[d, "\rho_{C,M}"'] & \Lambda\widehat{\otimes} M \arrow[d, "\rho_{C,\Lambda\widehat{\otimes} M}"] \\ C\widehat{\otimes} M \arrow[r, "1_C\otimes\rho_{\Lambda,M}"] & C\widehat{\otimes}\Lambda\widehat{\otimes} M. \end{tikzcd}$$

Furthermore, $\rho_{C,\Lambda\widehat{\otimes}M}\circ\rho_{\Lambda,M} = (1_C\otimes\rho_{\Lambda,M})\circ\rho_{C,M}$ *is the* $A_*^\vee(A)$*–comodule structure map* ρ_M*.*

Proof Let $f=(\pi_\Lambda\otimes\pi_C)\circ\psi\colon A_*^\vee(A)\to\Lambda\widehat{\otimes}C$. By the co-associativity of $A_*^\vee(A)$–comodule M, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\rho_{\Lambda,M}"] \arrow[d, "\rho_M"'] & \Lambda\widehat{\otimes}M \arrow[d, "1_\Lambda\otimes\rho_{C,M}"] \\ C\widehat{\otimes}\Lambda\widehat{\otimes}M \arrow[r, "f\otimes 1_M"] & \Lambda\widehat{\otimes}C\widehat{\otimes}M. \end{tikzcd}$$

Let $g=(1_C\otimes 1_\Lambda\otimes\varepsilon_C)\circ\rho_{C,\Lambda\widehat{\otimes}C}\colon\Lambda\widehat{\otimes}C\to C\widehat{\otimes}\Lambda$. Then we can check that $g\circ f$ is the identity map of $C\widehat{\otimes}\Lambda$. Since $(g\otimes 1_M)\circ(1_\Lambda\otimes\rho_{C,M})=\rho_{C,\Lambda\widehat{\otimes}M}$, we obtain that $\rho_M=\rho_{C,\Lambda\widehat{\otimes}M}\circ\rho_{\Lambda,M}$.

Let $h=(\pi_C\otimes\pi_\Lambda)\circ\psi\colon A_*^\vee(A)\to A_*^\vee(A)$. By the coassociativity of $A_*^\vee(A)$–comodule M, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\rho_{C,M}"] \arrow[d, "\rho_M"'] & C\widehat{\otimes}M \arrow[d, "1_C\otimes\rho_{\Lambda,M}"] \\ C\widehat{\otimes}\Lambda\widehat{\otimes}M \arrow[r, "h\otimes 1_M"] & C\widehat{\otimes}\Lambda\widehat{\otimes}M. \end{tikzcd}$$

But it is easy to check that h is the identity map of $A_*^\vee(A)$. Hence we obtain that $\rho_M=(1_C\otimes\rho_{\Lambda,M})\circ\rho_{C,M}$. This completes the proof. □

Definition 5.20 We say that a complete module M is a C–Λ–comodule if M is a C–comodule and also a Λ–comodule such that the structure map of Λ–comodule $\rho_{\Lambda,M}$ is a map of C–modules.

Corollary 5.21 *A complete* $A_*^\vee(A)$*–comodule has a natural* C*–*Λ*–comodule structure.*

Let $\rho_{\Lambda,C\widehat{\otimes}\Lambda}=(\pi_\Lambda\otimes 1\otimes 1)\circ\psi\colon C\widehat{\otimes}\Lambda\to\Lambda\widehat{\otimes}C\widehat{\otimes}\Lambda$.

Lemma 5.22 *The following diagram commutes:*

$$\begin{tikzcd} \Lambda \arrow[r, "\psi_\Lambda"] \arrow[d, "\rho_{C,\Lambda}"'] & \Lambda\widehat{\otimes}\Lambda \arrow[d, "1_\Lambda\otimes\rho_{C,\Lambda}"] \\ C\widehat{\otimes}\Lambda \arrow[r, "\rho_{\Lambda,C\widehat{\otimes}\Lambda}"] & \Lambda\widehat{\otimes}C\widehat{\otimes}\Lambda. \end{tikzcd}$$

Proof Note that this is a diagram of $\mathbf{F}_p$–algebras. So it is sufficient to show the equality $f(b_{(i)}) = g(b_{(i)})$ holds for all $0 \le i < n$, where $f = (1_\Lambda \otimes \rho_{C,\Lambda}) \circ \psi_\Lambda$ and $g = \rho_{\Lambda, C\otimes\Lambda} \circ \rho_{C,\Lambda}$. We easily obtain that

$$f(b_{(i)}) = b_{(i)} \otimes 1 \otimes 1 + 1 \otimes \sum_{j=0}^{i} \chi(s_{i-j})^{p^j} \otimes b_{(j)}.$$

On the other hand,

$$\begin{aligned} g(b_{(i)}) &= \sum_{j=0}^{i} (1 \otimes \chi(s_{i-j})^{p^j} \otimes 1) \cdot (1 \otimes 1 \otimes b_{(j)} + \sum_{k=0}^{j} b_{(k)} \otimes s_{j-k}^{p^k} \otimes 1) \\ &= \sum_{j=0}^{i} 1 \otimes \chi(s_{i-j})^{p^j} \otimes b_{(j)} + \sum_{k=0}^{i}\sum_{j=k}^{i} b_{(k)} \otimes \left(s_{j-k}\chi(s_{i-j})^{p^{j-k}} \right)^{p^k} \otimes 1 \\ &= \sum_{j=0}^{i} 1 \otimes \chi(s_{i-j})^{p^j} \otimes b_{(j)} + b_{(i)} \otimes 1 \otimes 1. \end{aligned}$$

This completes the proof. $\square$

Corollary 5.23 *If M is a complete C–comodule, then the following diagram commutes:*

(9)
$$\begin{tikzcd}
\Lambda \widehat{\otimes} M \arrow[r, "\psi_\Lambda \otimes 1_M"] \arrow[d, "\rho_{C,\Lambda\widehat{\otimes}M}"'] & \Lambda \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[d, "1_\Lambda \otimes \rho_{C,\Lambda\widehat{\otimes}M}"] \\
C \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[r, "\rho_{\Lambda,C\widehat{\otimes}\Lambda} \otimes 1_M"] & \Lambda \widehat{\otimes} C \widehat{\otimes} \Lambda \widehat{\otimes} M.
\end{tikzcd}$$

Lemma 5.24 *Let M be a complete C–Λ–comodule with C–comodule structure map $\rho_{C,M}\colon M \to C \widehat{\otimes} M$. Then the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "\rho_{\Lambda,M}"] \arrow[d, "\rho_{\Lambda,M}"] & \Lambda \widehat{\otimes} M \arrow[r, "\rho_{C,\Lambda\widehat{\otimes}M}"] \arrow[d, "1\otimes\rho_{\Lambda,M}"'] & C \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[d, "1\otimes1\otimes\rho_{\Lambda,M}"] \\
\Lambda \widehat{\otimes} M \arrow[r, "\psi_\Lambda\otimes 1"] \arrow[d, "\rho_{C,\Lambda\otimes M}"] & \Lambda \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[r, "\rho_{C,\Lambda\widehat{\otimes}\Lambda\widehat{\otimes}M}"] \arrow[d, "1\otimes\rho_{C,\Lambda\widehat{\otimes}M}"'] & C \widehat{\otimes} \Lambda \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[d, "1\otimes1\otimes\rho_{C,\Lambda\widehat{\otimes}M}"] \\
C \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[r, "\rho_{\lambda,C\widehat{\otimes}\Lambda}\otimes 1"] & \Lambda \widehat{\otimes} C \widehat{\otimes} \Lambda \widehat{\otimes} M \arrow[r, "\rho_{C,\Lambda\widehat{\otimes}C}\otimes1\otimes1"] & C \widehat{\otimes} \Lambda \widehat{\otimes} C \widehat{\otimes} \Lambda \widehat{\otimes} M
\end{tikzcd}$$

Proof The top left square commutes since M is a Λ–comodule. From the assumption that $\rho_{\Lambda,M}$ is a morphism of C–comodules, so is $1_\Lambda \otimes \rho_{\Lambda,M}$. Hence we see that the top right square commutes. The bottom left square commutes by Corollary 5.23. Since $\rho_{C,\Lambda\widehat{\otimes}M}$ is a morphism of C–comodules, so is $1_\Lambda \otimes \rho_{C,\Lambda\widehat{\otimes}M}$. Hence the bottom right square commutes. This completes the proof. □

Lemma 5.25 *The map* $(\rho_{C,\Lambda\widehat{\otimes}C} \otimes 1_\Lambda)\circ(\rho_{\Lambda,C\widehat{\otimes}\Lambda})$ *is the comultiplication* ψ.

Proof Let $f = (\rho_{C,\Lambda\widehat{\otimes}C} \otimes 1_\Lambda)\circ(\rho_{\Lambda,C\widehat{\otimes}\Lambda})$. Since f is a map of $\mathbf{F}_p$–algebras, it is sufficient to show that $f(c) = \psi(c)$ for all $c \in C$ and $f(b_{(i)}) = \psi(b_{(i)})$ for all $0 \le i < n$. It is easy to check that $f(c) = \psi(c)$. On the other hand,

$$\begin{aligned} f(b_{(i)}) &= \sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{i-j} \chi(s_{j-k})^{p^k} s_{i-j-l}^{p^j} \otimes b_{(k)} \otimes s_l^{p^{i-l}} \otimes 1 + 1\otimes 1\otimes 1\otimes b_{(i)} \\ &= \sum_{\substack{k,l\ge 0\\ k+l\le i}} \left(\sum_{j=k}^{i-l} \chi(s_{j-k}) s_{i-j-l}^{p^{j-k}}\right)^{p^k} \otimes b_{(k)} \otimes s_l^{p^{i-l}} + 1\otimes 1\otimes 1\otimes b_{(i)} \\ &= 1\otimes 1\otimes 1\otimes b_{(i)} + \sum_{k=0}^{i} 1\otimes b_{(k)} \otimes s_{i-k}^{p^k} \otimes 1. \end{aligned}$$

Hence $f(b_{(i)}) = \psi(b_{(i)})$. This completes the proof. □

Let M be a complete C–Λ–comodule with C–comodule structure map $\rho_{C,M}\colon M \to C\widehat{\otimes}M$. We define a map $\rho_M\colon M \to C\widehat{\otimes}\Lambda\widehat{\otimes}M$ by

$$\rho_M\colon M \xrightarrow{\rho_{\Lambda,M}} \Lambda\widehat{\otimes}M \xrightarrow{\rho_{C,\Lambda\widehat{\otimes}M}} C\widehat{\otimes}\Lambda\widehat{\otimes}M.$$

By Lemma 5.24 and Lemma 5.25, we see that ρ_M gives M a complete $A_*^\vee(A)$–comodule structure.

Proposition 5.26 *Let M be a complete C–Λ–comodule. Then M has a natural $A_*^\vee(A)$–comodule structure ρ_M such that the induced C–Λ–comodule structure coincides with the given one.*

Note that if M is a complete C–Λ–comodule obtained from a complete $A_*^\vee(A)$–comodule, then the induced $A_*^\vee(A)$–comodule structure coincides with the given one by Lemma 5.19.

By summarizing the results in this section, we obtain the following theorem.

Theorem 5.27 *There is an equivalence of symmetric monoidal categories between the category of complete $A^{\vee}_{*}(A)$–comodules and the category of complete C–Λ–comodules.*

6 Main theorem

6.1 Symmetric monoidal functor $\mathcal{F}$

In Proposition 4.22 we showed that $\mathcal{F}$ is a monoidal functor from the category of profinite C_{E_*}–precomodules to the category of profinite C_{K_*}–comodules. In this section we show that the functor $\mathcal{F}$ extends to a monoidal functor from the category of profinite $E^{\vee}_{*}(E)$–precomodules to the category of profinite $K_*(K)$–comodules.

We let M be a profinite $E^{\vee}_{*}(E)$–precomodule. Then M is a profinite C_{E_*}–precomodule and also a Λ_{E_*}–comodule such that the Λ_{E_*}–comodule structure map $\rho_M\colon M \to \Lambda_E \widehat{\otimes} M$ is a map of profinite C_{E_*}–precomodules by Corollary 5.21. We note that Λ_{E_*} is a C_{E_*}–precomodule and there is an isomorphism of C_{K_*}–comodules: $\mathcal{F}(\Lambda_{E_*}) \cong \Lambda_{K_*}$ by Theorem 5.14.

Lemma 6.1 *If M is a profinite C_{E_*}–precomodule, then the natural map*

$$\Lambda_{K_*} \widehat{\otimes}_{K_*} \mathcal{F}(M) \xrightarrow{\cong} \mathcal{F}(\Lambda_{E_*}) \widehat{\otimes}_{K_*} \mathcal{F}(M) \longrightarrow \mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} M)$$

is an isomorphism of C_{K_*}–comodules.

Proof Since $\mathcal{F}(\Lambda_{E_*})$ is isomorphic to Λ_{K_*} as a C_{K_*}–comodule, $\Lambda_{E_*} \widehat{\otimes}_{E_*} L_* \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} L_*$ as twisted L_*–$\mathcal{G}$–modules. Hence there are isomorphisms of twisted L_*–$\mathcal{G}$–modules:

$$\begin{aligned}
\Lambda_{E_*} \widehat{\otimes}_{E_*} M \widehat{\otimes}_{E_*} L_* &\cong \left(\Lambda_{E_*} \widehat{\otimes}_{E_*} L_*\right) \widehat{\otimes}_{L_*} \left(M \widehat{\otimes}_{E_*} L_*\right) \\
&\cong \left(\Lambda_{K_*} \widehat{\otimes}_{K_*} L_*\right) \widehat{\otimes}_{L_*} \left(M \widehat{\otimes}_{E_*} L_*\right) \\
&\cong \Lambda_{K_*} \widehat{\otimes}_{K_*} M \widehat{\otimes}_{E_*} L_*.
\end{aligned}$$

By taking S_{n+1}–invariant submodules, we obtain an isomorphism of twisted K_*–G_n–modules: $\mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} M) \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} \mathcal{F}(M)$. This implies that the above map is an isomorphism of profinite C_{K_*}–comodules. □

By Lemma 6.1, we obtain a map

$$\mathcal{F}(\psi_{\Lambda_{E_*}})\colon \Lambda_{K_*} \cong \mathcal{F}(\Lambda_{E_*}) \longrightarrow \mathcal{F}(\Lambda_{E_*} \widehat{\otimes}_{E_*} \Lambda_{E_*}) \cong \Lambda_{K_*} \widehat{\otimes}_{K_*} \Lambda_{K_*}.$$

Lemma 6.2 *The map $\mathcal{F}(\psi_{\Lambda_{E_*}})$ coincides with the comultiplication map $\psi_{\Lambda_{K_*}}$ on Λ_{K_*}.*

Proof This follows from the fact that the algebra generators $\widehat{b}^E_{(i)}$ of $\mathcal{F}(\Lambda_{E_*})$ are given by linear combinations of the algebra generators $b^E_{(i)}$ of Λ_{E_*} with coefficients in L_* (see Lemma 5.12). □

Corollary 6.3 *Let M be a profinite $E^\vee_*(E)$–precomodule with corresponding Λ_{E_*}–comodule structure map $\rho_M\colon M \to \Lambda_{E_*}\widehat{\otimes}_{E_*}M$. Then the map $\mathcal{F}(\rho_M)\colon \mathcal{F}(M) \to \mathcal{F}(\Lambda_{E_*}\widehat{\otimes}_{E_*}M) \cong \Lambda_{K_*}\widehat{\otimes}_{K_*}\mathcal{F}(M)$ defines a natural Λ_{K_*}–comodule structure on $\mathcal{F}(M)$.*

Proposition 6.4 *If M is a profinite $E^\vee_*(E)$–precomodule, then $\mathcal{F}(M)$ has a natural $K_*(K)$–comodule structure.*

Proof Since $\mathcal{F}$ is a functor and the Λ_{E_*}–comodule structure map $\rho_M\colon M \to \Lambda_E\widehat{\otimes}M$ is a map of C_{E_*}–precomodule, $\mathcal{F}(\rho_M)$ is a map of C_{K_*}–comodules. Hence the proposition follows from Theorem 5.27. □

Corollary 6.5 *$\mathcal{F}$ extends to a symmetric monoidal functor from the category of profinite $E^\vee_*(E)$–precomodules to the category of profinite $K_*(K)$–comodules.*

Proof By Proposition 6.4, we see that $\mathcal{F}$ extends to a functor from the category of profinite $E^\vee_*(E)$–precomodules to the category of profinite $K_*(K)$–comodules. It is easy to check that $\mathcal{F}$ respects the symmetric monoidal structures. □

6.2 Main theorem

In this section we prove the main theorem (Theorem 6.11). The theorem states that for any spectrum X, $\mathcal{F}(\mathbb{E}^*(X))$ is naturally isomorphic to $\mathbb{K}^*(X)$ as a cofiltered system of finitely generated discrete $K_*(K)$–comodules. Furthermore, if X is a space, then this equivalence respects the graded commutative ring structures.

Definition 6.6 Let $\mathcal{M}^f_{\mathbb{E}}$ (resp. $\mathcal{M}^f_{\mathbb{K}}$) be the category of finitely generated discrete $E^\vee_*(E)$–precomodules (resp. $K_*(K)$-(pre)comodules). We define $\mathcal{M}_{\mathbb{E}}$ (resp. $\mathcal{M}_{\mathbb{K}}$) to be the procategory of $\mathcal{M}^f_{\mathbb{E}}$ (resp. $\mathcal{M}^f_{\mathbb{K}}$).

By Corollary 6.5, we can extend the functor $\mathcal{F}$ from $\mathcal{M}_{\mathbb{E}}$ to $\mathcal{M}_{\mathbb{K}}$ by obvious way:

$$\mathcal{F}\colon \mathcal{M}_{\mathbb{E}} \longrightarrow \mathcal{M}_{\mathbb{K}}.$$

Note that $\mathcal{F}$ is a monoidal functor. As in the cases of $\mathcal{C}_{\mathbb{E}}$ and $\mathcal{C}_{\mathbb{K}}$, we have the following lemma.

Lemma 6.7 *For any spectrum X, $\mathbb{E}^*(X) \in \mathcal{M}_{\mathbb{E}}$ and $\mathbb{K}^*(X) \in \mathcal{M}_{\mathbb{K}}$.*

Hence $\mathcal{F}(\mathbb{E}^*(X)) \in \mathcal{M}_{\mathbb{K}}$. The natural Λ_{K_*}–comodule structure on $\mathcal{F}(\mathbb{E}^*(X))$ gives natural K_*–module homomorphisms $\widehat{Q}_i$ on $K^*(X) = \varprojlim \mathcal{F}(\mathbb{E}^*(X))$ for $0 \le i < n$ with respect to the algebra generators $b^K_{(i)}$ of Λ_{K_*}.

Lemma 6.8 *For $0 \le i < n$, $\widehat{Q}_i$ is a stable cohomology operation on $K^*(X)$.*

Proof It is sufficient to show that $\widehat{Q}_i$ commutes with the suspension isomorphism Σ. Let $s \in \widetilde{K}^1(S^1)$ the canonical generator $\Sigma(1)$. Then the suspension isomorphism is given by the (exterior) product with s. Since $\widehat{Q}_i$ is an odd degree operation, $\widehat{Q}_i$ acts on s trivially. Hence we see that $\widehat{Q}_i$ commutes with the product with s since $\widehat{Q}_i$ is a derivation. This completes the proof. □

Recall that $\mathcal{Y}$ is the lens space S^{2p^n-1}/C_p, and $K^*(\mathcal{Y}) = \Lambda(u_k) \otimes K_*[x_K]/(x_K^{p^n})$.

Lemma 6.9 *For $0 \le i < n$, $\widehat{Q}_i(u_K) = {x_K}^{p^i}$.*

Proof By Lemma 5.5, $\rho(y_E) = 1 \otimes y_E + b^E(x_E)$. Since $b_E(X) = \widehat{b}(\Phi(X))$ mod (X^{p^n}) by definition, we have $\rho(1 \otimes u_E) = 1 \otimes 1 \otimes u_E + \widehat{b}^E(\Phi(x_E))$. From that fact that $u_K = 1 \otimes u_E$ and $\Phi(x_E) = x_K$, we obtain that $\widehat{Q}_i(u_K) = {x_K}^{p^i}$. □

Corollary 6.10 *For $0 \le i < n$, $\widehat{Q}_i = Q_i^K$.*

Proof By Lemma 6.8, $\widehat{Q}_i$ is an odd degree stable cohomology operation, which is a derivation with respect to the (exterior) product. Hence $\widehat{Q}_i$ is characterized by the action on $u_K \in K^1(\mathcal{Y})$. Then the corollary follows from Lemma 6.9. □

Recall that the generalized Chern character (5)

$$\Theta\colon\ E^*(X) \longrightarrow L^*(X)$$

induces a natural isomorphism in $\mathcal{C}_{\mathbb{K}}$

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X)$$

by Theorem 4.25. The following is our main theorem of this note.

Theorem 6.11 *The generalized Chern character Θ induces a natural isomorphism in $\mathcal{M}_{\mathbb{K}}$:*

$$\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X).$$

If X is a space, then this is an isomorphism of cofiltered systems of finite $K_*(K)$–comodule algebras.

Proof By Theorem 4.25, there is a natural isomorphism $\mathcal{F}(\mathbb{E}^*(X)) \cong \mathbb{K}^*(X)$ in $\mathcal{C}_{\mathbb{K}}$. Corollary 6.10 implies that the isomorphism $\mathcal{F}(E^*(Z)) \cong K^*(Z)$ respects the Λ_{K_*}–comodule structures for all $Z \in \Lambda(X)$. Hence the theorem follows from Theorem 5.27. □

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